中国民航大学
Civil Aviation University of China

Structural Applications of Finite Elements

Chapter 9

Scalar field problems

2018-09-01

Outline

- 1D steady-state heat conduction
- 2D steady-state heat conduction
- Torsion

Helmholtz equation

$$\frac{\partial}{\partial x}\left(k_z\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial\phi}{\partial z}\right)+\lambda\phi+Q=0$$

Problem	Equation	Field variable	Parameter	Boundary conditions
Heat conduction	$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+Q=0$	Temperature, T	Thermal conductivity, k	$T=T_0$, $-k\frac{\partial T}{\partial n}=q_0$ $-k\frac{\partial T}{\partial n}=h(T-T_\infty)$
Torsion	$\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)+2=0$	Stress function, θ		$\theta=0$
Potential flow	$\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right)=0$	Stream function, ψ		$\psi=\psi_0$
Seepage and groundwater flow	$k\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)+Q=0$	Hydraulic potential, ϕ	Hydraulic conductivity, k	$\phi=\phi_0$ $\frac{\partial \phi}{\partial n}=0$ $\phi=y$
Electric potential	$\epsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=-\rho$	Electric potential, u	Permittivity, ϵ	$u=u_0$, $\frac{\partial u}{\partial n}=0$
Fluid flow in ducts	$\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+1=0$	Nondimensional velocity, W		$W=0$
Acoustics	$\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+k^2 p=0$	Pressure p (complex)	Wave number, $k^2=\omega^2/c^2$	$p=p_0$, $\frac{1}{ik\rho c}\frac{\partial p}{\partial n}=v_0$

Fourier law

$$q_x = -k\frac{\partial T}{\partial x} \qquad q_y = -k\frac{\partial T}{\partial y}$$

$T = T(x, y)$ is a temperature field in the medium,

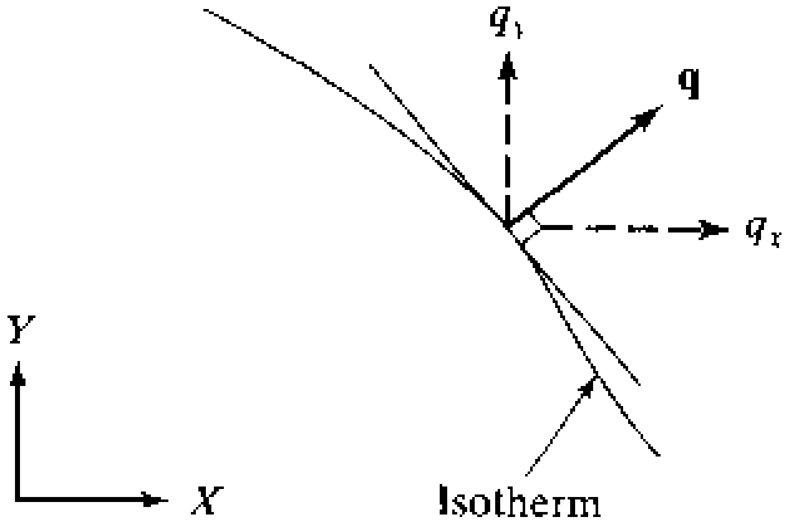
q_x and q_y are the components of the heat flux (W/m^2),

k is the thermal conductivity (W/m. °C)

$$q = h(T_s - T_\infty)$$

q is the convective heat flux (W/m^2),

h is the convection heat-transfer coefficient



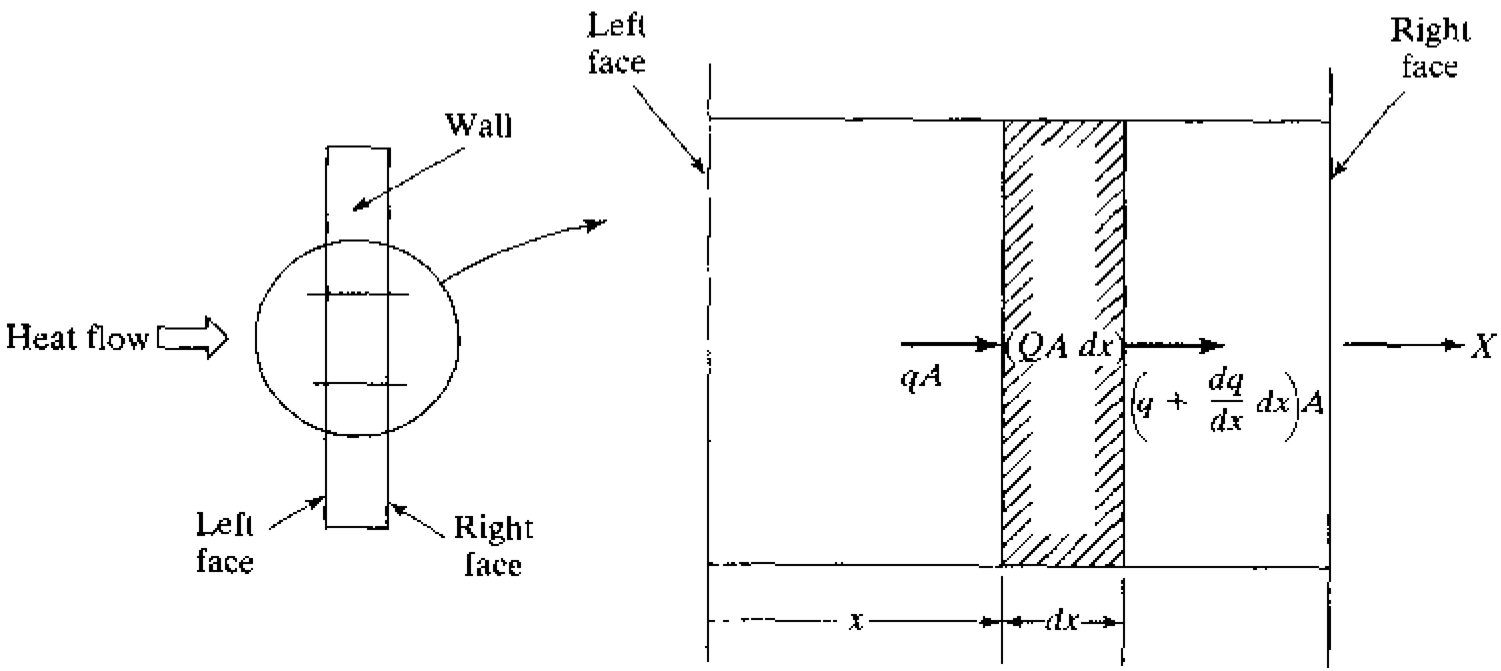
One-dimensional heat conduction

$$qA + QA\,dx = \left(q + \frac{dq}{dx}dx\right)A \qquad Q = \frac{dq}{dx}$$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0$$

