Analytical methods in elastic plane problems

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-Plane stress and plane strain problems

Plane Stress

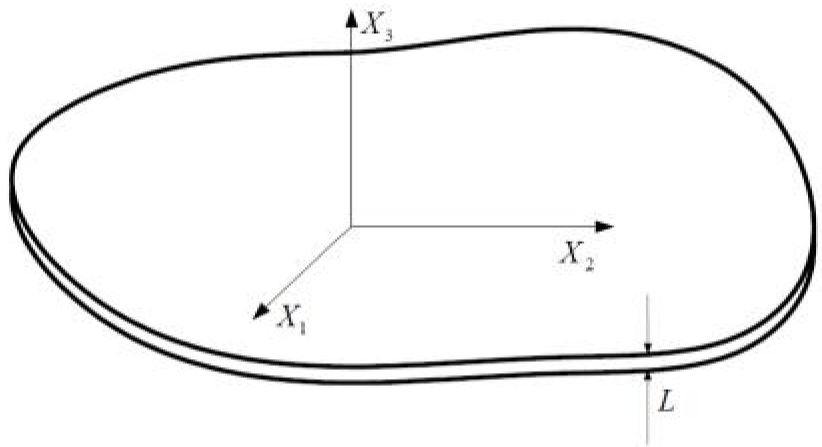
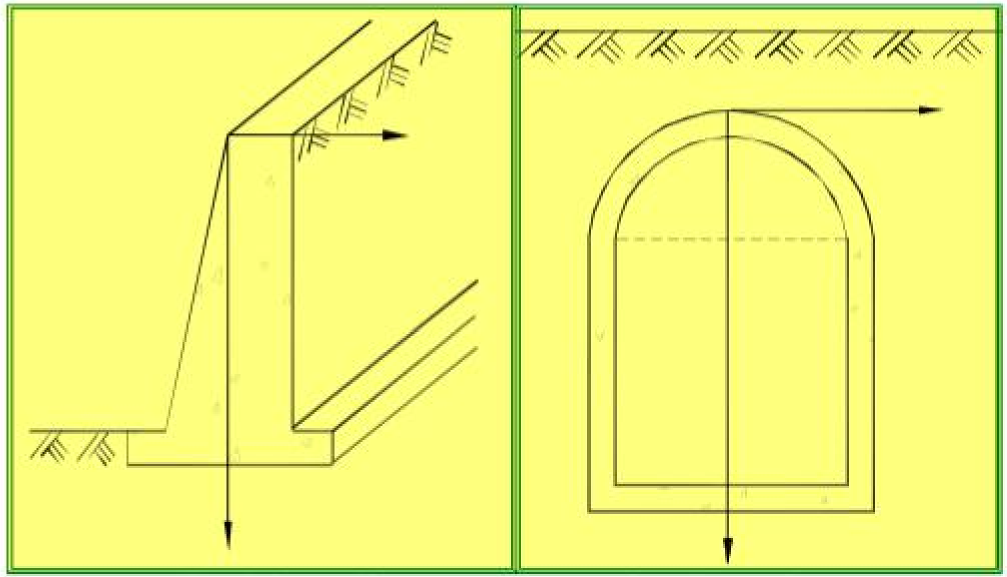


Figure 6.1 Illustration of the plane stress assumption.

(1) the thickness of the plate L, as depicted in Fig. 6.1, is much less than any in-plane dimensions, L <<R; (2) the loads are in-plane and their variation along the out-of-plane coordinate x_3 is neglected; and (3) the top and bottom planes of the body are traction free. The combination of the last two assumptions dictates that $\sigma_{x_3} \approx 0$ in the entire body.

-Plane strain problems



a. Retaining wall b. Tunnel

Figure 6.2 Examples of plane strain

The state of plane strain exists as a rigorous and non-conflicting case. Its essential assumptions can be listed as: (1) a prismatic body whose length is much larger than any in-plane dimension, $L >> R_{max}$; (2) in-plane loads independent of the out-of-plane coordinate x_3; and (3) absence of normal strain, $\varepsilon_{x_3} = 0$, in a direction perpendicular to the plane.

