Discrete Mechanical Vibrations SM32

Sino-European Institute of Aviation Engineering

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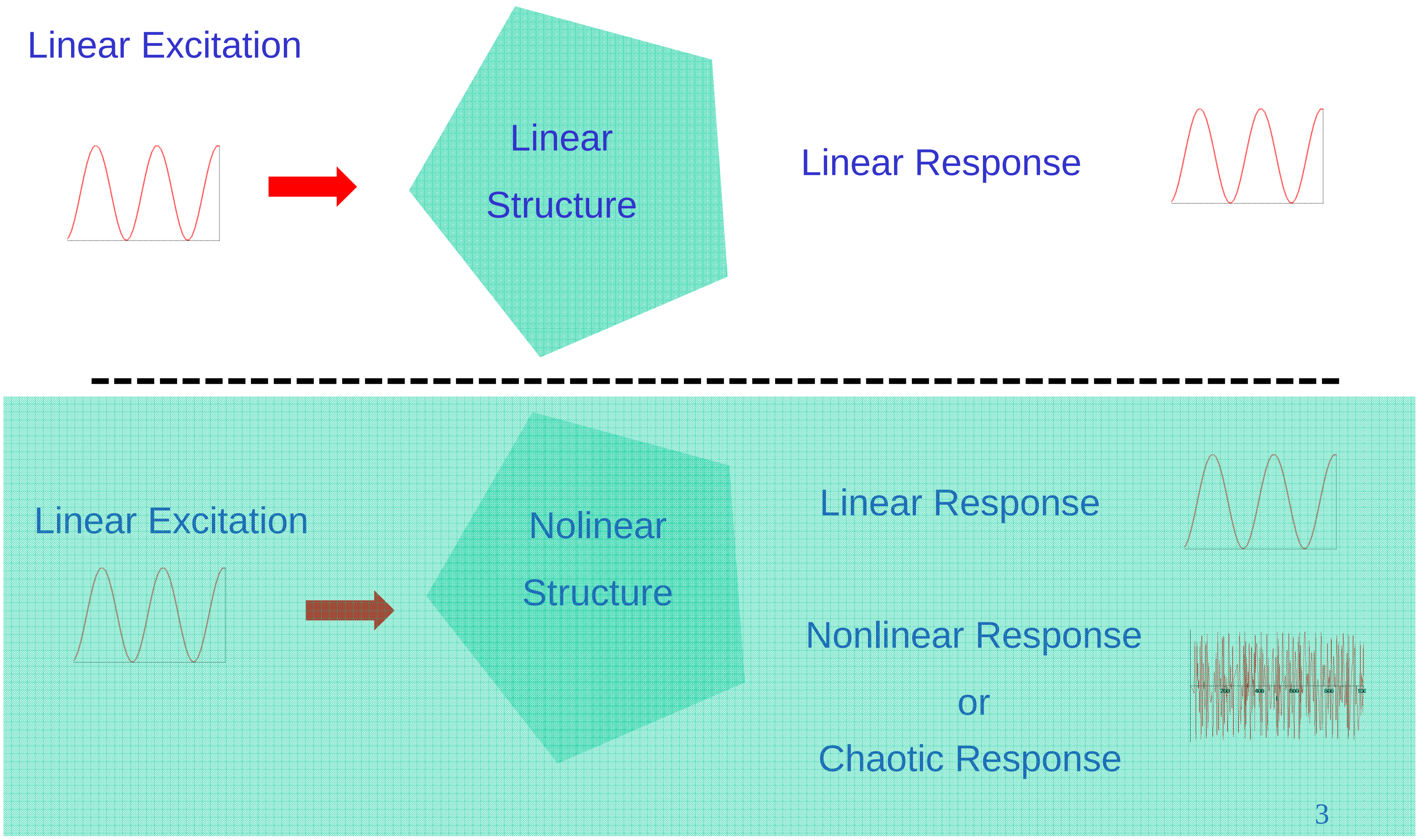
Dynamic Vibration can be defined as the study of the repetitive motion of structure about equilibrium positions.

Typical examples:

- Guitar string or cables
- Pendulum in gravity field
- Motorcycle , cars, …
- Airplane’s wings
- …

Linear vs Nonlinear



Prediction consists in several steps:

Definition of the

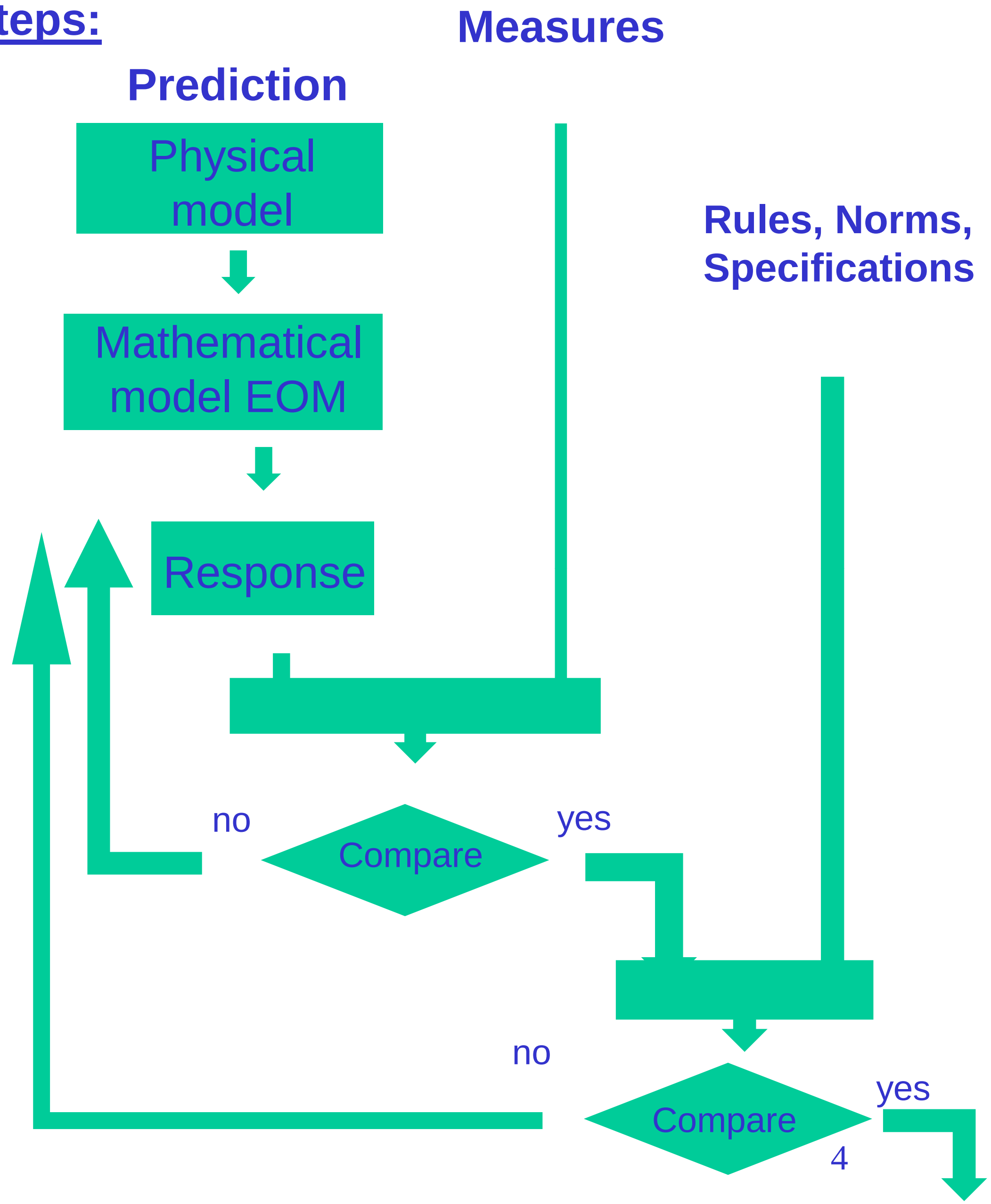
Mathematical Model

Derivation of the

Equation Of Motion

Studying

Response to Excitation,



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Single Degree of Freedom Systems

Free Conservative mass-spring system (undamped)

- Derivation of Equations of Motion by enery methods
- Non-conservative single degree of freedom model
- Vibration with dry friction damping (Bilinear)

Forced Harmonic Vibration

- Undamped system - Response and beating phenomenon)
- Damped System - Determination of damping with Half-Power Bandwith

Periodic Excitations

Energy dissipation per cycle - Damping in real systems

Applications

- System on a moving foundation
- Transmissibility
- Unbalanced Machine on a fixed foundation

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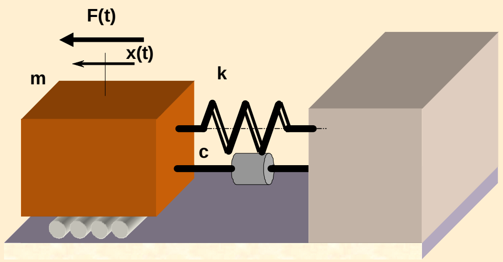
Single Degree of Freedom Systems

The study of single degree-of-freedom systems serves as a good introduction to basic phenomena of linear mechanical vibrations of structures.

It is also a good introduction for the presentation of several termes such as:



Resonance phenomena

Natural Frequency (angular)

Linear damping

Material damping (structural)

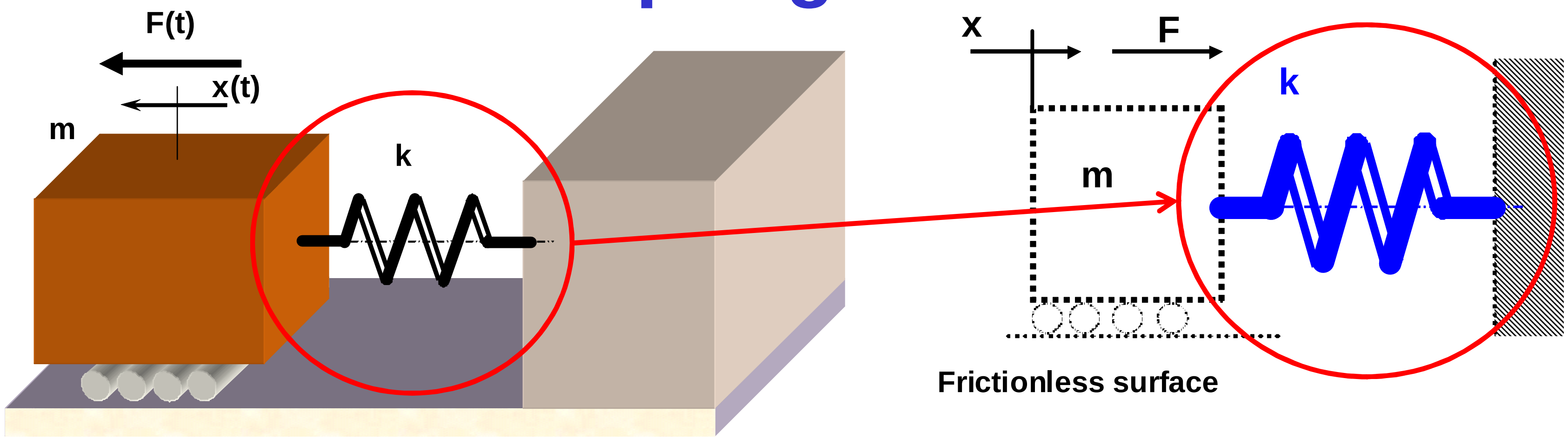
Damping ratio (factor)

Vibration Isolation

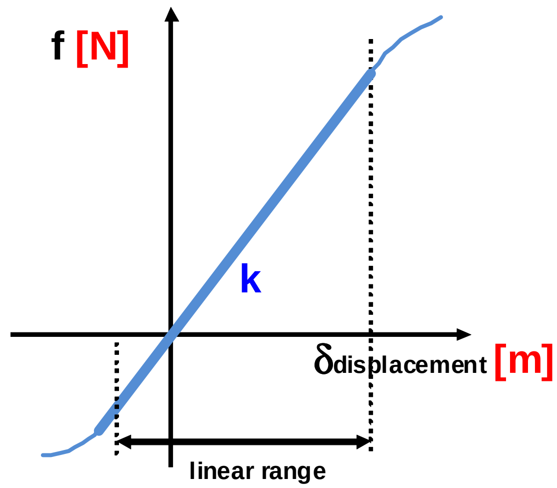
Shock effects

…

Spring Element

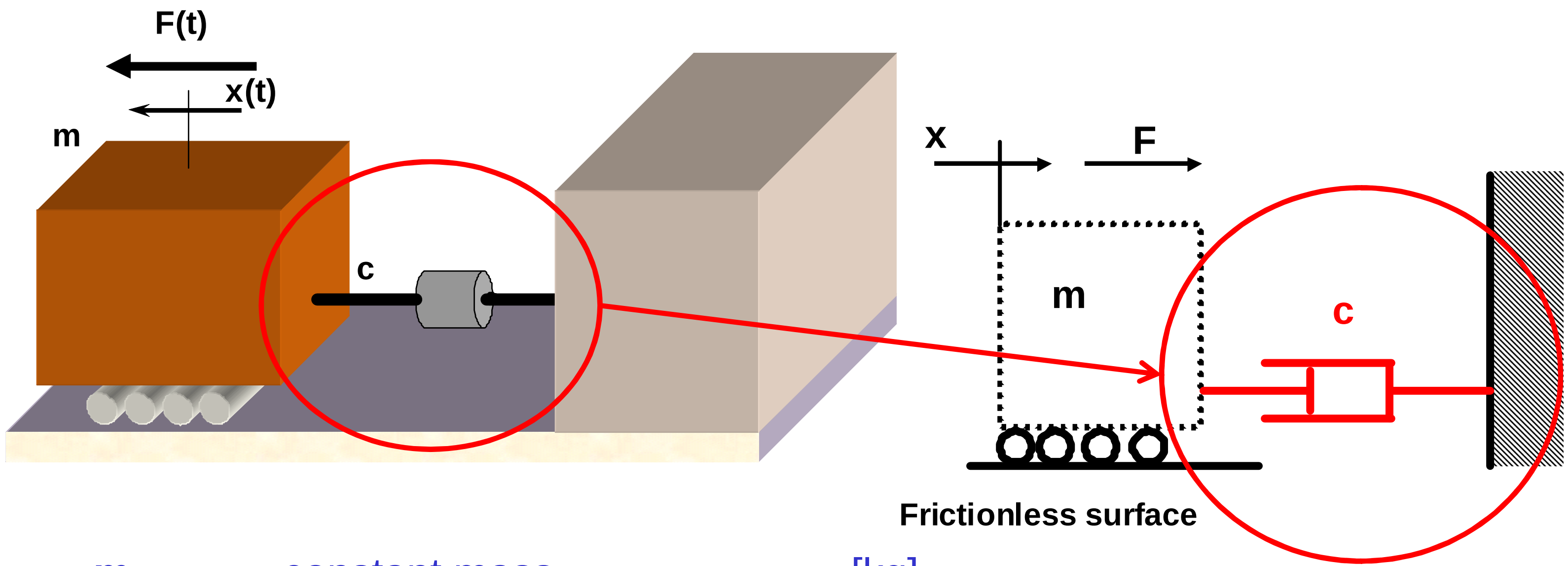


m	constant mass	[kg]
k	spring	[N/m]
F	external forcing sollicitation	[N]
x(t)	displacement	[m]

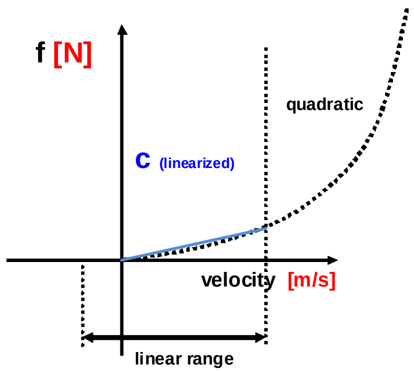


Linear behavior characteristic

Damper Element



m	constant mass	[kg]
c	spring	[Ns/m]
F	external forcing sollicitation	[N]
x(t)	displacement	[m]



Linear behavior characteristic

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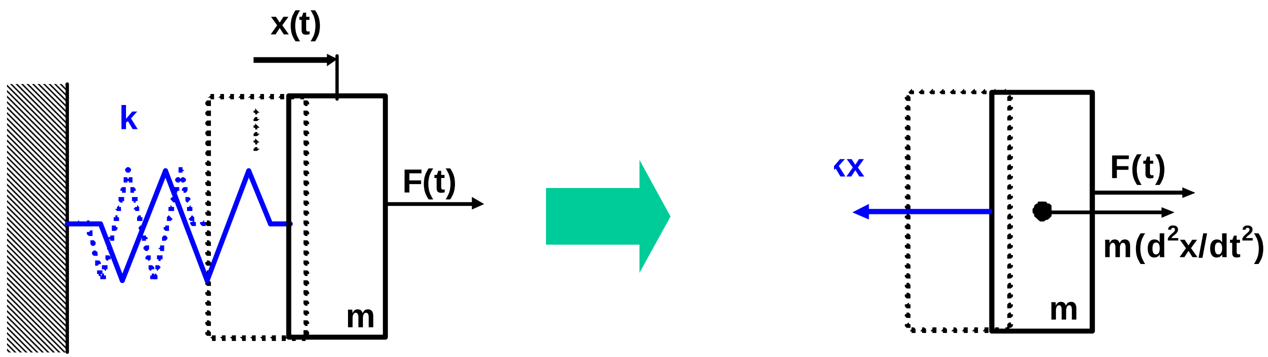
Free conservative mass-Spring System (undamped system):

From second Newton's law :



Inertia Force: $m\dfrac{d^2x(t)}{dt^2}$

Restoring Force:

spring (related to displacement) $-kx(t)$

External Force: (forcing term) $F(t)$

Free vibrations: F(t) = 0

$$m\ddot{x}+kx=0$$

Forced vibrations:

$$m\ddot{x}+kx=F(t)$$

Free Vibrations of the mass-spring system:

$$\boxed{\begin{gathered} m\frac{d^2x}{dt^2} + kx = 0 \\ m\ddot{x} + kx = 0 \end{gathered}}$$

Linear Ordinary Differential Equation with constant coefficients without second member (=> homogeneous).

