Computational Methods

Tutorial II

December 5, 2024

1. Suppose x0,x1,⋯ ,xn are n+1 equispaced points.

a) State the, forward difference operator, backward difference operator,

and central difference operator.

b) Show that Δ2y3 = y2 −y1.

(a) Operators

- **Forward Difference Operator** **(**Δ**)**: $\Delta y_i = y_{i+1} - y_i$
- **Backward Difference Operator** **(**∇**)**: $\nabla y_i = y_i - y_{i-1}$
- **Central Difference Operator** **(**δ**)**: $\delta y_i = \frac{y_{i+1} - y_{i-1}}{2}$

(b) Show $\Delta^2 y_3 = y_2 - y_1$**:**

Using the definition of the second forward difference:

$$\Delta^2 y_3 = \Delta(\Delta y_3) = \Delta(y_4 - y_3) = (y_5 - y_4) - (y_4 - y_3) = y_3 - y_2.$$

2. a) State the Newton's forward difference interpolation formula.

b) Using the result in (a), show that the Newton's interpolation polynomial for the points {(0,7),(10,18),(20,32),(30,51),(40,87)} is given by

Pn(x) = 0.0000416x4 −0.0022x3 +0.05x2 +1.26x+7

a) Newton's Forward Difference Interpolation Formula

The Newton's forward difference interpolation formula is used to estimate the value of a function at any given point using its known values at equally spaced points. The formula is:

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)...(p-n+1)}{n!}\Delta^n y_0$$

Where:

- $p = \frac{x - x_0}{h}$
- h is the interval size between the points.
- y_0 is the initial value of y.
- $\Delta y_0, \Delta^2 y_0, \ldots, \Delta^n y_0$ are the forward differences.

b) Deriving the Newton's Interpolation Polynomial for Given Points

Given points: $(0, 7)$, $(10, 18)$, $(20, 32)$, $(30, 51)$, $(40, 87)$.

To derive the Newton's interpolation polynomial $P_n(x)$ for the given points, we need to calculate the forward differences and then apply them into the Newton's formula.

Forward Difference Table

x	y	Δy	Δ²y	Δ³y	Δ⁴y
0	7	11	3	2	10
10	18	14	5	12	
20	32	19	17		
30	51	36			
40	87				

Step 2: Compute Step Size (h)

The x-values are equispaced with $h = 10$.

Step 3: Write Newton's Polynomial

Using the formula:

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x-0}{10} = \frac{x}{10}$$

Substitute the forward differences from the table:

$$P_n(x) = 7 + \frac{x}{10}(11) + \frac{\frac{x}{10}(\frac{x}{10}-1)}{2!}(3) + \frac{\frac{x}{10}(\frac{x}{10}-1)(\frac{x}{10}-2)}{3!}(2) + \frac{\frac{x}{10}(\frac{x}{10}-1)(\frac{x}{10}-2)(\frac{x}{10}-3)}{4!}(10)$$

Step 4: Simplify

Expand and simplify the terms to obtain:

$$P_n(x) = 7 + 1.1x + \frac{x(x-10)}{2 \cdot 100}(3) + \frac{x(x-10)(x-20)}{3 \cdot 1000}(2) + \frac{x(x-10)(x-20)(x-30)}{24 \cdot 10000}(10)$$

Simplifying each term step by step:

$$P_n(x) = 7 + 1.1x + 0.015x(x-10) + 0.0006667x(x-10)(x-20) + 0.00004167x(x-10)(x -$$

Further simplifying:

$$P_n(x) = 7 + 1.1x + 0.015(x^2 - 10x) + 0.0006667(x^3 - 30x^2 + 200x) + 0.00004167(x^4 - 60x^3$$

Combining like terms:

$$P_n(x) = 7 + 1.1x + 0.015x^2 - 0.15x + 0.0006667x^3 - 0.02x^2 + 0.1333x + 0.00004167x^4 - 0.0$$

Combining all coefficients:

$$P_n(x) = 0.00004167x^4 - 0.0022x^3 + 0.045x^2 + 0.6833x + 7$$

Thus, the Newton's interpolation polynomial is:

$$P_n(x) = 0.00004167x^4 - 0.0022x^3 + 0.045x^2 + 0.6833x + 7$$

3. a) Explain the principle of Numerical differentiation.

b) State the formulas for $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ based on the Newton's forward interpolation.

c) Find the value of $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ at $x = 1.5$ using the data given in the table.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

b) Formulas for $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$

First Derivative ($\frac{df}{dx}$) using Forward Difference:

$$\frac{df}{dx} \approx \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

Second Derivative ($\frac{d^2f}{dx^2}$) using Forward Difference:

\[\frac{d^2f}{dx^2} \approx \frac{y_{i+2} - 2y_{i+1} + y_i}{(x_{i+1} - x_i)^2} \]

c) Finding $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ at $x = 1.5$ using Newton's Forward Difference

Given data points:

x	y
1.5	3.375
2.0	7.000
2.5	13.625
3.0	24.000
3.5	38.875
4.0	59.000

We will use the given data to create a forward difference table and then apply the forward difference formulas to find the derivatives at $x = 1.5$.

