Quantum Zeno Dynamics

Supervised Learning Project

Reet Santosh Mhaske, 20D170032
Indian Institute of Technology, Bombay

Guide: Prof Sai Vinjanampaty
Department of Physics

Abstract

This report summarizes the learning outcomes of the supervised project I undertook in the autumn semester of the academic year 2022-2023. Aimed at studying the Quantum Zeno Dynamics as a technique to decouple a system from its environment, this report begins with the preliminary concepts in open systems theory. We then introduce the Quantum Zeno Effect, as a way of "freezing" system dynamics. We present its generalization to observe the partitioning of the Hilbert space, into what are called the Zeno Subspaces and the independent Quantum Zeno Dynamics within each subspace. We conclude with the application of QZD on a system-environment-meter model and study a specific condition, under which system -environment decoupling can be achieved.

Contents

List of Figures

1 Introduction

> "If everything, when it occupies an equal space, is at rest at that instant of time, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless at that instant of time and at the next instant of time but if both instants of time are taken as the same instant or continuous instant of time then it is in motion"
>
> - Aristotle, recounting Zeno's Arrow paradox

Thus stated, Zeno's Arrow paradox is not a paradox in our modern understanding of kinematics. While it intrigued Zeno's contemporaries to look for a resolution for the paradox, it served to be equally intriguing to the twentieth-century physicists E.C.G Sudarshan and B. Mishra. In their seminal work, they argued that if continuously monitored, a quantum mechanical system's state would not evolve. They also showed that "continuous monitoring" does possess operational meaning. This result was first presented as a Quantum analogue of Zeno's Paradox; the system's state 'arrow' would remain stationary, if one was to observe it continually. Lately, the 'paradox' has given way to what is now studied as the "Quantum Zeno Effect(Dynamics)". It has been treated as a characteristic of quantum mechanical evolution [5].

In this report, we visit this effect, with the aim of harnessing its potential to restrict the system dynamics. We begin with a brief introduction to Open systems theory, required for understanding the Quantum Zeno Effect. In sections 2 and 3, we introduce the Density matrix formalism and arrive and Liouville von-Neumann equation for density matrices. In sections 4,5 and 6 we first introduce the concept of two interacting systems, one called the 'system', the other *its* 'environment'. Realising that information can leak out of an 'open' system and that its evolution need not be unitary, we study the Operator-Sum Representation and Lindblad's mater equation in sections 7 and 8 respectively.
Having gained sufficient tools, we finally unveil the Quantum Zeno Effect with complete measurements in section 9. We extend the treatment to non-selective incomplete measurements in section 10. We describe the partitioning of the Hilbert space into Zeno subspaces in section 11. In section 12, we arrive at the main motive of this report, to harness Quantum Zeno Measurements to decouple a system from its environment. We observe that if the bases of the projection operators satisfy the condition of being mutually unbiased, then the interaction of the system with the environment can be cancelled. We then see if this decoupling is possible even if the said condition does not hold.
Throughout the report, I have tried to maintain the same chronology in which I learnt these topics, and have tried to elaborate on the niches I found while learning them.

2 Density Matrices

Consider a collection of spin 1/2 particles, half of them in spin up $|0\rangle$ and half of them in spin down $|1\rangle$(case A). How would one describe them? This is unlike a collection of particles, all in the superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (case B). Consider picking a particle at random from case A. The classical probability of finding it to be in state $|0\rangle$ is 0.5. Now picking up a random particle from case B, its state will always be $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, so the probability of measuring it in state $|0\rangle$ is also 0.5. It seems difficult to be able to distinguish these systems, probabilities of drawing out a particle and measuring it to be in state $|0\rangle$ (or $|1\rangle$) is 0.5 in both cases.
Suppose instead of measuring the system in state $\{|0\rangle, |1\rangle\}$ basis, we measure it in state $\{|+\rangle, |-\rangle\}$ basis. Now, in this bases, all particles in case B are in state $|+\rangle$. Case A is then a collection of particles, half of them in state $|+\rangle$, the other half in $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. A measurement of particles in case B will always give $|+\rangle$ with probability 1, whereas, the probability in case A is 0.5. Thus, for case B there exists a basis in which the measurement outcome is always 1, i.e., all particles of the collection are in that particular state. Whereas in case A, the particles may have different states, and such a collection of particles, each with different proportions is called an **ensemble mixture**.

Why restrict to a mixture of particles in only two possible states? Consider a large mixture in which the p_i fraction of particles can be in states $|\Psi_i\rangle$. How do we represent such a system?

We need a mathematical tool, compatible with Dirac's formalism, which also accounts for classical probability. Let's evaluate the expectation value of a Hermitian operator $\hat{A}$, on such a system, to see if it leads to something. The classical way to define such an expectation would be to sum, over all states Ψ, the expectation of $\hat{A}$ on Ψ "weighted" by the probability of a particle of the system to be in the state Ψ

$$\langle \hat{A} \rangle = \sum_i p_i \times \langle \Psi_i | \hat{A} | \Psi_i \rangle \tag{1}$$

Now, the classic trick, introduce $1 = \sum_k |k\rangle \langle k|$, identity operator as a summation over a complete bases set,
$\langle \hat{A} \rangle = \sum_i p_i \times \langle \Psi_i | \sum_k |k\rangle \langle k| \hat{A} | \Psi_i \rangle$
Rearranging a few terms, we obtain,

$$\langle \hat{A} \rangle = \sum_k \langle k| \hat{A} \sum_i p_i |\Psi_i\rangle \langle \Psi_i | k \rangle \tag{2}$$

Now, if we define

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i| \tag{3}$$

then it follows that

$$\langle \hat{A} \rangle = \sum_k \langle k| \hat{A}\rho |k\rangle = Tr(\hat{A}\rho) \tag{4}$$

where, $Tr(X)$, is the trace of the matrix X (since we are picking the diagonal elements $\langle k| X |k\rangle$ and summing over them)

We now have an expression for the expectation value of $\hat{A}$ on such a system. **We call ρ the density matrix representing the system.** It encodes which of the available states in the Hilbert space are populated and how densely so, because of the weights p_i, multiply the outer product of each state.

Now, because of this interpretation of density matrices, there arise some conditions for a matrix to be a valid density matrix.

1. $\rho^\dagger = \rho$, this is seen by taking the conjugate of Eq(2).
2. $Tr(\rho) = 1$, as the diagonal elements represent the probability of that state(*in the appropriate basis), and probabilities add to 1.
3. $\rho \geq 0$, i.e the eigenvalues of ρ are $\geq$ to 0. Intuitively, in the correct basis again, the eigenvalues are the diagonal elements, representing probabilities, and they can't be negative.

How do we know, by looking at a density matrix, how "mixed" is the mixture? Consider the correct basis*(the one in which the density matrix is diagonal; it need not be diagonal in every basis!). If the density matrix is not a mixture, and the all particles in the system are in only one state, there will only be one non-zero diagonal entry, which would be 1. If we then square such a diagonal matrix and take its trace, $Tr(\rho^2) = 1$. We call such states "pure" states. But had this been an impure "mixed" state, then in the diagonal basis, more than one of the diagonal entries are non-zero, and all of them must be less than 1 (because they need to add up to 1). Now, if we square such a matrix, and take its trace, then $Tr(\rho)^2 < 1$. (Why? Because say in a 2×2 case, $Tr(\rho) = a + b = 1$, and ,in diagonal basis$Tr(\rho^2) = a^2 + b^2$ then $a^2 + b^2 < (a + b)^2 = 1$).
Thus $Tr(\rho)^2$ **offers a way of measuring the purity of the system**.

Arguments on density matrices presented here and in the following sections, though not completely based on, are inspired from [2] and from recorded lectures of the PH534, IIT Bombay Spring 2019 course.

3 Evolution of a mixed system

Density matrices denote the most general way of describing any quantum system. Unlike the isolated H atom or a particle in a box, we can now use this treatment to study system-environment interaction, how a system of interest evolves when

coupled to an external environment (whose physical observables cannot usually be measured).

Now, let's find the ensemble mixture analogue for Schrodinger's equation. For simplicity, we consider a pure state, first. We set $\hbar = 1$ throughout this report.

$$\rho(t) = |\Psi(t)\rangle \langle\Psi(t)| \tag{5}$$

Using Schrodinger's equation,

$$|\psi(t)\rangle = -i\hat{H}(t) |\Psi(t)\rangle \tag{6}$$

If we take the derivative of Eq(4), and use Eq(5), we get

$$\boxed{\dot{\rho}(t) = -i[\hat{H}(t), \rho(t)]} \tag{7}$$

This is Liouville - von Neumann's equation, the analogue of Schrodinger's equation for a system.

The case for a mixed stat follows since we know that commutators have additivity of their arguments, and Eq(6) holds.

