中国民航大学
Civil Aviation University of China

Structural Applications of Finite Elements

Chapter 5
Three dimensional problems

2018-09-01

$$\mathbf{u} = [u, v, w]^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = [\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{yz}, \quad \tau_{xz}, \quad \tau_{xy}]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon} = [\epsilon_x, \quad \epsilon_y, \quad \epsilon_z, \quad \gamma_{yz}, \quad \gamma_{xz}, \quad \gamma_{xy}]^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

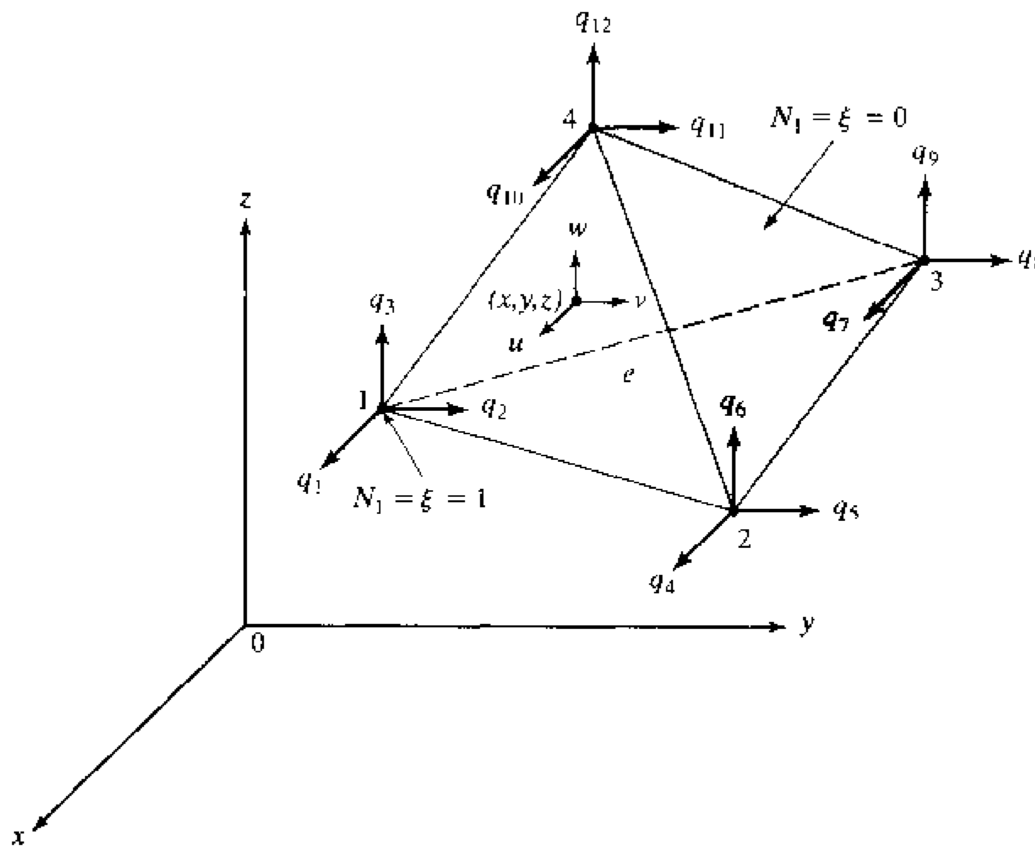
$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^{\mathrm{T}}$$

$$\mathbf{f} = [f_x, f_y, f_z]^{\mathrm{T}}$$

$$\mathbf{T} = [T_x, T_y, T_z]^{\mathrm{T}}$$

$$\mathbf{q} = [q_1, q_2, q_3, \ldots, q_{12}]^T$$
$$\mathbf{Q} = [Q_1, Q_2, Q_3, \ldots, Q_N]^T$$

	Nodes			
Element No.	1	2	3	4
e	I	J	K	L



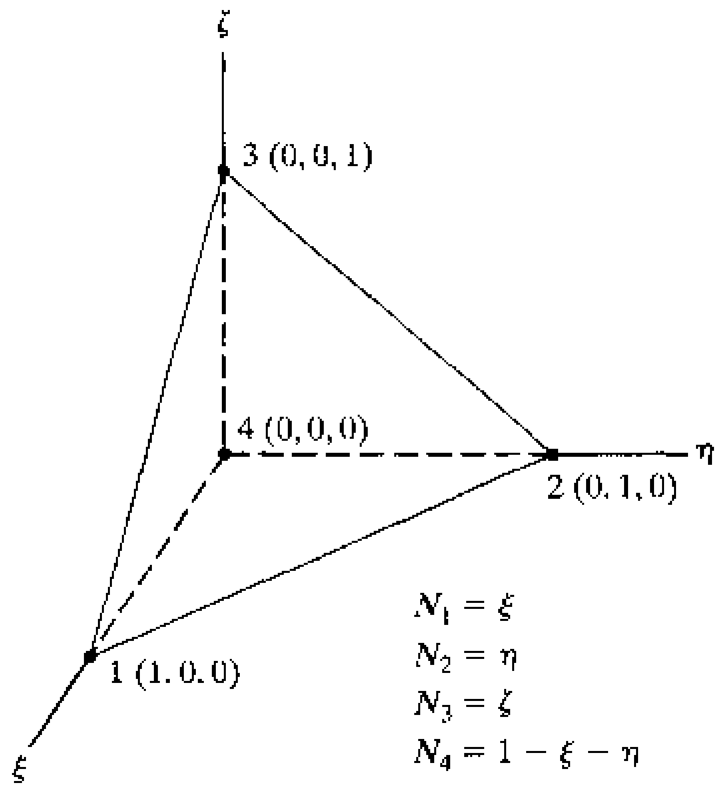
$$N_1 = \xi \quad N_2 = \eta \quad N_3 = \zeta \quad N_4 = 1 - \xi - \eta - \zeta$$

