Solution of plane problems in polar coordinate

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-Coordinates transformation of stress

-Polar expression of basic equations

-Stress function and compatibility equations in polar coordinate

-Axisymmetric stress and its displacement

Solution of plane problems in polar coordinate

-Coordinates transformation of stress

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\dfrac{y}{x} \end{cases} \tag{7.1}$$

Relations of two coordinate are

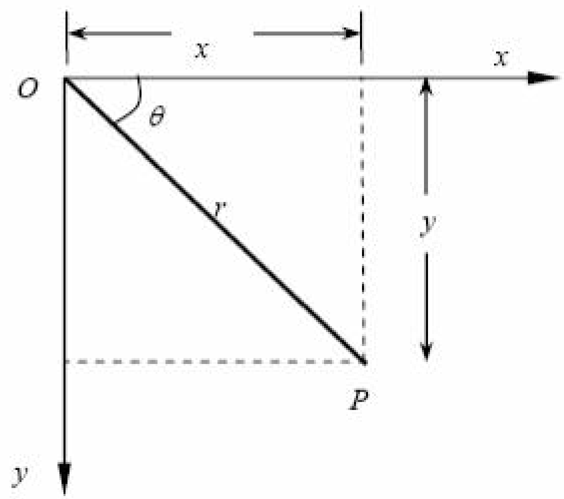


Figure 7.1 Transformation of stress description

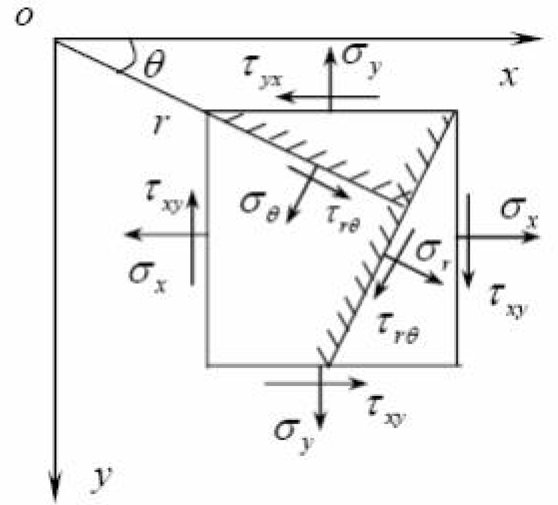


Figure 7.2 Transformation of stress components

$$\begin{cases} \sigma_x = \dfrac{\sigma_r + \sigma_\theta}{2} + \dfrac{\sigma_r - \sigma_\theta}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta \\ \sigma_y = \dfrac{\sigma_r + \sigma_\theta}{2} - \dfrac{\sigma_r - \sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta \\ \tau_{xy} = \dfrac{\sigma_r - \sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta \end{cases} \tag{7.2}$$

$$\begin{cases} \sigma_r = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \tau_{r\theta} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases} \tag{7.3}$$

-Polar expression of basic equations

a.Equilibrium differential equation in polar coordinate

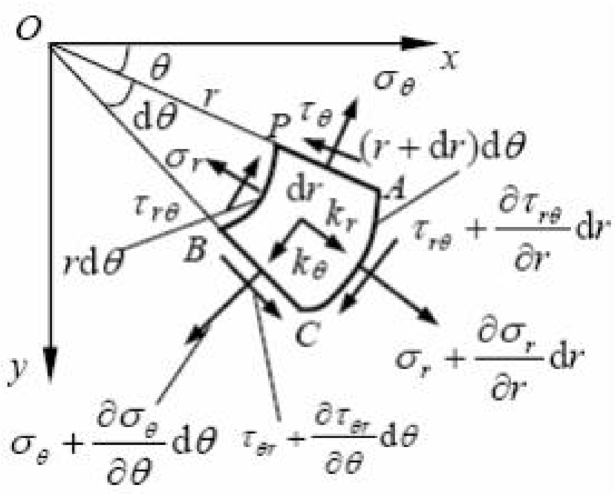


Figure 7.3 a balanced infinitesimal in polar coordinate

Stress components in surface ***AC*** and surface ***BC*** are

$$AC:\begin{cases}\sigma_r+\dfrac{\partial\sigma_r}{\partial r}\mathrm{d}r\\[2ex]\tau_{r\theta}+\dfrac{\partial\tau_{r\theta}}{\partial r}\mathrm{d}r\end{cases}$$

$$BC:\begin{cases}\sigma_\theta+\dfrac{\partial\sigma_\theta}{\partial\theta}\mathrm{d}\theta\\[2ex]\tau_{\theta r}+\dfrac{\partial\tau_{\theta r}}{\partial\theta}\mathrm{d}\theta\end{cases}$$

