Inferential Statistics **First semester, 2021**

Practice exercises on probability and random variables

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Example 1.1 *A positive integer N is selected with $P(N = n) = (1/2)^n$, $n = 1, 2, \dots,$. If N takes the value n, a coin with probability of heads e^{-n} is tossed once. Find the probability that the resulting toss will be a head.*

Solution.
Let $\mathcal{S} = \{\mathcal{H}, \mathcal{T}\}$ be the sample space of the random experiment "through a single coin". We need to compute $P(H)$. By the Total Laws of Probability,

$$P(H) = P(H|N = 1)p(N = 1) + P(H|N = 2)p(N = 2) + \dots = \sum_{n=1}^{\infty} P(H|N = n)p(N = n)$$

$$= \sum_{n=1}^{\infty} (1/2)^n e^{-n} = \tfrac{1}{2e-1}.$$

Example 1.2 *Let $X \sim \mathrm{Bin}(4, 1/2)$. Compute and plot its d.f., its quantile function and the density function.*

Solution.
Denoting the p.d.f. by $f(x)$, the d.f. by $F(x)$ and the quantile function by $Q(p)$, we have

$$f(x) = P(X = x) = \begin{cases} \frac{16}{81} & \textit{if } x = 0 \\ \frac{32}{81} & \textit{if } x = 1 \\ \frac{24}{81} & \textit{if } x = 2 \\ \frac{8}{81} & \textit{if } x = 3 \\ \frac{1}{81} & \textit{if } x = 4, \end{cases}, F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ \frac{16}{81} & \textit{if } 0 \le x < 1 \\ \frac{48}{81} & \textit{if } 1 \le x < 2 \\ \frac{72}{81} & \textit{if } 2 \le x < 3 \\ \frac{80}{81} & \textit{if } 3 \le x < 4 \\ 1 & \textit{if } 4 \le x, \end{cases}$$

$$Q(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq \frac{16}{81} \\ 1 & \text{if } \frac{16}{81} < p \leq \frac{48}{81} \\ 2 & \text{if } \frac{48}{81} < p \leq \frac{72}{81} \\ 3 & \text{if } \frac{72}{81} < p \leq \frac{80}{81} \\ 4 & \text{if } \frac{80}{81} < p. \end{cases}$$

See Figure 1.1.

Example 1.3 *Let X be a r.v. with p.d.f. $f(x) = Ke^{-\lambda x}$ if $x \geq 0$ and 0 otherwise. Compute:*

(i) K

(ii) $P(1 \leq X \leq 2)$

(iii) $P((R-1)(R-2) \geq 2)$.

Solution.
(i) $f(x)$ is continuous thus $F(x)$ is continuous and so X is continuous (exponential r.v.). We must have $\int_0^\infty Ke^{-x\lambda}dx = 1$, thus

$$K = \left(\int_0^\infty e^{-x\lambda}dx\right)^{-1} = (-\tfrac{1}{\lambda}e^{-y}|_0^\infty)^{-1} = \lambda,$$

where we have made the change of variable $y = \lambda x$.
(ii) X is a continuous r.v. thus

$$P(1 \leq X \leq 2) = \int_1^2 \lambda e^{-\lambda x}dx = \int_\lambda^{2\lambda} e^{-y}dy = -e^{-y}|_\lambda^{-2\lambda} = e^{-\lambda} - e^{-2\lambda}.$$

(iii) We need to consider only $X > 0$ since f is zero otherwise. The inequality $(X-1)(X-2) \geq 0$ holds if $X \leq 1$ or $X \geq 2$, thus

$$\begin{aligned} P((X-1)(X-2) \geq 0) &= P(\{X \leq 1\} \cup \{X \geq 2\}) = P(X \leq 1) + P(X \geq 2) \\ &= 1 - e^{-\lambda} + e^{-2\lambda}. \end{aligned}$$