2.Solution of plane prlblems in Cartesian coordinate

-Equilibrium differential equations

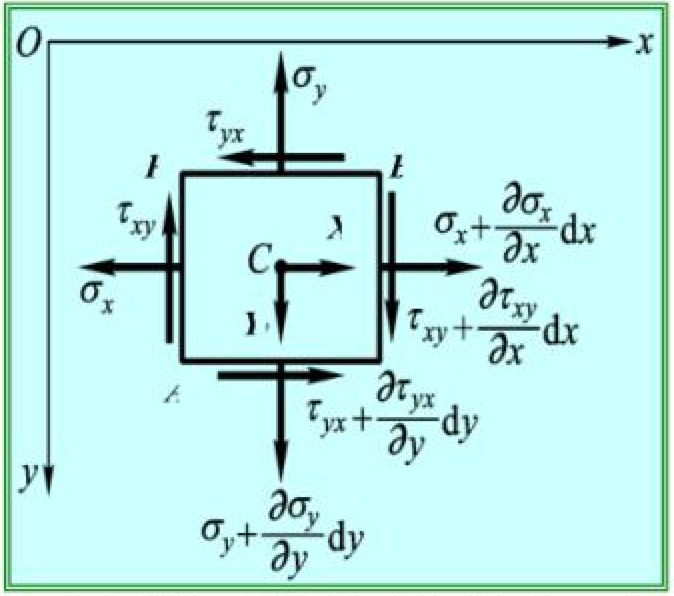


Figure 6.3 Equilibrium of an infinitesimal unit in plane problems

Hence, equilibrium equations are

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_C = 0 \end{cases}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0 \qquad (6.1a)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \qquad (6.1b)$$

$$\tau_{xy} = \tau_{yx} \qquad (6.1c)$$

Equation (6.1c) is a one more proof of reciprocal theorem of shear stress. An equation group including (6.1a) and (6.1b) is differential equilibrium differential equations in plane problems.

-Stress state in planle problems

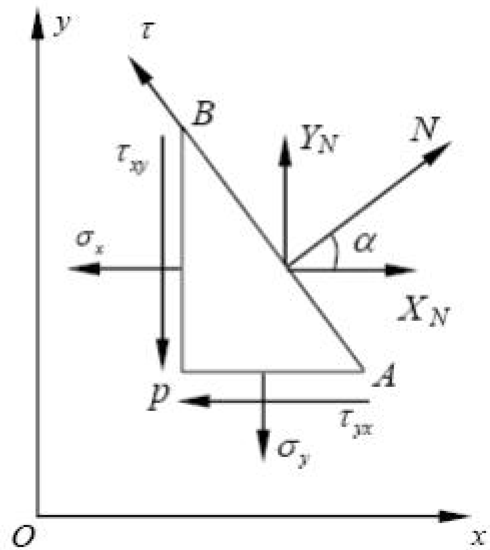


Figure 6.4 stress of tilted section in plane problems

$$\begin{cases} X_N = \sigma_x \cos\alpha + \tau_{yx} \sin\alpha \\ Y_N = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha \end{cases} \tag{6.2}$$

2. Normal stress and shear stress on tilted section are

$$\begin{cases} \sigma_N = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_N = \dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases} \tag{6.3}$$

3. Principal stresses locate on tilted section with zero shear stress, according to equations (6.2) or (6.3)

$$\left.\begin{aligned} &\begin{matrix}\sigma_1 \\ \sigma_2\end{matrix} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &\tan\alpha_1 = \frac{\sigma_1 - \sigma_x}{\tau_{xy}} \end{aligned}\right\} \tag{6.4}$$

4. Maximum and minimum of normal stress and shear stress can be worked out according to equations 6.3

$$\begin{aligned} &{}^{max}_{min}\sigma_n = \begin{matrix}\sigma_1 \\ \sigma_2\end{matrix} \\ &{}^{max}_{min}\tau_n = \pm\frac{\sigma_1 - \sigma_2}{2} \end{aligned} \tag{6.5}$$

Where planes with maximum and minimum of shear stress have angles of 45 degrees with plane with principal stresses.

