Bari Math
Bari Tutorials

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1 Question

Let ABCD be a square, and let E, F be points such that DA = DE = DF = DC, and $\angle ADE = \angle EDF = \angle FDC$. Prove that $\triangle BEF$ is equilateral.

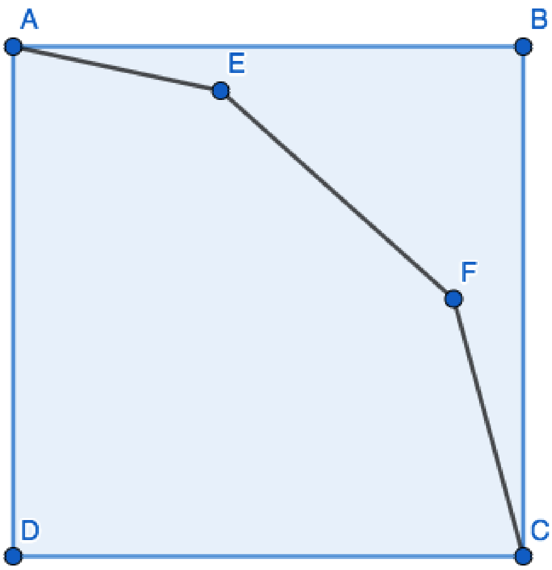

Figure 1: Square, and a bunch of nice points

Remember, try it on your own before you see my solution. If you see any errors in my solution, please let me know at Bari Tutorials.

2 What we know

Let ABCD be a square, and let E, F be points such that DA = DE = DF = DC, and $\angle ADE = \angle EDF = \angle FDC$. Prove that $\triangle BEF$ is equilateral.

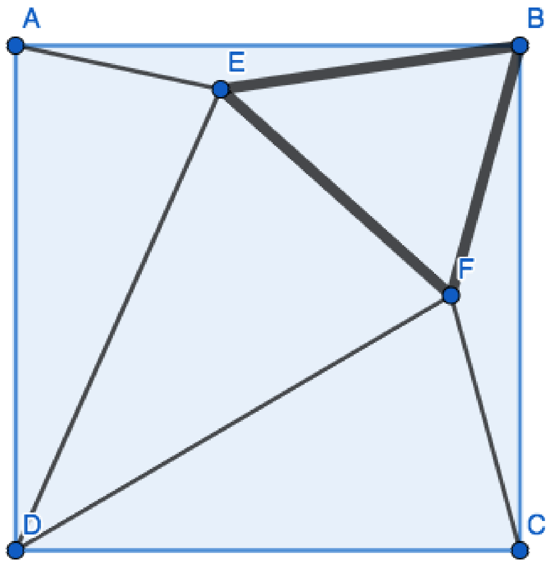

Figure 2: We've trying to prove $\triangle BEF$

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3 The Auxiliaries!

What can be better than this? To solve the problem, we need some auxiliary lines AKA, extra figures added to a problem to make it easier to solve. It took me a long time to figure this one out, but once you do, it's worth it. Let's first draw in line FA. Why? Well, the goal is to prove $\triangle FEA \cong \triangle BEA$. If we can do so, we can show that $BE = EF$!

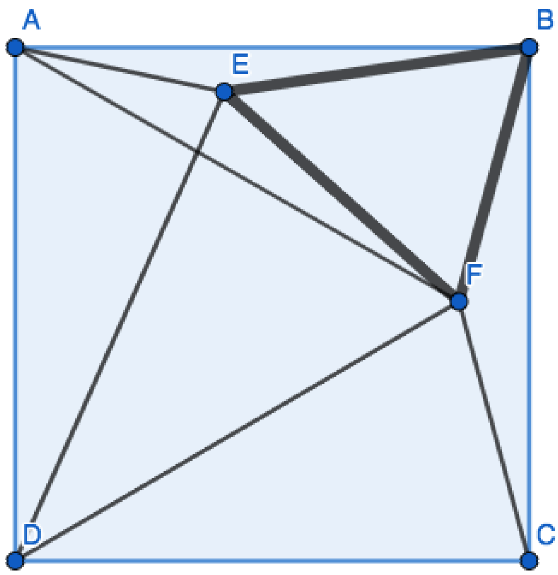

Figure 3: We've trying to prove $\triangle BEF$

But how do we prove $\triangle FEA \cong \triangle BEA$? Well, first things first, we're know that $\angle ADE = \angle EDF = \angle FDC$. Thus, we conclude that

$$\angle ADE = \angle EDF = \angle FDC = \frac{90}{3} = 30^O$$

Noice. Now, let's do some angle-chasing, and see what we can find that can help us prove $\triangle FEA \cong \triangle BEA$. Let us consider $\triangle AFD$. Realize that it is isoceles, since $AD = FD$. Thus,

