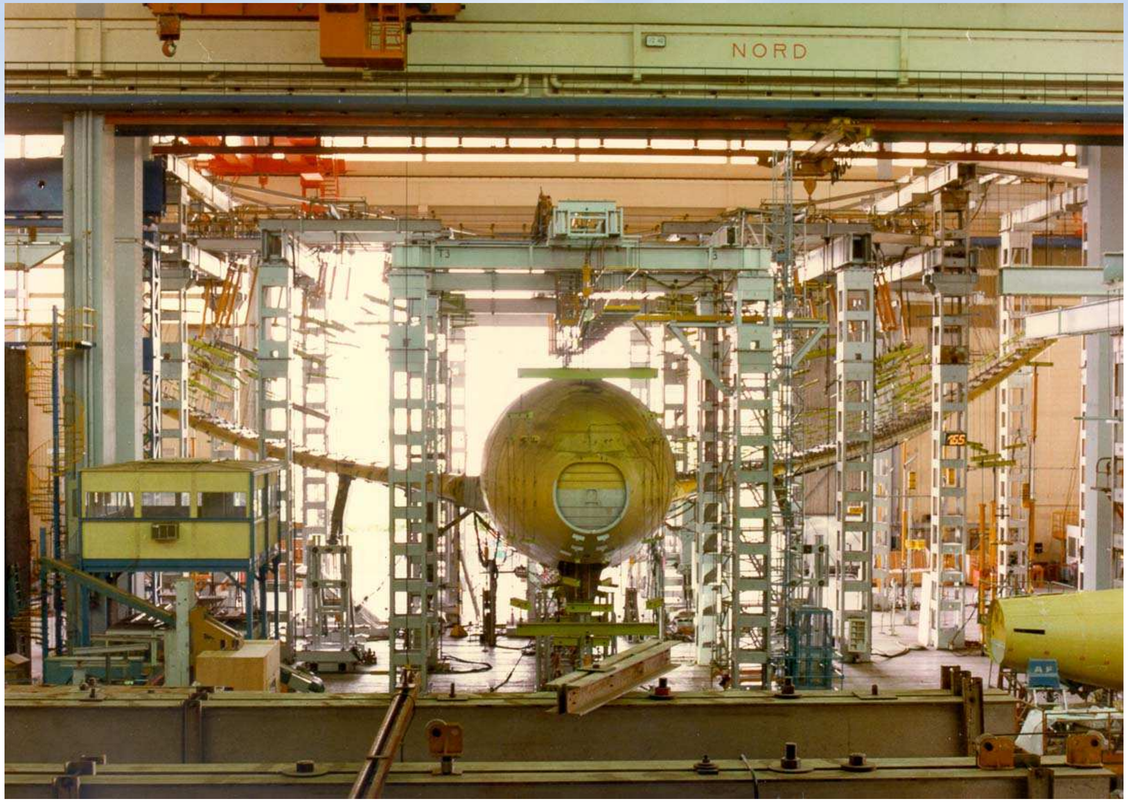
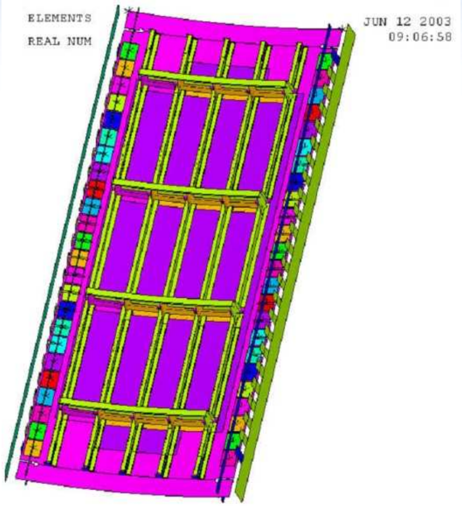
LE CALCUL DES STRUCTURES PAR ÉLÉMENTS