Now, for a initial density matrix $\rho(0) = \sum_i p_i |\psi_i(0)\rangle \langle\psi_i(0)|$ say each state of the system evolves unitarily under $U(t)$,i.e $|\psi_i(0)\rangle \longrightarrow U(t) |\psi_i(0)\rangle$, the density matrix at time t would naturally be $\rho(t) = \sum_i p_i U(t) |\psi_i(0)\rangle \langle\psi_i(0)| U^\dagger(t)$ or

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \tag{8}$$

A unitary evolution of the system states is incorporated by the left(right) multiplication of the density matrix with the unitary operator(its dagger).

A short word on higher dimensions. we can map a space of 2 density matrices to a space of 4 d column vectors, just by placing the elements of the density matrix, one below the other. In that case, we are in a new space, the density matrix given by $|\rho\rangle\rangle$, We then define the superoperator $\mathcal{H}(.) = [\hat{H}, .]$, This makes equation 6,

$$|\dot{\rho}\rangle\rangle = -i\mathcal{H} |\rho\rangle\rangle \tag{9}$$

with solution

$$|\rho(t)\rangle\rangle = e^{-i\mathcal{H}t} |\rho(0)\rangle\rangle \tag{10}$$

This evolution is in the superoperator space, and we will resort to it once we get to the Quantum Zeno Effect.

4 Representing more than one system

Consider two pure state systems A and B in state $|A\rangle$ and $|B\rangle$. We represent the state of the total system by as $|A\rangle |B\rangle$. There are some general properties to which we resort,

1. $(|V_1\rangle + |V_2\rangle) \otimes |W\rangle = |V_1\rangle \otimes |W\rangle + |V_2\rangle \otimes |W\rangle$
2. $(\hat{A} \otimes \hat{B})(|a\rangle \otimes |b\rangle) = \hat{A}\,|a\rangle \otimes \hat{B}\,|b\rangle$

Similar properties follow for a tensor product of density matrices. Consider two systems A and B, each represented by the density matrix ρ_A and ρ_B. The way to define the total state of the system A + B is by defining the tensor product $\rho_A \otimes \rho_B$. If $\hat{A}$ and $\hat{B}$ are operators for for respective systems A and B, we then the operation of $\hat{A} \otimes \hat{B}$ on $\rho_A \otimes \rho_B$ is same as $\hat{A}\rho_A \otimes \hat{B}\rho_B$. We can generalize the tensor product formalism for more than two systems, with the above properties getting extended appropriately. We have enough mathematical tools now, to discuss the simultaneous evolution of two systems.

5 Two Non-interacting systems

Suppose we have two systems A and B, each with independent Hamiltonians, $H_A$and H_B. The total Hamiltonians for the $A \otimes B$ system will be

$$H_0 = H_A \otimes 1 + 1 \otimes H_B \tag{11}$$

This is like acting H_A on A, and 1 on B, and 1 on A, and B on B; both evolutions don't affect the other system. This is unlike $H_A \otimes H_B$, as if $H_A \otimes H_B$ acts on this system, H_A and H_B would both act on A and B at the same time, making their evolution correlated.

Now, the total system will have the evolution operator

$$U(t) = e^{-iH_A \otimes 1 t - i1 \otimes H_B t} \tag{12}$$

Now, we use the Baker-Campbell Housdroff formula, i.e.
$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}}$ (if A and B have a constant commutator)

We use an additional relation

$$[A \otimes B, C \otimes D] = [A, C] \otimes \frac{1}{2}\{B, D\} + \frac{1}{2}\{A, C\} \otimes [B, D] \tag{13}$$

Now we can evaluate the commutator, to see that

$$[H_A \otimes 1, 1 \otimes H_B] = 0 \tag{14}$$

Now, consider,

$$e^{-i1 \otimes H_B t} = 1 - i(1 \otimes H_B t) - \frac{1}{2} 1 \otimes H_B^2 t^2 + ... = 1 \otimes e^{-iH_B t} \tag{15}$$

So, we end up having

$$e^{-iH_A \otimes 1 t - i1 \otimes H_B t} = e^{-iH_A t} \otimes e^{-iH_B t} \tag{16}$$

We observe that the total evolution of the non-interacting system-environment splits into the tensor product of the two individual evolution operators. Thus, the system and environment are evolving independently of each other and indeed non-interacting.

6 Partial Traces: Ignoring what the environment does

In the setup above, A can be thought of as the system of interest, the one whose observables can be measured; B is the environment that interacts with the system. Unlike the above case, however, if A and B interact then they add additional interaction terms to the Hamiltonian, of the form $\hat{A} \otimes \hat{B}$, where $\hat{A}, \hat{B}$ are the respective operators belonging to the Hilbert spaces of A and B. The most general interaction can be modelled as the sum of different such interaction terms i.e. the interaction Hamiltonian will be

$$H_I = \sum_k \hat{A}_k \otimes \hat{B}_k \tag{17}$$

Now, the total system (S) and environment (E) will evolve according to the total Hamiltonian $H_0 + H_I$. We are interested in the effect that such an evolution has only on the system. We need a way of extracting the systems density matrix, from the total S-E density matrix. Now, given ρ_S and ρ_E, the density matrices of S and E, it is easy to construct the total density matrix, but the reverse is not true, as different S and E matrices could result in the same SE matrix. Let's look at the structure of the tensor product, to see if that helps.

Let S be the system's density matrix ($m \times m$) and E be the environment's density matrix ($n \times n$). Then,

$$S \otimes E = \begin{bmatrix} S_{11}[E]_{n\times n} & S_{12}[E]_{n\times n} & S_{12}[E]_{n\times n} & ... \\ S_{21}[E]_{n\times n} & S_{22}[E]_{n\times n} & S_{23}[E]_{n\times n} & ... \\ ... & ... & ... & ... \end{bmatrix}_{m\times m}$$

$$S \otimes E = \begin{bmatrix} S_{11}E_{11} & S_{11}E_{12} & S_{11}E_{13} & ... & S_{12}E_{11} & S_{12}E_{12} & S_{12}E_{13} & ... \\ S_{11}E_{21} & S_{11}E_{22} & S_{11}E_{23} & ... & S_{12}E_{11} & S_{12}E_{12} & S_{12}E_{13} & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ S_{21}E_{11} & S_{21}E_{12} & S_{21}E_{13} & ... & S_{21}E_{11} & S_{21}E_{12} & S_{12}E_{13} & ... \end{bmatrix}_{mn\times mn}$$

Now, if we define the operation Tr_E the **partial trace** over environmental degrees of freedom. We look at the first $n \times n$ block, trace it, and replace the entire block with this trace. We move on to the next $n \times n$ block and do the same. The idea is that from the matrix above, the trace of the first $n \times n$ block would simply be $Tr(E)$, this by property of density matrices is 1. Thus, proceeding in this way, we get the

reduced density matrix S. Since we have the complete form of the matrix visible, it is easy to see that if we start from $S(0) \otimes E(0)$, and evolve the system under the action of some evolution operator generated by an interaction Hamiltonian, it mixes the terms, say for example $S_{11}E_{11}$ with $S_{12}E_{13}$. So even the reduced density matrix obtained by partial tracing will have effects because of cross-talking terms.

We can similarly define partial trace over the environment, Tr_E where we look at the $n \times n$ blocks lying along the diagonal on the $mn \times mn$ matrix and then add these $n \times n$ matrices (in a way, taking the trace, with each element being a nn matrix. We will end up with a $n \times n$ matrix, which is the reduced density matrix for the environment. Again referring to the above matrix, we expect this to give $Tr(S)[E]$, but again $Tr(S) = 1$, so we have E.

In a nutshell, to ignore the environment, starting from the S and E density matrices, (say initially uncorrelated), we construct the total SE density matrix, evolve it under the total Hamiltonian, and then trace out the environment degrees of freedom, to obtain the net effect of this evolution on the system.

