Exercises of Probability and Statistics (E05)

September 2014

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1. 随机游动

甲乙两人游戏，每一局甲赢1元的概率为p，输1元的概率为$q = 1 - p$。假设一开始甲带了0元钱，令S_n表示n局后甲所拥有的钱数。计算 $Pr(S_8 = 4|S_1 = 1, S_3 = 1, S_4 = 2)$ 和 $Pr(S_8 = 4|S_4 = 2)$，是否相等？

Solution:

$$\begin{aligned} & Pr(S_8 = 4 | S_1 = 1, S_3 = 1, S_4 = 2) \\ = \; & Pr(S_8 - S_4 = 2 | S_1 = 1, S_3 = 1, S_4 = 2) \\ = \; & Pr(S_8 - S_4 = 2) \\ = \; & 4p^3 q \end{aligned}$$

$$\begin{aligned} & Pr(S_8 = 4 | S_4 = 2) \\ = \; & Pr(S_8 - S_4 = 2 | S_4 = 2) \\ = \; & Pr(S_8 - S_4 = 2) \\ = \; & 4p^3 q \end{aligned}$$

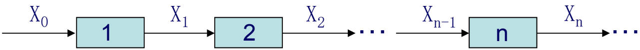
$$\therefore Pr(S_8 = 4 | S_1 = 1, S_3 = 1, S_4 = 2) = Pr(S_8 = 4 | S_4 = 2)$$

Remark

Markov性：$Pr(S_{n+1} = j | S_0 = i_0, \cdots, S_n = i_n) = Pr(S_{n+1} = j | S_n = i_n)$

2. 0–1传输系统

只传输0和1的串联系统中，设每一级的传真率为p，误码率为$q=1-p$。以X_0表示第一级的输入，X_n表示第n（$n\geqslant 1$）级的输出，请问$\{X_n\}$是否为时齐Markov链？其一步转移矩阵是什么，请画出相应的状态转移图。



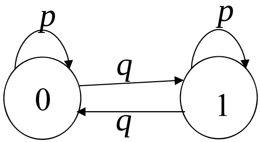
Solution:

$$
\begin{aligned}
\because p_{ij} &= Pr(X_{n+1} = j | X_n = i) \\
&= Pr(X_1 = j | X_0 = i) \\
&= \begin{cases} p & j = i \\ q & j \neq i \end{cases} \quad i, j = 0, 1
\end{aligned}
$$

So, it's time-homogeneous Markov Chain.

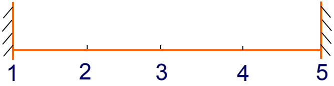
The transition matrix of the chain is

$$
\mathbf{P} = \begin{bmatrix} p & q \\ q & p \end{bmatrix}
$$



3. 双反射壁随机游动

设一醉汉在$I = \{1, 2, 3, 4, 5\}$作随机游动：如果现在位于点$i (1 < i < 5)$，则在下一时间他以等概率停留原处或向左、向右移动一步，如果当前位于点1或5，则下一时间一定转移到点2或点4。请问$\{X_n\}$是否为时齐Markov链？其一步转移矩阵是什么，请画出相应的状态转移图。

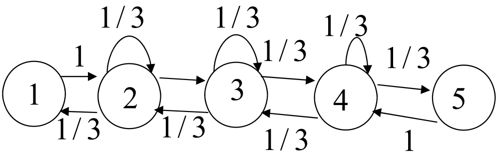


Solution:

The transition matrix of the chain is

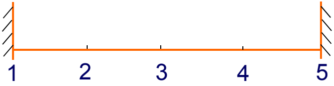
$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It doesn't change with n, so, the chain is time-homogeneous Markov Chain.



4. 吸收–反射壁随机游动

设一醉汉在$I = \{1, 2, 3, 4, 5\}$作随机游动：如果现在位于点$i (1 < i < 5)$，则在下一时间他以等概率停留原处或向左、向右移动一步，如果当前位于点5，则下一时间一定转移到点4，如果当前位于点1，则永远留在点1。求相应的一步转移矩阵是什么，请画出相应的状态转移图。



Solution:

The transition matrix of the chain is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

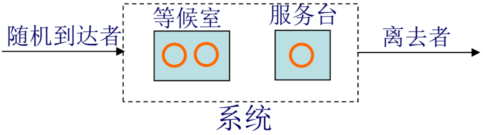
It doesn't change with n, so, the chain is time-homogeneous Markov Chain.