Principes et hypothèses de la MEF





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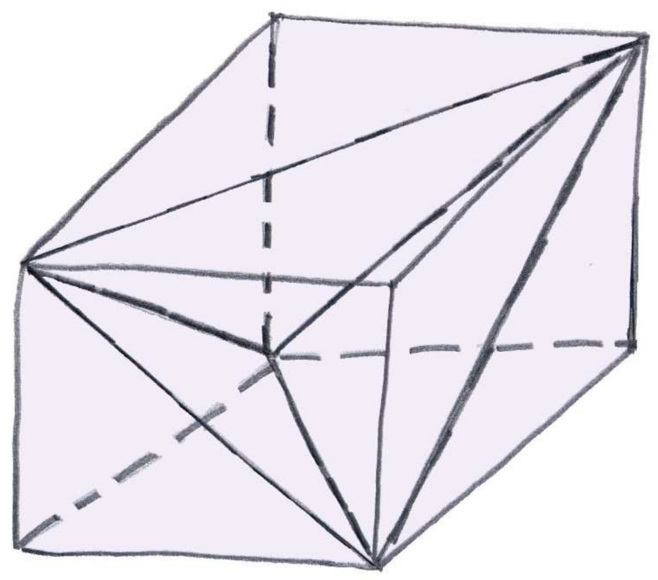
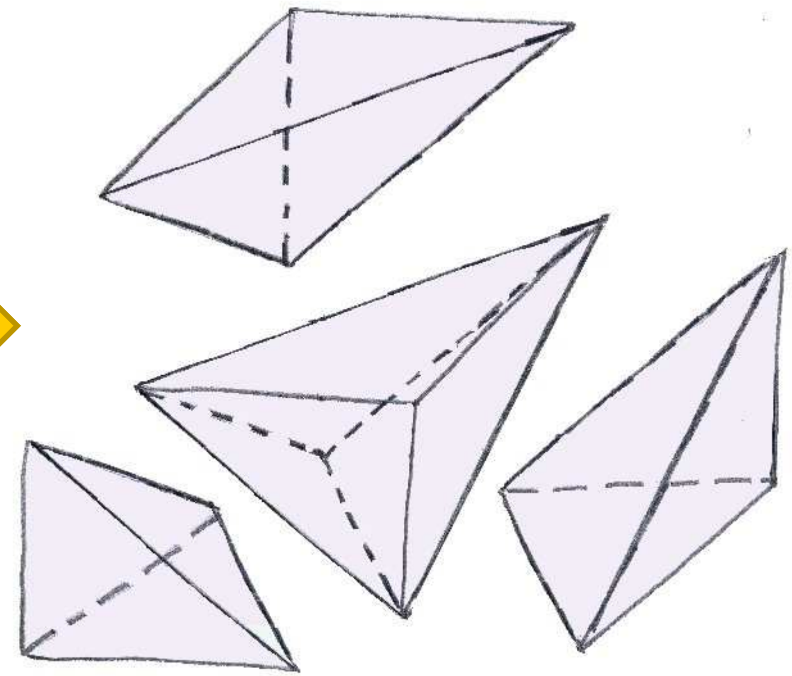
Eléments finis 3D

Division of a Cube in 4 Tetrahedrons





Notations

Each nodes is defined in the global axis system by its coordinates,
well known by the computer :

$$N_i \quad \left(x_i, \quad y_i, \quad z_i\right)$$

The displacement vector of each node is defined in the global axis system
by its components :

$$N_i N_i^{'} \quad \left(u_i, \quad v_i, \quad w_i\right)$$

Shape functions

The shape functions are an approximation of the field of displacement inside the finite element. These functions allow the computer to have for each point inside the finite element the displacment defined as an interpolated value of the nodal displacements

$$(ax+by+cy)^0 = C_0$$

$$(ax+by+cy)^1 = C_1^1 x + C_1^2 y + C_1^3 z$$

$$(ax+by+cy)^2 = C_2^1 x^2 + C_2^2 y^2 + C_2^3 z^2 + C_2^4 xy + C_2^5 yz + C_2^6 zx$$

$$(ax+by+cy)^3 = C_3^1 x^3 + C_3^2 y^3 + C_3^3 z^3 + C_3^4 x^2 y + C_3^5 x^2 z + C_3^5 y^2 z + C_3^6 y^2 x + C_3^7 z^2 x + C_3^8 z^2 y + C_3^9 xyz$$

