中国民航大学 Civil Aviation University of China

Structural Applications of Finite Elements

Chapter 5

2D problems

2018-09-01

Outline

- **2D constant strain triangles**
- **Axisymmetric solids subjected to axisymmetric loading**

Introduction

Displacement vector $\mathbf{u} = [u, v]^{\mathrm{T}}$

Stresses and strain

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \gamma_{xy}]^{\mathrm{T}}$$

The strain–displacement

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]^{\mathrm{T}}$$

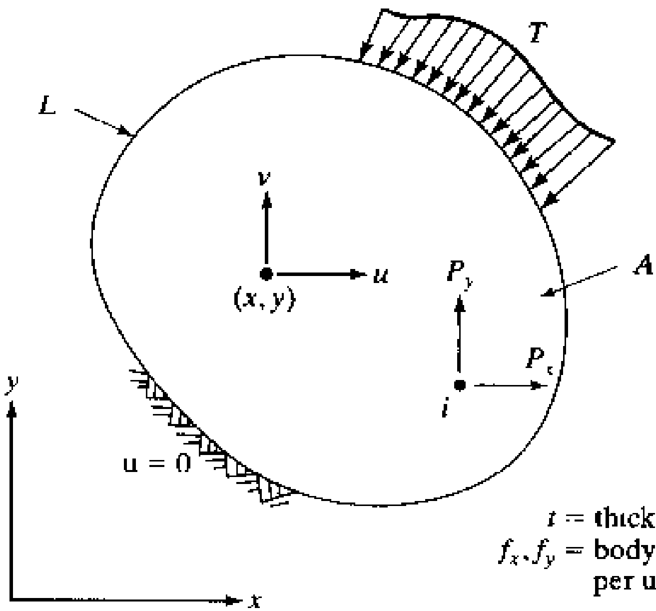
Body force, traction vector and element volume

$$\begin{cases} \mathbf{f} = [f_x, f_y]^{\mathrm{T}} \\ \mathbf{T} = [T_x, T_y]^{\mathrm{T}} \\ dV = t\,dA \end{cases}$$

Stresses and strains are related

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$



t = thickness at (x, y)
f_x, f_y = body force components per unit volume at (x, y)

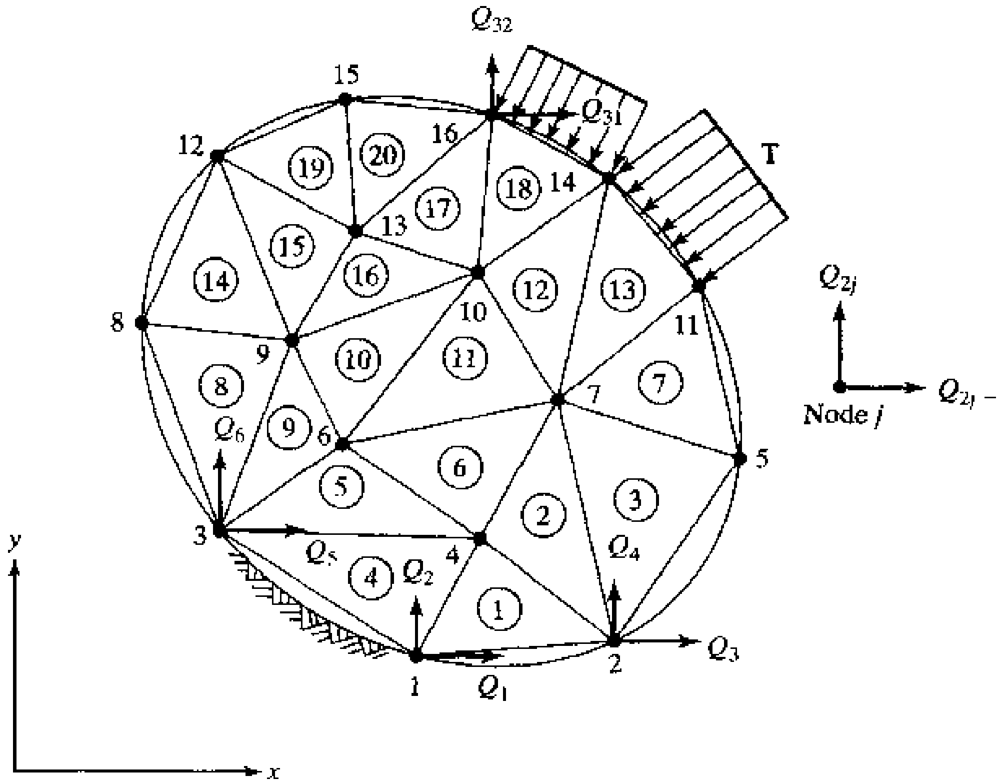
FIGURE 5.1 Two-dimensional problem.

Global displacement vector

$$\mathbf{Q} = [Q_1, Q_2, \ldots, Q_N]^{\mathrm{T}}$$

Element displacement vector

$$\mathbf{q} = [q_1, q_2, \ldots, q_6]^{\mathrm{T}}$$



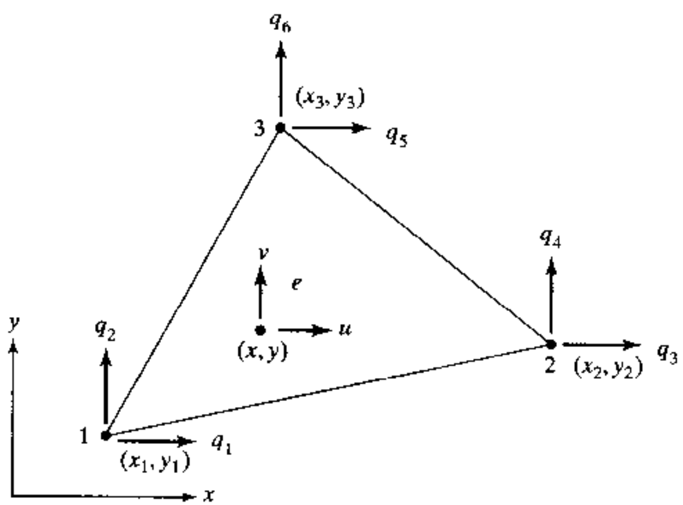


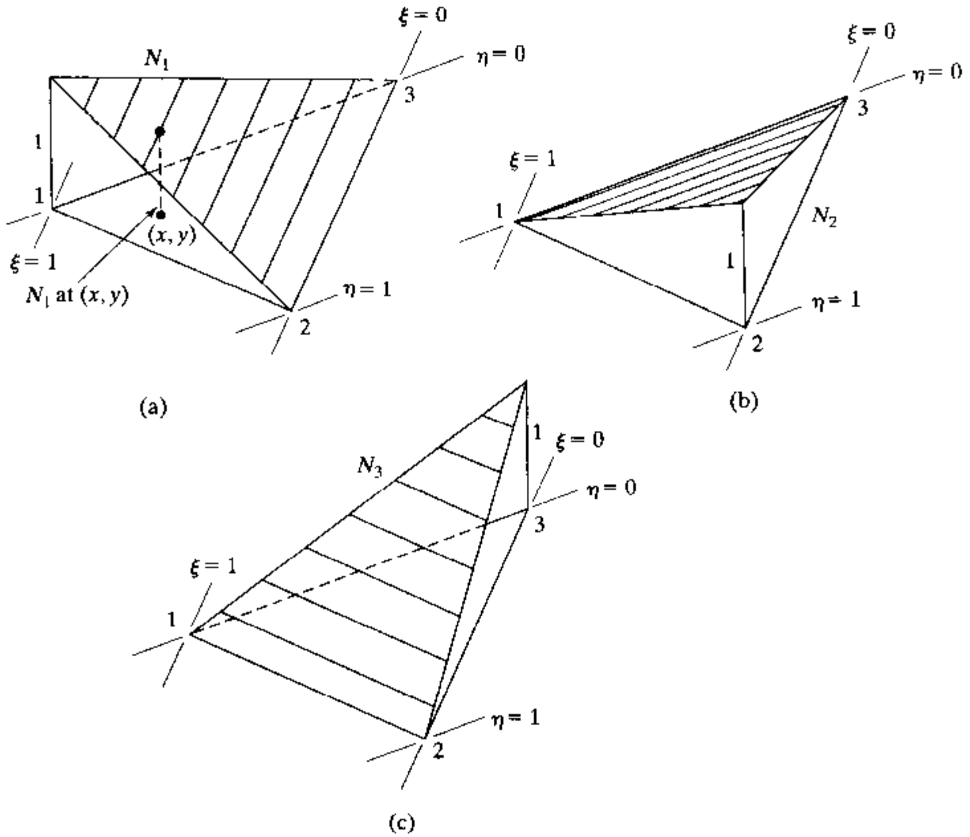
TABLE 5.1 Element Connectivity

Element number e	Three nodes 1	2	3
1	1	2	4
2	4	2	7
⋮			
11	6	7	10
⋮			
20	13	16	15

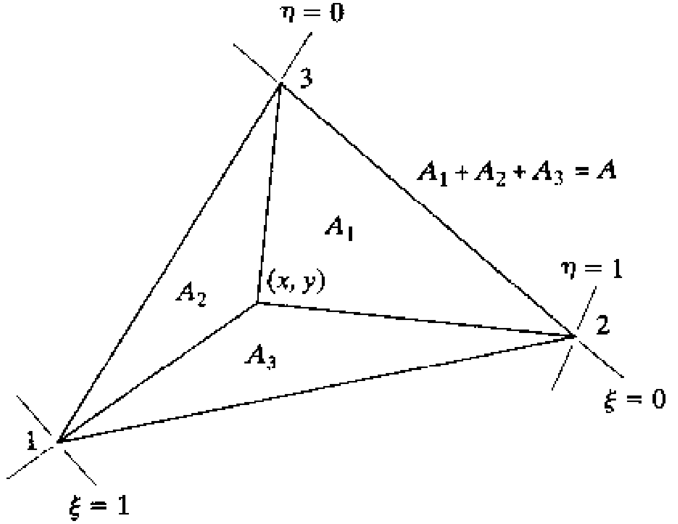
$$N_1 + N_2 + N_3 = 1$$

Natural coordinates

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$



Area coordinates



Isoparametric representation

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 \qquad N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$$u = (q_1 + q_5)\xi + (q_3 + q_5)\eta + q_5$$

$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \qquad \mathbf{u} = \mathbf{N}\mathbf{q}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

Using the notation, $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$,

$$x = x_{13}\xi + x_{23}\eta + x_3$$

$$y = y_{13}\xi + y_{23}\eta + y_3$$

Solution Using the isoparametric representation (Eqs. 5.15). we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

These two equations are rearranged in the form

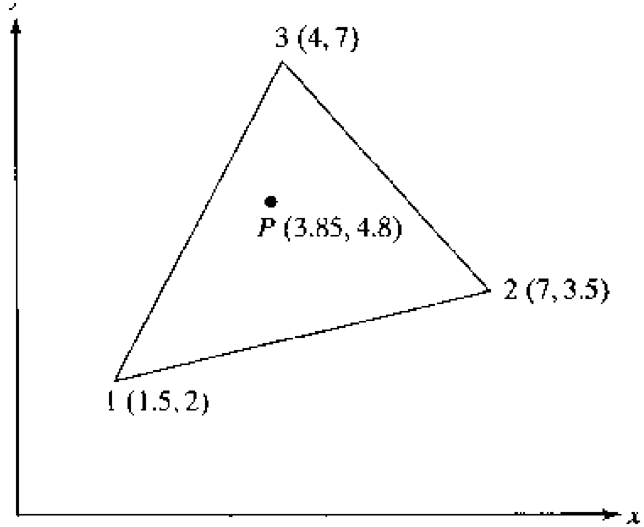
$$2.5\xi - 3\eta = 0.15$$
$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

$$x = N_1x_1 + N_2x_2 + N_3x_3$$
$$y = N_1y_1 + N_2y_2 + N_3y_3$$

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$



Jacobian matrix

$$u = u(x(\xi,\eta), y(\xi,\eta))$$

$$v = v(x(\xi,\eta), y(\xi,\eta))$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

$$x = x_{13}\xi + x_{23}\eta + x_3$$

$$y = y_{13}\xi + y_{23}\eta + y_3$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13}$$

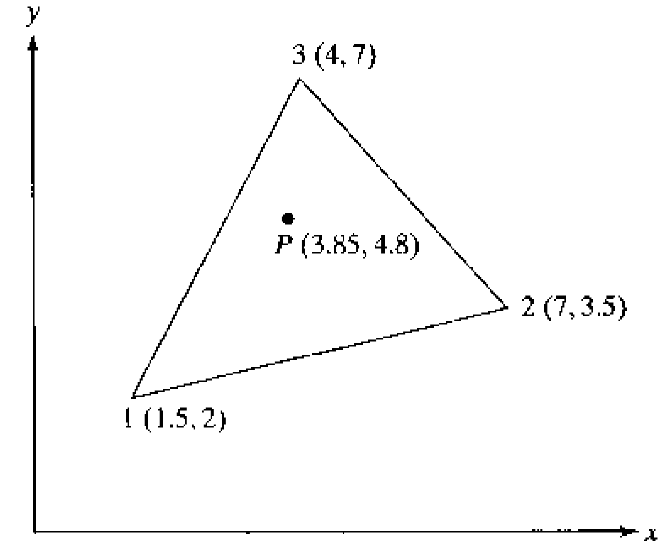
$$A = \tfrac{1}{2}|\det \mathbf{J}|$$

Example

Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, $\det \mathbf{J} = 23.75$ units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then $\det \mathbf{J}$ will be negative. ■



$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$u = (q_1 + q_5)\xi + (q_3 + q_5)\eta + q_5$$
$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6$$

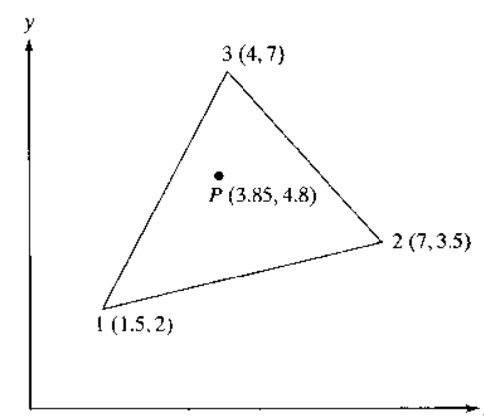
$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1}\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}}\begin{Bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} - y_{13}\dfrac{\partial u}{\partial \eta} \\ -x_{23}\dfrac{\partial u}{\partial \xi} + x_{13}\dfrac{\partial u}{\partial \eta} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}}\begin{Bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} - y_{13}\dfrac{\partial v}{\partial \eta} \\ -x_{23}\dfrac{\partial v}{\partial \xi} + x_{13}\dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{\det \mathbf{J}}\begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix}$$

$$y_{31} = y_{13} \text{ and } y_{12} = y_{13} - y_{23}$$

$$\boldsymbol{\epsilon} = \frac{1}{\det \mathbf{J}}\begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix}$$



$$\mathbf{B} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\boxed{\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}}$$

Example

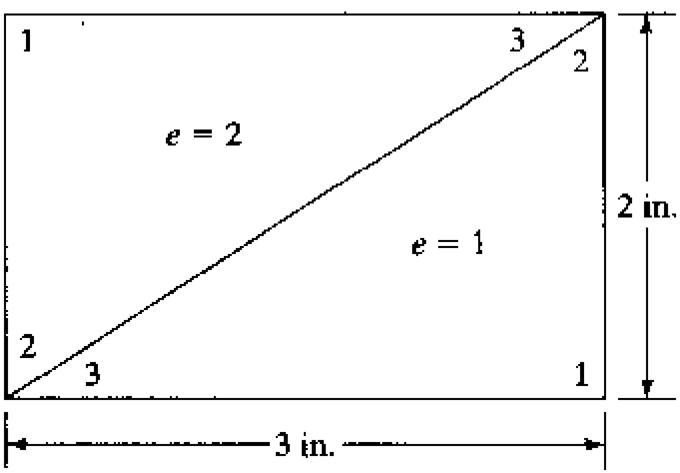
Solution We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where $\det \mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6} \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix} \quad \blacksquare$$



Potential energy approach

The potential energy of the system, Π, is given by

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon} t\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f} t\, dA - \int_L \mathbf{u}^{\mathrm{T}} t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon} t\, dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon} t\, dA$ is the element strain energy.