$$\angle DAF = \angle DFA = \frac{180^o - 60^o}{2} = 60^o$$

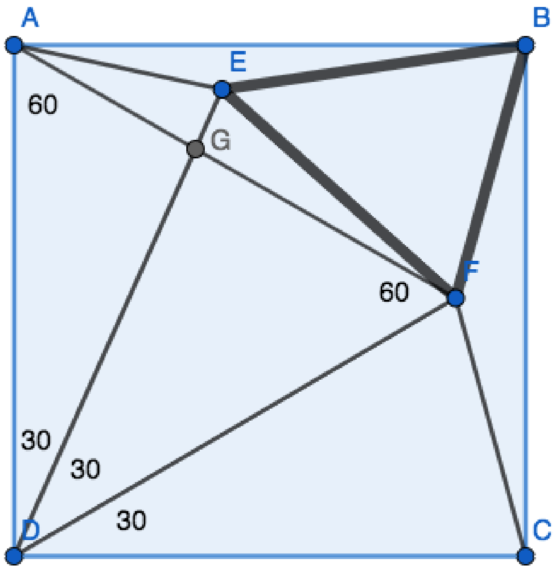

Figure 4: Point G is the intersection of lines FA and ED

Great. Now, it should be obvious that we have $\triangle DAG$ and $\triangle DFG$ as 30-60-90 triangles. Realize, also that $\triangle DEF$ and $\triangle DEA$ are both isosceles. Thus, we conclude that:

$$\angle DAE = \angle DEA = \angle DEF = \angle DFE = \frac{180^O - 30^O}{2} = 75^O$$

We can thus conclude that,

$$\angle GFE = \angle GAE = 75^O - 60^O = 15^O$$

Furthermore,

$$\angle BAE = \angle DAB - \angle EAD = 90^O - 75^O = 15^O$$

Noice! So now, we know that $\angle BAE \cong \angle EAF$. Great, one step closer to proving $\triangle FEA \cong \triangle BEA$. In fact, we also that $AE = AE$, by the Reflexive property (or by common sense).

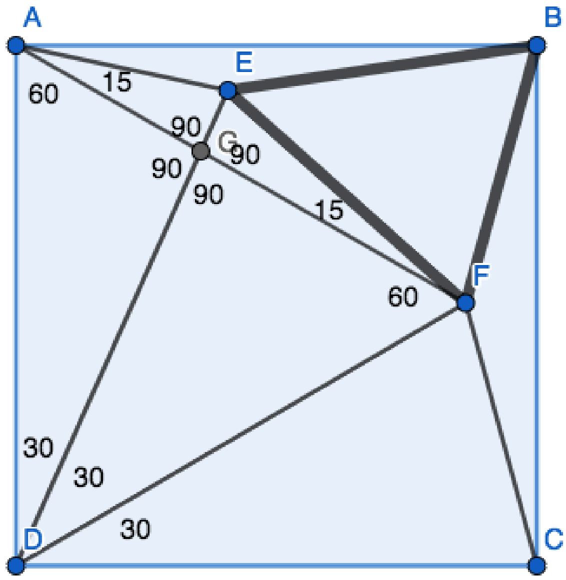

Figure 5: Woah.

OK, folks. Things are about to get greasy, so hold the railings. We know the $\triangle AFD$ is equilateral, so let's use that to our advantage. Call side AF x. Since $AF = AD = AB$, we now know that side $AB = AF$! Yes! Let's go! We can thus prove that $\triangle FEA \cong \triangle BEA$ by SAS (side $AF = AB$, $\angle BAE \cong \angle EAF$, and side $AE = AE$).

Thus, be CPCTC, sides $EF = EB$. But WAIT. That only shows that $\triangle EFB$ is isoceles, not equilateral! Well, that's why we must construct a second auxiliary line, as shown:

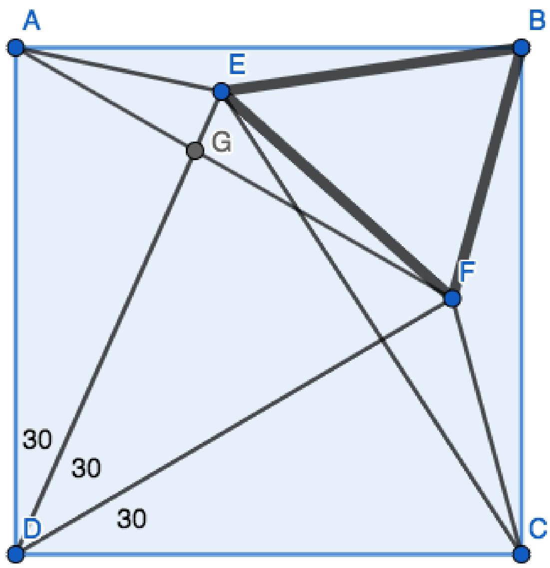

Figure 6: Magic.

Well, I'll leave you at that. It's up to you to show that $\triangle EFC \cong \triangle BFC$. Save the day!