Tribhuvan University
Institute of Science and Technology
Bachelor of Science in
Computer Science and Information Technology

3RD Semester

CSC209: Computer Graphics

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U4: 3Dimensional Geometric Transformation (5Hrs)

4.1 Three-Dimensional translation, Rotation, Scaling, Reflection and Shearing

4.2 Three-Dimensional Composite Transformations

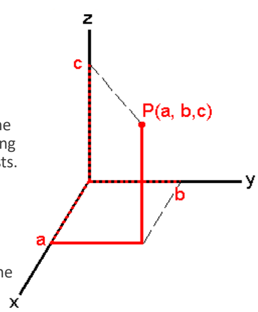
4.3 Three-Dimensional Viewing: Viewing pipeline, world to screen viewing transformation, Projection concepts(Orthographic, parallel, perspective projections)

3-Dimension

Three-dimensional space is a geometric 3-parameters model of the physical universe (without considering time) in which all known matter exists.

These three dimensions can be labeled by a combination of length, breadth, and depth.

Any three directions can be chosen, provided that they do not all lie in the same plane.



3D Transformations

Just as 2D-transfromtion can be represented by 3x3 matrices, using homogeneous co-ordinate; in 3D space; can be represented by 4x4 matrices

1. Translation
2. Rotation
3. Scaling
4. Reflection
5. Shear

Translation

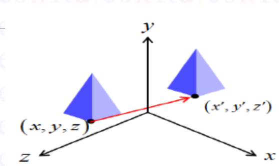
Translation in 3D is similar to translation in the 2D except that there is one more direction parallel to the z-axis.

If, tx, ty, and tz are used to represent the translation vectors then the translation of the position P(x, y, z) into the point P' (x', y', z') is done by

- x' = x + tx
- y' = y + ty
- z' = z + tz

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = T.P$$

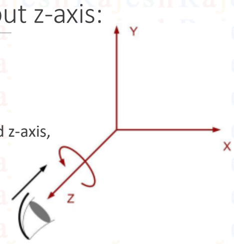
Rotation:- i) Rotation About z-axis:

Z-component does not change.

X' = X cosθ – Y sinθ
Y' = X sinθ + Y cosθ
Z' = Z

Matrix representation for rotation around z-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



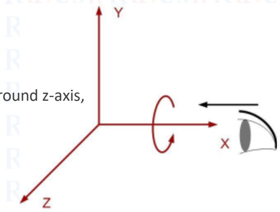
Rotation:- ii) Rotation About x-axis:

X-component does not change.

Y' = Y cosθ - Z sinθ
Z' = Y sinθ + Z cosθ
X' = X

Matrix representation for rotation around z-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



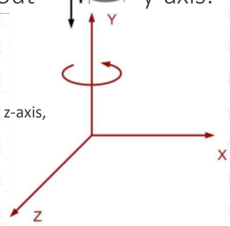
Rotation:- iii) Rotation About  y-axis:

Y-component does not change.

Z' = Z cosθ – X sinθ
X' = Z sinθ + X cosθ
Y' = Y

Matrix representation for rotation around z-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

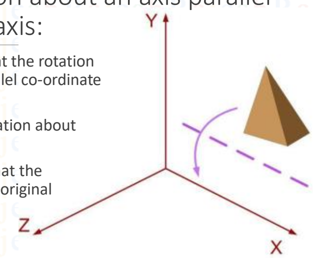


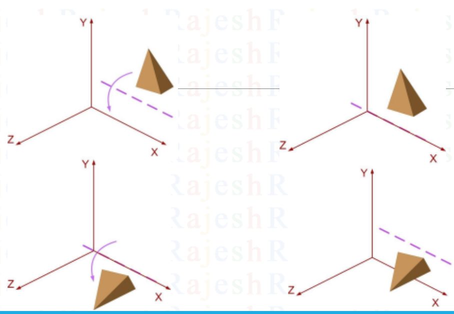
Rotation:- Rotation about an axis parallel to any of the co-axis:

i. Translate the object so that the rotation axis coincides with the parallel co-ordinate axis.

ii. Perform the specified rotation about the axis.

iii. Translate the object so that the rotation axis is moved to its original position.





Rotation:- Rotation about an axis not parallel to any of the co-axis:

i. Translate the object such that rotation axis passes through coordinate origin.