First Difference (Δy):

x	y	Δy
1.5	3.375	$7.000 - 3.375 = 3.625$
2.0	7.000	$13.625 - 7.000 = 6.625$
2.5	13.625	$24.000 - 13.625 = 10.375$
3.0	24.000	$38.875 - 24.000 = 14.875$
3.5	38.875	$59.000 - 38.875 = 20.125$
4.0	59.000	

Second Difference (Δ²y):

x	y	Δy	$\Delta^2 y$
1.5	3.375	3.625	$6.625 - 3.625 = 3.000$
2.0	7.000	6.625	$10.375 - 6.625 = 3.750$
2.5	13.625	10.375	$14.875 - 10.375 = 4.500$
3.0	24.000	14.875	$20.125 - 14.875 = 5.250$
3.5	38.875	20.125	
4.0	59.000		

Calculate First and Second Derivatives:

First Derivative ($\frac{df}{dx}$) at $x = 1.5$:

$$\frac{df}{dx} \approx \frac{\Delta y}{h} = \frac{3.625}{0.5} = 7.25$$

Second Derivative ($\frac{d^2f}{dx^2}$) at $x = 1.5$:

\[\frac{d^2f}{dx^2} \approx \frac{Δ²y}{h^2} = \frac{3.000}{0.5^2} = 12.00 \]

Thus, at $x = 1.5$:

- The first derivative $\frac{df}{dx} \approx 7.25$
- The second derivative $\frac{d^2f}{dx^2} \approx 12.00$

4. Suppose the function $y_i = f(x_i)$ is known at (n+1) points $x_0, x_1, x_2, \cdots, x_n$, $i = 0, 1, 2, \cdots n$ are known. Let $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$ and $p = \frac{x - x_n}{h}$

   a) State the Newton's backward difference formula.

   b) Using (a), write down the formulas for approximating the firs, second and third derivative at any point $x = x_n + ph$.

   c) A particle is moving along a straight line. The displacement x at some time increases t are given below:

a) State the Newton's Backward Difference Formula

The Newton's backward difference interpolation formula is used to estimate the value of a function at any given point using its known values at equally spaced points. The formula is:

$$P_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)...(p+n-}{n!}$$

Where:

- $p = \frac{x - x_n}{h}$
- h is the interval size between the points.
- y_n is the value of y at x_n.
- $\nabla y_n, \nabla^2 y_n, \ldots, \nabla^n y_n$ are the backward differences.

b) Formulas for Approximating First, Second, and Third Derivatives

Using Newton's backward difference formula, the derivatives can be approximated as follows:

1. **First Derivative ($\frac{df}{dx}$):**

$$\frac{df}{dx}\Big|_{x=x_n} \approx \frac{1}{h}\left(\nabla y_n - \frac{p}{2}\nabla^2 y_n + \frac{p(p+1)}{6}\nabla^3 y_n - \cdots\right)$$

2. **Second Derivative ($\frac{d^2f}{dx^2}$):**

\[\left.\frac{d^2f}{dx^2}\right|_{x = x_n} \approx \frac{1}{h^2} \left(∇^2 y_n - p ∇^3 y_n + \frac{p(p+1)}{12} ∇^4 y_n - \cdots \right) \]

3. **Third Derivative ($\frac{d^3f}{dx^3}$):**

\[\left.\frac{d^3f}{dx^3}\right|_{x = x_n} \approx \frac{1}{h^3} \left(∇^3 y_n - \frac{3p}{4} ∇^4 y_n + \frac{p(p+1)}{8} ∇^5 y_n - \cdots \right) \]

c) Finding Velocity and Acceleration at $t = 4$

To find the velocity and acceleration of the particle at $t = 4$, we first compute the backward difference table for the given data:

4(c) Velocity and Acceleration of the Particle at $t = 4$:

Data Table:

t	x
0	5
1	8
2	12
3	17
4	26

Step 1: Compute the Backward Difference Table

t	x	∇x	$\nabla^2 x$	$\nabla^3 x$
0	5			
1	8	3		
2	12	4	1	
3	17	5	1	0
4	26	9	4	3

Step 2: Compute Velocity (v)

Using the first derivative formula:

$$v \approx \frac{\nabla x_n}{h} - \frac{p\nabla^2 x_n}{h}.$$

Here, $n = 4$, $h = 1$, and $p = 0$ (since we are evaluating at $t = 4$).

$$v = \frac{\nabla x_4}{1} = 9\,\text{units}/\text{time}.$$

Step 3: Compute Acceleration (a)

Using the second derivative formula:

$$a \approx \frac{\nabla^2 x_n}{h^2}.$$

$$a = \frac{\nabla^2 x_4}{1^2} = 4\,\text{units}/\text{time}^2.$$

Final Results:

- **Velocity (v) at $t = 4$:** $9\,\text{units}/\text{time}$.
- **Acceleration (a) at $t = 4$:** $4\,\text{units}/\text{time}^2$.

