Introduction to Non-Parametric Tests

- Most of the hypothesis testing and confidence interval procedures so far we have learnt are based on the assumption that we are working with random samples drawn from normal population with some parameters.
- So these test procedures are parametric procedures because they depend on one or more parameters of the population distribution.
- A number of such test procedures have been developed which do not depend on parameters of population.
- They make no assumptions about the form of distribution of underlying population.
- Such procedures are called non-parametric test or sometime distribution free tests.
- Some benefits of non-parametric tests are that they are easy to carry out and they work with categorical data too.

Advantages

- Besides quantitative data they can be applied to non-numeric as well as categorical data.
- They are relatively simpler and quicker.
- They need no assumption on population from which sample is taken.
- They do not require complex sampling theory.

Disadvantages

- They cannot be used to estimate population parameters.
- They are less reliable and less efficient.
- Lots of information needs to be discarded.
- Lots of tables are to be observed.

Types of Non-parametric Tests

I. Sign Test
   a. Sign Test for Individual Observation
   b. Sign Test for Paired Observations
II. Wilcoxon's Signed Rank Test
   a. For Individual Observation
   b. For Paired Observations
III. Signed Rank Test
   a. Mann-Whitney's U-Test
   b. Kruskal-Wallis' H-Test

Sign Test

Sign Test for Individual Obsrvation

- It is a non-parametric test procedure that is used to test the hypothesis about the median $\tilde{\mu}$ of a population.
- In fact, median of a distribution is a value of the random variable such that the probability is 0.5 that an observed value of X is less than or equal to the median, and the probability is 0.5 that an observed value of X si greater than or equal to the median, i.e.,

$$P(X \leq \tilde{\mu}) = P(X \geq \tilde{\mu}) = 0.5$$

- The hypothesis underlying the sign test are:

$$H_0: \tilde{\mu} = \tilde{\mu}_0$$
$$H_1: \tilde{\mu} < \tilde{\mu}_0 \text{ or } \tilde{\mu} > \tilde{\mu}_0 \; or \; \tilde{\mu} \neq \tilde{\mu}_0$$

- For the test procedure a sample of size n is taken randomly from population of interest whose distribution may or may not be known. Let the sample be

$$X_1, X_2, \dots \dots \dots \dots \dots \dots \dots, X_n$$

- Then the differences $(X_i - \tilde{\mu})$ are computed for I = 1, 2, ….. , n.
- If the null hypothesis is true, any difference $(X_i - \tilde{\mu})$ is equally likely to be positive or negative.
- Let R^+ denote the number of differences $(X_i - \tilde{\mu})$ that are positive and let R^- be the number of these differences that are negative.
- If any value of X_i is exactly equal to $\tilde{\mu}_0$, then they should be discarded and test is applied to remaining data only.

Case I – <u>When sample size is small, i.e., when n <= 20</u>.

- Let $R = \min(R^+, R^-)$.
- Now we find a critical value R_α^* for given values of n and α from 'Sign Test' table (given below) and reject H_0 if

$$R \le R_\alpha^* \quad (for\ one\ tailed\ test)$$

Or,

$$R \le R_{\frac{\alpha}{2}}^* \quad (for\ two - tailed\ test)$$

Sign Test table for critical values-

n	One tailed, α = 0.005 Two tailed, α = 0.01	α = 0.01 α = 0.02	α = 0.025 α = 0.05	α = 0.05 α = 0.10
8	0	0	0	1
9	0	0	1	1
10	0	0	1	1
11	0	1	1	2
12	1	1	2	2
13	1	1	2	3
14	1	2	3	3
15	2	2	3	3
16	2	2	3	4
17	2	3	4	4
18	3	3	4	5
19	3	4	4	5
20	3	4	5	5
21	4	4	5	6
22	4	5	5	6
23	4	5	6	7
24	5	5	6	7
25	5	6	6	7

Solved Problem

18 measurements on coefficient of friction between two surfaces are recorded as 0.59, 0.56, 0.49, 0.55, 0.65, 0.55, 0.51, 0.60, 0.56, 0.47, 0.58, 0.61, 0.54, 0.68, 0.56, 0.50, 0.57, 0.53.
Use sign test at 5% level of significance to test that median of coefficient of friction between the surfaces considered is not different from 0.55.

Solution-

Here null and alternative hypotheses to test are:

$$H_0: \tilde{\mu} = 0.55$$
$$H_1: \tilde{\mu} \neq 0.55$$

Working Table:

Coefficient of Friction (X)	$X - \tilde{\mu}$	Sign
0.59	0.04	+
0.56	0.01	+
0.49	-0.06	-
0.55	0	
0.65	0.10	+
0.55	0	
0.51	-0.04	-
0.60	0.05	+
0.56	0.01	+
0.47	-0.08	-
0.58	0.03	+
0.61	0.06	+
0.54	-0.01	-
0.68	0.13	+
0.56	0.01	+
0.50	-0.05	-
0.57	0.02	+
0.53	-0.02	-

Here,
$Count\ of + (R^+) = 10$
$Count\ of - (R^-) = 6$
$n = R^+ + R^- = 10 + 6 = 16$

$R = \min(R^+, R^-) = \min(10,6) = 6$
From sign-test table for $n = 16, R^*_{\frac{0.05}{2}} = 3$ (since it is two-tailed test)
Since, $R \nless 3$, so H_0 is not rejected.