We comment that the reduced matrices obtained by partial tracing may not reconstruct the total SE matrix. If ρ_{SE} is the total S-E density matrix, ρ_S and rho_E are the reduced density matrices obtained by partial tracing, then $\rho_{SE} \neq \rho_S \times rho_E$ in general. Hence, it is wise to define a **correlation matrix** χ_{SE}, given by $\chi_{SE} = \rho_{SE} - \rho_S \times \rho_E$. If $\chi_{SE} = 0$, then the system and environment are currently uncorrelated, else there exists a correlation.
Mathematically, the partial trace is given as

$$Tr_E(\rho_{SE}) = \sum_k \langle E_k | \rho_{SE} | E_k \rangle \tag{18}$$

where $|E_k\rangle$ are the basis vectors for environment. Similarly, if $|S_k\rangle$ denote the basis states for system, partial trace over the system is given by

$$Tr_S(\rho_{SE}) = \sum_k \langle S_k | \rho_{SE} | S_k \rangle \tag{19}$$

7 Discrete Time Evolution: Operator Sum Representation

Consider the system $\rho_S = \rho_S(0)$, initially un-entangled with $\rho_E = \sum_n \mu_n |\mu_n\rangle \langle \mu_n|$. We then evolve $\rho_{SE} = \rho_S \otimes \rho_E$ under a unitary evolution $U(t)$ operated generated by the total Hamiltonian $H_0 + H_I$.
The total state state is given by $U(t)\rho_{SE}U^\dagger(t)$. To observe the effect of this evolution on the system, we partial trace using Tr_E So, the observable effect that the environment has on the system is incorporated in $\rho_S(t)$, which is given as

$$\rho_{SE}(t) = U(t)(\rho_S(0) \otimes \rho_E)U^\dagger(t)$$

$$\rho_{SE}(t) = U_{SE}(t)(\ \rho_S(0) \otimes \sum_n \mu_n \left|\mu_n\right\rangle \left\langle\mu_n\right|\ )U_{SE}^{\dagger}(t)$$

Now, say V_n^E is the operator that transforms $|0\rangle$ to $|\mu_n\rangle$, i.e. $|\mu_n\rangle = V_n^E |0\rangle$, $\forall \mu_n$

$$\rho_{SE}(t) = U_{SE}(t)(\ \rho_S(0) \otimes \sum_n \mu_n V_n^E \left|0\right\rangle \left\langle 0\right| V_n^{E\dagger}\ )U_{SE}^{\dagger}(t)$$

Now, we can extract out V_n^E, as $V_n^E \equiv 1 \otimes V_n^E$, so

$$\rho_{SE}(t) = \sum_n U_{SE}(t)\mu_n(1 \otimes V_n^E)(\ \rho_S(0) \otimes \left|0\right\rangle \left\langle 0\right| (1 \otimes V_n^{E\dagger})U_{SE}^{\dagger}(t)\ )$$

Now, $U_{SE}(t)(1 \otimes V_n^E)$ is a SE operator, i.e. it acts on the Hilbert space of $S \otimes E$. Let's call this

$$Z_n^{SE} = U_{SE}(t)(1 \otimes V_n^E) \tag{20}$$

So,

$$\rho_{SE}(t) = \sum_n Z_n^{SE}(\rho_S(0) \otimes \left|0\right\rangle \left\langle 0\right| Z_n^{SE\dagger})$$

We can finally take the partail Trace Tr_E:

$$\rho_S(t) = Tr_E(\rho_{SE}(t)) = Tr_E(\sum_n Z_n^{SE}(\rho_S(0) \otimes \left|0\right\rangle \left\langle 0\right| Z_n^{SE\dagger}))$$

$$\rho_S(t) = \sum_k \left\langle k^E\right| \sum_n Z_n^{SE}\ (\rho_S(0) \otimes \left|0\right\rangle \left\langle 0\right| Z_n^{SE\dagger})\ \left|k^E\right\rangle$$

where $\left|k^E\right\rangle$ are the basis vectors for the environment E

$$\rho_S(t) = \sum_k \sum_n \left\langle k^E\right| Z_n^{SE} \left|0\right\rangle^E)\ \rho_S(0)\ (\left\langle 0\right|^E Z_n^{SE\dagger} \left|k^E\right\rangle)$$

Now, $\left\langle k^E\right| Z_n^{SE} \left|0\right\rangle)$, is a system (S) operator. How? Let density matrices of S be of $s \times s$ dimensions and E be $n \times n$ dimensions. Then total density matrix is $(sn) \times (sn)$ dimensional. Now Z^{SE} is a $(s+n) \times (s+n)$ dimensional matrix, $\left|k\right\rangle^E Z^{SE} \left|0\right\rangle^E$ is *like* the partial trace element (without the summation), as if we are looking at the $(k,0)^{th}$ $s \times s$ block of the Z^{SE} matrix. Hence, $\left|k\right\rangle^E Z^{SE} \left|0\right\rangle^E$ is a $s \times s$ matrix , an operator for S.

having justified that it is an operator on the system alone, we can write $M^k(n) = \left\langle k\right|^E Z^{SE} \left|0\right\rangle^E$, so the above expression reduces to,

$$\rho_S(t) = \sum_k \sum_n M_k(n)\ \rho_S(0)\ M_k^{\dagger}(n)$$

Now, zoom out and look at the form, $M^k(n)$ are just operators, whose arguments just serve the purpose of labelling the operators. We may as well label them using some other single index, that runs over a range that accounts for all of these operators. Let's call this k. So we have arrived at the form

$$\boxed{\rho_S(t) = \sum_k M_k \; \rho_S(0) \; M_k^\dagger} \tag{21}$$

What have we arrived at? We have obtained an expression for the system state ρ after being subjected to the system-environment evolution, in terms of the *operators of the system alone*, the operators accounting for all the changes the environment could bring about in the system. This is known as the **Operator Sum Representation**, where M_k are called the Sudarshan or Kraus Operators. We note that to represent the evolution of a system in a d-dimensional Hilbert space using the operation sum representation, we need at most d^2 operators.

7.1 Some important assumptions :

We have made some subtle assumptions while arriving at this form of time evolution. These have some consequences.

1. The system and environment were initially uncorrelated.i.e $\chi_{SE} = 0$, as this enabled us to write the initial ρ_{SE} as tensor product of ρ_S and ρ_E. This fact helped us separate the dynamics of the environment from that of the system. Or else we should have something like $\rho_1^S \otimes \rho_1^E + \rho_2^S \otimes \rho_2^S$ to start with.

2. We have evolved the system for specific time $'t_0'$, found $\rho(t_0)$ and the found maps encode this evolution for the very specific time t_0. A particular set of Sudarshan-Kraus operators tells us of an initially uncorrelated system will look after time t_0. It does not tell us how it shall look at any other time t_1. The evolution of the system to time t_1 will be given by a different set of S-K operators.

3. We can't use S-K operators iteratively. We had assumed that ρ_S is initially uncorrelated, at time t_0, it may have become correlated with the environment, hence we **cannot** find $\rho(2t_0)$ from $\rho(t_0)$, or for that matter from $\rho(0)$, using the same S-K operators we used to find $\rho(t_0)$ from $\rho(0)$.

Now, this operation, of taking a density matrix $\rho(0)$ and mapping it to a fixed later time $\rho(t_0)$, is called a **Quantum Channel** or a **Completely Positive Trace Preserving (CPTP) Map**. CPTP maps can represent the non-unitary evolution of the system. A CPTP map Φ has the representation $\Phi(\rho_0) = \sum_k M^k \; \rho_S(0) \; M^k$.

7.2 Completeness of S-K operators

We need the CPTP map Φ given by the S-K operators to be trace-preserving. This implies,

$$Tr(\Phi(\rho)) = Tr(\rho)$$

$$Tr(\sum_k M_k(\rho)M_k^\dagger) = Tr(\rho) = Tr(\sum_k M_k^\dagger M_k(\rho))$$

This can hold only if

$$\sum_k M_k^\dagger M_k = 1$$

This is the completeness relation of Kraus operators. We'll use it to arrive at Lindblad's Master equation later.

7.3 Irreversibilty of CPTP maps

A CPTP map can account for decoherent processes, processes in which the system loses its purity i.e. $Tr(\rho^2)$ decreases. A general CPTP map may **not be time-reversal**. To see why, as mentioned above, a generic CPTP map may decrease the purity of the system. Now, here is an additional fact: $1 - Tr(\rho^2)$ is a measure of the entropy of the system. Hence, if a general map is introducing decoherence into the system, its time-reversed counterpart would amount to increasing $Tr(\rho)^2$ or decreasing the entropy of the system, which won't be compatible with the second law of thermodynamics. Another way to look at this is via the Bloch sphere. A pure state can be denoted by a vector originating at its centre and extending up to the surface of the Bloch sphere [2]. On the other hand, a mixed state is a vector with a length less than 1, hence the tip of this vector lies within the Bloch sphere [2]. As argued above, a CPTP can map a pure state to a mixed state, i.e. it can "*shrink*" the Bloch sphere. But it can't take a mixed state to a pure state, for then it would have to "expand" the Bloch sphere, but what would the surface, the pure states of the system map to then? We realize that expanding the Bloch sphere is not a valid operation.

7.4 Complete Positivity of CPTP maps

The "output" of a CPTP map is a density matrix, thus the map preserves trace (**T**race **P**reserving). Now, for the map to be a valid map, it should be positive i.e. if its input is positive, its output should be positive. We remember from the property of density matrices, that a matrix is called positive if its eigenvalues are positive semi-definite. These two conditions justify "trace-preserving" and "positive", What about "completely"? Now, if to our system, we add a few ancillary

redundant systems say α, which don't interact with any other system, nor with the environment. We can call our total system to then be $(S \otimes \alpha) \otimes E$. Now since the α doesn't interact with S or E, an evolution of $S \otimes \alpha$ under the CPTP map will be given as $\Phi \otimes 1$, i.e. act Φ on the system and 1 on the ancillary parts. Now, since we have not changed anything fundamentally, $\Phi \otimes 1$ should also be a positive map. In fact, we can add on many such ancillary parts. So, $\Phi \otimes 1^n$ should be positive for any $n \geq 1$. If Φ does satisfy such a property, it is called completely positive.

