Conservation and transport equations, method of characteristics.

MA31-Numerical analysis of Partial derivative equations: Courses 05-06

Manuel Samuelides

Institut Supérieur de l'Aéronautique et de l'Espace
Toulouse, France

October 2013

Outline

Objective of the course

-

Outline

1. Introduction
2. Linear transport equation on the line
   - Propagation along characteristics
   - Euler explicit numerical schemes
3. Burger's equation

Problem of the transport equation on a line segment

Problem

Consider the problem with initial condition and boundary condition on $\mathbb{R} \times \mathbb{R}^+$ where the speed function c is a bounded smooth function and where the initial condition u_0 and the creation term f are continuous and bounded.

$$(1) \begin{cases} \frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = f \\ u(x,0) = u_0(x) \end{cases}$$

- In physics, u may be temperature (without diffusion) or chemical concentration in a moving fluid.
- Let us show there is a unique solution and how we can compute it.

Definition of characteristics

Definition

Consider the unique solution $x(t) = \Phi(t, x_0)$ *of the ODE with IC*

$$\begin{cases} \frac{dx}{dt} = c(x,t) \\ x(0) = x_0 \end{cases}$$

This curve is called the ***characteristic curve issued from*** x_0

Proposition

Let $(x,t) \to u(x,t)$ *be a solution of the system (1) and let* $t \to \xi(t)$ *be the characteristic curve issued from* (t_0, x_0)*. Define* $\psi(t) = u(\xi(t), t)$ *then* ψ *is the solution of the 1st order IC ODE:*

$$(1) \begin{cases} \frac{d\psi}{dt} = f(\xi(t), t) \\ \psi(0) = u_0(x_0) \end{cases}$$

Example: Linear PDE with constant coefficient

- Consider the problem (2) $\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$
- The characteristic curve issued from x_0 is $\xi(t) = ct + x_0$.
- So solution u of (2) checks $u(x,t) = u_0(x - ct)$
- It is said that the solution u is propagated along the characteristics.

Outline

Euler explicit scheme

Choose a space step δx, a time step δt and set $x_i = i\delta x$ and $t_n = n\delta t$. The Euler explicit scheme is given by

$$\begin{cases} \frac{u_{i,n+1} - u_{i,n}}{\delta t} + c_{i,n}\frac{u_{i+1,n} - u_{i,n}}{\delta x} = f_{i,n} \\ u_{i,0} = u_0(x_i) \end{cases}$$

It gives the following iteration:

$$u_{i,n+1} = u_{i,n} + \frac{\delta t}{\delta x} c_{i,n}(u_{i+1,n} - u_{i,n}) + \delta t\ f_{i,n}$$

$$u_{i,n+1} = u_{i,n}\left(1 + c_{i,n}\frac{\delta t}{\delta x}\right) u_{i,n} - c_{i,n}\frac{\delta t}{\delta x} u_{i-1,n} + \delta t\ f_{i,n}$$

Proposition

This scheme is consistent with 1st order in space and in time. It is never stable

Method of characteristics

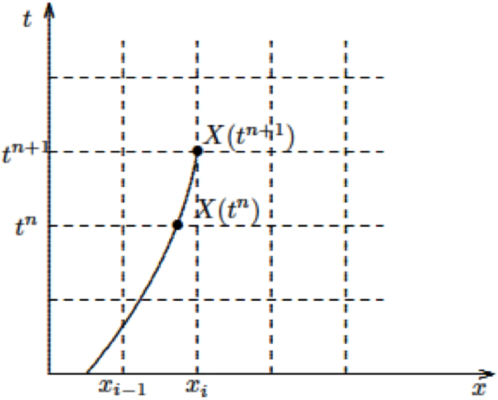

FIG. 5.1. Méthode des caractéristiques

2nd order decentered Euler explicit scheme

Let us propose the following decentred Euler scheme

$$\begin{cases} \frac{u_{i,n+1}-u_{i,n}}{\delta t} + c^+_{i,n}\frac{u_{i,n}-u_{i-1,n}}{\delta x} - c^-_{i,n}\frac{u_{i+1,n}-u_{i,n}}{\delta x} = f_{i,n} \\ u_{i,0} = u_0(x_i) \end{cases}$$

It gives the following iteration:

$$u_{i,n+1} = u_{i,n} - \frac{\delta t}{\delta x}[c^+_{i,n}(u_{i,n} - u_{i-1,n}) - c^-_{i,n}(u_{i+1,n} - u_{i,n})] + \delta t\ f_{i,n}$$

Proposition

This scheme is consistent with 1st order in space and in time.

It comes from the boundedness of u derivative.

