Don Bosco Institute of Technology
APPLIED MATHEMATICS - III

Sem III Comp/EXTC/IT/Mech **November 2020**

QUESTION BANK For IA-KT Nov-Dec 2020

Topics Covered: Laplace and Inverse Laplace Transforms

Revision of Laplace Transforms

Definition: Suppose $f(t)$ **is a function defined for** $t > 0$ **then the Laplace Transform of** $f(t)$ **is given by** $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$ **where** $Re(s) > 0$

Using the definition and the following properties we can solve problems in Laplace and Inverse Laplace Transforms. In what follows, you are given the properties and standard formulas which are required. You can solve the question bank problems by using the properties and formulas.

1. $\int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

2. $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

3. $\int \log x \ dx = x \log x - x$

4. **Generalised** uv **rule for integrating by parts**
$\int uv \ dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

   where 'dashes' represent differentiation and suffixes represent integration.

5. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

6. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

7. $\sinh x = \frac{e^{x} - e^{-x}}{2}$

8. $\cosh x = \frac{e^{x} + e^{x}}{2}$

9. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

10. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

11. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

12. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

13. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

14. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

15. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

16. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

17. $\sin 2x = 2 \sin x \cos x$
$[\Rightarrow \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}]$

18. $\cos 2x = \cos^2 x - \sin^2 x$
$[\Rightarrow \cos 2x = 2\cos^2 x - 1 \ (\ since \ \ \sin^2 x = 1 - \cos^2 x)]$
$[\Rightarrow \cos^2 x \quad = \quad \dfrac{1+\cos 2x}{2}]$
Also $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = 1 - 2\sin^2 x$
$\Rightarrow \sin^2 x \quad = \quad \dfrac{1-\cos 2x}{2}$

19. $\cos^3 x \quad = \quad \dfrac{1}{4}\left(3\cos x + \cos 3x\right)$

20. $\sin^3 x \quad = \quad \dfrac{1}{4}\left(3\sin x - \sin 3x\right)$

21. $\cosh^2 x = \dfrac{1+\cosh 2x}{2}$

22. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

1. **Linearity Property: If** $L\{f(t)\} = F(s)$ **and** $L\{g(t)\} = G(s)$**, then**
   $L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\} = c_1 F(s) + c_2 G(s)$

2. **First Shifting Theorem: If** $L\{f(t)\} = F(s)$**, then** $L\{e^{at} f(t)\} = F(s-a)$ **[That is, replace** s **by** $s-a$ **in the expression of Laplace Transform]**

   Example
   1. Find $L\{e^{2t}\sin 3t\}$
   Solution: We have
   $L\{e^{2t}\sin 3t\} = L\{\sin 3t\}_{s\to s-2}$ **by first shifting theorem**
   $\Rightarrow L\{e^{2t}\sin 3t\} = \dfrac{3}{s^2+9}_{s\to s-2}$
   $\Rightarrow L\{e^{2t}\sin 3t\} = \dfrac{3}{(s-2)^2+9}$

3. **Change of Scale Property: If** $L\{f(t)\} = F(s)$**, then** $L\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

4. **Multiplication by** t**: If** $L\{f(t)\} = F(s)$**, then** $L\{tf(t)\} = (-1)\dfrac{d}{ds}F(s)$, **In general,**
   $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}F(s)$, **where** $n = 1, 2, 3, \cdots$

5. **Laplace Transform of Derivatives: [Multiplication by** s**] Suppose** $f'(t)$**, the derivative of** $f(t)$ **is continuous and** $L\{f(t)\} = F(s)$**. Then** $L\{f'(t)\} = sF(s) - f(0)$.
   Similarly, $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.
   In general,
   $L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0). - \cdots, -sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n = 1, 2, 3, ...$

6. **Division by** t**: If** $L\{f(t)\} = F(s)$**, then** $L\left\{\dfrac{1}{t}f(t)\right\} = \displaystyle\int_s^\infty F(s)ds$,
   provided the integral exists.

7. **Laplace Transform of Integrals: [Division by** s**]**
   Suppose $L\{f(t)\} = F(s)$**. Then** $L\left\{\displaystyle\int_0^t f(u)du\right\} = \dfrac{1}{s}F(s)$
   Remark: $L\left\{\displaystyle\int_0^t \int_0^t \cdots \int_0^t f(u)du \ du \cdots n \ times\right\} = \dfrac{1}{s^n}F(s)$

8. **Second Shifting Theorem: If** $L\{f(t)\} = F(s)$**, and if** $g(t) = \begin{cases} 0, & t < a \\ f(t-a), & t > a \end{cases}$**, then** $L\{g(t)\} = e^{-as}F(s)$.