Constant shape function is not used

Linear Shape function is not accurate

Quadratic shape functions need 10 nodes for each tetrahedron

Cubic shape functions need 20 nodes for each tetrahedron

Exemple : Linear Shape Functions

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$$
$$v(x,y,z) = b_1 + b_2 x + b_3 y + b_4 z$$
$$w(x,y,z) = c_1 + c_2 x + c_3 y + c_4 z$$

The different coefficients of these equations are defined by writting the compatibility with the nodal displacements. We have 12 unknown coefficients, there are 3 components for the displacement of each node, there are 3 coordinates for each node

=> We need 4 nodes for each finite element to solve this problem.

Shape function : Bourdary conditions

$$\begin{cases} u_1 = a_1 + a_2 x_1 + a_3 y_1 + a_4 z_1 \\ u_2 = a_1 + a_2 x_2 + a_3 y_2 + a_4 z_2 \\ u_3 = a_1 + a_2 x_3 + a_3 y_3 + a_4 z_3 \\ u_4 = a_1 + a_1 x_4 + a_3 y_4 + a_4 z_4 \end{cases}$$

$$\begin{cases} v_1 = b_1 + b_2 x_1 + b_3 y_1 + b_4 z_1 \\ v_2 = b_1 + b_2 x_2 + b_3 y_2 + b_3 z_2 \\ v_3 = b_1 + b_2 x_3 + b_3 y_3 + b_4 z_3 \\ v_4 = b_1 + b_2 x_4 + b_3 y_4 + b_4 z_4 \end{cases}$$

$$\begin{cases} w_1 = c_1 + c_2 x_1 + c_3 y_1 + c_4 z_1 \\ w_2 = c_1 + c_2 x_2 + c_3 y_2 + c_4 z_2 \\ w_3 = c_1 + c_2 x_3 + c_3 y_3 + c_4 z_3 \\ w_4 = c_1 + c_2 x_4 + c_3 y_4 + c_4 z_4 \end{cases}$$

Each components of the displacement must be verified for each node.

-3 components for each nodes

- 4 nodes for the tetrahedron element

=>12 equations

Resolution with Cramer's formulas

$$a_1 = \frac{\begin{bmatrix} u_1 & x_1 & y_1 & z_1 \\ u_2 & x_2 & y_2 & z_2 \\ u_3 & x_3 & y_3 & z_3 \\ u_4 & x_4 & y_4 & z_4 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}} \quad a_2 = \frac{\begin{bmatrix} 1 & u_1 & y_1 & z_1 \\ 1 & u_2 & y_2 & z_2 \\ 1 & u_3 & y_3 & z_3 \\ 1 & u_4 & y_4 & z_4 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}} \quad a_3 = \frac{\begin{bmatrix} 1 & x_1 & u_1 & z_1 \\ 1 & x_2 & u_2 & z_2 \\ 1 & x_3 & u_3 & z_3 \\ 1 & x_4 & u_4 & z_4 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}} \quad a_4 = \frac{\begin{bmatrix} 1 & x_1 & y_1 & u_1 \\ 1 & x_2 & y_2 & u_2 \\ 1 & x_3 & y_3 & u_3 \\ 1 & x_4 & y_4 & u_4 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}}$$

Remark

$$\Delta = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} = 6 \quad V_{tetrahedron}$$

This determinent represent 6 time the volume of the tetrahedron

Resolution : exemple of development

$$a_1 = \frac{1}{\Delta}\left\{\begin{bmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z \end{bmatrix} u_1 - \begin{bmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z \end{bmatrix} u_2 + \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z \end{bmatrix} u_3 - \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} u_4 \right\}$$

$$a_1 = \frac{1}{\Delta}\left[\begin{bmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z \end{bmatrix}, \quad -\begin{bmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z \end{bmatrix}, \quad \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z \end{bmatrix}, \quad -\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$a_i = \left[A_{i1}, \quad A_{i2}, \quad A_{i3}, \quad A_{i4}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Shape functions for u

$$a_1 = \begin{bmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$a_2 x = \begin{bmatrix} A_{21}x, & A_{22}x, & A_{23}x, & A_{24}x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$a_3 y = \begin{bmatrix} A_{31}y, & A_{32}y, & A_{33}y, & A_{34}y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$a_4 z = \begin{bmatrix} A_{41}z, & A_{42}z, & A_{43}z, & A_{44}z \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

