Structural Applications of Finite Elements

Chapter 5

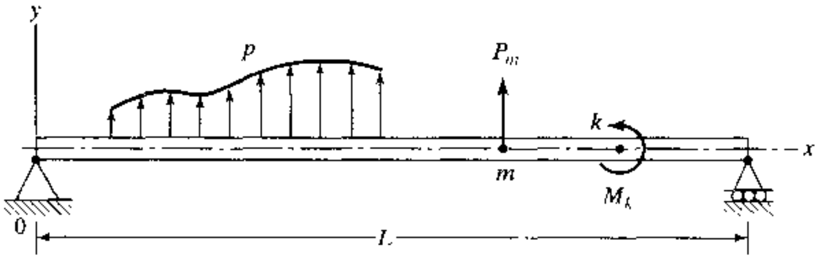
Beam and frames problems

2018-09-01

Outline

- 1D truss
- Plane truss
- 3D truss

Introduction

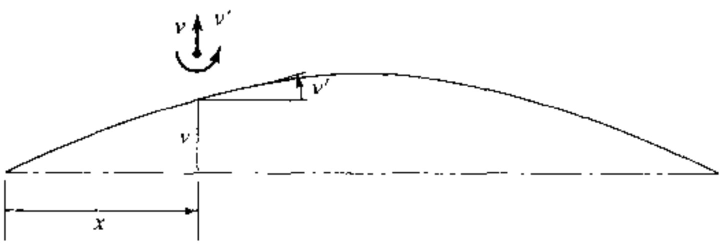


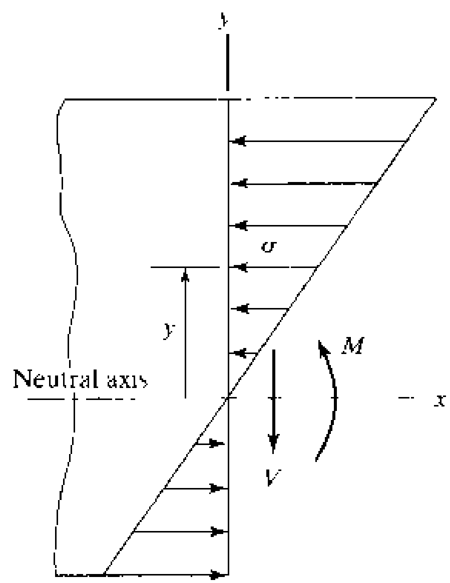
(a)

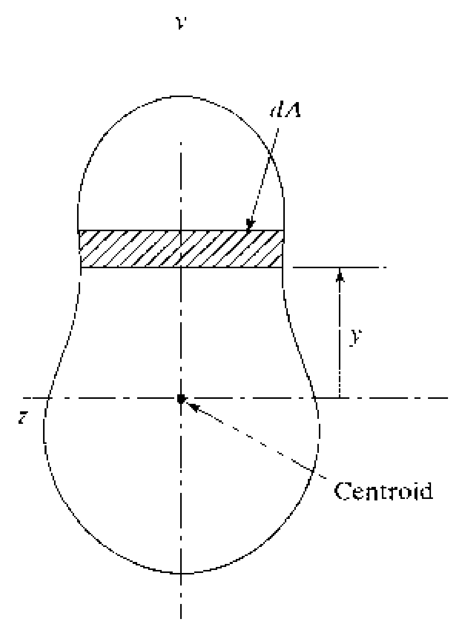
$$\sigma = -\frac{M}{I}y$$

$$\epsilon = \frac{\sigma}{E}$$

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$







Potential energy approach

$$\mathrm{d}U = \frac{1}{2}\int_A \sigma\epsilon \,\mathrm{d}A\,\mathrm{d}x = \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2\,\mathrm{d}A\right)\mathrm{d}x \qquad \int_A y^2\,\mathrm{d}A$$

$$\sigma = -\frac{M}{I}y$$

$$\epsilon = \frac{\sigma}{E}$$

$$\mathrm{d}U = \frac{1}{2}\frac{M^2}{EI}\mathrm{d}x$$

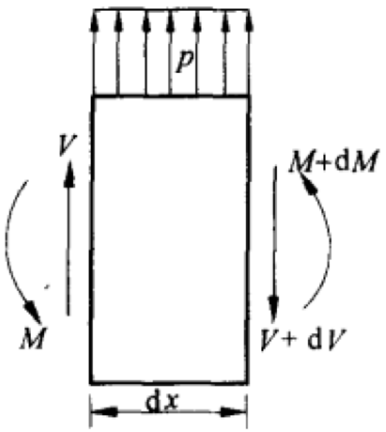
$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{M}{EI}$$

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2\mathrm{d}x$$

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L pv\,dx - \sum_m P_m v_m - \sum_k M_k v_k'$$

Galerkin approach

$$\frac{dV}{dx} = p \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V \qquad \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p = 0$$



$$\int_0^L \left[\frac{\mathrm{d}}{\mathrm{d}x^2}\left(EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) - p\right]\phi\,\mathrm{d}x = 0$$

$$\int_0^L EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}x - \int_0^L p\phi\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)\phi\Big|_0^{x_m} + \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)\phi\Big|_{x_m}^{L}$$
$$- EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_0^{x_k} - EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_{x_k}^{L} = 0$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{M}{EI} \qquad (d/dx)[EI(d^2v/dx^2)]$$