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$
$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4$$

$$x = x_4 + x_{14}\xi + x_{24}\eta + x_{34}\zeta$$
$$y = y_4 + y_{14}\xi + y_{24}\eta + y_{34}\zeta$$
$$z = z_4 + z_{14}\xi + z_{24}\eta + z_{34}\zeta$$



$$\left\{\begin{matrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial u}{\partial \zeta}\end{matrix}\right\} = \mathbf{J}\left\{\begin{matrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{matrix}\right\} \qquad \mathbf{J} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix} = \begin{bmatrix} x_{14} & y_{14} & z_{14}\\ x_{24} & y_{24} & z_{24}\\ x_{34} & y_{34} & z_{34}\end{bmatrix}$$

$$\det \mathbf{J} = x_{14}(y_{24}z_{34} - y_{34}z_{24}) + y_{14}(z_{24}x_{34} - z_{34}x_{24}) + z_{14}(x_{24}y_{34} - x_{34}y_{24})$$

$$V_e = \left| \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \det \mathbf{J}\, d\xi\, d\eta\, d\zeta \right| \qquad V_e = |\det \mathbf{J}| \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} d\xi\, d\eta\, d\zeta$$

$$V_e = \tfrac{1}{6}|\det \mathbf{J}|$$

$$\left\{\begin{matrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{matrix}\right\} = \mathbf{A}\left\{\begin{matrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial u}{\partial \zeta}\end{matrix}\right\} \qquad \mathbf{A} = \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{24}z_{34} - y_{34}z_{24} & y_{34}z_{14} - y_{14}z_{34} & y_{14}z_{24} - y_{24}z_{14}\\ z_{24}x_{34} - z_{34}x_{24} & z_{34}x_{14} - z_{14}x_{34} & z_{14}x_{24} - z_{24}x_{14}\\ x_{24}y_{34} - x_{34}y_{24} & x_{34}y_{14} - x_{14}y_{34} & x_{14}y_{24} - x_{24}y_{14}\end{bmatrix}$$

$$\boxed{\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}}$$

$$\mathbf{B} = \left[\begin{array}{cccccccccccc}
A_{11} & 0 & 0 & A_{12} & 0 & 0 & A_{13} & 0 & 0 & -\tilde{A}_1 & 0 & 0 \\
\hline
0 & A_{21} & 0 & 0 & A_{22} & 0 & 0 & A_{23} & 0 & 0 & -\tilde{A}_2 & 0 \\
\hline
0 & 0 & A_{31} & 0 & 0 & A_{32} & 0 & 0 & A_{33} & 0 & 0 & -\tilde{A}_3 \\
\hline
0 & A_{31} & A_{21} & 0 & A_{32} & A_{22} & 0 & A_{33} & A_{23} & 0 & -\tilde{A}_3 & -\tilde{A}_2 \\
\hline
A_{31} & 0 & A_{11} & A_{32} & 0 & A_{12} & A_{33} & 0 & A_{13} & -\tilde{A}_3 & 0 & -\tilde{A}_1 \\
\hline
A_{21} & A_{11} & 0 & A_{22} & A_{12} & 0 & A_{23} & A_{13} & 0 & -\tilde{A}_2 & -\tilde{A}_1 & 0
\end{array}\right]$$

$\tilde{A}_1 = A_{11} + A_{12} + A_{13}$, $\tilde{A}_2 = A_{21} + A_{22} + A_{23}$, and $\tilde{A}_3 = A_{31} + A_{32} + A_{33}$.

$$\begin{aligned}
U_e &= \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, dV \\
&= \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}\int_e dV \\
&= \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}V_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q} \\
&= \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q}
\end{aligned}$$

$$\mathbf{k}^e = V_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$$

$$\int_e \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}(\boldsymbol{\phi})\, dV = \boldsymbol{\psi}^{\mathrm{T}}V_e\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}$$

$$\int_e \mathbf{u}^T \mathbf{f}\, dV = \mathbf{q}^T \iiint \mathbf{N}^T \mathbf{f} \det \mathbf{J}\, d\xi\, d\eta\, d\zeta$$

