Eléments finis de Barre

A bar is a beam with a join ball on its two sides

Théorème : A bar can work only in traction or compression.

Démonstration : We use the equilibrium.

$$\overrightarrow{F_A} + \overrightarrow{F_B} = 0$$

$$\overrightarrow{AB} \wedge \vec{F}_B = \vec{0}$$

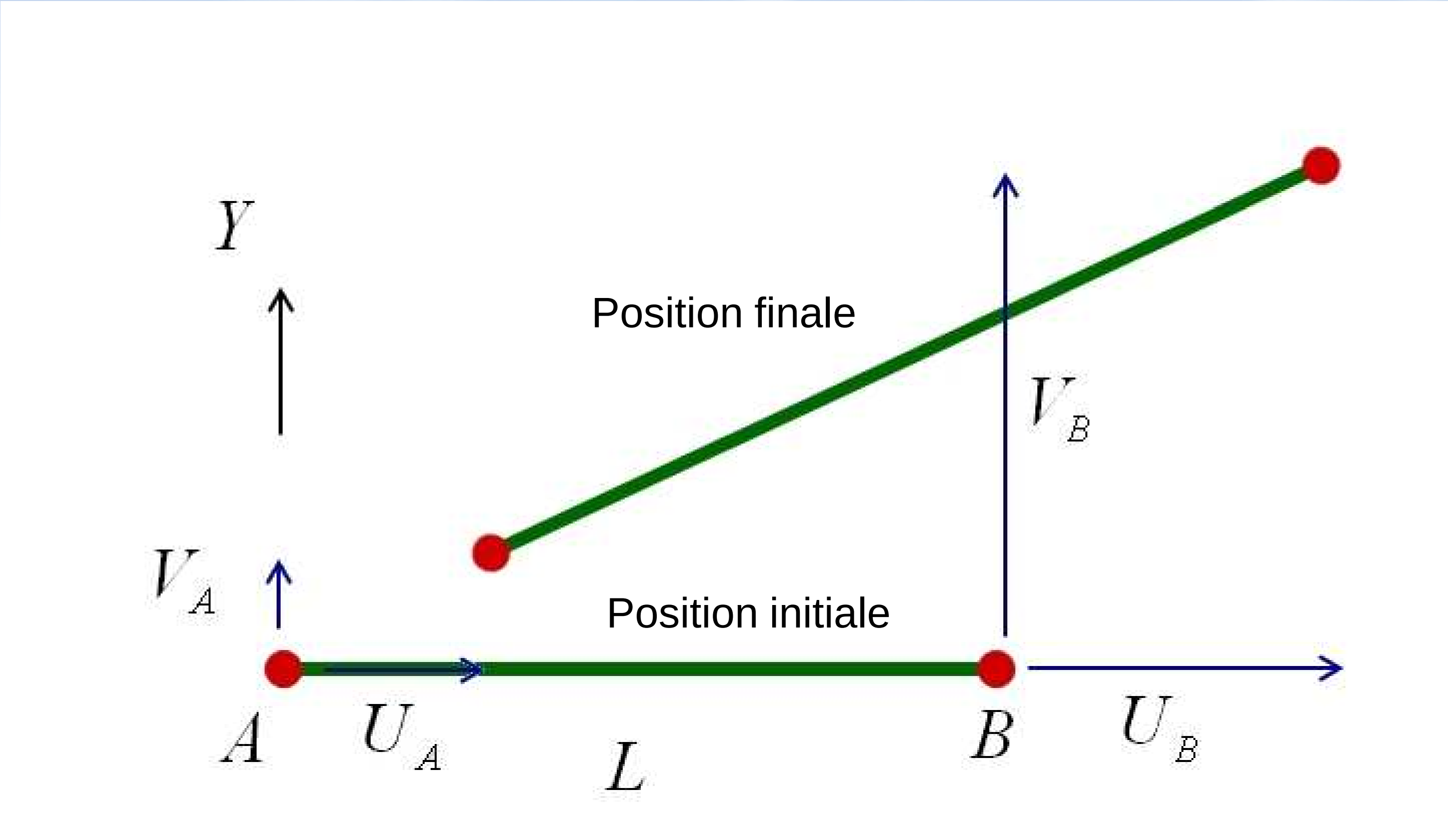
Eléments finis de Barre

$$X_A + X_B = 0$$

Matrice de rigidité d’une barre à section constante

Area : S

Young Modulus: E



Stiffness matrix of a bar

- Stress tensor

$$[\Sigma] = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Hooke law

$$\sigma_x = \frac{\varepsilon_x}{E}$$

- Strain

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

- Internal Energy

$$W = \iiint_{Vol} \frac{1}{2} trace\left([\Sigma][E]\right) dv$$

Stiffness matrix of a bar

$$W = \frac{1}{2}\begin{pmatrix} U_A & V_A & U_B & V_B \end{pmatrix}\begin{bmatrix} \frac{ES}{L} & 0 & -\frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} U_A \\ V_A \\ U_B \\ V_B \end{pmatrix}$$

The internal energy does not depend of the displacement in the Y direction

Th same for the stresses ans strains

$$K = \frac{ES}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \quad , \quad K = \frac{ES}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Work of the external forces

$$V = -\begin{pmatrix} U_A & V_A & U_B & V_B \end{pmatrix} \begin{pmatrix} X_A \\ 0 \\ X_B \\ 0 \end{pmatrix} \qquad V = -\begin{pmatrix} U_A & U_B \end{pmatrix} \begin{pmatrix} X_A \\ X_B \end{pmatrix}$$

