Finite element method for elliptic problems.

MA31-Numerical analysis of Partial Derivative Equations: Courses 11-12

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Outline

Objective of the course

- know the principle of finite element method approximation,
- be able to use it to solve a variational problem,
- be able to control approximation errors of the method.

Remark

This course is widely inspired from a similar course at "Ecole des Mines, ParisTech" from professor Michel Kern, senior searcher at INRIA who is quoted in reference.

Outline

Poisson problem in linear elastic deformation

- Let ϵ be the strain tensor which is the gradient of the displacement field u: $\epsilon = \nabla(u)$
- The Hooke law is governing the **stress tensor** σ: $\sigma = C\epsilon$ where C is the stiffness tensor ,[1].
- The elastic equilibrium of a solid in he hypothesis of small strains is given by the linear law: $\vec{\nabla}\textbf{.}\sigma + \rho f = 0$ [3]. So we get the Navier equation (vector version of Poisson equation $\Delta u + \rho f = 0$
- The boundary conditions on $\partial\Omega$ may be of two kinds
  - Fixed position constraint gives Dirichlet boundary condition on $\partial\Omega_D$: $\forall x \in \partial\Omega_D, u(x) = 0$
  - Applied force governs the stress on $\partial\Omega_N$ and gives Neumann boundary conditions $\forall x \in \partial\Omega_D, \sigma(x).\vec{n} = g(x)$

Linear elastic deformation equation

We shall study here as a simple example the linear elastic deformation of a 2d structure with a normal force (traction or compression). It gives the following Poisson problem with mixed boundary conditions:

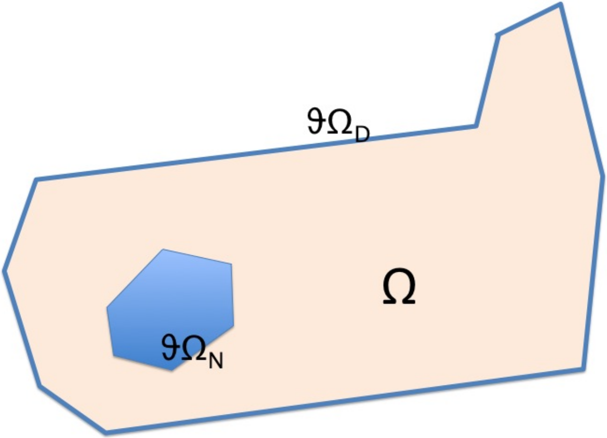
Problem

We give the external forces: the density one $f \in L^2(\Omega)$ and the Neumann boundary condition $g \in L^2(\partial\Omega_N)$

$$\begin{cases} \forall x \in \Omega, & -\vec{\nabla} \cdot \vec{\nabla}(ku)(x) = f \\ \forall x \in \partial\Omega_D, & u(x) = 0 \\ \forall x \in \partial\Omega_N, & k\frac{\partial u}{\partial \vec{n}}(x) = g(x) \end{cases}$$

A typical example

The polynomial boundary is given for further local approximation.



Variational formulation

Applying Stokes formula as in the last course, we get the variational formulation:

Problem

Let $\mathcal{H}_D = \{v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \partial\Omega_D\}$. *Then the problem is to find* $u \in \mathcal{H}_D$ *such that*

$$\forall v \in \mathcal{H}_D, a(u, v) = L(v)$$

with

- $a(u, v) = \int_\Omega k(x)\vec{\nabla}u(x) \,.\, \vec{\nabla}v(x)dx$
- $L(v) = \int_\Omega f(x)v(x)dx + \int_{\partial\Omega_N} g(x)v(x)ds(x)$

Recall Lax-Milgram theorem

Theorem

Let a be a bilinear symmetric continuous and coercive form on a Hilbert space $\mathcal{H}$ and L is a linear continuous form on $\mathcal{H}$:

$$\begin{cases} \forall(v,w) \in \mathcal{H}\times\mathcal{H}, & & a(v,w) = a(w,v) \\ \forall(v,w) \in \mathcal{H}\times\mathcal{H}, & \exists M > 0, & \mid a(v,w) \mid \leq M \parallel v \parallel\parallel w \parallel \\ \forall v \in \mathcal{H}, & \exists \alpha > 0, & a(v,v) \geq \alpha \parallel V \parallel^2 \\ \forall v \in \mathcal{H}, & \exists C > 0, & \mid L(v) \mid \leq C \parallel v \parallel \end{cases}$$

Then $\exists ! u$ *such that* $\forall v \in \mathcal{H}, a(u,v) = L(v)$ *and we have* $\parallel u \parallel \leq \frac{MC}{\alpha}$