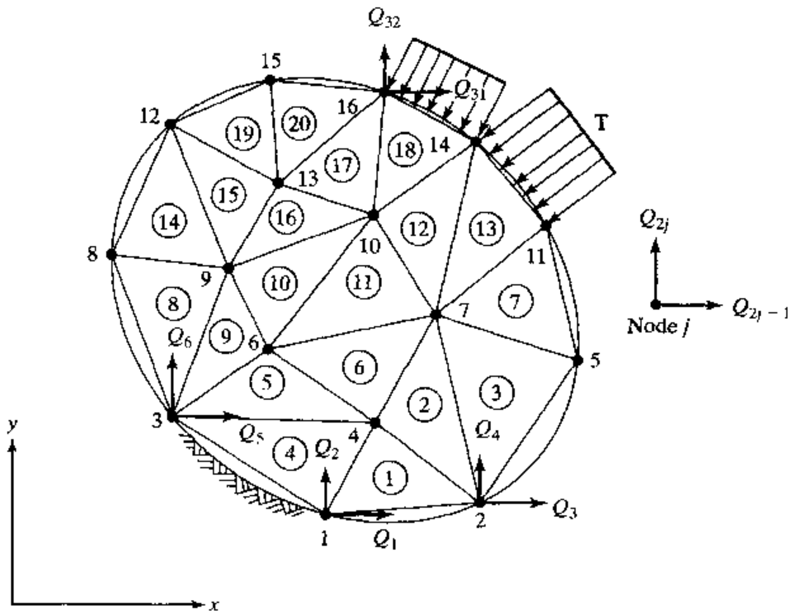


FIGURE 5.2 Finite element discretization.

Element stiffness

$$\boxed{\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}}$$

$$\begin{aligned} U_e &= \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA \\ &= \frac{1}{2}\int_e \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}t\,dA \end{aligned}$$

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t_e\left(\int_e dA\right)\mathbf{q}$$

$$\int_e dA = A_e \qquad U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}$$

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \qquad \mathbf{k}^e = t_eA_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$$

$$\boxed{\begin{aligned} U &= \sum_e \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \\ &= \frac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} \end{aligned}}$$

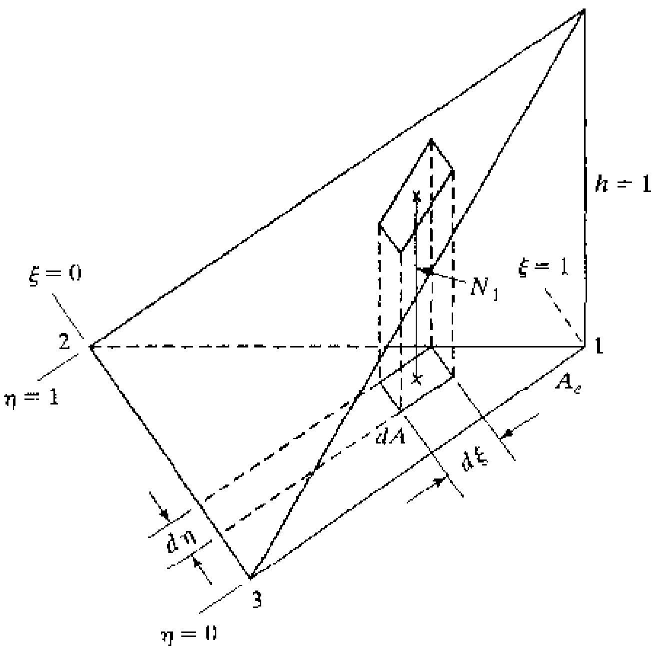
$$m_e = \max(|i_1 - i_2|, |i_2 - i_3|, |i_3 - i_1|) \qquad \mathrm{NBW} = 2\left(\max_{1 \le e \le \mathrm{NE}}(m_e) + 1\right)$$

Force terms

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$$\int_e \mathbf{u}^{\mathrm{T}} \mathbf{f} t \, dA = t_e \int_e (u f_x + v f_y) \, dA$$

$$\int_e \mathbf{u}^{\mathrm{T}} \mathbf{f} t \, dA = q_1\left(t_e f_x \int_e N_1 dA\right) + q_2\left(t_e f_y \int_e N_1 \, dA\right)$$
$$+ q_3\left(t_e f_x \int_e N_2 \, dA\right) + q_4\left(t_e f_y \int_e N_2 \, dA\right)$$
$$+ q_5\left(t_e f_x \int_e N_3 \, dA\right) + q_6\left(t_e f_y \int_e N_3 \, dA\right)$$



$$\int_e N_1 \, dA = \frac{1}{3} \cdot A_e h = \frac{1}{3} \cdot A_e$$

$$\text{or} \int_e N_1 \, dA = \int_0^1 \int_0^{1-\xi} N_1 \det \mathbf{J} \, d\eta \, d\xi = 2A_e \int_0^1 \int_0^{1-\xi} \xi \, d\eta \, d\xi = \frac{1}{3} \cdot A_e$$

$$\int_e N_i \, dA = \tfrac{1}{3} A_e \qquad \int_e N_2 \, dA = \int_e N_3 \, dA = \tfrac{1}{3} A_e$$

$$\int_e \mathbf{u}^{\mathrm{T}} \mathbf{f} t \, dA = \mathbf{q}^{\mathrm{T}} \mathbf{f}^e$$

$$\mathbf{f}^e = \frac{t_e A_e}{3} [f_x, f_y, f_x, f_y, f_x, f_y]^{\mathrm{T}}$$

$$\mathbf{F} \longleftarrow \sum_e \mathbf{f}^e$$

$$\int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell = \int_{\ell_{1-2}} (uT_x + vT_y)t\,d\ell$$

Using the interpolation relations involving the shape functions

$$u = N_1q_1 + N_2q_3$$

$$v = N_1q_2 + N_2q_4$$

$$T_x = N_1T_{x1} + N_2T_{x2}$$

$$T_y = N_1T_{y1} + N_2T_{y2}$$

and noting that

$$\int_{\ell_{1-2}} N_1^2\,d\ell = \frac{1}{3}\ell_{1-2}, \qquad \int_{\ell_{1-2}} N_2^2\,d\ell = \frac{1}{3}\ell_{1-2}, \qquad \int_{\ell_{1-2}} N_1N_2\,d\ell = \frac{1}{6}\ell_{1-2}$$

$$\ell_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

we get

$$\int_{\ell_{1-2}} \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell = [q_1, q_2, q_3, q_4]\mathbf{T}^e$$

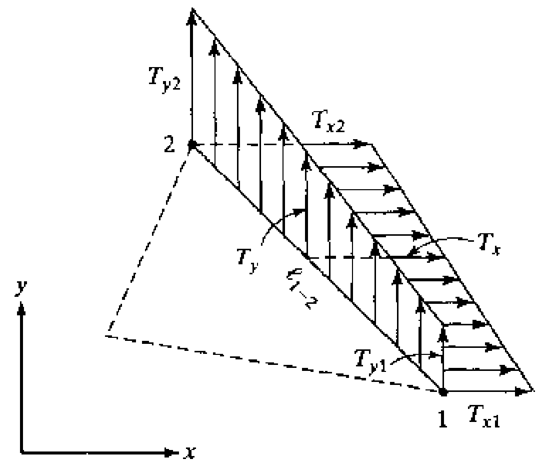
where $\mathbf{T}^e$ is given by

$$\mathbf{T}^e = \frac{t_e\ell_{1-2}}{6}[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}]^{\mathrm{T}}$$

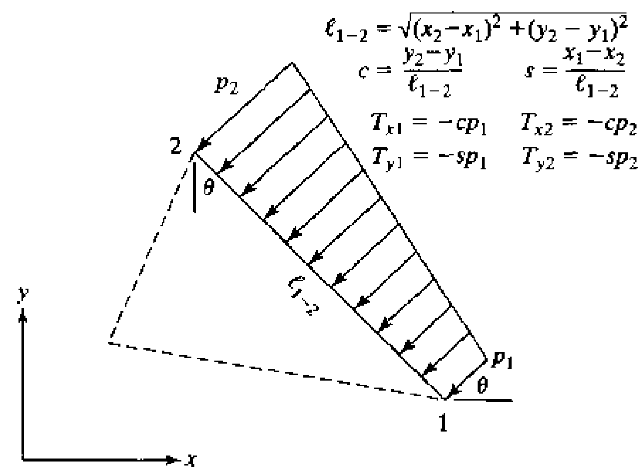
$$T_{x1} = -cp_1, \qquad T_{x2} = -cp_2, \qquad T_{y1} = -sp_1, \qquad T_{y2} = -sp_2$$

where

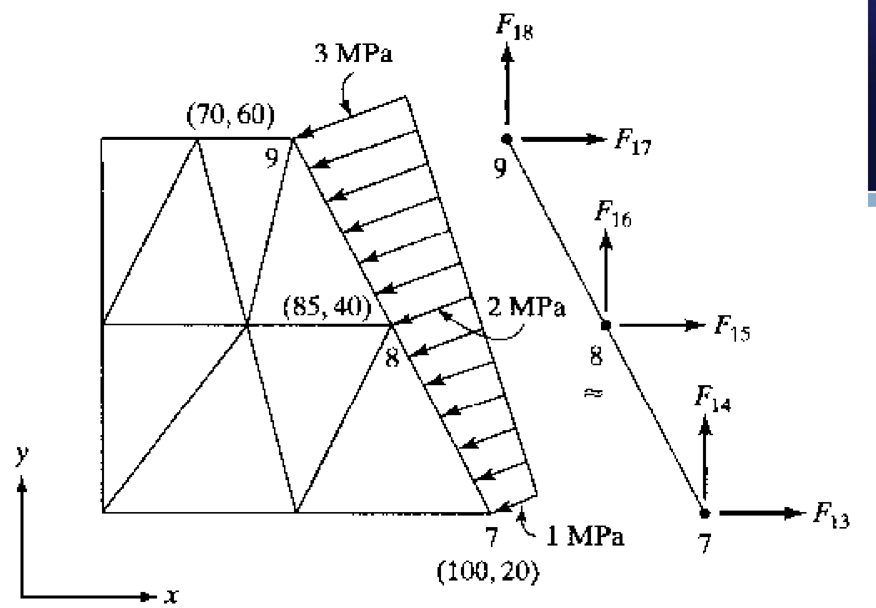
$$s = \frac{(x_1 - x_2)}{\ell_{1-2}} \quad \text{and} \quad c = \frac{(y_2 - y_1)}{\ell_{1-2}}.$$



(a) Component distribution



(b) Normal pressure



Thickness = 10 mm

Solution We consider the two edges 7–8 and 8–9 separately and then merge them.

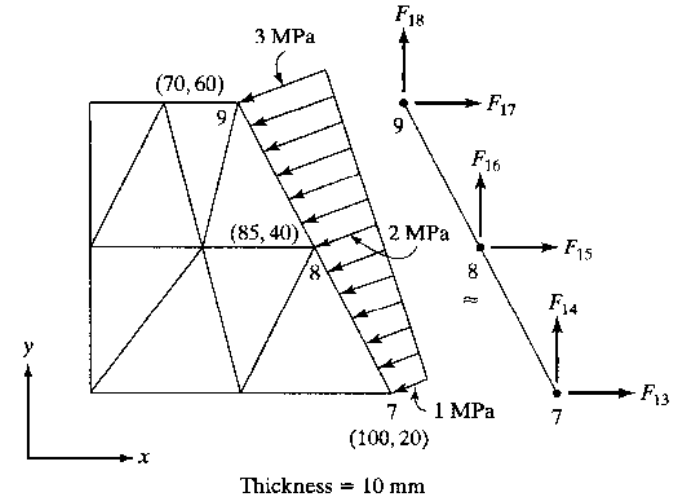
For edge 7–8

$p_1 = 1\,\text{MPa}, \quad p_2 = 2\,\text{MPa}, \quad x_1 = 100\,\text{mm}, \quad y_1 = 20\,\text{mm}, \quad x_2 = 85\,\text{mm}, \quad y_2 = 40\,\text{mm},$

$$\ell_{1-2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 25\,\text{mm}$$

$$c = \frac{y_2 - y_1}{\ell_{1-2}} = 0.8, \qquad s = \frac{x_1 - x_2}{\ell_{1-2}} = 0.6$$

$$T_{x1} = -p_1 c = -0.8, \qquad T_{y1} = -p_1 s = -0.6, \qquad T_{x2} = -p_2 c = -1.6,$$



$$T_{y1} = -p_2 s = -1.2$$

$$\mathbf{T}^1 = \frac{10 \times 25}{6}[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}]$$

$$= [-133.3, -100, -166.7, -125]^{\mathrm{T}}\,\mathrm{N}$$

These loads add to F_{13}, F_{14}, F_{15}, and F_{16}, respectively.