Solutions can be sought as:

$$x = A\cos\omega t + B\sin\omega t$$

Where A and B are constants which are found by considering the initial conditions and:

ω is the **circular (or angular) frequency** of the motion [**rd/s**]

Substituting solution in EOM leeds to:

$$\boxed{-\omega^2\left(A\cos\omega t + B\sin\omega t\right) + \frac{k}{m}\left(A\cos\omega t + B\sin\omega t\right) = 0}$$

Free Vibrations of the mass-spring system:

$$\boxed{m\frac{d^2x}{dt^2}+kx=0}$$

Other possible forms are:

$$x(t)=A\cos\omega t+B\sin\omega t$$

$$x(t)=a\cos(\omega t+\varphi)$$

$$\begin{aligned}x(t)&=\frac{a}{2}e^{i\varphi}e^{it}+\frac{a}{2}e^{-i\varphi}e^{-it}\\&=A(t)e^{it}+\bar{A}(t)e^{-it}\\&=A(t)e^{iT_0}+[c.c.]\end{aligned}$$

[c.c.] complex conjugate of previous part

Assuming $(A\cos \omega t + B\sin \omega t) \neq 0$

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

and

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$$

For initial conditions such as $x = x_0$ at $t = 0$

$$x_0 = A\cos 0 + B\sin 0$$

therefore $x_0=A$

Then $x^{\circ}=0$ at $t=0$ thus

$$0 = -A\sqrt{\frac{k}{m}}\sin 0 + B\sqrt{\frac{k}{m}}\cos 0$$

therefore $B=0$

Finally:

$$\boxed{x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right)}$$

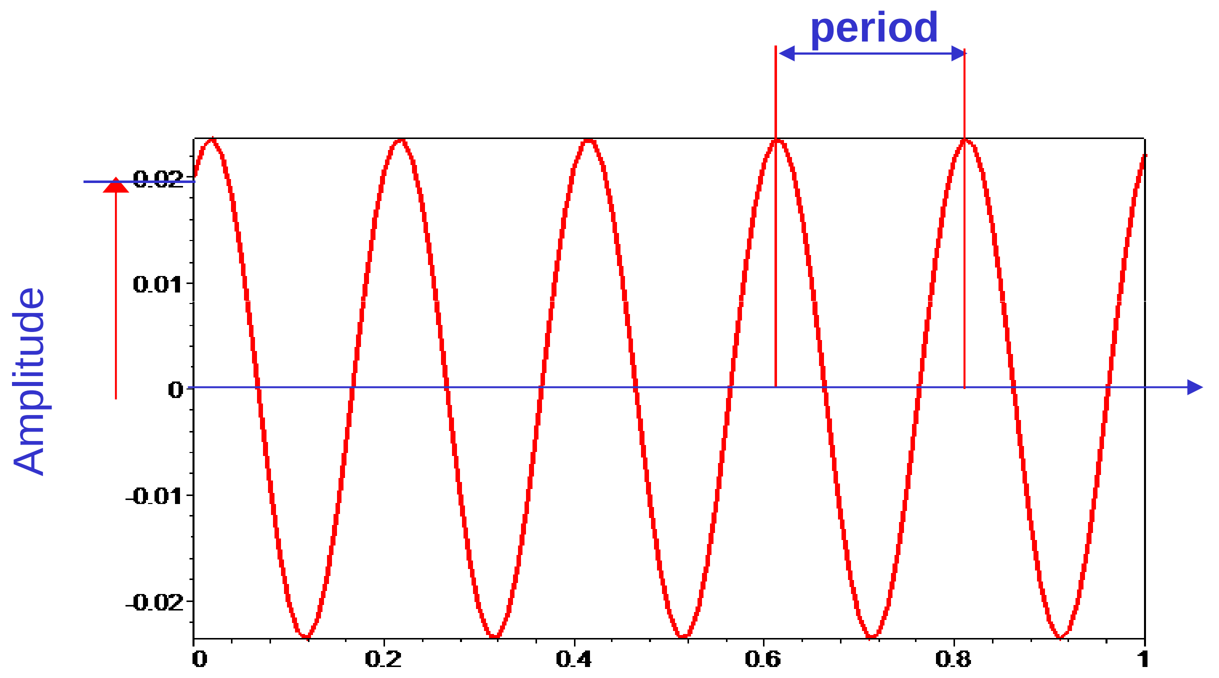
Conclusion:

The dynamic behavior of the system is controlled by :

ω which is related to **mechanical parameters**

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \quad \text{[rd/s]}$$

x_0 which is related to **initial conditions**



$$\boxed{f = \frac{\omega}{2\pi}}$$

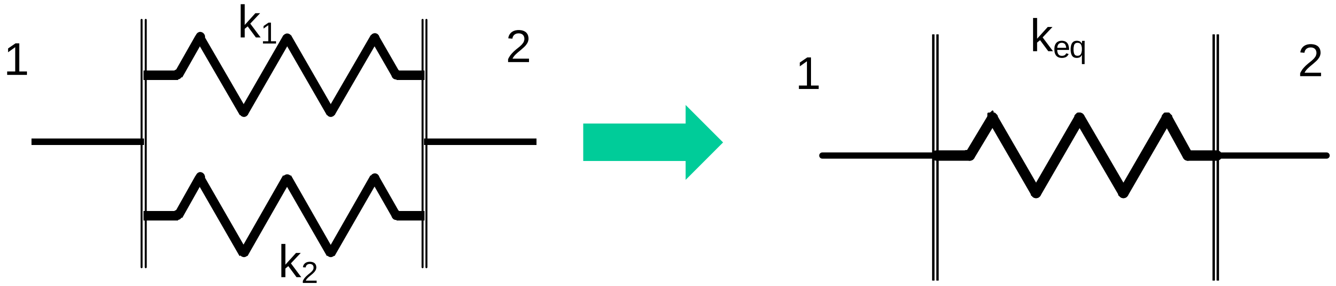
frequency of vibration [Hz]

$$\boxed{T = \frac{1}{f}}$$

period of vibration [s]

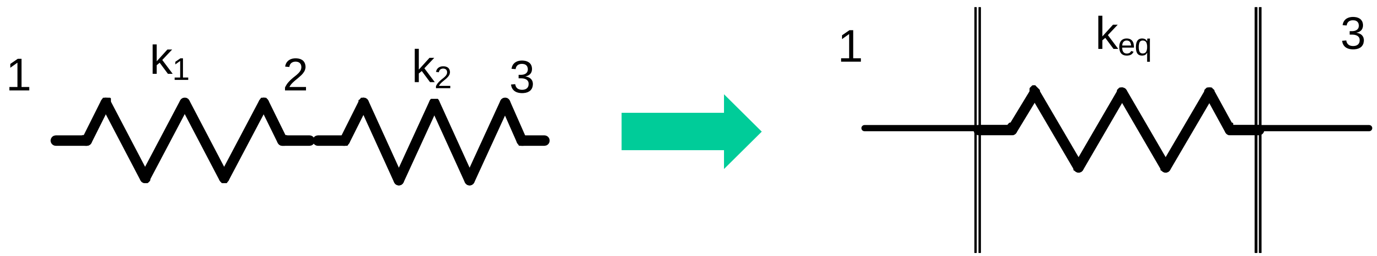
Practical configurations: (see TD)

Springs connected in series



$$k_{eq} = k_1 + k_2$$

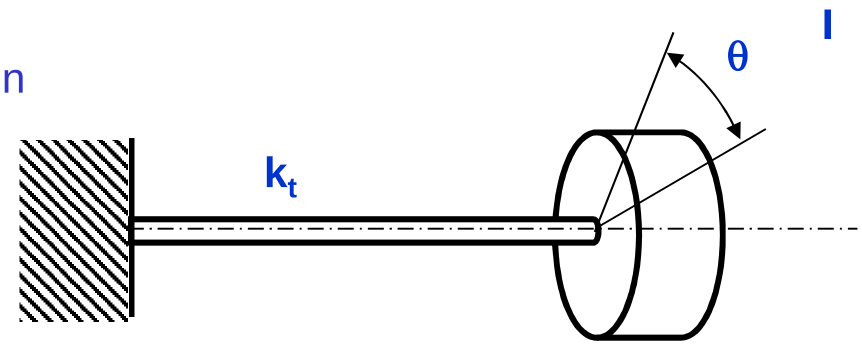
Springs connected in parallel (compound)



$$\frac{1}{k_{éq.}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Practical configuations:

Torsional vibration



$$\boxed{I\frac{d^2\theta}{dt^2}+k_t\theta=0}$$ [N.m]

$$\boxed{\frac{d^2\theta}{dt^2}+\omega^2\theta=0}$$

θ	deviation angle	[rd]
I	rotational mass moment of ineria	[kg.m^2]
k_t	torsion stiffness of the rod	[N.m/rd]
G	Coulomb modulus	[N/m^2]
J	polar inertia about axis (2nd moment of area)	[m^4]
ℓ	length of the shaft	

for a circular section shaft $k_t=\frac{GJ}{\ell}$

Hence

$$\boxed{\omega=\sqrt{\frac{GJ}{I\ell}}}$$

Derivation of equations of motion by energy method

Energy method

The ***kinetic Energy*** **T** is stored in the mass by its velocity and the ***potential energy*** **U** is stored in the form of strain energy in elastic deformation of the spring. For a conservative system, the total energy is constant.

$$T + U = \text{constant}$$

the rate of change is zero:

$$\boxed{\frac{d}{dt}\left(T + U\right) = 0}$$

For a conservative system, the total energy is constant.

The method resumes as follows, from a reasonable hypothesis about the motion of the system,

- calculate the approximate kinetic and strain energies,
- use the theorem of conservation of mechanical energy.

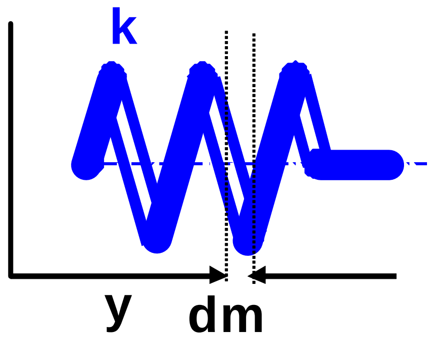
Application to Structure with heavy spring

An important mass of the spring (with respect to the mass of the system) can have a significant effect on the frequency of the vibration of the structure.

- Hypothesis about the motion of the system:

'Linear deformation of the whole spring'

The velocity of a specific point (at coordinate y) is:



$$\frac{dy}{dt} = \frac{y}{\ell}\frac{dx}{dt}$$

with ℓ and m_{spring} are the length and the mass of the spring respectively.

- Kinetic energy

Considering the basic mass-spring system it is possible to write for T the total kinetic energy:

$$T = T_{mass} + T_{spring} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\int_0^\ell \left(\frac{dy}{dt}\right)^2 dm_{spring}$$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\int_0^\ell \left(\frac{y}{\ell}\frac{dx}{dt}\right)^2 \frac{m_{spring}}{\ell}dy$$

$$= \frac{1}{2}\left[m + \frac{m_{spring}}{3}\right]\left(\frac{dx}{dt}\right)^2$$

Application to Structure with heavy spring

- Strain energy

$$U = \frac{1}{2}kx^2$$

- Conservation of mechanical energy

$$\frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2}\left(m + \frac{m_{spring}}{3}\right)\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2\right)$$

$$= 2\frac{1}{2}\left(m + \frac{m_{spring}}{3}\right)\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + 2\frac{1}{2}kx\left(\frac{dx}{dt}\right) = 0$$

Equation of motion

$$\left(m + \frac{m_{spring}}{3}\right)\frac{d^2x}{dt^2} + kx = 0$$

$$\boxed{\omega = \sqrt{\frac{k}{m + \frac{m_{spring}}{3}}}}$$

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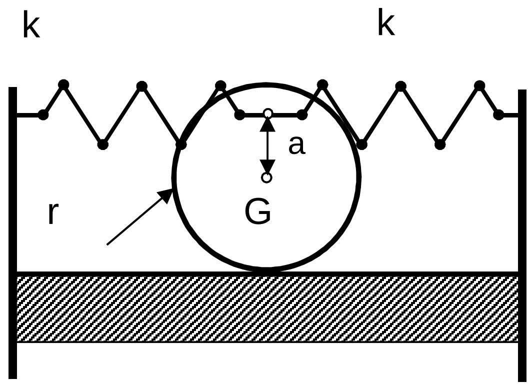
Application to rolling cylinder

A uniform cylinder with mass **m** and of rotational mass moment of ineria with respect its center I_G is rolling without slipping on a surface



Two springs of **k** characteristics are linked to that cylinder. The kinetic energy is composed of the energy of the cylinder during is motion of translation and of rotation around its center of mass.

$$T = T_{trans.} + T_{rot.} = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} I_c \left(\frac{d\theta}{dt} \right)^2$$

where I_G and m are the mass inertia and the mass of the mas of the cylinder respectively.

The angle is choosen as the parameter and assuming small rotations (***Hypothesis of*** Kinematic conditions), it leads to:

$$x = r\,\theta$$

Application to rolling cylinder

$$\dot{x} = r\,\dot{\theta} \qquad I_G = \frac{1}{2}mr^2 \qquad \Rightarrow \qquad T = \frac{3}{4}mr^2\left(\frac{d\theta}{dt}\right)^2$$

The strain energy is due to ,the deformation of the two springs, so that:

$$U = 2\,\frac{1}{2}k(\delta x)^2 = k(a+r)^2\,\theta^2$$

- ***Conservation of mechanical energy***

$$\frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{3}{4}mr^2\left(\frac{d\theta}{dt}\right)^2 + k(a+r)^2\,\theta^2\right)$$

$$= 2\frac{3}{4}mr^2\dot{\theta}\ddot{\theta} + 2k(a+r)^2\,\theta\dot{\theta} = 0$$

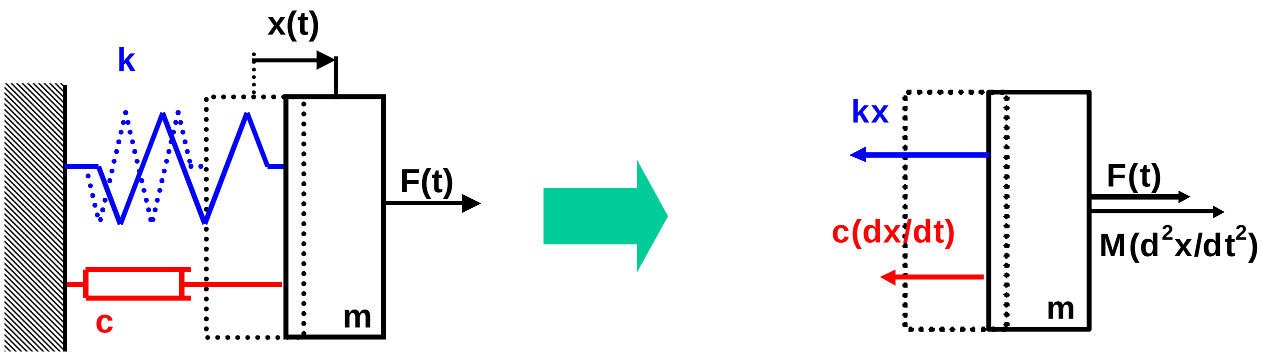
which is of the form :

$$\ddot{\theta} + \frac{4}{3}\frac{k(a+r)^2}{mr^2}\theta = 0$$

Hence the frequency of vibration is:

$$\boxed{\omega = \sqrt{\frac{4}{3}\frac{k(a+r)^2}{mr^2}}}$$

Non-conservative single degree of freedom model



Inertia Force: $$m\frac{d^2x(t)}{dt^2}$$

Contact Force:

spring (related to displacement) $$-kx(t)$$

damper (related to velocity) $$-c\frac{dx(t)}{dt}$$

External Force: (forcing term) $$F(t)$$

Free vibrations : F(t) = 0

$$m\ddot{x} + c\dot{x} + kx = 0$$

Forced vibrations :

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Non-conservative single degree of freedom model

Equation of motion

Second-order differential equation

$$\boxed{\begin{gathered} m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0 \\ m\ddot{x}+c\dot{x}+kx=0 \end{gathered}}$$

Assuming solution of the form:

$$x = Ae^{rt}$$

$$mr^2Ae^{rt}+crAe^{rt}+kAe^{rt}=0$$

Since $A \neq 0$ (otherwise no motion)

$$mr^2+cr+k=0$$

The roots are:

$$r_{1,2}=\frac{1}{2}\left[-\frac{c}{m}\pm\sqrt{(\frac{c}{m})^2-\frac{4k}{m}}\right]$$

Non-conservative single degree of freedom model

The dynamics of the system depends on the value of the radical

$$\left(\frac{c_c}{m}\right)^2 - \frac{4k}{m}$$

so this defines three types of behavior:

By definition :

The critical damping is define as the value wich makes the radical to be zero:

$$\left(\frac{c_c}{m}\right)^2 - \frac{4k}{m} = 0 \quad \Rightarrow \quad c_c = 2\sqrt{km}$$

Current or actual damping is related to c_c by:

$$\boxed{\alpha = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{\text{current damping}}{\text{critical damping}}}$$

Non-conservative single degree of freedom model

Equation of motion (other forms)

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega^2 x = 0$$

$$\alpha = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega}$$

$$\boxed{\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = 0}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \qquad \omega = 2\pi f \qquad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Cas I: (Oscillatory motion)

α **< 1** This means that the damping is less than critical value, this the most interesting case in vibrations applications.

$$r_{1,2} = -\alpha\omega \pm j\omega\sqrt{1-\alpha^2}$$

$$\omega_a = \omega\sqrt{1-\alpha^2}$$

ω_a is the frequency oscillation reduced by the damping. By substituting $r_{1,2}$ in EOM, the general solution becomes :

$$\begin{aligned} x &= A_1 e^{r_1 t} + A_2 e^{r_2 t} \\ &= A_1 e^{\left(-\alpha\omega + j\omega\sqrt{1-\alpha^2}\right)t} + A_2 e^{\left(-\alpha\omega - j\omega\sqrt{1-\alpha^2}\right)t} \\ &= e^{-\alpha\omega t}\left(A_1 e^{\left(j\omega\sqrt{1-\alpha^2}\right)t} + A_2 e^{\left(-j\omega\sqrt{1-\alpha^2}\right)t}\right) \end{aligned}$$

The following forms may be more suitable:

$$\boxed{x = Ae^{-\alpha\omega t}\sin(\omega\sqrt{1-\alpha^2}\,t + \psi)}$$

$$x = e^{-\alpha\omega t}\left(B_1\sin(\omega\sqrt{1-\alpha^2}\,t + \psi) + B2\sin(\omega\sqrt{1-\alpha^2}\,t + \psi)\right)$$

where the arbitrary constants A, ψ or B_1,B_2 are determined from initial conditions.

<u>Cas I: (Oscillatory motion)</u>

With initial condition such as:

$$x(0) = x_0 \qquad \text{and} \qquad \dot{x}(0) = \dot{x}_0$$

solution is

$$x = e^{-\alpha\omega t}\left(x_0 \cos\omega_a t + \frac{\dot{x}_0 + \alpha\omega x_0}{\omega_a}\sin\omega_a t\right)$$

then by putting:

$$A\sin\psi = x_0 \qquad A\cos\psi = \frac{\dot{x}_0 + \alpha\omega x_0}{\omega_a}$$

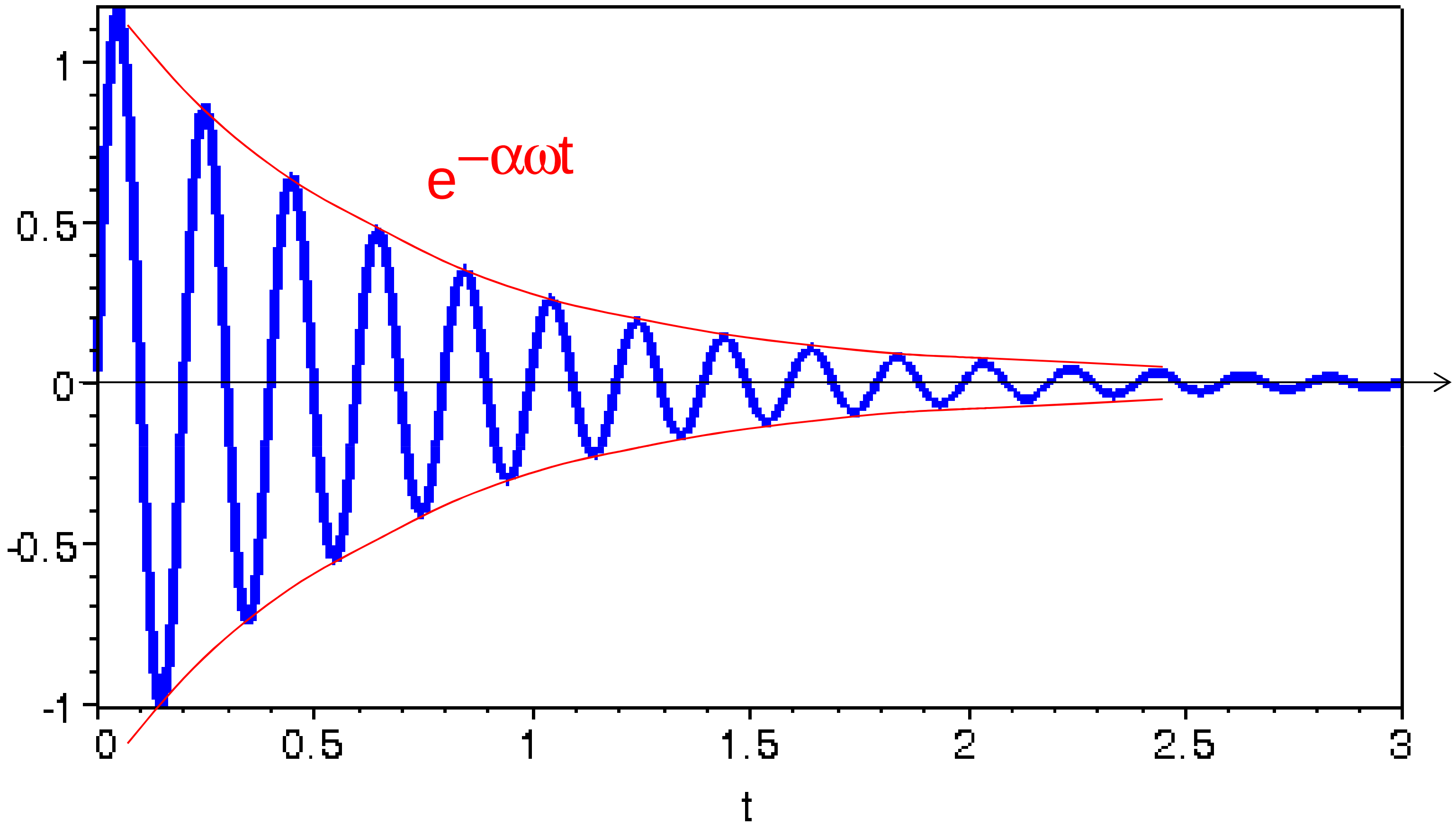
solution is reduced to

$$\boxed{x = Ae^{-\alpha\omega t}\sin(\omega_a t + \psi)}$$

This expression indicates that the motion of the mass is therefore an exponentially decaying harmonic oscillation at circular frequency ω_a