$$q = -k\frac{dT}{dx}$$

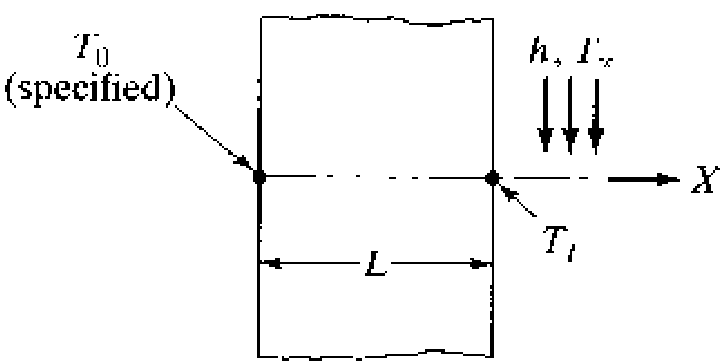
Q (W/m^3) be the internal heat generated per unit volume.

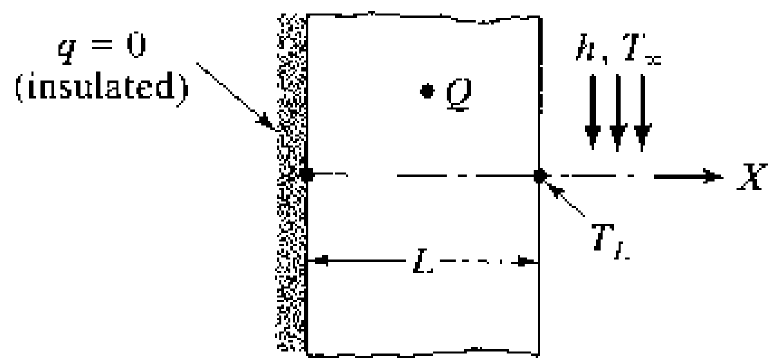


Boundary condition

$$T|_{x=0} = T_0$$
$$q|_{x=L} = h(T_L - T_\infty)$$

$$q|_{x=0} = 0 \qquad q|_{x=L} = h(T_L - T_\infty)$$





$$T(\xi) = N_1T_1 + N_2T_2 = \mathbf{NT}^e$$

$$N_1 = (1-\xi)/2, \quad N_2 = (1+\xi)/2.$$

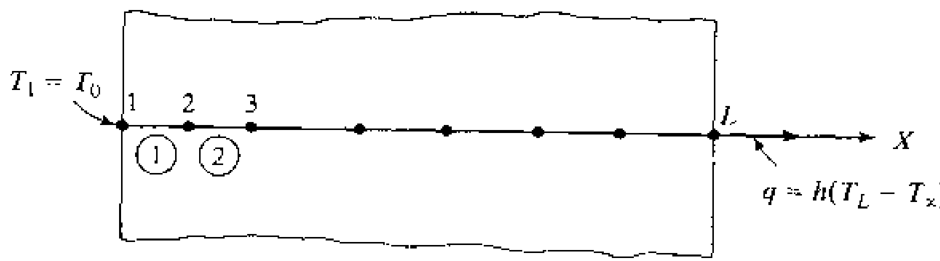
$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$d\xi = \frac{2}{x_2 - x_1}dx$$

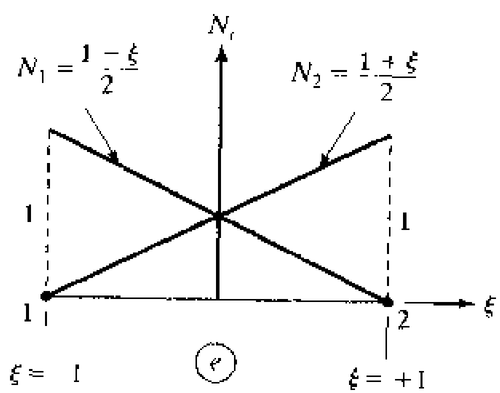
$$\frac{dT}{dx} = \frac{dT}{d\xi}\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{x_2 - x_1}[-1, \quad 1]\mathbf{T}^e$$

$$\frac{dT}{dx} = \mathbf{B}_T\mathbf{T}^e$$

$$\mathbf{B}_T = \frac{1}{x_2 - x_1}[-1, \quad 1]$$



(a)



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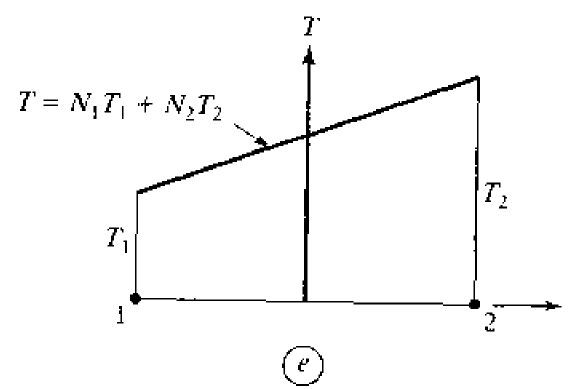


(c)

Galerkin’s approach

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0$$

$$T|_{x=0} = T_0 \qquad q|_{x=L} = h(T_L - T_\infty)$$

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q\right]dx = 0$$

$$\phi k\frac{dT}{dx}\bigg|_0^L - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q\,dx = 0$$

$$\phi k\frac{dT}{dx}\bigg|_0^L = \phi(L)k(L)\frac{dT}{dx}(L) - \phi(0)k(0)\frac{dT}{dx}(0)$$

Since $\phi(0) = 0$ and $q = -k(L)(dT(L)/dx) = h(T_L - T_\infty)$, we get

$$\phi k\frac{dT}{dx}\bigg|_0^L = -\phi(L)h(T_L - T_\infty)$$

$$-\phi(L)h(T_L - T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx + \int_0^L \phi Q\,dx = 0$$

$$\phi = \mathbf{N}\boldsymbol{\psi}$$

$$\frac{d\phi}{dx} = \mathbf{B}_T\boldsymbol{\psi} \qquad -\phi(L)h(T_L - T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}\,dx + \int_0^L \phi Q\,dx = 0$$

$$-\Psi_L h(T_L - T_\infty) - \sum_e \boldsymbol{\psi}^{\mathrm{T}}\left(\frac{k_e\ell_e}{2}\int_{-1}^{1}\mathbf{B}_T^{\mathrm{T}}\mathbf{B}_T\,d\xi\right)\mathbf{T}^e + \sum_e \boldsymbol{\psi}^{\mathrm{T}}\frac{Q_e\ell_e}{2}\int_{-1}^{1}\mathbf{N}^{\mathrm{T}}\,d\xi = 0$$

