Simulation of Continuous and Discrete System

Unit 2

Continuous System Models

- A continuous system is one in which the predominant activities of the system cause smooth changes in the attributes of the system entities.
- When such system is modeled mathematically, the variables of the model representing the attributes are controlled by continuous functions.
- More generally continuous system is described by the rates at which attributes change so that the model consists of differential equations.
- A continuous system is one in which important activities of the system completes smoothly without any delay i.e. no queue events, no shorting of time simulation.
- When such system is modeled mathematically, the variables representing the attributes are controlled by continuous functions.

- Differential equations, both linear, nonlinear, ordinary and partial occur repeatedly in scientific and engineering studies. The reason is that most physical and chemical processes involve rates of change, which require differential equations for their mathematical description.
- The simplest differential equation models have one or more linear differential equations with constant coefficients.
- The methods of applying simulation to continuous models can be developed by showing their application to models, where the differential equations are linear and have constant coefficients and then generalizing to more complex problems

Continuous Simulation

- It concerns the modeling over time of a system by a representation in which state variables change continuously with respect to time
- The simplest differential equation models have one or more linear differential equations with constant coefficients. It is then often possible to solve the model without the use of simulation. However, when nonlinearities are introduced in the model, it frequently becomes impossible or very difficult to model these systems.

Examples of Continuous Simulation

Example: Consider an easy predator-prey model. Let the prey population at time t be given by x(t), and the predator population by y(t). Assume that, in the absence of predators, the prey will grow exponentially according to $x' = ax$ for a certain a > 0. We also assume that the death rate of the prey due to interaction is proportional to x(t)y(t), with a positive proportionality constant. So:

$$x'(t) = ax(t) - bx(t)\, y(t)$$

Without prey, predators will die exponentially according to $y' = -cy$ for a certain c > 0.

Their birth strongly depends on both population sizes, so we finally find for a certain d > 0:

$$y'(t) = -cy(t) + dx(t)\,y(t)$$

$$x'(t) = ax(t) - bx(t)\,y(t)$$

$$y'(t) = -cy(t) + dx(t)\,y(t)$$

We immediately see that both $(e^{at}, 0)$ and $(0, e^{-ct})$ are solution of $(x(t), y(t))$. From this system we find that for every solution we must have

$$x' \frac{c}{x} - a + y' \frac{a}{y} - b = 0$$

$$x' \frac{c}{x} - a + y' \frac{a}{y} - b = 0$$

Integrating both sides of above equation gives us $c \log x(t) - dx(t) + a \log y(t) - by(t) = \text{constant}$ Solutions for $a = b = c = d = 1$.

The given model is considered very simple. Why? the integrating we did two above is possible, rest of the world has no influence, no randomness involved. Usually we cannot find closed-form solutions for the system of differential equations. How do we deal with this problem? Solve numerically.

Integrating both sides of above equation gives us

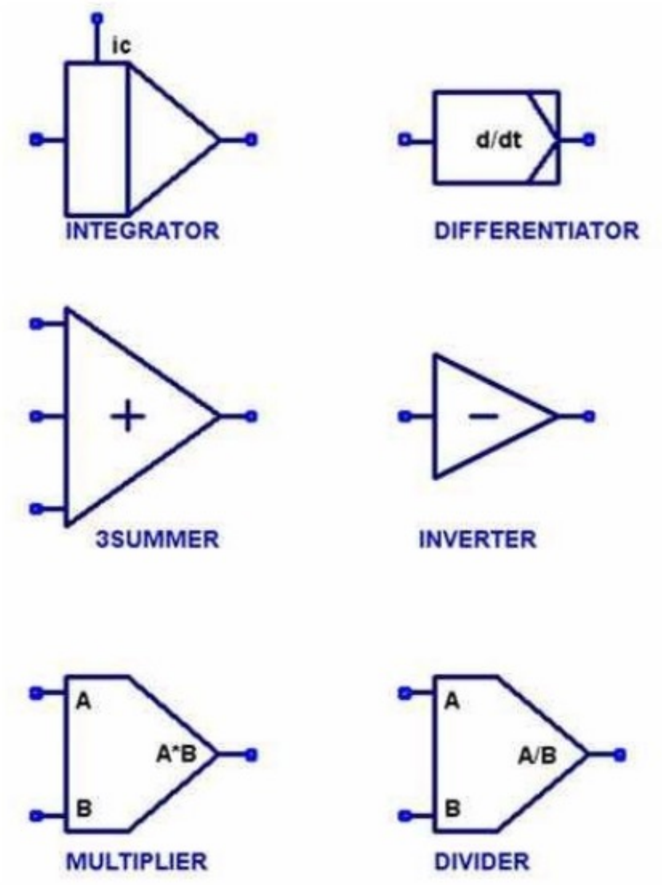
$c \log x(t) - dx(t) + a \log y(t) - by(t) = \text{constant}$ Solutions for $a = b = c = d = 1$