Recall: $\omega_a = \omega\sqrt{1-\alpha^2}$

Cas I: (Oscillatory motion)

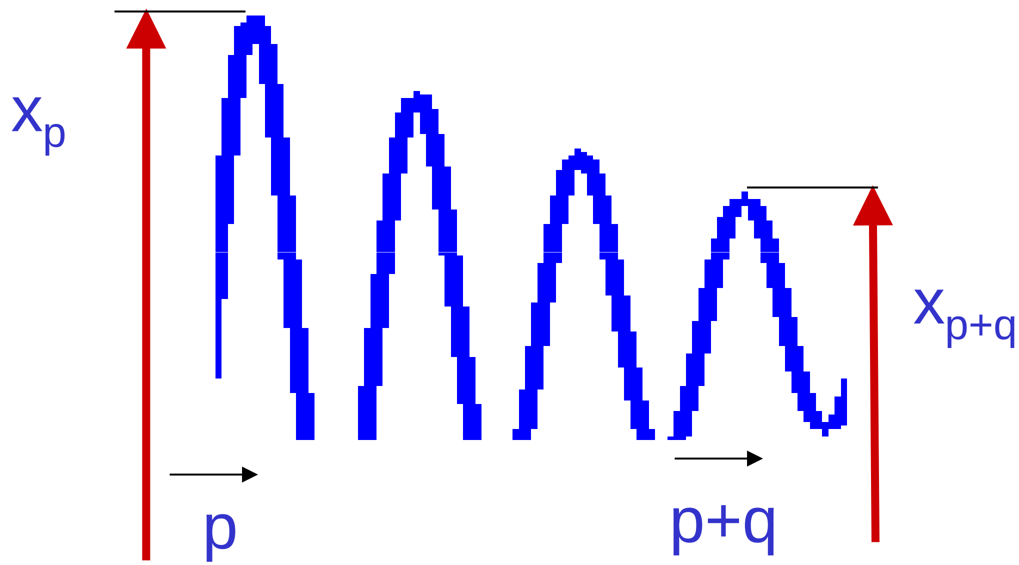


Cas I: (Oscillatory motion)

Logarithmic decrement

The amout of decay may be refered with the *constant of time.* In mechanical vibrations a common way to measure the amount of damping in the structure is to measure the rate of decay between to consecutive maxima of free oscillations x_p and x_{p+1} and to compute the ***Logarithmic decrement*** δ defined by

$$\delta = \ln \frac{x_p}{x_{p+1}}$$



The points of contacts with the exponential envelope curve do not coincide exactly with the maximum response points but from a practical use, the following approximtion is correct.

$$\delta \approx \ln \frac{e^{-\alpha\omega t}}{e^{-\alpha\omega(t+T)}}$$

Cas I: (Oscillatory motion)

Logarithmic decrement

Then,

$$\delta \approx \ln\left(e^{\alpha\omega T}\right) \approx \alpha\omega T$$

for usual cases

$$\delta = \frac{2\pi\alpha}{\sqrt{1-\alpha^2}}$$
$$\cong 2\pi\alpha$$

As α si very small, the ratio between to successive maxima approachs unity, it is better to measure response maxima which are separated by an integer number of period

$$\ln\frac{x_p}{x_{p+q}} = 2\pi\omega T$$
$$\approx 2\pi\alpha q$$

Finally.

$$\boxed{\alpha \cong \frac{1}{2\pi q}\ln\frac{x_p}{x_{p+q}}}$$

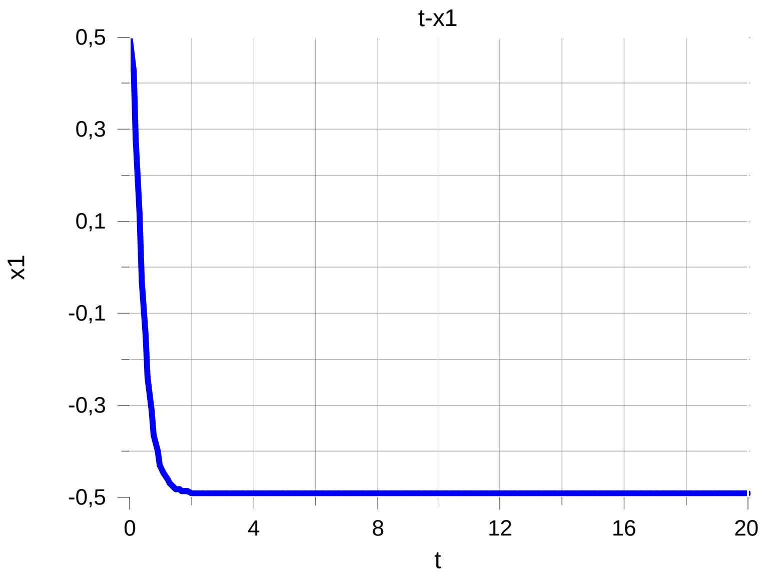
•Cas II: critical damping

•$\alpha = 1$

It represents the limit of periodic motion. The displaced mass is restored to equilibrium in the shortest time without oscillation and overshoot. Both values of roots are:

$$r_{1,2} = -\omega$$

Then, with the two constants obtained from initial conditions, the system response is aperiodic and given by:



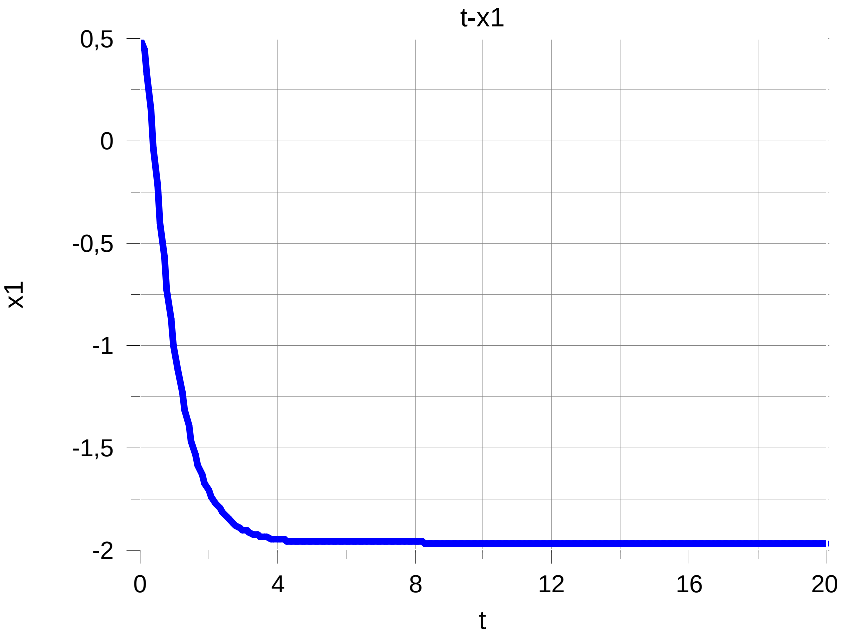
$$x = A_1 e^{-\omega t} + A_2 t e^{-\omega t}$$
$$= e^{-\omega t}(A_1 + A_2 t)$$

• Cas III: over critical damping

• $\alpha > 1$

This case is very rare in mechanical vibrations.The solution form is:

$$x = A_1 e^{\left(-\alpha\omega + \omega\sqrt{\alpha^2 - 1}\right)t} + A_2 e^{\left(-\alpha\omega - \omega\sqrt{\alpha^2 - 1}\right)t}$$



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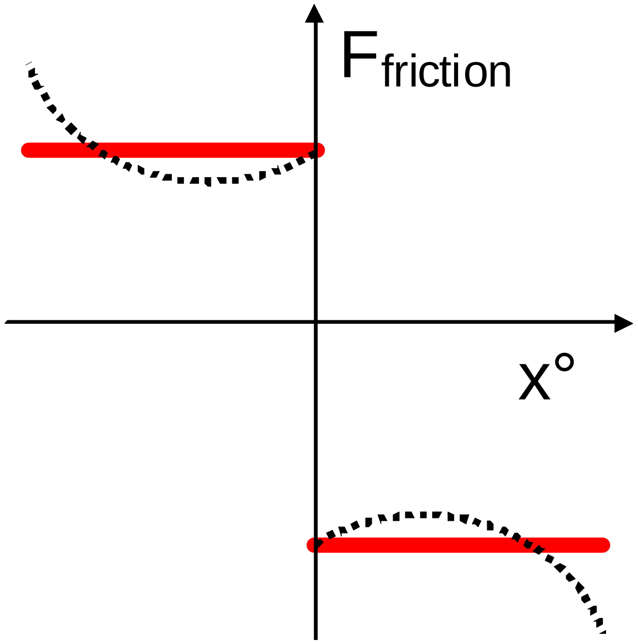
Vibration with dry friction damping (Coulomb):

Friction forces occur in many stuctures when relative displacement take place between adjacent components. These forces appear to be independant of amplitude and frequency but are always in opposition with the motion. Their magnitude is considered constant as a first approximation, that is the **Coulomb's** model.

Because the sign of the damping is always opposite to that of the velocity, the friction force is noted:

$$F_{friction} \Rightarrow \text{sign}(\dot{x})\, f$$

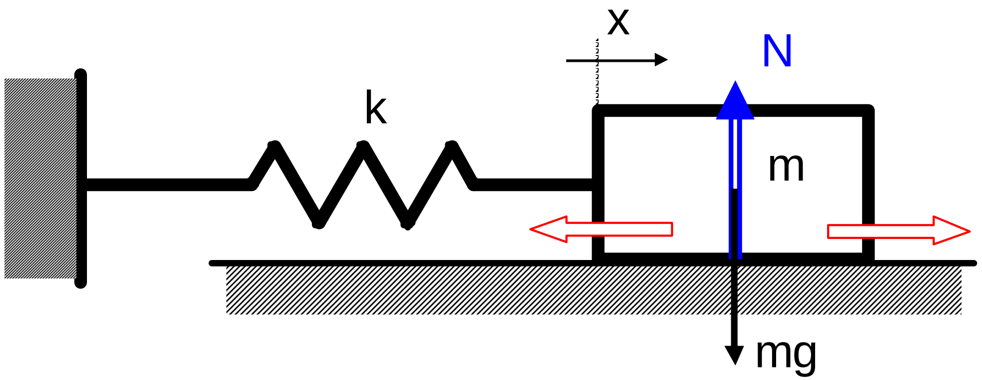


Piecewise linear

Bilinear system with Coulomb damping :

Let us consider the oscillator composed of a mass **m** and of a linear spring **k**. The mass is sliding on an horizontal plane in the gravity field (g).

Due to the weight of the mass and the dry friction between the mass and the plane, a friction force opposed to the motion exists.

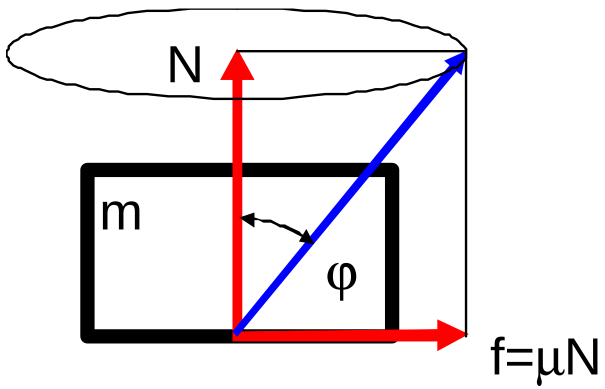


$$N = mg$$

That force which depond on: the material properties, the area of contact and of the type of surface is related to the normal force mg by:

$$F_{friction} = \mu N$$

$$\mu = tg\varphi$$

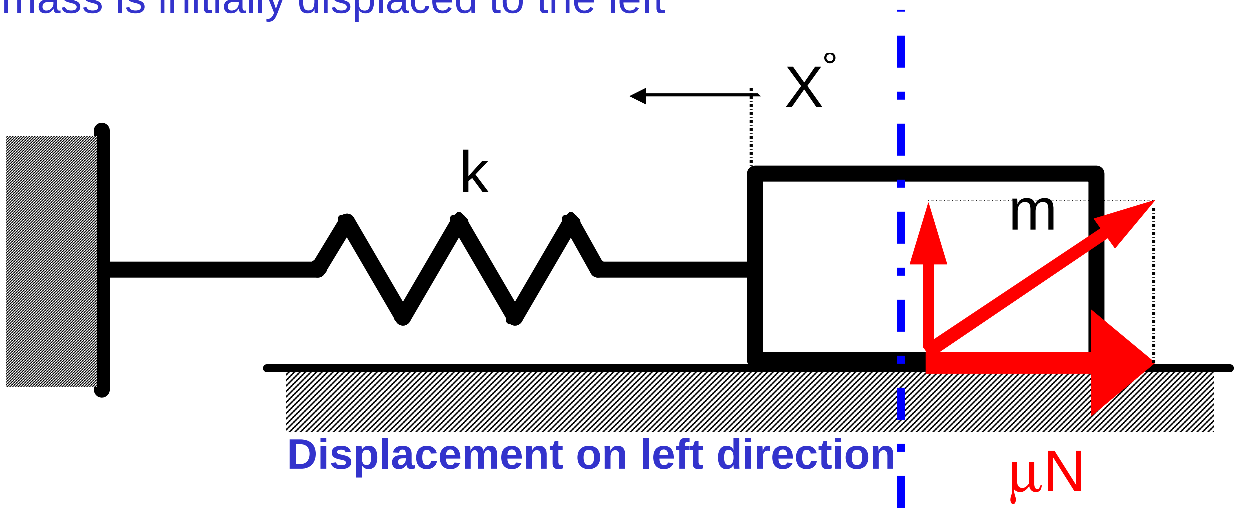


μ is called the coefficient of sliding friction.

Equations of motion

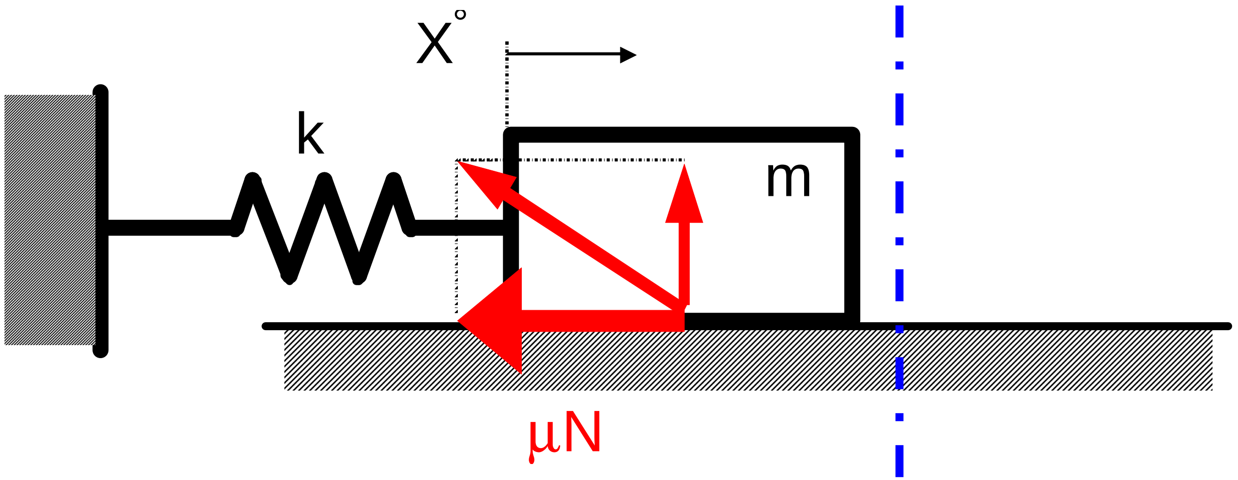
Two cases are considered:

The mass is initially displaced to the left



Displacement on left direction

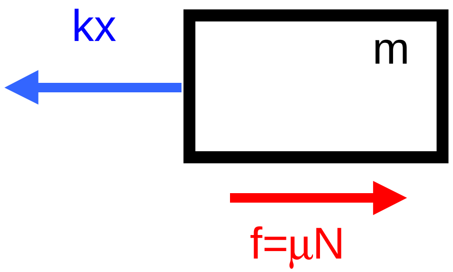
Then, the mass is going to move to the right



Displacement on right direction

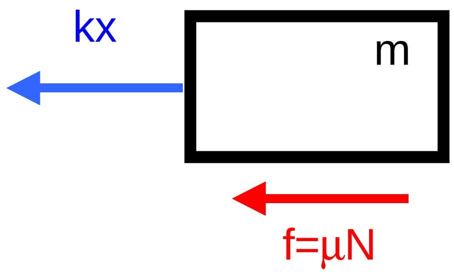
From second Newton's law

Displacement on left direction



$$m\frac{d^2x}{dt^2} - f + kx = 0$$

Displacement on Right direction



$$m\frac{d^2x}{dt^2} + f + kx = 0$$

The behavior of this system can be described by two linear differential equations with constant coefficients but with a **second member**. So such system is refered as bilinear.

$$m\frac{d^2x}{dt^2} + \text{sign}(\dot{x})f + kx = 0$$

or

$$\boxed{m\frac{d^2x}{dt^2} + \frac{\dot{x}}{|\dot{x}|}f + kx = 0}$$

Displacement on left direction

First differential equation

$$m\frac{d^2x}{dt^2}+kx=+f$$

$$x = x_H + x_P$$

$$\boxed{x(t) = A_1 \sin \omega t + B_1 \cos \omega t + \frac{f}{k}}$$ Displacement

$$\dot{x}(t) = \omega\left(A_1 \cos \omega t - B_1 \sin \omega t\right)$$ Velocity

Initial conditions:

$$x(0) = x_0 > 0 \quad \text{et} \quad \dot{x}(0) = 0$$

$$A_1 = 0 \qquad \text{and} \qquad B_1 = x_0 - \frac{f}{k}$$

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X_H homegenous solution

X_P particular solution

Displacement on left direction

First differential equation

$$
\begin{cases}
x(t) = \left(x_0 - \dfrac{f}{k} \right) \cos \omega t + \dfrac{f}{k} \\[2ex]
\dot{x}(t) = -\omega \left(x_0 - \dfrac{f}{k} \right) \sin \omega t
\end{cases}
$$

According to the sign of the velocity, this displacement occurs for:

$$t < \frac{\pi}{\omega}$$

Then, at the end of that first type of motion, (at $t = \pi/\omega$) , displacement and velocity are:

$$
\begin{cases}
x(\pi/\omega) = -x_0 + \dfrac{2f}{k} \\[2ex]
\dot{x}(\pi/\omega) = 0
\end{cases}
$$

!