$$-\Psi_L hT_L + \Psi_L hT_\infty - \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}_T\mathbf{T} + \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{R} = 0$$

$$\mathbf{k}_T = \frac{k_e}{\ell_e}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{r}_Q = \frac{Q_e\ell_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2L} \\ K_{32} & K_{33} & \cdots & K_{3L} \\ \vdots & & & \\ K_{L2} & K_{L3} & \cdots & (K_{LL} + h) \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ \vdots \\ T_L \end{Bmatrix} = \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ (R_L + hT_\infty) \end{Bmatrix} - \begin{Bmatrix} K_{21}T_0 \\ K_{31}T_0 \\ \vdots \\ K_{L1}T_0 \end{Bmatrix}$$

$$\begin{bmatrix} (K_{11} + C) & K_{12} & \cdots & K_{1L} \\ K_{21} & K_{22} & \cdots & K_{2L} \\ \vdots & & & \vdots \\ K_{L1} & K_{L2} & \cdots & (K_{LL} + h) \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_L \end{Bmatrix} = \begin{Bmatrix} (R_1 + CT_0) \\ R_2 \\ \vdots \\ (R_L + hT_\infty) \end{Bmatrix}$$

Solution A three-element finite element model of the wall is shown in Fig. E10.1b. The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global $\mathbf{K} = \Sigma \mathbf{k}_T$ is obtained from these matrices as

$$\mathbf{K} = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Now, since convection occurs at node 1, the constant $h = 25$ is added to the $(1, 1)$ location of $\mathbf{K}$. This results in

$$\mathbf{K} = 66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, the heat rate vector $\mathbf{R}$ consists only of hT_∞ in the first row. That is,

$$\mathbf{R} = [25 \times 800, \quad 0, \quad 0, \quad 0]^T$$

The specified temperature boundary condition $T_4 = 20$°C, will now be handled by the penalty approach. We choose C based on

$$\begin{aligned} C &= \max|\mathbf{K}_{ij}| \times 10^4 \\ &= 66.7 \times 8 \times 10^4 \end{aligned}$$

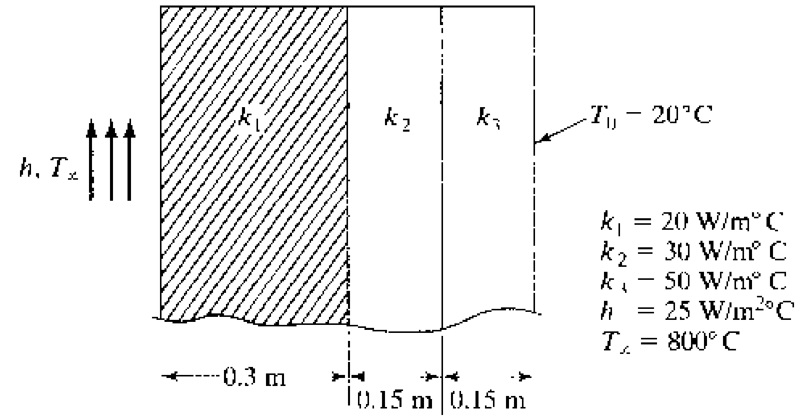
Now, C gets added to $(4, 4)$ location of $\mathbf{K}$, while CT_4 is added to the fourth row of $\mathbf{R}$. The resulting equations are

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 80\,005 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 25 \times 800 \\ 0 \\ 0 \\ 10\,672 \times 10^4 \end{Bmatrix}$$

The solution is

$$\mathbf{T} = [304.6, \quad 119.0, \quad 57.1, \quad 20.0]^T \text{ °C}$$

Comment. The boundary condition $T_4 = 20$°C can also be handled by the elimination approach. The fourth row and column of K is deleted, and $\mathbf{R}$ is modified according to Eq. 3.70. The resulting equations are



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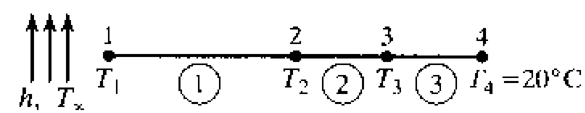


(b)

$$66.7\begin{bmatrix} 1.375 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 8 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 25 \times 800 \\ 0 \\ 0 + 6670 \end{bmatrix}$$

which yields

$$[T_1, \quad T_2, \quad T_3] = [304.6, \quad 119.0, \quad 57.1]^T\text{°C}$$

Heat transfer in thin fins

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + Q = 0 \qquad Q = -\frac{(P\,dx)h(T - T_\infty)}{A_c dx} = -\frac{Ph}{A_c}(T - T_\infty)$$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{Ph}{A_c}(T - T_\infty) = 0$$

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{Ph}{A_c}(T - T_\infty)\right]dx = 0$$

$$\phi k\frac{dT}{dx}\Big|_0^L - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx - \frac{Ph}{A_c}\int_0^L \phi T\,dx + \frac{Ph}{Ac}T_\infty\int_0^L \phi\,dx = 0$$

Using $\phi(0) = 0, k(L)[dT(L)/dx] = 0.$

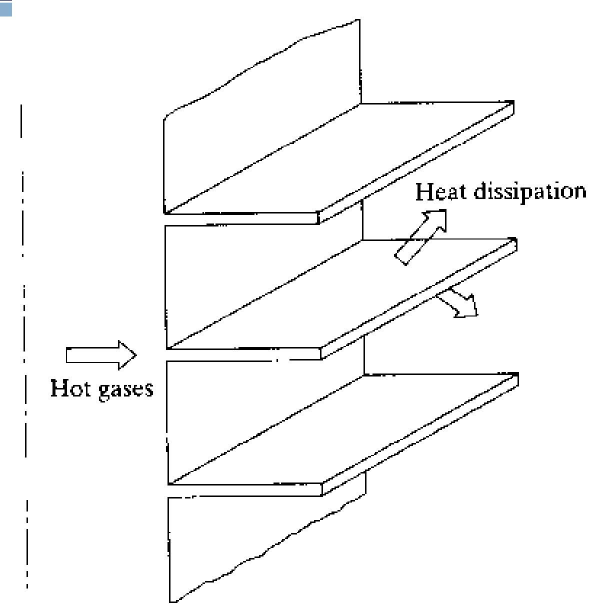
$$dx = \frac{\ell_e}{2}d\xi \qquad T = \mathbf{N}\mathbf{T}^e \qquad \phi = \mathbf{N}\psi \qquad \frac{dT}{dx} = \mathbf{B}_T\mathbf{T}^e \qquad \frac{d\phi}{dx} = \mathbf{B}_T\psi$$