-Geometric equations in plane problems

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{cases} \tag{6.6}$$

-Physical equations in plane problems

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}(\sigma_x - \mu\sigma_y) \\ \varepsilon_y = \dfrac{1}{E}(\sigma_y - \mu\sigma_x) \\ \gamma_{xy} = \dfrac{2(1+\mu)}{E}\tau_{xy} \end{cases} \tag{6.7}$$

Another equation can be utilized to compute change of thickness of thin plate

$$\varepsilon_z = -\frac{\mu}{E}(\sigma_x + \sigma_y) \qquad (6.8)$$

Similarly, plane strain problems have condition $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$, thus physical equation of plane strain problems are

$$\begin{cases} \varepsilon_x = \dfrac{1-\mu^2}{E}(\sigma_x - \dfrac{\mu}{1-\mu}\sigma_y) \\ \varepsilon_y = \dfrac{1-\mu^2}{E}(\sigma_y - \dfrac{\mu}{1-\mu}\sigma_x) \\ \gamma_{xy} = \dfrac{2(1+\mu)}{E}\tau_{xy} \end{cases} \qquad (6.9)$$

In the same moment,

$$\sigma_z = \mu(\sigma_x + \sigma_y) \qquad (6.10)$$

is used to get stress in direction z.

-Strain compatibility equations

Like we studied in Chapter 3, there exist a strain compatibility equation to insure the continuity condition of elasotmer, make displacement answers are feasible. It can be derivated from geometric equations (6.6)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (6.11)$$

It is usually called strain compatibility equation or consistency equation.

-Boudary conditions

1.stress boundary condition

According to equations (6.2), stress boundary condition can be obtained considering figure 6.4.

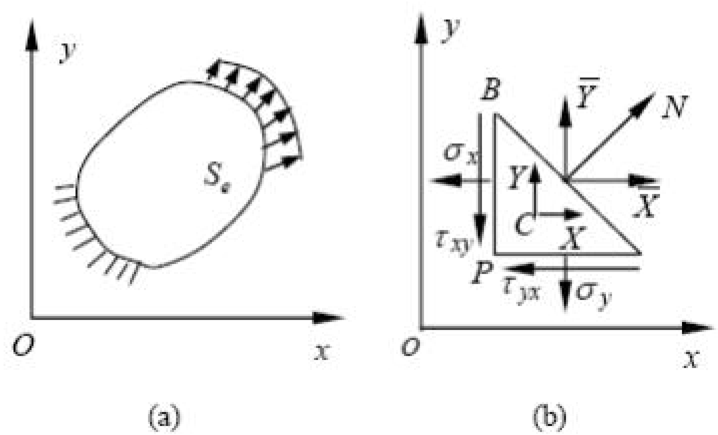


Figure 6.4 Analysis of stress boundary condition

$$\begin{cases} \overline{X} = \sigma_x \cos\alpha + \tau_{yx} \sin\alpha \\ \overline{Y} = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha \end{cases} \tag{6.12}$$

For example, (a) a border parallel with axis y,

$$\cos\alpha = \pm 1, \sin\alpha = 0 \Rightarrow \begin{cases} \sigma_x = \pm\overline{X} \\ \tau_{xy} = \pm\overline{Y} \end{cases}$$

Or (b) a border parallel with axis x,

$$\cos\alpha = 0, \sin\alpha = \pm 1 \Rightarrow \begin{cases} \sigma_x = \pm\overline{Y} \\ \tau_{xy} = \pm\overline{X} \end{cases}$$

2. displacement boundary condition

It refers that displacement components on points of boundary must equal the given values. To a planar cantilever beam (see figure 6.5), for example, the fixed border have relations

$$u_{(s)} = 0, v_{(s)} = 0$$

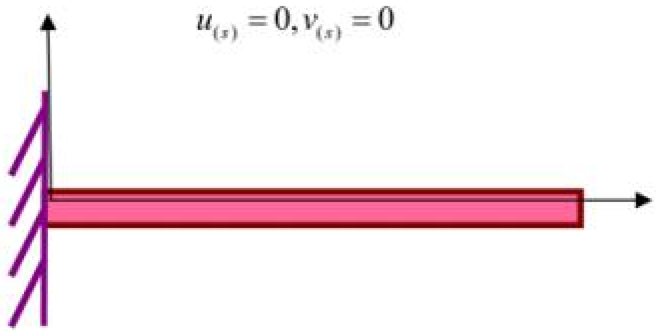


Figure 6.5 Displacement boundary condition of planar cantilever beam

3.mixed boundary condition

In most cases, there are displacement boundary conditions on some parts of boundary, while stress boundary conditions on other parts. This is called mixed boundary condition.

