[7th]

3, Solution of plane problems in polar coordinate (cylinder system)

-Coordinates transformation of stress

-Polar expression of basic equations

-Stress function and compatibility equations in polar coordinate

-Axisymmetric stress and its displacement

3, Solution of plane problems in polar coordinate (cylinder system)
-General
Essentially, selection of coordinate system does not affect solution of an elastostatics problem. However, selection of the coordinates directly affects description form of the boundary conditions, which relates to the difficulty of solving the problem. For the circle, shaped wedge and fan works, solving are more convenient in the polar coordinate system than in Cartesian coordinate system. Plus, a complexity method KM function will be introduced.

-Coordinates transformation of stress

In figure 7.1, a point ***P*** at Cartesian coordinate can also be expressed by radial coordinate ***r*** and angle coordinate θ. Positive direction of ***r*** take origin ***O*** as start point. Positive direction of θ means rotation from axis *x* to axis *y*.

Relations of two coordinate are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\dfrac{y}{x} \end{cases} \tag{7.1}$$

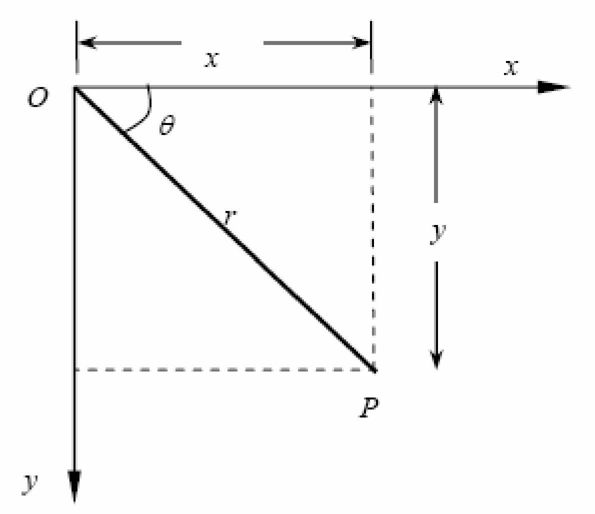


Figure 7.1 Transformation of stress description

Also, stress state of a point inside elastomer can be formulated in polar coordinate. It has to have a relationship with its expressions in Cartesian coordinate. This is called coordinates transformation of stress.

As shown in figure 7.2, an infinitesimal unit is given in Cartesian coordinate with thickness 1. According to equilibrium relationship of triangle unit, stress components can be expressed as following type,

$$\begin{cases} \sigma_x = \dfrac{\sigma_r + \sigma_\theta}{2} + \dfrac{\sigma_r - \sigma_\theta}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta \\ \sigma_y = \dfrac{\sigma_r + \sigma_\theta}{2} - \dfrac{\sigma_r - \sigma_\theta}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta \\ \tau_{xy} = \dfrac{\sigma_r - \sigma_\theta}{2}\sin 2\theta + \tau_{r\theta}\cos 2\theta \end{cases} \tag{7.2}$$

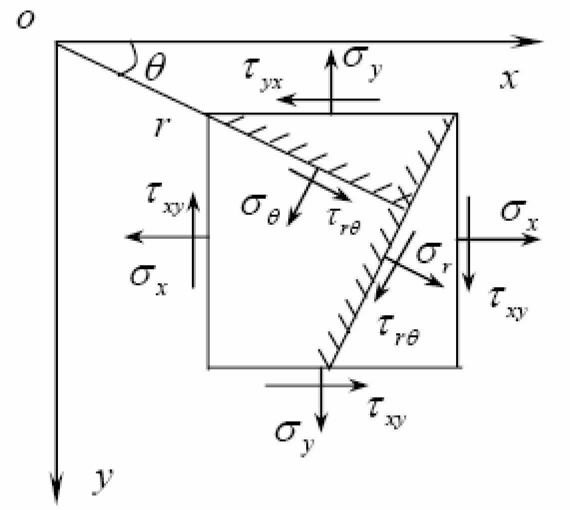


Figure 7.2 Transformation of stress components

Or, an inverse expression is

$$\begin{cases} \sigma_r = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \tau_{r\theta} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases} \tag{7.3}$$

-Polar expression of basic equations

a. Equilibrium differential equations in polar coordinate

In polar coordinate a point ***P***(***r***, $\boldsymbol{\theta}$) in a plane elastomer is considered as a differential unit with one unit thickness. As drawn in figure 7.3, radial and circumference volume force are denoted as K_r and K_θ, stress components are σ_r, σ_θ and $\tau_{r\theta}$. Positive direction is similar as in Cartesian coordinate.