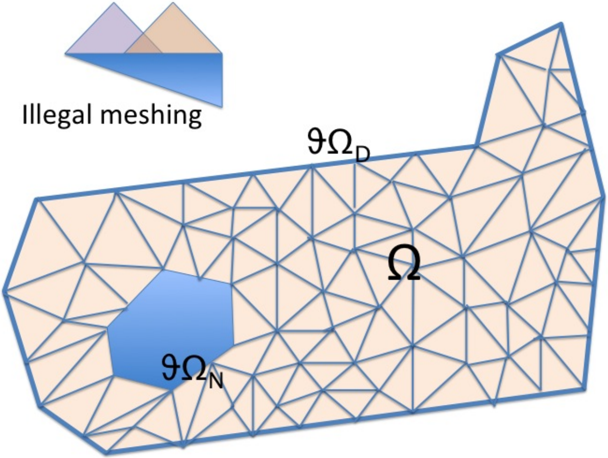
Outline

Triangular meshing

- Like in FD approximation scheme, we shall build an approximation meshing.
- Since we use integral formulation no need of a rectangular meshing to compute partial derivative.
- In 2d equations polygonal meshings are used, in 3d equations polyedral meshings.
- We shall use here the simpler one: triangular meshing.
- Triangular meshing is subject to regularity conditions: Two triangles have either no intersection or a point intersection: one single node) or a line intersection: a whole edge and one only.
- Let $\mathcal{T}$ be the set of triangles, $\mathcal{N} = \{x_j\}$ be the set of nodes and $\mathcal{N}_D$ be the subset which is included in $\partial\Omega_D$.

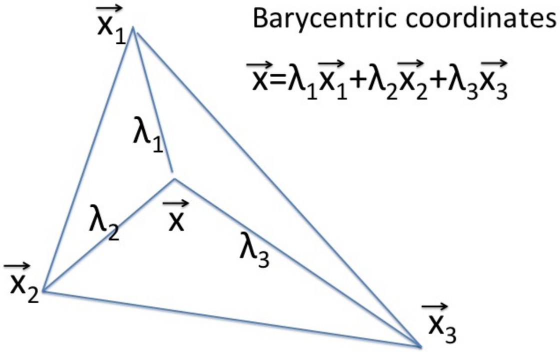
A typical meshing

The polynomial boundary is given for further local approximation.



Parametrization of a triangle

Each point of the convex triangle is defined by three positive barycentric coordinates $(\lambda_1, \lambda_2\lambda_3)$ subject to $\lambda_1 + \lambda_2 + \lambda_3 = 0$



Piecewise affine functions

- If $T_j = (\vec{x_1}, \vec{x_2}, \vec{x_3}) \in \mathcal{T}$, we consider the 3d vector space $\mathcal{F}_j$ of affine functions generated by the affine functions ϕ_1, ϕ_2, ϕ_3 such that $\phi_i(\vec{x_j}) = \delta_{ij}$.
- If $\vec{x} = \lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 + \lambda_3 \vec{x}_3$ then

$$\forall \phi \in \mathcal{F}_j, \phi(\vec{x}) = \lambda_1 \phi_1(\vec{x}_1) + \lambda_2 \phi_2(\vec{x}_2) + \lambda_3 \phi_3(\vec{x}_3)$$

- We shall associate to the meshing the $\mathcal{T}$ the subspace $\mathcal{H}_\mathcal{T} \in \mathcal{H}$ of continuous piecewise affine functions on the triangles of $\mathcal{T}$.
- We shall approximate elements of $\mathcal{H}$ by elements of $\mathcal{H}_\mathcal{T}$ with a uniform convergence when the diameter h of the meshing goes to 0.

Local affine approximation

Consider a continuous function u on Ω and a triangle $\mathcal{T} = (\vec{x_1}, \vec{x_2}, \vec{x_3}) \in \mathcal{T}$, and the affine function on $\mathcal{T}$, $\phi \in \mathcal{F}$ defined by

$$\phi(\lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 + \lambda_3 \vec{x}_3) = \lambda_1 u(\vec{x}_1) + \lambda_2 u(\vec{x}_2) + \lambda_3 u(\vec{x}_3)$$

Then

- ϕ is the unique affine interpolation of u on the three summits of T.
- Moreover since u is uniformly continuous on the bounded domain Ω, when $h \to 0$, the affine interpolation ϕ of u converges uniformly towards u

Outline

1. Introduction
2. Approximation of Poisson equation
   - Poisson problem
   - Local approximation
   - Internal approximation
3. Implementing FEM
   - Linear system assembling

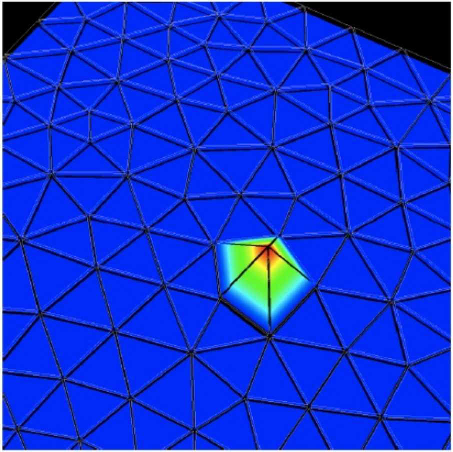
Space of internal approximation

Definition

Let $(\mathcal{T}, \mathcal{N})$ be a triangular meshing of Ω with diameter h, we define by $\mathcal{H}_{\mathcal{T}}$ the subspace of piecewise affine functions and we set $\mathcal{H}_{\mathcal{T},D} = \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_D$.
Let $\vec{x}_i \in \mathcal{N}$, and let ϕ_i be the only function of $\mathcal{H}_{\mathcal{T}}$ such that $\forall \vec{x}_j \in_{\mathcal{N}},\ \phi_i(\vec{x}_j) = \delta_{ij}$.
The interpolation of u in $\mathcal{H}$ is defined by the interpolation operator $I_{\mathcal{T}}$: $\quad I_{\mathcal{T}} u = \sum_{i \in \mathcal{N}} u(\vec{x}_i)\phi_i$