5. Consider the interval [a,b]. Say we wish to do seven function evalua tions, fk − f(xk) for k = 0,1,··· ,4,5,6. This implies six sub-intervals. [x0, x1], [x1,x2],[x2,x3],[x3,x4],[x4,x5] and [x5,x6]. a) State the composite Trapezoidal rule and the composite Simpson's rule for the six sub-intervals. b) Consider the integration of f(x) = 1 + e−xsin(4x) over [0,2]. Use exactly seven function evaluations and compare the result from the composite trapezoidal rule, composite Simpson's rule.

Soln

5(a) Composite Trapezoidal Rule and Composite Simpson's Rule

Composite Trapezoidal Rule

The composite trapezoidal rule is a numerical integration method that approximates the integral of a function over multiple subintervals by treating each subinterval as a trapezoid. The formula is:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right],$$

where:

- n: Number of subintervals ($n = 6$ here).
- $h = \frac{b-a}{n}$: Step size.
- $x_k = a + kh$: Points of evaluation.

b) Composite Simpson's Rule

The composite Simpson's rule is used to approximate the definite integral of a function by using parabolic arcs instead of straight lines to approximate the area under the curve.

For n sub-intervals (where n must be even, and $n+1$ points), the composite Simpson's rule formula is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{k=1,3,5,\ldots}^{n-1} f(x_k) + 2\sum_{k=2,4,6,\ldots}^{n-2} f(x_k) + f(x_n)\right]$$

Where:

- $h = \frac{b-a}{n}$
- $x_0, x_1, \ldots, x_n$ are the points at which the function is evaluated.

b) Integration of $f(x) = 1 + e^{-x}\sin(4x)$ over $[0, 2]$

Given function: $f(x) = 1 + e^{-x}\sin(4x)$ Interval: $[0, 2]$ Number of function evaluations: 7 (implies 6 sub-intervals)

Step 1: Calculate h

<u>Here, n must be even (it is 6 here) |||||| here is the comment for the point at which function evaluated</u>

5(b) Integration of $f(x) = 1 + e^{-x}\sin(4x)$ over $[0, 2]$

Step 1: Define Parameters

- Function: $f(x) = 1 + e^{-x}\sin(4x)$.
- Interval: $[a, b] = [0, 2]$.
- Number of evaluations: 7, so $n = 6$.
- Step size: $h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3} \approx 0.3333$.

$$h = \frac{2-0}{6} = \frac{2}{6} = \frac{1}{3}$$

Step 2: Evaluate the function at the points

$$x_k = 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$$

x_k	$f(x_k)$
0	$1 + e^{0}\sin(0) = 1$
$\frac{1}{3}$	$1 + e^{-\frac{1}{3}}\sin\left(\frac{4}{3}\right) \approx 1 + 0.7165 \cdot 0.9854 \approx 1.705$
$\frac{2}{3}$	$1 + e^{-\frac{2}{3}}\sin\left(\frac{8}{3}\right) \approx 1 + 0.5134 \cdot (-0.9472) \approx 0.514$
1	$1 + e^{-1}\sin(4) \approx 1 + 0.3679 \cdot (-0.7568) \approx 0.721$
$\frac{4}{3}$	$1 + e^{-\frac{4}{3}}\sin\left(\frac{16}{3}\right) \approx 1 + 0.2636 \cdot 0.1411 \approx 1.037$
$\frac{5}{3}$	$1 + e^{-\frac{5}{3}}\sin\left(\frac{20}{3}\right) \approx 1 + 0.1891 \cdot 0.9093 \approx 1.172$
2	$1 + e^{-2}\sin(8) \approx 1 + 0.1353 \cdot 0.9894 \approx 1.134$

Composite Trapezoidal Rule

$$\int_0^2 \left(1 + e^{-x}\sin(4x)\right) dx \approx \frac{n}{2}\left[f(x_0) + 2\sum_{k=1} f(x_k) + f(x_6)\right]$$

$$\approx \frac{1}{6}[1 + 2(1.705 + 0.514 + 0.721 + 1.037 + 1.172) + 1.134]$$

$$\approx \frac{1}{6}[1 + 2 \cdot 5.149 + 1.134]$$

$$\approx \frac{1}{6}[1 + 10.298 + 1.134]$$

$$\approx \frac{1}{6}[12.432] \approx 2.072$$

Composite Simpson's Rule

$$\int_0^2 \left(1 + e^{-x}\sin(4x)\right) dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{\text{odd } k} f(x_k) + 2\sum_{\text{even } k} f(x_k) + f(x_6)\right]$$

$$\approx \frac{1}{9}[1 + 4(1.705 + 0.721 + 1.172) + 2(0.514 + 1.037) + 1.134]$$

$$\approx \frac{1}{9}[1 + 4 \cdot 3.598 + 2 \cdot 1.551 + 1.134]$$

$$\approx \frac{1}{9}[1 + 14.392 + 3.102 + 1.134]$$

$$\approx \frac{1}{9}[19.628] \approx 2.181$$

Comparison

- **Composite Trapezoidal Rule Result:** ≈ 2.072
- **Composite Simpson's Rule Result:** ≈ 2.181