An intuitive argument for why the S-K Operators take this form is as follows. We have seen that if we have a density matrix ρ evolving unitarily, under $U(t)$, the density matrix after a late time would be $U(t)\rho U^\dagger(t)$. Now a CPTP map need not be unitary. A simple way to account for non-unitarity is to say that the system may evolve under the system's unitary U_1, or under U_2, with some probabilities. Such an evolution would look like $p_1\ U_1(t)\rho U_1^\dagger(t)\ +\ p_1\ U_2(t)\rho U_2^\dagger(t) + ...$ This resembles the operator sum form closely. Of course, this is not formal, since we have not demanded that the S-K operators be unitary.

To summarize, CPTP maps describe the stroboscopic evolution of the system. They don't describe how the system evolves in continuous time.Given a $\rho(0)$,it simply outputs $\rho(t_0)$ We can't comment on what happened during t_0.

8 Continuous Time Evolution: Lindblad's Master equation

Unlike the S-K operators, our goal now is to derive a differential equation which governs the system evolution, under the presence of the environment, by which we can find the state of the system at any instance of time.

We have seen that the evolution of S may not be unitary though that of SE (the total universe) is. We are looking for a "Markovian" differential equation, a relation that would give $\rho(t + dt)$, as a function of $\rho(t)$ alone, and not the previous states of the system. Such a relation may not always be possible. This is because the system is losing information in the environment, at after some time, it is possible that it returns to the system. In general, the state of the system may depend on the previous history of the system. We are interested in a Markovian process, one in which the system doesn't have "memory" about its previous states. There are certain assumptions under which the evolution can be approximated to be Markovian. To arrive at the desired form, we will for now assume it indeed is. The steps presented here are as followed in [4]. A more mathematically detailed proof was studied using [6].

This means that we can use the same S-K operators to continuously evolve the system. That is , we can have $\rho(t_0) = \Phi(\rho(0)$, $\rho(2t_0) = \Phi(\rho(t_0)$, and so on. This can be achieved by evolving the system by Φ, then replacing the environment with a new one, so that our system's state is uncorrelated with the environment.

We look at the S-K operators for infinitesimal evolution dt, we call the corresponding CPTP map ϵ_{dt}

$$\rho(t+dt) = \epsilon_{dt}(\rho(t))$$

Up to first order in dt, we can have $\epsilon_{dt} = 1 + dt\mathcal{L}$

$$\rho(t+dt) = (1+dt\mathcal{L})\rho(t)$$

Taking the limit $dt \longrightarrow 0$,

$$\dot{\rho}(t) = \mathcal{L}\rho(t)$$

$\mathcal{L}$ is called the **Liouvillian** or **Lindbladian**. Our aim now is to find a general form of $\mathcal{L}$. Had $\mathcal{L}$ been time-independent, the solution would have been (with $dt = \frac{t}{n}$)

$$\rho(t) = \lim_{n\to\infty}(1+\frac{t}{n}\mathcal{L})^n\rho(0) = e^{\mathcal{L}t}\rho(0)$$

Now, assume the operator sum representation $\epsilon_{dt}(\rho(0)) = \sum_k M_k\rho(0)M_k^\dagger$

We are interested in terms upto $\mathcal{O}(dt)$, such that $\rho(t+dt) = \rho(t) + \mathcal{O}(dt)$ We can demand M_k to have a particular form, which fits into this construct, without loss of generality.
We assume $M_0 = 1 + dt\hat{O}$, so that when it acts on $\rho(t)$, from left and right, up to first order, it produces the $\mathcal{O}(1)$ term which is $\rho(t)$, along with two other $\mathcal{O}(dt)$ terms. $\hat{O}$ can be any matrix. We can write it as a sum of Hermitian and non-Hermitian parts, as $\hat{O} = -i\hat{H} + \hat{K}$, where $\hat{H}$ and $\hat{K}$ are both Hermitian. (Note that we never demanded that an S-K operator be Hermitian).Now, we demand that other operators $M_k, k \neq 0$ must contribute only to $\mathcal{O}(dt)$ terms when acting from left and right, hence we can have $M_k = \sqrt{dt}L_k$. Now we use the completeness relation of density matrices,

$$\sum_k M_k^\dagger M_k = 1$$

Put M_0 and M_k of the form discussed above

$$(1+i\hat{H}dt + dt\hat{K})(1 - idt\hat{H} + dt\hat{K}) + \sum_{k\geq 1} M_k^\dagger M_k = 1$$

$$1 + 2dt\hat{K} + \sum_{k\geq 1} L_k^\dagger L_k(\sqrt{dt})^2 = 1$$

$$\hat{K} = \frac{-1}{2}\sum_{k\geq 1} L_k^\dagger L_k$$

So the Lindbladinan $\mathcal{L}$ can be found as $\epsilon_{dt}(\rho) - \rho$:

$$\mathcal{L}(\rho) = (1 - i\hat{H}dt + dt\hat{K})\rho(1 + idt\hat{H} + dt\hat{K}) + \sum_{k\geq 1} L_k^\dagger\rho L_k - \rho$$

$$\mathcal{L}(\rho) = -i[\hat{H}, \rho] - +\frac{1}{2}\rho \sum_{k\geq 1} L_k^\dagger L_k - \frac{1}{2}\sum_{k\geq 1} L_k^\dagger L_k \rho + \sum_{k\geq 1} L_k^\dagger \rho L_k$$

$$\boxed{\dot{\rho} = \mathcal{L}(\rho) = -i[\hat{H}, \rho] + \sum_k \Gamma_k \big(\frac{-1}{2}\{\rho, L_k^\dagger L_k\} + L_k^\dagger \rho L_k\big)}$$

This is the **Lindblad-Gorini-Sudarshan-Kossakowski Master equation**.

We compare this with von Neumann's equation for the evolution of a single system. We see that we have additional terms to the normal Hamiltonian. These are phenomenologically added damping terms, with Γ_k acting as the damping coefficients, which account for all the microscopic details of how the system interacts with the environment.

9 Quantum Zeno Effect

Having introduced the general formalism required necessary, we now motivate our way to the Quantum Zeno Effect.

For now, we look at an isolated system, without an environmental interaction, evolving under the action of a "well-behaved" Hamiltonian H. A state $|\Psi_0\rangle$ evolves under its action for a time $]\frac{t}{N}$. The probability that after time $\frac{t}{N}$, the system will still remain in state $|\rangle\, \Psi_0$ is given by

$$p(t) = ||\, \langle\Psi_0|\, e^{-iH\frac{t}{N}}\, |\Psi_0\rangle\, ||^2 \tag{22}$$

Now, if N is large enough, such that $\frac{t}{N} << 1$, we can Taylor expand the evolution operator $e^{-iH\frac{t}{N}}$, as $1 - iH\frac{t}{N} - H^2\frac{t^2}{N^2} + ...$, so we have:

$$p(\frac{t}{N}) = ||\, \langle\Psi_0|\, 1 - iH\frac{t}{N} - H^2\frac{t^2}{N^2}\, |\Psi_0\rangle\, ||^2$$

$$p(\frac{t}{N}) = \langle\Psi_0|\, 1 - iH\frac{t}{N} - \frac{1}{2}H^2\frac{t^2}{N^2}\, |\Psi_0\rangle\, \langle\Psi_0|\, 1 + iH\frac{t}{N} - \frac{1}{2}H^2\frac{t^2}{N^2}\, |\Psi_0\rangle$$

$$p(\frac{t}{N}) = (1 - i\, \langle\Psi_0|\, H\, |\Psi_0\rangle\, \frac{t}{N} - \langle\Psi_0|\, \frac{1}{2}H^2\, |\Psi_0\rangle\, \frac{t^2}{N^2})(1 + i\, \langle\Psi_0|\, H\, |\Psi_0\rangle\, \frac{t}{N} - \langle\Psi_0|\, \frac{1}{2}H^2\, |\Psi_0\rangle\, \frac{t^2}{N^2})$$

Keeping only terms up to order $\mathcal{O}(\frac{t^2}{N^2})$,

$$p(\frac{t}{N}) = 1 + (\langle\Psi_0|\, H\, |\Psi_0\rangle\, \frac{t}{N})^2 - \langle\Psi_0|\, H^2\, |\Psi_0\rangle\, \frac{t^2}{N^2}$$

So, we can write this as

$$\boxed{p(\frac{t}{N}) = 1 - \tau_Z^{-2}\frac{t^2}{N^2}} \tag{23}$$

where

$$\tau_Z^{-2} = \langle\Psi_0| H^2 |\Psi_0\rangle - \langle\Psi_0| H |\Psi_0\rangle^2 \tag{24}$$

τ_Z is called the Zeno Time scale. Intuitively for $t << \tau_Z$, our Taylor series expansion may hold.