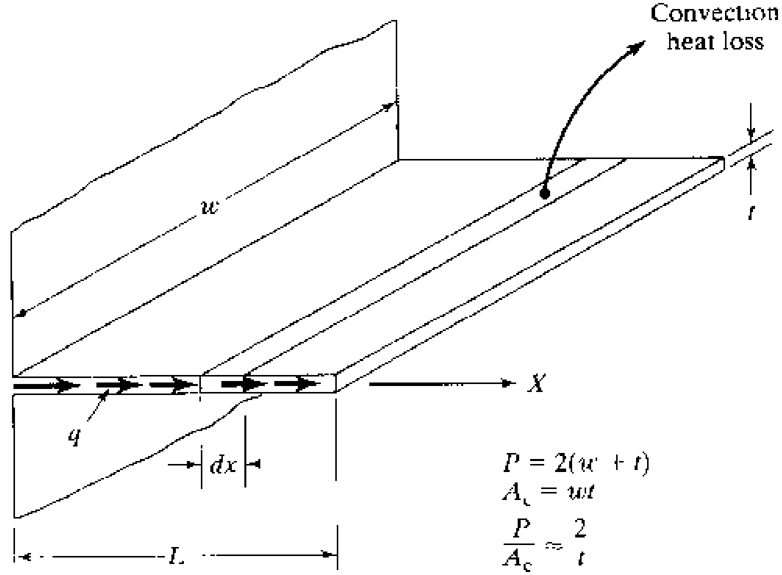
$$-\sum_e \psi^T\left[\frac{k_e\ell_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T\,d\xi\right]\mathbf{T}^e - \frac{Ph}{A_c}\sum_e \psi^T\int_{-1}^{1}\mathbf{N}^T\mathbf{N}\,d\xi\,\mathbf{T}^e$$

$$+\frac{PhT_\infty}{A_c}\sum_e \psi^T\frac{\ell_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi = 0$$

$$\mathbf{h}_T = \frac{Ph}{A_c}\frac{\ell_e}{2}\int_{-1}^{1}\mathbf{N}^T\mathbf{N}\,d\xi = \frac{Ph}{A_c}\frac{\ell_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

since $P/A_c \approx 2/t$ $\quad \mathbf{h}_T \approx \frac{h\ell_e}{3t}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$





$$-\sum_e \boldsymbol{\psi}^T\left[\frac{k_e\ell_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T\,d\xi\right]\mathbf{T}^e - \frac{Ph}{A_c}\sum_e \boldsymbol{\psi}^T\int_{-1}^{1}\mathbf{N}^T\mathbf{N}\,d\xi\,\mathbf{T}^e$$

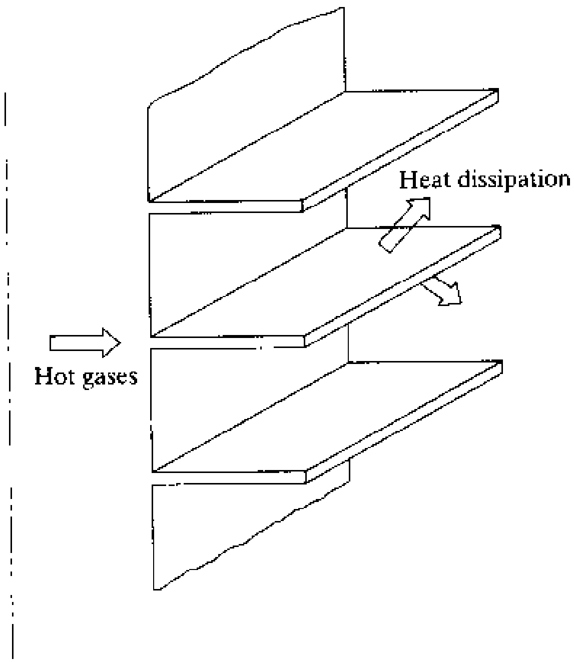
$$+\frac{PhT_\infty}{A_c}\sum_e \boldsymbol{\psi}^T\frac{\ell_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi = 0$$

$$\mathbf{r}_\infty = \frac{Ph}{A_c}T_\infty\frac{\ell_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi = \frac{PhT_\infty}{A_c}\frac{\ell_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\mathbf{r}_\infty \approx \frac{hT_\infty\ell_e}{t}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$-\sum_e \boldsymbol{\psi}^T(\mathbf{k}_T+\mathbf{h}_T)\mathbf{T}^e + \sum_e \boldsymbol{\psi}^T\mathbf{r}_\infty = 0$$

$$-\boldsymbol{\Psi}^T(\mathbf{K}_T+\mathbf{H}_T)+\boldsymbol{\Psi}^T\mathbf{R}_\infty = 0$$



Denoting $K_{ij} = (K_T + H_T)_{ij}$, we obtain

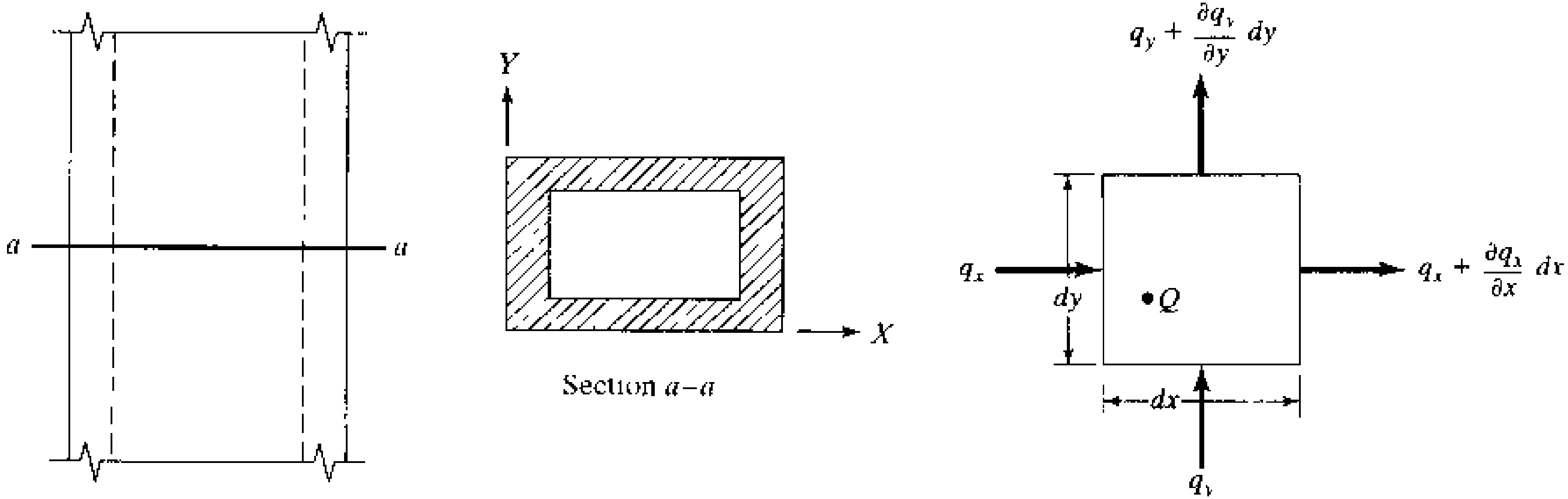
$$\begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2L}\\ K_{32} & K_{33} & \cdots & K_{3L}\\ \vdots & \vdots & & \vdots\\ K_{L2} & K_{L3} & \cdots & K_{LL}\end{bmatrix}\begin{Bmatrix}T_2\\T_3\\ \vdots\\ T_L\end{Bmatrix} = \begin{Bmatrix}\\ \mathbf{R}_\infty\\ \\ \end{Bmatrix} - \begin{Bmatrix}K_{21}T_0\\K_{31}T_0\\ \vdots\\ K_{L1}T_0\end{Bmatrix}$$