6. Find the least-squares line y = f(x) = mx + b for the data and calculate E2(f); (−4,−3),(−1,−1),(0,0),(2,1),(3,2)

Soln

6. Finding the Least-Squares Line $y = mx + b$

The least-squares line minimizes the sum of squared differences between the observed y-values and the predicted values from the line. The equation of the line is:

$$y = mx + b,$$

where:

- m: slope of the line.
- b: y-intercept of the line.

Step 1: Formulas for m and b

The slope m and intercept b are computed as:

$$m = \frac{n\sum(xy) - \sum x \sum y}{n\sum x^2 - (\sum x)^2},$$

$$b = \frac{\sum y - m\sum x}{n}.$$

Step 2: Compute Summations

Given data points:

$$(x, y) = \{(-4, -3), (-1, -1), (0, 0), (2, 1), (3, 2)\}.$$

x	y	xy	x^2
-4	-3	12	16
-1	-1	1	1
0	0	0	0
2	1	2	4
3	2	6	9

$$\sum x = 0, \quad \sum y = -1, \quad \sum xy = 21, \quad \sum x^2 = 30, \quad n = 5.$$

Step 3: Calculate m and b

1. Compute m:

$$m = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}.$$

$$m = \frac{5(21) - (0)(-1)}{5(30) - (0)^2} = \frac{105}{150} = 0.7.$$

2. Compute b:

$$b = \frac{\sum y - m \sum x}{n}.$$

$$b = \frac{-1 - 0.7(0)}{5} = \frac{-1}{5} = -0.2.$$

Thus, the least-squares line is:

$$y = 0.7x - 0.2.$$

Step 4: Calculate $E_2(f)$ (Error)

The error $E_2(f)$ is the sum of the squares of the residuals:

$$E_2(f) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2 .$$

Residuals for each point:

$$\text{For } (-4,-3) : y_{\text{pred}} = 0.7(-4) - 0.2 = -2.8 - 0.2 = -3, \quad \text{error} = (-3 - (-3))^2 = 0.$$

$$\text{For } (-1,-1) : y_{\text{pred}} = 0.7(-1) - 0.2 = -0.7 - 0.2 = -0.9, \quad \text{error} = (-1 - (-0.9))^2 = 0.01.$$

$$\text{For } (0,0) : y_{\text{pred}} = 0.7(0) - 0.2 = -0.2, \quad \text{error} = (0 - (-0.2))^2 = 0.04.$$

$$\text{For } (2,1) : y_{\text{pred}} = 0.7(2) - 0.2 = 1.4 - 0.2 = 1.2, \quad \text{error} = (1 - 1.2)^2 = 0.04.$$

$$\text{For } (3,2) : y_{\text{pred}} = 0.7(3) - 0.2 = 2.1 - 0.2 = 1.9, \quad \text{error} = (2 - 1.9)^2 = 0.01.$$

$$E_2(f) = 0 + 0.01 + 0.04 + 0.04 + 0.01 = 0.1.$$

Final Results

- Least-squares line:

$$y = 0.7x - 0.2.$$

- Sum of squared errors:

$$E_2(f) = 0.1.$$

7. Define the following as used in linear algebra a) A matrix. b) A system of linear equations. c) A linear combination

7(a) A Matrix

A **matrix** is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. It is commonly used to represent data or solve systems of linear equations.

A matrix is denoted as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} represents the element in the i-th row and j-th column, and the dimensions of the matrix are $m \times n$ (rows $\times$ columns).

7(b) A System of Linear Equations

A **system of linear equations** is a collection of one or more equations involving the same set of variables, where each equation is linear.

Example:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

This system can also be represented in matrix form as:

$$AX = B,$$

where A is the coefficient matrix, X is the vector of variables, and B is the vector of constants.

7(c) A Linear Combination

A **linear combination** is an expression made up of the sum of scalar multiples of vectors. In other words, for vectors $v_1, v_2, \ldots, v_n$ and scalars $c_1, c_2, \ldots, c_n$, the linear combination is:

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n.$$

Example:

If $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, a linear combination of v_1 and v_2 could be:

$$3v_1 + 2v_2 = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \end{bmatrix}.$$

8. Let A be a matrix. What do the following notations stand for:

a) aij.

b) |A|

8(a) a_{ij}

The notation a_{ij} represents the element of the matrix A located at the i-th row and j-th column.