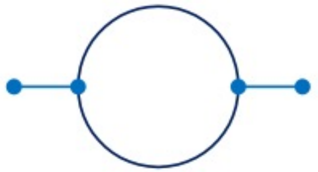
Analog Computers

- Computers that are unified with devices like adder and integrator so as to simulate the continuous mathematical model of the system, which generates continuous outputs
- The electronic analog computer based on the use of high gain DC Amplifiers are widely used analog computer. In such computers, voltages are equated to mathematical variables.
- The coefficient of the model equations are obtained by using the scale factors. The circuit can be arranged to produce integrator that provides integral with respect to time of a single input voltage or sum of input voltages.
- Sign inverter is used to reverse the sign of the input as per requirement of model equation.
- It provides limited accuracy

Analog methods

- Analog method of system simulation is for the use of analog computer and other analog devices in the simulation of continuous system.
- The analog computation is sometimes called differential analyzer. Electronic analog computers for simulation are based on the use of high gain dc amplifiers. In such computers, voltages are equated to mathematical variables. The proper configurations can handle addition of several input voltages each representing the input variables. It provide limited accuracy.
- The General methods to apply analog computers for the simulation of continuous system models involves following components:

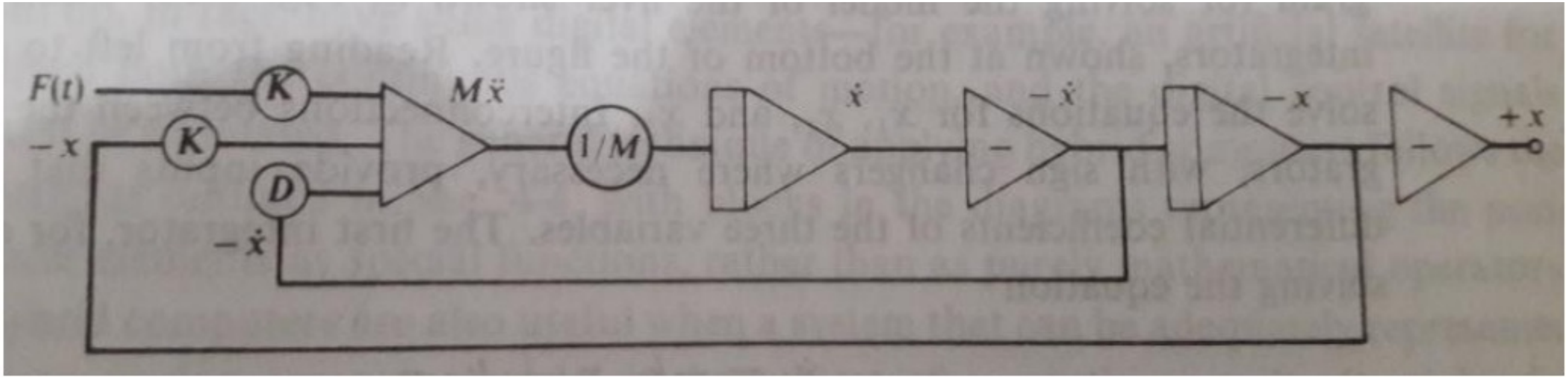


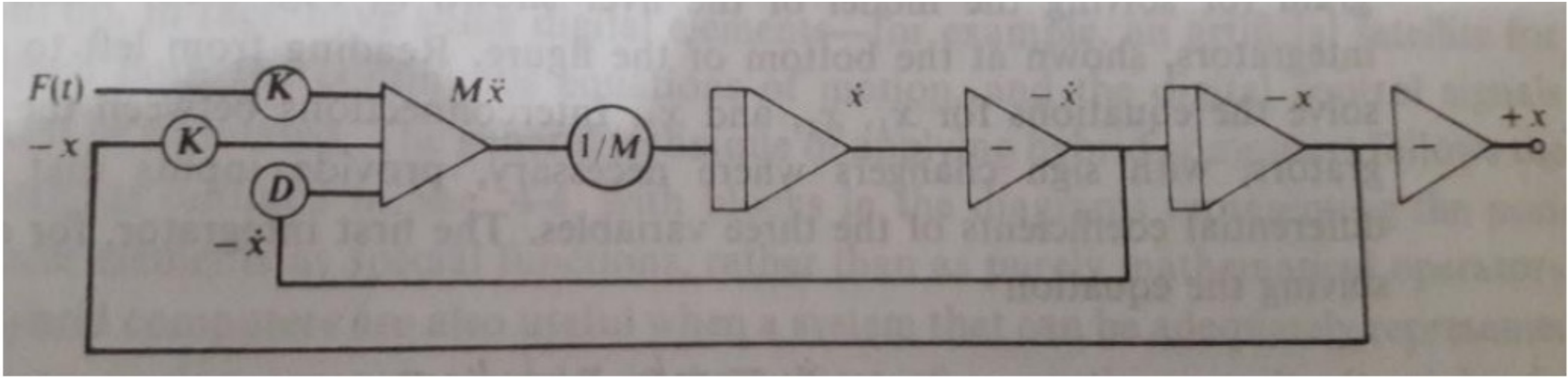


SINGLE MULTIPLIER

Automobile Suspension Problem : Example

- The model is defined by the second order differential equation as:
  M x’’ + D x’ + K x = K F(t)
  =>M x’’ = K F(t) – D x’ - K x
- The diagram analog method modeling is shown below:





- Suppose, a variable representing the input F(t) is supplied, assume there exist variables epresenting –x and –x’. These three variables can be scaled and added to produce Mx’’. Integrating it with a scale factor 1/M produces x’. Changing sign produces –x’, further integrating produces –x, a further sign inver is included to produce +x as output.