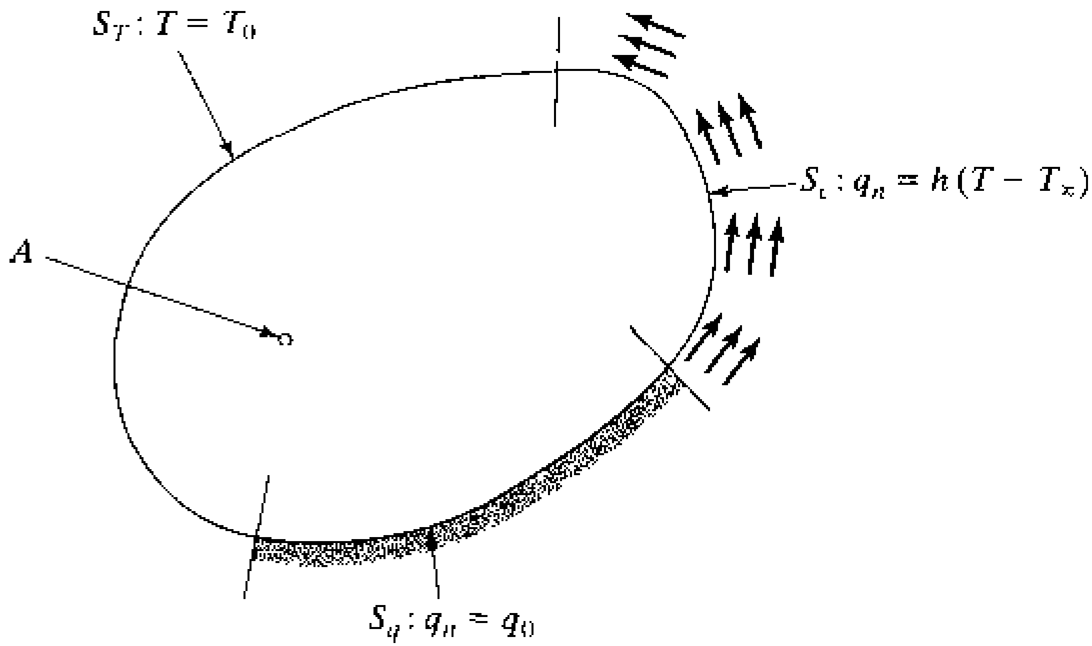
2D steady-state heat conduction

$$q_x\,dy\,\tau + q_y\,dx\,\tau + Q\,dx\,dy\,\tau = \left(q_x + \frac{\partial q_x}{\partial x}dx\right)dy\,\tau + \left(q_y + \frac{\partial q_y}{\partial y}dy\right)dx\,\tau$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - Q = 0 \qquad q_x = -k\frac{\partial T}{\partial x} \qquad q_y = -k\frac{\partial T}{\partial y}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = 0$$



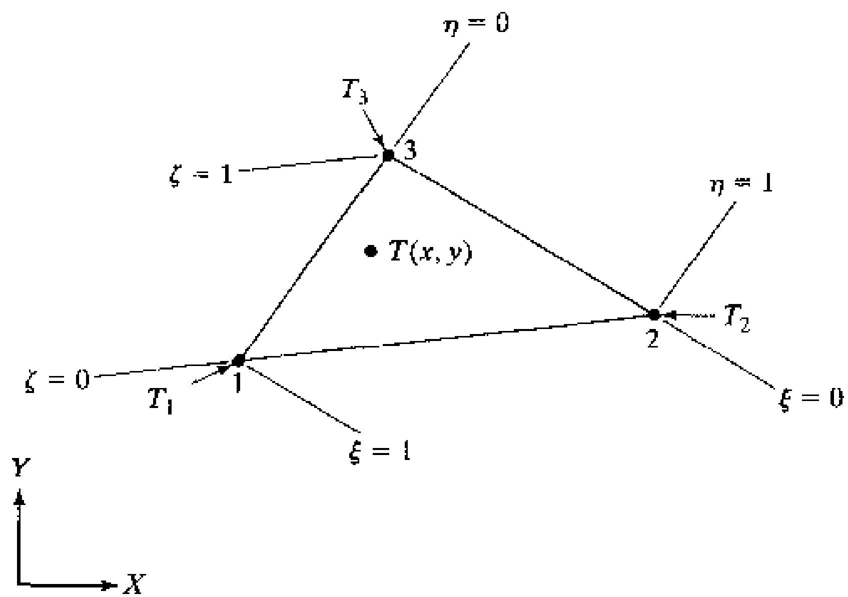


$$T = N_1T_1 + N_2T_2 + N_3T_3$$

$$T = \mathbf{N}\mathbf{T}^e$$

$$x = N_1x_1 + N_2x_2 + N_3x_3$$
$$y = N_1y_1 + N_2y_2 + N_3y_3$$

$$\frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \eta}$$



$$\begin{Bmatrix} \dfrac{\partial T}{\partial \xi} \\ \dfrac{\partial T}{\partial \eta} \end{Bmatrix} = \mathbf{J}\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} \qquad \mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1}\begin{Bmatrix} \dfrac{\partial T}{\partial \xi} \\ \dfrac{\partial T}{\partial \eta} \end{Bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\mathbf{T}^e$$