Question Bank in Laplace Transforms

1. $\frac{1}{s^2}$ **equals**

A.* $L\{t\}$ **B.** $\frac{1}{2}L\{t\}$ **C.**$L\{1\}$ **D.**$L\{t^2\}$

2. $\frac{2}{s^2+9}$ **equals**

A. $L\{\sin 3t\}$ **B.** $\frac{1}{3}L\{\sin 3t\}$ **C. *** $\frac{2}{3}L\{\sin 3t\}$ **D.**$\frac{1}{2}L\{\sin 3t\}$

3. $\frac{s}{s^2+25}$ **equals**

A. * $\frac{1}{5}L\{\frac{d}{dt}(\sin 5t)\}$ **B.** $\frac{1}{5}L\{\cos 5t\}$ **C.** $L\{\sin 5t\}$ **D.**$\frac{1}{5}L\{\sin 5t\}$

4. $L\{e^{3t}\sin 4t\}$ **equals**

A. $\frac{4}{(s+3)^2+16}$ **B. *** $\frac{4}{(s-3)^2+16}$ **C.** $\frac{3}{(s-4)^2+9}$ **D.** $\frac{3}{(s+4)^2+9}$

5. $L\{e^{-3t}\cos 2t\}$ **equals**

A. $\frac{s}{(s-3)^2+9}$ **B.** $\frac{s}{(s-2)^2+4}$ **C. *** $\frac{s}{(s+3)^2+4}$ **D.** $\frac{s}{(s+2)^2+4}$

6. $L\{(\sinh t - \cosh t)^{10}\}$ **equals**

A. $(\frac{10}{(s^2-100)} - \frac{s}{(s^2-100)})$ **B. *** $\frac{1}{(s+10)}$ **C.** $\frac{-2}{(s-10)}$ **D.** $(\frac{10}{(s^2+100)} - \frac{s}{(s^2+100)})$

7. $L\{\sin 2t \cos 3t\}$ **equals**

A. * $\frac{1}{2}\left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right)$ **B.** $\frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$ **C.** $\frac{1}{2}\left(\frac{s}{s^2+25} - \frac{s}{s^2+1}\right)$ **D.** $\frac{1}{2}\left(\frac{s}{s^2+25} + \frac{s}{s^2+1}\right)$

8. $L\{\frac{\sin^2 2t}{t}\}$ **equals**

A. $\frac{1}{2}log\left(\frac{s^2}{s^2+16}\right)$ **B.** $\frac{1}{4}\tan^{-1}(\frac{s}{4})$ **C.** $\frac{1}{2}\tan^{-1}(\frac{s}{2})$ **D. *** $\frac{1}{4}log\left(\frac{s^2+16}{s^2}\right)$

9. $\int_0^\infty e^{-2t} t\cos 5t$ **equals**

A. $\frac{1}{29}$ **B. *** $\frac{-21}{841}$ **C.** $\frac{-21}{29}$ **D.** $\frac{1}{841}$

Questions in Inverse Laplace transforms

Suppose $L\{f(t)\} = F(s)$, **then** $f(t)$ **is called the Inverse Laplace Transform (ILT) of** $F(s)$ **and we write** $f(t) = L^{-1}\{F(s)\}$

I. $L\{e^{at}\} = \dfrac{1}{s-a} \quad \Rightarrow L^{-1}\{\dfrac{1}{s-a}\} = e^{at}$

II. (i) $L\{\sin at\} = \dfrac{a}{s^2+a^2} \quad \Rightarrow L^{-1}\{\dfrac{a}{s^2+a^2}\} = \sin at \quad \Rightarrow L^{-1}\{\dfrac{1}{s^2+a^2}\} = \dfrac{1}{a}\sin at$

(ii) $L\{\cos at\} = \dfrac{s}{s^2+a^2} \quad \Rightarrow L^{-1}\{\dfrac{s}{s^2+a^2}\} = \cos at$