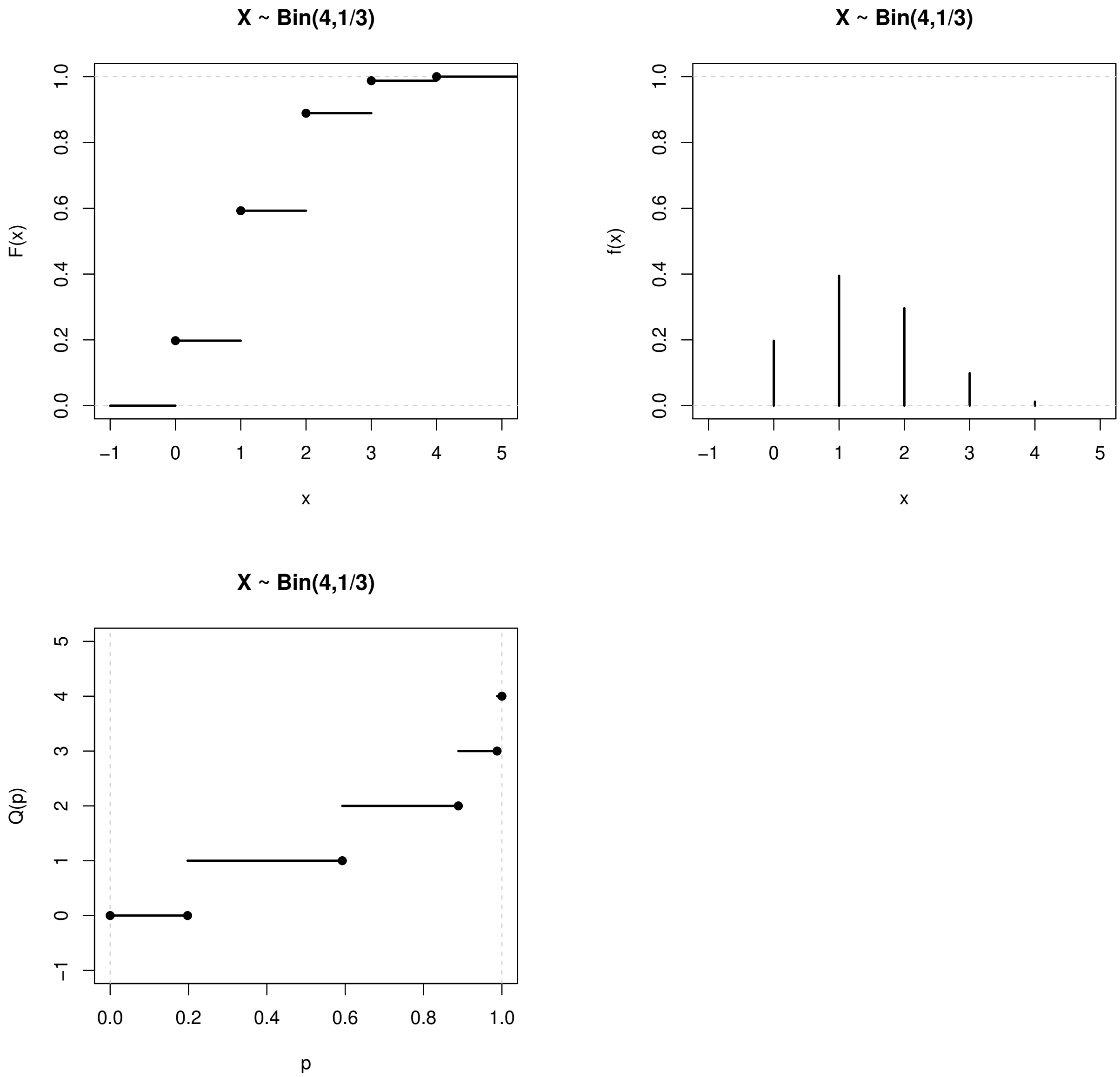

Figure 1.1: Exercise 1.2.

Example 1.4 *Let X be as in Example 1.3 and $Y = \sqrt{X}$. Derive the d.f. and p.d.f. of Y.*

Solution.

Note that $X \geq 0$ and $Y = g(X)$ with $g(x)$ being the square root function. Clearly, $y = g(x)$

is both bijective and differentiable, with $g^{-1}(y) = y^2$, *with* $\frac{dg^{-1}(y)}{dy} = 2y \geq 0$. *Now*

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}.$$

Application of Theorem 0.6 (Lecture 0) gives

$$F_Y(y) = F_x(g^{-1}(y)) = 1 - e^{-\lambda y^2},$$

We see that the properties of the distribution function (Theorem 0.5) hold (check!). Furthermore

$$f_Y(y) = \frac{dF_Y(y)}{dy} = 2\lambda y e^{-\lambda y^2},$$

and clearly $\int_0^\infty f_Y(y)dy = \int_0^\infty 2\lambda y e^{-\lambda y^2} dy = 1$.