As an infinitesimal part of balanced body, unit ***PACB*** satisfy statics equation

$$\sum F_r=0,\ \left(\sigma_r+\frac{\partial\sigma_r}{\partial r}dr\right)rd\theta-\sigma_r rd\theta+\left(\tau_{\theta r}+\frac{\partial\tau_{\theta r}}{\partial\theta}d\theta\right)dr-\tau_{\theta r}dr+K_r rd\theta dr-$$
$$\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}d\theta\right)dr\sin(\frac{d\theta}{2})-\sigma_\theta dr\sin(\frac{d\theta}{2})\ =0$$

As a differential unit, $\sin(\frac{d\theta}{2})\approx\frac{d\theta}{2}$, simplification is made

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}-\frac{\sigma_r-\sigma_\theta}{r}+K_r\ =0$$

As a differential unit, $\sin(\frac{d\theta}{2})\approx\frac{d\theta}{2}$, simplification is made

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta}-\frac{\sigma_r-\sigma_\theta}{r}+K_r \;=0$$

At another direction,

$$\sum F_\theta=0,\ \left(\sigma_\theta+\frac{\partial \sigma_\theta}{\partial \theta}d\theta\right)dr-\sigma_\theta dr+\left(\tau_{r\theta}+\frac{\partial \tau_{r\theta}}{\partial r}dr\right)(r+dr)d\theta-\tau_{r\theta}rd\theta+K_\theta rdrd\theta+$$
$$\left(\tau_{\theta r}+\frac{\partial \tau_{\theta r}}{\partial \theta}d\theta\right)dr\frac{d\theta}{2}+\tau_{\theta r}dr\frac{d\theta}{2}\;=0$$

The simplified type is

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{\partial \tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+K_\theta \;=0$$

According to equilibrium of momentum to centre of unit, reciprocal theorem of shear stress is proved again.

$$\boxed{\tau_{r\theta}=\tau_{\theta r}}$$

Thus, differential equilibrium equations in polar system are

$$\boxed{\begin{cases}\dfrac{\partial \sigma_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta}+\dfrac{\sigma_r-\sigma_\theta}{r}+K_r \;=0\\[2ex] \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta}+\dfrac{\partial \tau_{r\theta}}{\partial r}+\dfrac{2\tau_{r\theta}}{r}+K_\theta \;=0\end{cases}\qquad(7.4)}$$

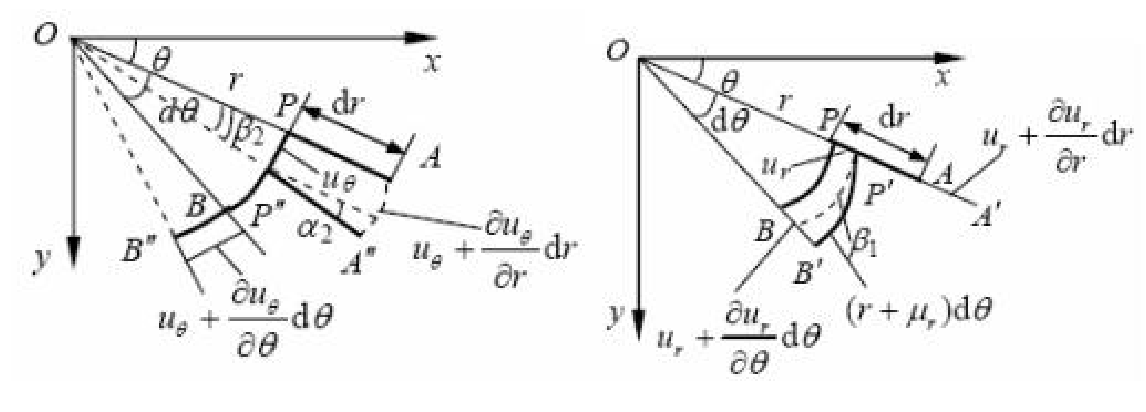
-Equilibrium differential equation in polar coordinate

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\sigma_r - \sigma_\theta}{r} + K_r = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{2\tau_{r\theta}}{r} + K_\theta = 0 \end{cases} \tag{7.4}$$

-reciprocal theorem of shear stress

$$\boxed{\tau_{r\theta} = \tau_{\theta r}}$$

-Geometric equations in polar coordinate



a. displacement in radial direction b. displacement in circumference direction

Figure 7.4 strain components under different displacements in polar system

To build up relation of strain and displacement, two kinds of displacement components μ_r and μ_θ are analyzed separately according to superposition principle.