Sometimes, there are both displacement boundary condition and stress boundary condition on the same border.

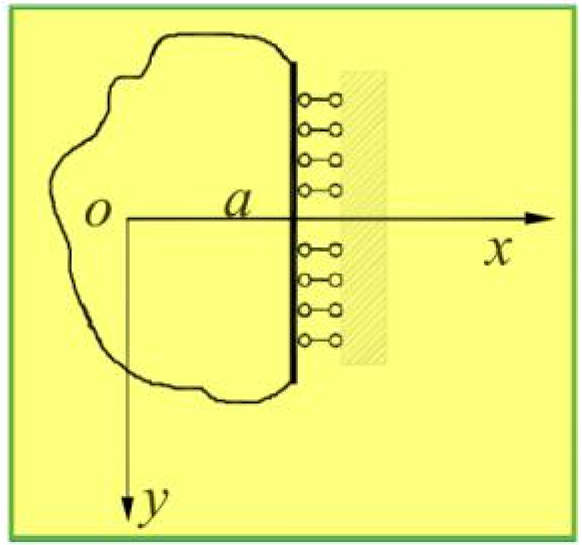


Figure 6.6 mixed boundary condition in plane problems

As shown in figure 6.6, the boundary condition is

$$x = a,$$
$$u_{(x=a)} = 0,$$
$$\tau_{xy(x=a)} = 0.$$

-Solve plane problems according to displacement

As introduced in Chapter 5, displacement have procedures:

a. Utilizing constitutive equations (6.7) and geometric equation (6.6), stress components can be described by displacements components by eliminating strain components. Combinate differential equilibrium equations (6.1) with new achieved equations, two equilibrium equations including two uncertain displacement components are formulated

$$\left.\begin{aligned}\frac{E}{1-\mu^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\partial y}\right)+X=0\\ \frac{E}{1-\mu^2}\left(\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\partial y}\right)+Y=0\end{aligned}\right\}\qquad(6.13)$$

b. In the process of solving group (6.13), undetermined coefficients will appear because of integral operation. These coefficients need satisfy the stress boundary conditions in displacement component's type

$$\left.\begin{aligned}\frac{E}{1-\mu^2}\left[l\left(\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)+m\frac{1-\mu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]=\overline{X}\\ \frac{E}{1-\mu^2}\left[m\left(\frac{\partial v}{\partial y}+\mu\frac{\partial u}{\partial x}\right)+l\frac{1-\mu}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]=\overline{Y}\end{aligned}\right\}\qquad(6.14)$$

Also, displacement boundary condition needs to be satisfied,

$$u_{(s)} = \bar{u}, v_{(s)} = \bar{v}$$

Since the similarity of equations (6.7) and (6.9), the above equations are also useful for solving plane strain problems, if we substitute E with $\dfrac{E}{1-\mu^2}$, and μ with $\dfrac{\mu}{1-\mu}$.

c.Other variable functions can be achieved according to basic equations.

It need to be pointed out that displacement method can handle all boundary condtions. Although function answers are hard to get, this method has a comprehensive application in approximate method such as variation method, difference method, and finite element method (known as FEM).

-Solve plane problems according to stress

Stress method regard stress components as basic variables. Two equations (6.1) and (6.11) are concerned.

Stress equations of compatibility can be formulated from physical equations (6.7) and strain compatibility equation (6.11)

$$\nabla^2(\sigma_x+\sigma_y) = -(1+\mu)(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}) \tag{6.15}$$

In plane strain problems, consistency equation has a similar type

$$\nabla^2(\sigma_x+\sigma_y) = -\frac{1}{1-\mu}(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}) \tag{6.16}$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is Laplace operator.

