Some exam-type exercises

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The following are some problems that you may find on a final exam session of Inferential Statistics. The solutions provided are kept short for the sake of brevity. During the exam session you must be as detailed as possible, by justifying and explaining the reasoning you are applying in orderd to solve the problem.

Exercise 1

Let $X \sim F_\theta$ be a discrete r.v. with $\theta \in \Theta$, where the parameter space is $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. The four possible distributions are as follows.

X	1	2	3
$f(x;\theta_1)$	.4	.1	.5
$f(x;\theta_2)$	.2	.1	.7
$f(x;\theta_3)$	.2	.4	.4
$f(x;\theta_4)$	.6	.3	.1

Let (X_1, X_2) be an i.i.d. random sample from F_θ.

(a) Compute the maximum likelihood estimator

(b) Compute the most powerful test of level $\alpha = .09$ for testing the hypthesis $H_0 : \theta = \theta_2$ against $H_1 : \theta = \theta_4$.

(c) Compute the power of the test under the hypothesis in (b).

(d) Perform a test of level $\alpha = .09$ for testing the hypthesis $H_0 : \theta = \theta_2$ against $H_1 : \theta \neq \theta_2$.

Solution

(a) We need to compute the likelihood function for every possible pair (X_1, X_2) and for every possible $\theta \in \Theta$. Thus consider the following tables

$L(\theta_1; x_1, x_2) = f(x_1;\theta_1)f(x_2;\theta_1)$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	.16	.04	.20
$x_1 = 2$	.04	.01	.05
$x_1 = 3$	.20	.05	.25

$L(\theta_2; x_1, x_2) = f(x_1;\theta_2)f(x_2;\theta_2)$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	.04	.02	.14
$x_1 = 2$	.02	.01	.07
$x_1 = 3$	.14	.07	.49

$L(\theta_3; x_1, x_2) = f(x_1;\theta_3)f(x_2;\theta_3)$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	.04	.08	.08
$x_1 = 2$	.08	.16	.16
$x_1 = 3$	.08	.16	.16

$L(\theta_4; x_1, x_2) = f(x_1; \theta_4) f(x_2; \theta_4)$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	.36	.18	.06
$x_1 = 2$	.18	.09	.03
$x_1 = 3$	.06	.03	.01

By looking at each of the possible pairs we have thus that the maximum likelihood estimator is given by the following table

$\hat{\theta}$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	θ_4	θ_4	θ_1
$x_1 = 2$	θ_4	θ_3	θ_3
$x_1 = 3$	θ_1	θ_3	θ_2

or more formally

$$\hat{\theta} = \begin{cases} \theta_1 & \text{if } (x_1, x_2) \in \{(1,3), (3,1)\} \\ \theta_2 & \text{if } (x_1, x_2) = (3,3) \\ \theta_3 & \text{if } (x_1, x_2) \in \{(2,2), (2,3), (3,2)\} \\ \theta_4 & \text{if } (x_1, x_2) \in \{(1,1), (1,2), (2,1)\}. \end{cases}$$

(b) By the Neyman-Pearson Lemma, the uniformly most powerful test or the best test is one that has rejection region $R = \{(x_1, x_2) : \frac{L(\theta_4; x_1, x_2)}{L(\theta_2; x_1, x_2)} \geq k\}$ for some positive k such that the size of R is equal to $\alpha = .09$. Consider thus the ratio of likelihoods for all possible pairs (x_1, x_2) as in the table below.

$L(\theta_4; x_1, x_2)/L(\theta_2; x_1, x_2)$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 1$	9	9	.4286
$x_1 = 2$	9	9	.4286
$x_1 = 3$	.4286	.4286	0.0204

Now we need to define k.

$$k = .4286 \implies R_1 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}.$$

Thus Under $H_0 : \theta = \theta_2$ we have

$$\begin{aligned} P_{\theta_2}((X_1, X_2) \in R_1) &= P_{\theta_2}(X_1 = 1, X_2 = 1) + P_{\theta_2}(X_1 = 1, X_2 = 2) + \cdots + P_{\theta_2}(X_1 = 3, X_2 = 2) \\ &= .04 + .02 + \cdots + .07 \\ &= .51, \end{aligned}$$

which is too high with respect to the desired $\alpha = 0.09$. The only other choice is then

$$k = 9 \implies R_2 = \{(1,1), (1,2), (2,1), (2,2)\}.$$

for this choice we see that

$$\begin{aligned} P_{\theta_2}((X_1, X_2) \in R_2) &= P_{\theta_2}(X_1 = 1, X_2 = 1) + \cdots + P_{\theta_2}(X_1 = 2, X_2 = 2) \\ &= .04 + 2 \cdot .02 + .01 \\ &= .09 \end{aligned}$$

thus the rejection region has the desired size $\alpha = .09$. The test is thus to reject H_0 if the observed sample follows in the rejection region R_2.