$$= \mathbf{q}^T \mathbf{f}^e$$

$$\mathbf{f}^e = \frac{V_e}{4}[f_x, f_y, f_z, f_x, f_y, f_z, \ldots, f_z]^T$$

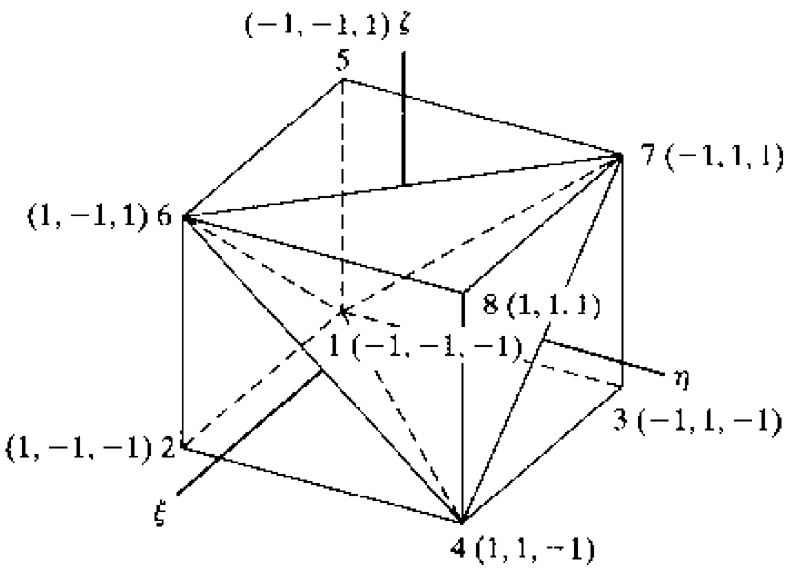
$$\mathbf{KQ} = \mathbf{F}$$

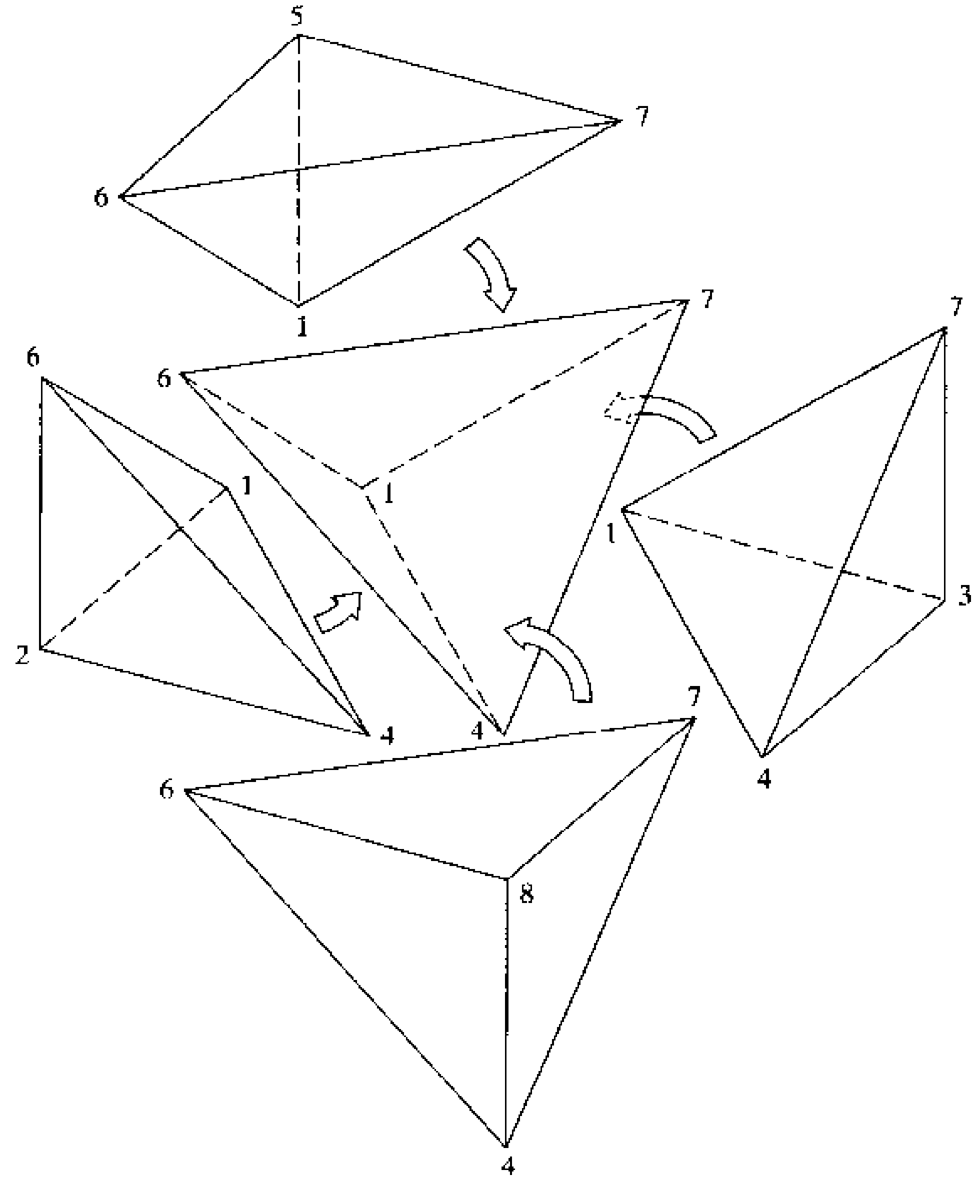
$$\int_{A_e} \mathbf{u}^T \mathbf{T}\, dA = \mathbf{q}^T \int_{A_e} \mathbf{N}^T \mathbf{T}\, dA = \mathbf{q}^T \mathbf{T}^e$$

$$\mathbf{T}^e = \frac{A_e}{3}[T_x, \quad T_y, \quad T_z, \quad T_x, \quad T_y, \quad T_z, \quad T_x, \quad T_y, \quad T_z, \quad 0, \quad 0, \quad 0]$$