Considering equations (6.1) and (6.15), or (6.16), stress component function can be introduced with help of boundary conditions.

-Simplification under constant volume force

As a expression of equations (5.3), Stress equations of compatibility (6.15) or (6.16) is more simple, in case of a zero or other constants volume force like gravity. Since

$$\frac{\partial X}{\partial x}=\frac{\partial Y}{\partial y}=0$$

A unified equation to express stress consistency in plane problems is

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{6.17}$$

Equation (6.17) is always called Lévy formulation. Under constant volume force, answers of elastostatics plane problems satisfy

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + X = 0 \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + Y = 0 \\ \nabla^2(\sigma_x + \sigma_y) = 0 \end{cases} \tag{6.18}$$

A specific answer need satisfy stress boundary condition (6.12) too. Satisfaction of (6.12) and (6.18) is a necessary and sufficient condition to a solution.

Furthermore, there have two important results that (a) solution of equation (6.12) and (6.18) is independent to material constants. And (b) two kinds of plane problems have same solution. Result (a) makes it possible to substitute a rare material with cheaper materials in experiments. Result (b) allows plane strain mode replaced by a plane stress mode in experiment.

-Stress function

To find equations (6.18), three partial differential equations need to be integrated, which is very difficult in mathematics. An indirectly method -stress function was developed. According to its thought, a stress function $\varphi(x, y)$ can be introduced. $\varphi(x, y)$ has relationships with all stress components. Equations (6.18) can be transformed into a formulation only include $\varphi(x, y)$, which is easier to analyze.

Investigating a homogeneous partial differential equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} = 0 \end{cases} \qquad \text{(a)}$$

There must exist a two variables function $A = A(x, y)$, which can satisfy first equation of (a), if suppose

$$\sigma_x = \frac{\partial A}{\partial y}, \tau_{yx} = \frac{\partial A}{\partial x} \qquad \text{(b)}$$

For same reason, a function $B = B(x, y)$ can feed the second equation of (a) only if

$$\sigma_y = \frac{\partial B}{\partial x}, \tau_{xy} = \frac{\partial B}{\partial y} \tag{c}$$

According to reciprocal theorem of shear stress

$$\tau_{yx} = \frac{\partial A}{\partial x} = \frac{\partial B}{\partial y} = \tau_{xy} \tag{d}$$

Similarly, $\varphi = \varphi(x, y)$ is introduced if presume

$$A = \frac{\partial \varphi}{\partial y}, B = \frac{\partial \varphi}{\partial x} \tag{e}$$

Expressing equation (b) and (c) with φ

$$\boxed{\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \qquad (6.19)}$$

Solution (6.19) is actually general solution of first equations of (6.18). It firstly presented by Airy. So it was called Airy stress function. Under circumstance of constant volume force, a particular solution could be

$$\sigma_x = -Xx, \quad \sigma_y = -Yy, \quad \tau_{xy} = 0$$

According to solution theory of differential equations, a general solution is

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} - Xx, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2} - Yy, \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \tag{6.20}$$

Apply solution (6.20) into Lévy formulation (6.17)

$$\nabla^2\nabla^2\varphi = 0 \qquad (6.21)$$

Formulation (6.21) is compatibility formulation in stress function type, also called biharmonic equation.

-Inverse method and semi-inverse solving method

General solution of equation (6.21) can be written as limited number of items. So, two indirect methods: inverse method and semi-inverse method are applied.

Inverse method finds out different stress functions which are qualified to equation (6.21). According to stress boundary condition, different surface force cases are classified in correspondent stress components.

To compute stress function, semi-inverse method presumes some or all stress components are known as certain types. If the calculated stress function and stress components are acceptable to boundary conditions and compatibility formulation, it is the right answer. Otherwise, a new guess should be tried.

-Polynomial solutions

Since surface forces in elastic problems are always uniform or linear distribution, stress functions are polynomial. A rectangular plane elastomer with zero volume force is observed.