The Hamiltonian must be "well-behaved", it should be bounded within the time $\frac{t}{N}$, for the expansion to hold. The convergence of the expansion series is also important. We ignore cases where higher-order terms play a role in this short-time evolution.

Now, after this measurement, the wave function collapses to $|\Psi_0\rangle$. The probability to find it in the state $|\Psi_0\rangle$, after time $2\frac{t}{N}$, is again $1 - \tau_Z^{-2}\frac{t^2}{N^2}$, since it is as if the dynamics have started anew.

We can what is the probability of finding the particle in $|\Psi_0\rangle$, after a time t, after N such measurements have been done.

$$p(t) = \lim_{N\longrightarrow\infty}(1 - \tau_Z^{-2}\frac{t^2}{N^2})^N \sim e^{-\frac{t^2}{N^2\tau_Z^2}} \xrightarrow{\frac{t}{N}<<1} 1$$

Thus we see that under the limit that the measurement is performed very rapidly and frequently. This limit is termed " the limit of continuous observation". This may sound like a mathematical abstraction, but for sufficiently large N, the system evolution has been observed to slow down.

Note: We are perturbing the system by the measurement at $\frac{t}{N}$. We assume at this is an infinitesimal duration measurement, i.e., instantaneous. But during the evolution period $\frac{t}{N}$, there is no such perturbation.

9.1 Measurement operators

We note that the measurement operators referred to above were the familiar measurement or Neumann projection operators of type $|\Phi_0\rangle\langle\Phi_0|$. We assume that the spectrum of projection operators is non-degenerate so that the measurement is **"complete"** or **selective**, i.e., given the outcome of the measurement, the state of the system is uniquely determined. We treat a more general measurement later.

9.2 Why does the probability limit tend to one

The fact that continuous observation "freezes" the system evolution is usually associated with the evolution characteristic in a short-time range. As we increase N, the time period between two measurements falls as $\mathcal{O}(\frac{1}{N})$, whereas from eq(23), we see that the decrease in probability falls as $\mathcal{O}(\frac{1}{N^2})$, i.e, faster than the change of state.

10 When the projection is finite dimensional: Quantum Zeno Dynamics

We look at what happens if the measurement operation, unlike the previous section, is "incomplete". If the projection operator has degenerate eigenvalues, then on the outcome corresponding to that eigenvalue, we can't assert that the system is in one of the corresponding eigenstates. But for sure, it is in the states in that subspace of eigenstates (superposition states included).

Here we consider an isolated system, without an environment, evolving under the Hamiltonian H. Now, suppose the projection operator P describes the measurement and projects the system into a subspace $\mathcal{H}_P$. Suppose P does not commute with Hamiltonian H. Now, if $[P, H] \neq 0$, then evolution caused by H, would lead a system, which starts out in $\mathcal{H}_P$, to evolve out of it. If $\mathcal{H}$ is the system's total Hilbert space, $\mathcal{H}_P = P\mathcal{H}$ is the subspace corresponding to the projector P. The dimension of $\mathcal{H}_P$ is less than that of $\mathcal{H}$.

Suppose we start of with a state in $\mathcal{H}_P$, say ρ_0, such that $P\rho_0 P = \rho_0$. (the projection of ρ_0, already in $\mathcal{H}_P$, is ρ_0 itself.) The probability of measuring ρ_0 in $(H)_P$ is given as $Tr(\rho_0 P) = 1$.

Now, under the evolution operator U generated by H and after time τ, the system state will be $U(\tau)\rho_0 U(\tau)$. We then perform an instantaneous projection P, so that up to a normalization factor, if the outcome does correspond to P, the state at $\rho(\tau)$ will be denoted by

$$\rho(\tau) = PU(\tau)\rho_0 U^\dagger P = PU(\tau)P\rho_0 PU^\dagger(\tau)P$$

We let $V(\tau) = PU(\tau)P$, so that

$$\rho(\tau) = V(\tau)\rho_0 V^\dagger(\tau) \tag{25}$$

The probability of measuring $rho(\tau)$ to be in the $\mathcal{H}_P$, is:

$$p_P(\tau) = Tr(V(\tau)\rho_0 V^\dagger(\tau))$$

Now, as $[H, P] \neq 0$, $p(\tau)$ is less than 1 in general. Had $[H, P] = 0$, we could have used the cyclic property of trace and the fact that $HP = PH$, to show that $p(\tau) = Tr(V(\tau)\rho_0 V^\dagger(\tau)) = 1$ always. $[H, P] = 0$ would also imply that $V\, V^\dagger = P$.

There also exists a probability of transitions completely out of $\mathcal{H}_P$, into $\mathcal{H}_P^\perp$, given by

$$p_{Q(\tau)} = Tr(V_{PQ}(\tau)Q\rho_0 QV_{PQ}^\dagger(\tau))$$

Here Q corresponds to the projection operator onto $\mathcal{H}_P^\perp$, and $V_{PQ} = QU(\tau)Q$

We focus on $p_P(\tau)$. Suppose within time t of the experiment, we measure the system N times. After the first measurement,

$$p_P(\frac{t}{N}) = Tr(PU(\frac{t}{N})P\rho_0 PU^\dagger(\frac{t}{N})P)$$

Up to the second measurement, the system has again evolved by $U(\frac{t}{N})$,
(We'll call $p_P \equiv p$)

$$p(2\frac{t}{N}) = Tr(PUPU\rho_0 U^\dagger PU^\dagger P) = Tr(\ (PUP)\ (PUP)\ \rho_0 (PU^\dagger P)\ (PU^\dagger P)\ )$$

$$p(2\frac{t}{N}) = Tr(\ (PUP)^2 \rho_0\ (PU^\dagger P)^2\ )$$

Similarly, the "survival probability" of having retained in state ρ_0 after time t, after N such projections have been performed is,

$$p(t) = Tr(\ (PU(\frac{t}{N})P)^N \rho_0\ (PU(\frac{t}{N})^\dagger P)^N\ )$$

$$p(t) = Tr(\ V_N(t)\rho_0 V_N^\dagger(t)\ ) \tag{26}$$

where we defined $V_N(t) = (PU(\frac{t}{N})P)^N$.

Note: $V_N(t)$ is not unitary, as P and U don't commute in general. We are interested in the net effect of this evolution-projection scheme.

Let $U_Z(t)$ denote $\lim_{N\longrightarrow\infty} V_N(t)$. We also know that $U(\frac{t}{N}) = e^{-iH\frac{t}{N}}$. In the limit that $\frac{t}{N} << 1$, we can expand as $e^{-iH\frac{t}{N}} = 1 - iH\frac{t}{N} + H^2\frac{t^2}{N^2} + ...$

$$U_Z(t) = \lim_{N\longrightarrow\infty}(Pe^{-iH\frac{t}{N}}P)^N = \lim_{N\longrightarrow\infty}(P\ (1 - iH\frac{t}{N} + H^2\frac{t^2}{N^2} + ...\ )P)^N$$

We can remove a P common using the fact that $P^m = P\ \forall\ m \in N$

$$U_Z(t) = \lim_{N\longrightarrow\infty}((P - iPHP\frac{t}{N} + PH^2P\frac{t^2}{N^2} + ...\ )P)^N = \lim_{N\longrightarrow\infty} P(\ (1 - iPHP\frac{t}{N} + ...\ )\ )^N$$

$$\boxed{U_Z(t) = P(e^{-iPHPt}) = Pe^{-iH_Z t}} \tag{27}$$

where, $H_Z = PHP$ is the effective **Zeno Hamiltonian**. The evolution is unitary within the subspace $\mathcal{H}_P$. We see that the system's evolution has not frozen, it is just governed by the new Zeno Hamiltonian H_Z, and is restricted to $\mathcal{H}_P$, once you enter $\mathcal{H}_P$. This is restricted dynamics is referred to as **Quantum Zeno Dynamics** , the subspace $\mathcal{H}_P$ is referred to as the **Quantum Zeno subspace** It is only in this continuous observation limit that the system's evolution can be represented by a unitary operator times the projection P.

Let's find the "survival probability" in this continuous observation limit.

$$p(t) = Tr(U_z(t)\rho_0 U_z^\dagger(t)) = Tr(\rho_0 P) = 1$$

Thus, we see that the system is restricted within $\mathcal{H}_P$

Again, the physical mechanism that makes this happen is that the probability of making a transition out of $\mathcal{H}_P$ in time τ falls as τ^2, which is subdued by the $N \longrightarrow \infty$ limit.

10.1 A note on Infinite Dimensional Projection

For an infinite dimensional case, $U_Z(t)$ is irreversible, in general, i.e., an inverse of the evolution operator may not exist. In fact, $U_Z(t)$, in infinite dimensional subspace, forms a semi-group, so reversibility of $U_Z(t)$ is not guaranteed. (Semi-group, because it doesn't have the existence of an inverse).