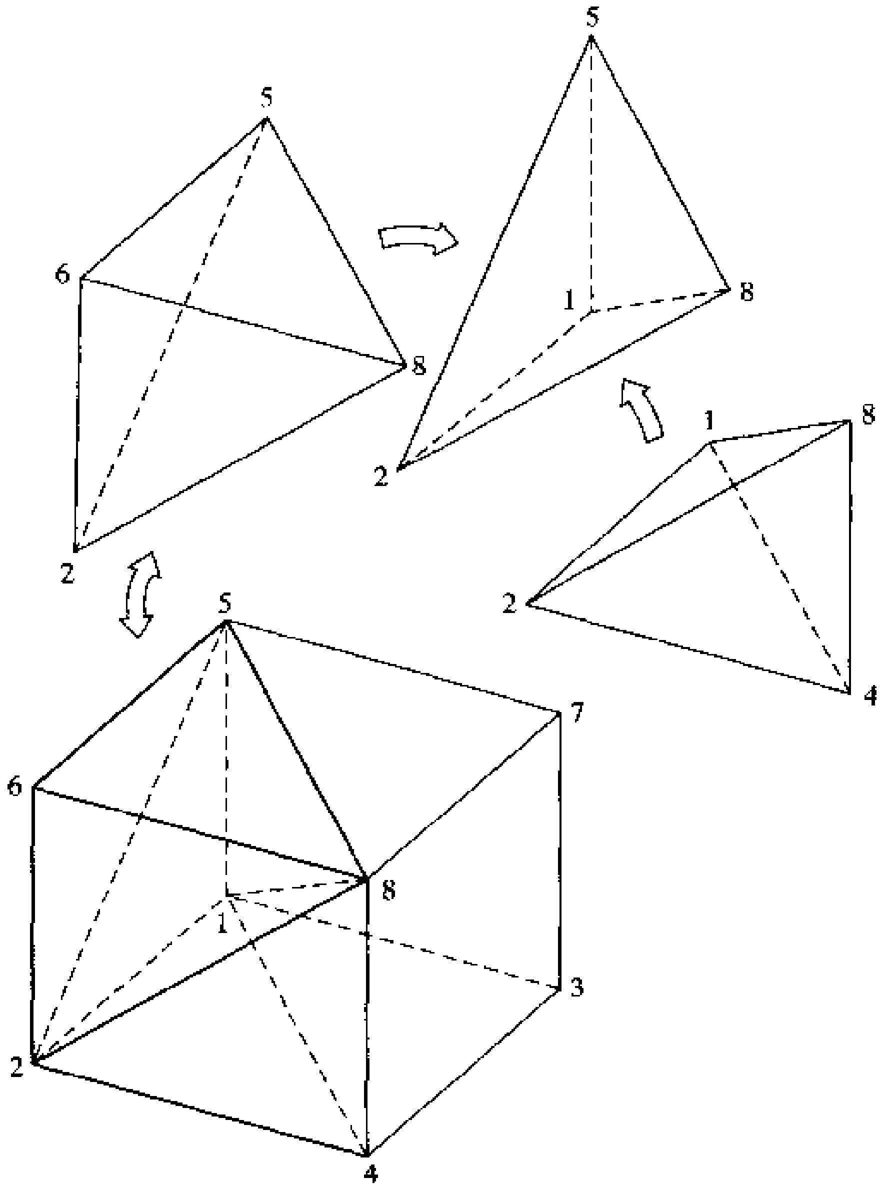
$$\boldsymbol{\sigma} = \mathbf{DBq}$$

Element No.	Nodes 1	2	3	4
1	1	4	2	6
2	1	4	3	7
3	6	7	5	1
4	6	7	8	4
5	1	4	6	7

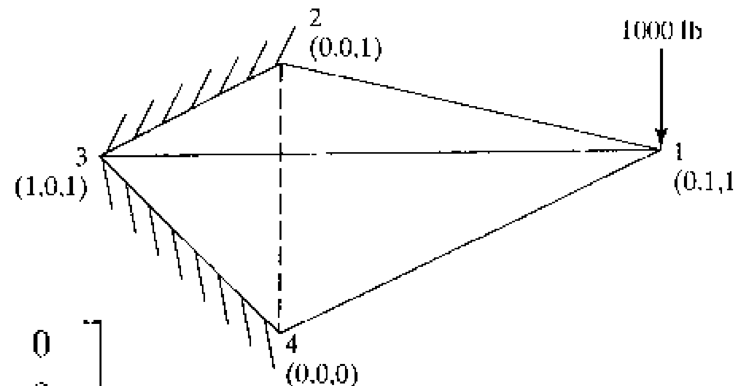




Element No.	Nodes 1	2	3	4
1	1	2	4	8
2	1	2	8	5
3	2	8	5	6
4	1	3	4	7
5	1	7	8	5
6	1	8	4	7



$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D} = 10^7 \begin{bmatrix} 4.038 & 1.731 & 1.731 & 0 & 0 & 0 \\ 1.731 & 4.038 & 1.731 & 0 & 0 & 0 \\ 1.731 & 1.731 & 4.038 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.154 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.154 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.154 \end{bmatrix}$$

$$\mathbf{K} = V_e \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 = 10^6 \begin{bmatrix} 1.923 & 0 & 0 \\ 0 & 6.731 & 0 \\ 0 & 0 & 1.923 \end{bmatrix}$$

$$N_i = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta) \qquad i = 1 \text{ to } 8$$

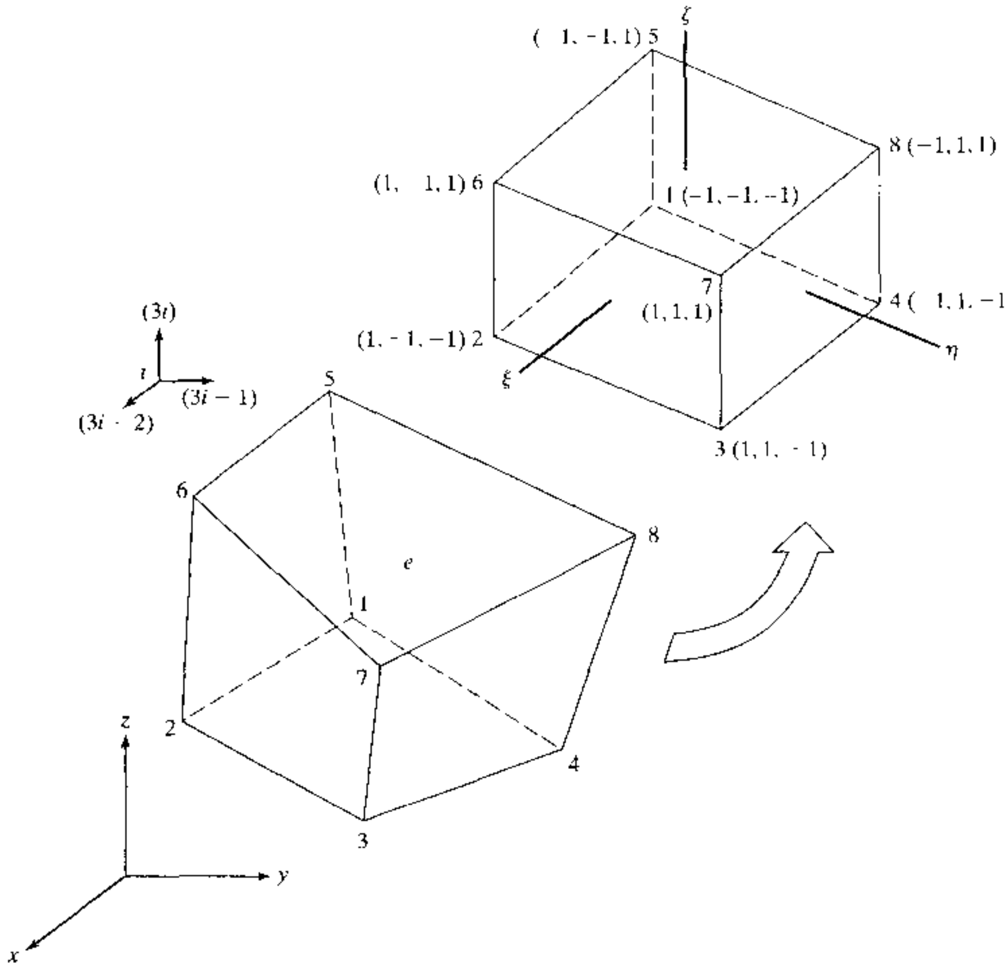
$$\mathbf{q} = [q_1, q_2, \ldots, q_{24}]^T$$