For example, in the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

- $a_{11} = 1$ (element in the 1st row, 1st column),
- $a_{23} = 6$ (element in the 2nd row, 3rd column).

b) $|A|$

Definition: The notation $|A|$ represents the determinant of the matrix A.

Explanation:

- **Determinant (**$|A|$**):** The determinant is a scalar value that can be computed from the elements of a square matrix. It provides important properties of the matrix, such as whether it is invertible.
- If $|A| \neq 0$, the matrix A is invertible (i.e., it has an inverse).
- If $|A| = 0$, the matrix A is singular (i.e., it does not have an inverse).

Example: For the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is calculated as:

$$|A| = ad - bc$$

For the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

the determinant is calculated as:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

9. Let A =[aij]m×n for 1 ≤ i ≤ 5 and 1 ≤ j ≤ 5.

a) Write A in expanded form.

b) Define the minor of aij

c) Using (a) write an expression for |A| for i = 3. Hint: apply cofactor.

<u>**Soln**</u>

9(a) Write $A = [a_{ij}]_{m\times n}$ in Expanded Form

Given that the matrix A is of size 5×5, where $1 \leq i \leq 5$ and $1 \leq j \leq 5$, the matrix A in expanded form can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Each element a_{ij} corresponds to the element in the i-th row and j-th column of the matrix.

9(b) Define the Minor of a_{ij}

The **minor** of an element a_{ij}, denoted as M_{ij}, is the determinant of the submatrix that remains after deleting the i-th row and j-th column from the matrix A.

For example, if A is a 5×5 matrix and we are considering the minor M_{ij} of the element a_{ij}, the minor is obtained by removing the row and column that contains a_{ij}, and then taking the determinant of the resulting 4×4 submatrix.

Mathematically:

$$M_{ij} = \det(A_{ij}),$$

where A_{ij} is the submatrix formed by deleting the i-th row and j-th column of A.

9(c) Expression for $|A|$ for $i = 3$ Using Cofactor Expansion

To compute the determinant $|A|$ of the 5×5 matrix using **cofactor expansion** along the third row (where $i = 3$), the formula is:

$$|A| = \sum_{j=1}^{5} (-1)^{3+j} a_{3j} M_{3j},$$

where:

- a_{3j} are the elements in the third row of A,
- M_{3j} is the minor of the element a_{3j},
- $(-1)^{3+j}$ is the cofactor sign.

So, the determinant can be expanded as:

So, the determinant can be expanded as:

$$|A| = (-1)^{3+1} a_{31} M_{31} + (-1)^{3+2} a_{32} M_{32} + (-1)^{3+3} a_{33} M_{33} + (-1)^{3+4} a_{34} M_{34} + (-1)^{3+5} a_{35} M_{35}.$$

This is the cofactor expansion for the determinant of A along the third row. You would compute each M_{3j} (the minors) and multiply by the corresponding a_{3j} and cofactor signs $(-1)^{3+j}$.

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10. Show that AX = B is equivalent to the upper-triangular system UX = Y

and find the solution

a) 2x1 +4x2 −6x3 = −4

x1 +5x2 +3x3 = 10

x1 +3x2 +2x3 = 5

b) −5x1+2x2−x3=−1

x1+0x2+3x3=5

3x1+x2+6x3=17

Soln

Part (a)

Given system of equations:

$$\begin{cases} 2x_1 + 4x_2 - 6x_3 = -4 \\ x_1 + 5x_2 + 3x_3 = 10 \\ x_1 + 3x_2 + 2x_3 = 5 \end{cases}$$

Step-by-Step Gaussian Elimination

1. **Write the augmented matrix:**

$$\begin{bmatrix} 2 & 4 & -6 & | & -4 \\ 1 & 5 & 3 & | & 10 \\ 1 & 3 & 2 & | & 5 \end{bmatrix}$$

2. **Make the pivot in the first column:**

Divide row 1 by 2:

$$\begin{bmatrix} 1 & 2 & -3 & | & -2 \\ 1 & 5 & 3 & | & 10 \\ 1 & 3 & 2 & | & 5 \end{bmatrix}$$

3. **Eliminate the first element in rows 2 and 3:**

Subtract row 1 from rows 2 and 3:

$$\begin{bmatrix} 1 & 2 & -3 & | & -2 \\ 0 & 3 & 6 & | & 12 \\ 0 & 1 & 5 & | & 7 \end{bmatrix}$$

4. **Make the pivot in the second column:**

Divide row 2 by 3:

$$\begin{bmatrix} 1 & 2 & -3 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & 1 & 5 & | & 7 \end{bmatrix}$$

5. **Eliminate the second element in row 3:**

Subtract row 2 from row 3:

$$\begin{bmatrix} 1 & 2 & -3 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 3 & | & 3 \end{bmatrix}$$

Now, the system is in upper-triangular form:

$$\begin{cases} x_1 + 2x_2 - 3x_3 = -2 \\ x_2 + 2x_3 = 4 \\ 3x_3 = 3 \end{cases}$$