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \mathbf{B}_T \mathbf{T}^e \qquad \mathbf{B}_T = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} & (y_{13} - y_{23}) \\ -x_{23} & x_{13} & (x_{23} - x_{13}) \end{bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = 0$$

$$\int_A\int \phi\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] dA + \int_A\int \phi Q\, dA = 0$$

$$\phi\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\phi k\frac{\partial T}{\partial x}\right) - k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x}$$

$$\int_A\int \left\{\left[\frac{\partial}{\partial x}\left(\phi k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi k\frac{\partial T}{\partial y}\right)\right] - \left[k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\right]\right\} dA + \int_A\int \phi Q\, dA = 0$$

$$\int_A\int\left\{\left[\frac{\partial}{\partial x}\left(\phi k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi k\frac{\partial T}{\partial y}\right)\right]-\left[k\frac{\partial\phi}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial\phi}{\partial y}\frac{\partial T}{\partial y}\right]\right\}dA+\int_A\int\phi Q\,dA=0$$

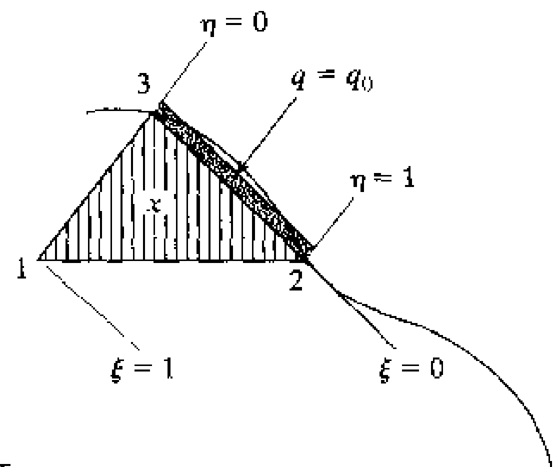
$$-\int_A\int\left[\frac{\partial}{\partial x}(\phi q_x)+\frac{\partial}{\partial y}(\phi q_y)\right]dA=-\int_S\phi[q_xn_x+q_yn_y]\,dS=-\int_S\phi q_n\,dS$$

$$-\int_{S_q}\phi q_0\,dS-\int_{S_c}\phi h(T-T_\infty)\,dS-\int_A\int\left(k\frac{\partial\phi}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial\phi}{\partial y}\frac{\partial T}{\partial y}\right)dA+\int_A\int\phi Q\,dA=0$$

$$\phi=\mathbf{N}\boldsymbol{\psi}\qquad\left[\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right]^{\mathrm{T}}=\mathbf{B}_T\boldsymbol{\psi}$$

$$\int_{S_q}\phi q_0\,dS=\sum_e\boldsymbol{\psi}^{\mathrm{T}}q_0\mathbf{N}^{\mathrm{T}}\,dS$$

$$\int_{S_q}\phi q_0\,dS=\sum_e\boldsymbol{\psi}^{\mathrm{T}}q_0\ell_{2-3}\int_0^1\mathbf{N}^{\mathrm{T}}\,d\eta=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_q$$



$$\mathbf{r}_q=\frac{q_0\ell_{2\text{-}3}}{2}[0\quad 1\quad 1]^{\mathrm{T}}$$

$$-\int_{S_q}\phi q_0\,dS-\int_{S_c}\phi h(T-T_\infty)\,dS-\int_A\int\left(k\frac{\partial\phi}{\partial x}\frac{\partial T}{\partial x}+k\frac{\partial\phi}{\partial y}\frac{\partial T}{\partial y}\right)dA+\int_A\int\phi Q\,dA=0$$

$$\int_{S_c}\phi h(T-T_\infty)\,dS=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\left[h\ell_{2-3}\int_0^1\mathbf{N}^{\mathrm{T}}\mathbf{N}\,d\eta\right]\mathbf{T}^e-\sum_e\boldsymbol{\psi}^{\mathrm{T}}hT_\infty\ell_{2-3}\int_0^1\mathbf{N}^{\mathrm{T}}\,d\eta=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{h}_T\mathbf{T}^e-\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_\infty$$

$$\mathbf{h}_T=\frac{h\ell_{2-3}}{6}\begin{bmatrix}0&0&0\\0&2&1\\0&1&2\end{bmatrix}\qquad\mathbf{r}_\infty=\frac{hT_\infty\ell_{2-3}}{2}[0\quad 1\quad 1]^{\mathrm{T}}$$

$$\int_A\int k\left(\frac{\partial\phi}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial T}{\partial y}\right)dA=\int_A\int k\left[\frac{\partial\phi}{\partial x}\ \frac{\partial\phi}{\partial y}\right]\begin{Bmatrix}\dfrac{\partial T}{\partial x}\\[2mm]\dfrac{\partial T}{\partial y}\end{Bmatrix}dA=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\left[k_e\int_e\mathbf{B}_T^{\mathrm{T}}\mathbf{B}_T\,dA\right]\mathbf{T}^e$$
$$=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}_T\mathbf{T}^e$$

$$\mathbf{k}_T=k_eA_e\mathbf{B}_T^{\mathrm{T}}\mathbf{B}_T$$

$$\int_A\int\phi Q\,dA=\sum_e\boldsymbol{\psi}^{\mathrm{T}}Q_e\int_e\mathbf{N}\,dA=\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_Q$$

$$\mathbf{r}_Q=\frac{Q_eA_e}{3}[1\quad 1\quad 1]^{\mathrm{T}}$$

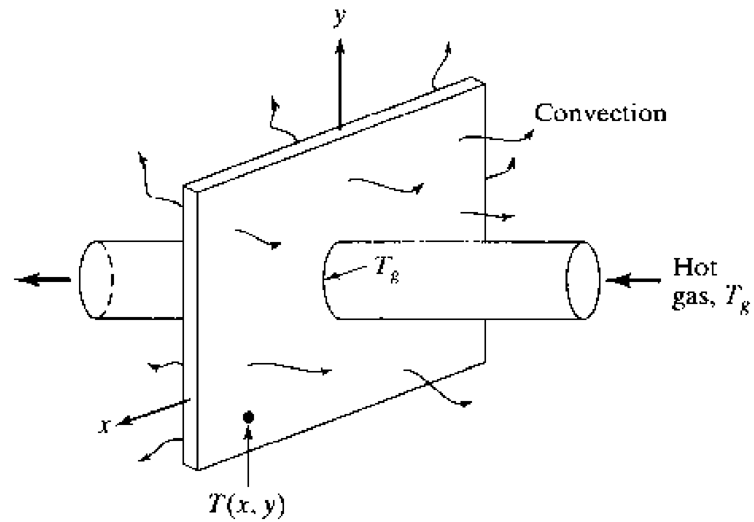
$$-\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_q-\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{h}_T\mathbf{T}^e+\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_\infty-\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}_T\mathbf{T}^e+\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{r}_Q=0$$

