[4th]-Elastic linear isotropic Hooke’s law [book1 7.6\book2 2.6\courseware reference CH4]

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1. Definition of “classical elastic” schema - Hooke's Law

-General

In this chapter, relation between stress and strain is researched. It is a special property of material. Mathematical equation to describe this kind of relation is called physical equation or constitutive equation. It communicates motion and force via another channel if we can remember the first channel F=ma. Two simple relations of strain and stress can be ensured by experiment, they are uniaxial tension and torsion (See figure 4.1). But, relation of stress and strain under complex stress state (See figure 4.2) can not be accomplished by experiment. A common and practical theorem is attempted to express this relation.

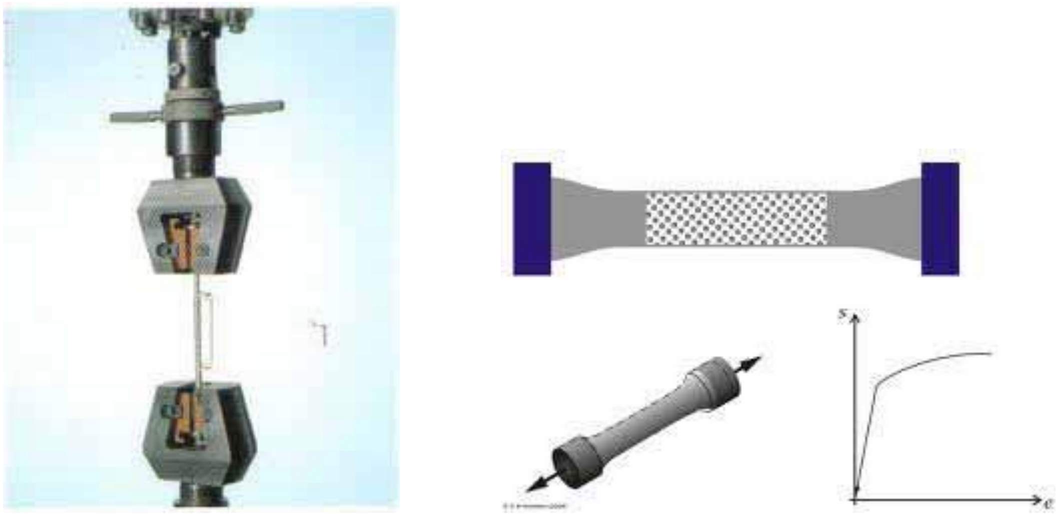

Figure 4.1a. Uniaxial tension

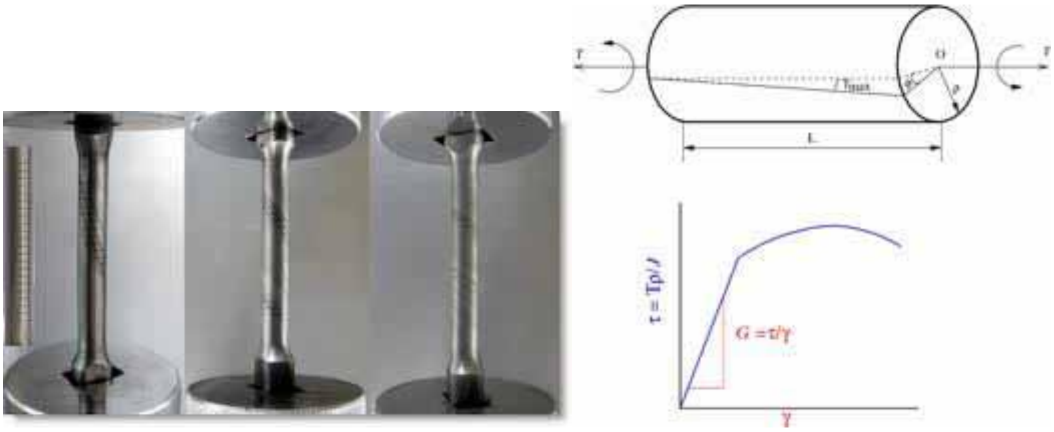

Figure 4.1b. Torsion

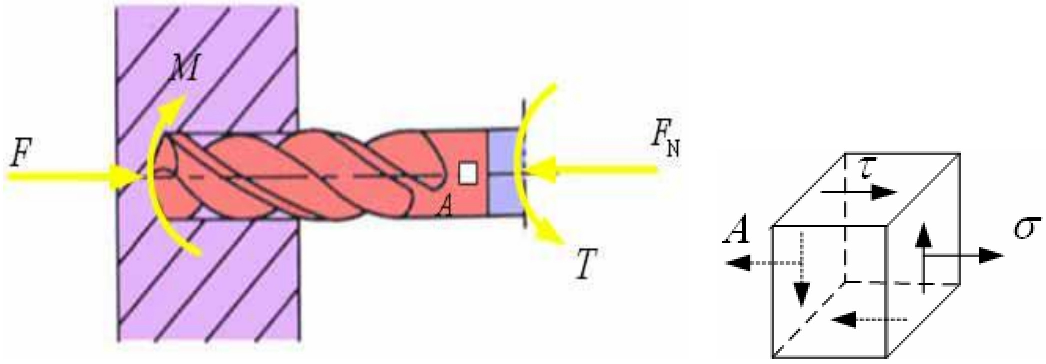

Figure 4.2. Complexe stress state

-Relation of stress and strain: one dimensional Hooke's law

Object during the loading process have stress, strain and displacement effects. There are relations between these effects.

All kinds of materials have a common first period during the loading. In this period, which is linear elastic period, stress is proportional to strain. Spring is a typical example (see figure 4.3).

Figure 4.3. Elasticity of spring

As we already knew, extension length Δl of spring depend on its tension force F, that is

$$F = k \cdot \Delta l \tag{4.1}$$

Object in elastic phase obey the same equation (4.1) as spring do except smaller deformation. Under hypothesis of small deformation, equation (4.1) can be rewritten as below. Divide two sides of equation (4.1) by original volume of object, we have