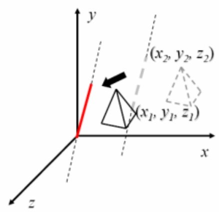
ii. Rotate the axis such that axis of rotation coincides with one of the co-ordinate axis.

iii. Perform the specific rotation about the ordinate axis.

iv. Apply inverse rotation to bring the rotation axis back to its original orientation.

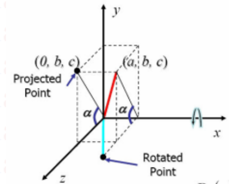
v. Apply inverse translation to bring the rotation axis back to its original position.

- Step 1. Translate



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 2. Rotate about x axis by α

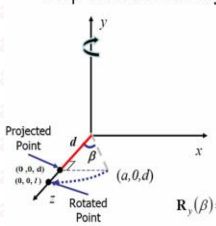


$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 3. Rotate about y axis by β (clockwise)

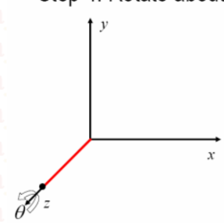


$$\sin\beta = \frac{a}{l}, \quad \cos\beta = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

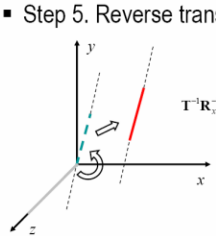
$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 4. Rotate about z axis by the angle θ



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 5. Reverse transformation



$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta) = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\theta) = \mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

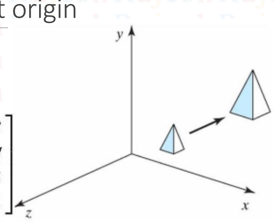
Scaling:- Scaling about origin

X' = X . Sx

Y' = Y . Sy

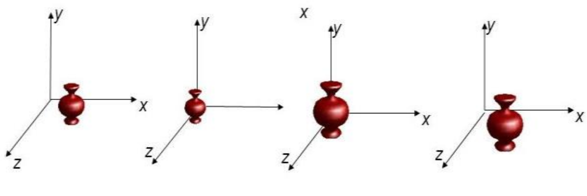
Z' = Z . Sz

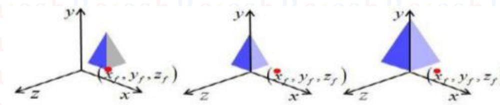
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

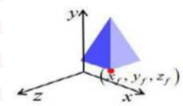


$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Scaling:- Scaling about an arbitrary point or Fixed point (xf, yf, zf)







$$T \cdot S \cdot T^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= T(x_f, y_f, z_f).S(s_x, s_y, s_z).T(-x_f, -y_f, -z_f)$$

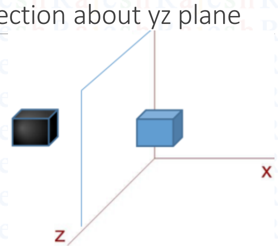
Reflection:- i) Reflection about yz plane

X' = -X

Y' = Y

Z' = Z

$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



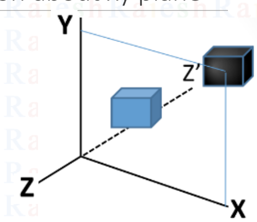
Reflection:- ii) Reflection about xy plane

X' = X

Y' = Y

Z' = -Z

$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



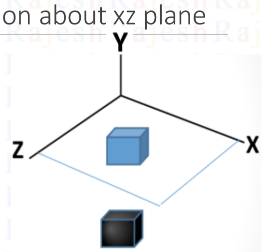
Reflection:- iii) Reflection about xz plane

X' = X

Y' = -Y

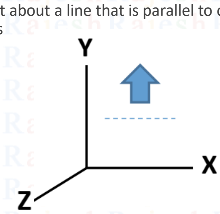
Z' = Z

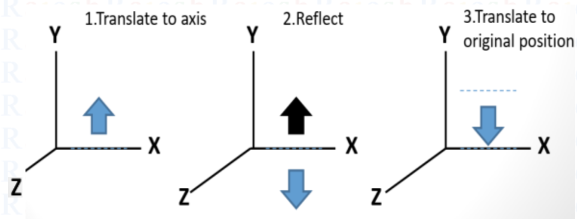
$$T_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Reflection

Reflection of an object about a line that is parallel to one of the major coordinate axes





Reflection:- About any plane:

Step-1: Translate the plane such that it passes through origin i.e. normal vector passes through origin.