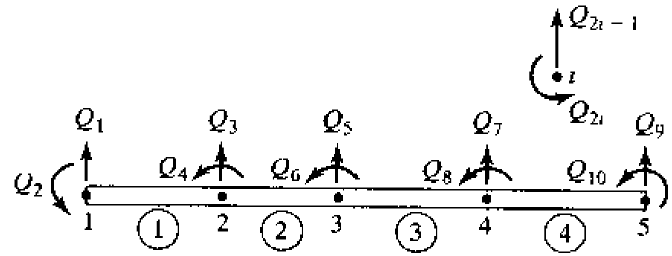
$$\int_0^L EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}x - \int_0^L p\phi\,\mathrm{d}x - \sum_m P_m\phi_m - \sum_k M_k\phi'_k = 0$$

Finite element formulation

$$\mathbf{Q} = [Q_1, Q_2, \ldots, Q_{10}]^{\mathrm{T}}$$

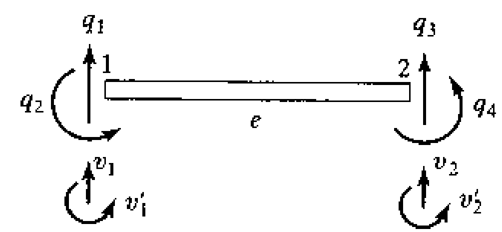
$$\mathbf{q} = [q_1, q_2, q_3, q_4]^{\mathrm{T}}$$

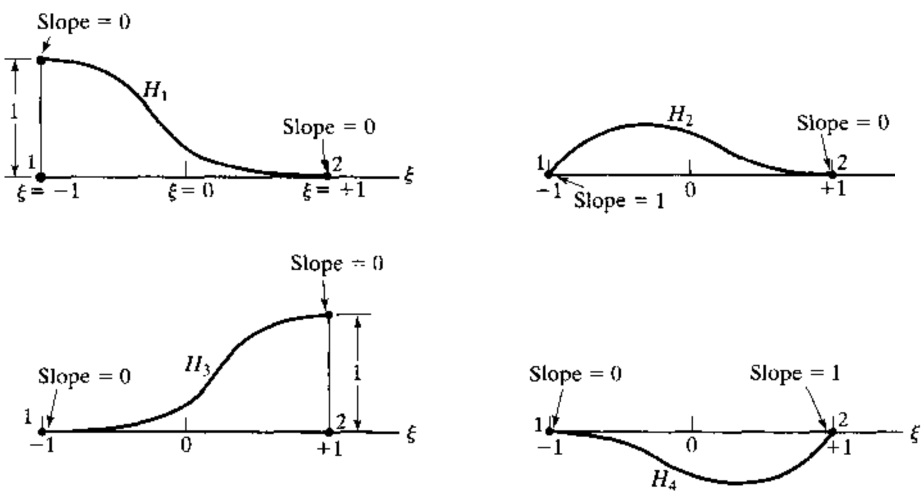
$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3, \quad i = 1,2,3,4$$



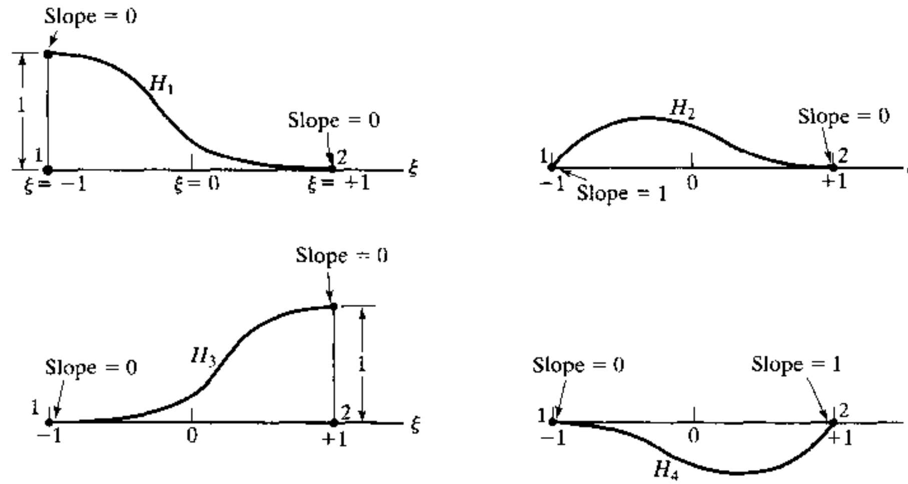
	Local	
e	1	2
1	1	2
2	2	3
3	3	4
4	4	5

Global





	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1



$$H_1 = \tfrac{1}{4}(1-\xi)^2(2+\xi) \quad \text{or} \quad \tfrac{1}{4}(2-3\xi+\xi^3)$$

$$H_2 = \tfrac{1}{4}(1-\xi)^2(\xi+1) \quad \text{or} \quad \tfrac{1}{4}(1-\xi-\xi^2+\xi^3)$$

$$H_3 = \tfrac{1}{4}(1+\xi)^2(2-\xi) \quad \text{or} \quad \tfrac{1}{4}(2+3\xi-\xi^3)$$

$$H_4 = \tfrac{1}{4}(1+\xi)^2(\xi-1) \quad \text{or} \quad \tfrac{1}{4}(-1-\xi+\xi^2+\xi^3)$$

$$v(\xi) = H_1 v_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3 v_2 + H_4\left(\frac{dv}{d\xi}\right)_2$$

