Gaussian processes

MA13-Probability and statistics: Courses 21-22

September 2014

Manuel SAMUELIDES[1] Zhigang SU[2]
[1]Professor
Institut Supereur de l'Aeronautique et de l'Espace

[2]Professor
Sino-European Institute of Aviation Engineering
Civil Aviation University of China

About stochastic processes

- A(second order)stochastic process is a measurable application $(t,\omega) \in \mathcal{T} \times \Omega \to X_t(\omega)$ such that $X_t \in L^2$.
- $\mathcal{T}$ is the time set which can be discrete $\mathcal{T} = \mathbb{N}$ or $\mathcal{T} = ZZ$ or continuous $\mathcal{T} = [a,b], \mathcal{T} = \mathbb{R}^+, \mathcal{T} = \mathbb{R}$.
- When T is many-dimensional as in images or in multi-sensor fusion or in fluid mechanics, the right term is random field
- We shall consider that time set $\mathcal{T}$ is continuous.
- Mean-square continuity of the process is the continuity of $t \in \mathcal{T} \to X_t \in L^2$. Hereafter, continuity is meaning "mean square continuity".
- In that case,

$$t \in \mathcal{T} \to m(t) = \mathrm{E}(X_t)$$

  and

$$(s,t) \in \mathcal{T} \times \mathcal{T} \to k(s,t) = Cov(X_s, X_t)$$

  are continuous.

Recall about Gaussian vectors

- A Gaussian vector $\mathbf{X}$ is a random vector such that the vector subspace of L^2 generated by its components $(X_1, \ldots, X_n)$ is a set of Gaussian random variables.
- The law of a Gaussian vector is completely determined by its expectation $\mathbf{m}$ and its covariance matrix $\mathbf{\Gamma}$. More precisely, its characteristic function is

$$\phi_X(\mathbf{u}) = \exp[i(\mathbf{m}|\mathbf{u}) - \frac{1}{2}(\mathbf{u}|\Gamma\mathbf{u})]$$

- If $\mathbf{X}$ is a Gaussian vector with law $\mathcal{N}(\mathbf{m}, \mathbf{\Gamma})$ then $A\mathbf{X} + \mathbf{b}$ is a Gaussian vector with law $\mathcal{N}(A\mathbf{m} + \mathbf{b}, A\mathbf{\Gamma}\widetilde{A})$
- When $\mathbf{\Gamma}$ is invertible,$\mathbf{X}$ has the following probability density

$$x \to \frac{1}{\sqrt{(2\pi)^n|\Gamma|}} \exp[-\frac{1}{2}(\mathbf{X} - \mathbf{m}|\Gamma^{-1}(\mathbf{X} - \mathbf{m}))]$$

Poisson process

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Definition

A Gaussian process is a stochastic process such that for any $(t_1, \dots, t_n) \in \mathcal{T}^n$ and any $(\alpha_1, \dots, \alpha^n) \in \mathbb{R}$ the random variable $\sum_{k=1}^n \alpha_k X_{t_k}$ is Gaussian.

The law of $(X_{t_1}, \dots, X_{t_n})$ is defined by the mean function $t \to m(t)$ and the covariance kernel $(s, t) \to k(s, t)$ of the process since

$$\mathrm{E}\begin{bmatrix} X_{t_1} \\ \dots \\ X_{t_n} \end{bmatrix} = \begin{bmatrix} m(t_1) \\ \dots \\ m(t_n) \end{bmatrix}$$

$$Cov\begin{bmatrix} X_{t_1} \\ \dots \\ X_{t_n} \end{bmatrix} = \begin{bmatrix} k(t_1, t_1) & k(t_1, t_2) & \dots & k(t_1, t_n) \\ k(t_2, t_1) & k(t_2, t_2) & \dots & k(t_2, t_1) \\ \dots & \dots & \dots & \dots \\ k(t_{n-1}, t_1) & k(t_{n-1}, t_2) & \dots & k(t_{n-1}, t_n) \\ k(t_n, t_1) & k(t_n, t_2) & \dots & k(t_n, t_n) \end{bmatrix}$$