As continuous functions, stress components are variables according to coordinate. Assume stress components around point ***P*** are: surface ***PA*** has σ_θ and $\tau_{\theta r}$, surface ***PB*** has σ_r and $\tau_{r\theta}$.

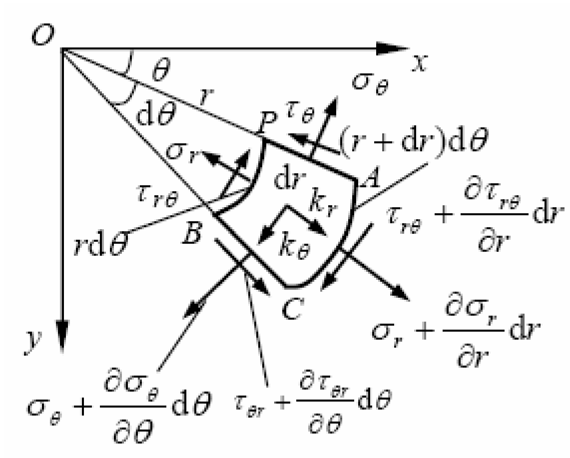


Figure 7.3 a balanced infinitesimal in polar coordinate

Stress components in surface ***AC*** and surface ***BC*** are

$$AC:\quad \begin{cases} \sigma_r + \dfrac{\partial \sigma_r}{\partial r}\mathrm{d}r \\ \tau_{r\theta} + \dfrac{\partial \tau_{r\theta}}{\partial r}\mathrm{d}r \end{cases}$$

$$BC:\quad \begin{cases} \sigma_\theta + \dfrac{\partial \sigma_\theta}{\partial \theta}\mathrm{d}\theta \\ \tau_{\theta r} + \dfrac{\partial \tau_{\theta r}}{\partial \theta}\mathrm{d}\theta \end{cases}$$

As an infinitesimal part of balanced body, unit ***PACB*** satisfy statics equation

$$\sum F_r = 0,\ \left(\sigma_r + \frac{\partial \sigma_r}{\partial r}dr\right)rd\theta - \sigma_r rd\theta + \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta}d\theta\right)dr - \tau_{\theta r}dr + K_r rd\theta dr -$$
$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}d\theta\right)dr\sin(\frac{d\theta}{2}) - \sigma_\theta dr\sin(\frac{d\theta}{2})\ = 0$$

As a differential unit, $\sin(\frac{d\theta}{2}) \approx \frac{d\theta}{2}$, simplification is made

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\sigma_r - \sigma_\theta}{r} + K_r\ = 0$$

At another direction,

$$\sum F_\theta = 0,\ \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}d\theta\right)dr - \sigma_\theta dr + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}dr\right)(r+dr)d\theta - \tau_{r\theta}rd\theta + K_\theta rdrd\theta +$$
$$\left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta}d\theta\right)dr\frac{d\theta}{2} + \tau_{\theta r}dr\frac{d\theta}{2}\ = 0$$

The simplified type is

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + K_\theta\ = 0$$

According to equilibrium of momentum to centre of unit, reciprocal theorem of shear stress is proved again.