Step-2: Rotate the plane such that normal vector lies on one of the co-ordinate axis.

Step-3: Perform reflection about the plane whose normal vector is one of the co-ordinate axis.

Step-4: Rotate back the plane such that normal vector takes its original orientation.

Step-5: Translate back the plane to its original position.

Shear: Z-axis Shear

Shearing transformations can be used to modify object shapes.

Z-axis Shear

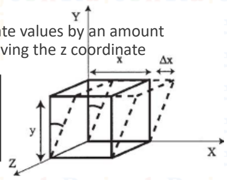
This transformation alters x- and y-coordinate values by an amount that is proportional to the z value, while leaving the z coordinate unchanged, i.e,

x' = x + Shx .z

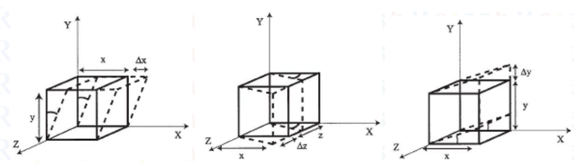
y '= y + Shy .z

z' = z

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & Sh_x & 0 \\ 0 & 1 & Sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$SH_z = \begin{bmatrix} 1 & 0 & Sh_x & 0 \\ 0 & 1 & Sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ Sh_y & 1 & 0 & 0 \\ Sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_y = \begin{bmatrix} 1 & Sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1) A homogenous coordinate point P(3, 2, 1) is translated in x, y, z direction by -2, -2 & -2 unit respectively followed by successive rotation of 60^0 about x- axis. Find the final position of homogenous coordinate.

2) A cube of length 10 units having one of its corner at origin (0, 0, 0) & three edges along principal axis. If the cube is to be rotated about z-axis by an angle of 45^0 in counter clockwise direction, calculate the new position of point.

A homogenous coordinate point P(3, 2, 1) is translated in x, y, z direction by -2, -2 & -2 unit respectively followed by successive rotation of 60^0 about x- axis. Find the final position of homogenous coordinate.

Here, $tx = -2$, $ty = -2$, $tz = -2$

$$T = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(60^0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos60 & -sin60 & 0 \\ 0 & sin60 & cos60 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

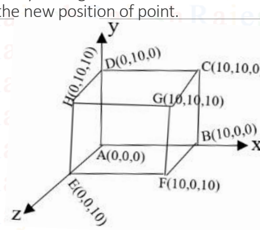
Composite transformation

$$R_x(60^0).T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -1+\sqrt{3} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3}-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$P' = M.P = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -1+\sqrt{3} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3}-1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} =$$

A cube of length 10 units having one of its corner at origin (0, 0, 0) & three edges along principal axis. If the cube is to be rotated about z-axis by an angle of 45^0 in counter clockwise direction, calculate the new position of point.



$$R_z(45^0) = \begin{bmatrix} cos45 & -sin45 & 0 & 0 \\ sin45 & cos45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{bmatrix} 0 & 10 & 10 & 0 & 0 & 10 & 10 & 0 \\ 0 & 0 & 10 & 10 & 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 & 10 & 10 & 10 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} =$$

3D-Viewing

In 3D viewing, we specify a view volume in the world, a projection onto a projection plane, and a viewport on the view surface.

Conceptually, objects in 3D world are clipped against the 3D view volume and are then projected.

The contents of the projection of the view volume onto the projection plane, called the window, are then transformed (mapped) into the viewport for display.

3D-Viewing

It involves the following considerations:

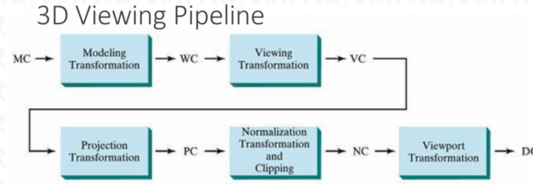
- We can view an object from any spatial position.

E.g. In front of an object, Behind the object, In the middle of a group of the objects, Inside an object.

- 3D descriptions of objects must be projected onto the flat viewing surface of the output device.

- The clipping boundaries enclose a volume of space.

3D Viewing Pipeline

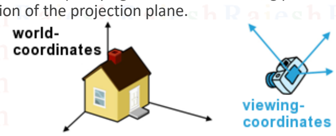


General three-dimensional transformation pipeline, from modeling coordinates to world coordinates to viewing coordinates to projection coordinates to normalized coordinates and, ultimately, to device coordinates.