1.First degree polynomial

Assume $\varphi(x, y) = ax + by + c$, investigate its validity

$$\nabla^4\varphi = \frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0$$

It obviously can be stress function.

According to equation (6.20),

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} = 0, \quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2} = 0, \quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} = 0 \qquad (6.22)$$

Results are (1) first degree polynomial point to state of zero volume force, zero stress, and zero surface force. (2) First degree polynomial has no influence to stress distribution.

2.Quadratic polynomial

Quadratic polynomial $\varphi(x,y)=ax^2+bxy+cy^2$ is eligible to biharmonic equation (6.21).

According to equation (6.20),

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=2c,\quad \sigma_y=\frac{\partial^2\varphi}{\partial x^2}=2a,\quad \tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=-b \tag{6.23}$$

Solution (6.23) indicates even distribution of stress. According to stress boundary condition (6.12), two cases are shown in figure 6.7.

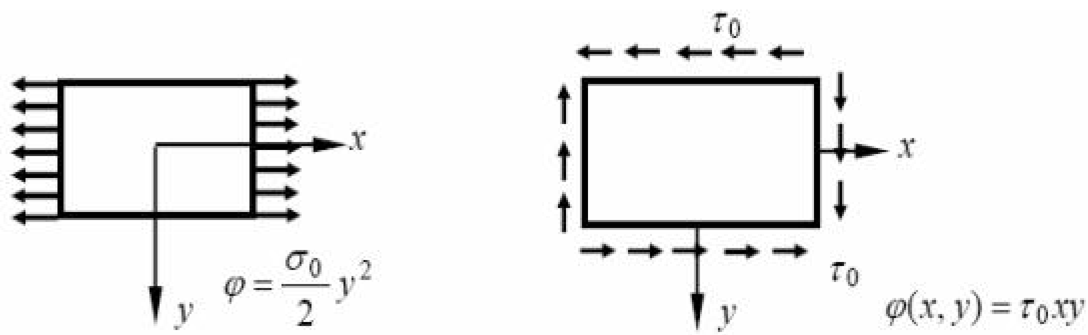


Figure 6.7 Quadratic polynomial cases of stress function

3.Cubic polynomial

Cubic polynomial $\varphi(x,y)=ax^3+bx^2y+cxy^2+dy^3$ is also receivable to biharmonic equation, expressions of stress are

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=2c+6dy,\quad \sigma_y=\frac{\partial^2\varphi}{\partial x^2}=2by+6ax,\quad \tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=-2bx-2cy \tag{6.24}$$

Answer (6.24) manifest stress components are proportional distributed.

If we let a=b=c=0,d≠0, stress components are $\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=6dy, \sigma_y=\tau_{xy}=0$. It delegate

dyxyxy∂2y

stress state and surface force of pure bending beam, which is drawn in figure 6.8.

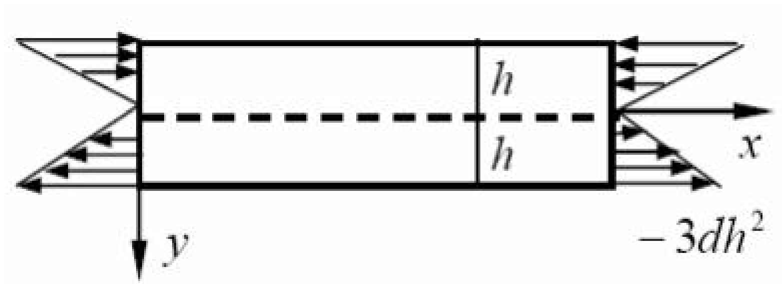


Figure 6.8 Pure bending beam

4.Quartic polynomial

To satisfy biharmonic equation (6.21), coefficients of quartic polynomial

$\varphi(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ have constraint equation

$$3a+ +c+3e=0$$

Stress components of quartic polynomial are second degree functions.

Utilizing superposition principle, a more complex problem can be researched. About polynomial solutions, two things deserve to be mentioned: (a) polynomial construction of stress function can handle some simple linear stress border problems. (b) When degrees of polynomial exceed 3, coefficients have to satisfy a condition to feed biharmonic equation. Plus, these polynomials are hardly applicable for practice.