A Puzzling Inequality

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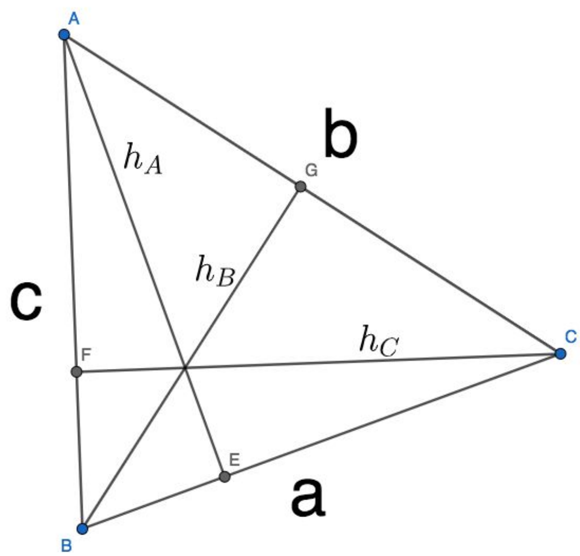
Q: Given h_A,h_B,h_C are the altitudes of a triangle, prove $\frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C}$.

Triangles. Immediately, you start thinking Pythagorean Theorem and the Triangle Inequality. It's inevitable, given their practicality, and usefulness. Yet many of the most important properties of triangles, such as the one above, are often overlooked. It turns out this theorem not only enables us to solve many geometric problems, but also paves the way for more sophisticated inequalities involving the medians and sides of a triangle. It even shares a startling resemblance to the triangle inequality, which shouldn't be taken for granted. But how can we go along proving this pivotal theorem?

As with all geometry problems, it's essential to begin with a diagram.



Much better! To begin proving the theorem, we brainstorm for formulas essential to all triangles. First thing to come to mind? The area for a triangle, given by $A = \frac{1}{2}bh$. Of course, the area of a triangle remains constant regardless of what pair of bases and heights we use. Immediately, we begin setting up a system of equations, as follows:

$$[\triangle ABC] = \frac{1}{2}a * (h_A) = \frac{1}{2}b * (h_B) = \frac{1}{2}c * (h_C)$$

Great, so we've got a system of equations that involve triangle altitudes. But how do we make it into an inequality? Immediately, one recalls the triangle inequality, which states that given three sides of a triangle, a, b, and c, a+b>c. But how can we apply the triangle inequality theorem to the area formula? It's simply a matter of rephrasing the above equation.

Let's begin by multiplying both sides by two, simply to remove all the fractions, as follows:

$$2[\triangle ABC] = a * (h_A) = b * (h_B) = c * (h_C)$$

Great! Now let's isolate the altitudes, since that's we're focusing on. This is the result:

$$h_A = \frac{2[\triangle ABC]}{a}$$
$$h_B = \frac{2[\triangle ABC]}{b}$$
$$h_C = \frac{2[\triangle ABC]}{c}$$

However, the inequality we're attempting to prove involves the reciprocal of the altitudes of a triangle. When we find the reciprocals of the altitudes, we find:

$$\frac{1}{h_A} = \frac{a}{2[\triangle ABC]}$$
$$\frac{1}{h_B} = \frac{b}{2[\triangle ABC]}$$
$$\frac{1}{h_C} = \frac{c}{2[\triangle ABC]}$$

Immediately, you should recognize the triangle inequality within these system of equations. By applying the triangle inequality theorem, we arrive at the following:

$$\frac{a}{2[\triangle ABC]} < \frac{b}{2[\triangle ABC]} + \frac{c}{2[\triangle ABC]}$$

Why? Well, simply take $a < b + c$, and divide both sides by $2[\triangle ABC]$. Great! Now, we simply substitute the reciprocal of the altitudes for each fraction to achieve our desired inequality:

$$\frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C}$$

And there you have it. A simple, beautiful solution to a simply powerful theorem.