$$\frac{F}{S \cdot l} = \frac{k \cdot \Delta l}{S \cdot l}$$

According to definitions of stress and strain

$$\sigma = \frac{F}{S} = \frac{k \cdot l}{S}\frac{\Delta l}{l} = \frac{k \cdot l}{S}\varepsilon \tag{4.2}$$

Where k is spring constant, S is aera of cross section, l is original length.
In the uniaxial tension stress state, elastic phase relationship between stress and strain is known as Hooke's Law

$$\sigma = E\varepsilon \tag{4.3}$$

Where E is modulus of elasticity, a material constant.

-Generalized Hooke’s law

Since the researched object is an ideal elastic body, which means it is absolute elastic body and satisfy hypothesis of small deformation, we can expend the one-dimensional Hooke’s law to complex stress state. Generally, value of stress only relies on strain state of material in elastic phase instead of loading process. Expression can be

$$\begin{cases} \sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{xy} + C_{15}\gamma_{yz} + C_{16}\gamma_{xz} \\ \sigma_y = C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z + C_{24}\gamma_{xy} + C_{25}\gamma_{yz} + C_{26}\gamma_{xz} \\ \sigma_z = C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\varepsilon_z + C_{34}\gamma_{xy} + C_{35}\gamma_{yz} + C_{36}\gamma_{xz} \\ \tau_{xy} = C_{41}\varepsilon_x + C_{42}\varepsilon_y + C_{43}\varepsilon_z + C_{44}\gamma_{xy} + C_{45}\gamma_{yz} + C_{46}\gamma_{xz} \\ \tau_{yz} = C_{51}\varepsilon_x + C_{52}\varepsilon_y + C_{53}\varepsilon_z + C_{54}\gamma_{xy} + C_{55}\gamma_{yz} + C_{56}\gamma_{xz} \\ \tau_{xz} = C_{61}\varepsilon_x + C_{62}\varepsilon_y + C_{63}\varepsilon_z + C_{64}\gamma_{xy} + C_{65}\gamma_{yz} + C_{66}\gamma_{xz} \end{cases} \tag{4.4}$$

Equation group (4.4) is called generalized Hooke’s law.