3D Viewing Pipeline

Modeling transformation and viewing transformation can be done by 3D transformations.

The viewing-coordinate system is used in graphics package as a reference for specifying the observer viewing position and the position of the projection plane.



3D Viewing Pipeline

Projection operations convert the viewing-coordinate description (3D) to coordinate position on the projection plane (2D).

With the operations like clipping, visual-surface identification, and surface rendering.

Normalization transformation & clipping and view port transformation maps the coordinate positions on the projection plane to the output device.

Projection

Transformation that changes a point in n-dimensional coordinate system into a point in a coordinate system that has dimension less than n.

Converts 3-D viewing co-ordinates to 2-D projection coordinates

View Plane or Projection Plane: Two dimensional plane in which 3D objects are projected is called the view plane or projection plane.

Simply it is a display plane on an output device

Types of Projection

1. Parallel Projection

    (On the basis of angle made by projection line with view plane, there are two types of parallel projection.)

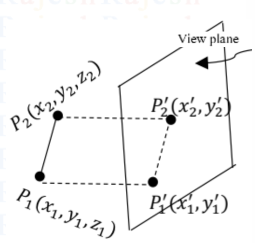
    a) Orthographic parallel projection

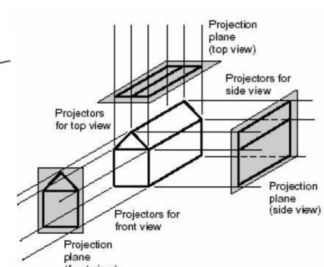
    b) Oblique parallel projection

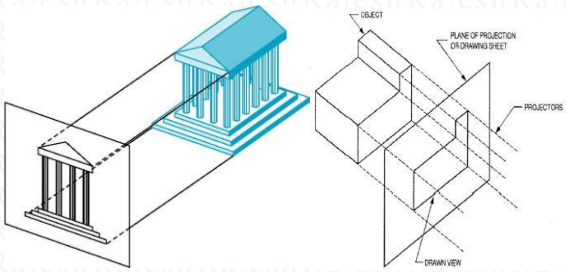
2. Perspective Projection

1 . Parallel Projection

- Coordinate positions are transformed to view plane along parallel lines (projection lines)
- Preserves relative proportions of objects
- Accurate views of various sides of an object are obtained.
- Doesn't give realistic representation of the appearance of the 3-D object

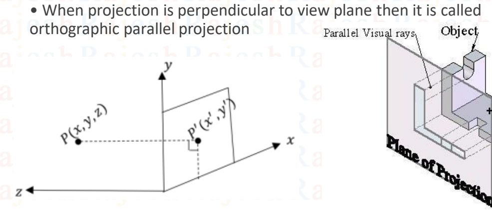






1.1. Orthographic Parallel Projection

- When projection is perpendicular to view plane then it is called orthographic parallel projection



1.1. Orthographic Parallel Projection

Here, after projection of $P(x, y, z)$ on XY-plane we get $P'(x', y')$ where,

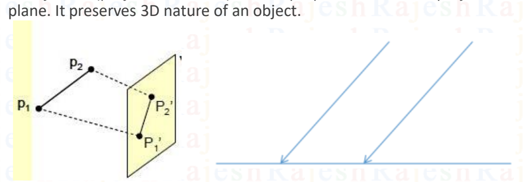
$xx' = x$, $y' = y$ & z=0

In homogenous coordinate form,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

1.2. Oblique Parallel Projection

- Projectors (projection vectors) are not perpendicular to the projection plane. It preserves 3D nature of an object.



1.2. Oblique Parallel Projection

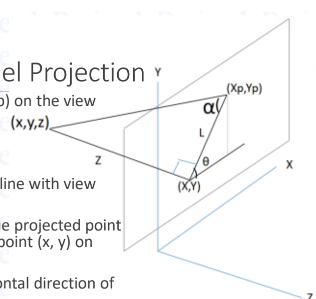
(x,y,z) is projected to position (Xp,Yp) on the view plane.

Oblique projection is specified with two angle 'α' & 'θ'

'α' is the angle made by projection line with view plane line (L)

Where L is formed by joining oblique projected point (xp, yp) and orthogonal projected point (x, y) on view plane

'θ' is the angle between 'L' & horizontal direction of view plane.



