Flight mechanics

Correction of tutorial n°1
Twin-engine aircraft performances

1 – Required thrust for steady level flight at sea level

– Limits of the thrust diagram Fn (V)

- Stall speed V_{S1g} :

  In level flight, just before stalling, lift equation says that : $mg = 1/2\,\rho S V_{S1g}^2 C_{Z\max}$

  At sea level (Zp=0): $\rho = \rho_0 = 1.225\,kg/m^3$

  Moreover, $C_{Z\max} = 1.3$, because M<0.3

  Therefore $V_{S_{1g}} = \sqrt{\dfrac{2mg}{\rho_0 S C_{Z\max}}} = \sqrt{\dfrac{2\times 61.2\times 10^3\times 9.81}{1.225\times 120\times 1.3}} = \mathbf{79.27\,m/s \approx 154.1Kts}$

  Remark: with $a = 340m/s$ at sea level, we check that $M = 0.233 < 0.3$

- Maximum speed VMO :

  The value of VMO is given : 350Kts *i.e.* 180m/s.

  (Reminder: VMO is the maximum operating speed, it is an indicated airspeed.)

  At Mach number corresponding to VMO ($M = 0.529$), we read on the curve of graph 1 that the maximum C_Z for which it is still possible to neglect compressibility effects is $C_{Zsuo} \approx 0.65$

  Let us calculate C_Z required for level flight at maximum speed :

  $$C_Z = \frac{2mg}{\rho_0 S (VMO)^2} = \frac{2\times 61.2\times 10^3\times 9.81}{1.225\times 120\times (180)^2} = 0.252$$

  This value is less than the maximum value C_{Zsup}, therefore we confirm we are in the incompressible domain of the polar. $C_x = C_{x_0} + kC_Z^2$

1.1 – Required thrust as a function of TAS

In steady level flight, thrust balances drag, therefore :

$$Fn(V) = 1/2\,\rho S V^2 C_x = 1/2\,\rho S V^2 \left(C_{x_0} + kC_z^2\right)$$

With $C_z = \dfrac{2mg}{\rho S V^2}$, we have: $Fn(V) = 1/2\,\rho S C_{x_0} V^2 + \dfrac{2km^2g^2}{\rho S}\dfrac{1}{V^2}$

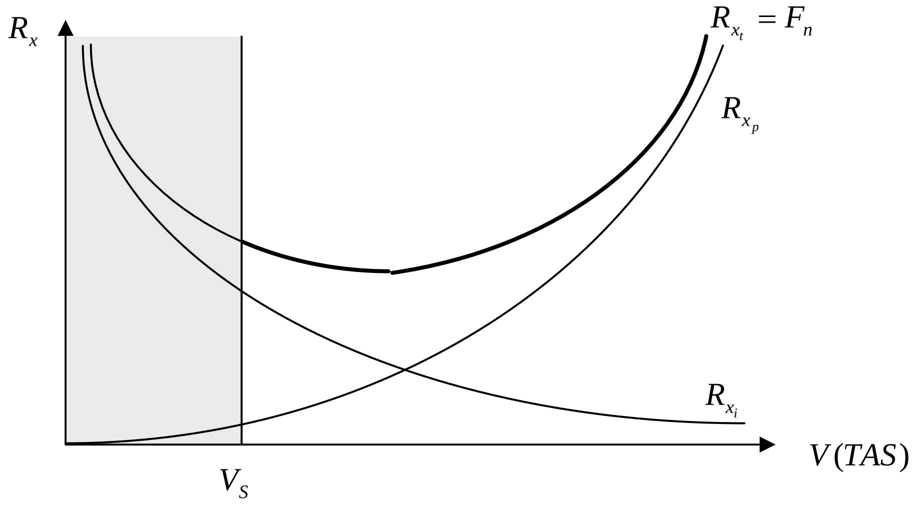
In steady level flight, required thrust balances total drag R_{x_t} .