Definition of a Gaussian process

Definition

Let (X_t) be a Gaussian process. The mapping $t \to X_t \in L^2$ is continuous if the covariance kernel is continuous since

$$||X_s - X_t||^2 = k(s,s) + k(t,t) - 2\mathrm{Re}\{k(s,t)\}$$

The continuity of the process allows to compute in the Hilbert space L^2. Notably, if $\mu \in \mathcal{M}([a,b])$, we can define

$$Z = \int_a^b X_t d\mu(t) = \lim_{\Delta t \to 0} \sum_{k=0}^{\frac{b-1}{\Delta t}} X_{a+k\Delta t}\mu([a+k\Delta t, a+(k+1)\Delta t])$$

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Stationary processes

Definition

A stochastic process with $\mathcal{T} = \mathbb{R}$ is said stationary if any finite dimensional law is invariant by time shift, i.e.

$$\forall (t_1, \ldots, t_n, t) \in \mathbb{R}^{n+1}, (X_{t_1}, \ldots, X_{t_n}) \sim (X_{t_1-t}, \ldots, X_{t_n-t})$$

Proposition

A Gaussian process is stationary if and only if its mean function is constant and his covariance kernel is of the form

$$k(t, s) = c(t - s)$$

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Application: finite filtering by moving average (MAn)

Proposition

Consider a stochastic process (X_t) and a finite set of times and weights $((\alpha_1, t-1), \ldots, (\alpha_n, t_n)) \in (\mathbb{R} \times \mathcal{T})^n$. The stochastic process (Y_t) defined by $Y_t = X_t - \alpha_1 X_{t-t_1} \ldots \alpha X_{t-t_n}$ is called amoving average process of order n (MAn) The stochastic process (Y_t) is Gaussian -resp. continuous, stationary- as soon as (X_t) is Gaussian, - stationary, continuous.

It is easy to compute the mean function and the covariance kernel of (Y_t) (left to exercise)

Basic example: Brownian motion alias Wiener process

Proposition

The Wiener processt $t \in \mathbb{R}^+ \to W_t$ is the stationary independent increase Gaussian process with mean function $m(t) = 0$ and covariance kernel $k(s, t) = min(s, t)$

Proposition

The Wiener process is continuous (in quadratic mean sense)

More advanced theory shows that we can build a probability space where almost all the trajectories of the Wiener process are continuous

Exercises on the Wiener process as limit of discrete random walks

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Exercise

We consider the discrete time process with stationary independent increase $X(n\epsilon) \sim \mathcal{N}(0, n\epsilon^2)$. Show that when $\epsilon \to 0$ it converges towards the Wiener process.

Exercise

Let $(Y_N$ a sequence of independent identically distributed binary variables such that $P(\mathbb{Y}_n = 1) = 1 - P(Y_n = -1) = p$.
Let $X(n\epsilon) = \epsilon(Y_1 + \cdots + Y_n)$.
① If $p = 0.5$, show that its limit in law is the Wiener process.
② What about the general case $0 < p < 1$?

Definition of Gaussian spaces

Proposition

Let $\{X_n\}$ a sequence of Gaussian random variables that converges in L^2 towards X . Then X is Gaussian too.

Proof We have $\mathrm{E}(X_n) \to \mathrm{E}(X)$ and $Var(X_n) \to Var(X)$. So

$$\phi_X(u) = \lim\ \phi_{X_n}(u) = \lim\ \exp(j\mathrm{E}(X_n)u - \frac{Var(X_n)u^2}{2})$$

$$= \exp(j\mathrm{E}(X)u - \frac{Var(X)u^2}{2})$$

Definition

Let $t \in \mathcal{T} \to X_t \in L^2$ a Gaussian process, then the associate Gaussian space is the Hilbert sub-space $\mathcal{H}_X \subset L^2$ generated by the Gaussian variables X_t.

It’s clear that all the elements of $\mathcal{H}_X$ are Gaussian random variables.