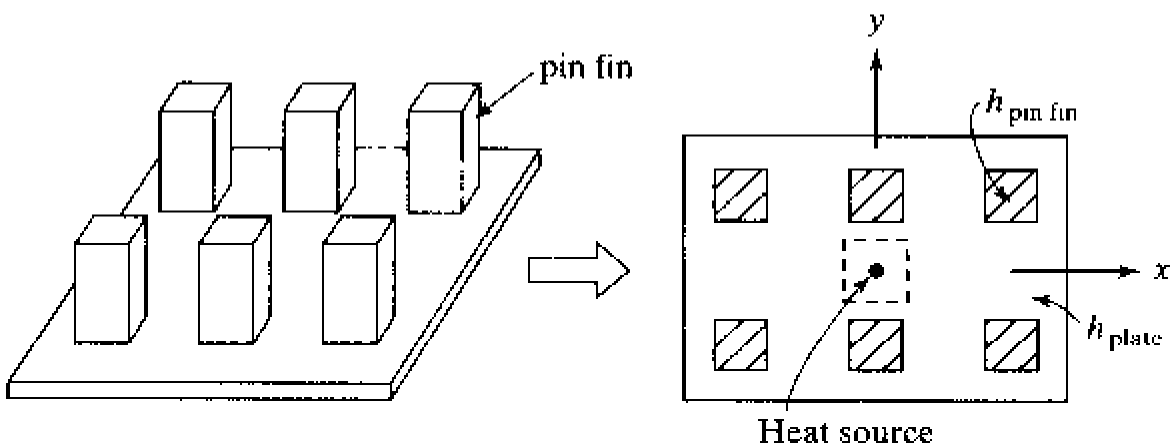
$$\boldsymbol{\Psi}^{\mathrm{T}}(\mathbf{R}_\infty-\mathbf{R}_q+\mathbf{R}_Q)-\boldsymbol{\Psi}^{\mathrm{T}}(\mathbf{H}_T+\mathbf{K}_T)\mathbf{T}=0$$

$$\mathbf{K}^E\mathbf{T}^E=\mathbf{R}^E$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - C(T - T_\infty) + Q = 0$$

$$C = -2h/t.$$

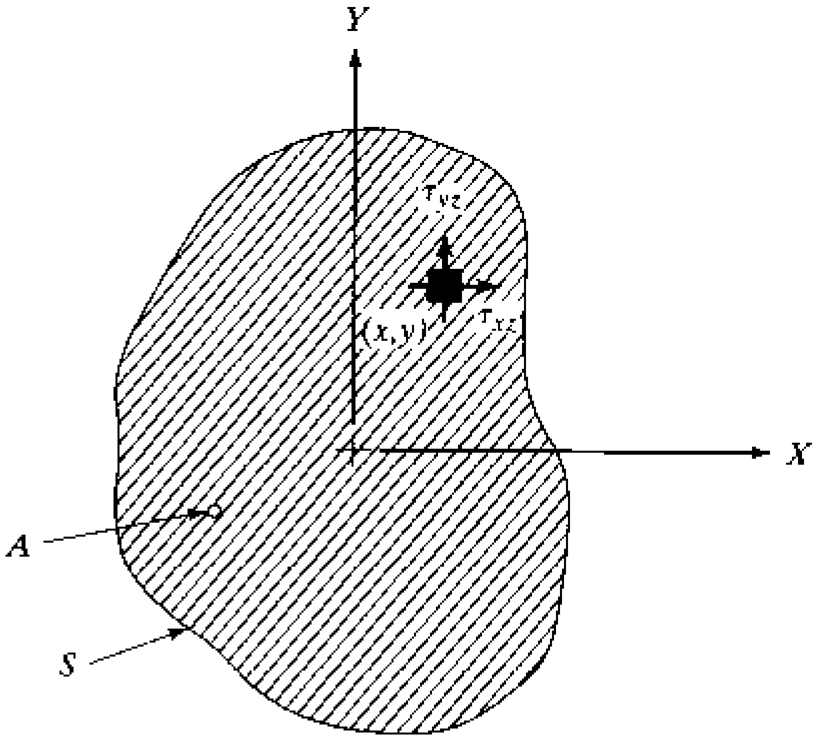


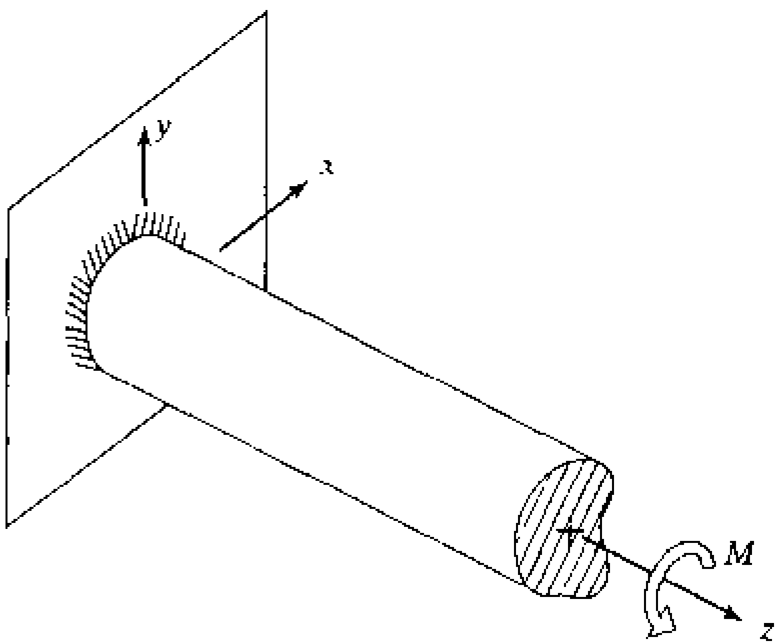


$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+2=0 \qquad \text{in } A$$

$$\theta=0 \qquad \text{on } S$$

$$\tau_{xz}=G\alpha\frac{\partial\theta}{\partial y} \qquad \tau_{yz}=-G\alpha\frac{\partial\theta}{\partial x} \qquad M=2G\alpha\int_A\int\theta\,dA$$





$$\theta = \mathbf{N}\boldsymbol{\theta}^e \qquad \begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned} \qquad \begin{Bmatrix} \dfrac{\partial \theta}{\partial \xi} \\ \dfrac{\partial \theta}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial \theta}{\partial y} \end{Bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \theta}{\partial \xi} & \dfrac{\partial \theta}{\partial \eta} \end{bmatrix}^{\mathrm{T}} = \mathbf{J} \begin{bmatrix} \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix}^{\mathrm{T}} = \mathbf{B}\boldsymbol{\theta}^e$$

$$[-\tau_{yz} \quad \tau_{xz}]^{\mathrm{T}} = G\alpha \mathbf{B}\boldsymbol{\theta}^e$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$\int_A\int \phi\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+2\right)dA=0 \qquad \phi\frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right)-\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x}$$

$$\int_A\int\left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\theta}{\partial y}\right)\right]dA-\int_A\int\left(\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y}\right)dA+\int_A\int 2\phi\,dA=0$$

$$\int_A\int\left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\theta}{\partial y}\right)\right]dA=\int_S\phi\left(\frac{\partial\theta}{\partial x}n_x+\frac{\partial\phi}{\partial y}n_y\right)dS=0$$ boundary condition $\phi=0$ on S.