Case II – <u>When n > 20</u>.

- In this case minimum of R^+ and R^- is determined. Let $R = \min(R^+, R^-)$
- Obviously, R has binomial distribution and if sample size is large, it can be approximated by using normal distribution.
- Thus R has normally distributed with mean np, where p = 0.5, i.e.,

$$\mu = n \times 0.5 = \frac{n}{2}$$

  and variance

$$\sigma^2 = np(1-p) = n \times \frac{1}{2} \times \left(1 - \frac{1}{2}\right) = \frac{n}{4}$$

  So that

$$\frac{R - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \sim N(0,1)$$

- Let

$$Z_0 = \frac{R - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$$

- Since R is discrete random variable and Z_0 is continuous random variable, so a correction of 0.5 is made on R by adding 0.5 if $R < \frac{n}{2}$ or by subtracting 0.5 if $R > \frac{n}{2}$, so that the test statistic is
  $$Z_0 = \frac{(R + 0.5) - \frac{n}{2}}{\sqrt{\frac{n}{4}}}, if\ R < \frac{n}{2}\ \ and\ Z_0 = \frac{(R - 0.5) - \frac{n}{2}}{\sqrt{\frac{n}{4}}}, if\ R > \frac{n}{2}$$
- Finally H_0 is rejected at α level of significance if
  $|Z_0| > Z_\alpha$ (for two one tailed test) and $|Z_0| > Z_{\alpha/2}$ (for two tailed test).

Solved Problem

Breaking strength of glass measured for 30 pieces of glass are recorded as 153, 159, 144, 160, 158, 153, 171, 162, 159, 137, 159, 159, 148, 162, 154, 159, 160, 157, 140, 168, 163, 148, 151, 153, 157, 155, 148, 168, 152, 149. Use sign test at 5% level to test whether median of breaking strength of glass is more than 150.

Solution-
Here,

$$R^+ = 23 \quad R^- = 7 \quad n = 23 + 7 = 30$$
$$R = \min(R^+, R^-) = 7$$

Since

$$R\ (i.e., 7) < \frac{n}{2}\ (i.e. 15)$$

So test statistic is

$$Z_0 = \frac{(R + 0.5) - \frac{n}{2}}{\sqrt{\frac{n}{4}}} = \frac{(7 + 0.5) - \frac{30}{2}}{\sqrt{\frac{30}{4}}} = -2.74$$
$$H_0\ is\ rejected.$$

<u>Sign Test for Paired Obsrvation</u>

- It is a non-parametric test procedure that is used to test equality of medians of two populations.
- Let the medians of two populations, the form of distribution of which may or may not be known be $\tilde{\mu}_1\ and\ \tilde{\mu}_2$.
- The hypothesis underlying the test are:
  $$H_0: \tilde{\mu}_1 = \tilde{\mu}_2$$
  $$H_1: \tilde{\mu}_1 < \tilde{\mu}_2 \text{ or } \tilde{\mu}_1 > \tilde{\mu}_2\ or\ \tilde{\mu}_1 \neq \tilde{\mu}_2$$
- For the test procedure two samples each from one population of same size 'n' are taken randomly from the populations.
- Then the differences of corresponding paired observations are computed.
- If the first observation is greater than the second then a '+' sing is assigned to it, otherwise '-' sign is assigned.
- Let R^+ denote the number of '+' signs and let R^- be the number of '-' signs.
- Remaining procedure is same as in 'Signed Test for Individual Observation' ………………..

Solved Problem-

The number of tickets issued by two police officers on 17 randomly selected days are recorded as follows:
First Officer (X1) : 7, 11, 14, 11, 12, 6, 9, 8, 10, 11, 13, 7, 8, 11, 9, 10, 13
Second Officer (X2) : 10, 13, 14, 15, 9, 10, 13, 11, 11, 15, 11, 10, 8, 12, 14, 9, 16
Use sign test at 5% level of significance to test whether second police officer issues more tickets than the first.

Solution-

Let $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$ denote mean number of tickets issued by Officer1 and Officer2 respectively.

To test

$$H_0: \tilde{\mu}_1 = \tilde{\mu}_2$$

$$H_1: \tilde{\mu}_1 < \tilde{\mu}_2$$

Working Table-

Officer1	Officer2	Difference	Sign
7	10	-3	-
11	13	-2	-
14	14	0	0
11	15	-4	-
12	9	3	+
6	10	-4	-
9	13	-4	-
8	11	-3	-
10	11	-1	-
11	15	-4	-
13	11	2	+
7	10	-3	-
8	8	0	0
11	12	-1	-
9	14	-5	-
10	9	1	+
13	16	-3	-

Here, $R^+ = 3 \quad R^- = 12 \quad n = 12 + 3 = 15$

Also, $R = \min(R^+, R^-) = \min(3,12) = 3$

Here n<20 so we use 'Sign-Test' table from which we see that for n = 15, $R^*_{0.05} = 3$.

Since it is true that $R \leq 3$, so null hypothesis is rejected.