Back-Substitution:

From the third equation:

$$x_3 = 1$$

Substitute $x_3 = 1$ into the second equation:

$$x_2 + 2(1) = 4 \implies x_2 = 2$$

Substitute $x_2 = 2$ and $x_3 = 1$ into the first equation:

$$x_1 + 2(2) - 3(1) = -2 \implies x_1 + 4 - 3 = -2 \implies x_1 = -3$$

So, the solution is:

$$(x_1, x_2, x_3) = (-3, 2, 1)$$

Part (b)

Given system of equations:

$$\begin{cases} -5x_1 + 2x_2 - x_3 = -1 \\ x_1 + 0x_2 + 3x_3 = 5 \\ 3x_1 + x_2 + 6x_3 = 17 \end{cases}$$

Step-by-Step Gaussian Elimination

1. **Write the augmented matrix:**

$$\begin{bmatrix} -5 & 2 & -1 & | & -1 \\ 1 & 0 & 3 & | & 5 \\ 3 & 1 & 6 & | & 17 \end{bmatrix}$$

2. **Make the pivot in the first column:**

Interchange row 1 and row 2:

$$\begin{bmatrix} 1 & 0 & 3 & | & 5 \\ -5 & 2 & -1 & | & -1 \\ 3 & 1 & 6 & | & 17 \end{bmatrix}$$

3. **Eliminate the first element in rows 2 and 3:**

Add 5 times row 1 to row 2, and subtract 3 times row 1 from row 3:

$$\begin{bmatrix} 1 & 0 & 3 & | & 5 \\ 0 & 2 & 14 & | & 24 \\ 0 & 1 & -3 & | & 2 \end{bmatrix}$$

4. **Make the pivot in the second column:**

Divide row 2 by 2:

$$\begin{bmatrix} 1 & 0 & 3 & | & 5 \\ 0 & 1 & 7 & | & 12 \\ 0 & 1 & -3 & | & 2 \end{bmatrix}$$

5. **Eliminate the second element in row 3:**

Subtract row 2 from row 3:

$$\begin{bmatrix} 1 & 0 & 3 & | & 5 \\ 0 & 1 & 7 & | & 12 \\ 0 & 0 & -10 & | & -10 \end{bmatrix}$$

Now, the system is in upper-triangular form:

$$\begin{cases} x_1 + 3x_3 = 5 \\ x_2 + 7x_3 = 12 \\ -10x_3 = -10 \end{cases}$$

Back-Substitution:

From the third equation:

$$x_3 = 1$$

Substitute $x_3 = 1$ into the second equation:

$$x_2 + 7(1) = 12 \implies x_2 = 5$$

Substitute $x_2 = 5$ and $x_3 = 1$ into the first equation:

$$x_1 + 3(1) = 5 \implies x_1 = 2$$

So, the solution is:

$$(x_1, x_2, x_3) = (2, 5, 1)$$

11. SolvethefollowinglinearsystemusingGaussianeliminationwithpartial pivoting.

x1+20x2−x3+0.001x4=0

2x1−5x2+30x3−0.1x4=1

5x1+x2−100x3−10x4=0

2x1−100x2−x3+x4=0

Soln

11. Solve the following linear system using Gaussian elimination with partial pivoting:

The system is:

$$x_1 + 20x_2 - x_3 + 0.001x_4 = 0 \quad (1)$$

$$2x_1 - 5x_2 + 30x_3 - 0.1x_4 = 1 \quad (2)$$

$$5x_1 + x_2 - 100x_3 - 10x_4 = 0 \quad (3)$$

$$2x_1 - 100x_2 - x_3 + x_4 = 0 \quad (4)$$

Step 1: Set up the augmented matrix for the system

The augmented matrix of the system is:

$$\left[\begin{array}{cccc|c} 1 & 20 & -1 & 0.001 & 0 \\ 2 & -5 & 30 & -0.1 & 1 \\ 5 & 1 & -100 & -10 & 0 \\ 2 & -100 & -1 & 1 & 0 \end{array}\right]$$

Step 2: Perform Partial Pivoting

- We need to select the largest element in column 1 for pivoting.
- In the first column, the largest absolute value is 5, which is in row 3. So, we swap row 1 with row 3.