III. (i) $L\{\sinh at\} = \dfrac{a}{s^2-a^2} \quad \Rightarrow L^{-1}\{\dfrac{a}{s^2-a^2}\} = \sinh at \quad \Rightarrow L^{-1}\{\dfrac{1}{s^2-a^2}\} = \dfrac{1}{a}\sinh at$

(ii) $L\{\cosh at\} = \dfrac{s}{s^2-a^2} \quad \Rightarrow L^{-1}\{\dfrac{s}{s^2-a^2}\} = \cosh at$

IV. $L\{t^b\} = \dfrac{\Gamma(b+1)}{s^{b+1}} \quad \Rightarrow L^{-1}\{\dfrac{\Gamma(b+1)}{s^{b+1}}\} = t^b \quad \Rightarrow L^{-1}\{\dfrac{1}{s^{b+1}}\} = \dfrac{t^b}{\Gamma(b+1)} \quad \Rightarrow L^{-1}\{\dfrac{1}{s^b}\} = \dfrac{t^{b-1}}{\Gamma(b)}$

1. **Obtain the Inverse Laplace Transform of the following functions:**

   (i) $\dfrac{1}{(s-5)}$ **(ii)** $\dfrac{1}{(2s-3)}$ **(iii)** $\dfrac{7}{(5s+2)}$ **(iv)** $\dfrac{8}{(6s-1)}$

   V. First Shifting Theorem $L\{e^{at}f(t)\} = \overline{f(s-a)} \quad \Rightarrow L^{-1}\overline{f(s-a)} = e^{at}f(t)$

2. **Obtain the Inverse Laplace Transform of the following functions:**

   (i) $\dfrac{s-2}{s^2-4s+5}$ **(ii)** $\dfrac{s+1}{s^2+1}$ **(i)** $\dfrac{s}{s^2+6s+25}$

3. **Obtain the Inverse Laplace Transform of the following functions:**

   (Use of Partial Fractions) (i) $\dfrac{1}{s(s-1)}$ **(ii)** $\dfrac{1}{s(s-2)}$ **(iii)** $\dfrac{1}{(s-2)(s-3)}$ **(iv)** $\dfrac{1}{s(s^2+4)}$ **(v)** $\dfrac{1}{s^2+42+13}$ **(vi)** $\dfrac{s+3}{s^2+6s+2}$

   (vii) $\dfrac{s}{s^2+5s+10}$ **(viii)** $\dfrac{1}{(s^2+4)(s^2+9)}$ **(ix)** $\dfrac{s}{(s^2+4)(s^2+9)}$ **(x)** $\dfrac{s-1}{(s^2-2s+5)}$

   (xi) $\dfrac{5s}{25s^2+3}$ **(xii)** $\dfrac{3s}{(9s^2+25)}$ **(xiii)** $\dfrac{2}{(s-1)^5}$ **(xiv)** $\dfrac{s+3}{(s^2+6s+1)}$

   VI. Inverse Laplace Transform using Derivatives

   We have $L\{tf(t)\} = -\dfrac{d}{ds}\overline{f(s)}$

   $\Rightarrow tf(t) = -L^{-1}\{\overline{f'(s)}\} \quad \Rightarrow f(t) = \dfrac{-1}{t}L^{-1}\{\overline{f'(s)}\}$

4. **Obtain the Inverse Laplace Transform of the following functions:**

   (i) $\log\{\dfrac{s+a}{s+b}\}$ **(ii)** $\cot^{-1}(s)$ **(iii)** $\log\{\dfrac{(s-2)^2}{s^2++1}\}$ **(iv)** $\log\{\dfrac{s^2+a^2}{s^2+b^2}\}$

   (v) $\log\sqrt{\dfrac{s-1}{s+1}}$ **(vi)** $\log\sqrt{\dfrac{s^2-a^2}{s^2}}$ **(vii)** $\log\sqrt{\dfrac{s^2+1}{s(s+1)}}$ **(viii)** $\tan^{-1}(\dfrac{2}{s^2})$

   (ix) $\tan^{-1}(\dfrac{2}{s})$ **(x)** $\tan^{-1}(\dfrac{s+a}{b})$ **(xi)** $2\tanh^{-1}s$ **(xii)** $\cot^{-1}(s-1)$

   VII. Inverse Laplace Transform using Second Shifting Theorem

   Recall that if $g(t) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \qquad (*) \end{cases}$