Q.> Design analog computer of

$$x\dot{1} = -k_{12}x_1 + k_{21}x_2$$

$$x\dot{2} = k_{12}x_1 - (k_{21} - k_{23})x_2$$

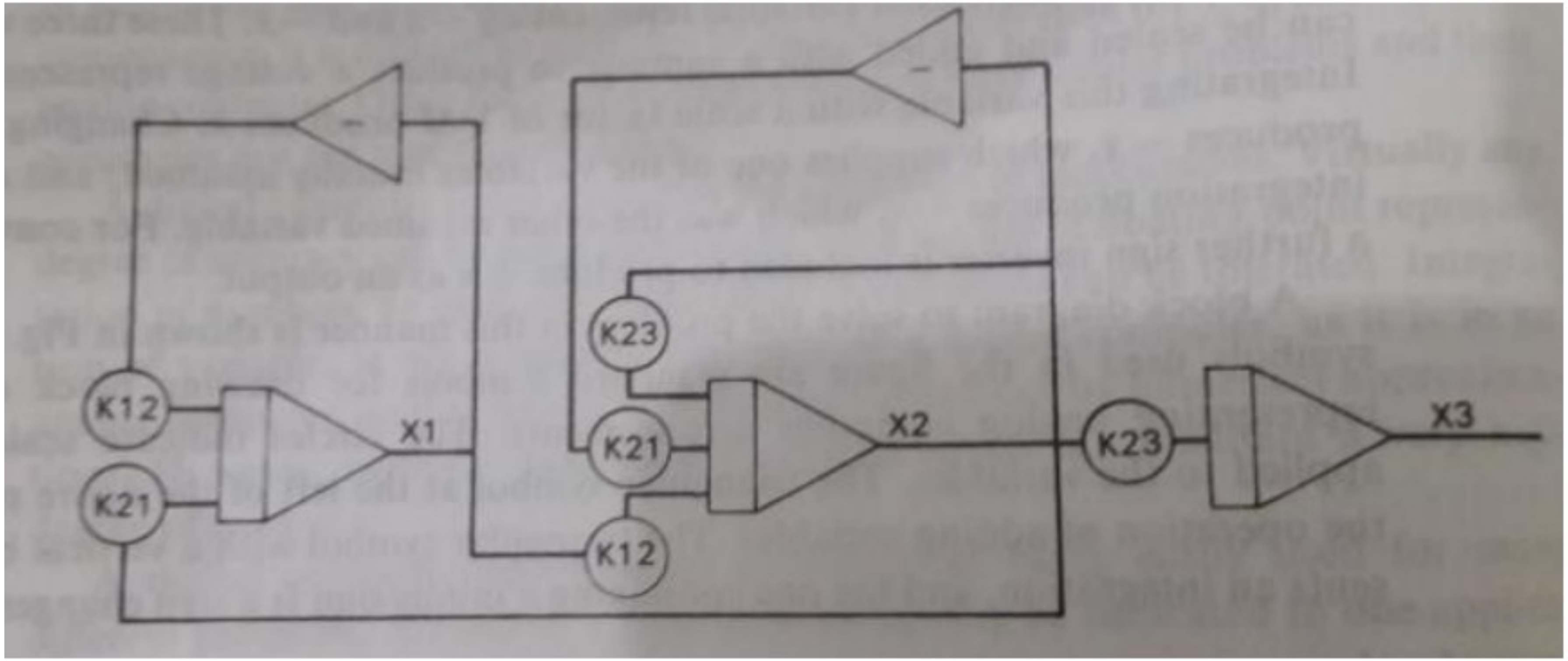
$$x\dot{3} = k_{23}x_2$$

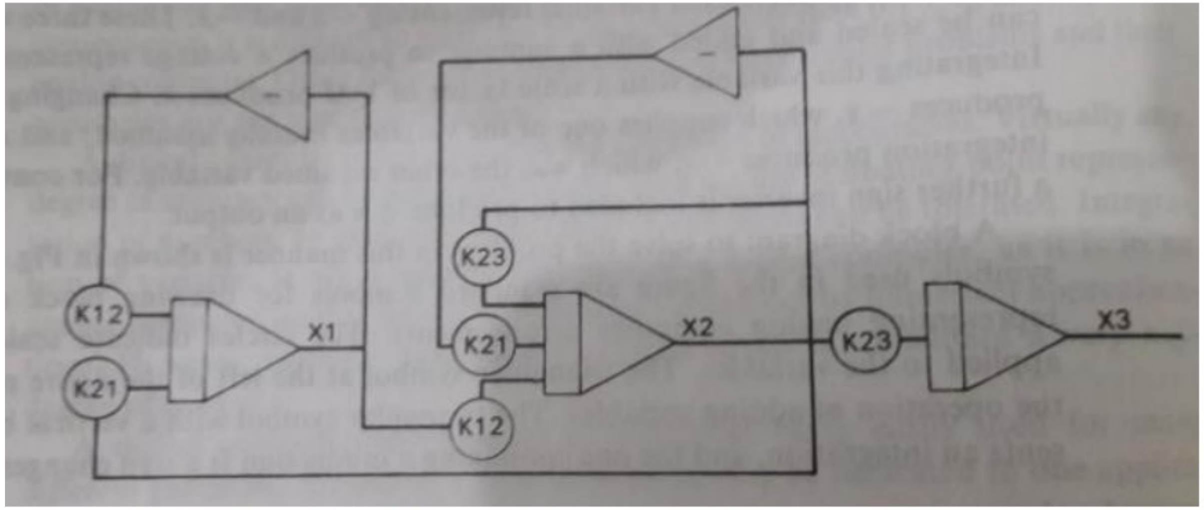
Q.> Design analog computer of

$$\dot{x1} = -k_{12}x_1 + k_{21}x_2$$

$$\dot{x2} = k_{12}x_1 - (k_{21} - k_{23})x_2$$

$$\dot{x3} = k_{23}x_2$$





There are three integrators. Reading from left to right, they solve the equations for $x1$, $x2$ & $x3$. Interconnections between the three integrators with sign changers where necessary provides inputs that define the differential coefficients of the three variables.

First integrator, for example is solving the equation,

$$x\dot{1} = -k_{12}x_1 + k_{21}x_2$$

The second integrator is solving the equation

$$x\dot{2} = k_{12}x_1 - (k_{21} + k_{23})x_2$$

In this case, the variable $x2$ is being used twice as an input to the integrator, so that the two coefficients $k21$ and $k23$ can be changed independently. The last integrator solves the equation

$$x\dot{3} = k_{23}x_2$$

Hybrid Simulation

- In reality, the system is of neither a pure continuous nor a pure discrete nature.
- For simulating such system, the combination of analog and digital computers are used. Such setup is known as hybrid computers.
- The simulation provided by the hybrid computers is known as hybrid simulation.
- The major difficulty in use of hybrid simulation is that it requires high speed converters to transform signals from analog to digital form and vice versa.

Digital Analog Simulator

- Digital Analog simulators indicates the use of programming languages in digital computer to simulate the continuous system.
- The language is composed of macro-instructions which are able to act as adder, integrator and sign-changer.
- A program is written to link these macro-instructions as per the necessity which are provide virtual connections of op amps in analog computers

Feedback System

The system takes feedback from the output i.e. input is coupled with output. Example can be; heat monitoring and control system.

- Issues – amplification and correction of feedback
- Negative feedback – control variable is proportional with output