We know that total drag is composed of :

- profile drag R_{x_p} due the airfoil shape, it varies with KV^2 (it is composed of form drag and friction drag)

- induced drag R_{x_i} due to wingtip vortices, it varies with $\frac{K'}{V^2}$

If we plot the curves $R_{x_p} = f(V)$, $R_{x_i} = f(V)$ and $R_{x_t} = R_{x_p} + R_{x_i} = f(V)$, we obtain :



If we represent minimum speed V_S , we recognize the appearance of the Fn curve. We remark that for low speeds, induced drag is predominant; for higher speeds, profile drag is the most important.

1.2 – Noticeable points on Fn (V) curve

We also write that lift balances weight, hence the two following equations :

$$\left.\begin{aligned} F_n(V) &= 1/2\,\rho S V^2 C_x \\ mg &= 1/2\,\rho S V^2 C_z \end{aligned}\right\} \quad \text{therefore} \quad F_n(V) = \frac{mg}{f} = mg\frac{C_{x_0} + kC_z^2}{C_z}$$

- Minimum F_n

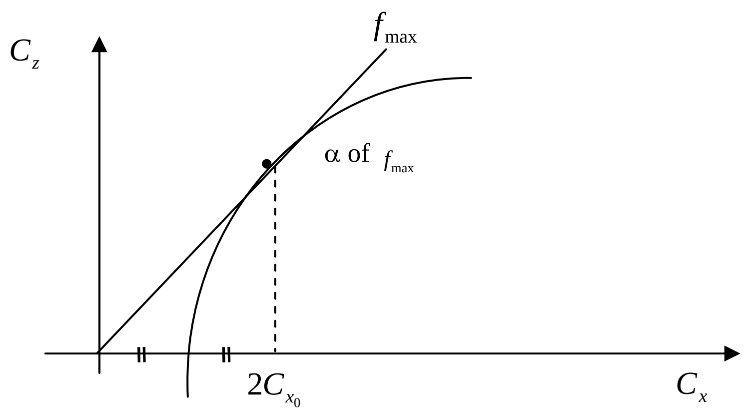
F_n is minimum when f is maximum, *i.e.* when $\frac{d}{dC_z}\left(\frac{C_z}{C_{x_0} + kC_z^2}\right) = 0$

This is true for C_z such that : $(C_{x_0} + kC_z^2) - C_z(2kC_z) = 0$

Therefore $C_{z_{f\,\max}} = \sqrt{\frac{C_{x_0}}{k}}$ and consequently $C_{x_{f\,\max}} = 2C_{x_0}$

Finally, $f_{\max} = \frac{1}{\sqrt{4kC_{x_0}}}$

On the polar curve :



So : $f_{\max} = \dfrac{1}{\sqrt{4 \times 3.881 \times 10^{-2} \times 1.753 \times 10^{-2}}} = 19.17$

Hence $F_{n_{\min i}} = \dfrac{61.2 \times 10^3 \times 9.81}{19.17} = \mathbf{31318N}$

And $V_{f\max} = \sqrt{\dfrac{2mg}{\rho S \sqrt{C_{x_0}/k}}} = \sqrt{\dfrac{2 \times 61.2 \times 10^3 \times 9.81}{1.225 \times 120 \times \sqrt{1.753/3.881}}} = \mathbf{110.2m/s \approx 214.3Kts}$