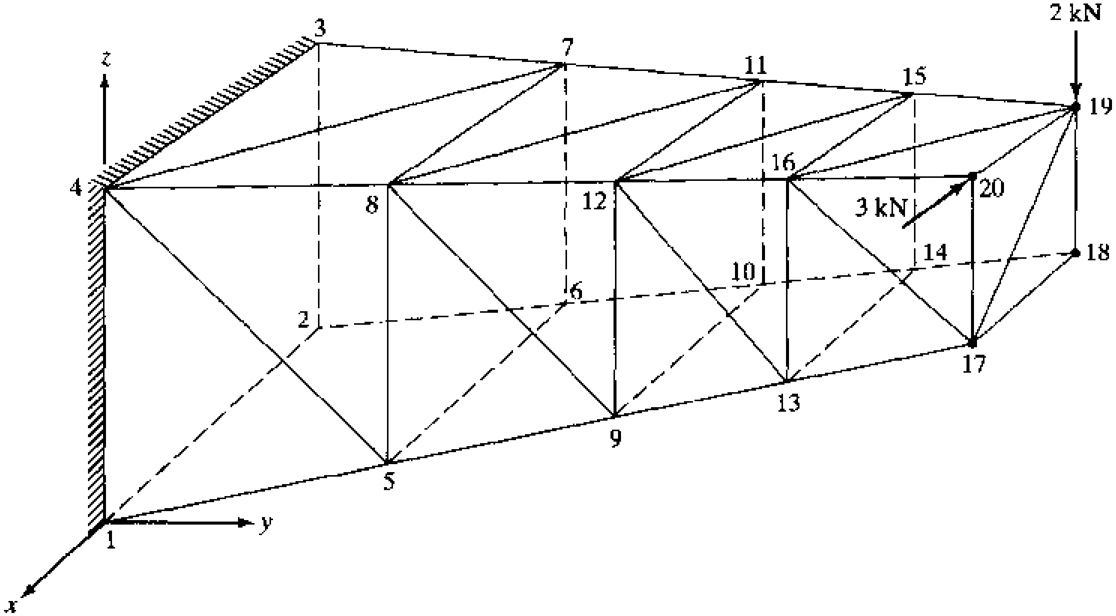
$$u = N_1 q_1 + N_2 q_4 + \cdots + N_8 q_{22}$$
$$v = N_1 q_2 + N_2 q_5 + \cdots + N_8 q_{23}$$
$$w = N_1 q_3 + N_2 q_6 + \cdots + N_8 q_{24}$$

$$x = N_1 x_1 + N_2 x_2 + \cdots + N_8 x_8$$
$$y = N_1 y_1 + N_2 y_2 + \cdots + N_8 y_8$$
$$z = N_1 z_1 + N_2 z_2 + \cdots + N_8 z_8$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$$

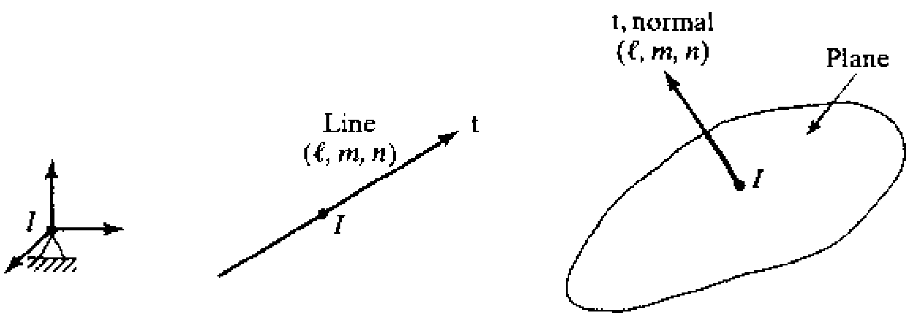
$$\mathbf{k}^e = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \mathbf{B}^T\mathbf{D}\mathbf{B}\,|\det \mathbf{J}|\, d\xi\, d\eta\, d\zeta$$