(1). Presuming point P of differential unit only has radial displacement u , strain components of unit $\varepsilon_{r1}, \varepsilon_{\theta 1}$ are relative elongation of differential segment PA and PB individually. Rotation angle of two segments are denoted as α_1 and β_1 .

$$\varepsilon_{r1} = \frac{P'A' - PA}{PA} = \frac{AA' - PP'}{PA}$$

$$= \frac{u_r + \frac{\partial u_r}{\partial r}\mathrm{d}r - u_r}{\mathrm{d}r} = \frac{\partial u_r}{\partial r}$$

And $\alpha_1 = 0$

At same moment

$$\varepsilon_{\theta 1} = \frac{P'B' - PB}{PB} = \frac{(r + u_r)\mathrm{d}\theta - r\mathrm{d}\theta}{r\mathrm{d}\theta} = \frac{u_r}{r}$$

And

$$\tan \beta_1 \approx \beta_1 = \frac{BB' - PP'}{PB} = \frac{\left(u_r + \frac{\partial u_r}{\partial \theta}\mathrm{d}\theta\right) - u_r}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

Then, Shear strain is

$$\gamma_{r\theta 1} = \alpha_1 + \beta_1 = \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

(2) Presuming point P of differential unit only has circumference displacement μ_θ, strain of unit $\varepsilon_{r2}, \varepsilon_{\theta 2}$ are relative elongation of differential segment PA and PB individually . Rotation angle of two segments are denoted as α_2 and β_2.

Relative elongation ε_{r2} is

$$\varepsilon_{r2} = \frac{P''A'' - PA}{PA} = \frac{\mathrm{d}r - \mathrm{d}r}{\mathrm{d}r} = 0$$

Rotation angle is

$$\alpha_2 = \frac{u_\theta + \frac{\partial u_\theta}{\partial r}\mathrm{d}r - u_\theta}{\mathrm{d}r} = \frac{\partial u_\theta}{\partial r}$$

Relative elongation $\varepsilon_{\theta 2}$ is

$$\varepsilon_{\theta 2} = \frac{P''B'' - PB}{PB} = \frac{BB'' - PP''}{PB} = \frac{u_\theta + \frac{\partial u_\theta}{\partial \theta}\mathrm{d}\theta - u_\theta}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

Angle of ***PB*** is

$$\beta_2 = -\frac{u_\theta}{r}$$

Thus,

$$\gamma_{r\theta 2} = \alpha_2 + \beta_2 = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

To sum up, if point P has displacement μ_θ and μ_r , strain components are

$$\boxed{\begin{cases} \varepsilon_r = \varepsilon_{r1} + \varepsilon_{r2} = \dfrac{\partial u_r}{\partial r} + 0 = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \varepsilon_{\theta 1} + \varepsilon_{\theta 2} = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \gamma_{r\theta 1} + \gamma_{r\theta 2} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}} \tag{7.5}$$

-Physical equations in polar coordinate

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \mu\sigma_\theta) \\ \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_r) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} = \dfrac{2(1+\mu)}{E}\tau_{r\theta} \end{cases} \tag{7.6}$$

To plane strain problems, physical equations are

$$\begin{cases} \varepsilon_r = \dfrac{1+\mu}{E}[(1-\mu)\sigma_r - \mu\sigma_\theta] \\ \varepsilon_\theta = \dfrac{1+\mu}{E}[(1-\mu)\sigma_\theta - \mu\sigma_r] \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} \end{cases} \tag{7.7}$$

-Stress function and compatibility equations in polar coordinate

According to equations (7.1), relations of derivatives of different systems are

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta, & \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r}, & \dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$

$$\nabla^2\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi = 0 \qquad (7.8)$$

or

$$\nabla^4\varphi = \nabla^2\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \varphi = 0 \qquad (7.9)$$

Thus, stress components can be written as

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \varphi}{\partial \theta^2} \\ \sigma_\theta = \dfrac{\partial^2 \varphi}{\partial r^2} \\ \tau_{r\theta} = \dfrac{1}{r^2}\dfrac{\partial \varphi}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 \varphi}{\partial r \partial \theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta}\right) \end{cases} \qquad (7.10)$$

-Axisymmetric stress and its displacement

In engineering, if shape and load of elastomer are all independent to coordinate θ, it can be classified as plane axisymmetric problems, which can solved by inverse method.