$$\tau_{r\theta} = \tau_{\theta r}$$

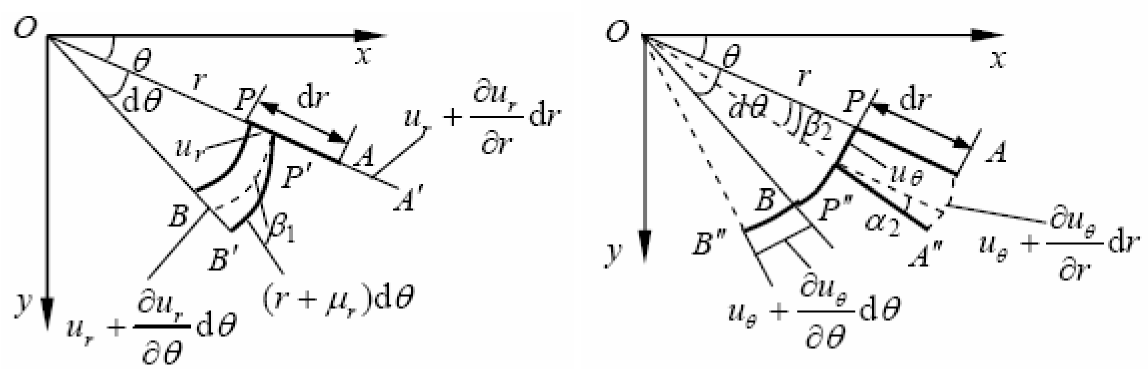
Thus, differential equilibrium equations in polar system are

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} - \dfrac{\sigma_r - \sigma_\theta}{r} + K_r = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{2\tau_{r\theta}}{r} + K_\theta = 0 \end{cases} \tag{7.4}$$

Equation (7.4) is a equation group with two equations including three variables, which means it is first-order hyperstatic question and need complementary equation.

b. Geometric equations in polar coordinate

To build up relation of strain and displacement, two kinds of displacement components u_r and u_θ are analyzed separately according to superposition principle.
(1). Presuming point ***P*** of differential unit only has radial displacement u_r, strain components of unit ε_{r1}, $\varepsilon_{\theta 1}$ are relative elongation of differential segment ***PA*** and ***PB*** individually.
Rotation angle of two segments are denoted as α_1 and β_1.



a. displacement in radial direction b. displacement in circumference direction

Figure 7.4 strain components under different displacements in polar system

According to figure 7.4a,

$$\varepsilon_{r1} = \frac{P'A' - PA}{PA} = \frac{AA' - PP'}{PA}$$

$$= \frac{u_r + \dfrac{\partial u_r}{\partial r}\mathrm{d}r - u_r}{\mathrm{d}r} = \frac{\partial u_r}{\partial r}$$

And

$$\alpha_1 = 0$$

At same moment

$$\varepsilon_{\theta 1} = \frac{P'B' - PB}{PB} = \frac{(r + u_r)\mathrm{d}\theta - r\mathrm{d}\theta}{r\mathrm{d}\theta} = \frac{u_r}{r}$$

And

$$\tan\beta_1 \approx \beta_1 = \frac{BB' - PP'}{PB} = \frac{\left(u_r + \frac{\partial u_r}{\partial \theta}\mathrm{d}\theta\right) - u_r}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

Then, Shear strain is

$$\gamma_{r\theta 1} = \alpha_1 + \beta_1 = \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

(2) Presuming point ***P*** of differential unit only has circumference displacement u_θ, strain components of unit ε_{r2}, $\varepsilon_{\theta 2}$ are relative elongation of differential segment ***PA*** and ***PB*** individually. Rotation angle of two segments are denoted as α_2 and β_2.

Relative elongation ε_{r2} is

$$\varepsilon_{r2} = \frac{P''A'' - PA}{PA} = \frac{\mathrm{d}r - \mathrm{d}r}{\mathrm{d}r} = 0$$

Rotation angle is

$$\alpha_2 = \frac{u_\theta + \frac{\partial u_\theta}{\partial r}\mathrm{d}r - u_\theta}{\mathrm{d}r} = \frac{\partial u_\theta}{\partial r}$$

Relative elongation $\varepsilon_{\theta 2}$ is

$$\varepsilon_{\theta 2} = \frac{P''B'' - PB}{PB} = \frac{BB'' - PP''}{PB} = \frac{u_\theta + \frac{\partial u_\theta}{\partial \theta}\mathrm{d}\theta - u_\theta}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

Angle of ***PB*** is

$$\beta_2 = -\frac{u_\theta}{r}$$

Thus,

$$\gamma_{r\theta 2} = \alpha_2 + \beta_2 = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