Displacement on right direction

Second differential equation

$$m\frac{d^2x}{dt^2}+kx=-f$$

$$x = x_H + x_P$$

$$\boxed{x(t) = A_2 \sin \omega t + B_2 \cos \omega t - \frac{f}{k}}$$

Displacement

$$\dot{x}(t) = \omega\left(A_2 \cos \omega t - B_2 \sin \omega t\right)$$

Velocity

Relative Initial conditions (at $t = \pi/\omega$)

$$A_2 = 0 \quad \text{and} \quad B_2 = +x_0 - \frac{3f}{k}$$

X_H homegenous solution

X_P particular solution

Displacement on right direction

$$x(t) = \left(x_0 - \frac{3f}{k}\right)\cos\omega t - \frac{f}{k}$$

$$\dot{x}(t) = -\omega\left(x_0 - \frac{3f}{k}\right)\sin\omega t$$

According to the sign of the velocity, this displacement occurs only for:

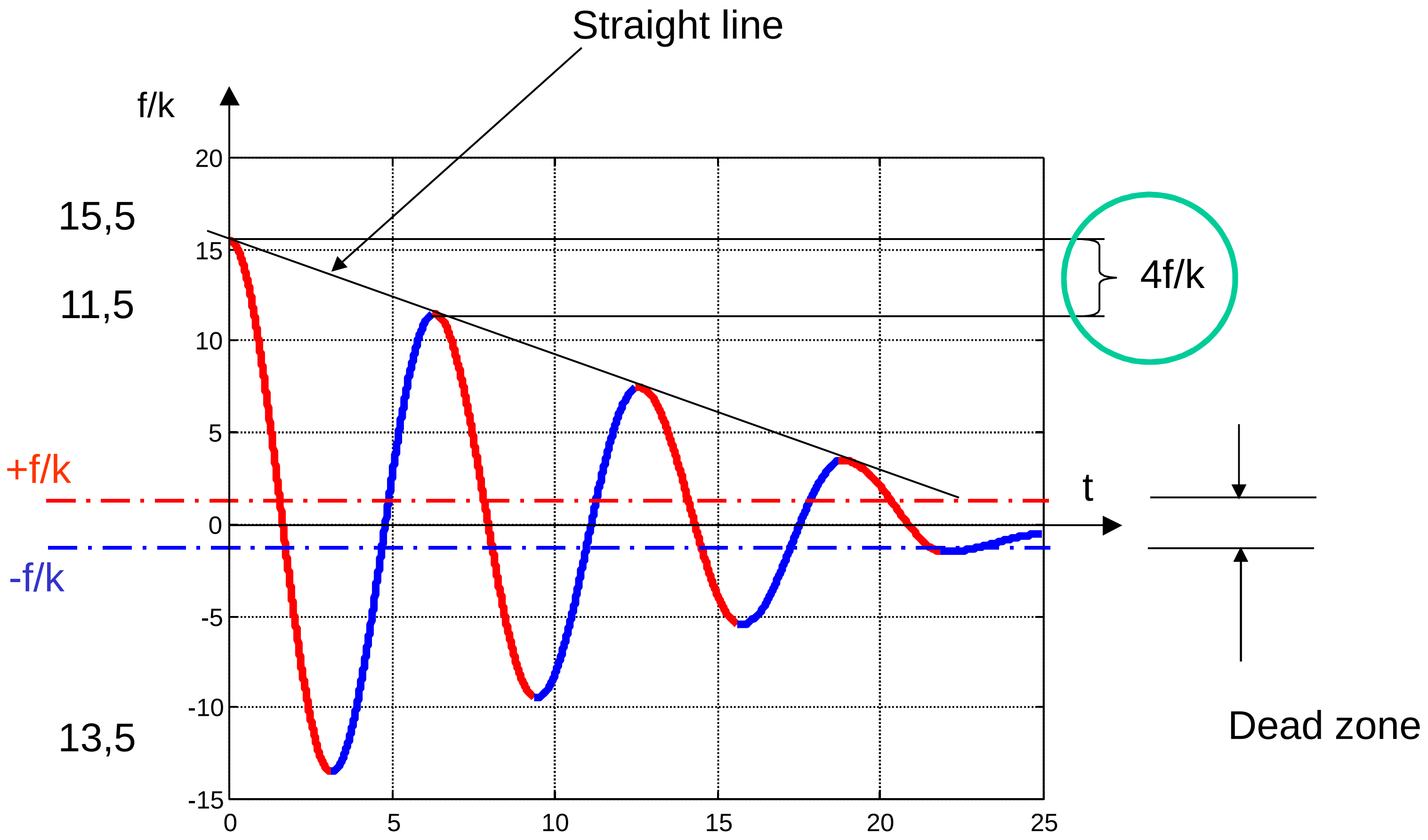
$$\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

Then, at the end of that new type of motion, (at t = π/ω) , displacement and velocity are :

$$x(2\pi/\omega) = x_0 - \frac{4f}{k}$$

!

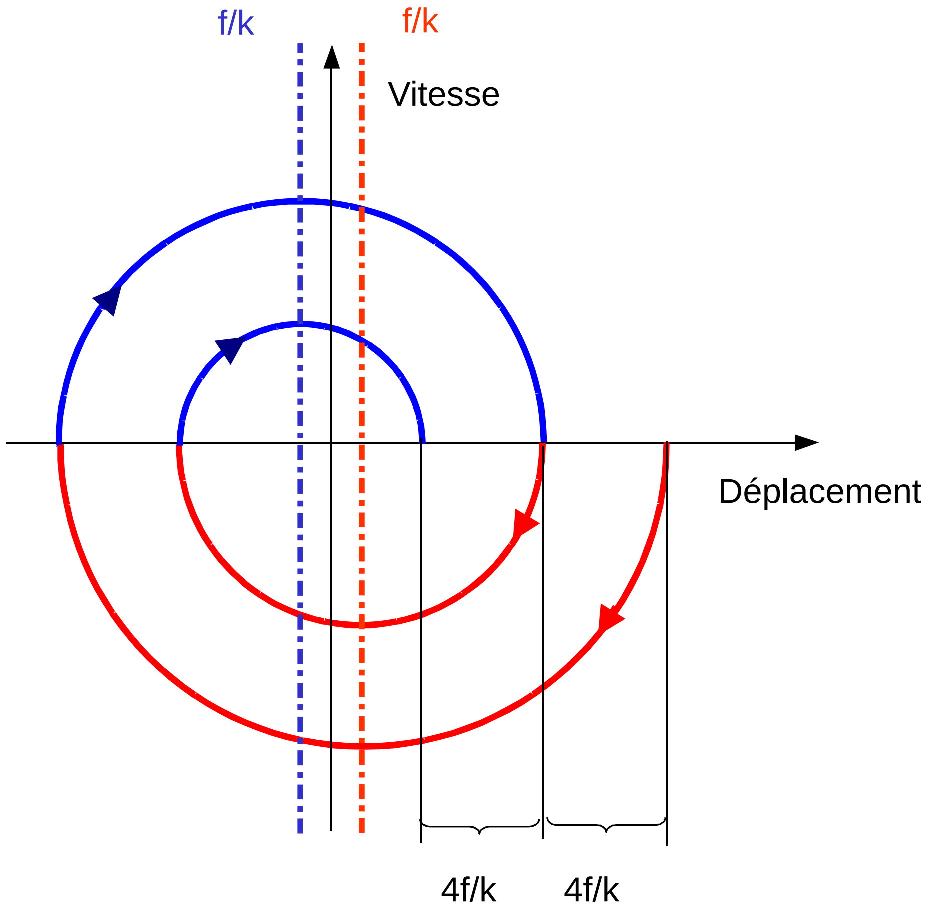
$$\dot{x}(2\pi/\omega) = 0$$



Displacement vs time

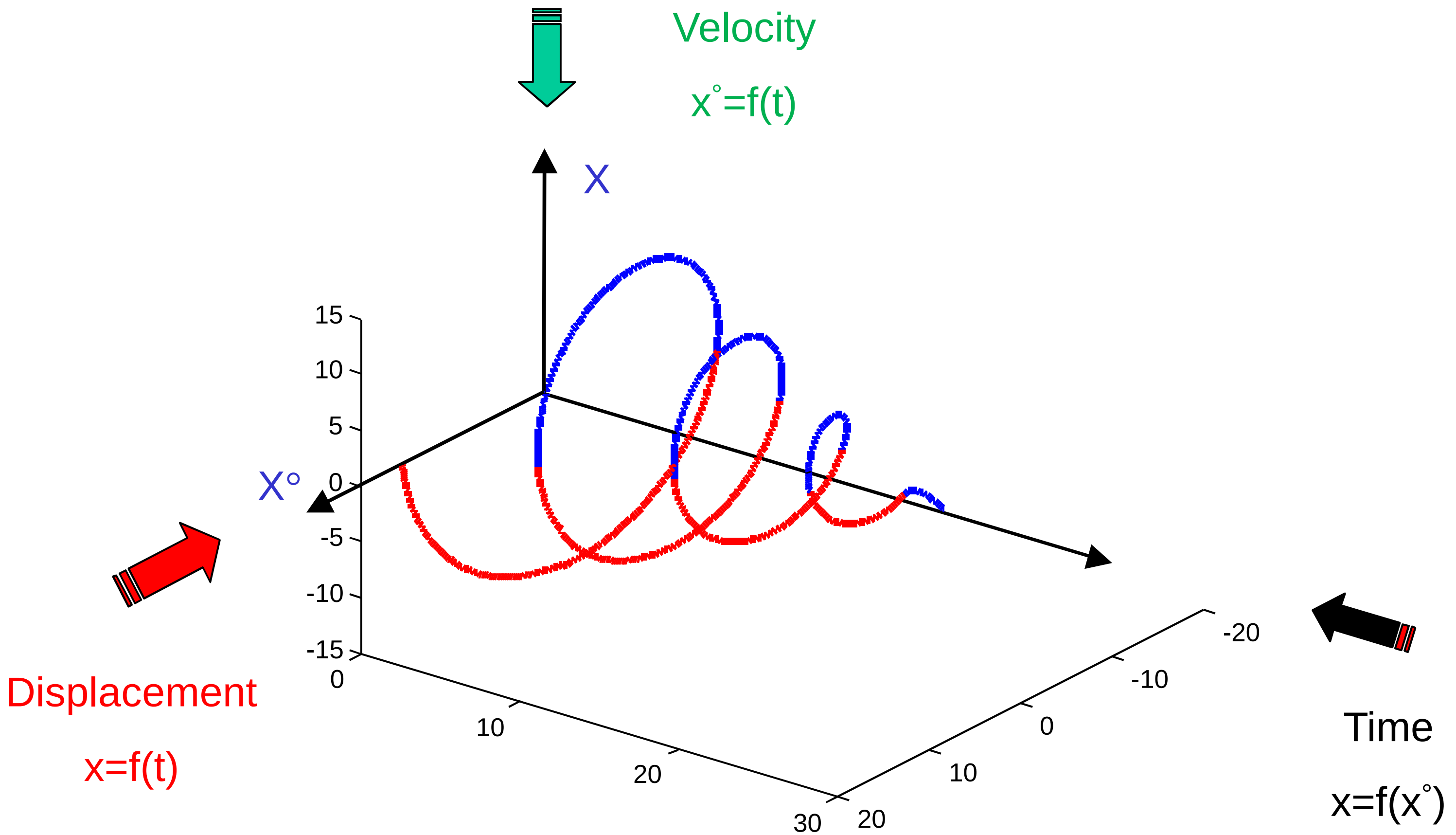
Note: frequency of oscillation is not affected by Coulomb friction

Phase plane presentation if very often used for the treatment of nonlinear systems. In that case the velocity is plotted as a function of the displacement. The following trajectory is obtained.



Phase plan presentation

(Phase plane portrait)



Centre3d.mws

State plane representation

trajectory in x,dx/dt and t

centre3da.mws

Forced Harmonic Vibration

Undamped system

$$\boxed{m\ddot{x} + kx = F\sin(\Omega t)}$$

This is a linear differential equations with constant coefficients but with a **second member**

Homegeneous solution + Particular solution

$\Leftrightarrow$ $x = A\cos\omega t + B\sin\omega t$

Homegenous solution is known (see previous part) and particular solution can be sought as:

$$x = X\sin(\Omega t - \phi)$$

where ϕ is a phase which will represent the decay between force and displacement.

By substituting this expression into EOM, it comes

$$(k - m\Omega^2)X\sin(\Omega t - \phi) = F\sin(\Omega t)$$

Forced Harmonic Vibration

Previous equation holds for any time, so :

$$\left[-(k-m\Omega^2)\sin(\phi)\right]X\cos(\Omega t)+$$
$$\left[(k-m\Omega^2)X\cos(\phi)-F\right]\sin(\Omega t)=0$$

is reduced to

$$(k-m\Omega^2)\sin(\phi)=0$$
$$\left[(k-m\Omega^2)\cos(\phi)\right]X-F=0$$

and

$$\sin(\phi)=0 \qquad \cos(\phi)=\frac{F}{X(k-m\Omega^2)} \qquad \Rightarrow \qquad \boxed{0<\phi<\pi}$$

Then

$$\boxed{X=\frac{F}{\sqrt{(k-m\Omega^2)^2}}}$$

or

$$\boxed{|X|=\frac{F}{k-m\Omega^2}}$$

Forced Harmonic Vibration

Damped system with $\alpha < 1$):

$$\boxed{m\ddot{x} + c\dot{x} + kx = F\sin(\Omega t)}$$

Similarly, this linear differential equations admits a general solution composed of an Homegenous solution which is known (see previous part) and of a particular solution can be sought as

Homegeneous solution

+

particular solution

$$x = Ae^{-\alpha\omega t}\sin(\omega\sqrt{1-\alpha^2}t + \phi)$$

Particular solution can be sought as:

$$x = X\sin(\Omega t - \phi)$$

where ϕ is a phase which will represent the decay between force and displacement.

Forced Harmonic Vibration

Substituting this expression into EOM becomes

$$(k - m\Omega^2)X\sin(\Omega t - \phi) + c\Omega X\cos(\Omega t - \phi) = F\sin(\Omega t)$$

and

$$\left[c\Omega\cos(\phi) - (k - m\Omega^2)\sin(\phi)\right]X\cos(\Omega t) +$$
$$\left[(k - m\Omega^2)X\cos(\phi) + c\Omega X\sin(\phi) - F\right]\sin(\Omega t) = 0$$

for any time:

$$c\Omega\cos(\phi) - (k - m\Omega^2)\sin(\phi) = 0$$
$$\left[(k - m\Omega^2)\cos(\phi) + c\Omega\sin(\phi)\right]X - F = 0$$

$$\sin(\phi) = \frac{c\Omega\cos(\phi)}{(k - m\Omega^2)} \qquad \cos(\phi) = \frac{F}{X}\frac{(k - m\Omega^2)}{(k - m\Omega^2)^2 + c^2\Omega^2}$$

$$\boxed{X = \frac{F}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}}$$

Recall: $x = X\sin(\Omega t - \phi)$

Forced Harmonic Vibration

Other computation ways are:

$$x = A\cos\Omega t + B\sin\Omega t$$

with

$$X = \sqrt{A^2 + B^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

So

$$\dot{x} = \Omega\left(-A\sin\Omega t + B\cos\Omega t\right)$$

$$\ddot{x} = -\Omega^2\left(A\cos\Omega t + B\sin\Omega t\right)$$

Substituting this expression into EOM and after some manipulations (grouping terms as coefficients of sin and cos)it becomes

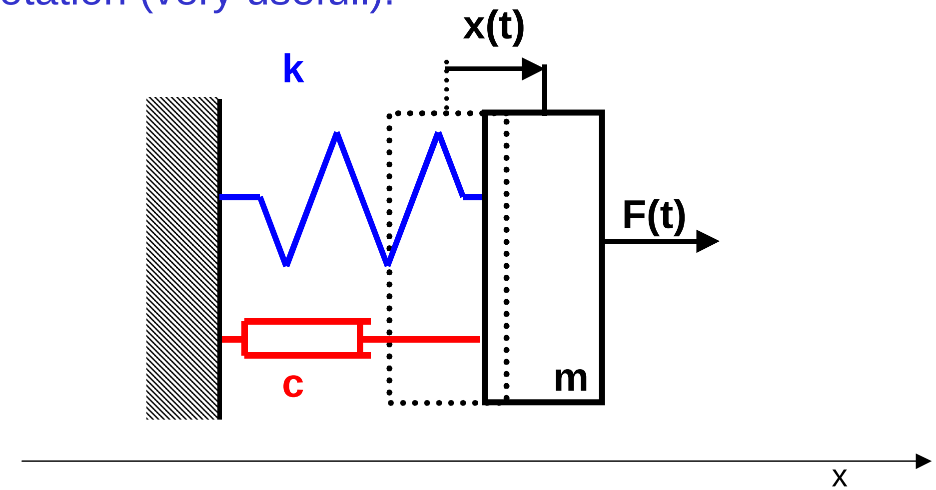
$$A = \frac{(k - m\Omega^2)F}{(k - m\Omega^2)^2 + c^2\Omega^2} \qquad B = \frac{-c\Omega F}{(k - m\Omega^2)^2 + c^2\Omega^2}$$

using condition on A and B leads to the same result.

Note: Complex notation is very usefull (see TD)

Forced Harmonic Vibration

Using complex notation (very usefull):



So

$$\left(m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F\sin\Omega t \right) \quad * \, j$$

$$+ \qquad m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F\cos\Omega t$$

$$= \qquad m\ddot{z}(t) + c\dot{z}(t) + kz(t) = Fe^{j\Omega t}$$

where

$$\boxed{z(t) = y(t) + j\,x(t)}$$

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New modified equation is:

$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = Fe^{j\Omega t}$$

Solution is sought as:

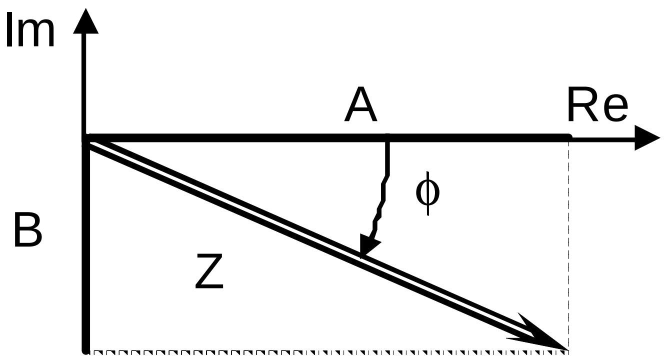
$$z = Ze^{j\Omega t}$$

then

$$\left(k - m\Omega^2 + jc\Omega\right)Ze^{j\Omega t} = Fe^{j\Omega t}$$

loocking at a graph and assuming for any time **t**

$$\boxed{\begin{aligned} Z &= \frac{F\left(\left(k - m\Omega^2\right) - jc\Omega\right)}{\left(\left(k - m\Omega^2\right) + jc\Omega\right)\left(\left(k - m\Omega^2\right) - jc\Omega\right)} \\ &= A - jB \end{aligned}}$$



or

$$\boxed{Z = |Z|e^{-j\phi}}$$

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Modulus determination:

$$Z = \frac{F\left(\left(k - m\Omega^2\right) - jc\Omega\right)}{\left(k - m\Omega^2\right)^2 + (c\Omega)^2} \quad \Rightarrow \quad |Z| = \frac{F}{\sqrt{\left(k - m\Omega^2\right)^2 + (c\Omega)^2}}$$

$$= A - jB$$

As the solution was

$$z = Ze^{j\Omega t} \quad \Rightarrow \quad z = |Z|e^{-j\varphi}e^{j\Omega t} \quad \Rightarrow \quad Z = |Z|e^{-j\phi}$$

Finally:

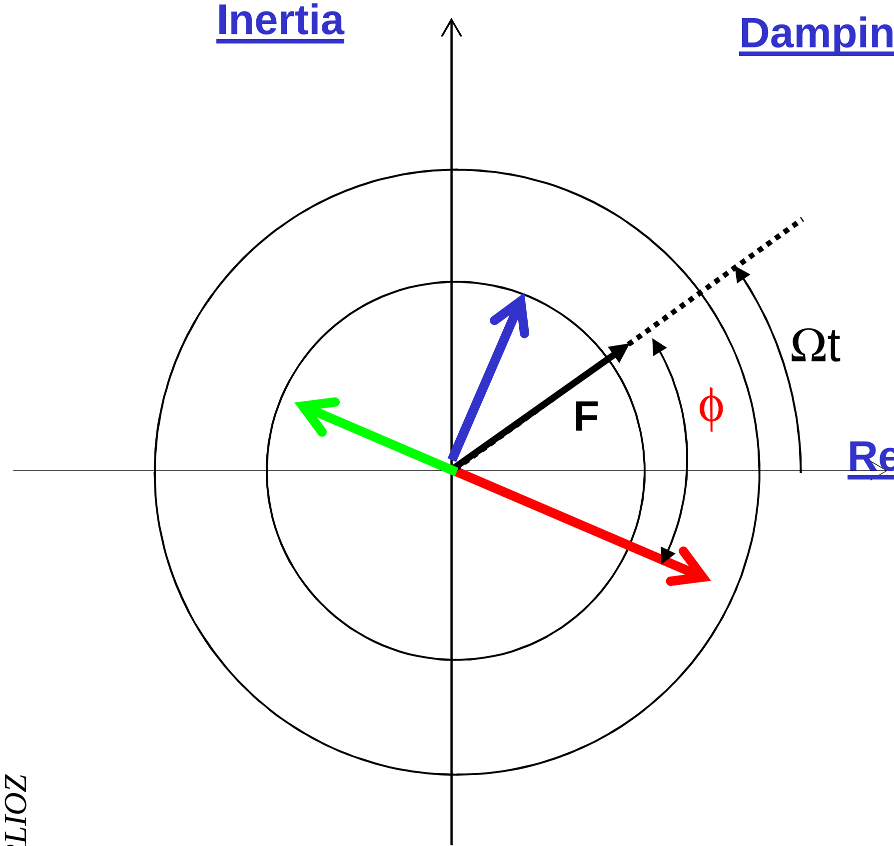
$$\boxed{\begin{aligned} x &= \text{Imaginary Part }\{z(t)\} \\ &= \text{Imaginary Part }\left\{|Z|e^{j(\Omega t - \phi)}\right\} \\ &= |Z|\sin(\Omega t - \phi) \\ &= \frac{F}{\sqrt{\left(k - m\Omega^2\right)^2 + (c\Omega)^2}}\sin(\Omega t - \phi) \end{aligned}}$$