-Isotropy of materials

In fact, physical characteristics of different material have big differences. Materials for engineering use are selected as different requirements. Metals have the widest application. They are isotropic materials. To isotropic materials, relationship of stress-strain is equal at any direction.

Currently, non-metallic materials and composites are gradually employed into many fields. These materials usually have some extremely special properties such as less mass, good ductibility, high strength, etc. Most of them are anisotropic materials though.

In aeronautic engineering, many materials such as new alloy, carbon fiber composite, and glass fiber composites are now tested to make the aircraft more efficient, lighter. In the far past, anisotropic materials also were tested to make an aircraft such as wood, and canvas. Anisotropic materials have more complicated properties than isotropic ones. As an entrance course, we just study isotropic materials here. To an isotropic material, it has same elasticity in all directions that means a complete symmetry of physical properties. In mathematics, this makes a very simple relationship of stress-strain in all coordinates.

-Physical equations

Coefficients of Equation (4.4) will become an asymmetric matrix, if an isotropic material is studied, which is called generalized Hooke’s law for isotropic material

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \\ \varepsilon_y = \dfrac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)] \\ \varepsilon_z = \dfrac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{xz} = \dfrac{1}{G}\tau_{xz} \end{cases} \tag{4.5}$$

Where G is shear modulus, μis Poisson ratio, also known transverse deformation coefficient. Equation (4.5) is physical equation in elasticity.

Adding the first three equations of equation (4.5),

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\mu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{4.6}$$

Volumetric stress is defined as

$$\theta = \frac{1-2\mu}{E}\Theta \tag{4.7}$$

Obviously, it is a linear relationship between volumetric stress and volumetric strain. A proportional constant $\frac{1-2\mu}{E}$ is known as volumetric modulus, also called bulk modulus.

2. Constants of materials

- Young's Modulus

Figure 4.4 Builders of elastic modulus

Elastic modulus E, also called Young's modulus, is defined by Thomas Young (1773-1829, British doctor and physicist) according to test result of 1807. Young's modulus is a physical quantity to describe performance of tension or compression of materials within the elastic limits, which is longitudinal elastic modulus. According to Hooke's law, the objects within the elastic limit, stress is proportional to strain. The ratio is known as the Young's modulus of the material, which depends only on the physical nature of the material itself. Value of Young's modulus of a kind of material indicate the rigidity. The bigger Young's modulus is, the harder it conduct deformation.T

Young's modulus is one of the most important references to decide material of a mechanical parts, which thus is an usual engineering design parameter. Determination of Young's modulus are significant to research mechanical properties of metallic materials, fiber materials, semiconductors, nano-materials, polymers, ceramics, rubber and other materials. It is also used in field of biological mechanics, geology, etc.

Measurement methods of Young's modulu are generally tensile method, beam bending, vibration method, internal friction and so on. There are even a use of optical fiber displacement sensor, Moire fringe, eddy current sensors and wave transmission technology (microwave or ultrasound), etc.

-Poisson ratio

Ratio of transverse strain to longitudinal strain is called Poisson ratio, also called transverse deformation coefficient, which reflects relation of two directions of materials.Within proportional limit of materials, it is absolute value of ratio for transverse strain to longitudinal strain caused by uniformly distributed longitudinal stress. For example, when a beam is on tension, the axial elongation is associated with transverse contraction, and vice versa. Poisson ration of materials can be measured by test.