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 = \frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\xi \qquad dx = \frac{l_e}{2}d\xi$$

$$dv/d\xi = (dv/dx)(dx/d\xi) \qquad \frac{dv}{d\xi} = \frac{l_e}{2}\frac{dv}{dx}$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2}H_2 q_2 + H_3 q_3 + \frac{l_e}{2}H_4 q_4$$

$$v = \boldsymbol{Hq}$$

$$\boldsymbol{H} = \left[H_1, \frac{l_e}{2}H_2, H_3, \frac{l_e}{2}H_4\right]$$

$$U_e = \frac{1}{2} EI \int_e \left(\frac{d^2 v}{dx^2} \right)^2 dx$$

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \frac{l_e}{2} \frac{\mathrm{d}v}{\mathrm{d}x} \qquad \frac{dv}{dx} = \frac{2}{\ell_e} \frac{dv}{d\xi} \quad \text{and} \quad \frac{d^2 v}{dx^2} = \frac{4}{\ell_e^2} \frac{d^2 v}{d\xi^2}$$

$$v = \boldsymbol{H}\boldsymbol{q} \qquad \left(\frac{d^2 v}{dx} \right)^2 = \mathbf{q}^{\mathrm{T}} \frac{16}{\ell_e^4} \left(\frac{d^2 \mathbf{H}}{d\xi^2} \right)^{\mathrm{T}} \left(\frac{d^2 \mathbf{H}}{d\xi^2} \right) \mathbf{q}$$

$$\left(\frac{d^2 \mathbf{H}}{d\xi^2} \right) = \left[\frac{3}{2}\xi, \frac{-1 + 3\xi}{2} \frac{\ell_e}{2}, -\frac{3}{2}\xi, \frac{1 + 3\xi}{2} \frac{\ell_e}{2} \right]$$

$$\mathrm{d}x = \frac{l_e}{2} \mathrm{d}\xi$$

$$U_e = \frac{1}{2} \mathbf{q}^{\mathrm{T}} \frac{8EI}{\ell_e^3} \int_{-1}^{+1} \begin{bmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1 + 3\xi)\ell_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1 + 3\xi)\ell_e \\ & \left(\frac{-1 + 3\xi}{4} \right)^2 \ell_e^2 & -\frac{3}{8}\xi(-1 + 3\xi)\ell_e & \frac{-1 + 9\xi^2}{16} \ell_e^2 \\ & \text{Symmetric} & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1 + 3\xi)\ell_e \\ & & & \left(\frac{1 + 3\xi}{4} \right)^2 \ell_e^2 \end{bmatrix} d\xi \, \mathbf{q}$$

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\frac{8EI}{\ell_e^3}\int_{-1}^{+1}\begin{bmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi)\ell_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1+3\xi)\ell_e \\ & \left(\frac{-1+3\xi}{4}\right)^2\ell_e^2 & -\frac{3}{8}\xi(-1+3\xi)\ell_e & \frac{-1+9\xi^2}{16}\ell_e^2 \\ & \text{Symmetric} & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)\ell_e \\ & & & \left(\frac{1+3\xi}{4}\right)^2\ell_e^2 \end{bmatrix} d\xi\,\mathbf{q}$$

$$\int_{-1}^{+1}\xi^2\,\mathrm{d}\xi = \frac{2}{3} \quad \int_{-1}^{+1}\xi\,\mathrm{d}\xi = 0 \quad \int_{-1}^{+1}\mathrm{d}\xi = 2$$

$$U_e = \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{k}^e\boldsymbol{q}$$

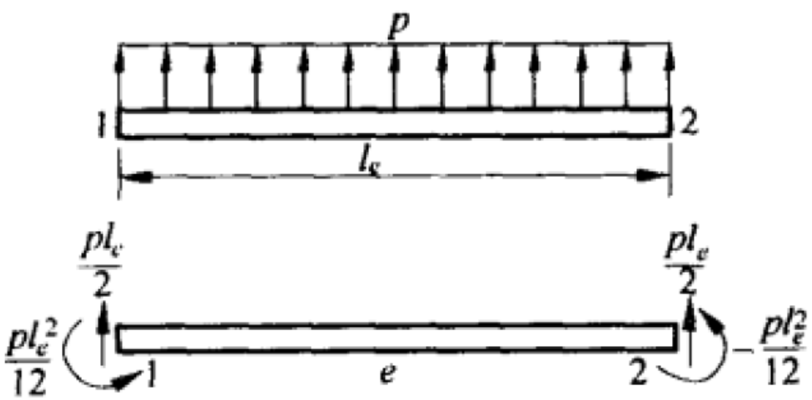
$$\boldsymbol{k}^e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\int_{l_e} pv\,\mathrm{d}x = \left(\frac{pl_e}{2}\int_{-1}^{1}\boldsymbol{H}\,\mathrm{d}\xi\right)\boldsymbol{q} \qquad \int_{l_e} pv\,\mathrm{d}x = \boldsymbol{f}^{e\mathrm{T}}\boldsymbol{q}$$

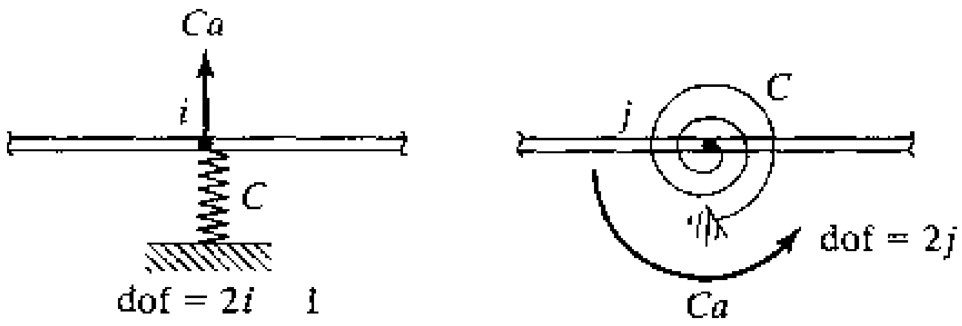
$$\boldsymbol{f}^e = \left[\frac{pl_e}{2}, \frac{pl_e^2}{12}, \frac{pl_e}{2}, \frac{-pl_e^2}{12}\right]^{\mathrm{T}}$$

$$\Pi = \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{Q} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}$$

$$\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{Q} - \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{F} = 0$$