- Positive feedback – control variable and output are inversely proportional

Other examples;

- Aircraft system
- Error Correction mechanism

Feedback Systems

- A significant factor in the performance of many systems is that coupling occurs between the input and output of the system. The term feedback is used to describe the phenomenon.
- A home heating system controlled by a thermostat is a simple example of a feedback system. The system has a furnace whose purpose is to heat a room, and the output of the system can be measured as room temperature. Depending upon whether the temperature is below or above the thermostat setting the furnace will be turned on or off
- Here the information is being feedback from the output to the input. In this case there are only two states , either furnace is on or off.

Discrete Event Simulation

- Model used in discrete system simulation has a set of numbers to represent the state of the system.
- A number used to represent some aspect of the system state is called a state description. Some state descriptors range over values that have physical significance. Other represent conditions such as flag.
- As the simulation proceeds, the state descriptors change value. We define a discrete event as a set of circumstances that cause the instantaneous change in one or more system state descriptors.
- It is possible that two different events occur simultaneously, or are modelled as being simultaneous so that not all changes of state descriptors occurring simultaneously necessarily belong to a single event.

Representation of Time

- The passage of time is recorded by a number referred to as clock time.
- It is usually set to zero at the beginning of simulation and subsequently indicates how many units of simulated time has passed since the beginning
- As a rule, there is no direct connection between simulated time and time taken to carry out computations.
- The simulation even when carried out by a high speed digital computer could easily take thousand times as long as the actual system operation. On the other hand, for the simulation of an economic system where events have been aggregated to occur once a year, a hundred years of operation could easily be performed in a few minutes of calculations.