11 Quantum Zeno Subspaces

In the above analysis, we say that a single projection operator limited the dynamics to the corresponding subspace. Now, we consider non-selective and incomplete measurements,i.e, measurements that don't project the state into a subspace, but rather destroy the correlation between different subspaces.[3][5] [1]. Let P_n be the set of complete projection operators, such that $\sum_n P_n = 1$ and $P_n P_m = \delta_{m,n}$ (Kronecker's delta). The non-selective measurement is then described by the superoperator

$$\hat{P}\rho = \sum_n P_n \rho P_n \tag{28}$$

Note that in the superoperator space, the density matrix behaves as a vector. We'll distinguish Superoperator by hats in this section (we have occasionally used hats on Hamiltonian in other sections, not to be confused). Now, we have different subspaces of the Hilbert space, corresponding to each P_n given as $\mathcal{H}_n = P_n\mathcal{H}$. We prepare the system in the initial state ρ_0, $\hat{P}\rho_0 = \sum_n P_n \rho_0 P_n$.
In the superoperator notation, the evolution operator may be written as
$\hat{U}_{\frac{t}{N}}\rho = U(t)\rho U^\dagger(t)$, with $U(t) = e^{-iHt}$ being the normal unitary evolution operator. We now evolve the system under the Hamiltonian, and make a non-selective measurement $\hat{P}$. The total sequence of operations is given by

$$\rho(t) = \hat{P}\hat{U}_{\frac{t}{N}} ... \hat{P}\hat{U}_{\frac{t}{N}}\rho(0) = \hat{V}_t^{(N)}\rho$$

where the $\hat{P}\hat{U}_{\frac{t}{N}}$ sequence is repeated N times. Here, we have defined,

$$\hat{V}_t^{(N)} = (\ \hat{P}\hat{U}_{\frac{t}{N}}\ )^N$$

In the standard operator form, this evolution can be written as

$$\rho(t) = \sum_{n_1} P_{n_1}(... \sum_{n_N} P_{n_{N-1}}(\ U(\frac{t}{N})\ \sum_{n_N}(P_{n_N}U(\frac{t}{N})\ \rho\ U^\dagger(\frac{t}{N})P_{n_N})\ U^\dagger(\frac{t}{N})\ P_{n_{N-1}})...)P_{n_1}$$

We have expressed the action of superoperator $\hat{P}$ using eq(28) above

$$\rho(t) = \sum_{n_1,n_2,...,n_N} P_{n_1}\ U(\frac{t}{N})\ ...P_{n_{N-1}}\ U(\frac{t}{N})\ P_{n_N}U(\frac{t}{N})\ \rho\ U^\dagger(\frac{t}{N})P_{n_N}\ U^\dagger(\frac{t}{N})\ P_{n_{N-1}}...P_{n_1}$$

To tidy this up a bit, define

$$V^{(N)}_{n_1,n_2,..n_N}(t) = P_{n_1}\ U(\frac{t}{N})\ ...P_{n_{N-1}}\ U(\frac{t}{N})\ P_{n_N}U(\frac{t}{N}) \tag{29}$$

Compare this with our earlier section, where all the projection operators were the same.

$$\rho(t) = \sum_{n_1,n_2,...n_N} V^{(N)}_{n_1,n_2,...n_N}(t)\ \rho\ {V^\dagger_{n_1,n_2,...n_N}}^{(N)}(t) \tag{30}$$

If we assume that the limit $\lim_{N\longrightarrow\infty} V^{(N)}_{n,n,...n}(t)$ exists, and is given as $U^n_Z(t)$

Now, consider eq(29), in the limit $\frac{t}{N} << 1$, we can Taylor expand $U(\frac{t}{N})$, as $1 - iH\frac{t}{N}$. Consider a particular term in the resulting expansion that has P_n and P_{n-1} surrounding it. Say $P_nU(\frac{t}{N})P_{n-1}$

$$P_nU(\frac{t}{N})P_{n-1} = P_n(1 - iH\frac{t}{N})P_{n-1} \tag{31}$$

Suppose $n_N \neq n_{N-1}$. Now, using the property of orthogonality f projection operators, $P_{n_N}P_{n_{N-1}} = 0$. So this particular combination contributes only $-P_{n_N}iH\frac{t}{N}P_{n_{N-1}}$. Other such combinations also contribute similar $\mathcal{O}(\frac{t}{N})$ terms that are multiplied together. In the end, on the left-hand side of ρ is the eq(29), we have something of the form $(-i)^N HP_{n_N}HP_{n_{N-1}}H\ (\frac{t^N}{N^N})$. In the limit $\frac{t}{N} << 1$, this goes to 0. Thus $V^{(N)}_{n_1,n_2,...n_N}(t)$, will have non-zero contribution in the large N limit only if the 1s in the term considered above survive, which will happen only if $n_1 = n_2 = ..n_N$, i.e while performing the N summations, only the terms with all same indices survive.

Thus, we see that,

$$\lim_{N\longrightarrow\infty} V_{n,...n',...} = 0\ if\ \ n \neq n'$$

In general, for finite N, this need not be true, as $U(\frac{t}{N})$ produces transitions between $\mathcal{H}_n$. Thus the net evolution is given by (from eq[30]) and the above argument,

$$\rho(t) = \sum_n U^n_Z(t)\ \rho_0\ {U^n}^\dagger_Z(t) \tag{32}$$

where U^n_Z was defined as the limiting case of $V^N_{n_1,n_2,..n_N}$

It can be shown by using properties of a semi-group that ${U^n}^\dagger_Z U^n_Z = P_n$. So, we have

$$\sum_n {U^n}^\dagger_Z(t) U^n_Z(t) = \sum_n P_n = 1 \tag{33}$$

Let's find the probability of the system being in the subspace $\mathcal{H}_n$ corresponding to P_n.

$$p_n(t) = Tr(\ P_n \rho(t)\ ) = Tr(\ P_n \sum_{n'} U^{n'}_Z(t)\ \rho_0\ {U^{n'}}^\dagger_Z(t)\ )$$

$$p_n(t) = \sum_{n'} Tr(\ P_n U^{n'}_Z(t)\ \rho_0\ {U^{n'}}^\dagger_Z(t)\ )$$

Knowing that U^n_Z is a limiting case of $v^{(N)_{n,n,..}}$, and from the form of $v^{(N)_{n,n,..}}$ that involves P_n, we can see that only $P_n U^n_Z$ term remains, that too, with the additional P absorbed in $U^n_Z(t)$

$$p_n(t) = Tr(\ U^n_Z(t) \rho_0 {U^n}^\dagger_Z(t)\ ) = Tr(\rho_0 P_n) = p_n(0)$$

Thus we see that the probability of the system being in each of the $\mathcal{H}_n$ subspaces remains conserved and does not leak into other subspaces. The correlation between different subspaces is completely destroyed (total decoherence).

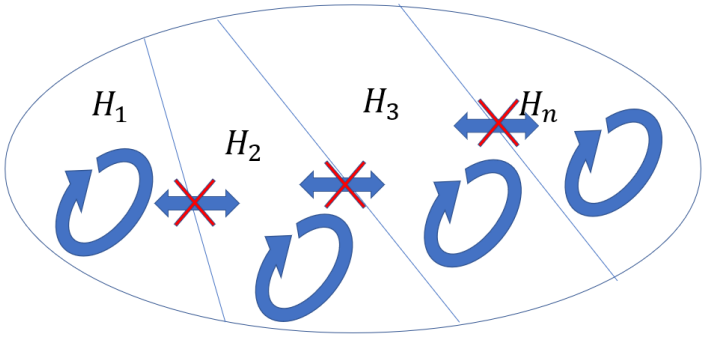

Figure 1: Illustration for the splitting of Hilbert space into Zeno subspaces. Probability can't leak into another subspace, but dynamics within the subspace are allowed.

If the Hamiltonian H is bounded, then each U^n_Z is **unitary within the subspace** $\mathcal{H}_n$, and given by (following our now familiar procedure of expanding the Hamiltonian in Taylor series up to first order in $\frac{t}{N}$),

$$U^n_Z(t) = \lim_{N \longrightarrow \infty} (\ P_n U(\frac{t}{N}) P_n\ )^N = P_n e^{-iP_n H P_n t}$$

Had we written this as $P_n e^{-i\sum_n P_n H P_n t}$, this would remain the same as P_n outside would select only the required term. So the total evolution of the state ρ_0 is given by

$$\boxed{\rho(t) = \sum_n P_n e^{-iH_Z t} \rho_0 e^{iH_Z t} P_n} \tag{34}$$

where

$$\boxed{H_z = \sum_n P_n H P_n} \tag{35}$$

H_Z is the global **Zeno Hamiltonian**.

To summarize, a non-selective measurement stops the probability from leaking out of one Zeno subspace into another, allowing for dynamics within each subspace.