Forced Harmonic Vibration

$x = X\sin(\Omega t - \phi)$ $\quad \dot{x} = \Omega X\cos(\Omega t - \phi)$ $\quad \ddot{x} = -\Omega^2 X\sin(\Omega t - \phi)$ $\quad F(t) = F\sin(\Omega t)$

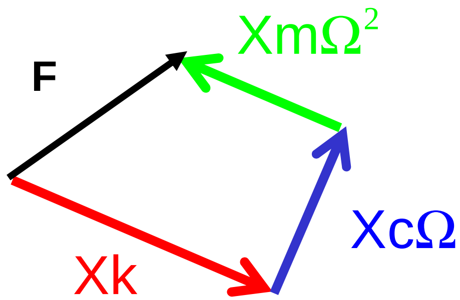
$$-m\Omega^2 X\sin(\Omega t - \phi) + c\Omega X\cos(\Omega t - \phi) + kX\sin(\Omega t - \phi) = F\sin(\Omega t)$$

Inertia **Damping** **Elastic**



Reference line

(Ω/ω<1)



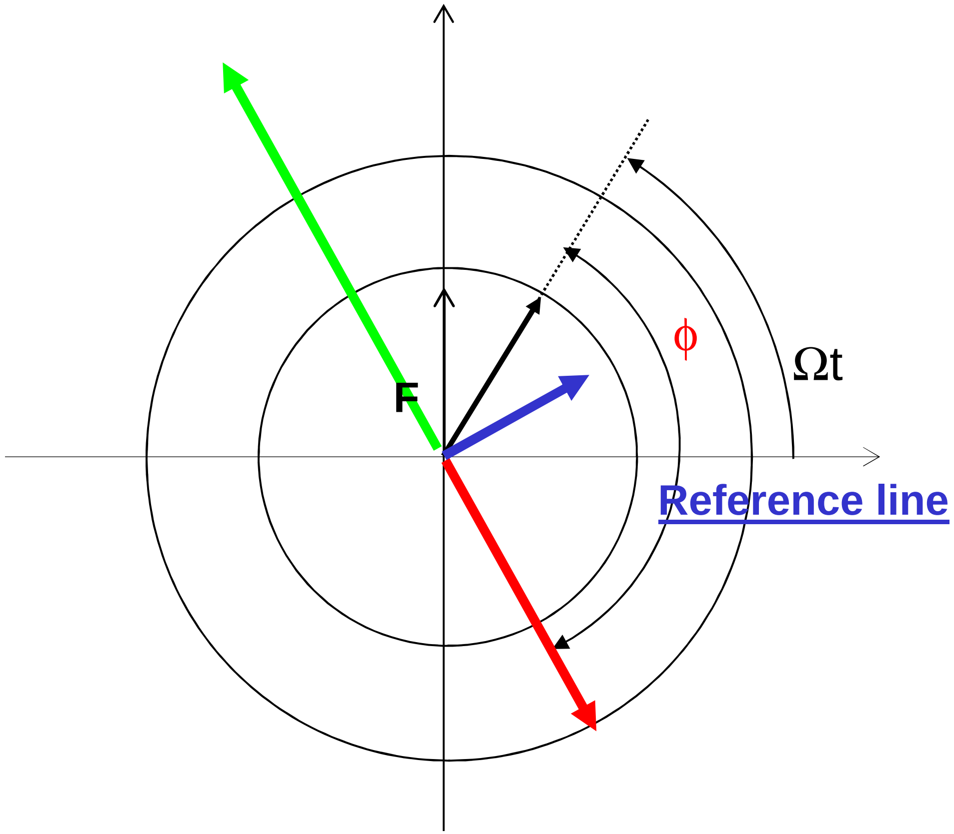
It is clear from this graph that

$$(k - m\Omega^2)^2 + c^2\Omega^2 = \frac{F^2}{X^2}$$

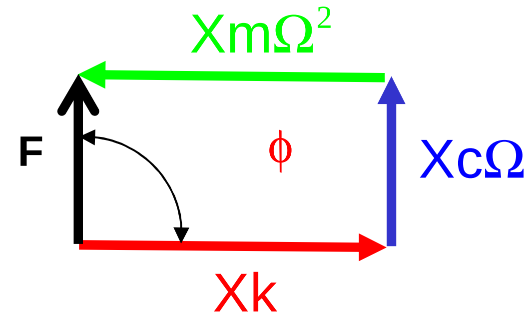
or

$$\boxed{X = \frac{F}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}}$$

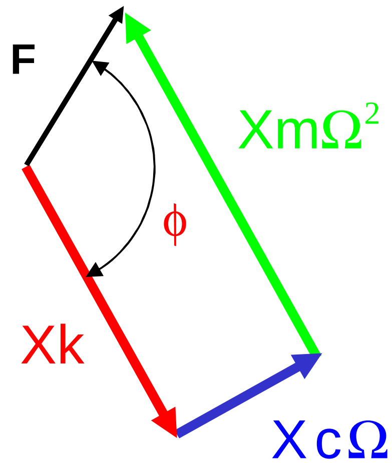
Response



(Ω/ω=1)



(Ω/ω > 1)



It is still clear from these graphs that

$$\boxed{X = \frac{F}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}}$$

Study of steady state motion under harmonic force

$$\boxed{x = Ae^{-\alpha\omega t}\sin(\omega\sqrt{1-\alpha^2}\,t+\psi) + \frac{F}{\sqrt{(k-m\Omega^2)^2 + c^2\Omega^2}}\sin(\Omega t-\phi)}$$

Looking at the complete solution s of x , it must be noted that for positive values of α the first term of solution will disappear with time. So the particular solution, leads to the steady state motion.

Putting the static deflection $X_{st} = \frac{F}{k}$

The particular solution, for damped system, gives the Nondimensonial forms with normalized amplitues and frequencies.

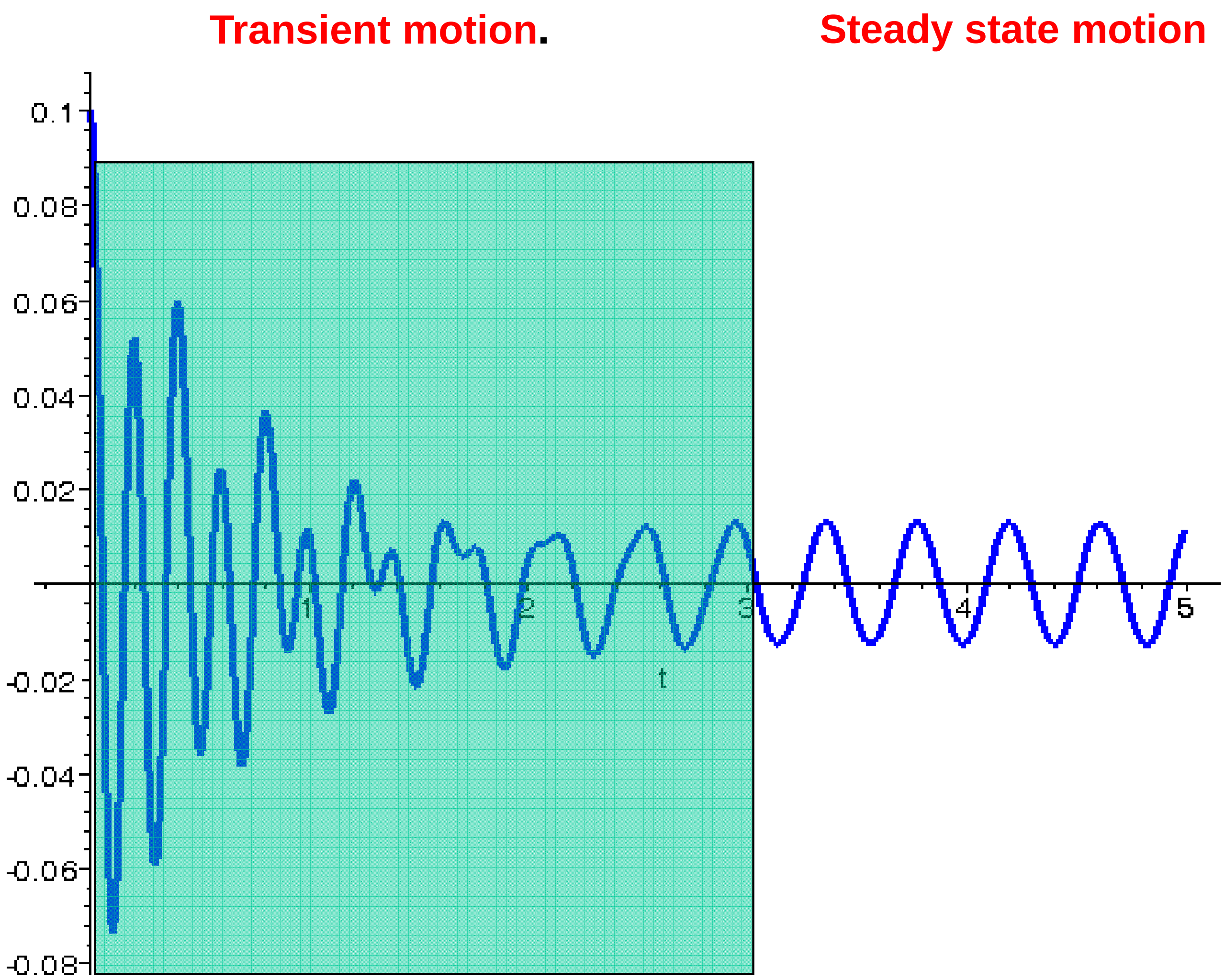
$$\boxed{X = \frac{X_{st}}{\sqrt{\left[1-(\Omega/\omega)^2\right]^2 + \left[2\alpha(\Omega/\omega)\right]^2}}}$$

$$\boxed{tg\phi = \frac{2\alpha\left(\frac{\Omega}{\omega}\right)}{1-\left(\frac{\Omega}{\omega}\right)^2}}$$

but for undamped system, the two motions are still surimposed.

$x_H = A\cdots\cos\omega t + B\sin\omega t$ and $$\boxed{x_P = \frac{X_{st}}{\left|1-\left(\frac{\Omega}{\omega}\right)^2\right|}\sin(\Omega t-\phi)}$$

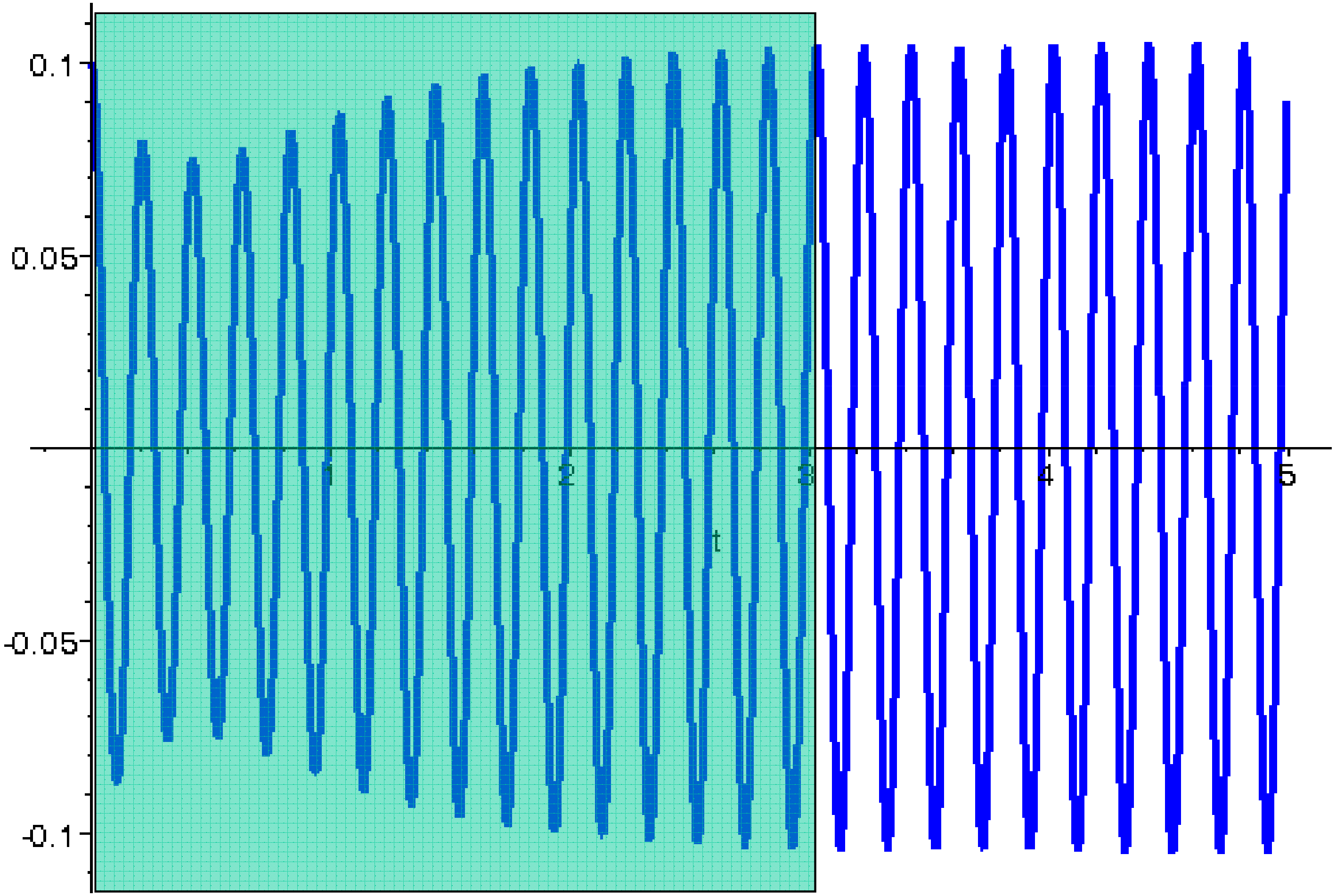
Forced Harmonic Vibration



Displacement versus time

Forced Harmonic Vibration

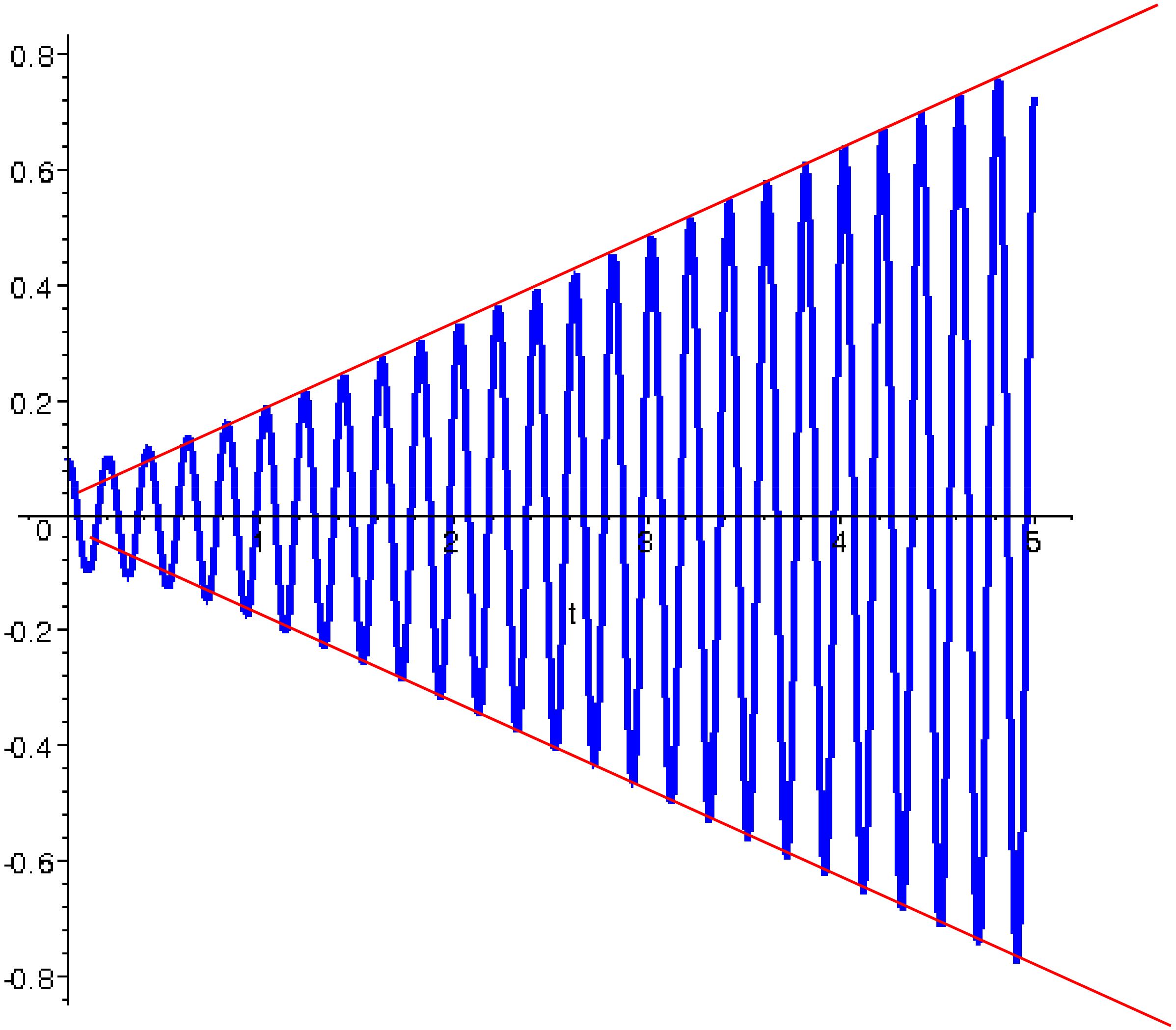
Transient motion. **Steady state motion**