Boundary conditions



a = known generalized displacement

Shear and bending moment

$$M = EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \quad V = \frac{\mathrm{d}M}{\mathrm{d}x} \quad 及 \quad v = \boldsymbol{Hq}$$

$$M = \frac{EI}{l_e^2}\left[6\xi q_1 + (3\xi - 1)l_e q_2 - 6\xi q_3 + (3\xi + 1)l_e q_4\right]$$

$$V = \frac{6EI}{l_e^3}(2q_1 + l_e q_2 - 2q_3 + l_e q_4)$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{\ell_e^3}\begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} \dfrac{-p\ell_e}{2} \\ \dfrac{-p\ell_e^2}{12} \\ \dfrac{-p\ell_e}{2} \\ \dfrac{p\ell_e^2}{12} \end{Bmatrix}$$

Solution We consider the two elements formed by the three nodes. Displacements Q_1, Q_2, Q_3, and Q_5 are constrained to be zero, and Q_4 and Q_6 need to be found. Since the lengths and sections are equal, the element matrices are calculated from Eq. 8.29 as follows:

$$\frac{EI}{\ell^3} = \frac{(200 \times 10^9)(4 \times 10^{-6})}{1^3} = 8 \times 10^5 \text{ N/m}$$

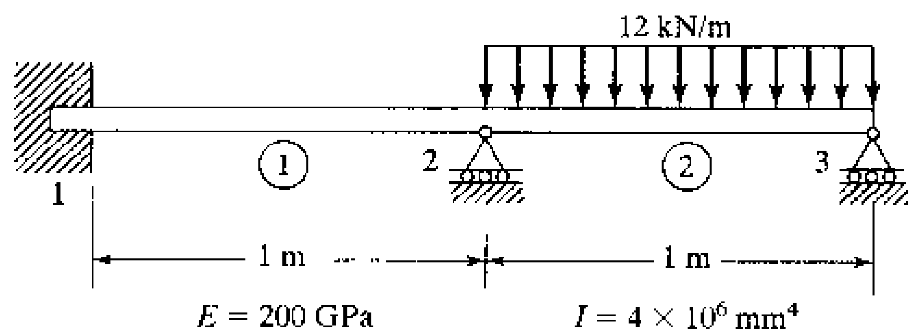
$$\mathbf{k}^1 = \mathbf{k}^2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

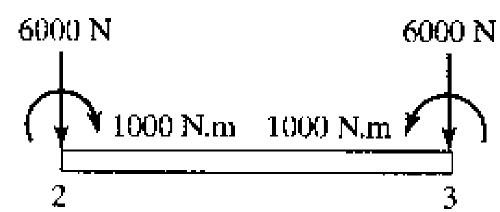
$$\begin{array}{lcccc} e = 1 & Q_1 & Q_2 & Q_3 & Q_4 \\ e = 2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array}$$

We note that global applied loads are $F_4 = -1000$ N.m and $F_6 = +1000$ N.m obtained from $p\ell^2/12$, as seen in Fig. 8.6. We use here the elimination approach presented in Chapter 3. Using the connectivity, we obtain the global stiffness after elimination:

$$\mathbf{K} = \begin{bmatrix} k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{42}^{(2)} & k_{44}^{(2)} \end{bmatrix}$$

$$= 8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$





The set of equations is given by

$$8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ +1000 \end{Bmatrix}$$

The solution is

$$\begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -2.679 \times 10^{-4} \\ 4.464 \times 10^{-4} \end{Bmatrix}$$

For element 2, $q_1 = 0$, $q_2 = Q_4$, $q_3 = 0$, and $q_4 = Q_6$. To get vertical deflection at the mid-point of the element, use $v = \mathbf{Hq}$ at $\xi = 0$:

$$\begin{aligned} v &= 0 + \frac{\ell_e}{2} H_2 Q_4 + 0 + \frac{\ell_e}{2} H_4 Q_6 \\ &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(-2.679 \times 10^{-4}) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{4}\right)(4.464 \times 10^{-4}) \\ &= -8.93 \times 10^{-5}\ \text{m} \\ &= -0.0893\ \text{mm} \end{aligned}$$

■

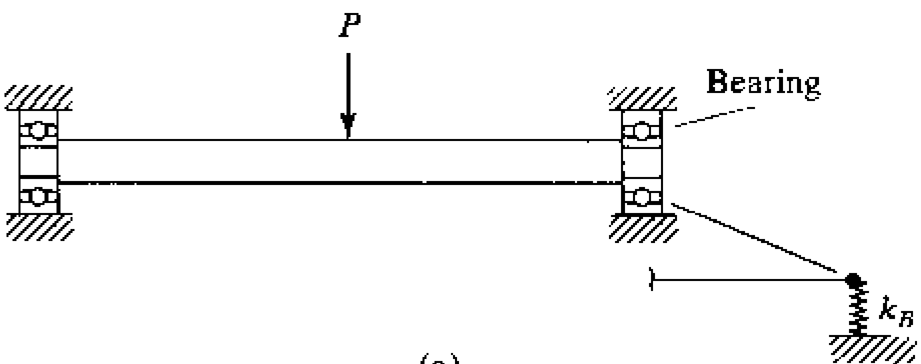
Beams on elastic supports

$$\frac{1}{2}\int_0^{\ell} s v^2\, dx$$

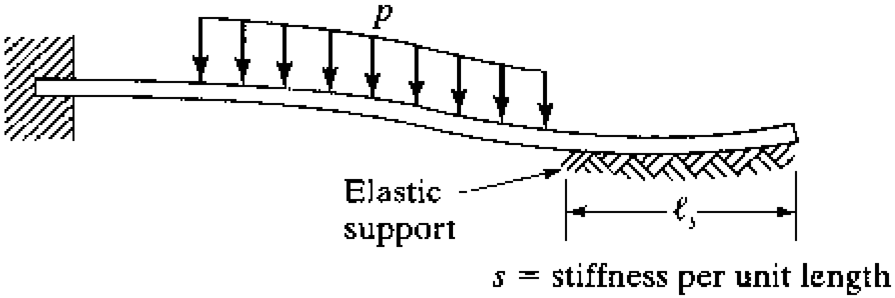
$$\frac{1}{2}\sum_e \mathbf{q}^{\mathrm{T}} s \int_e \mathbf{H}^{\mathrm{T}}\mathbf{H}\, dx\, \mathbf{q}$$