For edge 8–9

$p_1 = 2\,\mathrm{MPa}, \quad p_2 = 3\,\mathrm{MPa}, \quad x_1 = 85\,\mathrm{mm}, \quad y_1 = 40\,\mathrm{mm}, \quad x_2 = 70\,\mathrm{mm}, \quad y_2 = 60\,\mathrm{mm},$

$$\ell_{1-2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 25\,\mathrm{mm}$$

$$c = \frac{y_2 - y_1}{\ell_{1-2}} = 0.8, \qquad s = \frac{x_1 - x_2}{\ell_{1-2}} = 0.6$$

$$T_{x1} = -p_1 c = -1.6, \qquad T_{y1} = -p_1 s = -1.2, \qquad T_{x2} = -p_2 c = -2.4,$$

$$T_{y2} = -p_2 s = -1.8$$

$$\mathbf{T}^2 = \frac{10 \times 25}{6}[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}]^{\mathrm{T}}$$

$$= [-233.3, -175, -266.7, -200]^{\mathrm{T}}\,\mathrm{N}$$

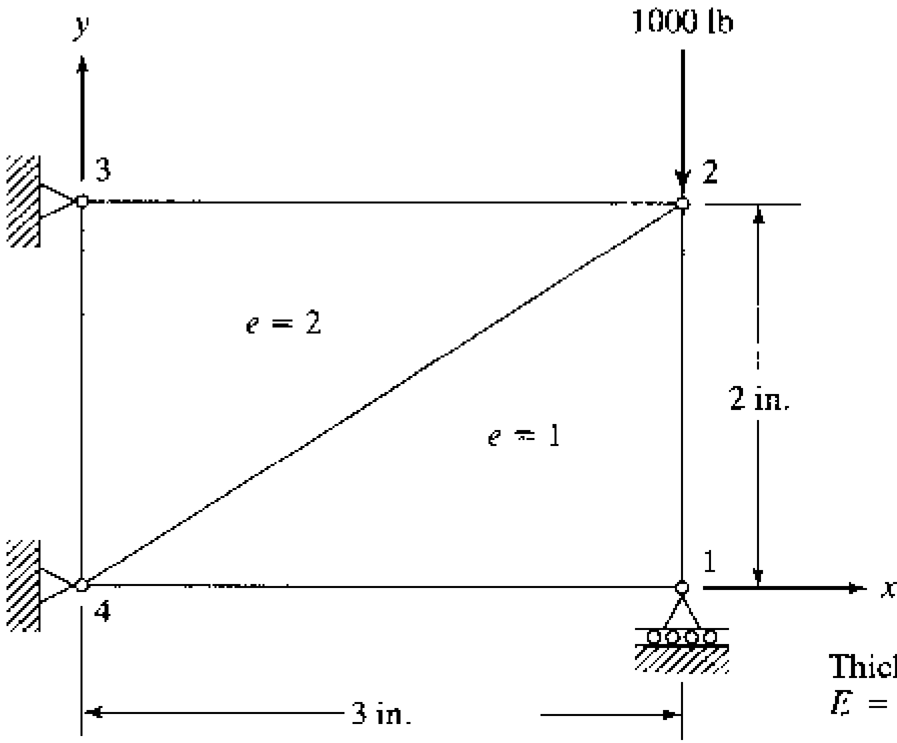
These loads add to F_{15}, F_{16}, F_{17}, and F_{18}, respectively. Thus,

$$[F_{13} \quad F_{14} \quad F_{15} \quad F_{16} \quad F_{17} \quad F_{18}] = [-133.3 \quad -100 \quad -400 \quad -300 \quad -266.7 \quad -200]\,\mathrm{N}$$

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i = Q_{2i-1}P_x + Q_{2i}P_y$$

$$\Pi = \frac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\mathbf{F}$$

$$\mathbf{K}\mathbf{Q} = \mathbf{F}$$



Thickness $t = 0.5$ in.,
$E = 30 \times 10^6$ psi, $\nu = 0.25$

Solution For plane stress conditions, the material property matrix is given by

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2\times10^7 & 0.8\times10^7 & 0 \\ 0.8\times10^7 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{bmatrix}$$

Using the local numbering pattern used in Fig. E5.3, we establish the connectivity as follows:

	Nodes		
Element No.	1	2	3
1	1	2	4
2	3	4	2

On performing the matrix multiplication $\mathbf{DB}^e$, we get

$$\mathbf{DB}^1 = 10^7\begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}$$

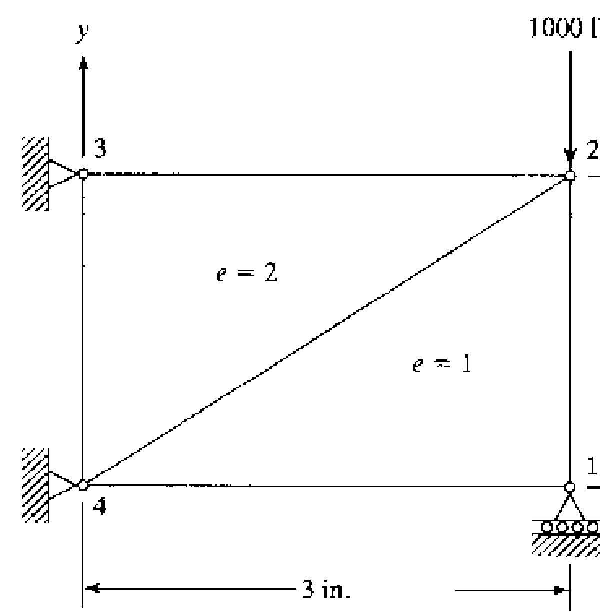
and

$$\mathbf{DB}^2 = 10^7\begin{bmatrix} -1.067 & 0.4 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.6 & 0 & -1.6 & 0.267 & 0 \\ 0.6 & -0.4 & -0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

These two relationships will be used later in calculating stresses using $\boldsymbol{\sigma}^e = \mathbf{DB}^e\mathbf{q}$. The multiplication $t_eA_e\mathbf{B}^{e^{\mathrm{T}}}\mathbf{DB}^e$ gives the element stiffness matrices,

$$\mathbf{k}^1 = 10^7 \begin{array}{c} \begin{matrix} \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 7 & \quad 8 \leftarrow \text{Global dof} \end{matrix} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ \text{Symmetric} & & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$

$$\mathbf{k}^2 = 10^7 \begin{array}{c} \begin{matrix} \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 3 & \quad 4 \leftarrow \text{Global dof} \end{matrix} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ \text{Symmetric} & & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$



In the previous element matrices, the global dof association is shown on top. In the problem under consideration, Q_2, Q_5, Q_6, Q_7, and Q_8, are all zero. Using the elimination approach discussed in Chapter 3, it is now sufficient to consider the stiffnesses associated with the degrees of freedom Q_1, Q_3, and Q_4. Since the body forces are neglected, the first vector has the component $F_4 = -1000$ lb. The set of equations is given by the matrix representation

$$10^7 \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}$$

Solving for Q_1, Q_3, and Q_4, we get

$$Q_1 = 1.913 \times 10^{-5} \text{ in.} \qquad Q_3 = 0.875 \times 10^{-5} \text{ in.} \qquad Q_4 = -7.436 \times 10^{-5} \text{ in.}$$

For element 1, the element nodal displacement vector is given by

$$\mathbf{q}^1 = 10^{-5}[1.913, 0, 0.875, -7.436, 0, 0]^T$$

The element stresses $\boldsymbol{\sigma}^1$ are calculated from $\mathbf{DB}^1\mathbf{q}$ as

$$\boldsymbol{\sigma}^1 = [-93.3, -1138.7, -62.3]^T \text{ psi}$$

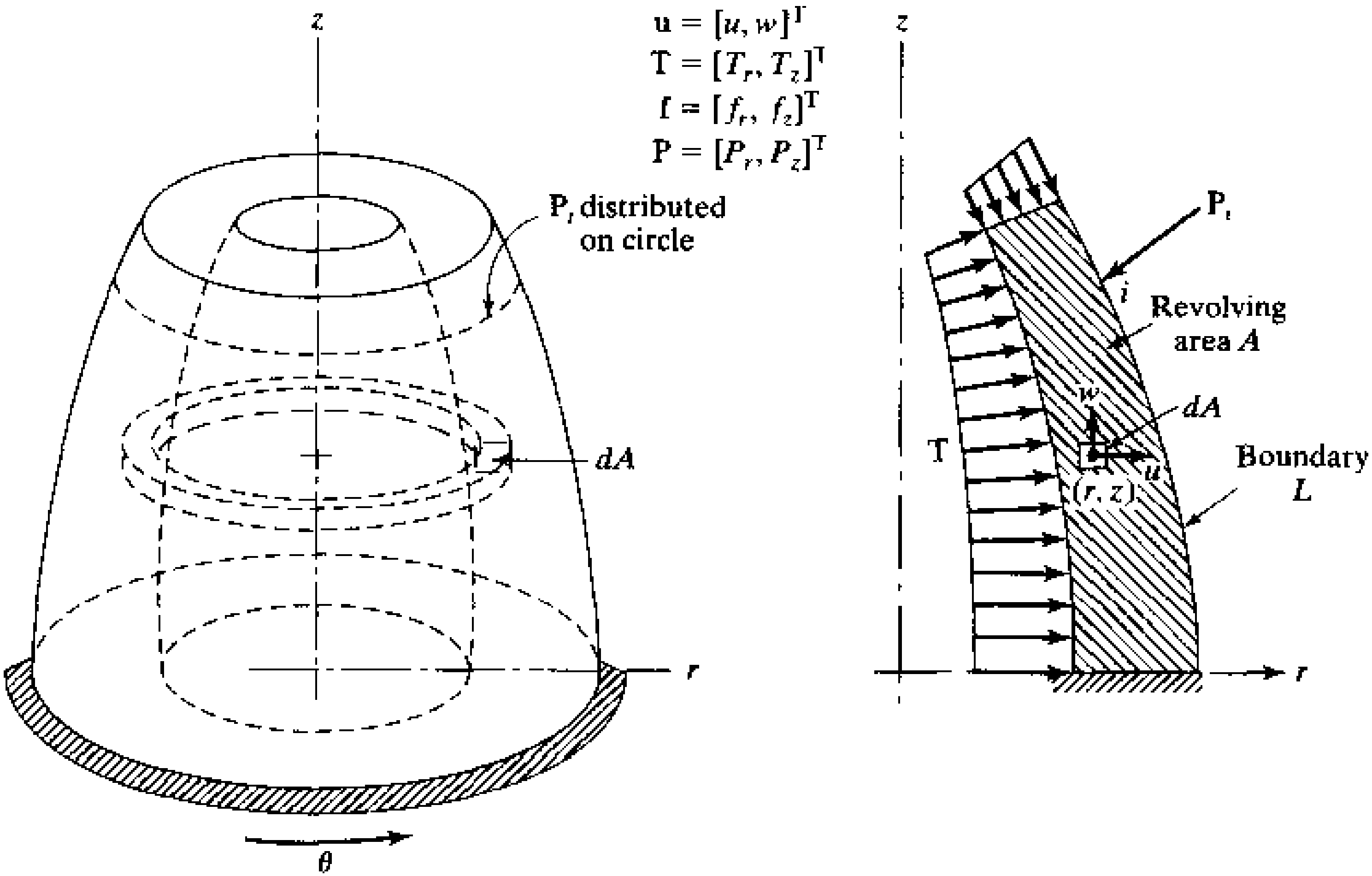
Similarly,

$$\mathbf{q}^2 = 10^{-5}[0, 0, 0. 0, 0.875, -7.436]^T$$

$$\boldsymbol{\sigma}^2 = [93.4, 23.4, -297.4]^T \text{ psi}$$

The computer results may differ slightly since the penalty approach for handling boundary conditions is used in the computer program. ■

Axisymmetric solids subjected to axisymmetric loading



$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon} r\,dA\,d\theta - \int_0^{2\pi}\int_A \mathbf{u}^{\mathrm{T}}\mathbf{f} r\,dA\,d\theta - \int_0^{2\pi}\int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} r\,d\ell\,d\theta - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

$$\Pi = 2\pi\left(\frac{1}{2}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon} r\,dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f} r\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T} r\,d\ell\right) - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

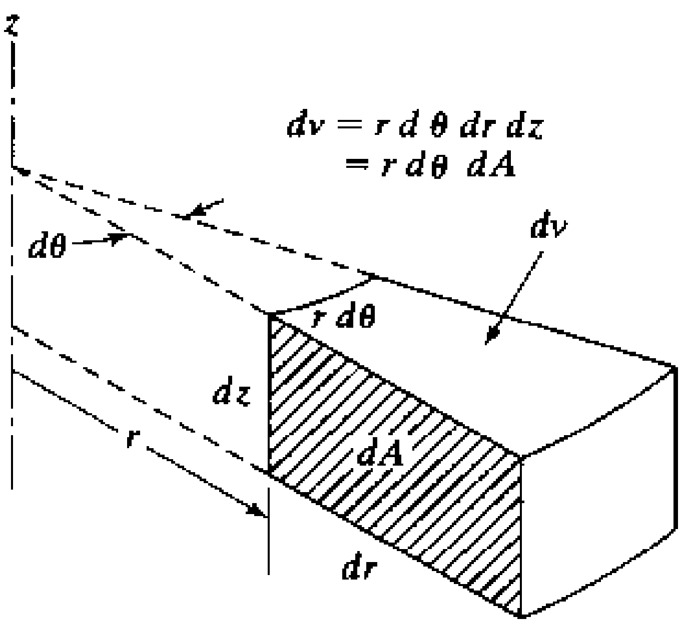
$$\mathbf{u} = [u, w]^{\mathrm{T}}$$

$$\mathbf{f} = [f_r, f_z]^{\mathrm{T}}$$

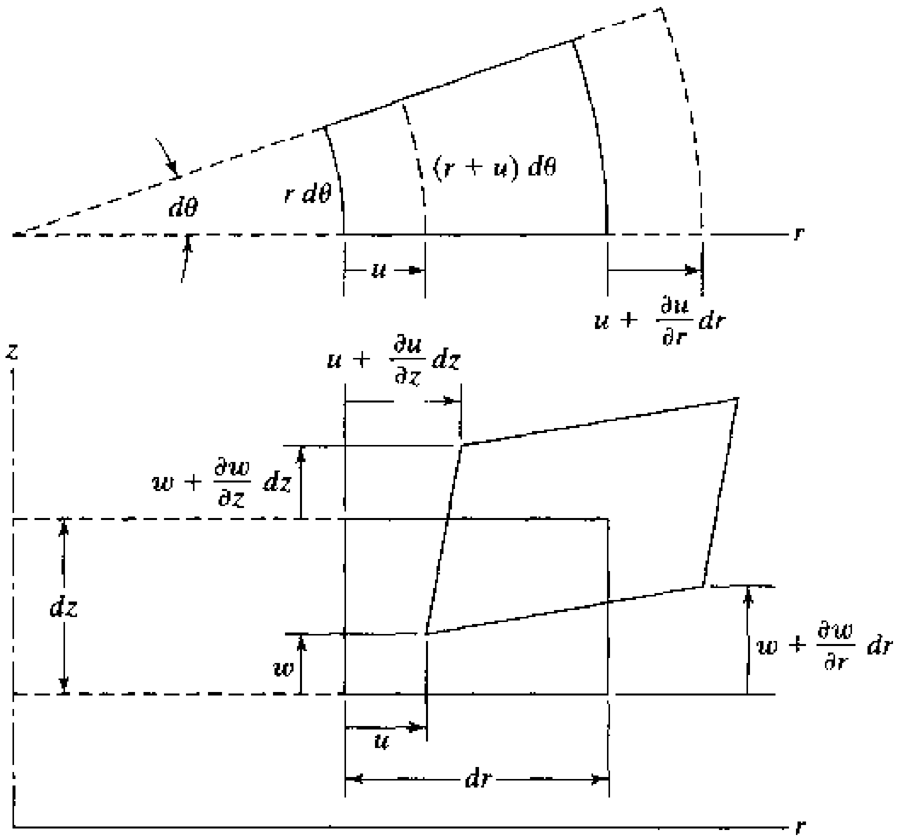
$$\mathbf{T} = [T_r, T_z]^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$