Displacement versus time

Forced Harmonic Vibration

Resonance Case $\Omega \approx \omega$



Displacement versus time

Study of steady state motion under harmonic force

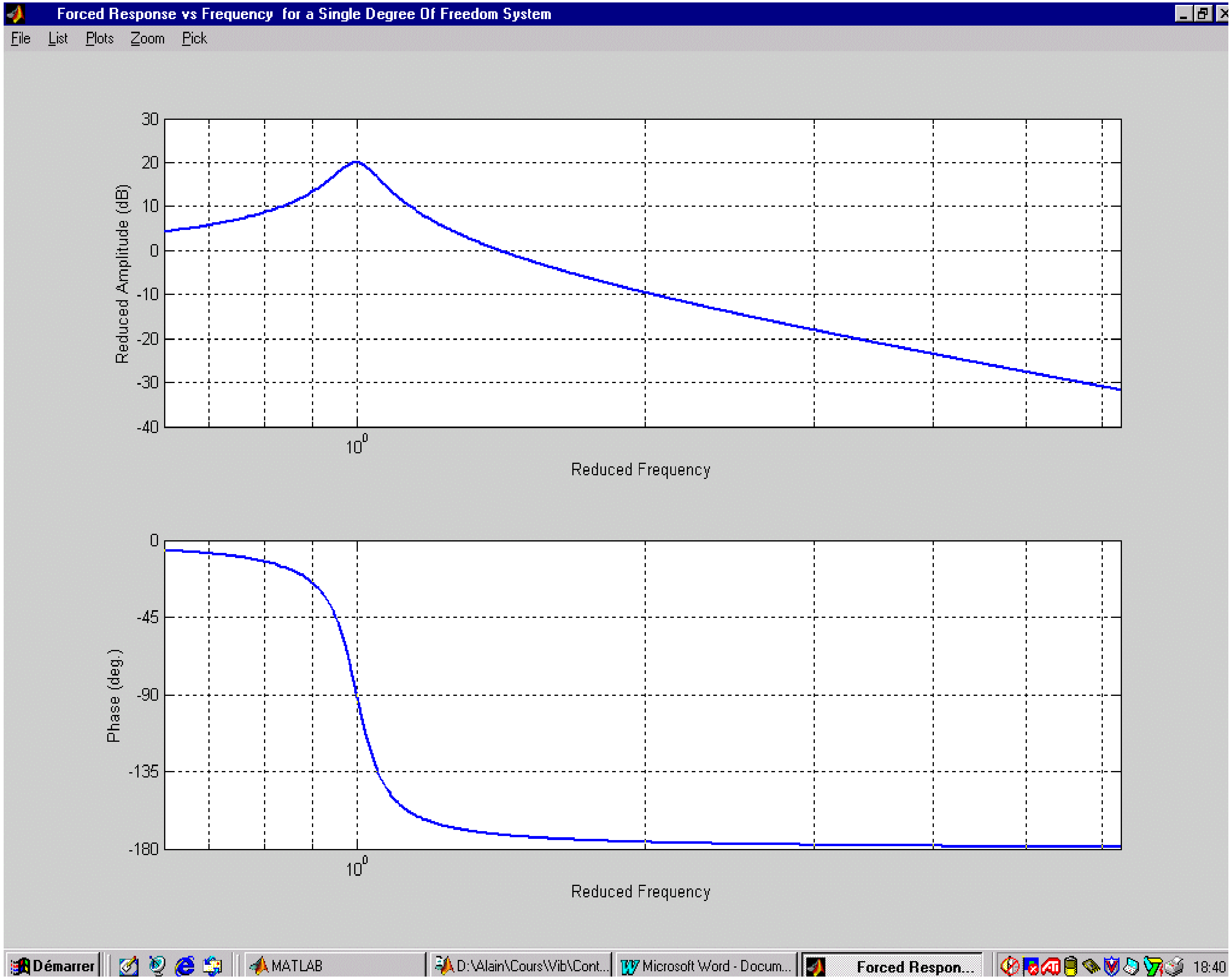
A mathematical study of the curve shows that:

maximum amplitude for X is found at $$\frac{\Omega}{\omega} = \sqrt{1 - 2\alpha^2}$$

corresponding value of X is $$X_r = \frac{X_{st}}{2\alpha\sqrt{1 - \alpha^2}}$$

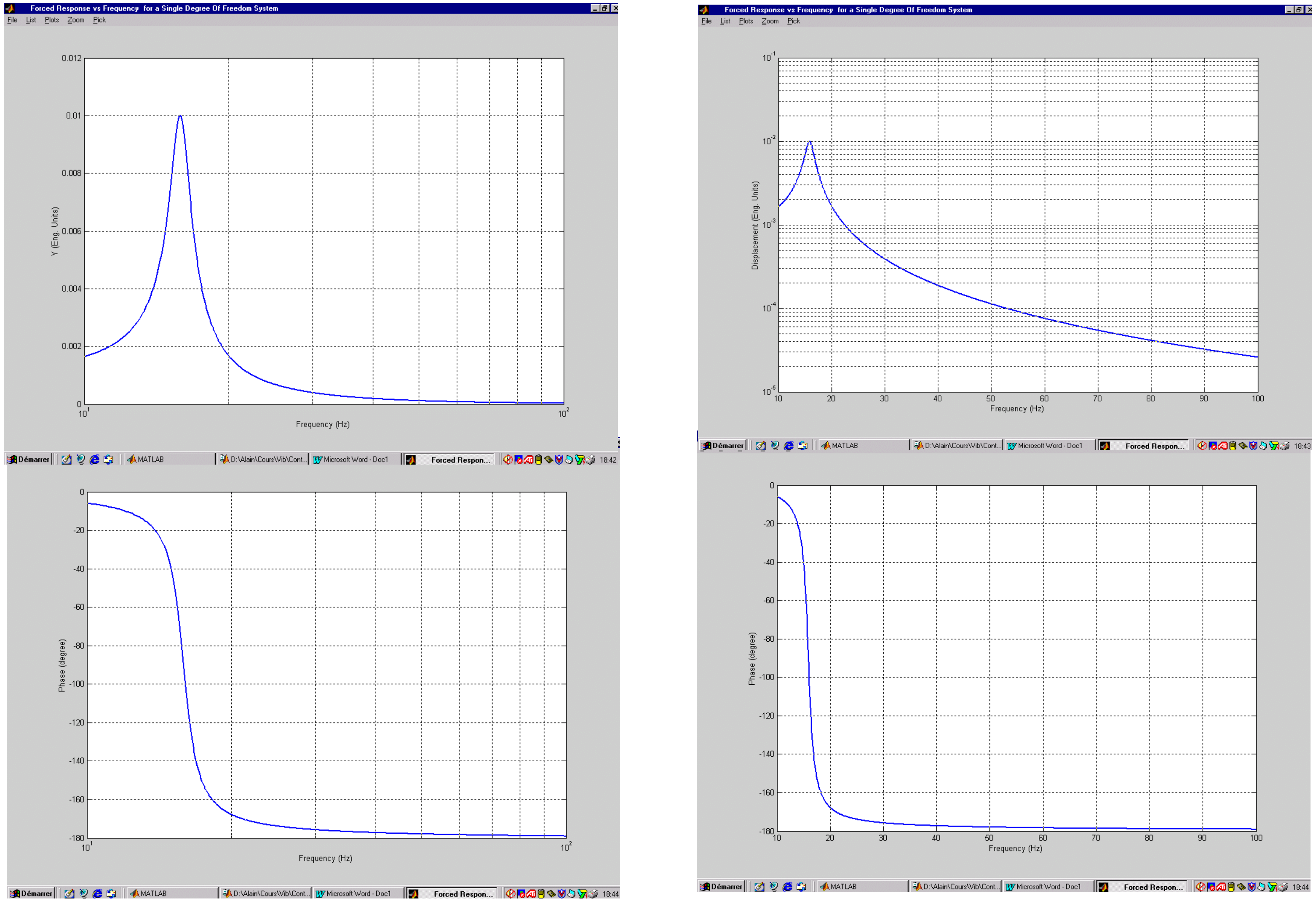
and phase angle is obtained from $$tg\phi = \frac{\sqrt{1 - 2\alpha^2}}{\alpha}$$

Study of steady state motion under harmonic force



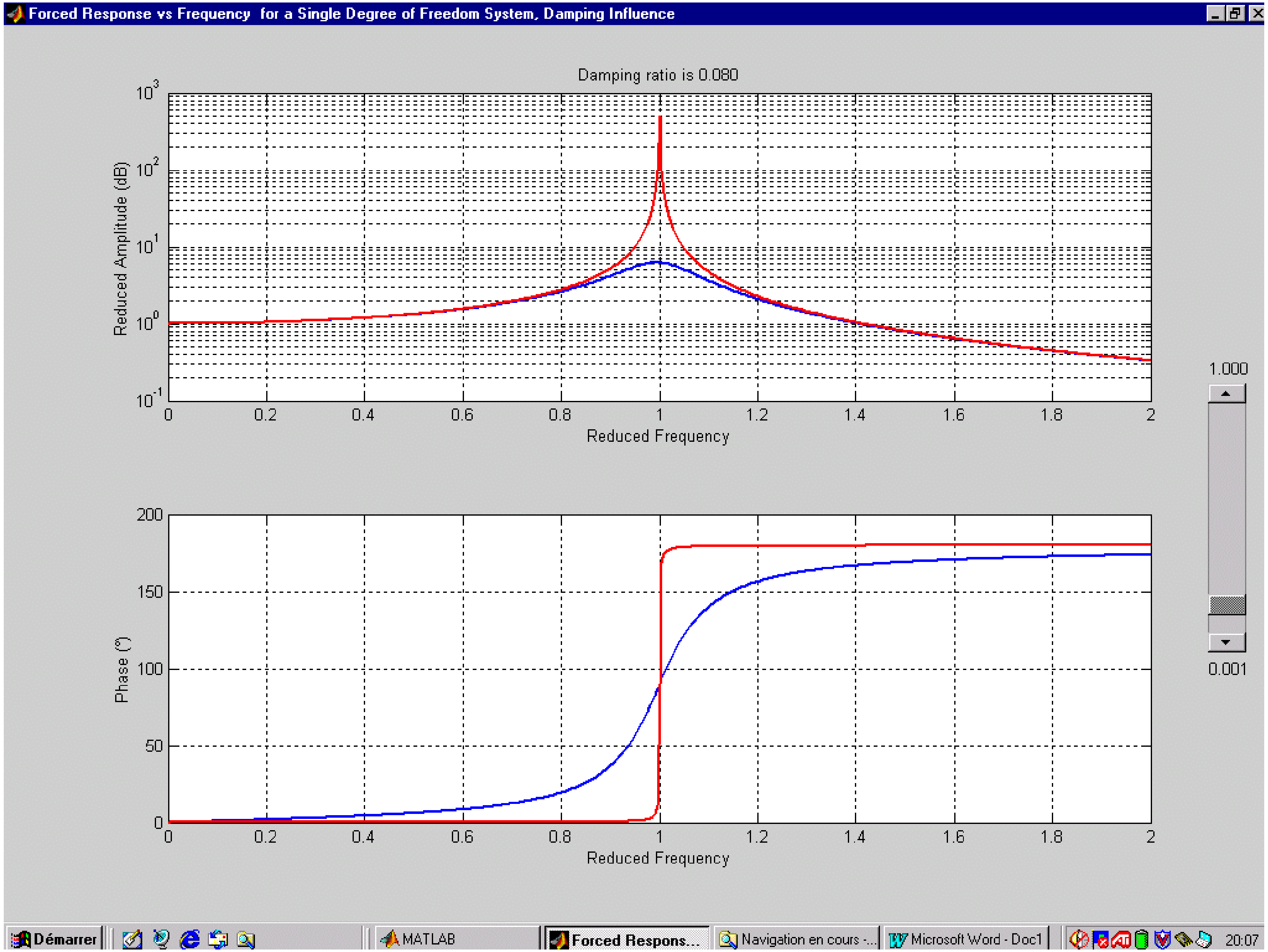
Bode Diagram

Study of steady state motion under harmonic force



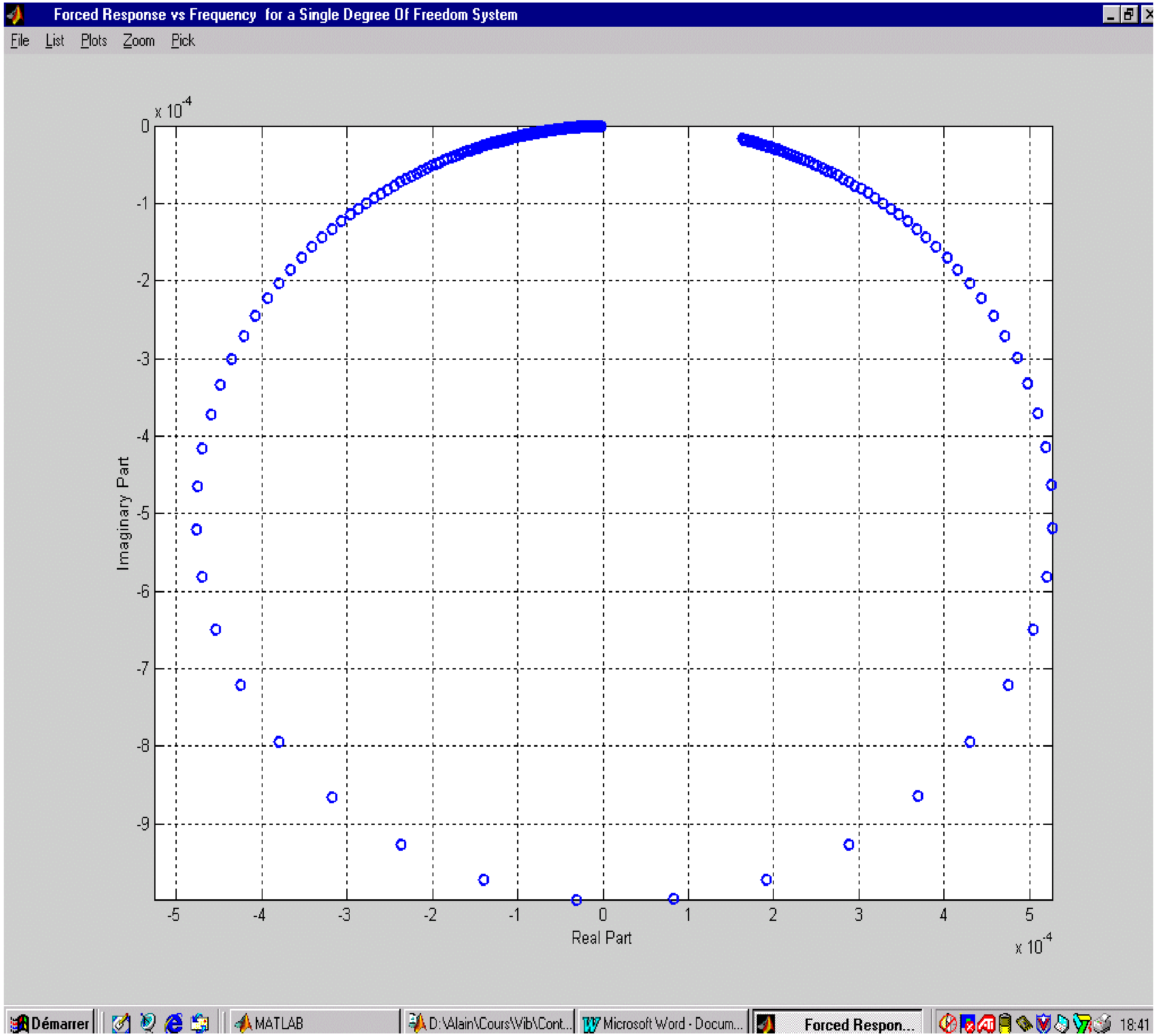
Bode Diagrams

Study of steady state motion under harmonic force



Bode Diagram

Study of steady state motion under harmonic force



Nyquist Diagram

Study of steady state motion under harmonic force

Amplification factor

At resonance, the displacement X_r is important and the force in the spring is very large (especially for small damping). The spring force amplitude is:

$$F_r = kX_r$$
$$= k\frac{X_{st}}{2\alpha\sqrt{1-\alpha^2}}$$

But, at resonance

$$\frac{\Omega}{\omega} = \sqrt{1-2\alpha^2} \approx 1-\alpha^2 \approx 1 \qquad \text{and} \qquad X_r = \frac{X_{st}}{2\alpha\sqrt{1-\alpha^2}} \approx \frac{X_{st}}{2\alpha}\left(1+\frac{\alpha^2}{2}\right) \approx \frac{X_{st}}{2\alpha}$$

It follows that

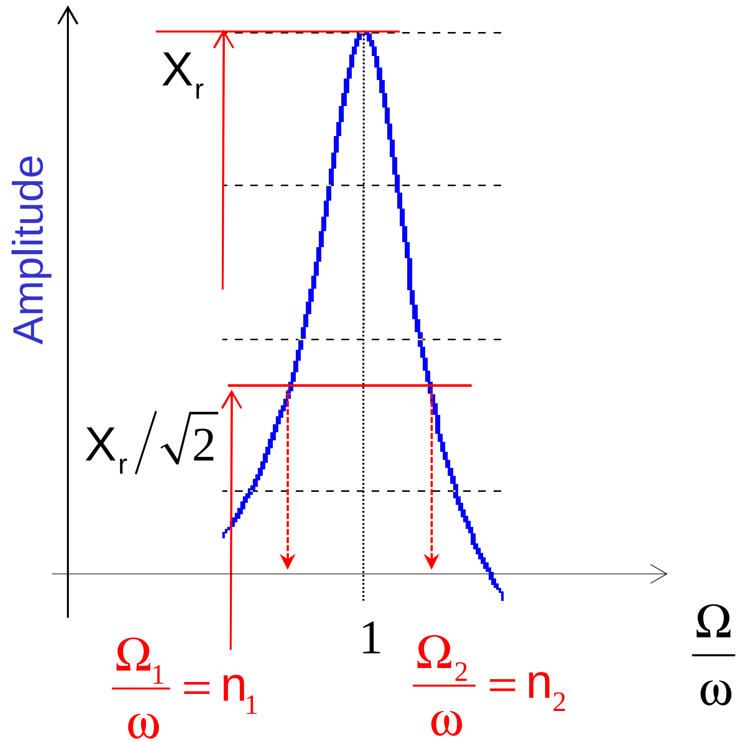
$$F_r = kX_r = \frac{kX_{st}}{2\alpha} = \frac{F}{2\alpha} = Q.F \qquad \text{and} \qquad Q = \frac{X_{st}}{2\alpha X_{st}} = \frac{1}{2\alpha}$$

$$Q = \frac{X_r}{X_{st}}$$

where Q is called the quality Q-factor (amplification factor at resonance).

Determination of damping with Half-Power Bandwith

For small damping, this method serves for the determination of the damping ratio. On a plot of amplitude versus frequency, the bandwith is measured at 0.707 of the maximun amplitude.



$$X(\Omega_1) = \frac{X_r}{\sqrt{2}} \approx \frac{1}{\sqrt{2}}\frac{X_{st}}{2\alpha} \qquad X(\Omega_2) = \frac{X_r}{\sqrt{2}} \approx \frac{1}{\sqrt{2}}\frac{X_{st}}{2\alpha}$$

$$N_{\text{décibel}} \cong 10\text{Log}\frac{X^2(F_1)}{X^2(F)} \cong 20\text{Log}\frac{X(F_1)}{X(F)} \cong 20\text{Log}\frac{X(F_2)}{X(F)} \cong -3\text{dB}$$

Determination of damping with Half-Power Bandwith

$$x(\Omega_{1,2}) = \frac{1}{\sqrt{2}}\frac{X_{st}}{2\alpha} = \frac{X_{st}}{\sqrt{\left(1-n_{1,2}^2\right)^2 + \left(2\alpha\, n_{1,2}\right)^2}}$$

Rewriting that equation gives:

$$2\sqrt{2}\alpha = \sqrt{\left(1-n_{1,2}^2\right)^2 + \left(2\alpha\, n_{1,2}\right)^2}$$

$n_{1,2}$ must be the solutions of:

$$n_{1,2}^4 + n_{1,2}^2\left(4\alpha^2 - 2\right) + 1 - 8\alpha^2 = 0$$

There are two roots $n_{1,2}$

$$n_{1,2}^2 = 1 - 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2+1}$$

neglecting α^2 for small value of α

$$n_{1,2}^2 \approx 1 \pm 2\alpha \qquad n_1 \approx 1-\alpha \qquad n_2 \approx 1+\alpha$$

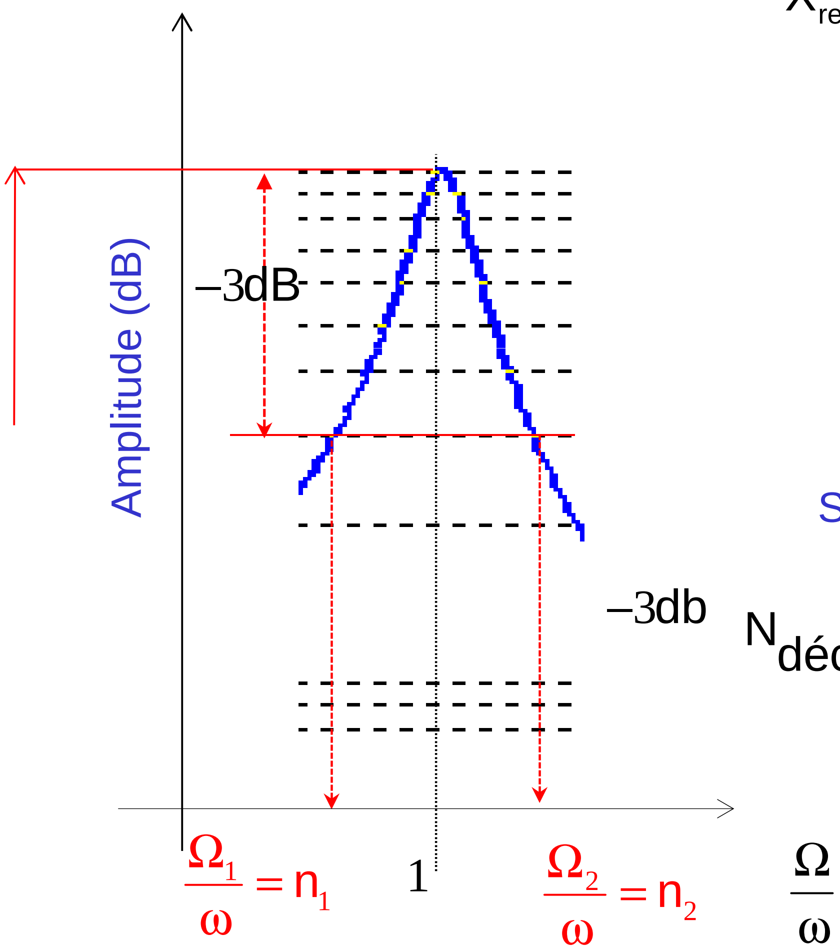
$$n_2 - n_1 = \frac{\Omega_2}{\omega} - \frac{\Omega_1}{\omega} = \frac{\Delta\Omega}{\omega} = \frac{\Delta f}{f} = 2\alpha = \frac{1}{Q}$$

It means that measuring bandwidth leads to the damping ratio α

Determination of damping with Half-Power Bandwith

Using decibel units is very usefull.

$$N_{\text{décibel}} \cong 10\ \text{Log}\frac{X^2}{X_{ref}^2} \cong 20\ \text{Log}\frac{X}{X_{ref}}$$



where

$$X_{ref} = X_r \qquad X = \frac{X_r}{\sqrt{2}}$$

So that

$$N_{\text{décibel}} \cong 20\ \text{Log}\frac{X_{ref}}{2X_{ref}} \cong 20\ \text{Log}\frac{1}{2} \cong -3\text{dB}$$

This technique is frequently named: half power bandwith method.