Minimum of total energy :

$$\begin{bmatrix} +\dfrac{ES}{L} & -\dfrac{ES}{L} \\ -\dfrac{ES}{L} & +\dfrac{ES}{L} \end{bmatrix} \begin{pmatrix} U_A \\ U_B \end{pmatrix} = \begin{pmatrix} X_A \\ X_B \end{pmatrix}$$

Bar Element

Displacement

$$\varepsilon_{xx} = \frac{du}{dx} = C^{te} \quad \Rightarrow \quad u(x) = ax + b$$

$$u(x) = \left(\frac{q_2 - q_1}{L} \right) x + q_1$$

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{q_2 - q_1}{L}$$

Rigid Body movement

- Mode propres
  - Mode rigide

$$\text{Premier Vecteur Propre} \quad \text{Mode RIGIDE} \quad \begin{pmatrix} U_A \\ U_B \end{pmatrix} = \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

$$\text{Valeur propre} \quad \lambda_1 = 0$$

  - Mode élastique

$$\text{Deuxième Vecteur Propre} \quad \text{Mode ELASTIQUE} \quad \begin{pmatrix} U_A \\ U_B \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$\text{Valeur propre} \quad \lambda_2 = 2\frac{ES}{L}$$

General axis system

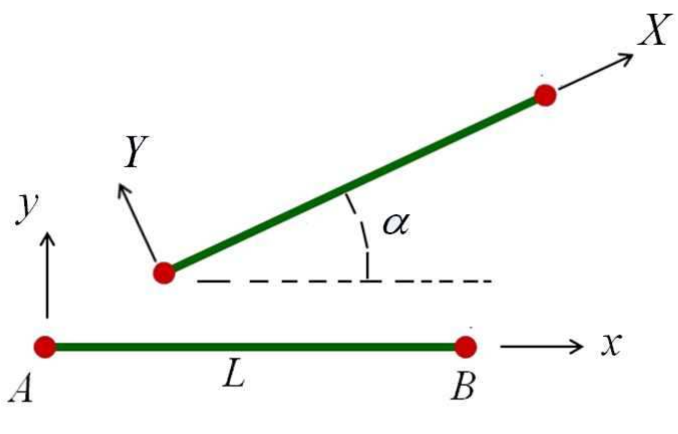
- All the bars have the same stiffness matrix
- We need to change of axis system

$$\begin{pmatrix} \vec{X} \\ \vec{Y} \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \end{pmatrix}$$

$$\begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{X} & \vec{Y} \end{pmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