$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$



$$\boldsymbol{\epsilon} = [\epsilon_r, \epsilon_z, \gamma_{rz}, \epsilon_\theta]^{\mathrm{T}} = \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]^{\mathrm{T}}$$

$$\boldsymbol{\phi} = [\phi_r, \phi_z]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \left[\frac{\partial \phi_r}{\partial r}, \frac{\partial \phi_z}{\partial z}, \frac{\partial \phi_r}{\partial z} + \frac{\partial \phi_z}{\partial r}, \frac{\phi_r}{r}\right]^{\mathrm{T}}$$

$$2\pi \int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}(\boldsymbol{\phi}) r\,dA - \left(2\pi \int_A \boldsymbol{\phi}^{\mathrm{T}}\mathbf{f} r\,dA + 2\pi \int_L \boldsymbol{\phi}^{\mathrm{T}}\mathbf{T} r\,d\ell + \sum \boldsymbol{\phi}_i^{\mathrm{T}}\mathbf{P}_i\right) = 0$$

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$\mathbf{q} = [q_1, \quad q_2, \quad q_3, \quad q_4, \quad q_5, \quad q_6]^{\mathrm{T}}$$

$$u = \xi q_1 + \eta q_3 + (1 - \xi - \eta) q_5$$

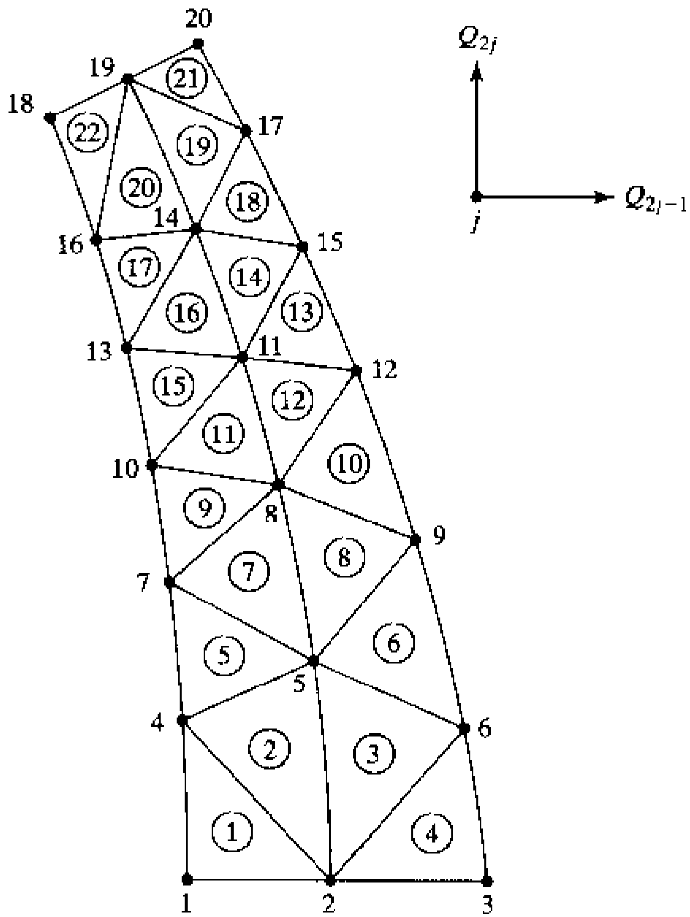
$$w = \xi q_2 + \eta q_4 + (1 - \xi - \eta) q_6$$

$$r = \xi r_1 + \eta r_2 + (1 - \xi - \eta) r_3$$

$$z = \xi z_1 + \eta z_2 + (1 - \xi - \eta) z_3$$

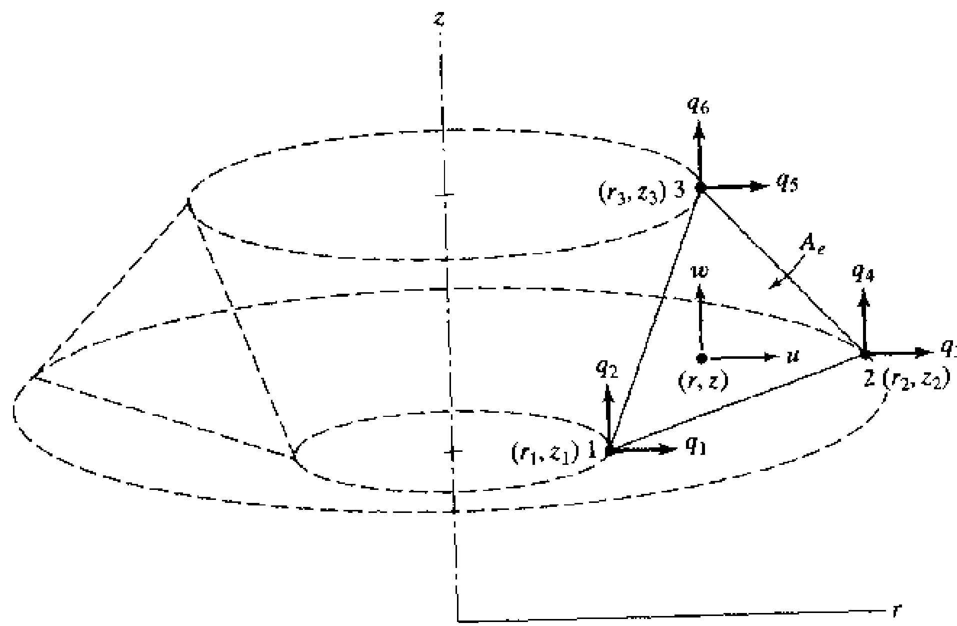
$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix}$$



$$\mathbf{J} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

$$\det \mathbf{J} = r_{13} z_{23} - r_{23} z_{13}$$



$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} \qquad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & z_{13} \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_2 - q_6) + r_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(q_1 - q_5) + r_{13}(q_3 - q_5) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6)}{\det \mathbf{J}} \\ \dfrac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{Bmatrix}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{z_{23}}{\det \mathbf{J}} & 0 & \dfrac{z_{31}}{\det \mathbf{J}} & 0 & \dfrac{z_{12}}{\det \mathbf{J}} & 0 \\ 0 & \dfrac{r_{32}}{\det \mathbf{J}} & 0 & \dfrac{r_{13}}{\det \mathbf{J}} & 0 & \dfrac{r_{21}}{\det \mathbf{J}} \\ \dfrac{r_{32}}{\det \mathbf{J}} & \dfrac{z_{23}}{\det \mathbf{J}} & \dfrac{r_{13}}{\det \mathbf{J}} & \dfrac{z_{31}}{\det \mathbf{J}} & \dfrac{r_{21}}{\det \mathbf{J}} & \dfrac{z_{12}}{\det \mathbf{J}} \\ \dfrac{N_1}{r} & 0 & \dfrac{N_2}{r} & 0 & \dfrac{N_3}{r} & 0 \end{bmatrix}$$

$$\Pi = \sum_e \left[\frac{1}{2}\left(2\pi \int_e \boldsymbol{\epsilon}^{\mathrm{T}} \mathbf{D} \boldsymbol{\epsilon} r\, dA\right) - 2\pi \int_e \mathbf{u}^{\mathrm{T}} \mathbf{f} r\, dA - 2\pi \int_e \mathbf{u}^{\mathrm{T}} \mathbf{T} r\, d\ell\right] - \sum \mathbf{u}_i^{\mathrm{T}} \mathbf{P}_i$$

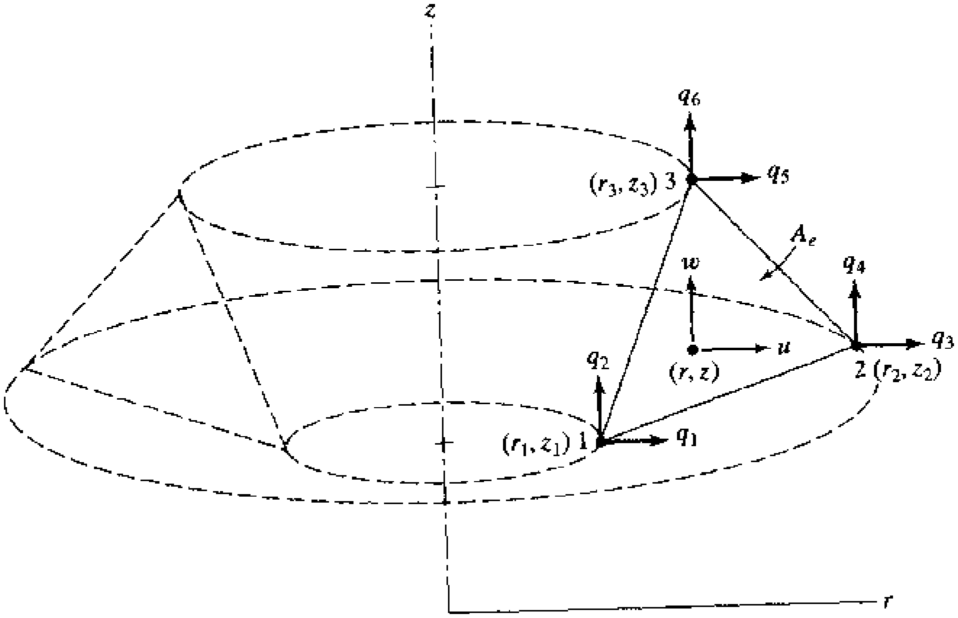
$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(2\pi \int_e \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} r\, dA\right)\mathbf{q} \qquad \mathbf{k}^e = 2\pi \int_e \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} r\, dA$$

$$N_1 = N_2 = N_3 = \tfrac{1}{3} \qquad \bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

$$\mathbf{k}^e = 2\pi \bar{r}\, \overline{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{B}} \int_e dA$$

$$\mathbf{k}^e = 2\pi \bar{r} A_e \overline{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{B}}$$

$$A_e = \tfrac{1}{2}\left|\det \mathbf{J}\right|$$



$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA = 2\pi \int_e (uf_r + wf_z) r\, dA$$

$$= 2\pi \int_e [(N_1 q_1 + N_2 q_3 + N_3 q_5) f_r + (N_1 q_2 + N_2 q_4 + N_3 q_6) f_z] r\, dA$$

$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f} r\, dA = \mathbf{q}^{\mathrm{T}}\mathbf{f}^e$$

$$\mathbf{f}^e = \frac{2\pi \bar{r} A_e}{3} [\bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z]^{\mathrm{T}}$$

Rotating flywheel

$$\mathbf{f} = [f_r, f_z]^{\mathrm{T}} = [\rho r\omega^2, -\rho g]^{\mathrm{T}}$$

$$\bar{f}_r = \rho\bar{r}\omega^2, \bar{f}_z = -\rho g$$

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3$$

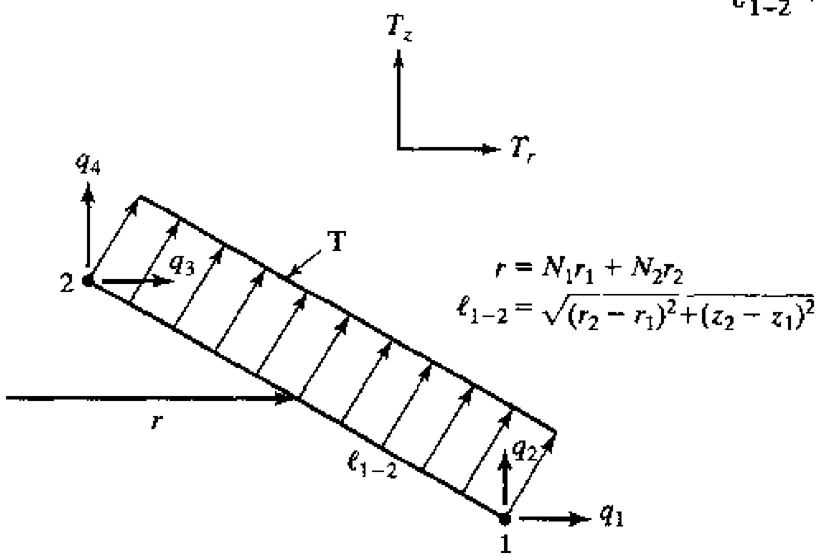
$$2\pi \int_e \mathbf{u}^{\mathrm{T}}\mathbf{T} r \, d\ell = \mathbf{q}^{\mathrm{T}}\mathbf{T}^e$$