The augmented matrix after swapping rows 1 and 3:

$$\left[\begin{array}{cccc|c} 5 & 1 & -100 & -10 & 0 \\ 2 & -5 & 30 & -0.1 & 1 \\ 1 & 20 & -1 & 0.001 & 0 \\ 2 & -100 & -1 & 1 & 0 \end{array}\right]$$

Step 3: Eliminate x_1 in rows 2, 3, and 4

- To eliminate x_1 in row 2, we subtract $\frac{2}{5}$ of row 1 from row 2:

$$R_2 \to R_2 - \frac{2}{5}R_1$$

$$
\left[\begin{array}{cccc|c}
5 & 1 & -100 & -10 & 0 \\
0 & -5.4 & 50.4 & 3.9 & 1 \\
1 & 20 & -1 & 0.001 & 0 \\
2 & -100 & -1 & 1 & 0
\end{array}\right]
$$

- To eliminate x_1 in row 3, we subtract $\frac{1}{5}$ of row 1 from row 3:

$$
R_3 \to R_3 - \frac{1}{5}R_1
$$

$$
\left[\begin{array}{cccc|c}
5 & 1 & -100 & -10 & 0 \\
0 & -5.4 & 50.4 & 3.9 & 1 \\
0 & 19.8 & 19 & 2.001 & 0 \\
2 & -100 & -1 & 1 & 0
\end{array}\right]
$$

- To eliminate x_1 in row 4, we subtract $\frac{2}{5}$ of row 1 from row 4:

$$
R_4 \to R_4 - \frac{2}{5}R_1
$$

$$
\left[\begin{array}{cccc|c}
5 & 1 & -100 & -10 & 0 \\
0 & -5.4 & 50.4 & 3.9 & 1 \\
0 & 19.8 & 19 & 2.001 & 0 \\
0 & -101.2 & 39 & 5 & 0
\end{array}\right]
$$

Step 4: Partial Pivoting in the Second Column

- Now, we look for the largest element in column 2 (in absolute value). The largest value is 19.8, which is in row 3. We swap row 2 and row 3:

The augmented matrix after swapping rows 2 and 3:

$$
\left[\begin{array}{cccc|c}
5 & 1 & -100 & -10 & 0 \\
0 & 19.8 & 19 & 2.001 & 0 \\
0 & -5.4 & 50.4 & 3.9 & 1 \\
0 & -101.2 & 39 & 5 & 0
\end{array}\right]
$$

Step 5: Eliminate x_2 in rows 3 and 4

- To eliminate x_2 in row 3, we add $\frac{5.4}{19.8}$ times row 2 to row 3:

$$R_3 \to R_3 + \frac{5.4}{19.8} R_2$$

$$\left[\begin{array}{cccc|c} 5 & 1 & -100 & -10 & 0 \\ 0 & 19.8 & 19 & 2.001 & 0 \\ 0 & 0 & 56.1 & 4.141 & 1 \\ 0 & -101.2 & 39 & 5 & 0 \end{array}\right]$$

- To eliminate x_2 in row 4, we add $\frac{101.2}{19.8}$ times row 2 to row 4:

$$R_4 \to R_4 + \frac{101.2}{19.8} R_2$$

$$\left[\begin{array}{cccc|c} 5 & 1 & -100 & -10 & 0 \\ 0 & 19.8 & 19 & 2.001 & 0 \\ 0 & 0 & 56.1 & 4.141 & 1 \\ 0 & 0 & 138.3 & 10.2 & 0 \end{array}\right]$$

Step 6: Partial Pivoting in the Third Column

- The largest element in the third column is 138.3, which is in row 4. We swap rows 3 and 4:

The augmented matrix after swapping rows 3 and 4:

$$\left[\begin{array}{cccc|c} 5 & 1 & -100 & -10 & 0 \\ 0 & 19.8 & 19 & 2.001 & 0 \\ 0 & 0 & 138.3 & 10.2 & 0 \\ 0 & 0 & 56.1 & 4.141 & 1 \end{array}\right]$$

Step 7: Eliminate x_3 in row 4

- To eliminate x_3 in row 4, we subtract $\frac{56.1}{138.3}$ times row 3 from row 4:

$$R_4 \to R_4 - \frac{56.1}{138.3} R_3$$

$$\left[\begin{array}{cccc|c} 5 & 1 & -100 & -10 & 0 \\ 0 & 19.8 & 19 & 2.001 & 0 \\ 0 & 0 & 138.3 & 10.2 & 0 \\ 0 & 0 & 0 & -0.9 & 1 \end{array}\right]$$

Now, the system is in upper-triangular form. We can proceed with back-substitution.

Step 8: Back Substitution

Step 8: Back Substitution

1. From the fourth row, we get:

$$x_4 = \frac{1}{-0.9} = -\frac{10}{9}$$

2. Substitute $x_4 = -\frac{10}{9}$ into the third row:

$$138.3x_3 + 10.2x_4 = 0$$

$$138.3x_3 + 10.2 \times -\frac{10}{9} = 0$$

$$138.3x_3 - 11.333 = 0 \quad \Rightarrow \quad x_3 = \frac{11.333}{138.3} \approx 0.082$$

3. Substitute $x_3 \approx 0.082$ and $x_4 \approx -\frac{10}{9}$ into the second row:

$$19.8x_2 + 19x_3 + 2.001x_4 = 0$$

$$19.8x_2 + 19 \times 0.082 + 2.001 \times -\frac{10}{9} = 0$$

$$19.8x_2 + 1.558 - 2.223 = 0 \quad \Rightarrow \quad x_2 \approx -\frac{-0.335}{19.8} \approx 0.017$$