Methods for updating clock time

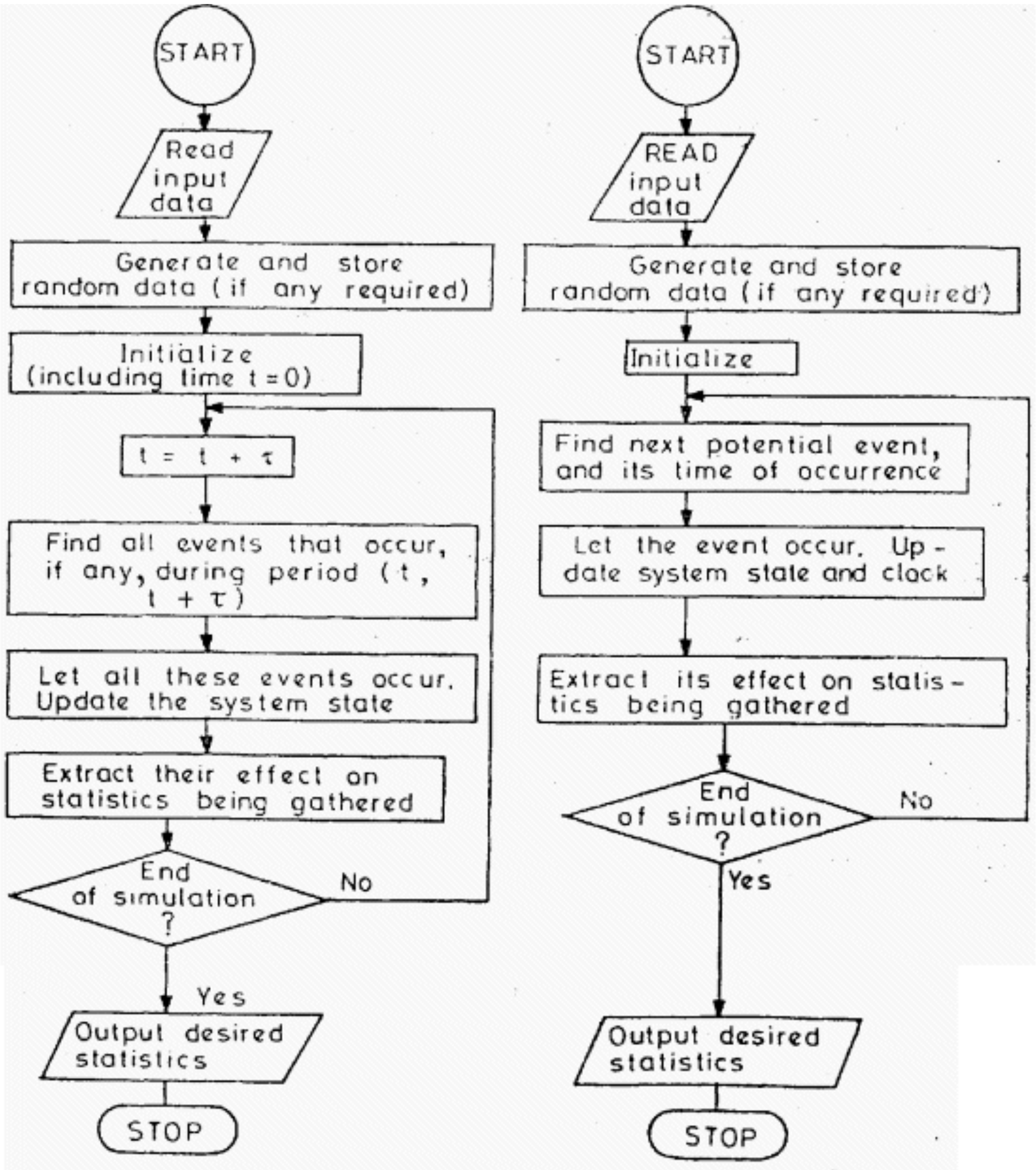
Clock time is updated based on the following two models:

1. <u>Fixed time-step model</u>

- In this the timer simulated by the computer is updated at a fixed time interval. The system is checked to see if any event has taken place during that interval. All the events which take place during the time interval are considered to have occurred simultaneously at the end of the interval.

2. <u>Event-to-event model</u>

- It is also known as the next-event model. In this the computer advances the time to the occurrence of the next event. So it shifts from one event to the next event and the system state does not change in between. A track of the current time is kept when something interesting happens to the system.



(a) Fixed time-step simulation (b) Next-event simulation

Arrival Processes:-

The simulation time starts when first customer arrives to the system. The arrival time is random and hence inter-arrival time is also random. If arrivals vary stochastically, it is necessary to define the probability function of the inter-arrival times. Two or more arrivals may be simultaneous. If n variables are simultaneous then (n-1) of them have zero inter-arrival times.

Customer	Inter-arrival time	Arrival time on clock
1	–	0
2	2	2
3	4	6
4	1	8
5	2	9
6	6	15

<u>Notation</u>

τ_a = mean inter-arrival time.

λ = mean arrival time

They are related as, $\lambda = \frac{1}{\tau_a}$

Example: Office

- Five days a week
- Eight hours a day
- 800 calls a week.

Model the office using time of minutes.

Solution:

Here, 1 week = 40 working hours.

$\therefore$ Inter-arrival time $= \frac{40 \times 60}{800}$

$= 3$ minutes.

3 minutes = 1 call

1 min $= \left(\frac{1}{3}\right)$ call

$= 0.333$ calls

i.e, 0.333 calls per minutes.