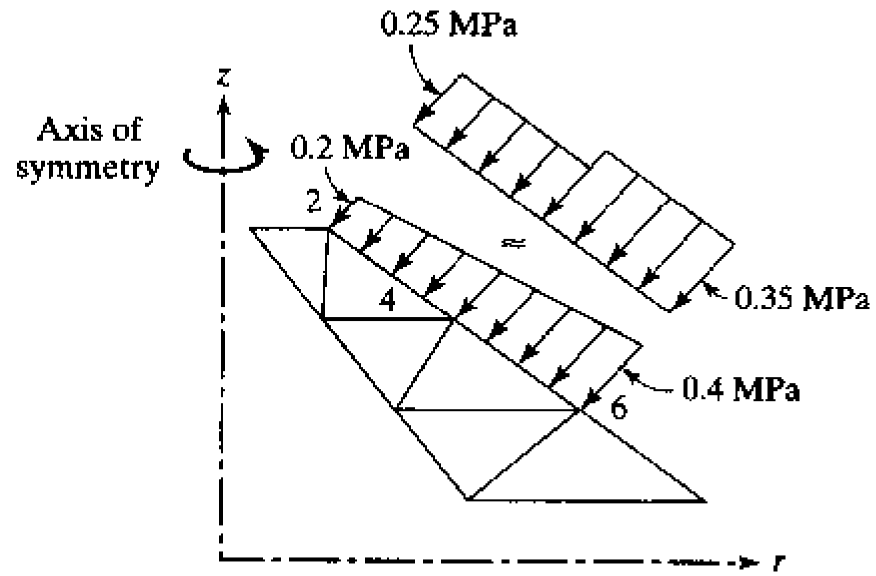
$$\mathbf{q} = [q_1, q_2, q_3, q_4]^{\mathrm{T}}$$

$$\mathbf{T}^e = 2\pi\ell_{1-2}[aT_r, aT_z, bT_r, bT_z]^{\mathrm{T}}$$

$$a = \frac{2r_1 + r_2}{6} \qquad b = \frac{r_1 + 2r_2}{6}$$

$$\ell_{1-2} = \sqrt{(r_2 - r_1)^2 + (z_2 - z_1)^2}$$





For edge 6–4

$$p = 0.35\text{ MPa}, \quad r_1 = 60\text{ mm}, \quad z_1 = 40\text{ mm}, \quad r_2 = 40\text{ mm}, \quad z_2 = 55\text{ mm}$$

$$\ell_{1-2} = \sqrt{(r_1 - r_2)^2 + (z_1 - z_2)^2} = 25\text{ mm}$$

$$c = \frac{z_2 - z_1}{\ell_{1-2}} = 0.6, \qquad s = \frac{r_1 - r_2}{\ell_{1-2}} = 0.8$$

$$T_r = -pc = -0.21, \qquad T_z = -ps = -0.28$$

$$a = \frac{2r_1 + r_2}{6} = 26.67, \qquad b = \frac{r_1 + 2r_2}{6} = 23.33$$

$$\begin{aligned} \mathbf{T}^1 &= 2\pi\ell_{1-2}[aT_r \quad aT_z \quad bT_r \quad bT_z]^{\mathrm{T}} \\ &= [-879.65 \quad -1172.9 \quad -769.69 \quad -1026.25]^{\mathrm{T}} \quad \mathrm{N} \end{aligned}$$

These loads add to F_{11}, F_{12}, F_7, and F_8, respectively.

For edge 4–2

$$p = 0.25\,\text{MPa}, \quad r_1 = 40\,\text{mm}, \quad z_1 = 55\,\text{mm}, \quad r_2 = 20\,\text{mm}, \quad z_2 = 70\,\text{mm}$$

$$\ell_{1-2} = \sqrt{(r_1 - r_2)^2 + (z_1 - z_2)^2} = 25\,\text{mm}$$

$$c = \frac{z_2 - z_1}{\ell_{1-2}} = 0.6, \qquad s = \frac{r_1 - r_2}{\ell_{1-2}} = 0.8$$

$$T_r = -pc = -0.15, \qquad T_z = -ps = -0.2$$

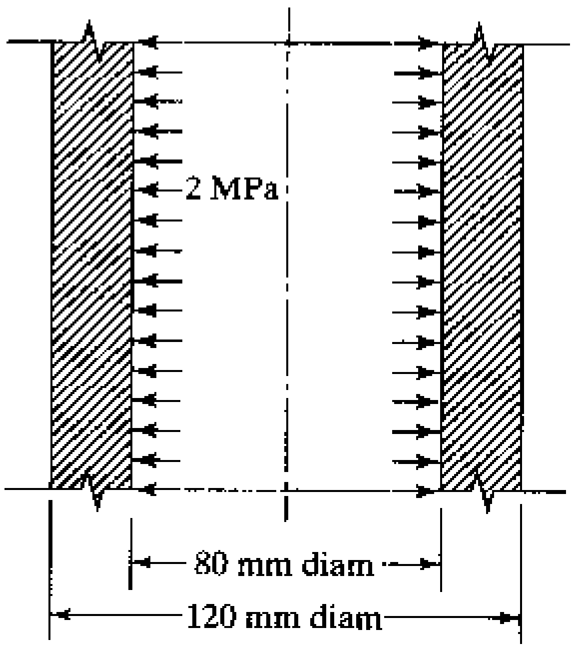
$$a = \frac{2r_1 + r_2}{6} = 16.67, \qquad b = \frac{r_1 + 2r_2}{6} = 13.33$$

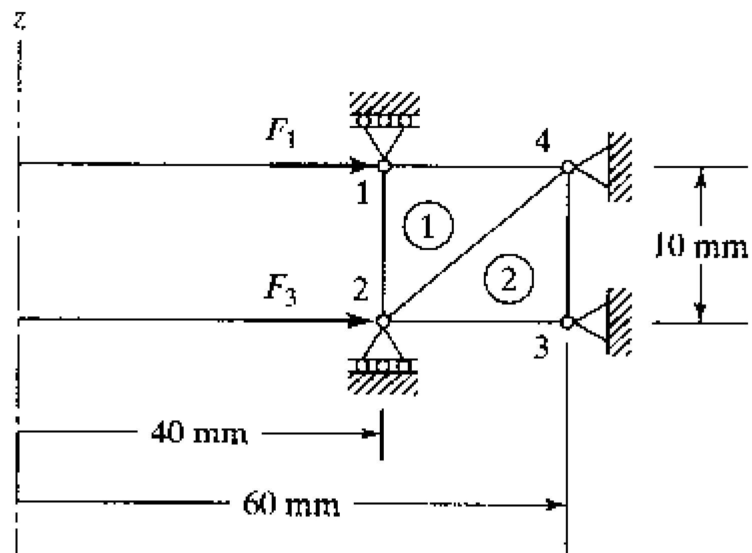
$$\mathbf{T}^1 = 2\pi\ell_{1-2}[aT_r \quad aT_z \quad bT_r \quad bT_z]^{\mathrm{T}}$$

$$= [-392.7 \quad -523.6 \quad -314.16 \quad -418.88]^{\mathrm{T}} \quad \text{N}$$

These loads add to F_7, F_8, F_3, and F_4, respectively. Thus,

$$[F_3 \quad F_4 \quad F_7 \quad F_8 \quad F_{11} \quad F_{12}] = [-314.2 \quad -418.9 \quad -1162.4 \quad -1696.5 \quad -879.7 \quad -1172.9] \quad \text{N}$$





E = 200 GPa

ν = 0.3

Element	Connectivity 1	2	3
1	1	2	4
2	2	3	4

Node	Coordinates r	z
1	40	10
2	40	0
3	60	0
4	60	10

$$\mathbf{D} = \begin{bmatrix} 2.69\times10^5 & 1.15\times10^5 & 0 & 1.15\times10^5 \\ 1.15\times10^5 & 2.69\times10^5 & 0 & 1.15\times10^5 \\ 0 & 0 & 0.77\times10^5 & 0 \\ 1.15\times10^5 & 1.15\times10^5 & 0 & 2.69\times10^5 \end{bmatrix}$$

$$F_1 = F_3 = \frac{2\pi r_1 \ell_e p_i}{2} = \frac{2\pi(40)(10)(2)}{2} = 2514\ \mathrm{N}$$

$$\overline{\mathbf{B}}^1 = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$\overline{\mathbf{B}}^2 = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.00625 & 0 & 0.00625 & 0 & 0.00625 & 0 \end{bmatrix}$$

$$\mathbf{D}\overline{\mathbf{B}}^1 = 10^4\begin{bmatrix} -1.26 & 1.15 & 0.082 & -1.15 & 1.43 & 0 \\ -0.49 & 2.69 & 0.082 & -2.69 & 0.657 & 0.1 \\ 0.77 & -0.385 & -0.77 & 0 & 0 & 0.385 \\ -0.384 & 1.15 & 0.191 & -1.15 & 0.766 & 0 \end{bmatrix}$$

$$\mathbf{D}\overline{\mathbf{B}}^2 = 10^4\begin{bmatrix} -1.27 & 0 & 1.42 & -1.15 & 0.072 & 1.15 \\ -0.503 & 0 & 0.647 & -2.69 & 0.072 & 2.69 \\ 0 & -0.385 & -0.77 & 0.385 & 0.77 & 0 \\ -0.407 & 0 & 0.743 & -1.15 & 0.168 & 1.15 \end{bmatrix}.$$

The stiffness matrices are obtained by finding $2\pi \bar{r} A_e \overline{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{B}}$ for each element:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{Global dof} \rightarrow 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\
\mathbf{k}^1 = 10^7 & \begin{bmatrix} 4.03 & -2.58 & -2.34 & 1.45 & -1.932 & 1.13 \\ & 8.45 & 1.37 & -7.89 & 1.93 & -0.565 \\ & & 2.30 & -0.24 & 0.16 & -1.13 \\ & & & 7.89 & -1.93 & 0 \\ \text{Symmetric} & & & & 2.25 & 0 \\ & & & & & 0.565 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{Global dof} \rightarrow 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\mathbf{k}^2 = 10^7 & \begin{bmatrix} 2.05 & 0 & -2.22 & 1.69 & -0.085 & -1.69 \\ & 0.645 & 1.29 & -0.645 & -1.29 & 0 \\ & & 5.11 & -3.46 & -2.42 & 2.17 \\ & & & 9.66 & 1.05 & -9.01 \\ \text{Symmetric} & & & & 2.62 & 0.241 \\ & & & & & 9.01 \end{bmatrix}
\end{array}
$$

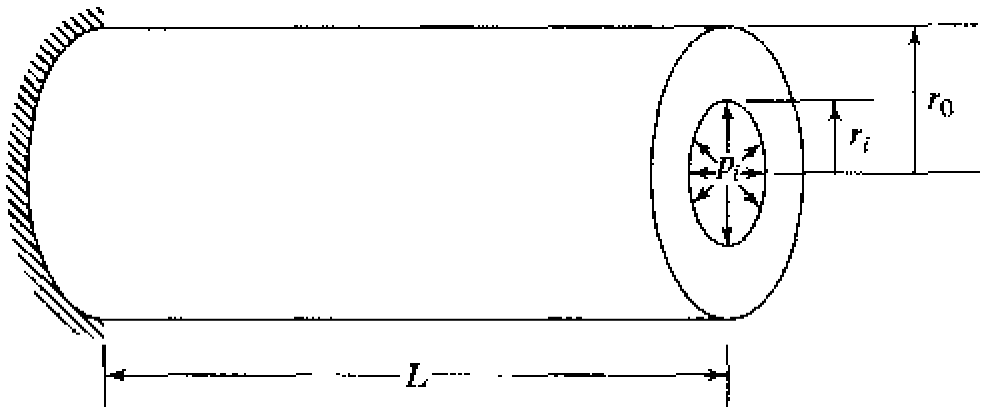
Using the elimination approach, on assembling the matrices with reference to the degrees of freedom 1 and 3, we get

$$
10^7 = \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 2514 \\ 2514 \end{Bmatrix}
$$

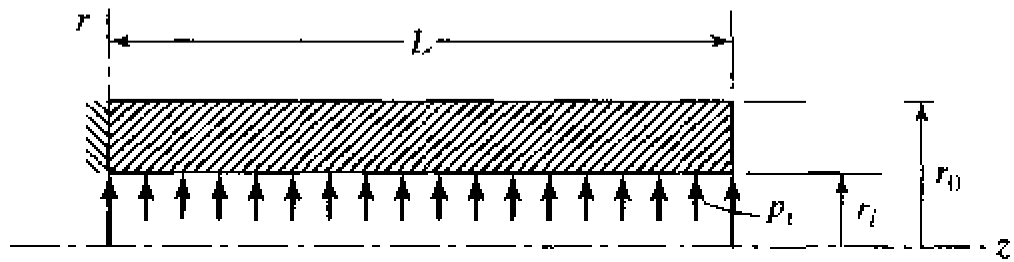
so that

$$
Q_1 = 0.014 \times 10^{-2}\,\text{mm}
$$

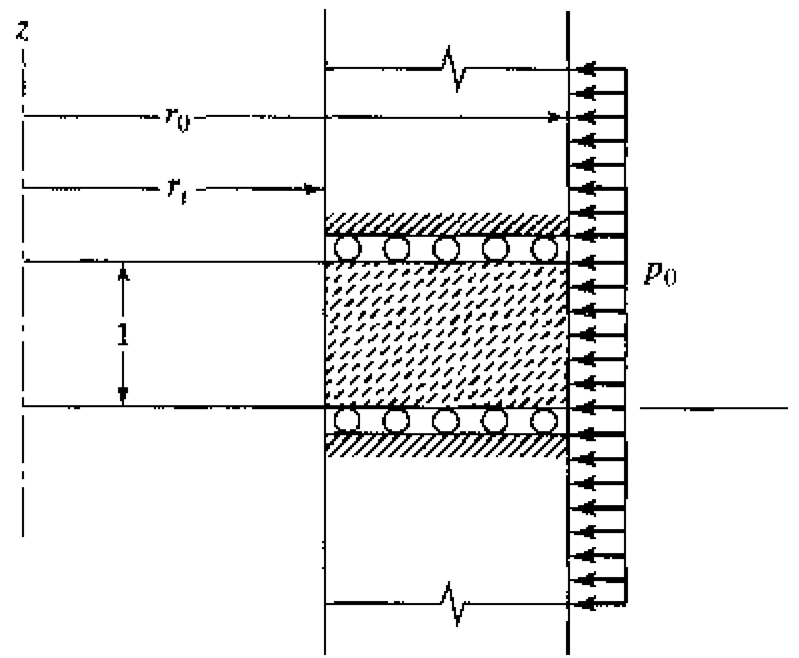
$$
Q_3 = 0.0133 \times 10^{-2}\,\text{mm} \qquad \blacksquare
$$