$$\mathbf{k}_s^e = s\int_e \mathbf{H}^{\mathrm{T}}\mathbf{H}\, dx = \frac{s\ell_e}{2}\int_{-1}^{+1} \mathbf{H}^{\mathrm{T}}\mathbf{H}\, d\xi$$

$$\mathbf{k}_s^e = \frac{s\ell_e}{420}\begin{bmatrix} 156 & 22\ell_e & 54 & -13\ell_e \\ 22\ell_e & 4\ell_e^2 & 13\ell_e & -3\ell_e^2 \\ 54 & 13\ell_e & 156 & -22\ell_e \\ -13\ell_e & -3\ell_e^2 & -22\ell_e & 4\ell_e^2 \end{bmatrix}$$



(a)

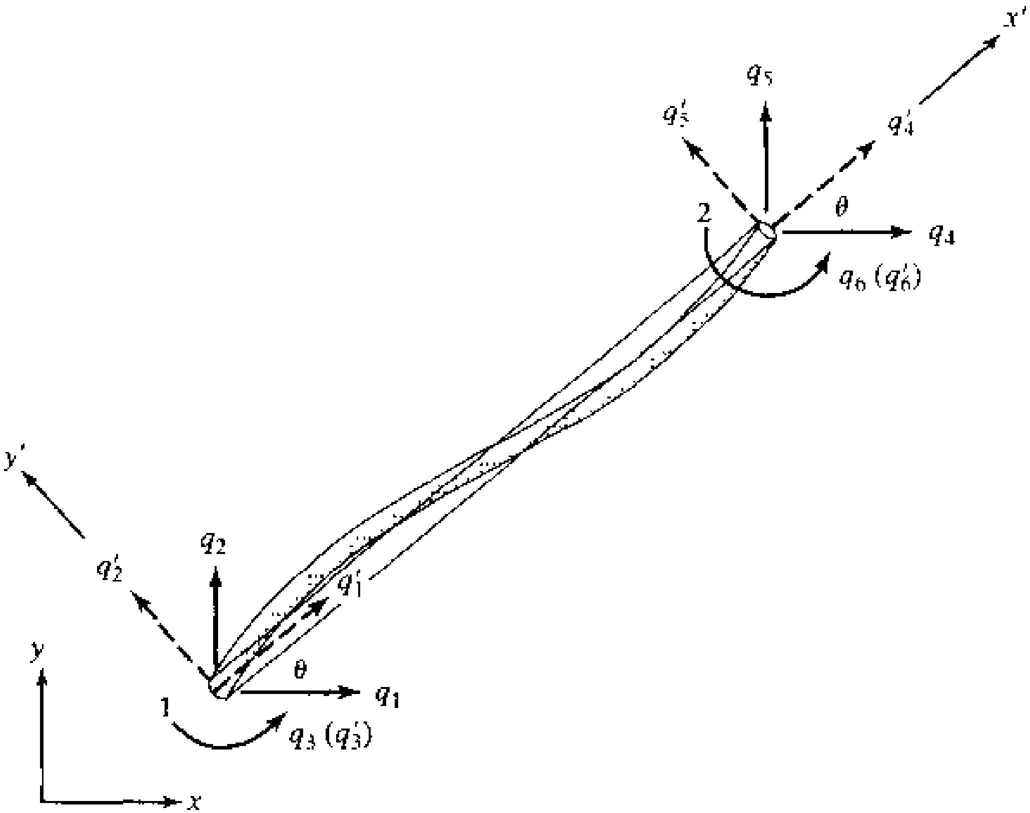


(b)

Plane frames

$$\boldsymbol{q} = [q_1, q_2, q_3, q_4, q_5, q_6]^{\mathrm{T}}$$

$$\boldsymbol{q}' = [q'_1, q'_2, q'_3, q'_4, q'_5, q'_6]^{\mathrm{T}}$$



$$\boldsymbol{q}' = \boldsymbol{L}\boldsymbol{q}$$

$$= \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\mathbf{k}'^{e} = \begin{bmatrix}
\frac{EA}{l_e} & 0 & 0 & \frac{-EA}{l_e} & 0 & 0 \\
0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & \frac{-12EI}{l_e^3} & \frac{6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{2EI}{l_e} \\
\frac{-EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\
0 & \frac{-12EI}{l_e^3} & \frac{-6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & \frac{-6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{4EI}{l_e}
\end{bmatrix}
$$