12 Decoupling a System by non-selective Measurements

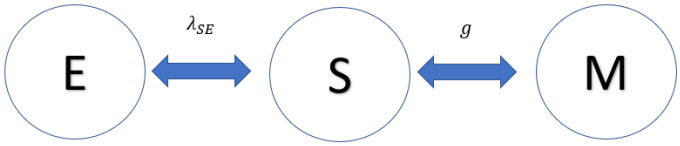

Figure 2: Representative model of a system, a meter and environment

Consider a setup wherein we have 3 entities, system S, a meter M, and the system's environment E (excluding the meter). S can interact with E and M, but E and M don't interact with each other (this isn't necessary but simplifies our analysis). Without interaction terms, the systems would evolve independently under their own Hamiltonians as $H_E \otimes 1 \otimes 1 + 1 \otimes H_s \otimes 1 + 1 \otimes 1 \otimes H_M$. Let the coupling between S and E be captured by the interaction $\lambda_{SE} A_E \otimes X_s \otimes 1$ and the interaction between S and M be modelled as $g 1 \otimes Z_s \otimes Y_M$, where $_{SE}$ and g denote the respective coupling strengths. A_E, Z_S, X_S, and Y_M, are operators in respective spaces.
Thus the total Hamiltonian of the system is :

$$H = H_E \otimes 1 \otimes 1 + 1 \otimes H_s \otimes 1 + 1 \otimes 1 \otimes H_M + \lambda_{SE}(A_E \otimes X_S \otimes 1) + g(1 \otimes Z_S \otimes Y_M) \tag{36}$$

In typical settings of Weak Value Measurement Experiments, we are interested in finding an estimate for g. We won't involve ourselves in weak-value measurements here, but the need for such experiments is that there are two interacting systems and no environment E. We try to use the technique of non-selective projection measurement on S, to decouple S from E, but retain its interaction with M,i.e we don't want the $A_E \otimes X_S \otimes 1$ term.
A projection operator on system S, say P_n, must now be denoted as $1 \otimes P_n \otimes 1$.

Now, we have seen in the previous section that the dynamics of a system subject to repeated projective measurements are controlled by the Zeno Hamiltonian $H_Z = \sum_n P_n H P_n$.

Before that, we can go to the rotating frame of the non-interacting part of the Hamiltonian, so that we only have to deal with the interaction terms. We now project the interaction terms of the Hamiltonian in eq(36),

$$H_z = \lambda_{SE}(A_E \otimes \sum_\mu P_\mu X_S P_\mu \otimes 1) + g(1 \otimes \sum_\mu P_\mu Z_S P_\mu \otimes Y_M)$$

We are interested in somehow setting $(A_E \otimes \sum_\mu P_\mu X_S P_\mu \otimes 1)$ to 0.

Let the projection operators $P_\mu = |\mu\rangle \langle\mu|$

Consider,

$$X_Z = \sum_\mu |\mu\rangle \langle\mu| X_S |\mu\rangle \langle\mu|$$

Expanding X_S in its eigenbasis as $X_S = \sum_\nu \chi_\nu |\nu\rangle \langle\nu|$,

$$X_Z = \sum_\mu \sum_\nu |\mu\rangle \langle\mu| \ (\chi_\nu |\nu\rangle \langle\nu|) \ |\mu\rangle \langle\mu| \tag{37}$$

Rearranging (note that χ_ν is a scalar),

$$X_Z = \sum_\mu \sum_\nu \chi_\nu || \langle\mu|\nu\rangle ||^2 |\mu\rangle \langle\mu|$$

Now, the crucial step. What if $|| \langle\mu|\nu\rangle ||^2$ is a constant? It can be pulled out of the summation. Let's say it is a constant, we'll see when it is so. Let this constant be $\frac{1}{d^2}$.

$$X_Z = \frac{1}{d^2} \sum_\mu \sum_\nu \chi_\nu |\mu\rangle \langle\mu|$$

We can now perform the summations over ν and μ individually as there are no cross terms. Recall that χ_ν are just the eigenvalues of X_S in its eigenbasis. So a summation over them is equivalent to its trace in its diagonal form (X_S will be diagonal in its eigenbasis). So $\sum_\nu \chi_\nu = Tr(X_S)$. And since $|\mu\rangle$ form a complete basis, $\sum_\mu |\mu\rangle \langle\mu| = 1$. So,

$$X_Z = \frac{1}{d^2} Tr(X_S) 1$$

We have not been able to destroy this term but have at least been able to reduce it to identity times a scalar. Now, here's the point. This term can be accounted for in the non-interacting Hamiltonian. We came to the interaction picture to look at X_S. For, stated otherwise, we can choose to set $Tr(X) = 0$, by accounting for it in the non-interacting part of H.

Thus, X_Z can be set to 0, under the assumption that $|| \langle\mu|\nu\rangle ||^2$ is a constant for all μ and ν. This condition is equivalent to demanding that the $|\mu\rangle$ basis is such that it has equal "contribution" from all vectors of $|\nu\rangle$ basis. In fact, such bases are called **Mutually Unbiased Bases** (MUBs).

The most common example for MUBs in 2-dimensional Hilbert space are the bases $\{|0\rangle, |1\rangle\}$ and $\{\frac{1}{\sqrt{2}}(\ |0\rangle + |1\rangle\ ),\ \frac{1}{\sqrt{2}}(\ |0\rangle - |1\rangle\ )\}$. Each vector of one basis has an equal component $\frac{1}{2}$ along a vector of the other basis. In general, if the Hilbert space is d dimensional, then the projection value would be $\frac{1}{d}$. (Since each basis has d vectors, and each "unbiased" implies the component of the other bases' vector must be equally split)

Thus to decouple S from E, we only need to perform non-selective measurements on S in the "correct" basis.Tracing backwards, H_Z doesn't have the contribution of X_S now, and we have been able to decouple S from E. In doing so, we have retained the interaction of S with M, of course, under the assumption and X_S (the interaction term for $E - S$) does not commute with Z_S, (the interaction term for $S - M$). We must note that we have assumed that the interaction term between $E - S$ could be more general, say a summation over non-commuting operators, in which case our analysis needs modification.

To re-emphasize, **our projection basis must be mutually unbiased w.r.t the eigenbasis of the interaction term (X) in the Hamiltonian.**

12.1 A note on the existence of MUBs

If the d has the prime factor decomposition given as $d = p_1^{n_1}...p_m^{n_m}$, where $\{\ p_1, .., p_m\}$ are prime numbers. Let $p_1 = min(p_1, p_2, .., p_n)$ Then if M denotes the maximal number of MUBs in this d-dimensional space, we have the following relation:

$$p_1^{n_1} + 1 \leq M \leq d + 1 \tag{38}$$

So, for 2 d case, the maximal number of MUBs is bounded as $2^1 + 1 \leq M \leq 2 + 1$. Thus we have at most 3 MUBs in 2d (The first two mentioned above, the third set being $\{\frac{1}{\sqrt{2}}(\ |0\rangle + i\,|1\rangle\ ),\ \frac{1}{\sqrt{2}}(\ |0\rangle - i\,|1\rangle\ )\}$) It turns out that by this way of bounds on M, one can precisely find the maximal number of MUBs in a dimension, say for 3 dimensions it is $3^1 + 1$=3+1=4, for 4 likewise, it is 5. There is a catch in dimension 6. $6 = 2 \times 3$, so $p = 2$. The bounds are $2^1 + 1 \leq M \leq 6 + 1$. We can't precisely pin down on the number of MUBs in dimension 6, using this method. In fact, this is an open question.

For our purpose, however, it is always possible to find at least one pair of MUBs, since we can always use the Hadamard transform to get to one of the possible MUBs.

13 If we are not in a Mutually unbiased basis

Suppose our projection operators are not exactly in the MUB of X_S's eigenvectors. Let $\{|\nu\rangle\}$ be the eigenvector basis of X_S, and $\{|\mu\rangle\}$ be the true MUB basis. Now instead of projecting in $|\mu\rangle$, we are off by a constant offset generated by the generator

G. So, the projection operators we use are in fact $e^{iG\epsilon}\left|\mu\right\rangle$, where ϵ is a measure of the error by which we are off. We assume that the generator G is Hermitian, i.e., $G = G^{\dagger}$. We replace $X_S = X$, just for convenience.