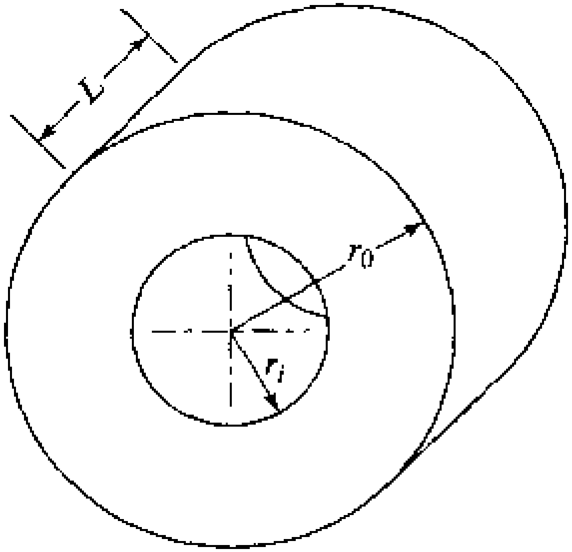
r_0
r_i
p_i
L



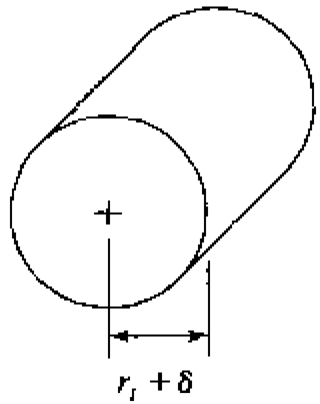
r
L
p_i
r_i
r_0
z



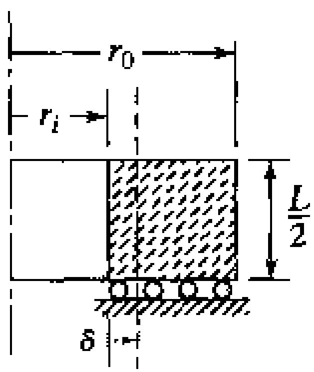
z
r_0
r_i
1
p_0
r



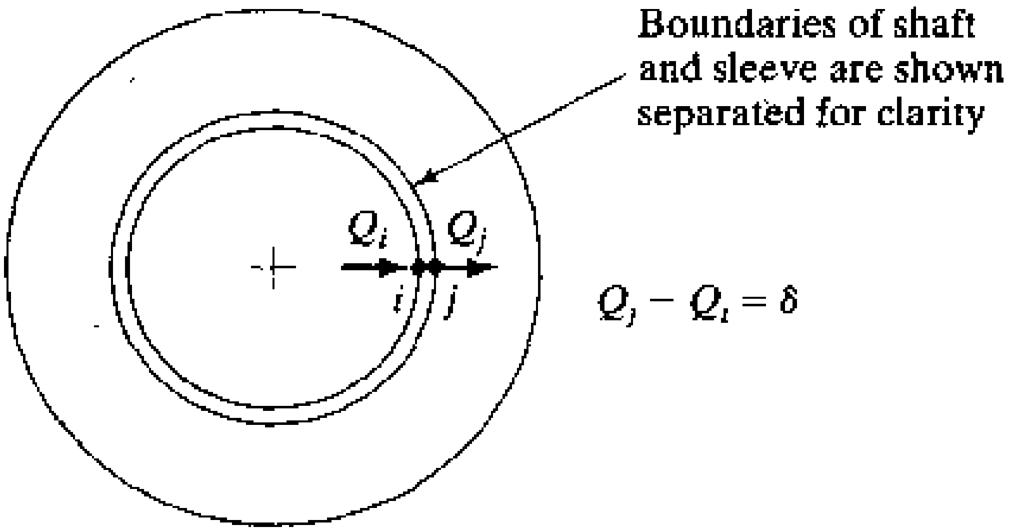
Ring of length L

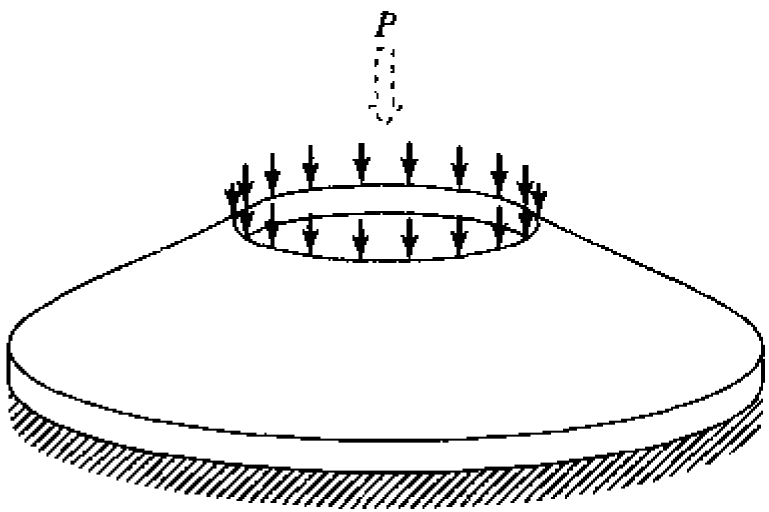


Rigid shaft

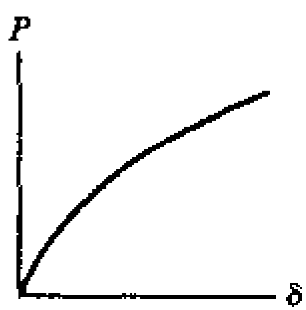


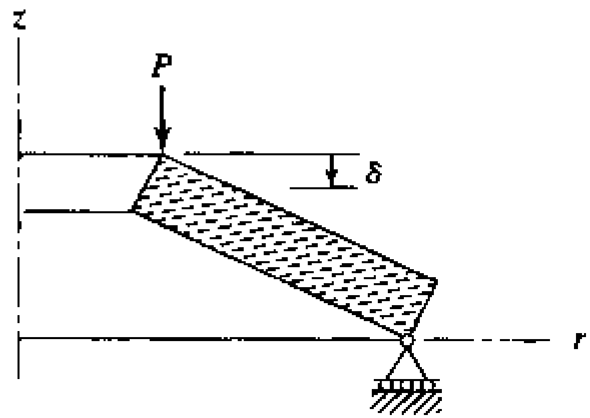
Model

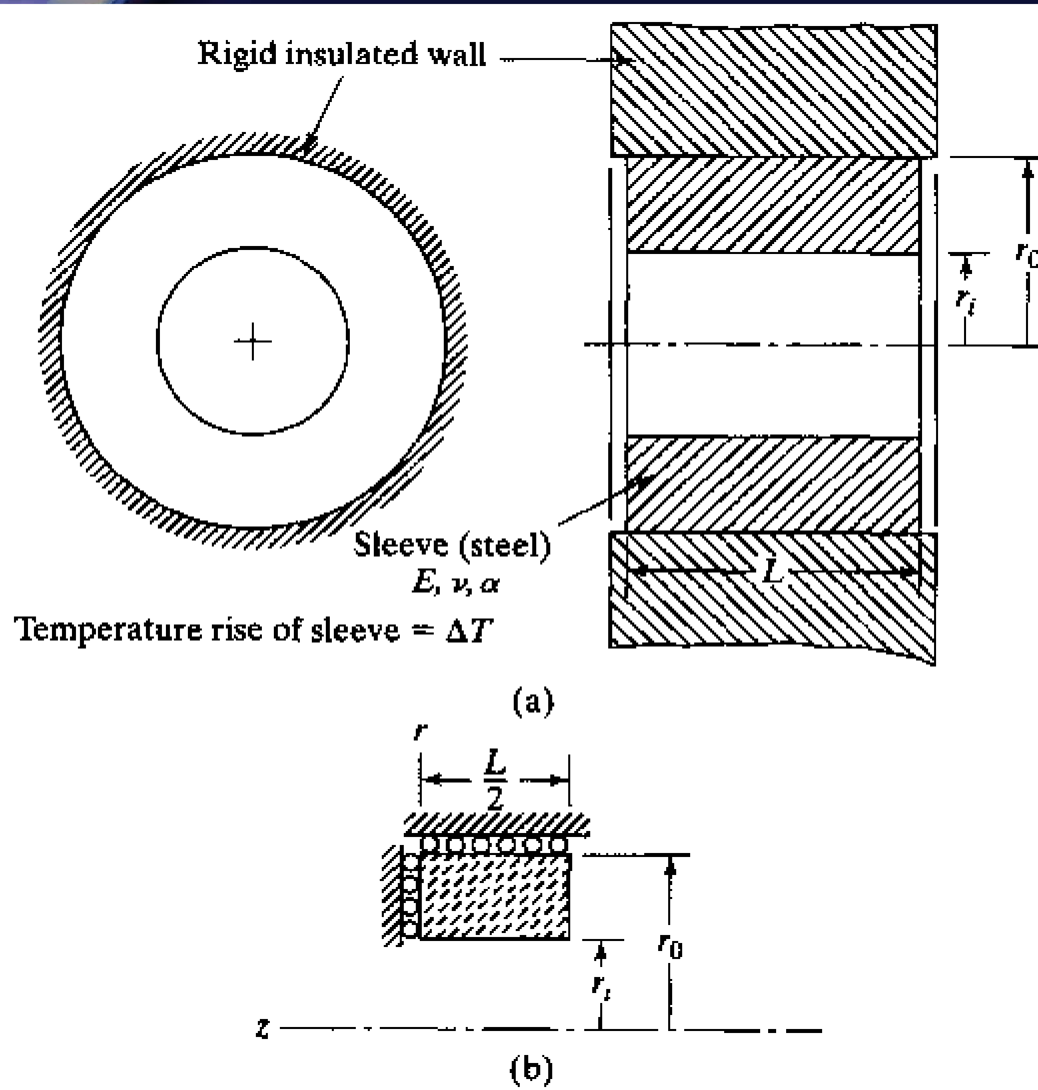

Boundaries of shaft and sleeve are shown separated for clarity
Q_i
Q_j
i
j
$Q_j - Q_i = \delta$



(a)







Rigid insulated wall
Sleeve (steel)
E, ν, α
Temperature rise of sleeve = ΔT
r_0
r_i
L
(a)
r
$\frac{L}{2}$
r_0
r_i
z
(b)

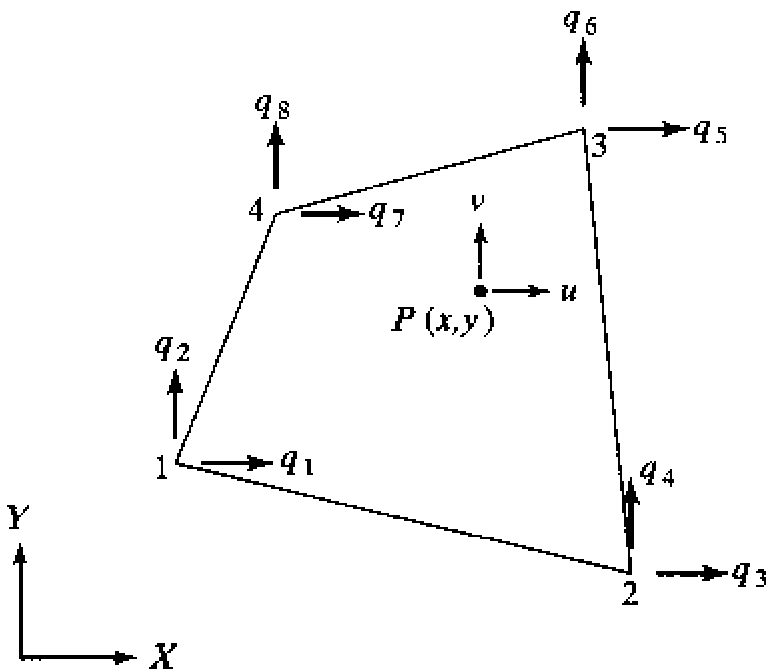
Isoparametric elements

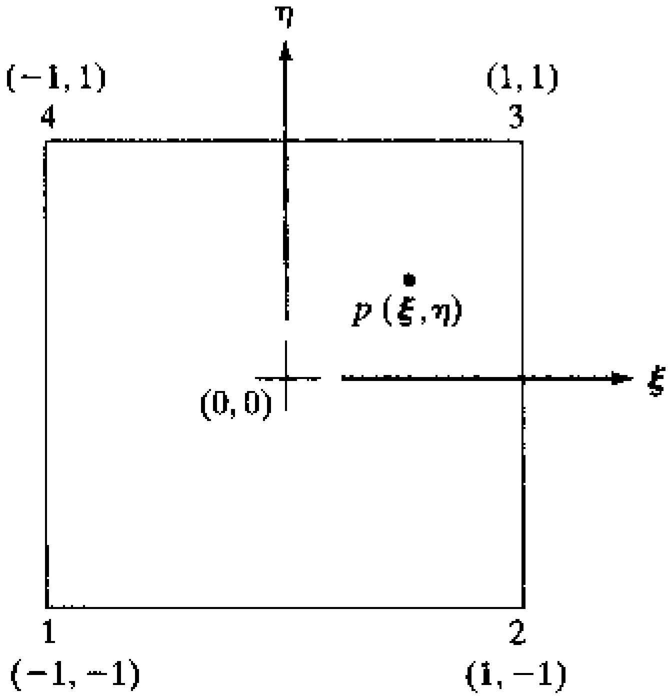
$$N_1 = 1 \quad \text{at node 1}$$
$$\quad = 0 \quad \text{at nodes 2, 3, and 4}$$

$$N_1 = c(1-\xi)(1-\eta) \qquad 1 = c(2)(2) \qquad N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$

$$\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$





$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8$$

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$f = f[x(\xi,\eta), y(\xi,\eta)].$$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J}\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1}\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

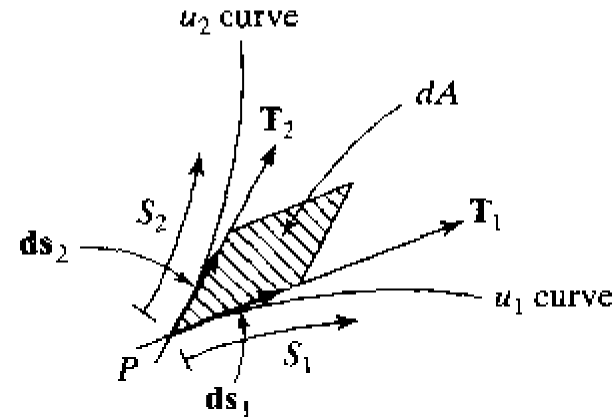
$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right]$$

$$dx\,dy = \det \mathbf{J}\, d\xi\, d\eta$$

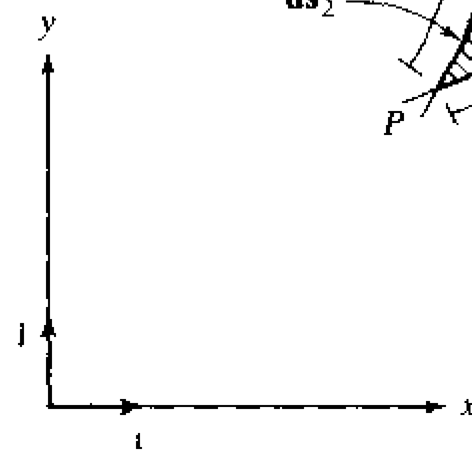
$$x = x(u_1, u_2) \qquad y = y(u_1, u_2) \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$$\mathbf{T}_1 = \frac{\partial x}{\partial u_1}\mathbf{i} + \frac{\partial y}{\partial u_1}\mathbf{j} \qquad \mathbf{T}_2 = \frac{\partial x}{\partial u_2}\mathbf{i} + \frac{\partial y}{\partial u_2}\mathbf{j} \qquad \mathbf{T}_1 = \frac{\partial \mathbf{r}}{\partial u_1} \qquad \mathbf{T}_2 = \frac{\partial \mathbf{r}}{\partial u_2}$$