Considering $\varphi = \varphi(r)$, solution (7.10) is changed as

$$\sigma_r = \frac{1}{r}\frac{d\varphi}{dr} \quad , \quad \sigma_\theta = \frac{d^2\varphi}{dr^2} \quad , \quad \tau_{r\theta} = 0$$

While, compatibility equation (7.9) will be

$$\nabla^4\varphi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \varphi = 0 \qquad (7.11)$$

Expand (7.11)

$$\frac{d^4\varphi}{dr^4} + \frac{2}{r}\frac{d^3\varphi}{dr^3} - \frac{1}{r^2}\frac{d^2\varphi}{dr^2} + \frac{1}{r^3}\frac{d\varphi}{dr} = 0$$

It is a fourth- order variable coefficient homogeneous differential equation, which can be re-written as Euler homogeneous differential equation

$$r^4\frac{d^4\varphi}{dr^4} + 2r^3\frac{d^3\varphi}{dr^3} - r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} = 0 \qquad (7.12)$$

Equation (7.12) has a common solution

$$\varphi = A\ln r + Br^2\ln r + Cr^2 + D \qquad (7.13)$$

Where A, B, C, and D are undecided coefficients.

Thus, stress components are

$$\begin{cases} \sigma_r = \dfrac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_\theta = -\dfrac{A}{r^2} + B(3 + 2\ln r) + 2C \\ \tau_{r\theta} = \tau_{\theta r} = 0 \end{cases} \tag{7.14}$$

To plane stress problems, solution (7.14) can be taken into physical equation (7.6)

$$\begin{cases} \dfrac{\partial u_r}{\partial r} = \varepsilon_r = \dfrac{1}{E}(\sigma_r - \mu\sigma_\theta) \\ \qquad = \dfrac{1}{E}\left[(1+\mu)\dfrac{A}{r^2} + (1-3\mu)B + 2(1-\mu)B\ln r + 2(1-\mu)C\right] \\ \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} = \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_r) \\ \qquad = \dfrac{1}{E}\left[-(1+\mu)\dfrac{A}{r^2} + (3-\mu)B + 2(1-\mu)B\ln r + 2(1-\mu)C\right] \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} = 0 \end{cases} \tag{a}$$

Integrating the first equation of (a)

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\ln r - 1) + (1-3\mu)Br + 2(1-\mu)Cr\right] + f(\theta) \tag{b}$$

Where $f(\theta)$ can be any function.

Taking equation (b) into the second equation of (a)

$$\frac{\partial u_\theta}{\partial \theta} = \frac{r}{E}\left[-(1+\mu)\frac{A}{r^2} + (3-\mu)B + 2(1-\mu)B\ln r + 2(1-\mu)C\right] - u_r$$

$$= \frac{4Br}{E} - f(\theta)$$

Integrating this equation

$$u_\theta = \frac{4Br\theta}{E} - \int f(\theta)\mathrm{d}\theta + f_1(r) \tag{c}$$

Where $f_1(r)$ can be any function.

Taking equation (b) and (c) into the third equation of (a)

$$f_1(r) - r\frac{\mathrm{d}f_1(r)}{\mathrm{d}r} = \frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta} + \int f(\theta)\mathrm{d}\theta$$

To make above equation possible whatever r and θ are, both sides have to equal a constant, namely

$$f_1(r) - r\frac{\mathrm{d}f_1(r)}{\mathrm{d}r} = F \qquad \text{(d)}$$

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta} + \int f(\theta)\mathrm{d}\theta = F \qquad \text{(e)}$$

Solution of (d) is

$$f_1(r) = Hr + F \qquad \text{(f)}$$

Differentiate (e)

$$\frac{\mathrm{d}^2 f(\theta)}{\mathrm{d}\theta^2} + f(\theta) = 0$$

Its solution is

$$f(\theta) = I\cos\theta + K\sin\theta \qquad \text{(g)}$$

According to equation (b) and (c)

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\ln r - 1) + (1-3\mu)Br + 2(1-\mu)Cr \right] + I\cos\theta + K\sin\theta$$

$$u_\theta = \frac{4Br\theta}{E} + Hr - I\sin\theta + K\cos\theta \qquad (7.15)$$

Where A, B, C, H, I, and K are decided by boundary conditions.

According to (7.15), axisymmetric stress components must not mean axisymmetric displacement components. Only in case of axisymmetric shape, load, and constraints, axisymmetric stress components result in axisymmetric displacement components. Under this circumstance, $\mu_\theta = 0$, and $B = H = I = K = 0$. We have

$$\begin{cases} u_r = \dfrac{1}{E}\left[-(1+\mu)\dfrac{A}{r} + 2(1-\mu)Cr \right] \\ u_\theta = 0 \end{cases} \qquad (7.16)$$

To plane strain problems, equation (7.15) is same valid, only if substitute E and μ with

$\dfrac{E}{1-\mu^2}$ and $\dfrac{\mu}{1-\mu}$.