- The set of (ϕ_i) is clearly a basis of $\mathcal{H}_{\mathcal{T}}$ since a continuous piecewise affine function is equal to its interpolation.
- We have a uniform convergence of the interpolates $I_{\mathcal{T}} u$ when the diameter h of $\mathcal{T}$ goes to 0.
- Let $\mathcal{N}_{\mathcal{D}}^c = (\mathcal{N}_D)^c$, the set$(\phi_i)_{i \in \mathcal{N}_{\mathcal{D}}^c}$ is a basis of $\mathcal{H}_{\mathcal{T},D}$

Representing one ϕ_i



Internal approximation of a variational formulation

Consider the variational problem on the Hilbert space $\mathcal{H}$ with the bilinear symmetric continuous coercive form a and the linear continuous form L and a subspace $\mathcal{H}_h$.
We intend to approximate the solution u of

$$\text{Find } u \in \mathcal{H} \text{ such that } \forall v \in \mathcal{H}, a(u, v) = L(v)$$

by the solution u_h of an internal approximate problem

$$\text{Find } u_h \in \mathcal{H}_h \text{ such that } \forall v \in \mathcal{H}_h, a(u, v) = L(v)$$

We show the following projection theorem

Theorem

*The bilinear form a defines an equivalent norm on the Hilbert space $\mathcal{H}$ (**the energy norm**). The approximate solution u_h is the orthogonal projection for this norm of $\mathcal{H}$ onto $\mathcal{H}_h$.*

Proof of the projection theorem

- The properties of the energy product a gives
$$\forall v \in \mathcal{H}, \alpha \parallel v \parallel^2 \leq a(v, v) \leq M \parallel v \parallel^2$$
So the energy norm is equivalent to the original norm.
- We have
$$\forall v \in \mathcal{H}_h, a(u, v) = a(u_h, v) \Rightarrow a(u - u_h, v) = 0$$
It proves that u_h is the orthogonal projection of u onto $\mathcal{H}_h$.
- More over the projection property
$a(u - u_h, u - h) = \inf_{v \in \mathcal{H}_h} a(u - v, u - v)$ implies
$\alpha \parallel u - u_h \parallel^2 \leq M \inf_{v \in \mathcal{H}_h} \parallel u - v \parallel^2$
We use it to show the convergence of internal approximation.

Outline

Linear system assembling

The solution of the approximated variational problem

Problem

Find $u \in \mathcal{H}_{\mathcal{T},D}$ *such that* $\forall v \in \mathcal{H}_{\mathcal{T},D}, a(u,v) = L(v)$

amounts to solve the linear system $\forall j \in \mathcal{N}, \sum_{i \in \mathcal{N}} a_{ji} u_i = b_j$ with $a_{i,j} = a(\phi_i, \phi_j)$ and $b_j = L(\phi_j)$

- If the nodes i and j are not the summits of one common element $a_{i,j} = 0$
- The computation is done element by element to avoid complexity.
- Each element adds its contribution to the $a_{i,j}$ and the b_j associated to the summits of that element.

Dirichlet boundary conditions

- The previous linear system does not take into account the Dirichlet boundary conditions. Actually the values of the functions $\phi_i \in \partial\Omega_D$ are constrained to 0 (in case of Dirichlet homogeneous boundary condition on $\partial\Omega_D$
- If we separate from the other nodes the nodes of $\mathcal{N}_D$, the matrix decomposition of the previous system

$$\begin{pmatrix} A_{II} & A_{ID} \\ A_{DI} & A_{DD} \end{pmatrix} \begin{pmatrix} u_I \\ u_D \end{pmatrix} = \begin{pmatrix} b_I \\ b_D \end{pmatrix}$$

becomes

$$\begin{pmatrix} A_{II} & 0 \\ 0 & A_{DD} \end{pmatrix} \begin{pmatrix} u_I \\ u_D \end{pmatrix} = \begin{pmatrix} b_I - A_{ID} g_D \\ g_D \end{pmatrix}$$

- the matrix A is called the **stiffness matrix** of the meshing.

For Further Reading I

- Yves Debard
  Méthode des éléments finis: élasticité plane .
  IUT du Mans, 2006-2007(free internet available).
- Michel Kern
  Introduction à la méthode des éléments finis .
  Ecole Nationale Supérieure des Mines de Paris, 2004-2005(free internet available).
- Christian Weilgosz, Bernard Peseux, Yves Lecointe
  Formulations mathématiques et résolution numérique en mécanique .
  M2R Université de Nantes, 2004(free internet available).

For Further Reading II

Stéphanie Salmon
Calcul scientifique .
M1Université de Strasbourg, 2009(free internet available).