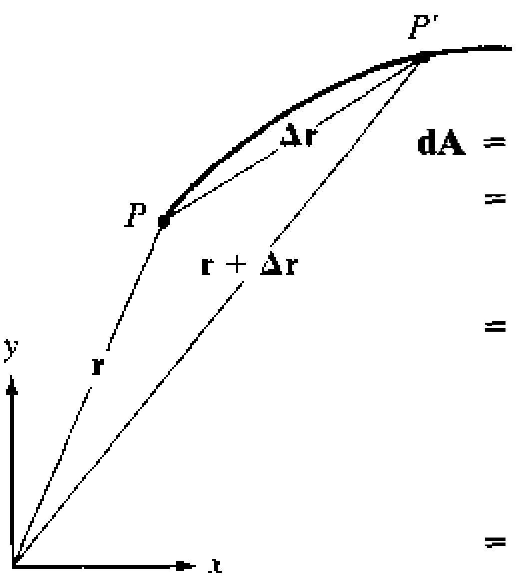
$$\frac{\partial \mathbf{r}}{\partial u_1} = \lim_{\Delta u_1 \to 0} \frac{\Delta \mathbf{r}}{\Delta u_1} \Longrightarrow \frac{\partial \mathbf{r}}{\partial u_1} = \frac{\partial \mathbf{r}}{\partial s_1}\frac{ds_1}{du_1} \Longrightarrow \frac{\partial \mathbf{r}}{\partial s_1} = \lim_{\Delta s_1 \to 0} \frac{\Delta \mathbf{r}}{\Delta s_1}$$

$$\mathbf{T}_1 = \frac{\partial \mathbf{r}}{\partial u_1} \qquad \mathbf{T}_2 = \frac{\partial \mathbf{r}}{\partial u_2} \Longrightarrow \begin{array}{l} \mathbf{T}_1 = \left(\dfrac{ds_1}{du_1}\right)\mathbf{t}_1 \\ \mathbf{T}_2 = \left(\dfrac{ds_2}{du_2}\right)\mathbf{t}_2 \end{array}$$



$$\mathbf{ds}_1 = \mathbf{t}_1\, ds_1 = \mathbf{T}_1\, du_1$$

$$\mathbf{ds}_2 = \mathbf{t}_2\, ds_2 = \mathbf{T}_2\, du_2$$



$$\begin{aligned} \mathbf{dA} &= \mathbf{ds}_1 \times \mathbf{ds}_2 \\ &= \mathbf{T}_1 \times \mathbf{T}_2\, du_1\, du_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & 0 \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & 0 \end{vmatrix} du_1\, du_2 \\ &= \left(\frac{\partial x}{\partial u_1}\frac{\partial y}{\partial u_2} - \frac{\partial x}{\partial u_2}\frac{\partial y}{\partial u_1}\right) du_1\, du_2\, \mathbf{k} \end{aligned}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} \end{bmatrix}$$

$$dA = \det \mathbf{J}\, d\xi\, d\eta$$

$$dV = \det \mathbf{J}\, d\xi\, d\eta\, d\zeta$$

$$dA = \det \mathbf{J}\, du_1\, du_2$$

$$U = \int_V \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \qquad U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{G}\mathbf{q}$$

$$\mathbf{G} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \mathbf{A}\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{G}\mathbf{q}$$

$$\boxed{\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}} \qquad \mathbf{B} = \mathbf{A}\mathbf{G}$$

$$\boxed{\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{q}}$$

$$U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA$$

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta \right]\mathbf{q}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{1}\mathbf{k}^e\mathbf{q}$$

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta}$$

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV \qquad \int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV = \sum_e \mathbf{q}^{\mathrm{T}}\mathbf{f}^e$$

$$\mathbf{f}^e = t_e\left[\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^{\mathrm{T}}\det\mathbf{J}\,d\xi\,d\eta\right]\begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

$$\mathbf{T}^e = \frac{t_e\ell_{2-3}}{2}\begin{bmatrix}0 & 0 & T_x & T_y & T_x & T_y & 0 & 0\end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta$$

$$\mathbf{f}^e = t_e\left[\int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^{\mathrm{T}} \det \mathbf{J}\, d\xi\, d\eta\right]\begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) + \cdots + w_n f(\xi_n)$$

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) \qquad f(\xi) = a_0 + a_1\xi,$$

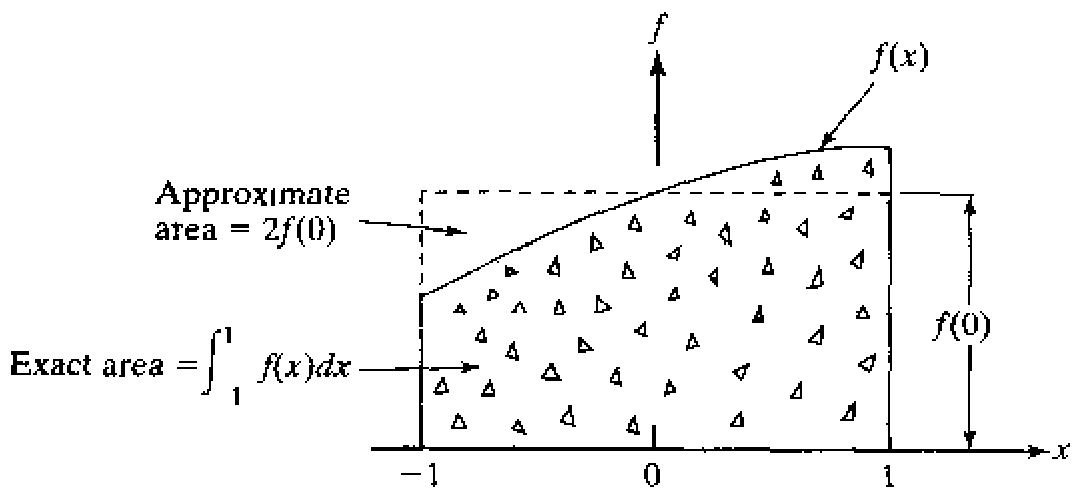
$$\text{Error} = \int_{-1}^{1} (a_0 + a_1\xi)\, d\xi - w_1 f(\xi_1) = 0$$

$$\text{Error} = 2a_0 - w_1(a_0 + a_1\xi_1) = 0$$

$$\text{Error} = a_0(2 - w_1) - w_1 a_1 \xi_1 =$$

$$w_1 = 2 \qquad \xi_1 = 0$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx 2f(0)$$



Two-Point Formula. Consider the formula with $n = 2$ as

$$\int_{-1}^{1} f(\xi)\, d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2) \qquad f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

$$\text{Error} = \left[\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3)\, d\xi\right] - [w_1 f(\xi_1) + w_2 f(\xi_2)]$$

$$\begin{aligned} w_1 + w_2 &= 2 \\ w_1\xi_1 + w_2\xi_2 &= 0 \\ w_1\xi_1^2 + w_2\xi_2^2 &= \tfrac{2}{3} \\ w_1\xi_1^3 + w_2\xi_2^3 &= 0 \end{aligned}$$

$$w_1 = w_2 = 1 \qquad -\xi_1 = \xi_2 = 1/\sqrt{3} = 0.5773502691\ldots$$

$$\int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i)$$

Number of points, n	Location, ξ_i	Weights, w_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	±0.7745966692	0.5555555556
	0.0	0.8888888889
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549
5	±0.9061798459	0.2369268851
	±0.5384693101	0.4786286705
	0.0	0.5688888889
6	±0.9324695142	0.1713244924
	±0.6612093865	0.3607615730
	±0.2386191861	0.4679139346

Evaluate

$$I = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{(x+2)} \right] dx$$

using one-point and two-point Gauss quadrature.

Solution For $n = 1$, we have $w_1 = 2$, $x_1 = 0$, and

$$I \approx 2f(0)$$
$$= 7.0$$

For $n = 2$, we find $w_1 = w_2 = 1$, $x_1 = -0.57735\ldots$, $x_2 = +0.57735\ldots$, and $I \approx 8.7857$. This may be compared with the exact solution

$$I_{\text{exact}} = 8.8165$$ ■

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta$$

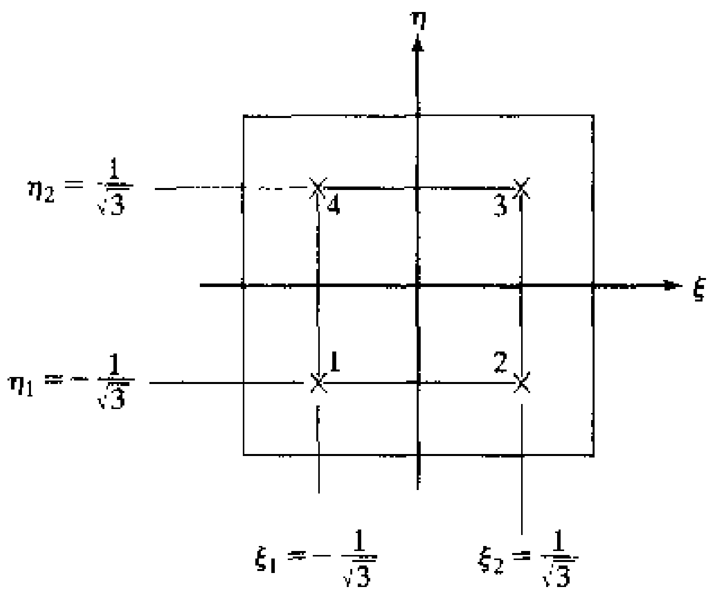
$$I \approx \int_{-1}^{1}\left[\sum_{i=1}^{n} w_i f(\xi_i,\eta)\right] d\eta$$

$$\approx \sum_{j=1}^{n} w_j\left[\sum_{i=1}^{n} w_i f(\xi_i,\eta_j)\right]$$

$$I \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(\xi_i,\eta_j)$$

$$\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\det \mathbf{J}\, d\xi\, d\eta$$

$$\phi(\xi,\eta) = t_e(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\det\mathbf{J})_{ij}$$



$$\boxed{w_1 = w_2 = 1}$$

$$k_{ij} \approx w_1^2\phi(\xi_1,\eta_1) + w_1 w_2\phi(\xi_1,\eta_2) + w_2 w_1\phi(\xi_2,\eta_1) + w_2^2\phi(\eta_2,\eta_2)$$

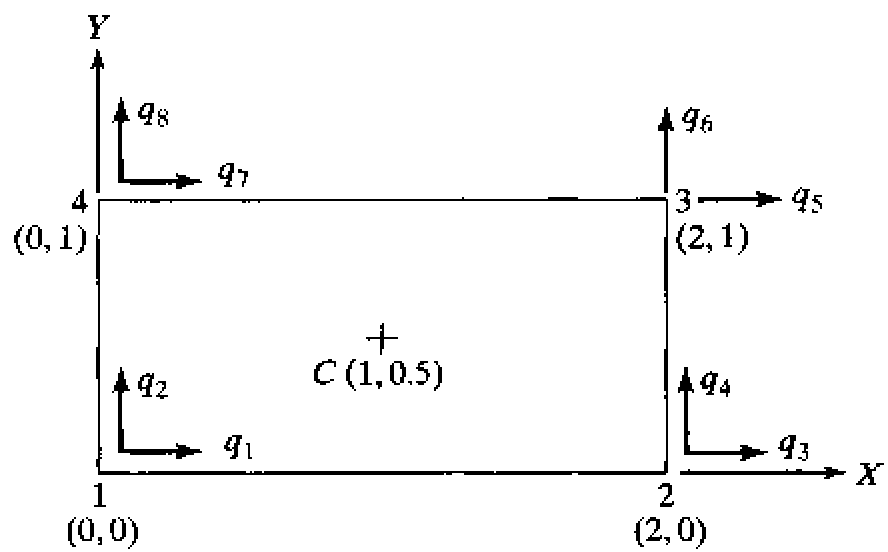
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx w_1^2 f(\xi_1,\eta_1) + w_2 w_1 f(\xi_2,\eta_1) + w_2^2 f(\xi_2,\eta_2) + w_1 w_2 f(\xi_1,\eta_2)$$

Consider a rectangular element as shown in Fig. E7.1. Assume plane stress condition, $E = 30 \times 10^6$ psi, $\nu = 0.3$, and $\mathbf{q} = [0, 0, 0.002, 0.003, 0.006, 0.0032, 0, 0]^T$ in. Evaluate $\mathbf{J}$, $\mathbf{B}$, and $\boldsymbol{\sigma}$ at $\xi = 0$ and $\eta = 0$.