4. Substitute $x_2 \approx 0.017$, $x_3 \approx 0.082$, and $x_4 \approx -\frac{10}{9}$ into the first row:

$$5x_1 + x_2 - 100x_3 - 10x_4 = 0$$

$$5x_1 + 0.017 - 100 \times 0.082 - 10 \times -\frac{10}{9} = 0$$

$$5x_1 + 0.017 - 8.2 + 11.111 = 0 \quad \Rightarrow \quad x_1 \approx \frac{-2.928}{5} \approx -0.586$$

Final Solution:

$$x_1 \approx -0.586, \quad x_2 \approx 0.017, \quad x_3 \approx 0.082, \quad x_4 \approx -\frac{10}{9}.$$

12. Find the triangular factorization $A = LU$ for the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}$$

Soln

12. Find the triangular factorization $A = LU$ for the matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{pmatrix}$$

We want to find the **LU decomposition** of matrix A, where $A = LU$, with L being a **lower triangular matrix** and U being an **upper triangular matrix**.

Step 1: Set up the matrices L and U

Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Step 2: Perform Gaussian elimination to reduce A to upper triangular form

Eliminate the elements below the pivot in the first column (using $u_{11} = 1$)

- **Row 2**: Subtract $2 \times \text{row } 1$ from row 2:

$$R_2 \to R_2 - 2 \times R_1$$

$$\begin{bmatrix} 2 & -1 & 5 & 0 \\ \to & 2-2(1) & -1-2(1) & 5-2(0) & 0-2(4) \end{bmatrix} = \begin{bmatrix} 0 & -3 & 5 & -8 \end{bmatrix}$$

- **Row 3**: Subtract $5 \times \text{row } 1$ from row 3:

$$R_3 \to R_3 - 5 \times R_1$$

$$= [0, -3, 5, -8]$$

We are performing Gaussian elimination to transform A into an upper triangular matrix U while tracking the multipliers in the lower triangular matrix L.

Given matrix A:

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{pmatrix}$$

Step 1: Eliminate elements below the pivot in the first column

Row 2 (R2):

Subtract $2 \times \text{Row 1}$ from Row 2:

$$R_2 \to R_2 - 2 \times R_1$$

$$\begin{pmatrix} 2 & -1 & 5 & 0 \end{pmatrix} - 2 \times \begin{pmatrix} 1 & 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 5 & -8 \end{pmatrix}$$

Thus, $L_{21} = 2$.

Row 3 (R3):

Subtract $5 \times \text{Row 1}$ from Row 3:

$$R_3 \to R_3 - 5 \times R_1$$

$$\begin{pmatrix} 5 & 2 & 1 & 2 \end{pmatrix} - 5 \times \begin{pmatrix} 1 & 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 1 & -18 \end{pmatrix}$$

Thus, $L_{31} = 5$.

Row 4 (R4):

Add $3 \times \text{Row 1}$ to Row 4:

$$R_4 \to R_4 + 3 \times R_1$$

$$\begin{pmatrix} -3 & 0 & 2 & 6 \end{pmatrix} + 3 \times \begin{pmatrix} 1 & 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 2 & 18 \end{pmatrix}$$

Thus, $L_{41} = -3$.

Now, the matrix becomes:

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & -3 & 1 & -18 \\ 0 & 3 & 2 & 18 \end{pmatrix}$$

Step 2: Eliminate elements below the pivot in the second column

Row 3 (R3):

Subtract $\frac{-3}{-3} \times \text{Row } 2$ from Row 3:

$$R_3 \to R_3 - R_2$$

$$\begin{pmatrix} 0 & -3 & 1 & -18 \end{pmatrix} - \begin{pmatrix} 0 & -3 & 5 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -4 & -10 \end{pmatrix}$$

Thus, $L_{32} = -1$.

Row 4 (R4):

Add $\frac{3}{-3} \times \text{Row } 2$ to Row 4:

Share

$$R_4 \to R_4 + R_2$$

$$\begin{pmatrix} 0 & 3 & 2 & 18 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 5 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 7 & 10 \end{pmatrix}$$

Thus, $L_{42} = 1$.

Now, the matrix becomes:

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 7 & 10 \end{pmatrix}$$

Step 3: Eliminate elements below the pivot in the third column

Row 4 (R4):

Add $\frac{7}{-4} \times \text{Row } 3$ to Row 4:

$$R_4 \to R_4 + \frac{7}{4} \times R_3$$

$$\begin{pmatrix} 0 & 0 & 7 & 10 \end{pmatrix} + \frac{7}{4} \times \begin{pmatrix} 0 & 0 & -4 & -10 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $L_{43} = \frac{7}{4}$.

Now, the matrix becomes:

$$A = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Final LU Decomposition

Thus, the LU decomposition of matrix A is:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & -1 & 1 & 0 \\ -3 & 1 & \frac{7}{4} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the triangular factorization of the given matrix A.