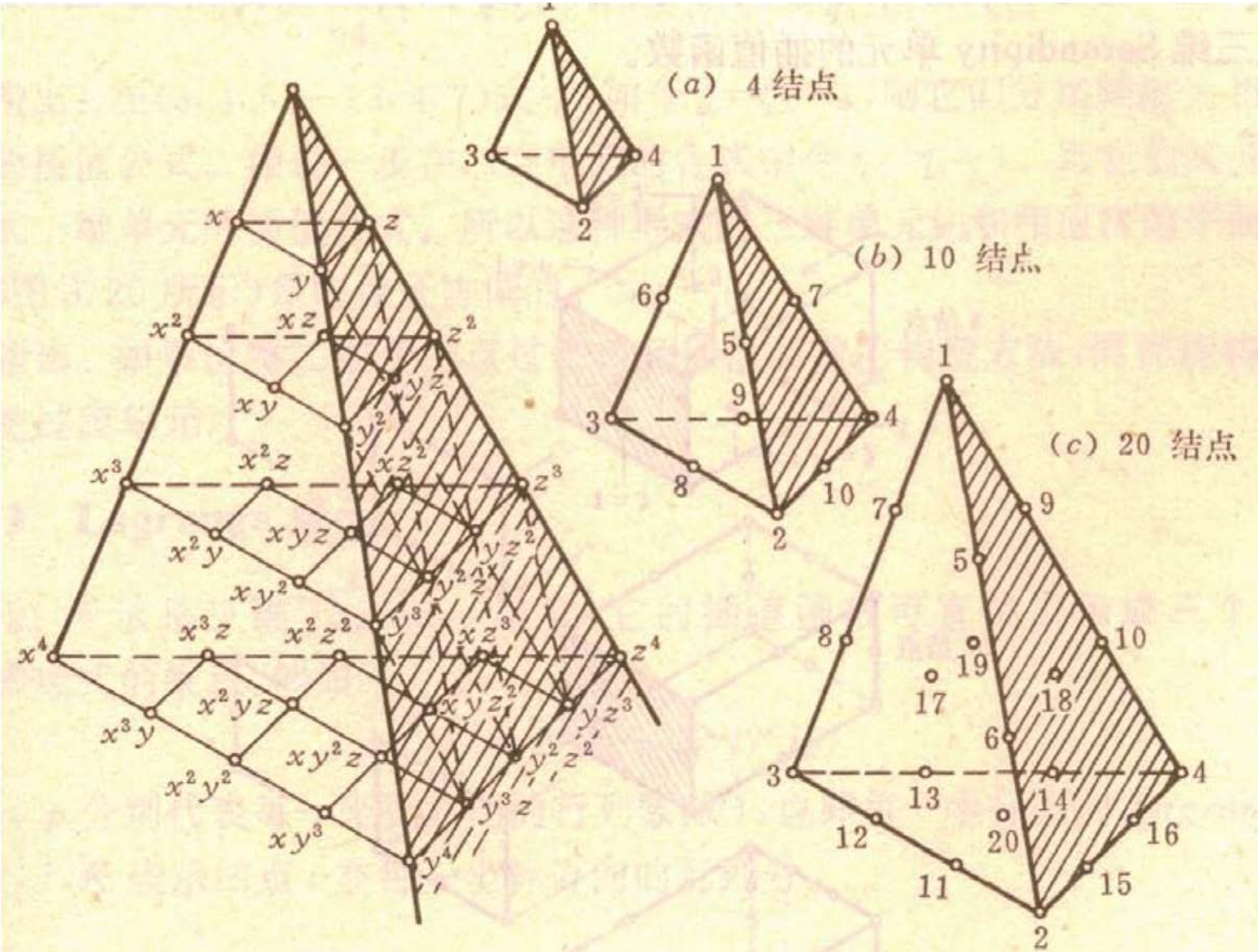




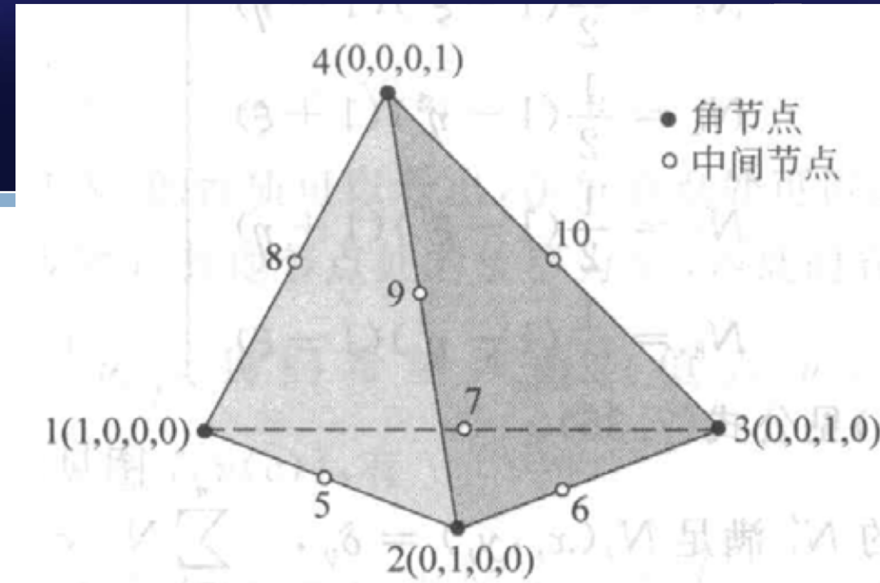
$$\begin{array}{c|ccc} & 3I-2 & 3I-1 & 3I \\ \hline 3I-2 & C(1-\ell^2) & -C\ell m & -C\ell n \\ 3I-1 & & C(1-m^2) & -Cmn \\ 3I & \text{Symmetric} & & C(1-n^2) \end{array}$$

$$\begin{array}{c|ccc} & 3I-2 & 3I-1 & 3I \\ \hline 3I-2 & C\ell^2 & C\ell m & C\ell n \\ 3I-1 & & Cm^2 & Cmn \\ 3I & \text{Symmetric} & & Cn^2 \end{array}$$




(a) 4 结点
(b) 10 结点
(c) 20 结点
x
z
y
x²
xz
z²
xy
yz
y²
x²z
xz²
x³
z³
xyz
yz²
x²y
y²z
xy²
y³
x²z²
xz³
x³z
x⁴
z⁴
x²yz
xyz²
x³y
yz³
xy²z
y²z²
x²y²
y³z
xy³
y⁴
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

$$\{\delta\}^e = \begin{bmatrix} u_1 & v_1 & w_1 & \cdots & u_{10} & v_{10} & w_{10} \end{bmatrix}^T$$