Now, we evaluate the X_Z, from eq(37)

$$X_Z = \sum_{\mu}\sum_{\nu} e^{iG\epsilon}\left|\mu\right\rangle\left\langle\mu\right| e^{-iG\epsilon}\left(\chi_\nu\left|\nu\right\rangle\left\langle\nu\right|\right) e^{iG\epsilon}\left|\mu\right\rangle\left\langle\mu\right| e^{-iG\epsilon}$$

We can expand $e^{iG\epsilon} = 1 + iG\epsilon - \frac{1}{2}G^2\epsilon^2 + ...$, as $\epsilon << 1$, up to $\mathcal{O}(\epsilon^2)$,

$$X_Z = \sum_{\mu}\sum_{\nu} e^{-iG\epsilon}\left|\mu\right\rangle\left\langle\mu\right|\left(\,1+iG\epsilon-\frac{1}{2}G^2\epsilon^2\right)\left(\chi_\nu\left|\nu\right\rangle\left\langle\nu\right|\right)\,1-iG\epsilon-\frac{1}{2}G^2\epsilon^2\left|\mu\right\rangle\left\langle\mu\right| e^{-iG\epsilon}$$

Now, we retain terms only up to $\mathcal{O}(\epsilon^2)$,

$$X_Z = \sum_{\mu}\sum_{\nu}\chi_\nu e^{iG\epsilon}\left|\mu\right\rangle\ \left(\ ||\left\langle\mu|\nu\right\rangle||^2 - i\left\langle\mu\right| G\left|\nu\right\rangle\left\langle\nu|\mu\right\rangle\epsilon - \frac{1}{2}\left\langle\mu\right| G^2\left|\nu\right\rangle\left\langle\nu|\mu\right\rangle\epsilon^2 + i\left\langle\mu|\nu\right\rangle\left\langle\nu\right| G\left|\mu\right\rangle\epsilon +$$

$$\left\langle\mu\right| G\left|\nu\right\rangle\left\langle\nu\right| G\left|\mu\right\rangle\epsilon^2 - \tfrac{1}{2}\left\langle\mu|\nu\right\rangle\left\langle\nu\right| G^2\left|\mu\right\rangle\epsilon^2\ )\ \left\langle\mu\right| e^{-iG\epsilon}$$

We can take χ_ν inside and evaluate the $\sum_\nu \chi_\nu\left|\nu\right\rangle\left\langle\nu\right| = X_s$ and $||\left\langle\nu|\mu\right\rangle||^2 = \frac{1}{d^2}$

$$X_Z =$$

$$\sum_\mu e^{iG\epsilon}\left|\mu\right\rangle\ \left(\ Tr(X_s)\tfrac{1}{d^2} - i\left\langle\mu\right| GX\left|\mu\right\rangle\epsilon - \tfrac{1}{2}\left\langle\mu\right| G^2X\left|\mu\right\rangle\epsilon^2 + i\left\langle\mu\right| XG\left|\mu\right\rangle\epsilon + \left\langle\mu\right| GXG\left|\mu\right\rangle\epsilon^2 -$$
$$\tfrac{1}{2}\left\langle\mu\right| XG^2\left|\mu\right\rangle\epsilon^2\ )\ \left\langle\mu\right| e^{-iG\epsilon}$$

$$= \sum_\mu e^{iG\epsilon}\left|\mu\right\rangle\ \left(\ Tr(X_s)\frac{1}{d^2} + i\left\langle\mu\right| XG - GX\left|\mu\right\rangle\epsilon - \frac{1}{2}\left\langle\mu\right| G^2X - 2GXG + XG^2\left|\mu\right\rangle\epsilon^2\ )\ \left\langle\mu\right| e^{-iG\epsilon}$$

$$= \sum_\mu e^{iG\epsilon}\left|\mu\right\rangle\ \left(\ Tr(X_s)\frac{1}{d^2} + i\left\langle\mu\right|\left[X,G\right]\left|\mu\right\rangle\epsilon - \frac{1}{2}\left\langle\mu\right| G^2X - 2GXG + XG^2\left|\mu\right\rangle\epsilon^2\ )\ \left\langle\mu\right| e^{-iG\epsilon}$$

Now, $2[[G,X],G] = 2GXG - XG^2 - G^2X$; So we have,

$$X_Z = \sum_\mu e^{iG\epsilon}\left|\mu\right\rangle\ \left(\ Tr(X_s)\frac{1}{d^2} + i\left\langle\mu\right|\left[X,G\right]\left|\mu\right\rangle\epsilon + \left\langle\mu\right|\left[[G,X],G\right]\left|\mu\right\rangle\epsilon^2\ )\ \left\langle\mu\right| e^{-iG\epsilon} \tag{41}$$

We group different orders of terms,

$\mathcal{O}(1) : \sum_\mu e^{iG\epsilon} |\mu\rangle \langle\mu| e^{-iG\epsilon} Tr(X_S) \frac{1}{d^2}$
We argued earlier that $Tr(X_S)$ could be set to 0. Hence, this is not a problem.

$\mathcal{O}(\epsilon) : \mathrm{i}\sum_\mu e^{iG\epsilon} |\mu\rangle \langle\mu| [X, G] |\mu\rangle \langle\mu| e^{-iG\epsilon}$ This is the sum of the diagonal entries of the commutator $[X, G]$. If we have $[X, G] = 0$, this term goes to zero.

$\mathcal{O}(\epsilon^2) : \mathrm{i}\sum_\mu e^{-iG\epsilon} |\mu\rangle \langle\mu| [[G, X], G] |\mu\rangle \langle\mu| e^{-iG\epsilon}$
If we have demanded $[X, G] = 0$ to set the first order term to 0, the second erroneous term also goes to 0.

What does it mean to demand $[G, X] = 0$? If X and G commute, so do $e^{iG\epsilon}$ and X, so they have a simultaneous eigenbasis. Let $G = \sum_\nu G_\nu |\nu\rangle \langle\nu|$. So, it follows that $e^{-iG\epsilon} = \sum_\nu e^{iG_\nu\epsilon} |\nu\rangle \langle\nu|$. We started by saying that $\{|\nu\rangle\}$ and $\{|\mu\rangle\}$ are MUB, which means $|| \langle\nu|\mu\rangle ||^2 = \frac{1}{d^2}\ \forall \nu, \mu$. So in our new erroneous bases $\{e^{iG\epsilon} |\nu\rangle\}$,

$$|| \langle\mu| e^{iG\epsilon} |\nu\rangle ||^2 = || \langle\mu| \sum_{\nu'} e^{iG_{\nu'}\epsilon} |\nu'\rangle \langle\nu'|\nu\rangle ||^2 = ||e^{iG_\nu} \langle\mu|\nu\rangle ||^2 = \frac{1}{d^2} \qquad (42)$$

Thus, we see that demanding that G and X commute, is equivalent to saying that the new erroneous projection basis produced using the generator G is also a MUB w.r.t the eigenbasis of X. **We have not escaped the condition of being in MUB.** It seems that being in MUB is required for us to be able to set the interaction term to 0.

14 Conclusions

In this report, we presented the theoretical background required to describe the evolution of a quantum system. We introduced the Quantum Zeno Effect, first for a 1-D selective projection, then generalised it to allow Zeno dynamics within a Zeno subspace, using non-selective incomplete measurements. It was seen that the effective Zeno Hamiltonian governs the dynamics within each subspace. Finally, we have described the idea of using Quantum Zeno Dynamics, to decouple a system from its environment. We have arrived at the condition that if the projection operators are mutually unbiased w.r.t the eigenbasis of the environment-system interaction term, we can ignore the interaction term in the effective Zeno Hamiltonian. We then looked at a perturbation to this MUB created using a generator and evaluated the higher order correction terms to the effective Zeno Hamiltonian. We found that if the generator commutes with the interaction term, the errors go to 0. Following that, we said that if this holds, then the operator has just taken you from one MUB to another MUB and the condition of being in a MUB is still required for decoupling.

For further studying this rather absurd condition of MUBs, we plan to numerically analyse how the errors in the effective Zeno Hamiltonian propagate if we do not project using a MUB basis.

References

[1] P. Facchi et al. "Control of decoherence: Analysis and comparison of three different strategies". In: *Phys. Rev. A* 71 (2 Feb. 2005), p. 022302. DOI: 10.1103/PhysRevA.71.022302. URL: https://link.aps.org/doi/10.1103/PhysRevA.71.022302.

[2] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011. ISBN: 9781107002173. URL: https://www.cambridge.org/in/academic/subjects/physics/quantum-physics-quantum-information-and-quantum-computation/quantum-computation-and-quantum-information-10th-anniversary-edition?format=HB&isbn=9781107002173.

[3] Daniel Burgarth et al. "Quantum Zeno Dynamics from General Quantum Operations". In: *Quantum* 4 (July 2020), p. 289. ISSN: 2521-327X. DOI: 10.22331/q-2020-07-06-289. URL: https://doi.org/10.22331/q-2020-07-06-289.

[4] John Preskill. *PH219 Notes, Quantum Computation, Caltech,2021*. 2021. URL: http://theory.caltech.edu/~preskill/ph229/.

[5] P Facchi1 and S Pascazio. "Quantum Zeno dynamics: mathematical and physical aspects". In: *arXiv:0903.3297[math-ph]* (). DOI: 10.1088/1751-8113/41/49/493001. URL: https://doi.org/10.48550/arXiv.0903.3297.

[6] Daniel Manzano. "A short introduction to the Lindblad Master Equation". In: *arXiv:1906.04478 [quant-ph]* (). DOI: 10.1063/1.5115323. URL: https://doi.org/10.48550/arXiv.1906.04478.

I also express deep gratitude towards my guide Prof. Vinjanampaty for having guided me patiently throughout this project.