(c) The power under H_0 is equal to α, the power under H_1 is

$$\gamma(\theta_4) = P_{\theta_4}((X_1, X_2) \in R_2) = .81.$$

(d) The only feasible test that we can conduct here is the likelihood ratio test $\lambda_n = 2\log\left(\frac{\sup_{\theta\in\Theta} L(\theta;x_1,x_2)}{\sup_{\theta\in\Theta_0} L(\theta;x_1,x_2)}\right)$. Now, under H_0 we have that $\sup_{\theta\in\Theta_0 L(\theta;x_1,x_2)} = L(\theta_2;x_1,x_2)$ which was given earlier in point (a). Under H_1 we have

$\sup_{\theta\in\Theta} L(\theta;x_1,x_2)$	$x_2=1$	$x_2=2$	$x_2=3$
$x_1=1$	.36	.18	.20
$x_1=2$	.18	.16	.16
$x_1=3$	.20	.16	.49

Thus the likelihood ratio is

$\sup_{\theta\in\Theta} L(\theta;x_1,x_2)/L(\theta_2;x_1,x_2)$	$x_2=1$	$x_2=2$	$x_2=3$
$x_1=1$	9	9	5
$x_1=2$	9	16	2.2857
$x_1=3$	.5	2.2857	1

Now we need to define c such that under H_0 $P_\theta(\lambda_n \geq c) = 0.09$. We see again that $c = 9$ and thus the rejection region must be $R = \{(1,1),(1,2),(2,1),(2,2)\}$.

Exercise 2

Let $X_i \overset{iid}{\sim}$ Wei$(\alpha, 1/\lambda)$, for $i = 1,\ldots,n$, $\alpha > 0, \lambda > 0$. Note that Wei$(\alpha,1/\lambda)$ has p.d.f.

$$f(x;\alpha,\lambda) = \alpha\lambda x^{\alpha-1}e^{-\lambda x^\alpha},\ x > 0,$$

and

$$E(X) = \frac{\lambda^{-1/\alpha}}{\alpha}\Gamma(1/\alpha),\ E(X^2) = \lambda^{-2/\alpha}\Gamma[(2+a)/a].$$

Assume $\alpha = 1$.

(a) Find $\hat{\lambda}_{MM}$, the method of moments estimator for λ.

(b) Compute the bias and the variance of $\hat{\lambda}_{MM}$.

(c) Is $\hat{\lambda}_{MM}$ consistent?

(d) Is $\hat{\lambda}_{MM}$ efficient ?

(e) Compute $\hat{\lambda}$, the maximum likelihood estimator for λ.

(f) Compute the exact and an approximate distribution of $\hat{\lambda}$.

(g) If possible find a UMP test for $H_0 : \lambda \leq \lambda_0$ aginst $H_1 : \lambda > \lambda_0$ with size α.

(h) Does there exist an UMP test for $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda \neq \lambda_0$? Why ?

(i) Compute an approximate confidence interval for λ.

(j) In a study about the lifetime of washing machines (measured in years), the observed sample of size $n = 20$ led to $\sum_{i=1}^{20} x_i = 9.849$. Get the p-value of the hypothesis $H_0 : \lambda = 1$ vs $H_1 : \lambda \neq 1$ using an exact test of size $\alpha = .05$ and compare it by the p-value obtained by an approximate test of the same size.

Solution (a) A method of moments estimator can be obtained by solving in λ the following equation

$$E(X) = \sum_{i=1}^{n} X_i/n \Longleftrightarrow \frac{\lambda^{-1/\alpha}}{\alpha}\Gamma(1/\alpha) = \overline{X},$$

The solution is $\hat{\lambda}_{MM} = \left(\frac{\alpha}{\Gamma(1/\alpha)}\overline{X}\right)^{-\alpha}$, thus for $\alpha = 1$, we have $\hat{\lambda}_{MM} = \frac{1}{\overline{X}}$.

(b) Note that $f(x;1,\lambda)$ is the density of an Exponential r.v. with parameter λ and thus $\sum_{i=1}^n X_i \sim \text{Ga}(n,\lambda)$

$$E(\hat{\lambda}_{MM}) = E\left(\frac{n}{\sum_{i=1}^n X_i}\right) = n\int_0^\infty s^{-1}\frac{s^{n-1}\lambda^n}{\Gamma(n)}e^{-\lambda s}ds = \frac{n\lambda}{n-1},$$

assuming $n > 1$. Thus $\text{bias}(\hat{\lambda}_{MM};\lambda) = \frac{\lambda}{n-1}$.