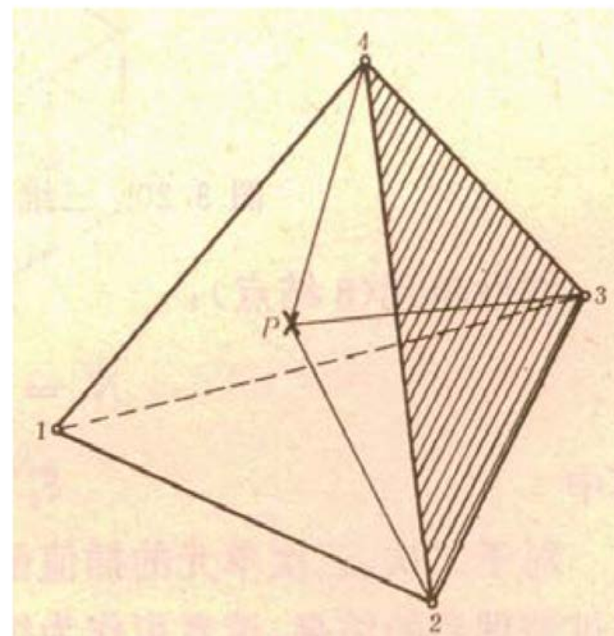
$$N_i = (2L_i - 1)L_i \qquad (i = 1, 2, 3, 4)$$