5. 排队模型

设服务系统由一个服务员和只可以容纳两个人的等候室组成。服务规则为：先到先服务，后来者需在等候室依次排队，假设一个需要服务的顾客到达系统时发现系统内已有3个顾客，则该顾客立即离去。

设时间间隔⊿t内有一个顾客进入系统的概率为q，有一接受服务的顾客离开系统(即服务完毕)的概率为p,又设当⊿t充分小时，在这时间间隔内多于一个顾客进入或离开系统实际上是不可能的，再设有无顾客来到与服务是否完毕是相互独立的。

求相应的一步转移矩阵



Solution:

Let X_n be the number of the customers at time n, then the statuses are $I = \{0, 1, 2, 3\}$. The transition matrix of the chain is

$$\mathbf{P} = \begin{bmatrix} 1-q & q & 0 & 0 \\ p(1-q) & pq+(1-p)(1-q) & q(1-p) & 0 \\ 0 & p(1-q) & pq+(1-p)(1-q) & q(1-p) \\ 0 & 0 & p(1-q) & pq+(1-p) \end{bmatrix}$$

It doesn't change with n, so, the chain is time-homogeneous Markov Chain.

6. Polya罐子模型

设一罐子装有r个红球，t个黑球，现随机从罐中取出一球，记录其颜色，然后将球放回，并加入a个同色球。持续进行这一过程，X_n表示第n次试验结束时罐中的红球数，$n=0,1,2,\cdots$。请问这一过程是否为Markov链，是否为时齐的，一步转移概率是什么？

Solution:

$\{X_n, n = 0, 1, 2, \cdots\}$是一随机过程，状态空间$I = \{r, r+a, r+2a, \cdots\}$，当$X_n = i$时，$X_{n+1} = j$的概率只与$i$有关，与$n$时刻之前如何取到$i$值是无关的，这是一Markov链。

一步转移概率为：

$$Pr(X_{n+1} = j | X_n = i) = \begin{cases} \frac{i}{r+t+na} & j = i + a \\ 1 - \frac{i}{r+t+na} & j = i \\ 0 & otherwise \end{cases}$$

因为$Pr(X_{n+1} = j | X_n = i)$与$n$有关，所以不是时齐的。

7. 掷骰子

独立重复地掷骰子，用X_n表示第n次掷出的点数，
令$Y_n = X_{n+1} + X_{n+2}$，$n \geqslant 0$，问：

1. 计算 $Pr(Y_2 = 12|Y_0 = 2, Y_1 = 7)$ 和 $Pr(Y_2 = 12|Y_1 = 7)$
2. 判断$\{Y_n\}$是否是Markov链。

Solution:

(1)

$$
\begin{aligned}
& Pr(Y_2 = 12 | Y_0 = 2, Y_1 = 7) \\
= \; & Pr(X_3 = X_4 = 6 | X_1 = X_2 = 1, X_3 = 6) \\
= \; & Pr(X_4 = 6) \\
= \; & \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
& Pr(Y_2 = 12 | Y_1 = 7) \\
= \; & \frac{Pr(Y_2 = 12, Y_1 = 7)}{Pr(Y_1 = 7)} \\
= \; & \frac{Pr(X_2 = 1, X_3 = X_4 = 6)}{Pr((X_2 + X_3 = 7)} \\
= \; & \frac{1}{36}
\end{aligned}
$$

(2) 因为 $Pr(Y_2 = 12 | Y_0 = 2, Y_1 = 7) \neq Pr(Y_2 = 12 | Y_1 = 7)$，所以$\{Y_n\}$不是Markov链。

8.

设$\{X_n, n \geqslant 0\}$是具有三个状态$0, 1, 2$的时齐Markov链，一步转移矩阵为：

$$\mathbf{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

再设初始分布$Pr(X_0 = i) = \frac{1}{3}$，$i = 0, 1, 2$，试求：

1. $Pr\{X_0 = 0, X_2 = 1, X_4 = 1\}$;
2. $Pr\{X_2 = 1, X_4 = 1, X_5 = 0 | X_0 = 0\}$;
3. $Pr\{X_2 = 1, X_4 = 1, X_5 = 0\}$;

Solution:

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{1}{4} \end{bmatrix}$$

(1)

$$\begin{aligned} & Pr\{X_0 = 0, X_2 = 1, X_4 = 1\} \\ = \; & Pr\{X_0 = 0\} p_{01}^{(2)} p_{11}^{(2)} \\ = \; & \frac{1}{3} \times \frac{5}{16} \times \frac{1}{2} \\ = \; & \frac{5}{96} \end{aligned}$$

(2)

$$
\begin{aligned}
& Pr\{X_2 = 1, X_4 = 1, X_5 = 0 | X_0 = 0\} \\
= \; & p_{01}^{(2)} p_{11}^{(2)} p_{10} \\
= \; & \frac{5}{16} \times \frac{1}{2} \times \frac{1}{4} \\
= \; & \frac{5}{128}
\end{aligned}
$$

(3)

$$
\begin{aligned}
& Pr\{X_2 = 1, X_4 = 1, X_5 = 0\} \\
= \; & Pr\{X_2 = 1\} p_{11}^{(2)} p_{10} \\
= \; & \{Pr(X_0 = 0) p_{01}^{(2)} + Pr(X_0 = 1) p_{11}^{(2)} + Pr(X_0 = 2) p_{21}^{(2)}\} p_{11}^{(2)} p_{10} \\
= \; & \frac{11}{192}
\end{aligned}
$$