Furthermore, since $\text{var}(\hat{\lambda}_{MM}) = E(\hat{\lambda}^2_{MM}) - E(\hat{\lambda}_{MM})^2$, and

$$E(\hat{\lambda}^2_{MM}) = E\left(\frac{n^2}{(\sum_{i=1}^n X_i)^2}\right) = n^2\int_0^\infty s^{-2}\frac{s^{n-1}\lambda^n}{\Gamma(n)}e^{-\lambda s}ds = \frac{n^2\lambda^2}{(n-1)(n-2)},$$

assuming $n > 2$, thus

$$\text{var}(\hat{\lambda}_{MM}) = \frac{n^2\lambda^2}{(n-1)^2(n-2)}.$$

(c) Yes. Indeed $\lim_{n\to\infty} \text{bias}(\hat{\lambda}_{MM};\lambda) = 0$ and $\lim_{n\to\infty} \text{var}(\hat{\lambda}_{MM};\lambda) = 0$, and the result follows by Theorem 4.5.

(d) For this we need to compute the Cramer-Rao lower bound. Since for each i, $X_i \sim \text{Exp}(\lambda)$, which has p.d.f. $f(x;\lambda) = \lambda e^{-\lambda x}$, we have that

$$\left(\frac{d\log f(X;\lambda)}{d\lambda}\right)^2 = \left(\frac{1}{\lambda} - X\right)^2,$$

and thus

$$I_n(\lambda) = nI(\lambda) = E\left(\frac{1}{\lambda} - X\right)^2 = \frac{1}{\lambda^2} - \frac{2}{\lambda^2} + \frac{2}{\lambda^2} = \frac{n}{\lambda^2}.$$

Therefore

$$\text{eff}(\hat{\lambda}_{MM};\lambda) = \frac{(n-1)^2(n-2)}{n^3} < 1.$$

(e) Let $\ell(\lambda)$ denote the log-likelihood function given by

$$\ell(\lambda) = n\log(\lambda) - \lambda\sum_{i=1}^n X_i.$$

Setting its first derivative to zero and solving for λ we have

$$\frac{d\ell(\lambda)}{d\lambda} = \ell'(\lambda) = n/\lambda - \sum_{i=1}^n X_i = 0 \implies \hat{\lambda} = \frac{1}{\overline{X}}.$$

Since $\ell(\lambda)$ is a strictly concave function (for $\lambda > 0$) then the solution found is a maximum (actually it's a global one), so the solution $\hat{\lambda}$ is the maximum likelihood estimator for λ.

(f) Note that $\overline{X} = \frac{\sum_{i=1}^n X_i}{n} \sim \text{Ga}(n, n\lambda)$. Let $Y = \frac{\sum_{i=1}^n X_i}{n}$ and consider first the exact distribution of $R = \frac{1}{Y}$. For any $y > 0$, we have

$$F_R(y) = P(R \leq y) = P(1/Y \leq y) = P(Y \geq 1/y) = 1 - P(Y < 1/y) = 1 - F_Y(1/y),$$

where F_Y is the d.f. of the r.v. Y, i.e. the d.f. of a $\mathrm{Ga}(n, n\lambda)$. Let $f_Y(y) = dF_Y(y)/dy$, then

Thus

$$\begin{aligned} f_R(y) &= \frac{dF_R(y)}{dy} \\ &= -f_Y(1/y)\frac{d(1/y)}{dy} \\ &= f_Y(1/y)/y^2. \end{aligned}$$

On the other hand, by the large-sample properties of the maximum likelihood estimator we have that

$$\hat{\lambda} \sim N(\lambda, \lambda^2/n), \quad \text{as} \quad n \to \infty.$$

The result can also be obtained by the Delta Method (see Theorem 0.16).

(g) Let's check the monotone likelihood ratio (MLR) property. Let $0 < \lambda' < \lambda$ and consider the likelihood ratio

$$\frac{L(\lambda)}{L(\lambda')} = \frac{\lambda^n e^{-\lambda \sum_{i=1}^n X_i}}{(\lambda')^n e^{-\lambda' \sum_{i=1}^n X_i}} = \left(\frac{\lambda}{\lambda'}\right)^n \exp\left[-(\lambda - \lambda')n\overline{X}\right].$$

The likelihood ratio as a function of the statistic $-\overline{X}$ is non decreasing, thus the MLR property is satisfied. Let $\Theta_0 = (0, \lambda_0]$ and $\Theta_1 = (\lambda_0, \infty)$. Now we apply the NP Lemma by which the UMP test is the one which has rejection region

$$R = \{(X_1, \ldots, X_n) : L(\theta_1)/L(\theta_0) \geq c, \quad \forall \theta_0 \in \Theta_0, \quad \forall \theta_1 \in \Theta_1\}.$$