Poisson Distribution

- The Poisson distribution is a discrete probability function that means the variable can only take specific values in a given list of numbers, probably infinite. A Poisson distribution measures how many times an event is likely to occur within "n" period of time.
- The formula for the Poisson distribution function is given by:

$$\mathbf{P(n) = P\ (N=n) = (e^{-\lambda}\ \lambda^{n})/n!}$$

Where,

- e is the base of the logarithm
- n is a Poisson random variable
- λ is an average rate of value and must be positive

Poisson Process

Consider random events that can be described by a counting function $N(t)$ defined for all $t \geq 0$. This counting function represents the number of events that occured in $[0, t]$. For each interval $[0, t]$ the value $N(t)$ is an observation of a random variable where the only possible values are the integers $0, 1, 2, \ldots$

The counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process with mean rate λ if the following assumptions are fulfilled:

i) $N(0) = 0$

ii) It has independent increments

iii) No. of events in any interval of length t is a Poisson random variable with parameter λt.

Therefore,

$$\boxed{P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}}$$

Generation of Poisson variate:

- Procedures for generating a Poisson variate N:

Step1: Set $n=0$, $P=1$

Step2: Generate a random number R_{n+1} and replace P by $P \cdot R_{n+1}$.

Step3: If $P < e^{-\alpha}$, then accept $N=n$. Otherwise reject the current n, increase n by one, and return to step 2.

Note: If $N=n$, then $n+1$ random numbers are requested. So the average number is given by $E(N+1) = \alpha+1$.

Example:- Generate 3 Poisson variates with mean $\alpha = 0.2$ for the random numbers $R = 0.4357, 0.4146, 0.8353, 0.9952, 0.8004$.

Solution:

Given: $e^{-\alpha} = e^{-0.2} = 0.8187$

Step1: Set $n=0, P=1$

Step2: $R_1 = 0.4357$, $P = P.R_{n+1} = 1 * R_1 = 1 \times 0.4357 = 0.4357$

Step3: $P = 0.4357$

$e^{-\alpha} = 0.8187$

Since $P < e^{-\alpha}$, accept $N=0$ and goto step1, for second random number.

Step1: Set $n=0$, $P=1$

Step2: $R_1 = 0.4146$, $P = 1 * R_1 = 0.4146$

Step3: $P = 0.4146$, $e^{-\alpha} = 0.8187$

Since, $P < e^{-\alpha}$, accept $N=0$ and goto ~~s~~tep 1.

reject greater या equals to भए हुन्छ । reject भए step1 मा लगाई n+1 गरि step2 मा जाने

Step1: set $n=0, P=1$

Step2: $R_1 = 0.8353$, $P = 1 * R_1 = 0.8353$

Step3: Since $P > e^{-\alpha}$, reject $n=0$ and return to step2 with $n=1$.

Step3: $P = 0.4140$, $e^{-\alpha} = 0.8187$
Since, $P < e^{-\alpha}$, accept $N=0$ and goto step 1.

reject greater या equals हो
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मा जाने

Step1: set $n=0$, $P=1$

Step2: $R_1 = 0.8353$; $P = 1 * R_1 = 0.8353$

Step3: Since $P > e^{-\alpha}$, reject $n=0$ and return to step2 with $n=1$.

Step2: $n=1$, Generating random no. R_{n+1} and replacing P by $P \cdot R_{n+1}$.

$$R_2 = 0.9952$$
$$P = R_1 * R_2 = 0.8353 * 0.9952 = 0.8313$$

Step3: Since $P > e^{-\alpha}$, reject $n=1$ and return to step2 with $n=2$.

Step2: $n=2$, $R_3 = 0.8004$

$$P = R_1 * R_2 * R_3 = 0.8353 * 0.9952 * 0.8004$$
$$= 0.6654$$

Step3: Since $P < e^{-\alpha}$, accept $N=2$.

So all the numbers are done so the summary will be as follows:-

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (Accept)	$N=0$
0	0.4146	0.4146	$P < e^{-\alpha}$ (Accept)	$N=0$
0	0.8353	0.8353	$P \geq e^{-\alpha}$ (Reject)	
1	0.9952	0.8313	$P \geq e^{-\alpha}$ (Reject)	
2	0.8004	0.6654	$P < e^{-\alpha}$ (Accept)	$N=2$

⊛. Nonstationary Poisson Process:

→ Theory upto arrival rate: maybe imp asked in model set.

A non-stationary Poisson process is a Poisson process for which the arrival rate varies with time. More specifically, it can be defined as follows:

The counting process $N(t)$ is a non-stationary Poisson process if:

i) The process has independent increments.

ii) $Pr[N(t+dt) - N(t) \begin{cases} =0] = 1-\lambda(t)\,dt \\ =1] = \lambda(t)\,dt \\ >1] = 0 \end{cases}$

where, $\lambda(t)$ = the arrival rate at time t.
dt = differential sized interval.

The definition is identical to the stationary Poisson process, with the exception that the arrival rate $\lambda(t)$ is now a function of time.