Figure 4.5 Siméon Denis Poisson

In 1798, Siméon Denis Poisson (1781-1840) entered the École Polytechnique in Paris as first in his year, and immediately began to attract the notice of the professors of the school, who left him free to make his own choices as to what he would study. In 1800, less than two years after his entry, he published two memoirs, one on Étienne Bézout's method of elimination, the other on the number of integrals of a finite difference equation. The latter was examined by Sylvestre-François Lacroix and Adrien-Marie Legendre, who recommended that it should be published in the Recueil des savants étrangers, an unprecedented honour for a youth of eighteen. This success at once procured entry for Poisson into scientific circles. Joseph Louis Lagrange, whose lectures on the theory of functions he attended at the École Polytechnique, recognized his talent early on, and became his friend (the Mathematics Genealogy Project lists Lagrange as his advisor, but this may be an approximation); while Pierre-Simon Laplace, in whose footsteps Poisson followed, regarded him almost as his son. The rest of his career, till his death in Sceaux near Paris, was almost entirely occupied by the composition and publication of his many works and in fulfilling the duties of the numerous educational positions to which he was successively appointed.

Immediately after finishing his studies at the École Polytechnique, he was appointed répétiteur (teaching assistant) there, a position which he had occupied as an amateur while still a pupil in the school; for his schoolmates had made a custom of visiting him in his room after an unusually difficult lecture to hear him repeat and explain it. He was made deputy professor (professeur suppléant) in 1802, and, in 1806 full professor succeeding Jean Baptiste Joseph Fourier, whom Napoleon had sent to Grenoble. In 1808 he became astronomer to the Bureau des Longitudes; and when the Faculté des Sciences was instituted in 1809 he was appointed professor of rational mechanics (professeur de mécanique rationelle). He went on to become a member of the Institute in 1812, examiner at the military school (École Militaire) at Saint-Cyr in 1815, graduation examiner at the École Polytechnique in 1816, councillor of the university in 1820, and geometer to the Bureau des Longitudes succeeding Pierre-Simon Laplace in 1827.

-Shear modulus

The shear modulus describes the material's response to shearing strains. Modulus of rigidity, it is ratio of shear stress to shear strain, an important index of materials'mechanical properties. It indicates the capacity of materials to resist shear strain. The reciprocal of shear modulus is called shear compliance.

The shear modulus is concerned with the deformation of a solid when it experiences a force parallel to one of its surfaces while its opposite face experiences an opposing force (such as friction). In the case of an object that's shaped like a rectangular prism, it will deform into a parallelepiped. Anisotropic materials such as wood and paper exhibit differing material response to stress or strain when tested in different directions. In this case, when the deformation is small enough so that the deformation is linear, the elastic moduli, including the shear modulus, will then be a tensor, rather than a single scalar value. Shear modulus can be detected through thin-walled tube torsion experiment.

The shear modulus is one of several quantities for measuring the stiffness of materials. All of them arise in the generalized Hooke's law (4.5).

-Lamé parameters

In linear elasticity, the Lamé parameters are the two parameters λ, also called Lamé's first parameter. ν, the shear modulus or Lamé's second parameter. Also denoted G.

Transform physical equation (4.5), express stress components with strain components,

$$\begin{cases} \sigma_x = \lambda\theta + 2\nu\varepsilon_x \\ \sigma_y = \lambda\theta + 2\nu\varepsilon_y \\ \sigma_z = \lambda\theta + 2\nu\varepsilon_z \\ \tau_{xy} = \nu\gamma_{xy} \\ \tau_{yz} = \nu\gamma_{yz} \\ \tau_{xz} = \nu\gamma_{xz} \end{cases} \tag{4.8}$$

Where λ is Lamé parameter,

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)} \tag{4.9}$$

-Relations of constants

There have relationships between engineering elastic constants and Lamé parameter,

$$E = \frac{\lambda + \nu}{\nu(2\lambda + 2\nu)}, \qquad \mu = \frac{\lambda}{2(\lambda + \nu)}, \qquad G = \nu$$

Two of three engineering elastic constants are actually independently, they have relation

$$G = \frac{E}{2(1+\mu)} \tag{4.10}$$

According to equation (4.10),

$$\lambda = \frac{2\mu G}{(1-2\mu)} \tag{4.11}$$

Now, we already obtained 15 basic formulations to solve elastostatics problems.