To sum up, if point ***P*** has displacement u_θ and u_r, strain components are

$$\begin{cases} \varepsilon_r = \varepsilon_{r1} + \varepsilon_{r2} = \frac{\partial u_r}{\partial r} + 0 = \frac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \varepsilon_{\theta 1} + \varepsilon_{\theta 2} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \gamma_{r\theta 1} + \gamma_{r\theta 2} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \end{cases} \tag{7.5}$$

b. Physical equations in polar coordinate

Considering isotropic mode, physical equations are no different as in polar system. In plane stress problems

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \mu\sigma_\theta) \\ \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_r) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} = \dfrac{2(1+\mu)}{E}\tau_{r\theta} \end{cases} \tag{7.6}$$

To plane strain problems, physical equations are

$$\begin{cases} \varepsilon_r = \dfrac{1+\mu}{E}\left[(1-\mu)\sigma_r - \mu\sigma_\theta\right] \\ \varepsilon_\theta = \dfrac{1+\mu}{E}\left[(1-\mu)\sigma_\theta - \mu\sigma_r\right] \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} \end{cases} \tag{7.7}$$

-Stress function and compatibility equations in polar coordinate
As we mentioned in Chapter 6, there have other methods to achieve the solution of elastostatics. Expressions of stress function and compatibility equations in polar coordinate also can be used.

According to equations (7.1), relations of derivatives of different systems are

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta, \qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r}, \quad \dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$

Derivatives of stress function φ about variable r and θ can be formulated through composite function derivative rules.

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial\varphi}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \cos\theta\dfrac{\partial\varphi}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial\varphi}{\partial\theta} = \left(\cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}\right)\varphi \\ \dfrac{\partial\varphi}{\partial y} = \dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial\varphi}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \sin\theta\dfrac{\partial\varphi}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial\varphi}{\partial\theta} = \left(\sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}\right)\varphi \end{cases}$$

$$\frac{\partial^2\varphi}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial\varphi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$= \cos^2\theta \frac{\partial^2\varphi}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\frac{\partial^2\varphi}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial\varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$= \sin^2\theta \frac{\partial^2\varphi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\frac{\partial^2\varphi}{\partial x\partial y} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial\varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$= \sin\theta\cos\theta\frac{\partial^2\varphi}{\partial r^2} + \frac{\cos^2\theta - \sin^2\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial\varphi}{\partial r}$$

$$- \frac{\cos^2\theta - \sin^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

Adding two second-order partial derivatives

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

It is Laplace operator expressed in polar system, namely

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi$$

So, compatibility equation (6.17) is modified as

$$\nabla^2\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi = 0 \tag{7.8}$$

or

$$\nabla^4\varphi = \nabla^2\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\varphi = 0 \tag{7.9}$$

Thus, stress components can be written as

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\varphi}{\partial\theta^2} \\ \sigma_\theta = \dfrac{\partial^2\varphi}{\partial r^2} \\ \tau_{r\theta} = \dfrac{1}{r^2}\dfrac{\partial\varphi}{\partial\theta} - \dfrac{1}{r}\dfrac{\partial^2\varphi}{\partial r\partial\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta}\right) \end{cases} \tag{7.10}$$

It is easily proved that equations (7.10) satisfy differential equilibrium equations (7.4) and stress compatibility equation (7.9). Finally, equations (7.10) can be completely decided by boundary conditions.

-Axisymmetric stress and its displacement

In engineering, if shape and load of elastomer are all independent to coordinate θ, it can be classified as plane axisymmetric problems, which can solved by inverse method.

Considering $\varphi = \varphi(r)$, solution (7.10) is changed as

$$\sigma_r = \frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} \quad , \quad \sigma_\theta = \frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} \quad , \quad \tau_{r\theta} = 0$$

While, compatibility equation (7.9) will be

$$\nabla^4\varphi = \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 \varphi = 0 \tag{7.11}$$

Expand (7.11)

$$\frac{\mathrm{d}^4\varphi}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3\varphi}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0$$

It is a fourth- order variable coefficient homogeneous differential equation, which can be re-written as Euler homogeneous differential equation

$$r^4\frac{\mathrm{d}^4\varphi}{\mathrm{d}r^4} + 2r^3\frac{\mathrm{d}^3\varphi}{\mathrm{d}r^3} - r^2\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + r\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0 \tag{7.12}$$