⇒ A counting process $N(t)$ is a stationary Poisson process with rate λ if

i) The process has independent increments.

ii) The process has stationary increments.

iii) $Pr[N(t+dt) - N(t) \begin{cases} =0] = 1-\lambda dt \\ =1] = \lambda dt \\ >1] = 0 \end{cases}$

⇒ A non-stationary Poisson process can be transformed into a stationary Poisson process with arrival rate 1.

Generation of non-stationary Poisson process:

Step1: Let $\lambda^* = \max_{0 \le t \le T} \lambda(t)$ be the maximum of the arrival rate function and set $t = 0$ and $i = 1$.

Step2: Generate E from the exponential distribution with rate λ^* and let $t = t + E$ (this is the arrival time of the stationary Poisson process).

Step3: Generate random number R from the $U(0,1)$ distribution.
If $R \le \lambda(t)/\lambda^*$ then, $T_i = t$ and $i = i+1$.

Step4: Go to step 2.

Example: For the arrival-rate function table given below, generate the first two arrival times.

t (min)	Mean Time between Arrivals (min)	Arrival rate $\lambda(t)$ (arrivals/min)
0	15	1/15
60	12	1/12
120	7	1/7
180	5	1/5
240	8	1/8
300	10	1/10
360	15	1/15
420	20	1/20
480	20	1/20

480	20		1/20

Solution:

$\therefore E = -5\ln(R)$

Step1: $\lambda^* = max_{0 \le t \le T} \lambda(t) = 1/5$, $t=0$, and $i=1$.

Step2: For random number $R=0.2130$, $E=-5\ \ln(0.213) = 13.13$
and $t=0+13.13 = 13.13$

Step3: Generate $R=0.8830$. Since $R=0.8830 \nleq \lambda(13.13)/\lambda^* = (1/15)/(1/5)$
$= 1/3$, do not generate the arrival.

Step4: Go to step 2.

Step2: For random number $R=0.5530$, $E=-5\ln(0.553)=2.96$.
and $t=13.13+2.96=16.09$

Step3: Generate $R=0.0240$. Since $R=0.0240 \le \lambda(16.09)/\lambda^* = (1/15)/(1/5)=1/3$,
set $T_1 = t = 16.09$ and $i=i+1=2$.

Step4: Go to step 2.

Step2: For random number $R=0.0001$, $E=-5\ln(0.0001)=46.05$
and $t=16.09+46.05=62.14$.

Step3: Generate $R=0.1443$. Since $R=0.1443 \le \lambda(62.14)/\lambda^* = (1/12)/(1/5)$
set $T_1 = t = 62.14$ and $i=i+1=3$.

Step4: Go to step 2.

⊛. Batch arrivals :

If arriving customers to a queue occur in "batches" such as busloads, then we can model this by a point process $\Psi = \{t_n\}$ in which the arrival times of customers can coincide: $t_0 \leq t_1 \leq t_2 \leq \ldots$, where $\lim_{n \to \infty} t_n = \infty$. Since the limit is infinite, we conclude that the inequalities with a finite number of equalities in between.

<u>For example:</u> $0 = t_0 = t_1 = t_2 < 1 = t_3 = t_4 = t_5 = t_6 < 3 = t_7 = t_8 < t_9 \ldots$ means that a batch of size 3 occured at the origin, followed by a batch of size 4 at time $t=1$ followed by a batch of size 2 at time $t=3$, and so on.

⊛. Models of Gathering statistics:

Commonly used statistics parameters (included in report).

i) <u>Counts</u> → Number of entities of particular type or number of times some event occured.

ii) <u>Summary measures</u> → Extreme values, mean values, standard deviation.

iii) <u>Utilization</u> → fraction of time some entity is engaged.

iv) <u>Occupancy</u> → Fraction of a group of entities in use on the average.

v) <u>Distribution</u> → Queue length, waiting times etc.

vi) <u>Transit times</u> → Time taken for an entity to move from one part of the system to some other part.

Monte Carlo Simulation

- Is a computerized mathematical technique to generate random sample data based on some known distribution for numerical experiments.
- It is applied in quantitative risk analysis and decision making.
- It is used by professionals of various fields such as finance, project management, energy, manufacturing, engineering, R&D, insurance, petroleum engineering, transportation, etc.

Characteristics of Monte Carlo Method

- Its output must generate random samples
- Its input distribution must be known
- Its result must be known while performing an experiment

