MATH303-22S2: Assignment 1

Instructions

- This assignment is due at **5pm on Monday 19 September**.
- It is to be submitted, as a single pdf file, to the dropbox folder on the Learn page.
- This assignment is worth **20%** of your final grade.
- You may work by yourself or with one other person. If you hand in a joint assignment, you will each be given the same mark.
- You must write your name (and your partner's name if you are working as a pair) on your submission.
- Your assignment does not need to be typed, but you should expect to lose marks for poorly presented working and/or incomplete working.

Questions

1. Let

$$A = \begin{bmatrix} 3 & 4 \\ 0 & -2 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

   (a) Find the QR factorisation of A using Householder reflections.

   (b) Find the QR factorisation of A using Givens rotations.

2. For a matrix $A \in \mathbb{R}^{2\times 3}$, the QR factors of A^T have been calculated as

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   (a) Compute the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

   (b) State any other solution to $A\mathbf{x} = \mathbf{b}$.

3. Consider a nonsingular matrix A (i.e., A^{-1} exists). A new matrix $B = A + \mathbf{u}\mathbf{v}^T$ is formed by making a rank-1 change to A. Verify that B^{-1} is given by

$$B^{-1} = A^{-1} - \alpha A^{-1}\mathbf{u}\mathbf{v}^T A^{-1},$$

   for some scalar α, and give an expression for α.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & a & -2 \\ 0 & -2 & 10 \end{bmatrix},$$

   where $a \in \mathbb{R}$.

   (a) For what values of a is A positive definite?

   (b) When is A positive semi-definite?

   (c) For the particular case for $a = 5$, find the modified Cholesky factorisation of A.

   (d) Hence solve the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & -2 \\ 0 & -2 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

5. The 100m sprint is held at the Olympic Games. The winning times each Olympics are

Year (Y)	**Time (T)**
1948	10.3
1952	10.4
1956	10.5
1960	10.2
1964	10.0
1968	9.95
1972	10.14
1976	10.06
1980	10.25
1984	9.99

Year (Y)	**Time (T)**
1988	9.92
1992	9.96
1996	9.84
2000	9.87
2004	9.85
2008	9.69
2012	9.63
2016	9.81
2020	9.80

Before you start: You should use a computer to help with this question; do not attempt to solve it by hand. You can use whatever programming language you like, but I am giving you the relevant MATLAB commands. It is helpful to plot the data before you begin; the 'Year' goes on the 'x-axis', while the 'Time' goes on the 'y-axis'.

(a) i. Fit this data with a polynomial of degree 1. That is, form a system of equations $A_1\mathbf{x}_1 = \mathbf{b}$, where each equation is of the form

$$T = c_0 + c_1 Y.$$

Form the QR factorization of A_1 using the command $[Q, R]$ = qr(A_1) in MATLAB. State the least squares solution $\mathbf{x}_1$ for the system $A_1\mathbf{x}_1 = \mathbf{b}$.

ii. Now, state the least squares solution for the system using the backslash command: $\mathbf{x}_1' = A_1 \setminus \mathbf{b}$.

iii. State the error in each of these fits.

(b) Repeat 5(a) i – iii, but using a polynomial of degree 2:

$$A_2\mathbf{x}_2 = \mathbf{b} \quad \Longleftrightarrow \quad T = c_0 + c_1 Y + c_2 Y^2.$$

(c) Repeat 5(a) i – iii, but using a polynomial of degree 3:

$$A_3\mathbf{x}_3 = \mathbf{b} \quad \Longleftrightarrow \quad T = c_0 + c_1 Y + c_2 Y^2 + c_3 Y^3.$$

(d) On a single set of axes, plot the data, the linear (degree 1) fit, the quadratic (degree 2) fit, and the cubic (degree 3) fit.

(e) For each fit (degree 1, degree 2, and degree 3), predict the winning time for the Paris 2024 Olympic games.