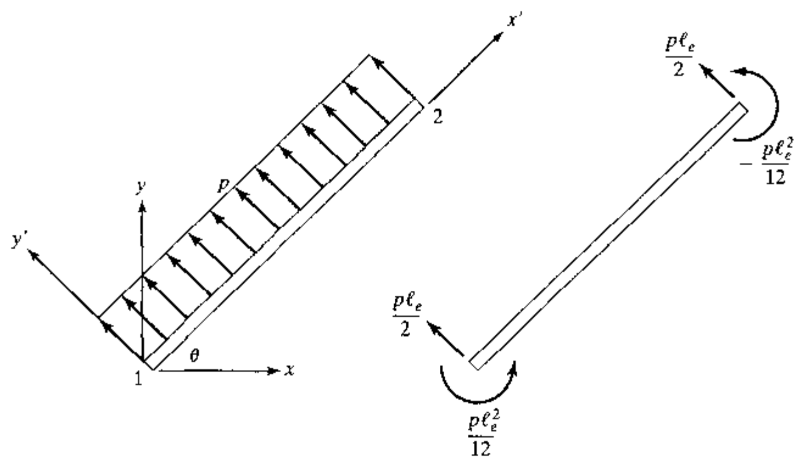
$$U_e = \frac{1}{2}\mathbf{q}'^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{q}' = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}\mathbf{q}$$

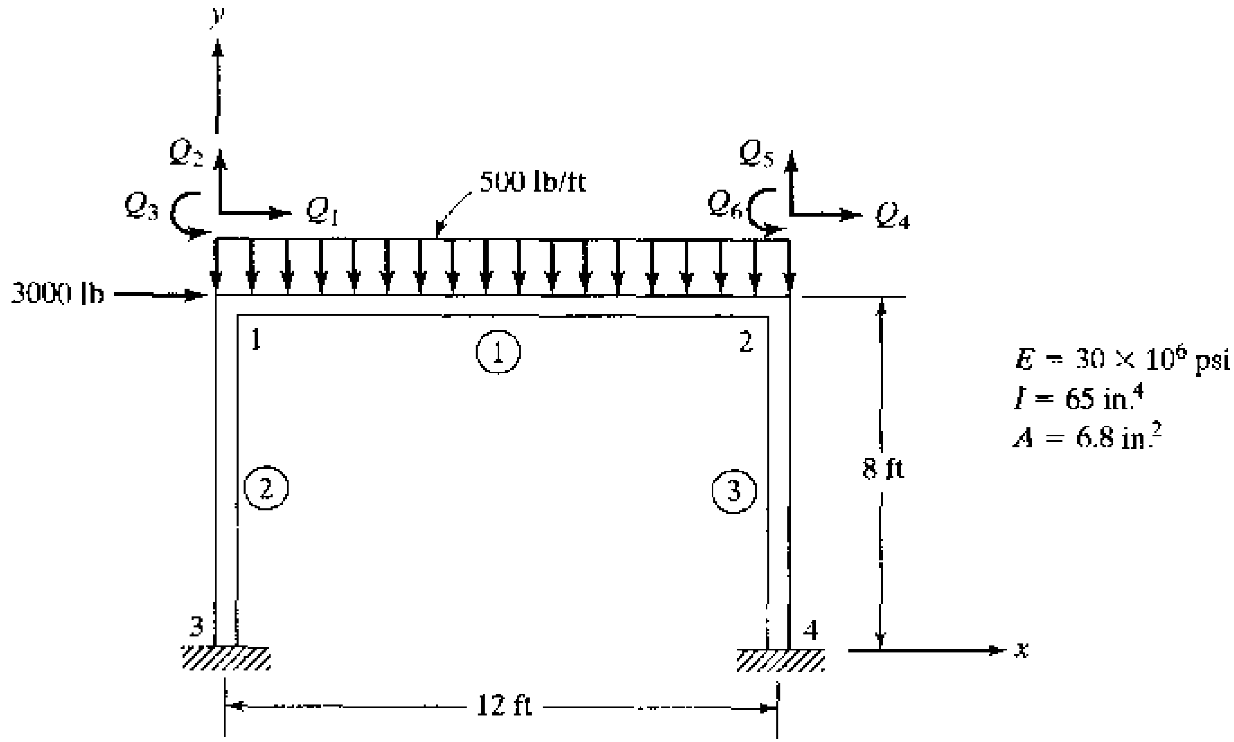
$$\boxed{\mathbf{k}^{e} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}}$$

$$W_e = \boldsymbol{\psi}'^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{q}' = \boldsymbol{\psi}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}\mathbf{q}$$

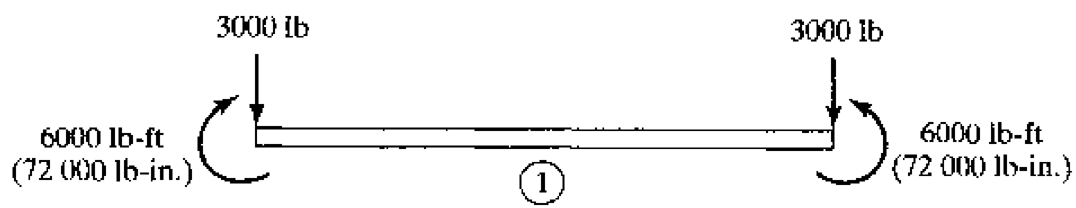
$$\mathbf{q}'^{\mathrm{T}}\mathbf{f}' = \mathbf{q}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{f}'$$

$$\mathbf{f}' = \left[0,\ \frac{p\ell_e}{2},\ \frac{p\ell_e^2}{12},\ 0,\ \frac{p\ell_e}{2},\ -\frac{p\ell_e^2}{12}\right]^{\mathrm{T}}$$





(a) Portal frame



(b) Equivalent load for element 1

Solution We follow the steps given below:

Step 1. Connectivity

The connectivity is as follows:

	Node	
Element No.	1	2
1	1	2
2	3	1
3	4	2

Step 2. Element Stiffnesses

Element 1. Using the matrix given in Eq. 8.45 and noting that $\mathbf{k}^1 = \mathbf{k}'^1$, we find that

$$\mathbf{k}^1 = 10^4 \times \begin{array}{c} \begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array} \\ \begin{bmatrix} 141.7 & 0 & 0 & -141.7 & 0 & 0 \\ 0 & 0.784 & 56.4 & 0 & -0.784 & 56.4 \\ 0 & 56.4 & 5417 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 141.7 & 0 & 0 \\ 0 & -0.784 & -56.4 & 0 & 0.784 & -56.4 \\ 0 & 56.4 & 2708 & 0 & -56.4 & 5417 \end{bmatrix} \end{array}$$

