MACS 201 : Hilbert spaces and probability

1 Hilbert spaces

Def. Let $\mathcal{H}$ be a complex linear space. An **inner-product** on $\mathcal{H}$ is a function $\langle \cdot \mid \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{C}$ which satisfies the following properties :

(i) $\forall (x, y) \in \mathcal{H} \times \mathcal{H}, \langle x \mid y \rangle = \overline{\langle y \mid x \rangle}$,

(ii) $\forall x, y, z \in \mathcal{H} \forall (\alpha, \beta) \in \mathbf{C} \times \mathbf{C}, \langle \alpha x + \beta y \mid z \rangle = \alpha \langle x \mid z \rangle + \beta \langle y \mid z \rangle$,

(iii) $\forall x \in \mathcal{H}, (\langle x \mid x \rangle = 0) \iff (x = 0)$

Then $\|\cdot\| : x \mapsto \sqrt{\langle x \mid x \rangle} \geqslant 0$ defines a norm on $\mathcal{H}$. Both are continuous.

Th. *For all $x, y \in \mathcal{H}$, we have :*

a) Cauchy-Schwarz inequality : $|\langle x \mid y \rangle| \leqslant \|x\| \cdot \|y\|$,

b) triangular inequality : $|\|x\| - \|y\|| \leqslant \|x - y\| \leqslant \|x\| + \|y\|$,

c) Parallelogram inequality : $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Def. An inner-product space $\mathcal{H}$ is called an Hilbert space if it is complete.

Prop. *For all measured space $(\Omega, \mathcal{F}, \mu)$, the space $L^2(\Omega, \mathcal{F}, \mu)$ endowed with $\langle f \mid g \rangle = \int f \bar{g} \, \mathrm{d}\mu$ is a Hilbert space.*

Def. Two vectors $x, y \in \mathcal{H}$ are orthogonal if $\langle x \mid y \rangle = 0$ which we denoted by $x \perp y$. If $\mathcal{S}$ is a subspace of $\mathcal{H}$, we write $x \perp \mathcal{S}$ if $\forall s \in \mathcal{S}, x \perp s$. Also we write $\mathcal{S} \perp \mathcal{T}$ if all vectors in $\mathcal{S}$ are orthogonal to $\mathcal{T}$.

Not. If $\mathcal{H} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \perp \mathcal{B}$ we will denote $\mathcal{H} = \mathcal{A} \overset{\perp}{\oplus} \mathcal{B}$.

Def. Let $\mathcal{E}$ be a subset of an Hilbert space $\mathcal{H}$. The orthogonal set of $\mathcal{E}$ is defined as $\mathcal{E} = \{x \in \mathcal{H} \mid \forall y \in \mathcal{E}, \langle x \mid y \rangle = 0\}$.

Th. *If $\mathcal{E}$ is a subset of an Hilbert space $\mathcal{H}$, then $\mathcal{E}^\perp$ is closed.*

Def. Let E be a subset of $\mathcal{H}$. It is an orthogonal set if for all $(x, y) \in E \times E, x \neq y, x \perp y$. If moreover $\forall x \in E, \|x\| = 1$, we say that E is orthonormal.

Th. *Let $(e_i)_{i \geqslant 1}$ be an orthonormal sequence of an Hilbert space $\mathcal{H}$ and let $(\alpha_i)_{i \geqslant 1} \in \mathbf{C}^{\mathbf{N}}$. The series $\sum_{i=1}^\infty \alpha_i e_i$ converges in $\mathcal{H}$ if and only if $\sum_i |\alpha_i|^2 < \infty$, in which case $\|\sum_{i=1}^\infty \alpha_i e_i\|^2 = \sum_{i=1}^\infty |\alpha_i|^2$.*

2 Probability

Th (π - λ theorem). *If $\mathcal{A} \subset \mathcal{C}$ with $\mathcal{A}$ a π-system and $\mathcal{C}$ a λ-system, then $\sigma(\mathcal{A}) = \mathcal{C}$.*

Th. *Let $\mathcal{C}$ be a π-system on Ω and $\mathcal{F} = \sigma(\mathcal{C})$ the smallest σ-field containing $\mathcal{C}$. Then a probability measure μ on $(\Omega, \mathcal{F})$ is uniquely characterized by $\mu(A)$ on $A \in \mathcal{C}$.*

Def. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{G}$ a sub-σ-field of $\mathcal{F}$.