1.2. Oblique Parallel Projection

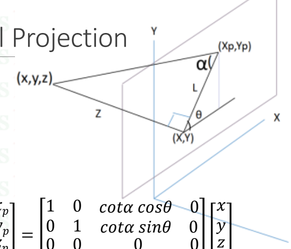
Cos θ = Xp / L
Xp = L Cos θ

But exact position is
Xp = X + L Cos θ

Similarly, Sin θ = Yp / L
Yp= Y + L Sin θ

L depend on angle α
Tan α = Z / L
Xp = X + $Zcot\alpha\ cos$θ
Yp = Y + $Zcot\alpha\ sin$θ

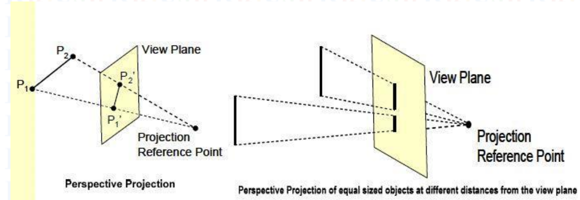
$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & cot\alpha\ cos\theta & 0 \\ 0 & 1 & cot\alpha\ sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



2. Perspective Projection

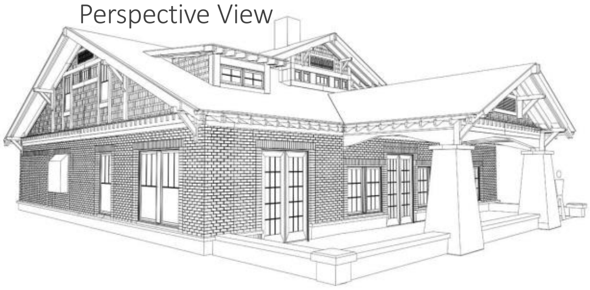
- Coordinate positions are transformed to view plane along lines (projection lines) that converges to a point called **projection reference point** (center of projection or convergence point)
- Produce realistic view
- Does not preserve relative proportions
- Equal sized object appears in different size according as distance from view plane

2. Perspective Projection

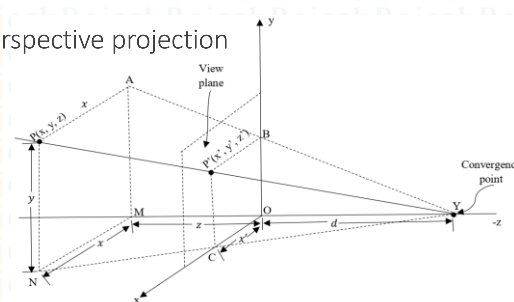


Perspective View

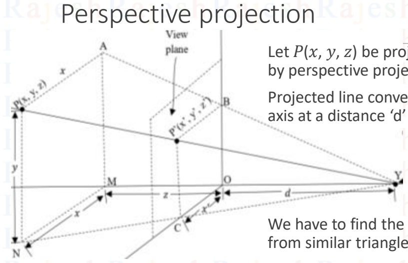
Perspective View



Perspective projection



Perspective projection

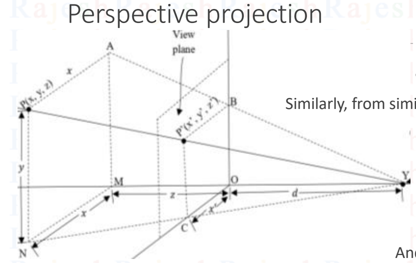


Let $P(x, y, z)$ be projected on X-Y view plane by perspective projection at $P'(x', y', z')$

Projected line converges at point 'y' on z-axis at a distance 'd' from XY view plane.

We have to find the value of $P'(x', y', z')$ from similar triangle NMY & COY.

Perspective projection



$$\frac{NM}{MY} = \frac{CO}{OY}$$

$$\frac{x}{z+d} = \frac{x'}{d}$$

$$\therefore x' = x\frac{d}{z+d}$$

Similarly, from similar triangle AMY & BOY

$$\frac{AM}{MY} = \frac{BO}{BY}$$

$$\frac{y}{z+d} = \frac{y'}{d}$$

$$\therefore y' = y\frac{d}{z+d}$$

And, $z' = 0$

Perspective projection

Now in homogenous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \frac{1}{z+d}\begin{bmatrix} x \cdot d \\ y \cdot d \\ 0 \\ z+d \end{bmatrix} = \frac{1}{z+d}\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

END OF UNIT