   Then $L\{g(t)\} = e^{-as}\overline{f(s)}$

   1. Find $f(t) = \overline{f(s)}$**. 2. Then obtain** $g(t)$ **as in (*)**

5. **Obtain the Inverse Laplace Transform of the following functions:**

   (i) $\dfrac{e^{-3s}}{(s-2)^4}$ **(ii)** $\dfrac{se^{-\frac{4\pi s}{5}}}{s^2+25}$ **(iii)** $\dfrac{e^{3-2s}}{(s+4)^{5/2}}$ **(iv)** $\dfrac{e^{-s}}{(s+1)^2}$

   (v) $\dfrac{se^{-as}}{(s^2+3s+2)}$ **(vi)** $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$ **(vii)** $\dfrac{e^{-4s}}{\sqrt{2s+7}}$ **(viii)** $e^{-s}(\dfrac{1-\sqrt{s}}{s^2})^2$

   VIII. Convolution Theorem: Suppose $L^{-1}\{\overline{f_1(s)}\} = f_1(t)$ **and** $L^{-1}\{\overline{f_2(s)}\} = f_2(t)$.

   Then, $L^{-1}\{\overline{f_1(s)} * \overline{f_2(s)}\} = \displaystyle\int_0^t f_1(u)f_2(t-u)du$

6. Obtain the ILT by convolution theorem:[Choose the simpler function as $\overline{f_2(s)}$]

(i) $\dfrac{1}{s(s+a)}$ **(ii)** $\dfrac{1}{(s-2)^2(s+3)}$ **(iii)** $\dfrac{1}{s\sqrt{s+4}}$ **(iv)** $\dfrac{s}{(s^2+a^2)^2}$ **(v)** $\dfrac{1}{(s-2)^4(s+3)}$ **(vi)** $\dfrac{1}{(s+1)(s^2+1)}$

(vii) $\dfrac{1}{(s^2+1)^3}$ **(viii)** $\dfrac{s+29}{(s+4)(s^2+9)}$ **(ix)** $\dfrac{s+2}{s^2(s-1)^2}$

7. Solve the following differential equations using Laplace Transforms:

(i) $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+5y=e^{-t}\sin t$ **(ii)** $y''-3y'+2y=4t+e^{3t}$, **when** $y(0)=1, y'(0)=-1.$

(iii) $\dfrac{d^2y}{dt^2}+4\dfrac{dy}{dt}+8y=1,$ **where** $y(0)=0, y'(0)=1$

(iv) $2\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-3y=0,$ **given that, when** $x=0,\ y=4$ **and** $\dfrac{dy}{dx}=9$

(v) $\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+13y=0,$ **where** $y(0)=3, y'(0)=7$

(vi) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}=9,$ **given that, when** $x=0,\ y=0$ **and** $\dfrac{dy}{dx}=0$

(vii) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+10y=e^{2x}+20,$ **given that, when** $x=0,\ y=0$ **and** $\dfrac{dy}{dx}=-1/3$

(viii) $\dfrac{d^2y}{dt^2}+9y=18t,$ **where** $y(0)=0, y(\pi/2)=0$

(ix) $\dfrac{dy}{dt}+2y+\displaystyle\int_0^t y\,dt=\sin t,$ **where** $y(0)=1$

(x) $\dfrac{d^2y}{dt^2}+4y=f(t),$ **where** $f(t)=\begin{cases} 1, & 0<t<1 \\ 0, & t>1 \end{cases}$ **and** $y(0)=0, y'(0)=1$

Comparison Table of Laplace and Inverse Laplace Transforms

S.No	Function	Laplace transform	Function	Inverse Laplace Transform
1	k	$\frac{k}{s}$	$\frac{k}{s}$	k
2	e^{at}	$\frac{1}{s-a}$	$\frac{1}{s-a}$	e^{at}
3	$\sin at$	$\frac{a}{s^2+a^2}$	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
4	$\cos at$	$\frac{s}{s^2+a^2}$	$\frac{s}{s^2+a^2}$	$\cos at$
5	t^b	$\frac{\Gamma(b+1)}{s^{b+1}}$	$\frac{1}{s^{b+1}}$	$\frac{t^b}{\Gamma(b+1)}$
			$\frac{1}{s^b}$	$\frac{t^{b-1}}{\Gamma(b)}$
6	$\cosh at$	$\frac{s}{s^2-a^2}$	$\frac{s}{s^2-a^2}$	$\cosh at$
7	$\sinh at$	$\frac{a}{s^2-a^2}$	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$