Solution Referring to Eq. 7.13a, we have

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} 2(1-\eta) + 2(1+\eta) & (1+\eta) - (1+\eta) \\ -2(1+\xi) + 2(1+\xi) & (1+\xi) + (1-\xi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$



For this rectangular element, we find that $\mathbf{J}$ is a constant matrix. Now, from Eqs. 7.21,

$$\mathbf{A} = \frac{1}{1/2}\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Evaluating $\mathbf{G}$ in Eq. 7.23 at $\xi = \eta = 0$ and using $\mathbf{B} = \mathbf{QG}$, we get

$$\mathbf{B}^0 = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

The stresses at $\xi = \eta = 0$ are now given by the product

$$\boldsymbol{\sigma}^0 = \mathbf{D}\mathbf{B}^0\mathbf{q}$$

For the given data, we have

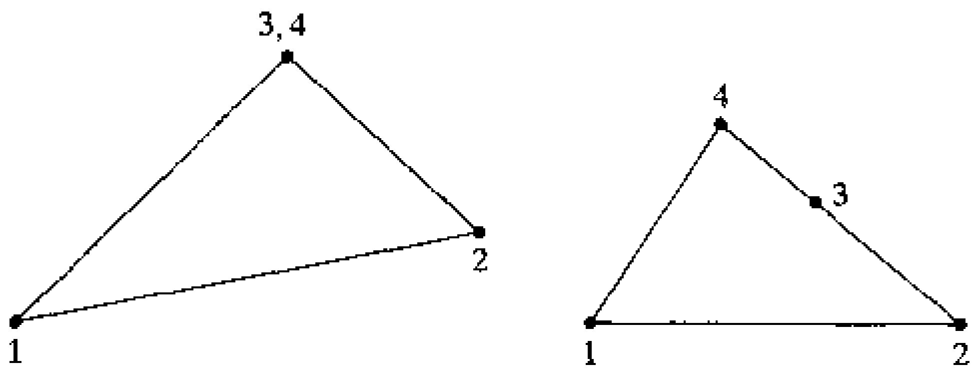
$$\mathbf{D} = \frac{30 \times 10^6}{(1 - 0.09)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.03 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Thus,

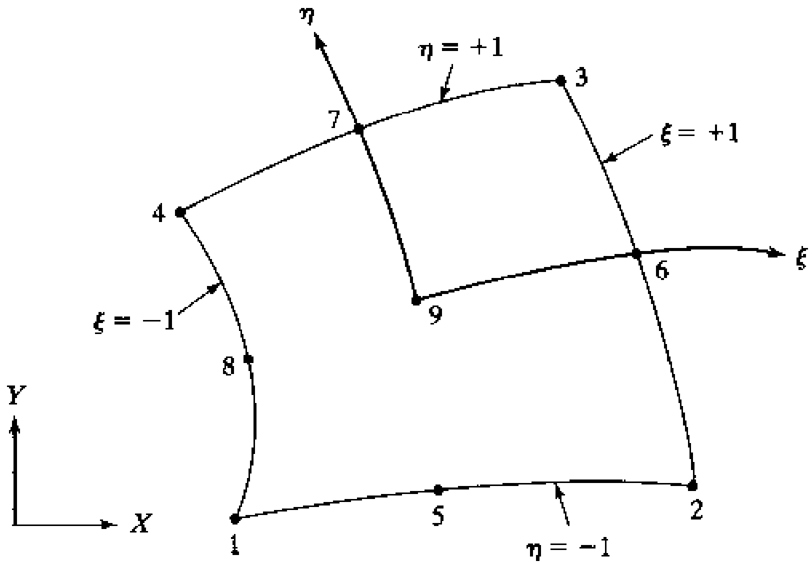
$$\boldsymbol{\sigma}^0 = [66\,920, 23\,080, 40\,960]^{\mathrm{T}} \text{ psi}$$

Stress Calculations

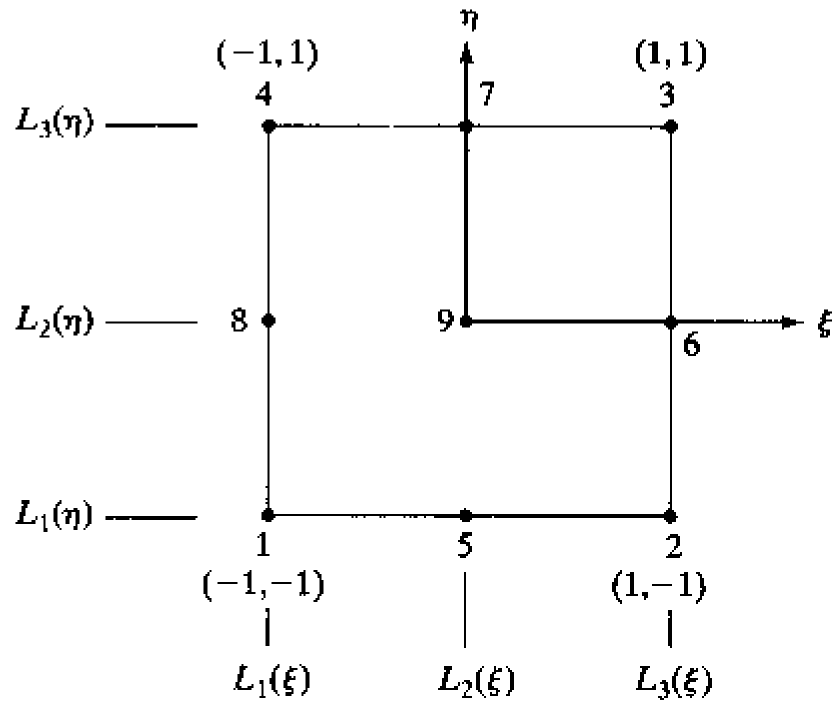
Unlike the constant-strain triangular element (Chapters 5 and 6), the stresses $\boldsymbol{\sigma} = \mathbf{DBq}$ in the quadrilateral element are not constant within the element; they are functions of ξ and η, and consequently vary within the element. In practice, the stresses are evaluated at the Gauss points, which are also the points used for numerical evaluation of $\mathbf{k}^e$, where they are found to be accurate. For a quadrilateral with 2×2 integration, this gives four sets of stress values. For generating less data, one may evaluate stresses at one point per element, say, at $\xi = 0$ and $\eta = 0$. The latter approach is used in the program QUAD.



(a) (b)



η
η = +1
3
7
ξ = +1
4
6
ξ
ξ = −1
9
8
Y
X
5
2
1
η = −1

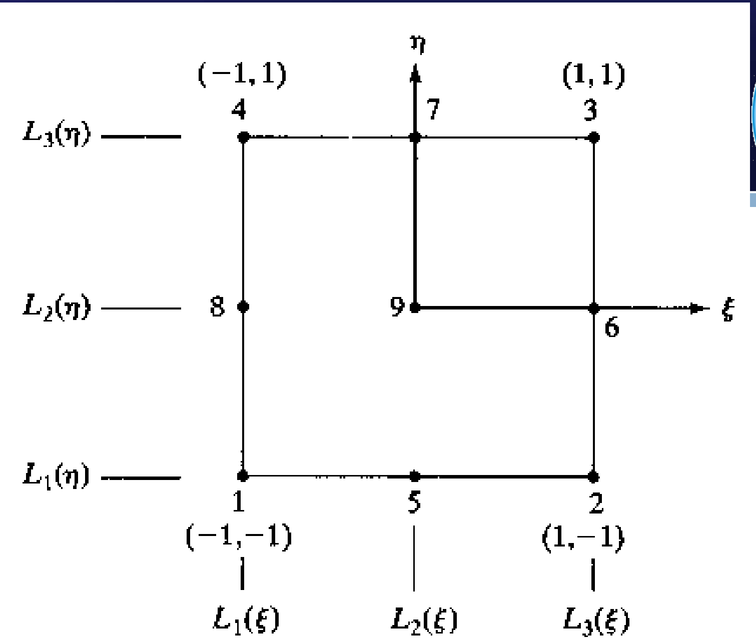


η
(−1, 1)
(1, 1)
4
7
3
L3(η)
L2(η)
8
9
6
ξ
L1(η)
1
5
2
(−1,−1)
(1,−1)
L1(ξ)
L2(ξ)
L3(ξ)

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$$

$$\mathbf{k}^e = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \det \mathbf{J}\, d\xi\, d\eta$$

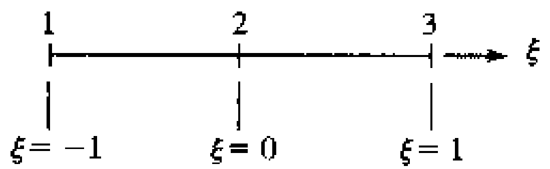


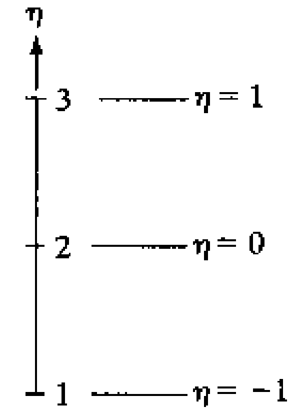
$$L_i(\xi) = 1 \quad \text{at node } i$$
$$= 0 \quad \text{at other two nodes}$$

$$L_1(\xi) = -\frac{\xi(1-\xi)}{2}$$

$$L_2(\xi) = (1+\xi)(1-\xi)$$

$$L_3(\xi) = \frac{\xi(1+\xi)}{2}$$





$$L_1(\eta) = -\frac{\eta(1-\eta)}{2}$$

$$L_2(\eta) = (1+\eta)(1-\eta)$$

$$L_3(\eta) = \frac{\eta(1+\eta)}{2}$$

$$N_1 = L_1(\xi)L_1(\eta) \qquad N_5 = L_2(\xi)L_1(\eta) \qquad N_2 = L_3(\xi)L_1(\eta)$$

$$N_8 = L_1(\xi)L_2(\eta) \qquad N_9 = L_2(\xi)L_2(\eta) \qquad N_6 = L_3(\xi)L_2(\eta)$$

$$N_4 = L_1(\xi)L_3(\eta) \qquad N_7 = L_2(\xi)L_3(\eta) \qquad N_3 = L_3(\xi)L_3(\eta)$$

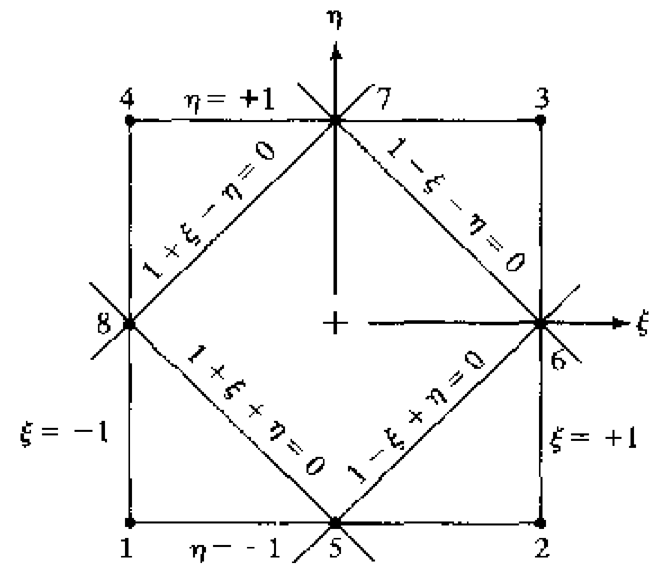
$$N_1 = c(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_1 = -\frac{(1-\xi)(1-\eta)(1+\xi+\eta)}{4}$$

$$N_2 = -\frac{(1+\xi)(1-\eta)(1-\xi+\eta)}{4}$$

$$N_3 = -\frac{(1+\xi)(1+\eta)(1-\xi-\eta)}{4}$$

$$N_4 = -\frac{(1-\xi)(1+\eta)(1+\xi-\eta)}{4}$$



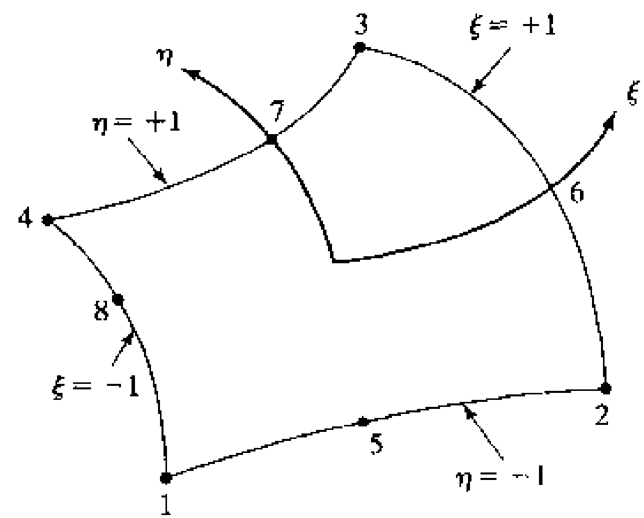
$$N_5 = c(1-\xi)(1-\eta)(1+\xi)$$
$$= c(1-\xi^2)(1-\eta)$$

$$N_5 = \frac{(1-\xi^2)(1-\eta)}{2}$$

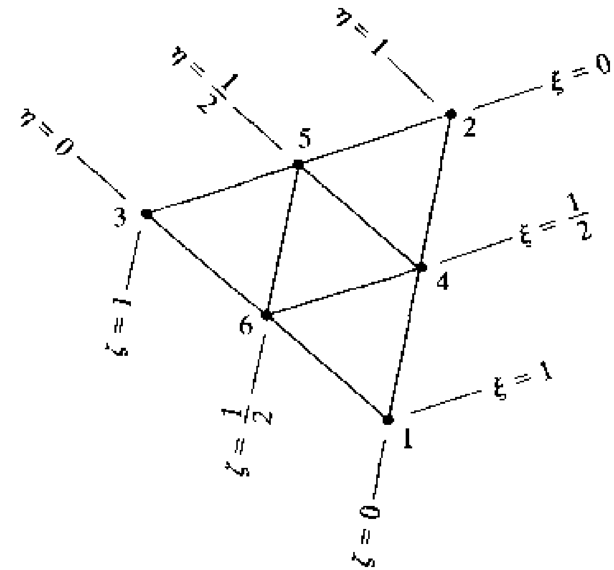
$$N_6 = \frac{(1+\xi)(1-\eta^2)}{2}$$

$$N_7 = \frac{(1-\xi^2)(1+\eta)}{2}$$

$$N_8 = \frac{(1-\xi)(1-\eta^2)}{2}$$



$$N_1 = \xi(2\xi - 1) \qquad N_4 = 4\xi\eta$$
$$N_2 = \eta(2\eta - 1) \qquad N_5 = 4\zeta\eta$$
$$N_3 = \zeta(2\zeta - 1) \qquad N_6 = 4\xi\zeta$$



$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$x = \sum N_i x_i \qquad y = \sum N_i y_i$$

$$\mathbf{k}^e = t_e \int_A \int \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \det \mathbf{J} \, d\xi \, d\eta$$

$$\mathbf{k}^e \approx \frac{1}{2} t_e \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{D}} \, \mathbf{B} \det \overline{\mathbf{J}}$$

$$\int_0^1 \int_0^{1-\xi} f(\xi, \eta) \, d\eta \, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i)$$

No. of points, n	Weight, w_i	Multiplicity	ξ_i	η_i	ζ_i
One	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Three	$\frac{1}{6}$	3	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
Three	$\frac{1}{6}$	3	$\frac{1}{2}$	$\frac{1}{2}$	0
Four	$-\frac{9}{32}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{25}{96}$	3	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
Six	$\frac{1}{12}$	6	0.6590276223	0.2319333685	0.1090390090