The sum of theses 4 terms represent the displacement in the direction x of all the points inside the tetrahedron finite element

Shape functions for u

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z = \begin{bmatrix} \Phi_1, & \Phi_2, & \Phi_3, & \Phi_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\Phi_i = A_{1i} + A_{2i} x + A_{3i} y + A_{4i} z$$

Shape functions for v

$$v(x,y,z) = b_1 + b_2 x + b_3 y + b_4 z = \begin{bmatrix} \quad , & \quad , & \quad , & \quad \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Write the terms of the matrix line that give the field of displacement in the tetrahadron finite element

Shape functions for w

$$w(x,y,z) = c_1 + c_2 x + c_3 y + c_4 z = \begin{bmatrix} \Phi_1, & \Phi_2, & \Phi_3, & \Phi_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

Method to find the Nodal Displacement

- Find the total energy as the sum of :
  - Internal energy
  - Potential energy
- Write du minimum total energy
- Use the equation F=Kq

Internal energy

$$W = \iiint_{Volume} \left(\frac{1}{2} trace\left([\Sigma][\mathrm{E}]\right) \right) dv$$

$$[\Sigma] = \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)[I] + 2\mu[\mathrm{E}]$$

$$W = \iiint_{Volume} \left(\frac{1}{2} trace\left(\left(\lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)[\mathrm{E}][I] + 2\mu[\mathrm{E}]\right).[\mathrm{E}]\right) \right) dv$$

$$trace\left([E][I]\right) = trace\left([E]\right) = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$trace\left([E][E]\right) = trace\left(\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} . \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} \right)$$

Internal Energy

$$W = \frac{1}{2}\iiint_{Volume}\left(\lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)^2+2\mu\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2\right)+4\mu\left(\varepsilon_{xy}^2+\varepsilon_{xz}^2+\varepsilon_{yz}^2\right)\right)dv$$

$$W = \frac{1}{2}\left\{\iiint_{Volume} 2\lambda\left(\varepsilon_x\varepsilon_y+\varepsilon_y\varepsilon_z+\varepsilon_z\varepsilon_x\right)dv+4\mu\iiint_{Volume}\left(\varepsilon_{xy}^2+\varepsilon_{xz}^2+\varepsilon_{yz}^2\right)dv+\left(\lambda+2\mu\right)\iiint_{Volume}\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2\right)dv\right\}$$

There are 9 terms to compute such as :

$$\iiint_{Volume}\varepsilon_x\varepsilon_y dv \quad or \quad \iiint_{Volume}\varepsilon_x^2 dv \quad or \quad \iiint_{Volume}\varepsilon_{xy}^2 dv$$

Relations between strains and displacements

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

These relations are only usable in the linear theory

Internal Energy

$$W = \iiint_{Volume} \left(\frac{1}{2} \left(\begin{array}{l} \lambda \left(\left(\frac{\partial u}{\partial x} \right) + \left(\frac{\partial v}{\partial y} \right) + \left(\frac{\partial w}{\partial z} \right) \right)^2 + 2\mu \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right) \\ + \mu \left(\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \right) \end{array} \right) \right) dv$$

λ and μ are the Lamé's Coefficients

Strains defined as functions of the displacements

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z = [\Phi_1, \quad \Phi_2, \quad \Phi_3, \quad \Phi_4] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \left[\frac{\partial \Phi_1}{\partial x}, \quad \frac{\partial \Phi_2}{\partial x}, \quad \frac{\partial \Phi_3}{\partial x}, \quad \frac{\partial \Phi_4}{\partial x} \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \left[\frac{\partial \Phi_1}{\partial x} \quad \frac{\partial \Phi_2}{\partial x} \quad \frac{\partial \Phi_3}{\partial x} \quad \frac{\partial \Phi_4}{\partial x} \quad 0 \quad 0 \quad 0 \quad 0 \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