Response and beating phenomenon (undamped system)

For small damping, when driving frequency Ω is very close by the natural frequecy ω the structure may exhibit a particular motion which is called the phenomenon of beats.

$$x = A\cos\omega t + B\sin\omega t + \frac{X_{st}}{\left|1-\left(\frac{\Omega}{\omega}\right)^2\right|}\sin(\Omega t - \phi)$$

Putting all initial conditions to 0 leads to:

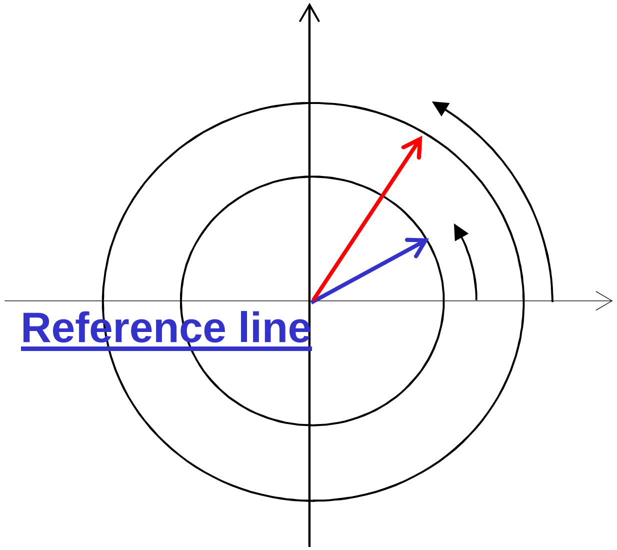
$$x(t) = B_0\sin\omega t + \frac{X_{st}}{\left|1-\left(\frac{\Omega}{\omega}\right)^2\right|}\sin(\Omega t - \phi)$$

Total solution of motion is:

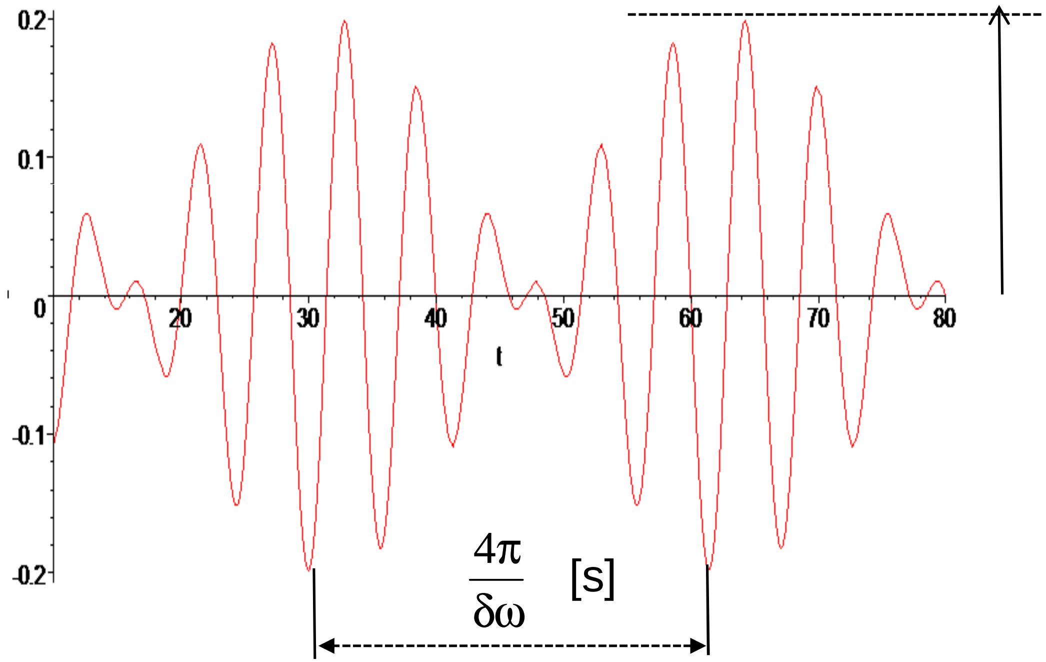
Response: Beat Phenomenon

Let x1= $x_1 = X\sin(\omega)t \qquad x_2 = X\sin(\omega+\delta\omega)t$

$$\boxed{\begin{aligned} x_1 + x_2 &= X\left(\sin(\omega)t + \sin(\omega+\delta\omega)t\right) \\ &= 2X\left(\sin(\omega+\delta\omega/2)t.\cos(\delta\omega/2)t\right) \end{aligned}}$$



$$\boxed{f_b = \frac{\delta\omega/2}{2\pi} = \frac{\delta\omega}{4\pi} \quad [\mathrm{Hz}]}$$



The amplitude of the result is seen to fluctuate between -2X and + 2X according to the 2X cos(δω/2.t) term. In the same time, the general motion sin(…) has an angular frequency of (ω+δω/2).

Periodic Excitations $F(t) = F(t+T)$

The excitation can be developed in a ***Fourier*** series by:

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{+\infty}\left(a_p \cos p\Omega t + b_p \sin p\Omega t\right)$$

p = harmonic order $F_p = \left(a_p^2 + b_p^2\right)^{1/2}$ = amplitude of harmonic

$$a_p = \frac{2}{T}\int_0^T F(t)\cos p\Omega t \, dt$$

p = 0,1,2, …

$$b_p = \frac{2}{T}\int_0^T F(t)\sin p\Omega t \, dt$$

p = 01,2,3 …

In steady-state motion, the response to each harmonic component is calculated separately and these responses are then added to obtain the complete solution.

$$x(t) = \text{oscillation libre} + x_1\cos(\Omega t + \psi_1) + x_2\cos(2\Omega t + \psi_2) + \cdots$$

- La réponse du système est la somme des réponses de chaque harmonique : principe de superposition (en linéaire).

- La période du mouvement est la même période que l'excitation.

- Il y a autant de résonance que d'harmonique

Periodic Excitations

Example : mass-spring system under squared waves exscitation.

Fourier series of such system is:

$$F(t) = \frac{F_0}{2} + \frac{2F_0}{\pi} \sum_{p=1,3,5,\dots}^{+\infty} \left(\frac{\sin p\Omega t}{p} \right)$$

The system becomes:

$$m\ddot{x} + kx = \frac{F_0}{2} + \frac{2F_0}{\pi} \sum_{p=1,3,5,\dots}^{+\infty} \left(\frac{\sin p\Omega t}{p} \right)$$

So solutions are composed like

$$\boxed{x = \frac{F_0}{2k} + \frac{2F_0}{\pi} \sum_{p=1,3,5,\dots}^{+\infty} \left(\frac{\sin p\Omega t}{p\left(k - m(p\Omega)^2\right)} \right)}$$

And resonance occur for :

$$k - m(p\Omega)^2 = 0$$ which is similar to $$\boxed{\Omega = \frac{1}{p}\sqrt{\frac{k}{m}} = \frac{1}{p}\omega}$$

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Energy dissipation per cycle

Energy dissipation per cycle

The energy supplied by the external force F(t) during one cycle of vibration is:

$$E = \int_0^T F(t)\frac{dx}{dt}dt$$

$$E = \int_0^{2\pi/\Omega} F(t)\Omega X\cos(\Omega t-\phi)dt$$

$$= F\Omega X\int_0^{2\pi/\Omega}\sin(\Omega t)\cos(\Omega t-\phi)dt$$

$$= F\Omega X\left[\int_0^{2\pi/\Omega}\sin(\Omega t)\cos(\Omega t)\cos(\phi)dt + \int_0^{2\pi/\Omega}\sin^2(\Omega t)cdt\right]$$

$$= \pi XF\sin(\phi)$$

Combining previous results (for linear damping)

$$\sin(\phi) = \frac{c\Omega\cos(\phi)}{(k-m\Omega^2)}$$ and $$\cos(\phi) = \frac{F}{X}\frac{(k-m\Omega^2)}{(k-m\Omega^2)^2+c^2\Omega^2}$$

Energy dissipation per cycle

leads to

$$\sin(\phi) = \frac{c\Omega}{(k - m\Omega^2)} \cos(\phi) = \frac{c\Omega F}{X\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]}$$

$$E = \pi X F \sin(\phi)$$

$$= \pi \frac{c\Omega F^2}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]}$$

From previous results

$$X = \frac{F}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

Then the expression for energy dissipation in one cycle of motion becomes

$$E = \pi c \Omega X^2$$

It is often interesting to present this result by considering the energy dissipated per cycle in the damper:

$$\boxed{E = \int_0^{2\pi/\Omega} c\dot{x} \frac{dx}{dt} dt}$$

Damping in real systems

Systems always have some damping but rarely is this damping viscous. Among the most common forms of damping are **structural** damping and **Coulomb** damping.

Structural damping is a material characteristic whose value can be strongly dependent on both temperature and forcing frequency.

Coulomb damping arises from the relative motion between dry surfaces in contact; it is quite difficult to quantify this phenomenon because it depends on so many parameters.

An equivalent viscous damping coefficient can be defined for the case of harmonic excitation by using the previous expression for energy dissipated per cycle.

For **structural damping** it has been observed that. the energy dissipated per cycle has the form

$$E = aX^2$$

over a limited range of frequency and temperature. X is the displacement amplitude and a is a constant of proportionality.

The coefficient of equivalent viscous damping is found from.

$$E = aX^2 = \pi\, c_{eq}\, \Omega X^2$$

Damping in real systems

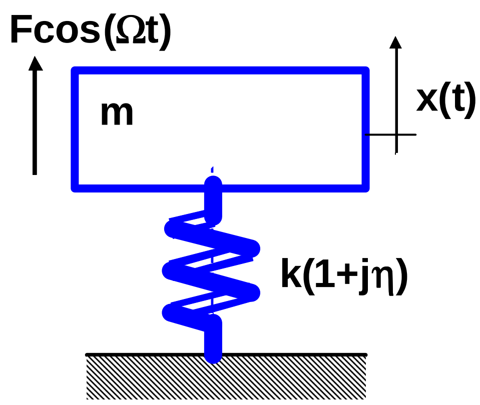
$$c_{eq} = \frac{a}{\pi\Omega}$$

The calculation of systems with structural damping subjected to harmonic excitation is more conveniently achieved with the use of complex notation.

A single degree-of-freedom system with structural damping and excited by the force F cos Ωt.

It has the equation

$$m\ddot{x} + c\dot{x} + kx = F\cos(\Omega t)$$



and with the equivalent viscous damping

$$m\ddot{x} + \frac{a}{\pi\Omega}\dot{x} + kx = F\cos(\Omega t)$$

Using the complex quantity z define as z=x+j.y previous equation is conveniently written as:

$$m\ddot{z} + \frac{a}{\pi\Omega}\dot{z} + kz = Fe^{j\Omega t}$$

Damping in real systems (see TD)

Solution are sought:

$$z(t) = Ze^{j\Omega t}$$

which gives

$$\left(k - m\Omega^2\right)Z + j\frac{a}{\pi}Z = F$$

or

$$-m\Omega^2 Z + k(1 + j\eta)Z = F$$

where :

$$\eta = \frac{a}{\pi k}$$

is called the structural damping factor. In addition, the complex stiffness is defined as

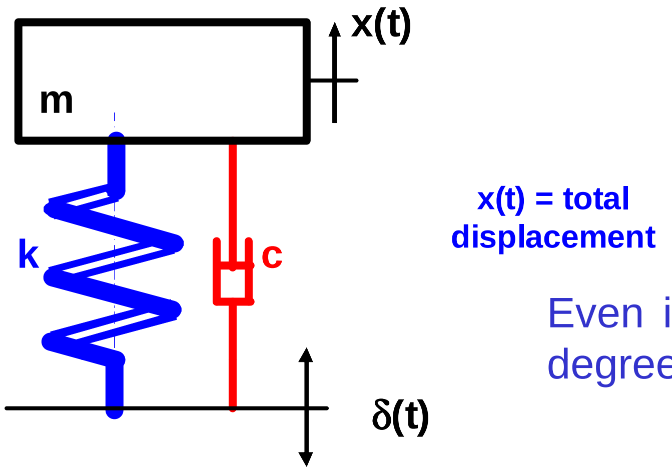
$$k^* = k(1 + j\eta)$$

and η is referred to as the loss factor.

APPLICATIONS

System on a moving foundation

The system is composed of a machine mounted onto a foundation and supported by a spring and viscous damper. The foundation has a imposed displacement.



$$\delta(t) = \Delta \cos \Omega t$$

Even if there is an imposed displacement, that is a single degree-of-freedom system.

It is desired to keep the machine motion, that is the motion of the mass m, to the smallest possible value. This situation also arises for a vehicle going over a rough road or for a container of delicate electronics attached to a vibrating surface. The movement of the mass can be deduced from the equation of motion which can be obtained with a direct application of the 2nd Newton's law

$$\boxed{m\ddot{x} = k(\delta - x) + c(\dot{\delta} - \dot{x})}$$

APPLICATIONS
System on a moving foundation

Previous equation is restated

$$m\ddot{x} + c\dot{x} + kx = k\delta + c\dot{\delta}$$

Using $\delta(t) = \Delta \cos \Omega t$ and $\dot{\delta}(t) = -\Omega\Delta \sin \Omega t$

$$m\ddot{x} + c\dot{x} + kx = \Delta(k \cos \Omega t - c\Omega \sin \Omega t)$$

which is of the type

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

where

$$F(t) = \Delta(k \cos \Omega t - c\Omega \sin \Omega t)$$

APPLICATIONS

System on a moving foundation

From the previous result for of a single damped system in steady state motion:

$$x = X\sin(\Omega t - \phi) \qquad \text{and} \qquad X = \frac{F(t)}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

Then $$F(t) = \Delta(k\cos\Omega t - c\Omega\sin\Omega t)$$

$$|X|\,\Delta\frac{\sqrt{k^2 + c^2\Omega^2}}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

the **nondimensional** form with normalized amplitude and frequency is:

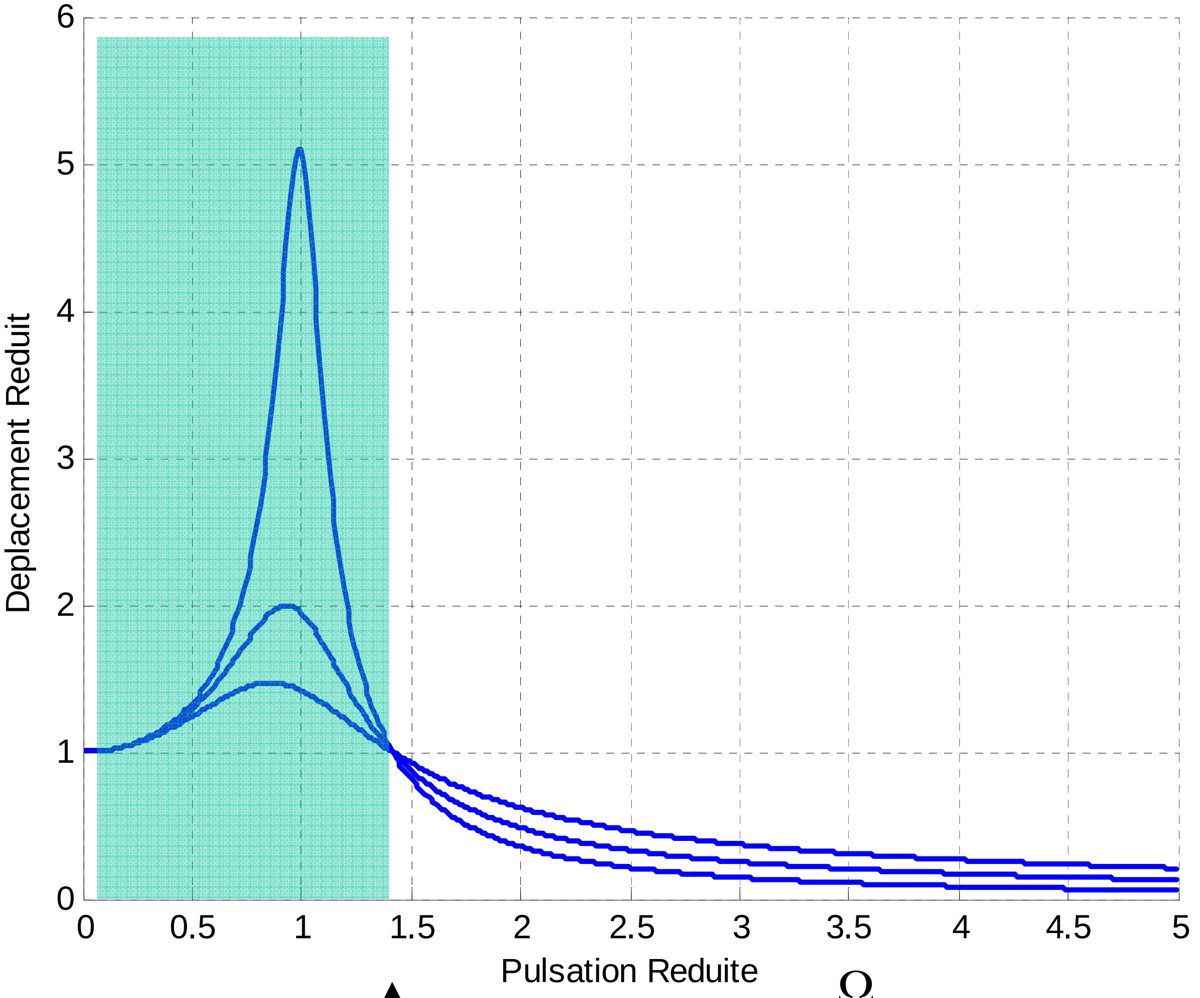
$$|X| = \Delta\sqrt{\frac{1 + [2\alpha(\Omega/\omega)]^2}{[1 - (\Omega/\omega)^2]^2 + [2\alpha(\Omega/\omega)]^2}}$$

The ratio X/X_{st} (the reduced displacement) is plotted in next figure as a function of Ω/ω_0 (the reduced frequency) with α as a parameter. In order to have a small motion of the mass, that is, good isolation, it is required that Ω/ω_0 must be much greater than 1. In other words, the resonant frequency of the system, ω_0, must be as low as possible.

In practice, this is limited by the value of X_{st} which is related to the gravity.

APPLICATIONS
System on a moving foundation

Nice isolation leads to the smallest possible value for the displacements:



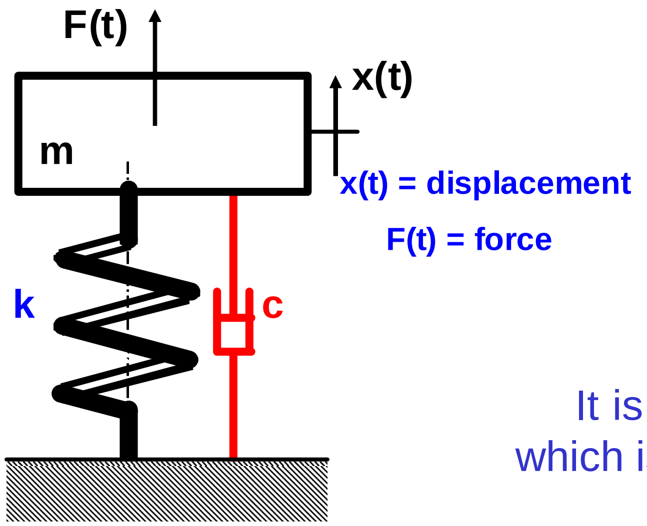
$$\frac{\Omega}{\omega} = \sqrt{2}$$

$$\frac{\Omega}{\omega} >> 1$$

rorot01 (!X/Xst !)

APPLICATIONS
Transmissibility

Now let the mass of the previous system just considered be subjected to the force:



$$F(t) = F\sin(\Omega t)$$

The foundation has a nil displacement.

It is required that the force transmitted to the foundation, which is now fixed, be as small as possible.