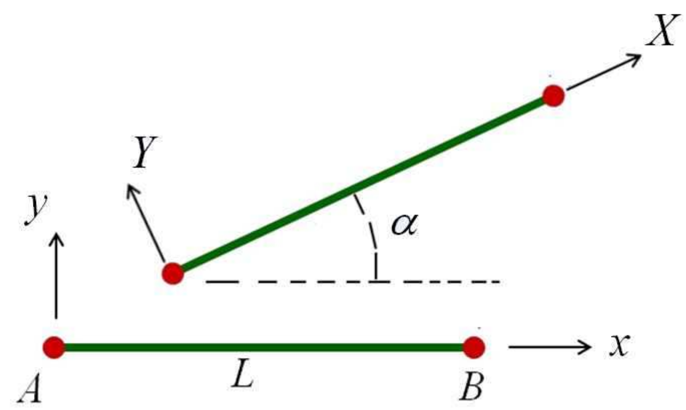
$$\begin{pmatrix} \vec{X} & \vec{Y} \end{pmatrix} = \begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$



$$P = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$P^{-1} = P^{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

General Axis system



$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}$$

$$\begin{pmatrix} \vec{X} \\ \vec{Y} \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \end{pmatrix}$$

$$\begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{X} & \vec{Y} \end{pmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\begin{pmatrix} \vec{X} & \vec{Y} \end{pmatrix} = \begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$P = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$P^{-1} = P^{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

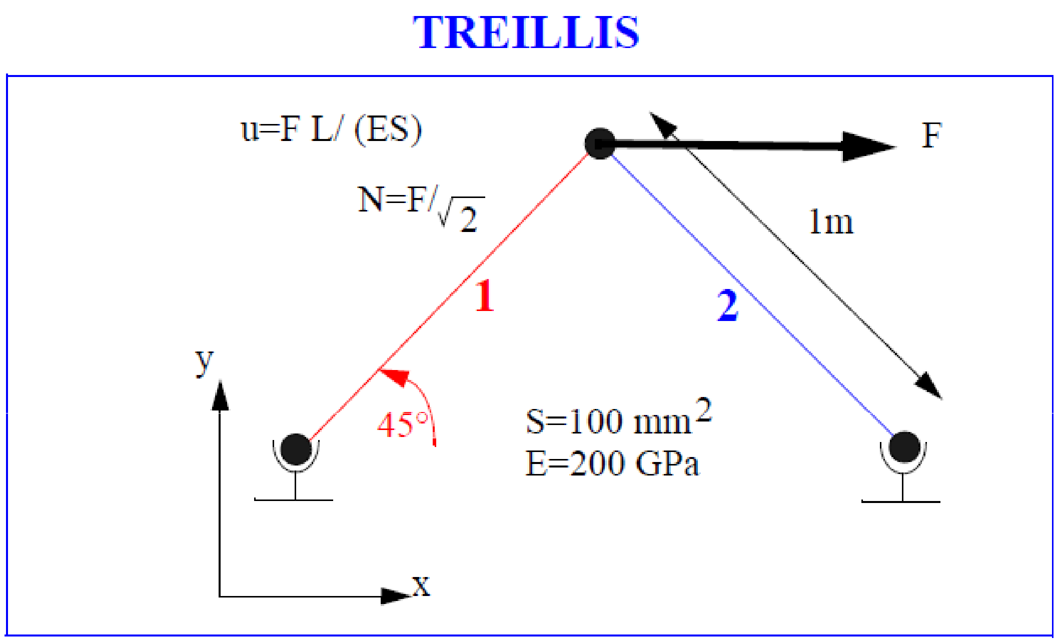
Stiffness Matrix in the global axix system