- Minimum F_n/V

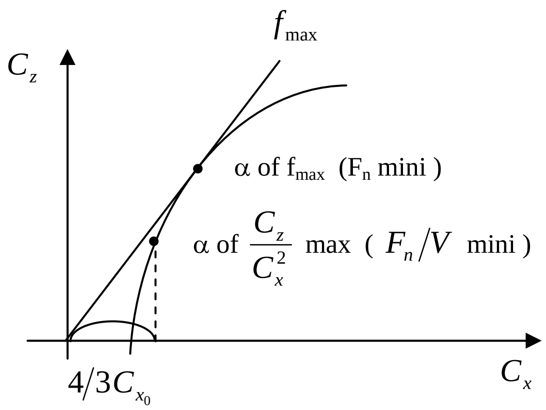
$$\frac{F_n}{V} = \frac{1/2\,\rho S V^2 C_x}{V} = 1/2\,\rho S V C_x = 1/2\,\rho S \sqrt{\frac{2mg}{\rho S C_z}} C_x = \sqrt{\frac{\rho S mg}{2}} \frac{C_x}{\sqrt{C_z}}$$

$\dfrac{F_n}{V}$ is minimum when $\dfrac{C_x}{\sqrt{C_z}}$ is minimum (or $\dfrac{C_z}{C_x^2}$ maximum)

$$\frac{d}{dC_z}\left(\frac{C_x}{\sqrt{C_z}}\right) = \frac{d}{dC_z}\left(\frac{C_{x_0} + kC_z^2}{\sqrt{C_z}}\right) = -1/2\,C_{x_0} C_z^{-3/2} + 3/2\,kC_z^{1/2} = -\frac{C_z^{-3/2}}{2}\left(C_{x_0} - 3kC_z^2\right)$$

$\dfrac{F_n}{V}$ is minimum when $C_z = \sqrt{\dfrac{C_{x_0}}{3k}}\left(= \dfrac{1}{\sqrt{3}} C_{z_{f\max}}\right)$ and $C_x = \dfrac{4}{3} C_{x_0}$

On the polar curve :



So : $C_z = \sqrt{\dfrac{1.753}{3 \times 3.881}} = 0.388$ $C_x = \dfrac{4}{3} \times 1.753 \times 10^{-2} = 0.0234$ and $f = 16.58$

Hence $F_n(F_n/V \min) = \dfrac{61.2 \times 10^3 \times 9.81}{16.58} = \mathbf{36211N}$

And $V(F_n/V \min) = \sqrt{\dfrac{2 \times 61.2 \times 10^3 \times 9.81}{1.225 \times 120 \times 0.388}} = \mathbf{145.1m/s \approx 282Kts}$

1.3 – Curve $F_n(V)$

We have the following table for some values of V :

	V_S		F_n mini	F_n/V mini	VMO
$V_{(m/s)}$	79.3	92.6	110.2	145.1	180
$V_{(Kts)}$	154.1	180	214.3	282	350
C_z	1.3	0.953	0.672	0.388	0.252
f	15.64	18.06	19.17	16.58	12.60
$F_{n(N)}$	38387	33243	31318	36211	47649

See curve in annex.

2 – Influence of altitude and mass

2.1 – We know that $F_n = \dfrac{mg}{f} = \dfrac{mg * C_x}{C_z} = \dfrac{mg\left(C_{x_0} + kC_z^2\right)}{C_z}$

Thanks to the lift equation, we can express $C_z = \dfrac{2mg}{\rho S V^2}$ at any pressure altitude, $C_z = \dfrac{2mg}{\rho_0 S (EAS)^2}$ at $PA = 0$, with ρ_0 density at $PA = 0$.

(Therefore $EAS = V \times \sqrt{\dfrac{\rho}{\rho_0}}$)

In the incompressible domain, we can therefore express F_n, function of C_z, as a function of EAS and mass, which are independent of pressure altitude and temperature.

$$F_n = \frac{\rho_0 S (EAS)^2}{2}\left(C_{x_0} + k\left(\frac{2mg}{\rho_0 S (EAS)^2}\right)^2\right)$$

2.2 – Influence of altitude

Let : $F_{n_0}(EAS, m)$ be the required thrust when considering incompressible flow

$F_{n_{Zp}}(EAS, m)$ be the required thrust at pressure altitude Z_P, taking compressibility into account

In compressible conditions, the polar curve equation becomes :