The transmitted force is:

For the spring ➡ **k x**

For the damper ➡ **c dx/dt**

then:

$$\boxed{F_t = kx + c\dot{x}}$$

APPLICATIONS
Transmissibility

Recalling the previous result for of a single damped system in steady state motion:

$$x = X\sin(\Omega t - \phi)$$

The amplitude of F(t) is shown to be:

$$F_t = X[k\sin(\Omega t - \phi) + c\Omega\cos(\Omega t - \phi)]$$

Hence the modulus is:

$$|F_t| = X\sqrt{k^2 + c^2\Omega^2}$$

$$= kX\sqrt{1 + [2\alpha(\Omega/\omega)]^2}$$

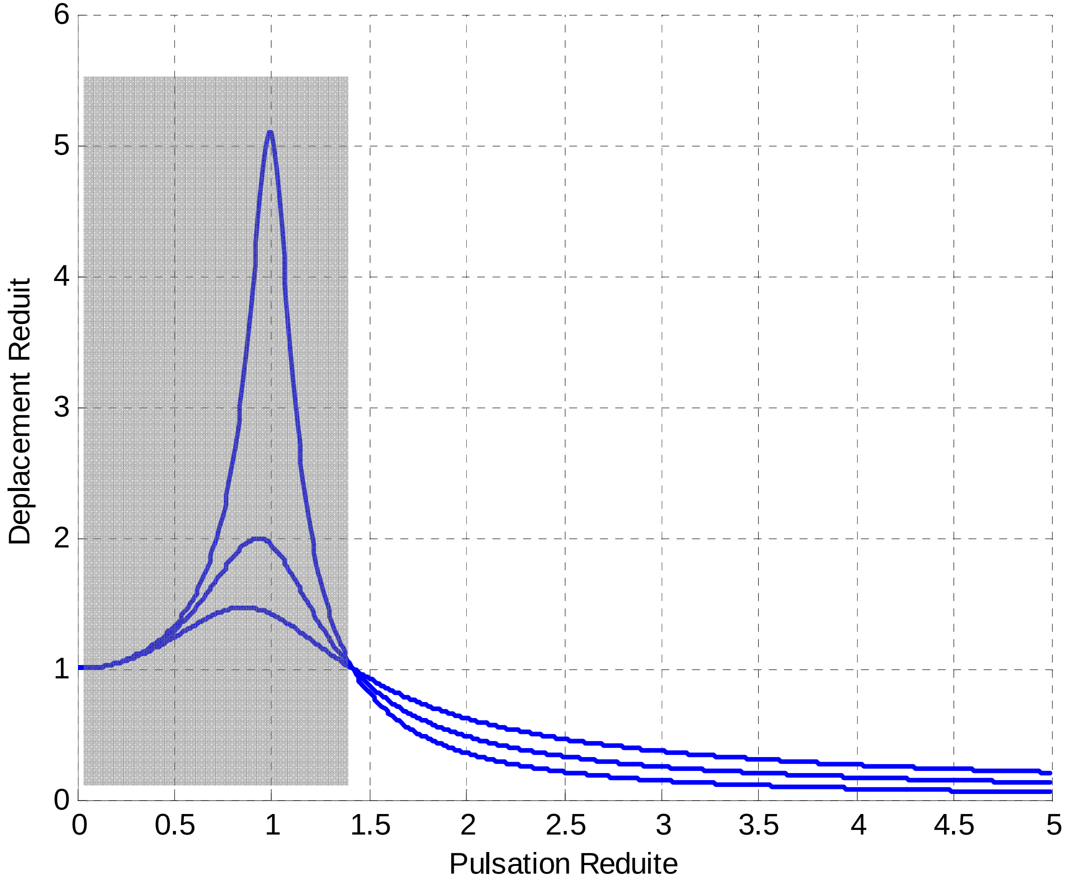
using:

$$\alpha = \frac{c}{2\sqrt{km}}$$

$$\boxed{|F_t| = F\sqrt{\frac{1 + [2\alpha(\Omega/\omega)]^2}{[1 - (\Omega/\omega)^2]^2 + [2\alpha(\Omega/\omega)]^2}}}$$

Expression of the ratio IF_tI/F is identical to ratio X/X_{st} and the conclusion is therefore: to limit the transmitted force, it is necessary that Ω/ω_0 be much greater than 1.

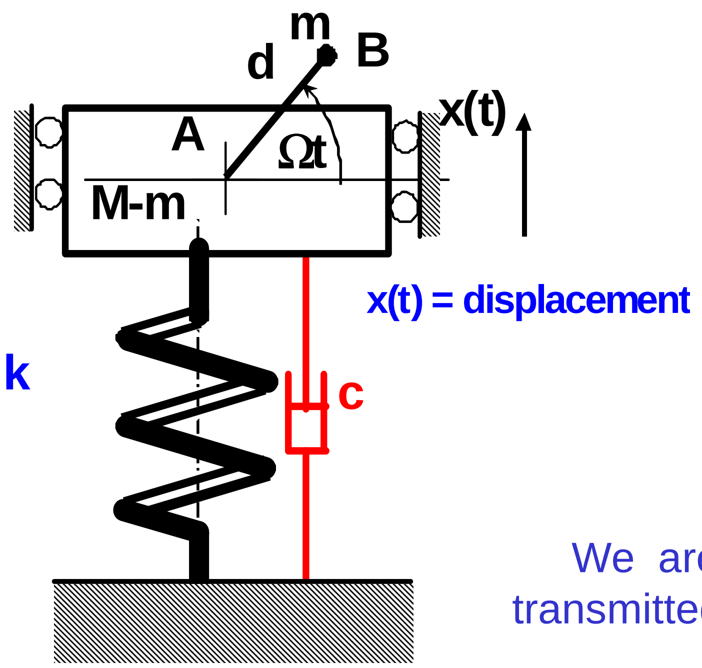
ω_0, must be as low as possible however the restriction due to the value of X_{st} is still active.



$$\frac{\Omega}{\omega} = \sqrt{2}$$

APPLICATIONS

Unbalanced Machine on a fixed foundation



The following system represents an elementary model of a rotating machine of mass **(M-m)** which is attached to a foundation by a spring **k** and by a damper **c**.

The machine has a rotating unbalance of mass **m** which is located at radial distance **d** from an axis of rotation about which the radius has a constant angular velocity **Ω**.

We assume that only motion in possible on the **x** direction.

We are interested on the displacement **X** and on force transmitted to the foundation $\mathbf{F_t}$.

Equation of motion can be obtained with application of the 2nd Newton'law to the whole system A+B:

$$(M-m)\ddot{x} + m\frac{d^2}{dt^2}(x + d\sin\Omega t) =$$

$$=$$

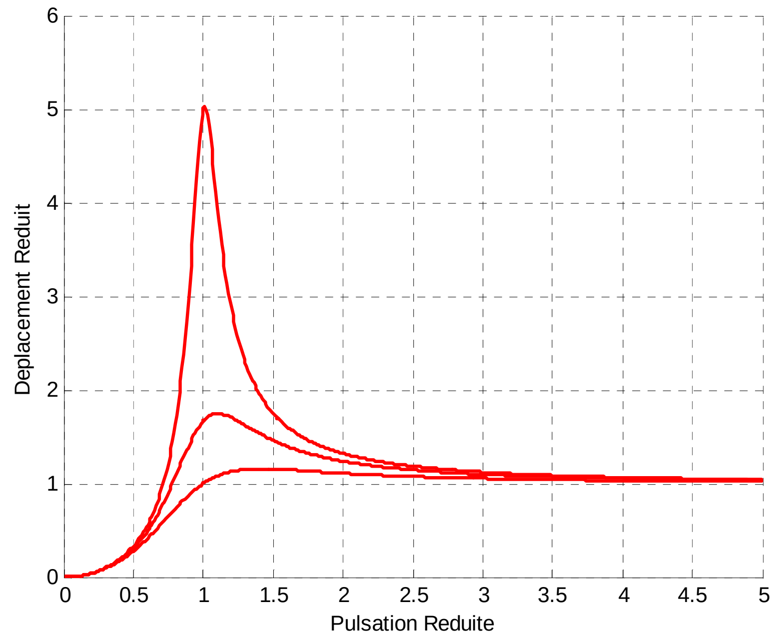
Unbalanced Machine on a fixed foundation (displacement)

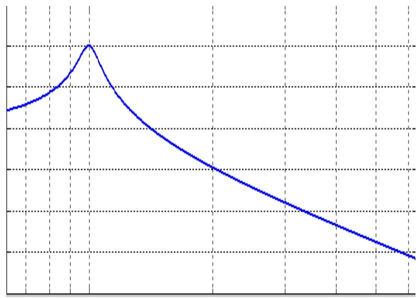
then

$$M\ddot{x} + c\dot{x} + kx = md\Omega^2 \sin \Omega t$$

so, from previous results:

$$\boxed{X = \frac{F}{\sqrt{(k - M\Omega^2)^2 + c^2\Omega^2}} = \frac{md\Omega^2}{\sqrt{(k - M\Omega^2)^2 + c^2\Omega^2}}}$$





rorotm1

Unbalanced Machine on a fixed foundation (force transmitted)

The force transmitted to the foundation F_t.is expressed as:

$$F_t = kx + c\dot{x}$$

with

$$k\,x = k\,X\sin(\Omega t-\phi) = k\frac{md\Omega^2}{\sqrt{(k-M\Omega^2)^2+c^2\Omega^2}}\sin(\Omega t-\phi)$$

$$c\dot{x} = c\Omega X\cos(\Omega t-\phi) = c\Omega\frac{md\Omega^2}{\sqrt{(k-M\Omega^2)^2+c^2\Omega^2}}\cos(\Omega t-\phi)$$

$$|F_t| = md\Omega^2\,\frac{\sqrt{k^2+c^2\Omega^2}}{\sqrt{(k-M\Omega^2)^2+c^2\Omega^2}}$$

or

$$\boxed{|F_t| = md\Omega^2\sqrt{\frac{1+(2\alpha\Omega/\omega)^2}{(1-(\Omega/\omega)^2)^2+(2\alpha\Omega/\omega)^2}}}$$

Unbalanced Machine on a fixed foundation

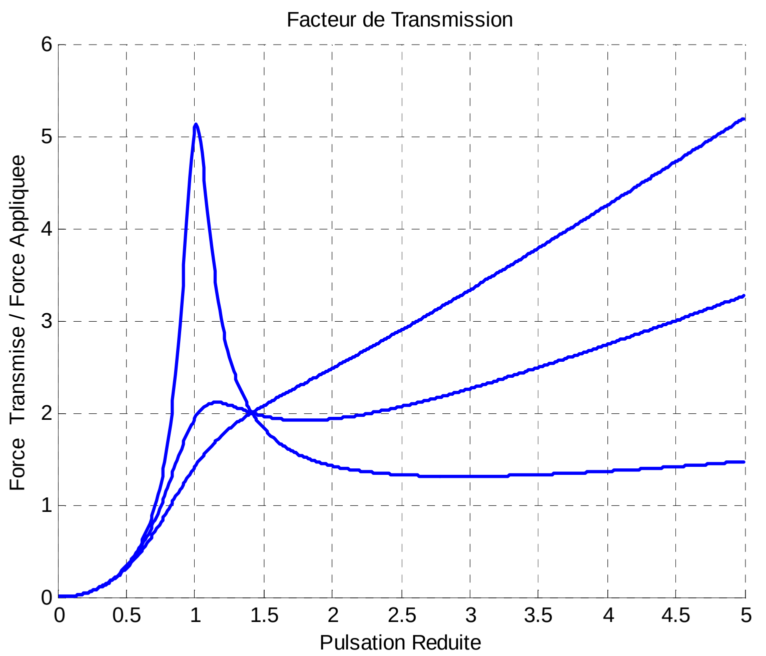
The nondimensional form with normalized amplitudes and frequencies is:

$$\boxed{\frac{M|F_t|}{mdk} = \left(\frac{\Omega}{\omega}\right)^2 \sqrt{\frac{1+(2\alpha\Omega/\omega)^2}{(1-(\Omega/\omega)^2)^2+(2\alpha\Omega/\omega)^2}}}$$

Note that putting the mass directly on the fondation (this meens that M=infinity) leads to a transmitted force proportional to $md^2\,\Omega$:

$$\frac{M|F_t|}{mdk} = \left(\frac{\Omega}{\omega}\right)^2 * (\approx 1)$$

$$|F_t| \approx md\frac{k}{M}\left(\frac{\Omega}{\omega}\right)^2 = md\Omega^2$$

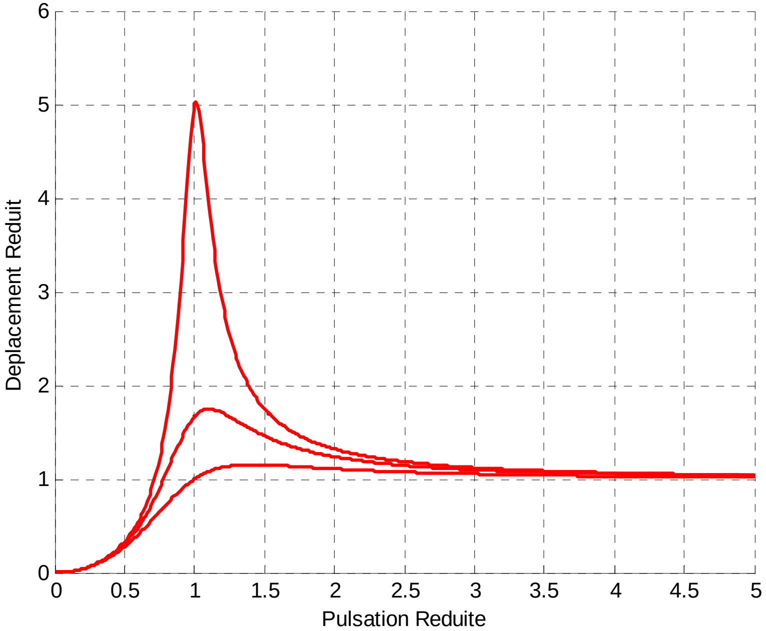


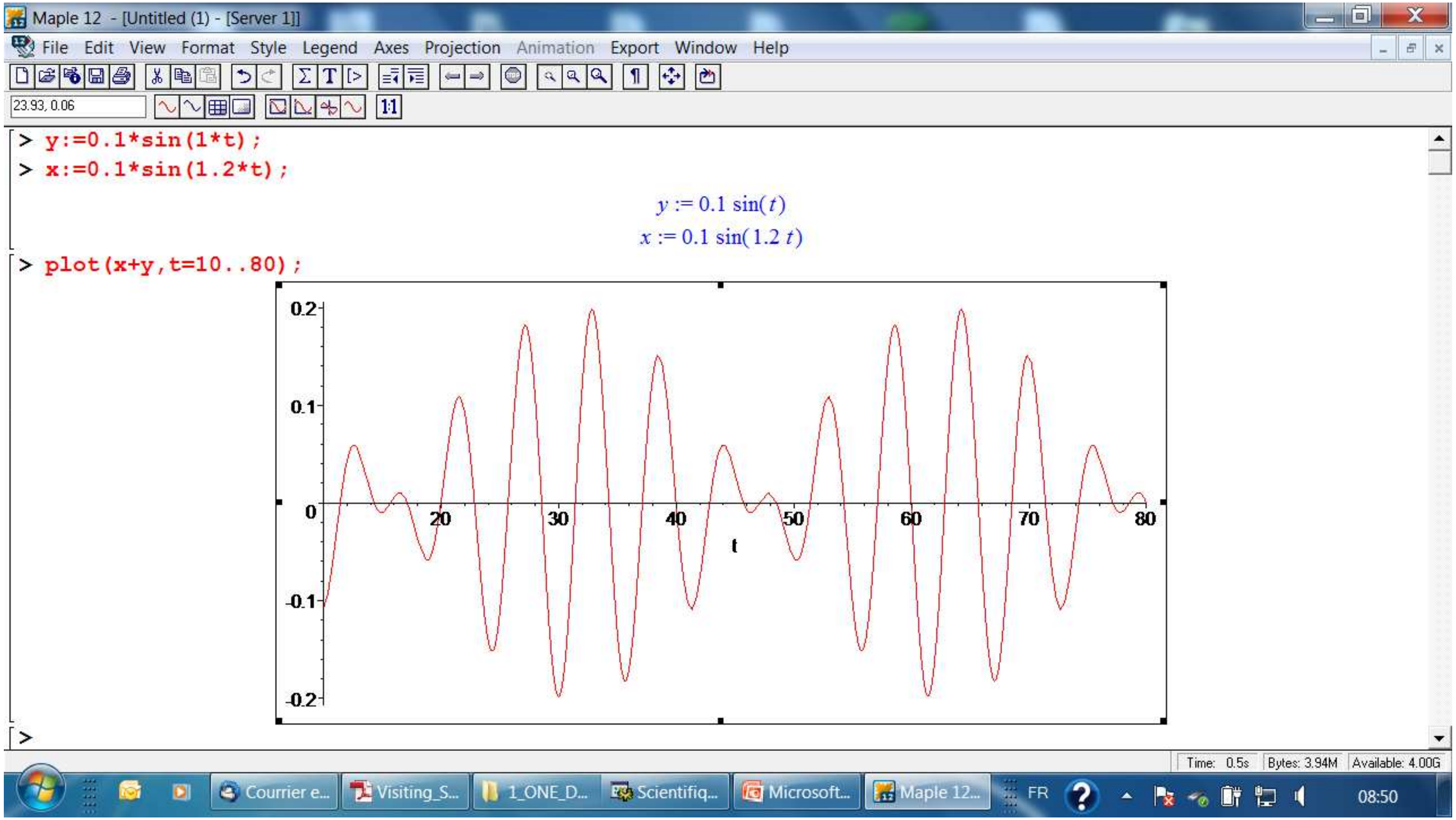
Unbalanced Machine on a fixed foundation

It can be concluded that a good isolation will be achieved with a very soft suspension.

In many applications, the damper is not suitable so that: the amplitude X is nearly constant:

$$\frac{MX}{md} = \frac{MF_t}{mdk} = 1$$




Maple 12 - [Untitled (1) - [Server 1]]
File Edit View Format Style Legend Axes Projection Animation Export Window Help
> y:=0.1*sin(1*t);
> x:=0.1*sin(1.2*t);
y := 0.1 sin(t)
x := 0.1 sin(1.2 t)
> plot(x+y,t=10..80);
0.2
0.1
0
-0.1
-0.2
20
30
40
50
60
70
80
t
Time: 0.5s
Bytes: 3.94M
Available: 4.00G

Hamilton's Principle

The evolution of many physical systems needs the minimization of a physical quantities.

The minimization approach to physics was formalized in detail by Hamilton. This principle can be used to derive the equation of motion of mechanical vibrating systems. The Hamilton's Principle which states:

" *Of all the possible paths along which a dynamical system may more from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* "

It can be can expressed (with external forces) as:

$$\delta\int_{t_1}^{t_2}\left(T-U\right)dt+\int_{t_1}^{t_2}\delta W\;dt=0$$

Where:

U = potential energy

T = kinetic energy

δW = virtual worked expressed with generalized coordinates

Hamilton's Principle – Lagrange's equations

In this approach, we suppose that the number of degrees of freedom is equal to the number of independent coordinates (**Generalized Coordinates**) which are required to describe the system configuration. It follows:

$$U = U\left(q_1, q_2, \cdots, q_n\right)$$

$$T = T\left(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n\right)$$

$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_n \delta q_n$$

Equations of Motion in Generalized Coordinates

Using the derivation procedure:

$$\delta T = \frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \cdots + \frac{\partial T}{\partial q_n}\delta q_n + \boxed{\frac{\partial T}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\delta \dot{q}_2 + \cdots + \frac{\partial T}{\partial \dot{q}_n}\delta \dot{q}_n}$$

$$\delta U = \frac{\partial U}{\partial q_1}\delta q_1 + \frac{\partial U}{\partial q_2}\delta q_2 + \cdots + \frac{\partial U}{\partial q_n}\delta q_n$$

$$\delta W = Q_1\, \delta q_1 + Q_2\, \delta q_2 + \cdots + Q_n\, \delta q_n$$

Hamilton's Principle – Lagrange's equations

A typical term of T related to velocity:

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i\right)dt$$

can be integrated by parts using:

$$\int_{x_1}^{x_2}[u\,v]'dt = \int_{x_1}^{x_2}u'\,v\,dx + \int_{x_1}^{x_2}u\,v'\,dx$$

with:

$$u = \frac{\partial T}{\partial \dot{q}_i} \Rightarrow \dot{u} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$$

$$\dot{v} = \delta\dot{q}_i \Rightarrow v = \delta q_i$$

$$\int_{t_1}^{t_2}u\,v'\,dt = [u\,v]_{t_1}^{t_2} - \int_{t_1}^{t_2}u'\,v\,dt$$

becomes:

$$\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i dt = \left[\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i dt$$

Hamilton's Principle – Lagrange's equations

where the first term of second vanishes (i.e. $\delta q_i(t_1) = \delta q_i(t_2) = 0$). Then substituting in previous equation leads to:

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial q_i} + Q_i \right] \delta q_i \right\} dt$$

$\delta q_i(t_1)$ and $\delta q_i(t_2) = 0$) are arbitrary so, a typical condition is that terms in angle brackets are zeros.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i}$$

This is the so-called Lagrange's equation.

Other form is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

with **L** being the Lagrangian defined as:

$$L = T - U$$