PDE Model Finite Difference Derivation for a plane

Let the diffusion be fixed by a scalar $d \in \mathbb{R}+$. The original model is then

$$
\begin{aligned}
&(1) \quad \frac{\partial \rho G}{\partial t} + \nabla(-\rho d \nabla G) = -\rho \alpha A G \\
&(2) \quad \frac{\partial \rho A}{\partial t} + \nabla(-\rho d \nabla A) = \rho(\alpha A G - \tau^{-1} A) \\
&(3) \quad \frac{\partial \rho K}{\partial t} = \rho(\kappa \tau^{-1} A - \sigma^{-1} K) \\
&(4) \quad \frac{\partial \rho R}{\partial t} + \nabla(-\rho d \nabla A) = \rho\big((1-\kappa)\tau^{-1} A + \sigma^{-1}(1-\theta) K\big) \\
&(5) \quad \frac{\partial \rho D}{\partial t} = \sigma^{-1} \rho \theta K
\end{aligned}
$$

Let the mass term be set with $\rho = 1$.

The geometry is assumed to be a plane with dimensions $plane_x, plane_y \in \mathbb{R}^+$. Assume a discretization step size of h. There are then $N_x = \frac{plane_x}{h} + 1$ grid points in the x direction of each row and $N_y = \frac{plane_y}{h} + 1$ rows overall. Representing grid points by

$G_{i,j}(t) = G\big(t; x_i, y_j\big),\ A_{,j}(t) = A\big(t; x_i, y_j\big), K_{,j}(t) = K\big(t; x_i, y_j\big),\ R_{,j}(t) = R\big(t; x_i, y_j\big),\ D_{i,j}(t) = D\big(t; x_i, y_j\big)$

the following finite differences approximation can be made:

$$
\frac{dG_{i,j}}{dt} = \underbrace{d\left(\frac{G_{i-1,j} - 2G_{i,j} + G_{i+1,j}}{h^2} + \frac{G_{i,j-1} - 2G_{i,j} + G_{i,j+1}}{h^2}\right)}_{FS\ formulation\ of\ \nabla(\rho d \nabla G)} - \alpha A_{i,j} G_{i,j}
$$

$$
\frac{dA_{i,j}}{dt} = d\left(\frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{h^2} + \frac{A_{i,j-1} - 2A_{i,j} + A_{i,j+1}}{h^2}\right) + \alpha A_{i,j} G_{i,j} - \tau^{-1} A_{i,j}
$$

$$
\frac{dK_{i,j}}{dt} = \left(\kappa \tau^{-1} A_{i,j} - \sigma^{-1} K_{i,j}\right)
$$

$$
\frac{dR_{i,j}}{dt} = -d\left(\frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{h^2} + \frac{A_{i,j-1} - 2A_{i,j} + A_{i,j+1}}{h^2}\right) + \alpha(1-\kappa)\tau^{-1} A_{i,j} + \sigma^{-1}(1-\theta) K_{i,j}
$$

$$
\frac{dD}{dt} = \sigma^{-1} \theta K_{i,j}
$$

This is a system of $N = N_x \cdot N_y \cdot 5$ ordinary differential equations.