$$W=\frac{1}{2}\begin{pmatrix}U_A & V_A & U_B & V_B\end{pmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}U_A\\ V_A\\ U_B\\ V_B\end{pmatrix}$$

$$W=\frac{1}{2}\begin{pmatrix}u_A & v_A & u_B & v_B\end{pmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha & 0 & 0\\ \sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & -\sin\alpha\\ 0 & 0 & \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha & 0 & 0\\ -\sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & \sin\alpha\\ 0 & 0 & -\sin\alpha & \cos\alpha\end{bmatrix}\begin{pmatrix}u_A\\ v_A\\ u_B\\ v_B\end{pmatrix}$$

$$W=\frac{1}{2}\begin{pmatrix}u_A & v_A & u_B & v_B\end{pmatrix}\begin{bmatrix}P & O\\ O & P\end{bmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P^T & O\\ O & P^T\end{bmatrix}\begin{pmatrix}u_A\\ v_A\\ u_B\\ v_B\end{pmatrix}$$

Exemple



ENERGIE BARRE 1

ENERGIE BARRE 2

Example

Energie barre 1

$$W = \frac{1}{2}\begin{pmatrix} u_A & v_A & u_B & v_B \end{pmatrix}\begin{bmatrix} P & O \\ O & P \end{bmatrix}\begin{bmatrix} \frac{ES}{L} & 0 & -\frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P^T & O \\ O & P^T \end{bmatrix}\begin{pmatrix} u_A \\ v_A \\ u_B \\ v_B \end{pmatrix}$$

Exemple

Energie barre 2

$$W=\frac{1}{2}\begin{pmatrix}u_A & v_A & u_B & v_B\end{pmatrix}\begin{bmatrix}P & O\\ O & P\end{bmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P^T & O\\ O & P^T\end{bmatrix}\begin{pmatrix}u_A\\ v_A\\ u_B\\ v_B\end{pmatrix}$$

Stiffness matrix of the system

$$W_1 = \frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}^T \frac{2ES}{L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

$$W_1 + W_2 = \frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}^T \frac{2ES}{L}\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

$$W_2 = \frac{1}{2}\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}^T \frac{2ES}{L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Equation résolvante

$$\frac{2ES}{L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1+1 & 1-1 & -1 & 1 \\ -1 & -1 & 1-1 & 1+1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}\begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{pmatrix}$$

- La Matrice de raideur est singulière car les mouvements de corps rigides n'ont pas été supprimés.

- Conditions aux limites :

$$\begin{pmatrix} u_1 = \\ v_1 = \\ u_2 = \\ v_2 = \\ u_3 = \\ v_3 = \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 = \\ Y_1 = \\ X_2 = \\ Y_2 = \\ X_3 = \\ Y_3 = \end{pmatrix}$$