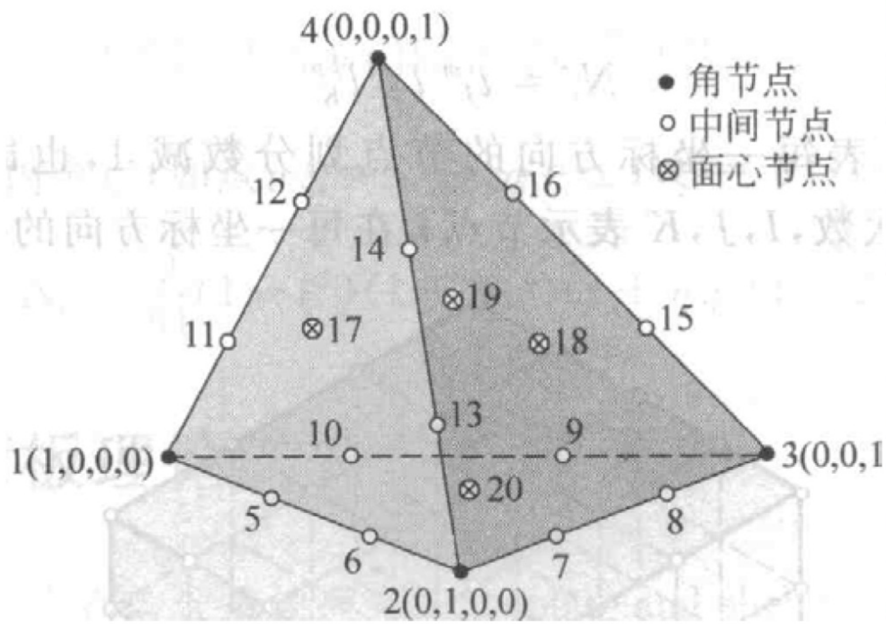
$$N_5 = 4L_1L_2 \qquad N_6 = 4L_3L_2$$

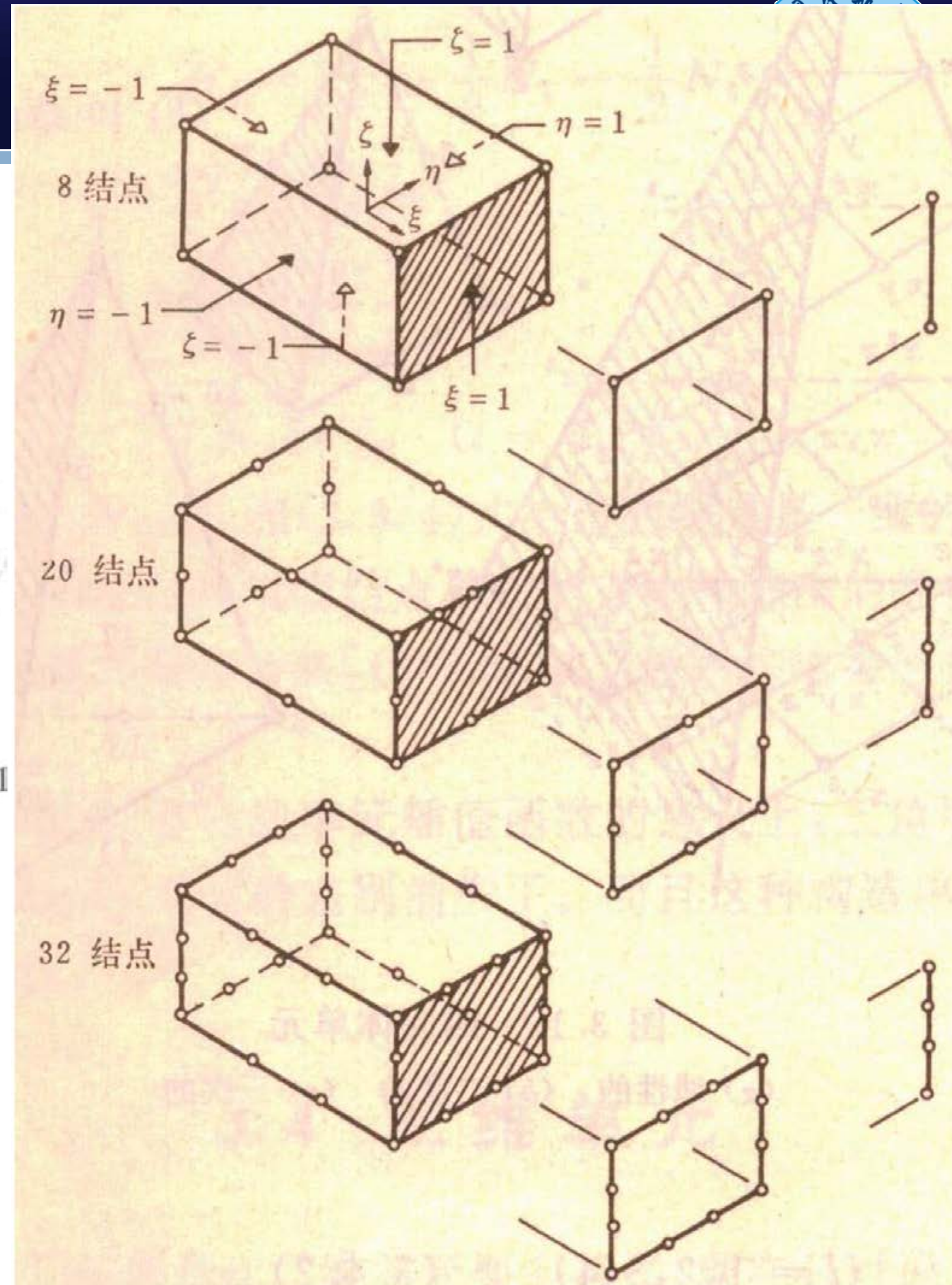
$$N_7 = 4L_1L_3 \qquad N_8 = 4L_1L_4$$

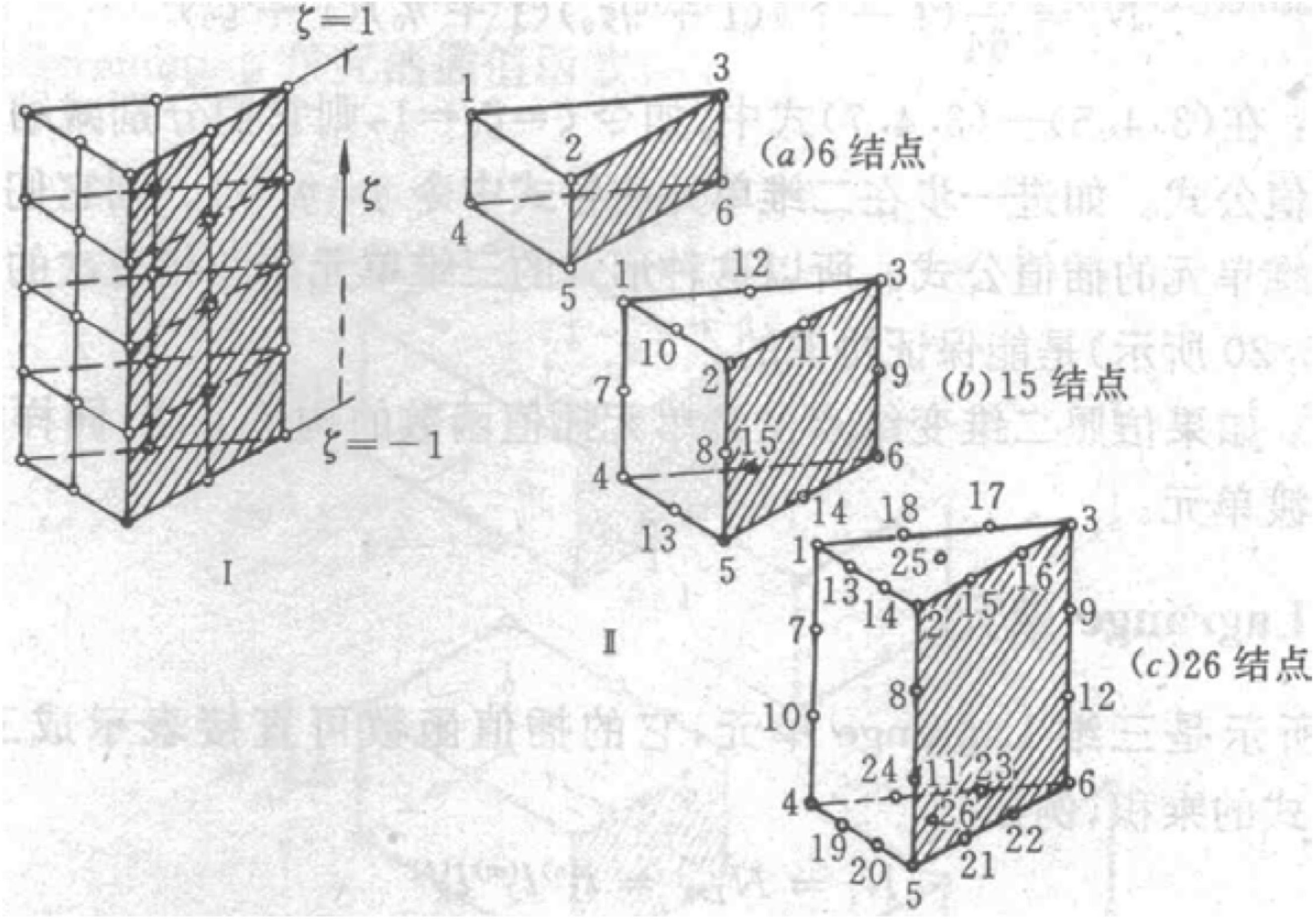
$$N_9 = 4L_4L_2 \qquad N_{10} = 4L_3L_4$$

$$L_1 = \frac{vol(P234)}{vol(1234)}, \quad L_2 = \frac{vol(P341)}{vol(1234)}, \quad L_3 = \frac{vol(P412)}{vol(1234)}, \quad L_4 = \frac{vol(P123)}{vol(1234)}$$




4(0,0,0,1)
1(1,0,0,0)
2(0,1,0,0)
3(0,0,1
● 角节点
○ 中间节点
⊗ 面心节点
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20


ξ = −1
ζ = 1
η = 1
η = −1
ζ = −1
ξ = 1
ζ
η
ξ
8 结点
20 结点
32 结点



ζ=1
ζ
ζ=−1
I
(a)6 结点
(b)15 结点
(c)26 结点
II