

MACS 201 : Hilbert spaces and probability

1 Hilbert spaces

Def. Let \mathcal{H} be a complex linear space. An **inner-product** on \mathcal{H} is a function $\langle \cdot | \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{C}$ which satisfies the following properties :

- (i) $\forall (x, y) \in \mathcal{H} \times \mathcal{H}, \langle x | y \rangle = \overline{\langle y | x \rangle},$
- (ii) $\forall x, y, z \in \mathcal{H} \forall (\alpha, \beta) \in \mathbf{C} \times \mathbf{C}, \langle \alpha x + \beta y | z \rangle = \alpha \langle x | z \rangle + \beta \langle y | z \rangle,$
- (iii) $\forall x \in \mathcal{H}, (\langle x | x \rangle = 0) \iff (x = 0)$

Then $\|\cdot\| : x \mapsto \sqrt{\langle x | x \rangle} \geq 0$ defines a norm on \mathcal{H} . Both are continuous.

Th. For all $x, y \in \mathcal{H}$, we have :

- a) *Cauchy-Schwarz inequality* : $|\langle x | y \rangle| \leq \|x\| \cdot \|y\|,$
- b) *triangular inequality* : $|\|x\| - \|y\|| \leq \|x - y\| \leq \|x\| + \|y\|,$
- c) *Parallelogram inequality* : $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$

Def. An inner-product space \mathcal{H} is called an Hilbert space if it is complete.

Prop. For all measured space $(\Omega, \mathcal{F}, \mu)$, the space $L^2(\Omega, \mathcal{F}, \mu)$ endowed with $\langle f | g \rangle = \int f \bar{g} d\mu$ is a Hilbert space.

Def. Two vectors $x, y \in \mathcal{H}$ are orthogonal if $\langle x | y \rangle = 0$ which we denoted by $x \perp y$. If \mathcal{S} is a subspace of \mathcal{H} , we write $x \perp \mathcal{S}$ if $\forall s \in \mathcal{S}, x \perp s$. Also we write $\mathcal{S} \perp \mathcal{T}$ if all vectors in \mathcal{S} are orthogonal to \mathcal{T} .

Not. If $\mathcal{H} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \perp \mathcal{B}$ we will denote $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$.

Def. Let \mathcal{E} be a subset of an Hilbert space \mathcal{H} . The orthogonal set of \mathcal{E} is defined as $\mathcal{E}^\perp = \{x \in \mathcal{H} \mid \forall y \in \mathcal{E}, \langle x | y \rangle = 0\}$.

Th. If \mathcal{E} is a subset of an Hilbert space \mathcal{H} , then \mathcal{E}^\perp is closed.

Def. Let E be a subset of \mathcal{H} . It is an orthogonal set if for all $(x, y) \in E \times E, x \neq y, x \perp y$. If moreover $\forall x \in E, \|x\| = 1$, we say that E is orthonormal.

2 Probability

Th (π - λ theorem). If $\mathcal{A} \subset \mathcal{C}$ with \mathcal{A} a π -system and \mathcal{C} a λ -system, then $\sigma(\mathcal{A}) = \mathcal{C}$.

Th. Let \mathcal{C} be a π -system on Ω and $\mathcal{F} = \sigma(\mathcal{C})$ the smallest σ -field containing \mathcal{C} . Then a probability measure μ on (Ω, \mathcal{F}) is uniquely characterized by $\mu(A)$ on $A \in \mathcal{C}$.

Def. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbf{P})$ and \mathcal{G} a sub- σ -field of \mathcal{F} .