Résultats

$$u_2 = FL / ES$$

$$v_2 = 0$$

$$X_1 = -F / 2$$

$$X_3 = -F / 2$$

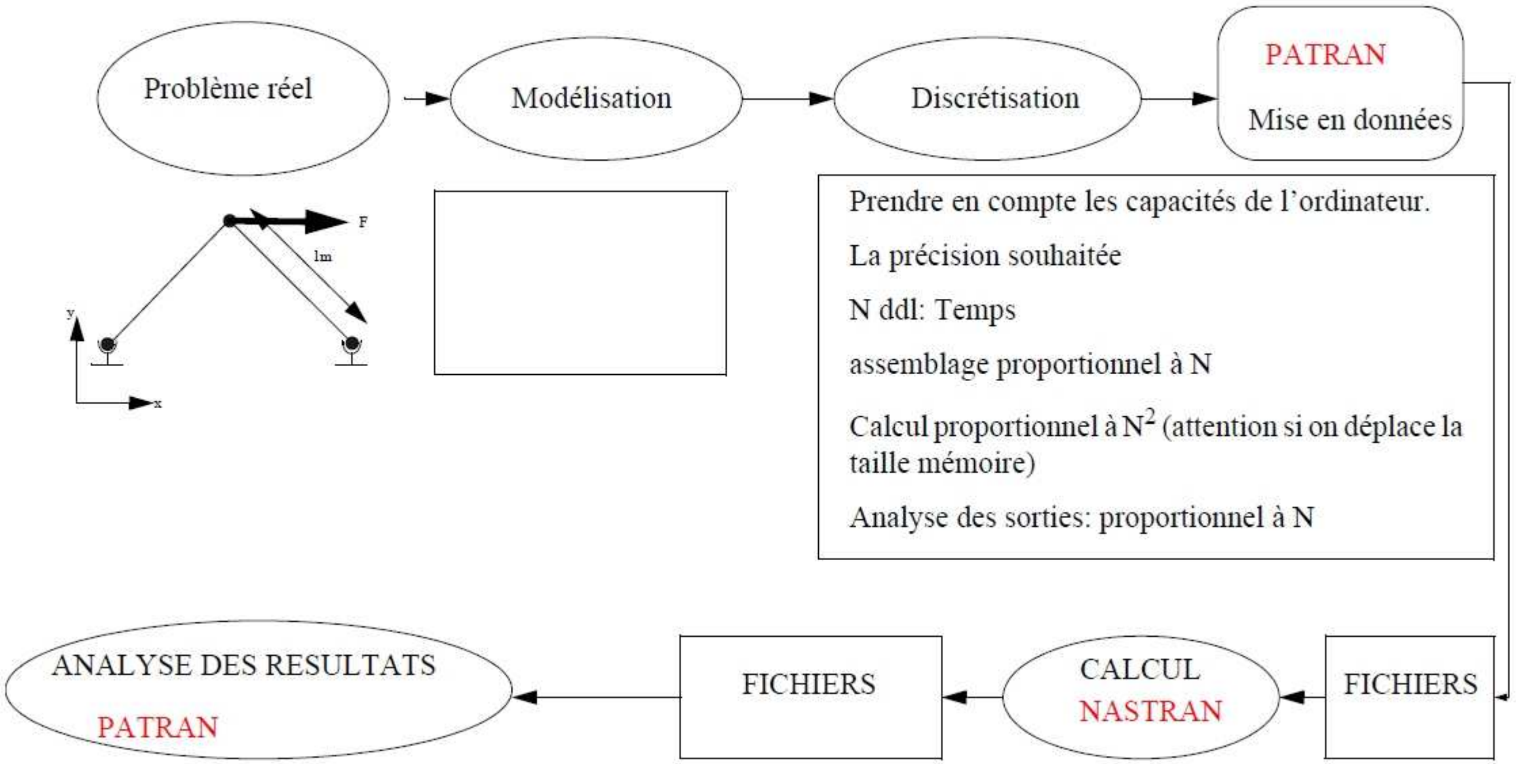
$$Y_1 = -F / 2$$

$$Y_2 = F / 2$$

From these results we must verify that the structure is under a state of equilibrium

Etude de ce problème avec NASTRAN

ETUDE NASTRAN DU PROBLEME PRECEDENT



La carte GRID pour la création de nœuds

GRID **Grid Point**

Defines the location of a geometric grid point, the directions of its displacement, and its permanent single-point constraints.

Format:

1	2	3	4	5	6	7	8	9	10
GRID	ID	CP	X1	X2	X3	CD	PS	SEID	

Example:

GRID	2	3	1.0	-2.0	3.0		316		

Field	Contents
ID	Grid point identification number. (0 < Integer < 100000000)
CP	Identification number of coordinate system in which the location of the grid point is defined. (Integer $\geq$ 0 or blank*)
X1, X2, X3	Location of the grid point in coordinate system CP. (Real; Default = 0.0)
CD	Identification number of coordinate system in which the displacements, degrees-of-freedom, constraints, and solution vectors are defined at the grid point. (Integer $\geq$ -1 or blank)*
PS	Permanent single-point constraints associated with the grid point. (Any of the Integers 1 through 6 with no embedded blanks, or blank*.)
SEID	Superelement identification number. (Integer $\geq$ 0; Default = 0)

*See the GRDSET entry for default options for the CP, CD, PS, and SEID fields.