Equation (7.12) has a common solution

$$\varphi = A\ln r + Br^2\ln r + Cr^2 + D \tag{7.13}$$

Where A, B, C, and D are undecided coefficients.
Thus, stress components are

$$\begin{cases} \sigma_r = \dfrac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_\theta = -\dfrac{A}{r^2} + B(3 + 2\ln r) + 2C \\ \tau_{r\theta} = \tau_{\theta r} = 0 \end{cases} \tag{7.14}$$

To plane stress problems, solution (7.14) can be taken into physical equation (7.6)

$$\begin{cases}\dfrac{\partial u_r}{\partial r}=\varepsilon_r=\dfrac{1}{E}(\sigma_r-\mu\sigma_\theta)\\ \qquad =\dfrac{1}{E}\left[(1+\mu)\dfrac{A}{r^2}+(1-3\mu)B+2(1-\mu)B\ln r+2(1-\mu)C\right]\\ \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{u_r}{r}=\varepsilon_\theta=\dfrac{1}{E}(\sigma_\theta-\mu\sigma_r)\\ \qquad\qquad =\dfrac{1}{E}\left[-(1+\mu)\dfrac{A}{r^2}+(3-\mu)B+2(1-\mu)B\ln r+2(1-\mu)C\right]\\ \gamma_{r\theta}=\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}=0\end{cases} \quad \text{(a)}$$

Integrating the first equation of (a)

$$u_r=\frac{1}{E}\left[-(1+\mu)\frac{A}{r}+2(1-\mu)Br(\ln r-1)+(1-3\mu)Br+2(1-\mu)Cr\right]+f(\theta) \quad \text{(b)}$$

Where $f(\theta)$ can be any function.

Taking equation (b) into the second equation of (a)

$$\frac{\partial u_\theta}{\partial\theta}=\frac{r}{E}\left[-(1+\mu)\frac{A}{r^2}+(3-\mu)B+2(1-\mu)B\ln r+2(1-\mu)C\right]-u_r$$

$$=\frac{4Br}{E}-f(\theta)$$

Integrating this equation

$$u_\theta=\frac{4Br\theta}{E}-\int f(\theta)\mathrm{d}\theta+f_1(r) \quad \text{(c)}$$

Where $f_1(r)$ can be any function.

Taking equation (b) and (c) into the third equation of (a)

$$f_1(r)-r\frac{\mathrm{d}f_1(r)}{\mathrm{d}r}=\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}+\int f(\theta)\mathrm{d}\theta$$

To make above equation possible whatever r and θ are, both sides have to equal a constant, namely

$$f_1(r)-r\frac{\mathrm{d}f_1(r)}{\mathrm{d}r}=F \quad \text{(d)}$$

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}+\int f(\theta)\mathrm{d}\theta=F \quad \text{(e)}$$

Solution of (d) is

$$f_1(r)=Hr+F \quad \text{(f)}$$

Differentiate (e)

$$\frac{\mathrm{d}^2 f(\theta)}{\mathrm{d}\theta^2}+f(\theta)=0$$

Its solution is

$$f(\theta) = I\cos\theta + K\sin\theta \tag{g}$$

According to equation (b) and (c)

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\ln r - 1) + (1-3\mu)Br + 2(1-\mu)Cr\right] + I\cos\theta + K\sin\theta$$

$$u_\theta = \frac{4Br\theta}{E} + Hr - I\sin\theta + K\cos\theta \quad (\tag{7.15}$$

Where *A*, *B*, *C*, *H*, *I*, and *K* are decided by boundary conditions.

According to (7.15), axisymmetric stress components must not mean axisymmetric displacement components. Only in case of axisymmetric shape, load, and constraints, axisymmetric stress components result in axisymmetric displacement components. Under this circumstance, $u_\theta = 0$, and $B = H = I = K = 0$. We have

$$\begin{cases} u_r = \dfrac{1}{E}\left[-(1+\mu)\dfrac{A}{r} + 2(1-\mu)Cr\right] \\ u_\theta = 0 \end{cases} \tag{7.16}$$

To plane strain problems, equation (7.15) is same valid, only if substitute E and μ with $\frac{E}{1-\mu^2}$ and $\frac{\mu}{1-\mu}$.