Example 1.5 *Let* $X \sim N(0,1)$ *and* $Y = X^2$. *Derive the p.d.f. of* Y.

Solution.
This time $X \in \mathbb{R}$ *and* $Y = g(X)$ *with* $g(x) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *being the square root function. Clearly,* $y = g(x)$ *is surjective and differentiable but not injective, so the Theorem 0.6 cannot be used for the p.d.f. (though it can be used for the d.f.). Let* $F_X(x)$ *be the d.f. of* X

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Since Y *is a continuous r.v., we can get its p.d.f. by differentiating the d.f. (assuming* $y > 0$*),*

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) \\ &= \frac{y^{1/2-1}e^{-y/2}}{2^{1/2}\sqrt{\pi}}, \end{aligned}$$

which is the p.d.f. of a χ^2_1 *random variable.*

Example 1.6 *Let* X_1 *and* X_2 *be independent r.v. each with distribution as in Example 1.3 with* $\lambda = 1$. *Find the density function of* $Y = X_1 + X_2$.

Solution.

We consider two approaches.

Approach 1: integration over a constrained space.

For any $y \geq 0$ (see Figure 1.2),

$$F_Y(y) = P(Y \leq y) = P(X_1 + X_2 \leq y) = P(X_2 \leq y - X_1) = \int\int_{(x_1,x_2)\in E} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$
$$= \int_0^y \int_0^{y-x_1} e^{-x_1-x_2} dx_2 dx_1$$
$$= 1 - e^{-y} - ye^{-y},$$

and

$$f_Y(y) = \frac{dF_Y(y)}{dy} = ye^{-y}.$$

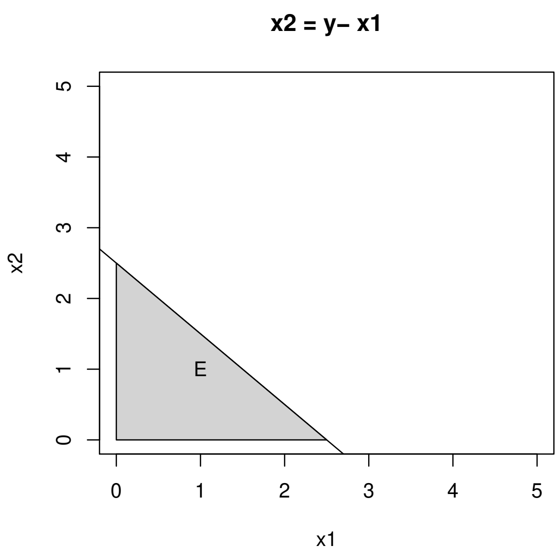

Figure 1.2: Exercise 1.6.

Approach 2: using Theorem 1.4.

Let $Y_1 = g_1(X_1+X_2) = X_1+X_2$ *and* $Y_2 = g_2(X_1+X_2) = X_2$. *Then* $X_1 = g_1^{-1}(Y_1, Y_2) = Y_1-Y_2$

and $X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$*, so*

$$g^{-1}(y_1, y_2) = (g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)).$$

The partial derivatives of g^{-1} *are*

$$\tfrac{dg^{-1}(y_1,y_2)}{dy_1} = (1, 0), \quad \tfrac{dg^{-1}(y_1,y_2)}{dy_2} = (-1, 1).$$

The Jacobian is thus

$$\det(J(y)) = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1.$$

The joint p.d.f. of (Y_1, Y_2) *is*

$$f_{Y_1,Y_2}(y_1, y_2) = e^{y_2 - y_1} e^{-y_2} = e^{-y_1}.$$

Note that this p.d.f. is positive in the space $E = \{(y_1, y_2) \in \mathbb{R}^2 : 0 \leq y_2 \leq y_1\}$ *and zero elsewhere, since* Y_2 *cannot be larger that* Y_1*. The required distribution is the marginal density of* Y_1*, which is*

$$f_{Y_1}(y_1) = \int_{y_2 \in E} f_{Y_1,Y_2}(y_1, y_2) dy_2 = \int_0^{y_1} e^{-y_1} dy_2 = y_1 e^{-y_1}.$$