La carte GRID pour la création de noeuds

Remarks:

1. All grid point identification numbers must be unique with respect to all other structural, scalar, and fluid points.
2. The meaning of X1, X2, and X3 depends on the type of coordinate system CP as follows (see the CORDij entry descriptions):

Type	X1	X2	X3
Rectangular	X	Y	Z
Cylindrical	R	θ(degrees)	Z
Spherical	R	θ(degrees)	φ(degrees)

Extrait du fichier BDF : Création des nœuds (3 dans notre exemple)

```
$ Nodes of the Entire Model
GRID     1               0.      0.      0.
GRID     2               1000.   1000.   0.
GRID     4               2000.   0.      0.
```

-Un $ en tête de ligne signifie un commentaire non pris en compte par le compilateur

-Il y a 10 champs de 8 caractères par ligne. Attention de ne pas mettre une donnée sur plusieurs champs

La carte Crod création élément et connectivité

CROD **Rod Element Connection**

Defines a tension-compression-torsion element.

Format:

1	2	3	4	5	6	7	8	9	10
CROD	EID	PID	G1	G2					

Example:

CROD	12	13	21	23					

Field	Contents
EID	Element identification number. (Integer > 0)
PID	Property identification number of a PROD entry. (Integer > 0; Default = EID)
G1, G2	Grid point identification numbers of connection points. (Integer > 0; $G1 \neq G2$)

Remarks:

1. Element identification numbers should be unique with respect to all other element identification numbers.
2. See CONROD, 1183 for alternative method of rod definition.
3. Only one element may be defined on a single entry.

```
$ Pset: "Propertyrod" will be imported as: "prod.1"
CROD       1       1       1       2
CROD       2       1       2       4
```

Carte Property – Propriété de l'élement

PROD Rod Property

Defines the properties of a rod element (CROD entry).

Format:

1	2	3	4	5	6	7	8	9	10
PROD	PID	MID	A	J	C	NSM			

Example:

PROD	17	23	42.6	17.92	4.2356	0.5			

Field	Contents
PID	Property identification number. (Integer > 0)
MID	Material identification number. See Remarks 2. and 3. (Integer > 0)
A	Area of the rod. (Real)
J	Torsional constant. (Real)
C	Coefficient to determine torsional stress. (Real; Default = 0.0)
NSM	Nonstructural mass per unit length. (Real)

Remarks:

1. PROD entries must all have unique property identification numbers.
2. For structural problems, MID must reference a MAT1 material entry.
3. For heat transfer problems, MID must reference a reference MAT4 or MAT5 entry.
4. The formula used to calculate torsional stress is

$$\tau = \frac{C M_\theta}{J}$$

where M_θ is the torsional moment.

L'ordre des cartes n'a pas d'importance

```
$ Elements and Element Properties for region : Propertyrod
PROD      1        1      1000.
```

Method of resolution

- We take in account the boundary conditions
  - We create two set for the displacment ans s

$$q = \begin{Bmatrix} q_l \\ q_r \end{Bmatrix} \qquad F = \begin{Bmatrix} F_l \\ F_r \end{Bmatrix}$$

  - Degree of Freedom: l
    - Unknown displacements: q_l
    - Applied load known: F_l

$$q_l = \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad F_l = \begin{Bmatrix} X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{Bmatrix}$$

  - Degree restrained: r
    - Dispacement known (imposeds) : q_r
    - Unknown reaction : F_r

$$q_r = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad F_r = \begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix}$$