Elements 2 and 3. Local element stiffnesses for elements 2 and 3 are obtained by substituting for E, A, I and ℓ_2 in matrix $\mathbf{k}'$ of Eq. 8.49:

$$\mathbf{k}'^2 = 10^4 \times \begin{bmatrix} 212.5 & 0 & 0 & -212.5 & 0 & 0 \\ 0 & 2.65 & 127 & 0 & -2.65 & 127 \\ 0 & 127 & 8125 & 0 & -127 & 4063 \\ -212.5 & 0 & 0 & 212.5 & 0 & 0 \\ 0 & -2.65 & -127 & 0 & 2.65 & 127 \\ 0 & 127 & 4063 & 0 & -127 & 8125 \end{bmatrix}$$

Transformation matrix $\mathbf{L}$. We have noted that for element 1, $\mathbf{k} = \mathbf{k}$. For elements 2 and 3, which are oriented similarly with respect to the x- and y-axes, we have $\ell = 0$, $m = 1$. Then,

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Noting that $\mathbf{k}^2 = \mathbf{L}^{\mathrm{T}}\mathbf{k}'^2\mathbf{L}$, we get

$$\begin{array}{llll} e = 3 & Q_4 & Q_5 & Q_6 \\ e = 2 \rightarrow & Q_1 & Q_2 & Q_3 \end{array}$$

$$\mathbf{k} = 10^4 \times \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & \mathbf{2.65} & \mathbf{0} & \mathbf{127} \\ 0 & -212.5 & 0 & \mathbf{0} & \mathbf{212.5} & \mathbf{0} \\ -127 & 0 & 4063 & \mathbf{127} & \mathbf{0} & \mathbf{8125} \end{bmatrix}$$

Stiffness $\mathbf{k}^1$ has all its elements in the global locations. For elements 2 and 3, the shaded part of the stiffness matrix shown previously is added to the appropriate global locations of $\mathbf{K}$. The global stiffness matrix is given by

$$\mathbf{K} = 10^4 \times \begin{bmatrix} 144.3 & 0 & 127 & -141.7 & 0 & 0 \\ 0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\ 127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 144.3 & 0 & 127 \\ 0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\ 0 & 56.4 & 2708 & 127 & -56.4 & 13542 \end{bmatrix}$$

From Fig. E8.2, the load vector can easily be written as

$$\mathbf{F} = \begin{Bmatrix} 3\,000 \\ -3\,000 \\ -72\,000 \\ 0 \\ -3\,000 \\ +72\,000 \end{Bmatrix}$$