Numerical diffusive perturbation

Proposition

The scheme can be viewed as a 2nd order approximation of

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \frac{\delta x \mid c \mid}{2}\frac{\partial^2 u}{\partial x^2} = f$$

Proof

$$\begin{aligned} u_{i,n+1} = \quad & u_{i,n} + \delta t\ f_{i,n} - \tfrac{\delta t}{2\delta x}(c^+_{i,n} - c^-_{i,n})(u_{i,n+1} - u_{i,n-1}) \\ & + \tfrac{\delta t}{2\delta x}(c^+_{i,n} + c^-_{i,n})(u_{i,n+1} + u_{i,n-1} - 2u_{i,n}) \end{aligned}$$

$$\begin{aligned} u_{i,n+1} = \quad & u_{i,n} + \delta t\ f_{i,n} + \tfrac{\delta t}{2\delta x}c^+_{i,n}(2u_{i,n-1} - 2u_{i,n}) \\ & + \tfrac{\delta t}{2\delta x}c^-_{i,n}(2u_{i,n+1} - 2u_{i,n}) \end{aligned}$$

$$u_{i,n+1} = u_{i,n} - \frac{\delta t}{\delta x}[c^+_{i,n}(u_{i,n} - u_{i-1,n}) - c^-_{i,n}(u_{i+1,n} - u_{i,n})] + \delta t\ f_{i,n}$$

Numerical diffusive perturbation

Remark

This is a small diffusive perturbation of the original equation. So the scheme can be viewed as a stabilization of Euler original explicit scheme by introducing a numerical diffusive perturbation.
Then the scheme is of second order.

Stability of the scheme

Definition

The following condition on the steps $\delta x > \sup \mid c(x,t) \mid \delta t$ means that the scheme is incremented faster than the solution physical propagation. It is called the Courant-Friedrich-Levy ***CFL condition***

Theorem

Under the CFL condition, we have

$$\sup_{i, 0 \leq i \leq N} \mid u_{i,n} \mid \leq \sup_x \mid u_0(x) \mid + N \delta t \sup_{x,t} \mid f(x,t) \mid$$

Proof of stability

$$u_{i,n+1} = u_{i,n} - \frac{\delta t}{\delta x}[c^{+}_{i,n}(u_{i,n} - u_{i-1,n}) - c^{-}_{i,n}(u_{i+1,n} - u_{i,n})] + \delta t\ f(x_i, t_n)$$
$$u_{i,n+1} \leq [1 - \frac{\delta t}{\delta x} \mid c_{i,n} \mid] \sup U_{i,n} + \frac{\delta t}{\delta x} \mid c_{i,n} \mid] \sup U_{i,n} + \delta t \sup_{x,t} \mid f(x, t$$
$$u_{i,n+1} \leq \sup u_{i,n} + \delta t \sup_{x,t} \mid f(x,t) \mid$$
$$\sup_i u_{i,n+1} \leq \sup u_{i,0} + n\delta t \sup_{x,t} \mid f(x,t) \mid$$

Remark

The inequality shows the conservation of u taking into account the production terms $f(x,t)$.
Moreover, if u_0 and f are non negative, the computed solution is non-negative. So we prove the convergence of the scheme under the CFL condition.

Method of characteristics

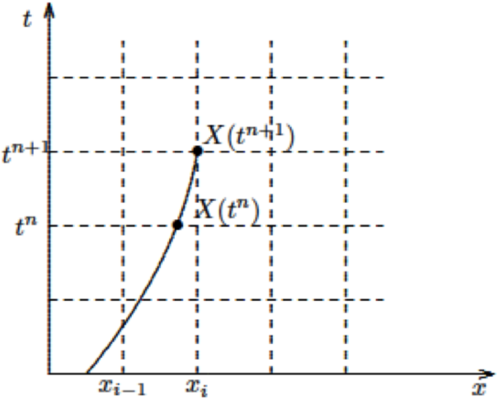

FIG. 5.1. Méthode des caractéristiques

Quasi-linear system

$$\begin{cases} \frac{\partial u}{\partial t} + c(u).\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Burger's equation

Burger's equation is a quasi-linear transport equation:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + u.\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{array} \right.$$

For Further Reading I

- Stanford university
  First order equations: Method of characteristics .
  CME 204: Partial Differential Equations in Engineering (ME 300B) (free internet available).

- Rachid Touzani.
  Méthodes numériques pour les équations aux dérivées partielles .
  http://math.univ-bpclermont.fr/ touzani/teaching.html (free internet available).

- Lawrence C.Evans.
  Partial Differential equations .
  Graduate Studies in Mathematics (19), American Mathematical Society, 2010.

For Further Reading II

Nelly Point and Jacques-Hervé Saiac .
Equations aux dérivées partielles .
CNAM, ESPCI, GM2, 2005-2006 (free internet available).