The strains must be defind as a linear function of all the nodal displacements

Different matrix writting for the strains

$$\varepsilon_x = \frac{\partial u}{\partial x} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x}, & \frac{\partial \Phi_2}{\partial x}, & \frac{\partial \Phi_3}{\partial x}, & \frac{\partial \Phi_4}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_3}{\partial x} & \frac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

- The strain can be written under theses 2 matrix form
- We need to add the components in the direction z to be exact

$$\varepsilon_x = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi_1}{\partial x} \\ \frac{\partial \Phi_2}{\partial x} \\ \frac{\partial \Phi_3}{\partial x} \\ \frac{\partial \Phi_4}{\partial x} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Strains as quadratic functions of nodal displacements

$$\varepsilon_x^2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial \Phi_1}{\partial x} \\ \dfrac{\partial \Phi_2}{\partial x} \\ \dfrac{\partial \Phi_3}{\partial x} \\ \dfrac{\partial \Phi_4}{\partial x} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi_1}{\partial x} & \dfrac{\partial \Phi_2}{\partial x} & \dfrac{\partial \Phi_3}{\partial x} & \dfrac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Partial Stiffness Matrix

$$\varepsilon_x^2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T \begin{bmatrix} \frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi_3}{\partial x} & \frac{\partial \Phi_1}{\partial x}\frac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \\ \frac{\partial \Phi_2}{\partial x}\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_2}{\partial x}\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial x}\frac{\partial \Phi_3}{\partial x} & \frac{\partial \Phi_2}{\partial x}\frac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \\ \frac{\partial \Phi_3}{\partial x}\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_3}{\partial x}\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_3}{\partial x}\frac{\partial \Phi_3}{\partial x} & \frac{\partial \Phi_3}{\partial x}\frac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \\ \frac{\partial \Phi_4}{\partial x}\frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_4}{\partial x}\frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_4}{\partial x}\frac{\partial \Phi_3}{\partial x} & \frac{\partial \Phi_4}{\partial x}\frac{\partial \Phi_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Contribution of one term of the internal energy

$$\frac{1}{2}\iiint_{Vol}\varepsilon_x^2 dv = \frac{1}{2}\iiint_{Vol}[q]^T\left[\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_j}{\partial x}\right][q].dv$$

But we have : $\Phi_i = A_{i1} + A_{i2}x + A_{i3}y + A_{i4}z$

$$\begin{cases} \dfrac{\partial \Phi_i}{\partial x} = A_{i2} \\ \dfrac{\partial \Phi_j}{\partial x} = A_{j2} \end{cases}$$

Contribution of one term of the internal energy

$$\frac{1}{2}\iiint_{Vol}\varepsilon_x^2 dv = \frac{1}{2}\iiint_{Vol}[q]^T\left[\frac{\partial\Phi_i}{\partial x}\frac{\partial\Phi_j}{\partial x}\right][q].dv$$

$$= \frac{1}{2}[q]^T\left\{A_{2i}A_{2j}\left(\iiint_{Vol} dv\right)\right\}[q] = \frac{1}{2}[q]^T\left\{A_{2i}A_{2j}V\right\}[q]$$

Contribution of one term of the internal energy

$$W = \frac{1}{2}[q]^T \left[\frac{A_{i2}A_{j2}}{6} \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \right] [q]$$

Potential Energy

$$V = -\begin{bmatrix} u_1 & u_2 & u_3 & u_4 & v_1 & \ldots & w_2 & w_3 & w_4 \end{bmatrix} \begin{bmatrix} F_x^1 \\ F_x^2 \\ F_x^3 \\ F_x^4 \\ \cdot \\ \cdot \\ F_z^3 \\ F_z^4 \end{bmatrix} = -[q]^T [F]$$

Minimisation of the total Energy

$$E_t = W + V = \frac{1}{2}[q]^T[K][q] - [q]^T[F]$$

$$[F] = [K][q]$$

Tetrahedron element

- L’élément volumique tétraèdre a quatre nœuds existe mais est rarement employée.
- Contrainte constante dans l’élément.
- Déformation constante dans l’élément.
- On lui préfère l’élément tétraèdre à 10 nœuds
- Ou mieux encore
- L’éléments hexaèdre à 8 nœuds.