$C_x(C_z, M) = C_{x_0} + kC_z^2 + \Delta C_x(C_z, M)$

If we note $C_x(C_z, 0)$ the value of C_x in incompressible conditions (*i.e.* the quantity $C_{x_0} + kC_z^2$), we can write :

$$F_n = mg\frac{C_x(C_z, M)}{C_z} = mg\frac{1}{C_z}\left[C_x(C_z, 0) + \Delta C_x(C_z, M)\right] = mg\frac{C_x(C_z, 0)}{C_z}\left(1 + \frac{\Delta C_x(C_z, M)}{C_x(C_z, 0)}\right)$$

We recognize F_{n_0} : $F_{n_{Zp}}(EAS, m) = F_{n_0}(EAS, m)\left(1 + \dfrac{\Delta C_x(C_z, M)}{C_x(C_z, 0)}\right)$

ΔC_x is a function of C_z and Mach number :

- $M = \sqrt{\dfrac{C_z M^2}{C_z}}$ so $\Delta C_x(C_z, M) = \Delta C_x(C_z, C_z M^2)$
- we saw (§2.1) that $C_z = \dfrac{2mg}{\rho_0 S (EAS)^2} = C_z(EAS, m)$
- the lift equation can be written : $mg = 1/2\, \rho S a^2 M^2 C_z$

  therefore $C_z M^2 = \dfrac{mg}{S} \times \dfrac{2}{\rho a^2} = \dfrac{mg}{S} \times \dfrac{2}{\rho \gamma r T} = \dfrac{mg}{S} \times \dfrac{2}{\gamma P} = \dfrac{mg}{S} \times \dfrac{1}{0{,}7 P(Z_P)}$

  So $C_z M^2 = C_z M^2 (Z_P, m)$

To conclude, we have : $F_{n_{Zp}}(EAS, m) = F_{n_0}(EAS, m)\left(1 + \dfrac{\Delta C_x}{C_x}\left(C_z(EAS, m), C_z M^2 (Z_P, m)\right)\right)$.

Because of compressibility, the required thrust increases with pressure altitude. This increase is all the more important that the EAS is high.

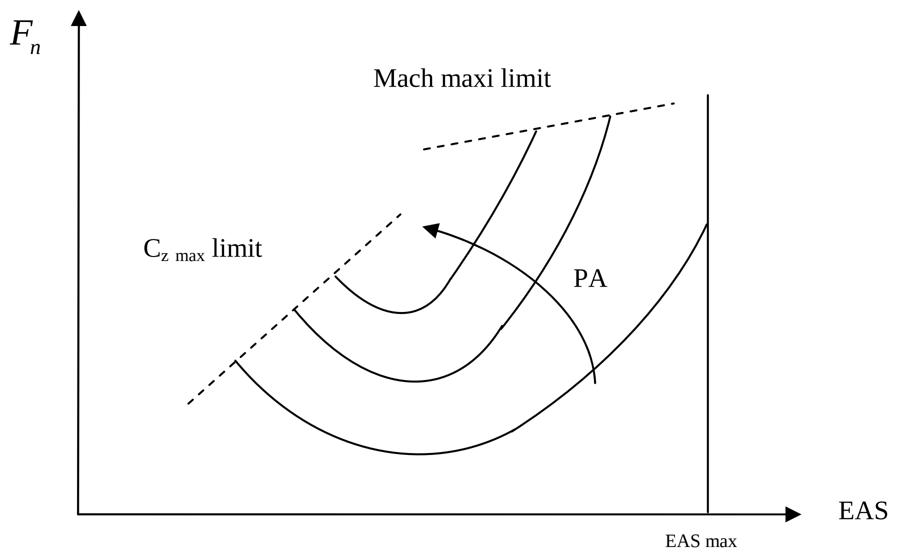
Note : at PA=13000ft (*i.e.* P=61943Pa), when the Mach number increases, there comes a time when available C_z is less than the C_z required for level flight.