But

$$\begin{aligned} L(\theta_1)/L(\theta_0) \geq c &\Longleftrightarrow \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left[(\theta_0 - \theta_1)n\overline{X}\right] \geq c \\ &\Longleftrightarrow \exp\left[(\theta_0 - \theta_1)n\overline{X}\right] \geq \underbrace{c\left(\frac{\theta_1}{\theta_0}\right)^{-n}}_{=b} \\ &\Longleftrightarrow \underbrace{(\theta_0 - \theta_1)}_{<0} n\overline{X} \geq \log b \\ &\Longleftrightarrow \overline{X} \leq \frac{\log b}{n(\theta_1 - \theta_0)} = a, \end{aligned}$$

for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$, thus the rejection region can be also written as

$$R = \{\overline{X} : \overline{X} \leq a\}, \quad \text{for some constant } a.$$

However, since we require a test of size α, we have to determine the value for a such that $\sup_{\theta \in \Theta_0} P_\theta((X_1, \ldots, X_n) \in R) \leq \alpha$. Note that, under $H_0 : \lambda \leq \lambda_0$

$$\overline{X} \sim \text{Ga}(n, n\lambda_0) \Longleftrightarrow T_n = 2n\lambda_0\overline{X} \sim \chi^2_{2n},$$

so a must be such that

$$\sup_{\lambda \leq \lambda_0} P_\lambda((X_1, \ldots, X_n) \in R)) = \sup_{\lambda \leq \lambda_0} P_\lambda(T_n \leq a) \leq \alpha.$$

But when the true parameter is λ we see that $\overline{X} \sim \text{Ga}(n, n\lambda) \sim \chi^2_{2n}$, so the the probability $P_\lambda(T_n \leq a)$ equals

$$P_\lambda\left(2n\lambda_0\overline{X} \leq a\right) = P_\lambda\left(2n\lambda\overline{X} \leq \frac{a\lambda}{\lambda_0}\right) = P_\lambda\left(\chi^2_{2n} \leq \frac{a\lambda}{\lambda_0}\right) = F_{\chi^2_{2n}}\left(\frac{a\lambda}{\lambda_0}\right),$$

where $F_{\chi^2_\nu}(\cdot)$ denotes the d.f. of a χ^2_{2n} r.v. This function is increasing in λ so that the supremum $\sup_{\lambda \leq \lambda_0} P_\lambda(T_n \leq a)$ is taken on at the greatest λ, that is $\lambda = \lambda_0$, thus the condition on the size of the test reduces to

$$P_{\lambda_0}\left(2n\lambda_0\overline{X} \leq a\right) \leq \alpha.$$

Thus the desired value is $a = \chi^2_{2n,1-\alpha}$. The UMP test is thus to reject $H_0 : \lambda \leq \lambda_0$ if in an observed sample we happen to observe $2n\lambda_0\bar{x} \leq \chi^2_{2n,1-\alpha}$. Note that taking $a \leq \chi^2_{2n,1-\alpha}$ would lead to a valid rejection region since the size would still be $\leq \alpha$ but the power would be lower so this new rejection region is not optimal and the test is not UMP.

(h) No. Because this is a case with two-sided alternative hypothesis problem for which the NP lemma does not hold.

(i) An approximate confidence interval of level $1 - \alpha$ for λ can be obtained by the result in point (f). Indeed since $\hat{\lambda} \dot{\sim} N(\lambda, \lambda^2/n)$, then

$$\frac{(\sqrt{n}\hat{\lambda} - \lambda)}{\lambda} \dot{\sim} N(0, 1),$$

which is asymptotically equivalent to

$$\frac{\sqrt{n}(\hat{\lambda} - \lambda)}{\hat{\lambda}} \dot{\sim} N(0, 1).$$

Thus a $1 - \alpha$ confidence interval for λ is

$$[\hat{\lambda}(1 - z_{\alpha/2})/\sqrt{n}, \hat{\lambda}(1 + z_{\alpha/2})/\sqrt{n}].$$

(j) For the exact test we have that under H_0, $2\lambda_0 \sum_{i=1}^{20} X_i \sim \chi^2_{40}$. The observed value of the test statistic is $t(x) = 2\lambda_0 \sum_i x_i = 2 \cdot 1 \cdot 9.849 = 19.698$ and thus the exact p-value is equal to $2\min[P(\chi^2_{40} \leq 19.698), 1 - P(\chi^2_{40} \leq 19.698)] \approx .0059$. On the other hand an approximate p-value can be obtained through the Wald test; the observed Wald test statistic is $\sqrt{n}(\hat{\lambda} - \lambda_0)/\hat{\lambda} = \sqrt{20}(2.031 - 1)/2.031 = 2.27$ and an approximate p-value for the given H_0 vs H_1 is $2P(Z > 2.27) = 0.0232$.