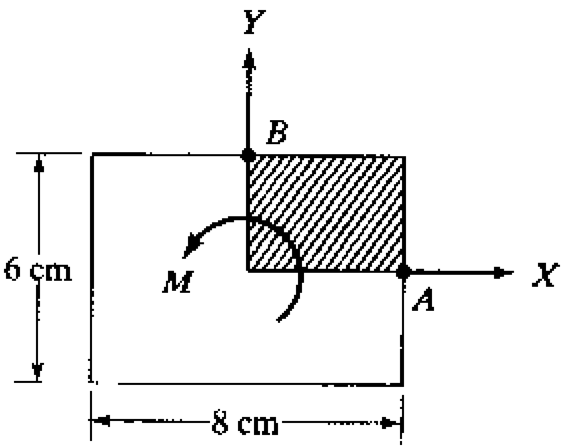
$$\int_A\int\left[\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y}\right]dA-\int_A\int 2\phi\,dA=0$$

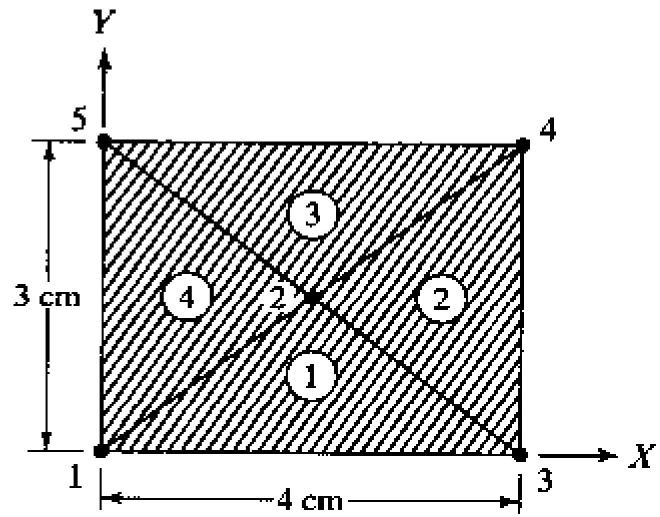
$$\phi=\mathbf{N}\boldsymbol{\psi} \qquad \left[\frac{\partial\phi}{\partial x}\quad\frac{\partial\phi}{\partial y}\right]^{\mathrm{T}}=\mathbf{B}\boldsymbol{\psi} \qquad \left(\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y}\right)=\left(\frac{\partial\phi}{\partial x}\quad\frac{\partial\phi}{\partial y}\right)\begin{Bmatrix}\dfrac{\partial\theta}{\partial x}\\[2mm]\dfrac{\partial\theta}{\partial y}\end{Bmatrix}$$

$$\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}\boldsymbol{\theta}^e-\sum_e\boldsymbol{\psi}^{\mathrm{T}}\mathbf{f}=0$$

$$\mathbf{k}=A_e\mathbf{B}^{\mathrm{T}}\mathbf{B} \qquad \boldsymbol{\Psi}^{\mathrm{T}}(\mathbf{K}\boldsymbol{\Theta}-\mathbf{F})=0$$

$$\mathbf{f}=\frac{2A_e}{3}[1,\quad 1,\quad 1]^{\mathrm{T}}$$





Element	1	2	3
1	1	3	2
2	3	4	2
3	4	5	2
4	5	1	2

Using the relations

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

and

$$\mathbf{k} = A_e \mathbf{B}^{\mathrm{T}} \mathbf{B}$$

we get

$$\mathbf{B}^{(1)} = \frac{1}{6} \begin{bmatrix} -1.5 & 1.5 & 0 \\ -2 & -2 & 4 \end{bmatrix} \qquad \mathbf{k}^{(1)} = \frac{1}{2} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} 1.042 & 0.292 & -1.333 \\ & 1.042 & -1.333 \\ \text{Symmetric} & & 2.667 \end{bmatrix} \end{array}$$

Similarly,

$$\mathbf{k}^{(2)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 3 & 4 & 2 \end{array} \\ \begin{bmatrix} 1.042 & -0.292 & -0.75 \\ & 1.042 & -0.75 \\ \text{Symmetric} & & 1.5 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(3)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 4 & 5 & 2 \end{array} \\ \begin{bmatrix} 1.042 & 0.292 & -1.333 \\ & 1.042 & -1.333 \\ \text{Symmetric} & & 2.667 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(4)} = \frac{1}{2}\begin{array}{c} \begin{array}{ccc} 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 1.042 & -0.292 & -0.75 \\ & 1.042 & -0.75 \\ \text{Symmetric} & & 1.5 \end{bmatrix} \end{array}$$

Similarly, the element load vector $\mathbf{f} = (2A_e/3)[1, \quad 1, \quad 1,]^T$ for each element is

$$\mathbf{f}^{(i)} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix} \qquad i = 1, \quad 2, \quad 3, \quad 4$$

We can now assemble **K** and **F**. Since the boundary conditions are

$$\Theta_3 = \Theta_4 = \Theta_5 = 0$$

we are interested only in degrees of freedom 1 and 2. Thus, the finite element equations are

$$\frac{1}{2}\begin{bmatrix} 2.084 & -2.083 \\ -2.083 & 8.334 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 8 \end{Bmatrix}$$

The solution is

$$[\Theta_1, \quad \Theta_2] = [7.676, \quad 3.838]$$

Consider the equation

$$M = 2G\alpha \int_A \int \theta \, dA$$

Using $\theta = \mathbf{N}\boldsymbol{\theta}^e$, and noting that $\int_e \mathbf{N}\, dA = (A_e/3)[1, \quad 1, \quad 1]$, we get

$$M = 2G\alpha\left[\sum_e \frac{A_e}{3}(\theta_1^e + \theta_2^e + \theta_3^e)\right] \times 4$$

This multiplication by 4 is because the finite element model represents only one-quarter of the rectangular cross section. Thus, we get the angle of twist per unit length to be

$$\alpha = 0.004\frac{M}{G}$$