The set of equations is given by

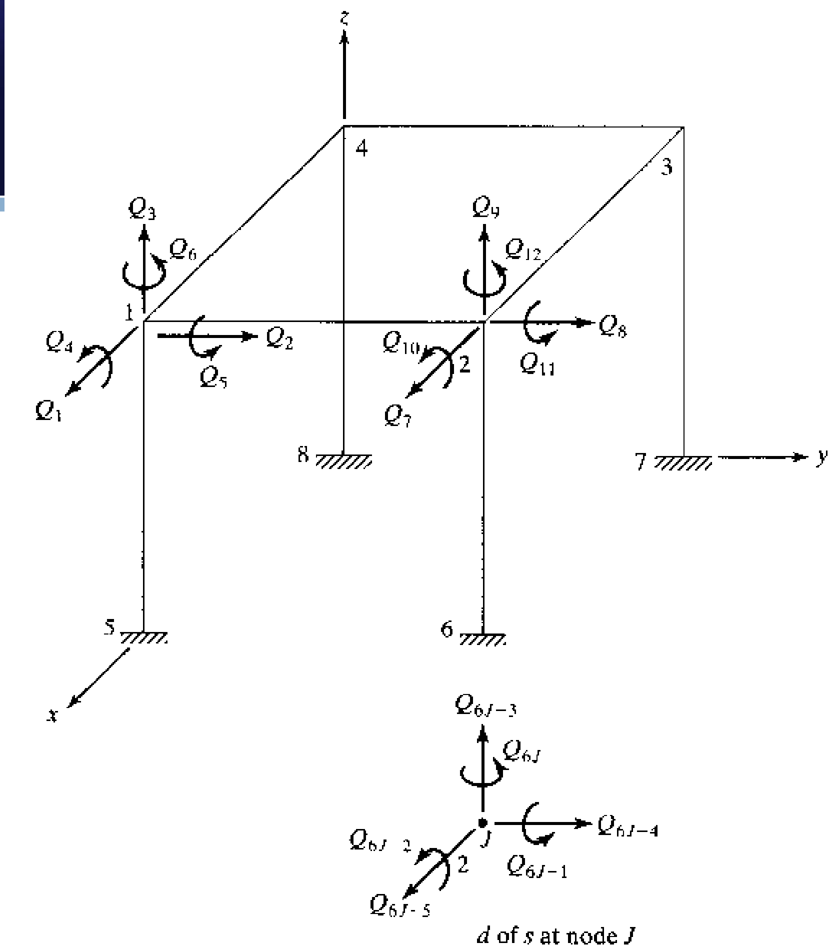
$$\mathbf{KQ} = \mathbf{F}$$

On solving, we get

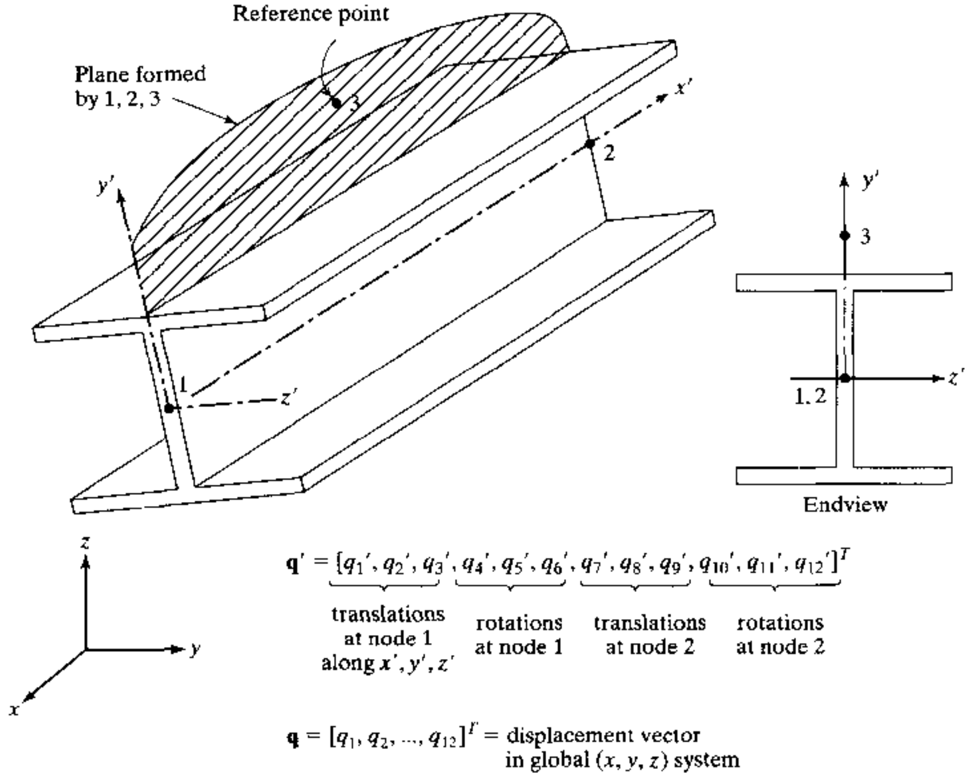
$$\mathbf{Q} = \begin{Bmatrix} 0.092 \text{ in.} \\ -0.00104 \text{ in.} \\ -0.00139 \text{ rad} \\ 0.0901 \text{ in.} \\ -0.0018 \text{ in.} \\ -3.88 \times 10^{-5} \text{ rad} \end{Bmatrix}$$

■

3D frames



z
y
x
1
2
3
4
5
6
7
8
Q_1, Q_2, Q_3, Q_4, Q_5, Q_6
Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}
Q_{6J-3}
Q_{6J}
Q_{6J-4}
Q_{6J-1}
Q_{6J-5}
J
d of s at node J



Reference point
Plane formed by 1, 2, 3
x'
y'
z'
1
2
3
Endview
1, 2
z
y
x
$\mathbf{q}' = [q_1', q_2', q_3', q_4', q_5', q_6', q_7', q_8', q_9', q_{10}', q_{11}', q_{12}']^T$
translations at node 1 along x', y', z'
rotations at node 1
translations at node 2
rotations at node 2
$\mathbf{q} = [q_1, q_2, \ldots, q_{12}]^T$ = displacement vector in global (x, y, z) system

$$
\mathbf{k}' = \begin{bmatrix}
AS & 0 & 0 & 0 & 0 & 0 & -AS & 0 & 0 & 0 & 0 & 0 \\
 & a_{z'} & 0 & 0 & 0 & b_{z'} & 0 & -a_{z'} & 0 & 0 & 0 & b_{z'} \\
 & & a_{y'} & 0 & -b_{y'} & 0 & 0 & 0 & -a_{y'} & 0 & -b_{y'} & 0 \\
 & & & TS & 0 & 0 & 0 & 0 & 0 & -TS & 0 & 0 \\
 & & & & c_{y'} & 0 & 0 & 0 & b_{y'} & 0 & d_{y'} & 0 \\
 & & & & & c_{z'} & 0 & -b_{z'} & 0 & 0 & 0 & d_{z'} \\
 & & & & & & AS & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & a_{z'} & 0 & 0 & 0 & -b_{z'} \\
 & & & & & & & & c_{y'} & 0 & b_{y'} & 0 \\
 & & & & & & & & & TS & 0 & 0 \\
\text{Symmetric} & & & & & & & & & & c_{y'} & 0 \\
 & & & & & & & & & & & c_{z'}
\end{bmatrix}
$$

$$\mathbf{q}' = \mathbf{L}\mathbf{q} \qquad \mathbf{L} = \begin{bmatrix} \boldsymbol{\lambda} & & & \mathbf{0} \\ & \boldsymbol{\lambda} & & \\ & & \boldsymbol{\lambda} & \\ \mathbf{0} & & & \boldsymbol{\lambda} \end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$l_1 = \frac{x_2 - x_1}{l_e} \qquad m_1 = \frac{y_2 - y_1}{l_e} \qquad n_1 = \frac{z_2 - z_1}{l_e}$$

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\mathbf{k} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}$$

$$\mathbf{f}' = \left[0, \frac{w_{y'}l_e}{2}, \frac{w_{z'}l_e}{2}, 0, \frac{-w_{z'}l_e^2}{12}, \frac{w_{y'}l_e^2}{12}, 0, \frac{w_{y'}l_e}{2}, \frac{w_{z'}l_e}{2}, 0, \frac{w_{z'}l_e^2}{12}, \frac{-w_{y'}l_e^2}{12}\right]^{\mathrm{T}}$$