If we read the available C_z on figure 1 in the text, and we compute required C_z ($C_z = \dfrac{61.2 \times 10^3 \times 9.81}{120 \times 0.7 \times P \times M^2}$), we have the following values :

Mach number	0.35	0.4	0.45	0.5	0.55	0.6	0.65
Required C_z	0.942	0.721	0.570	0.462	0.381	0.321	0.273
Available C_z	1.3	0.9	0.78	0.72	0.64	0.52	0.26

At 13000ft, it is impossible to fly level at Mach 0.65.

We can draw the evolution of the curve F_n (EAS) as a function of pressure altitude.



2.3 – Influence of mass

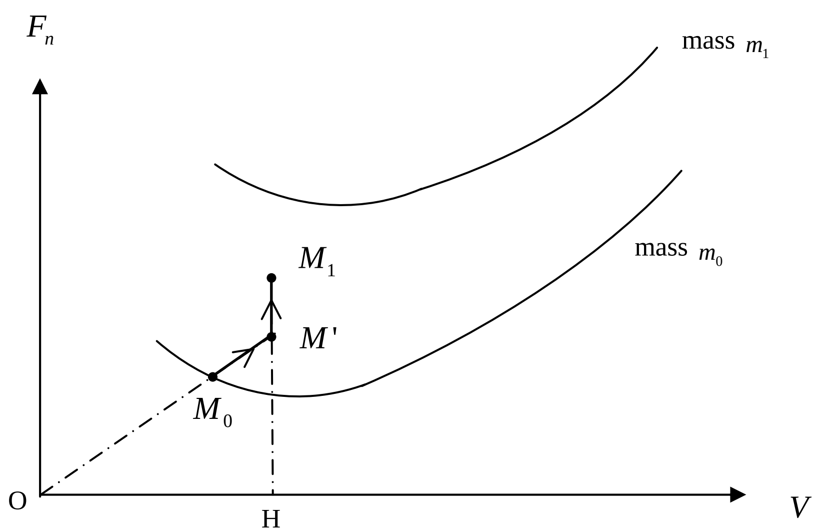
If we make the hypothesis that the polar is independent of the Mach number (incompressible conditions), at a given pressure altitude Z_P and at the same $\underline{C_{z_0}}$, the speed is :

$V_1 = \sqrt{\dfrac{m_1}{m_0}} V_0$ (because $V = \sqrt{\dfrac{2mg}{\rho S C_z}}$)

Hence the corresponding thrust : $F_n(V_1, m_1, Z_P) = m_1 g \dfrac{C_x}{C_z} = \dfrac{m_1}{m_0} F_n(V_0, m_0, Z_P)$.

Furthermore, if $C_z = cste = C_{z_0}$, then $C_x = C_{x_0} + kC_z^2 = cste$

We thus deduce the curve relative to mass m_1 from the curve relative to reference mass m_0 thanks to an homothetic transformation of ratio $\sqrt{\dfrac{m_1}{m_0}}$ followed by an affine transformation of same ratio on the axis of ordinates F_n.

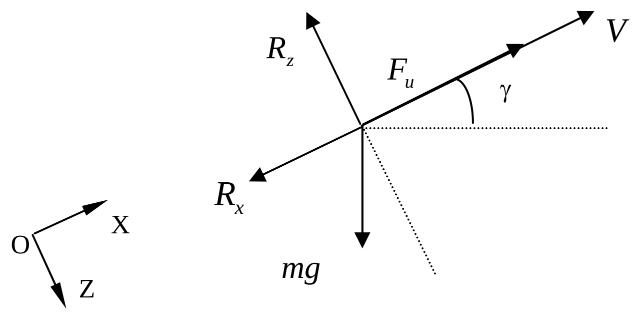


$$OM' = \sqrt{\frac{m_1}{m_0}} OM_0$$

$$HM_1 = \sqrt{\frac{m_1}{m_0}} HM'$$

3 – Climb performances

Let us consider the case of the climb with a constant climb gradient and a variable speed $\left(\dfrac{dV}{dt} \neq 0\right)$. We remind the present forces :