Méthodes de résolution

- Prise en compte des conditions aux limites
  - Partitionnement du système linéaire

$$\begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & K_{rr} \end{bmatrix} \begin{Bmatrix} q_l \\ q_r \end{Bmatrix} = \begin{Bmatrix} F_l \\ F_r \end{Bmatrix}$$

  - Etape 1

$$K_{ll} q_l + K_{lr} q_r = F_l \qquad \Rightarrow \qquad q_l = K_{ll}^{-1} \left(F_l - K_{lr} q_r \right)$$

  - Etape 2

$$F_r = K_{rl} q_l + K_{rr} q_r$$

Méthodes de résolution

- Prise en compte des conditions aux limites
  - Cas particulier : degrés restreints bloqués
    - – Encastrement
    - – Appui simple
    - – …

$$\Rightarrow \quad q_r = 0$$

  - Etape 1

$$q_l = K_{ll}^{-1} F_l$$

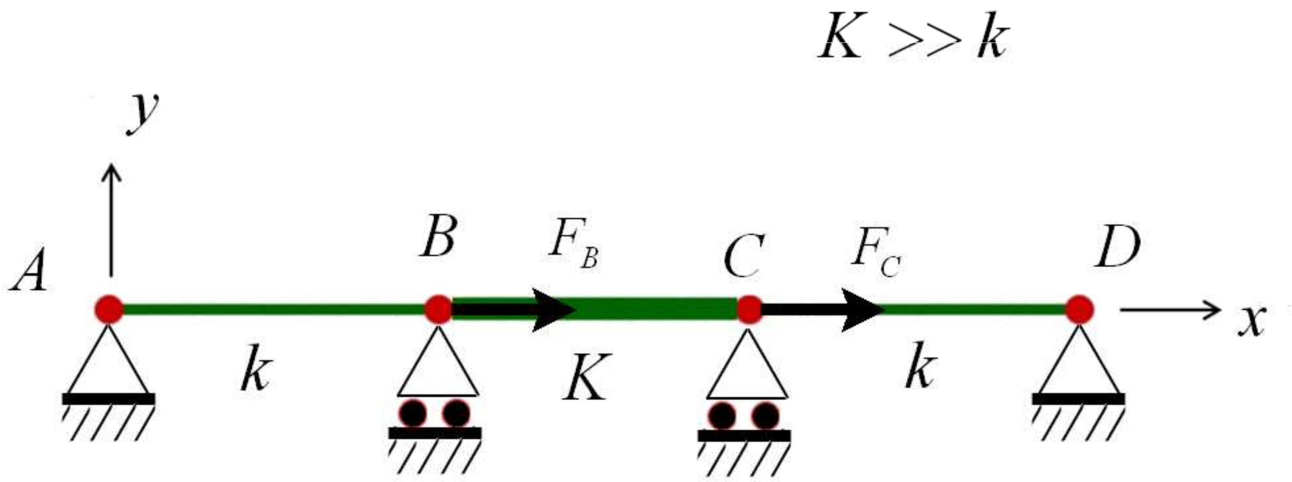
  - Etape 2

$$F_r = K_{rl} q_l$$

Liaison cinématique

Lorsque des éléments ont des rigidités très différentes la matrice de raideur pourra être mal conditionnée. On préférera créer une liaison cinématique.

Exemple.



Liaison cinématique

Matrice de raideur

$$W = \frac{1}{2}\begin{pmatrix} u_A \\ u_B \\ u_C \\ u_D \end{pmatrix}^T \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{pmatrix} u_A \\ u_B \\ u_C \\ u_D \end{pmatrix}$$

Liaison cinématique : On écrit que le déplacement du point B est le même que le déplacement du point C :

$$u_B = u_C$$

$$\begin{pmatrix} u_A \\ u_B \\ u_C \\ u_D \end{pmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{bmatrix} \begin{pmatrix} u_A \\ u_B \\ u_D \end{pmatrix}$$