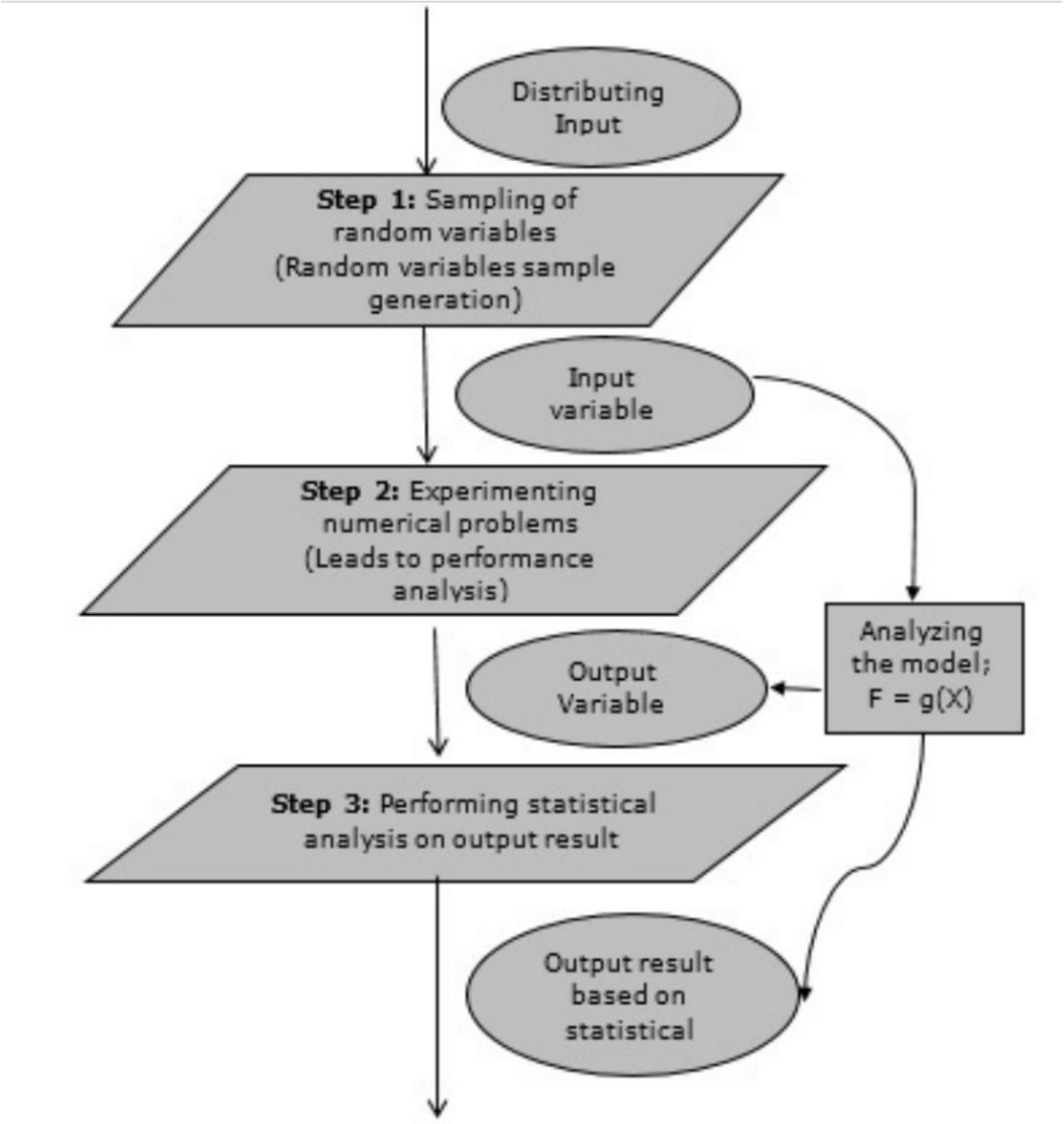
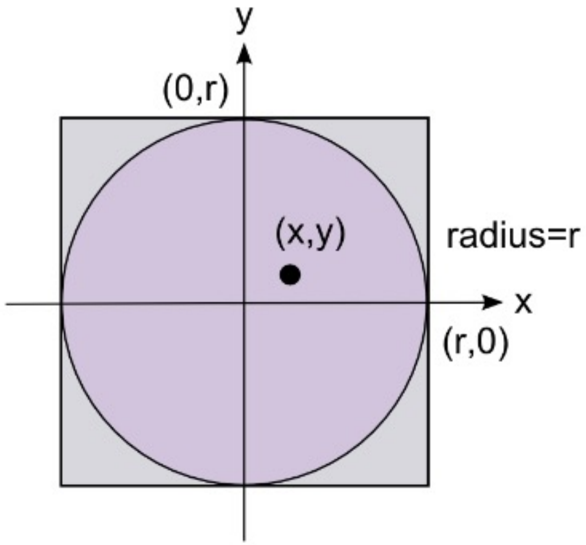


Fig: Flowchart of Monte Carlo simulation

Example:

Determining the value of PI using Monte Carlo method:



$$\frac{Area\ of\ quadrant\ of\ circle}{Area\ of\ Rectangle} = \frac{Number\ of\ points\ inside\ the\ curve}{Number\ of\ points\ inside\ the\ rectangle}$$

$$or, \frac{{}^{1}/_{4}\,\pi r^2}{r^2} = \frac{n}{N}$$

$$\therefore\ \pi = \frac{4n}{N}$$

We use random number generation method to determine the sample points that lie inside or outside the curve. Let (x_0, y_0) be an initial guess for the sample point than from a linear congruential method of random number generation:

$X_{i+1} = (ax_i+c) \bmod m$

$Y_{i+1} = (ay_i+c) \bmod m$

Where a & c are constants, m is the upper limit of generated random number. If $y<=y_i$ then increment n.