F_u is the available thrust delivered by the engines (depends on the thrust lever position)

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Along (Ox) : $m\dfrac{dV}{dt} = F_u - R_x - mg\sin\gamma$ (1)

Along (Oz) : $R_z = mg\cos\gamma$ (2)

The slope γ being small, we take : $\sin\gamma \approx \gamma_{rd}$ and $\cos\gamma \approx 1$, which gives us :

$$m\frac{dV}{dt} + mg\gamma \approx F_u - R_{x(\text{climb})} \qquad (1')$$

$$R_{z(\text{climb})} \approx mg \qquad (2')$$

In level flight, we saw that $mg = R_{z(level)}$. With (2'), we deduce that $R_{z(\text{climb})} \approx R_{z(level)}$, hence, by making the hypothesis that the angle of attack (and therefore the C_z) do not change, $V_{\text{climb}} \approx V_{level}$.

We deduce that : $R_{x(\text{climb})} = 1/2\rho S V_{\text{climb}}^2 C_x \approx 1/2\rho S V_{level}^2 C_x = R_{x(level)}$

In level flight, we had $R_{x(level)} = F_{n(level)}$, therefore $R_{x(climb)} \approx F_{n(level)}$

Equation (1') finally becomes : $m\frac{dV}{dt} + mg\gamma = F_u - F_{n(level)}$ (3)

3.1 – Climb at constant TAS

In the case of the constant speed climb, we use the above result with $\frac{dV}{dt} = 0$; the climb gradient equals : $\gamma = \frac{F_u - F_n}{mg}$

γ will be maximum when F_n is minimum, *i.e.* for α of $f_{\max}$ (ref §1.3) and therefore for the corresponding speed.

Numerical application : at $Z_P = 0$, we found $V_{f_{\max}} = 110.2 m/s$ and $F_{n_{\min i}} = 31318N$

If maximum thrust is available (full throttle), we have $F_u = F_{u_0} = 150kN$

Therefore $\gamma_{\max} = \frac{150000 - 31318}{61200 \times 9.81} = 0.1977 rd = 11.3°$

3.2 – Climb at constant EAS

We remind that $EAS = TAS\sqrt{\sigma}$.

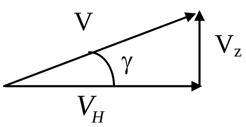
If we perform a climb at constant EAS, relative density σ decreases, so the TAS increases. We have to introduce in the equations a term $\frac{dV}{dt}$.

Equation (3) becomes : $\gamma + \frac{1}{g}\frac{dV}{dt} = \frac{F_u - F_n}{mg}$; we recognize the expression of climb gradient in case of a climb at constant TAS.

We therefore have : $\gamma + \frac{1}{g}\frac{dV}{dt} = \gamma$ (constant TAS)

i.e. : $\gamma\left(1 + \frac{1}{\gamma g}\frac{dV}{dH}\frac{dH}{dt}\right) = \gamma$ (constant TAS)

$\frac{dH}{dt} = V_Z = V\sin\gamma \approx V\gamma$



(Vz : rate of climb)

So : $\gamma\left(1 + \frac{V}{g}\frac{dV}{dH}\right) = \gamma$ (constant TAS)

We finally have :

$$\boxed{\gamma\,(\text{constant EAS}) = \frac{1}{1 + \frac{V}{g}\frac{dV}{dH}}\gamma\,(\text{constant TAS})}$$

We saw that when pressure altitude increases, with a constant EAS, density decreases and speed increases, therefore $\frac{dV}{dH} > 0$.

It means that during a constant EAS climb, the climb gradient decreases as the altitude increases.

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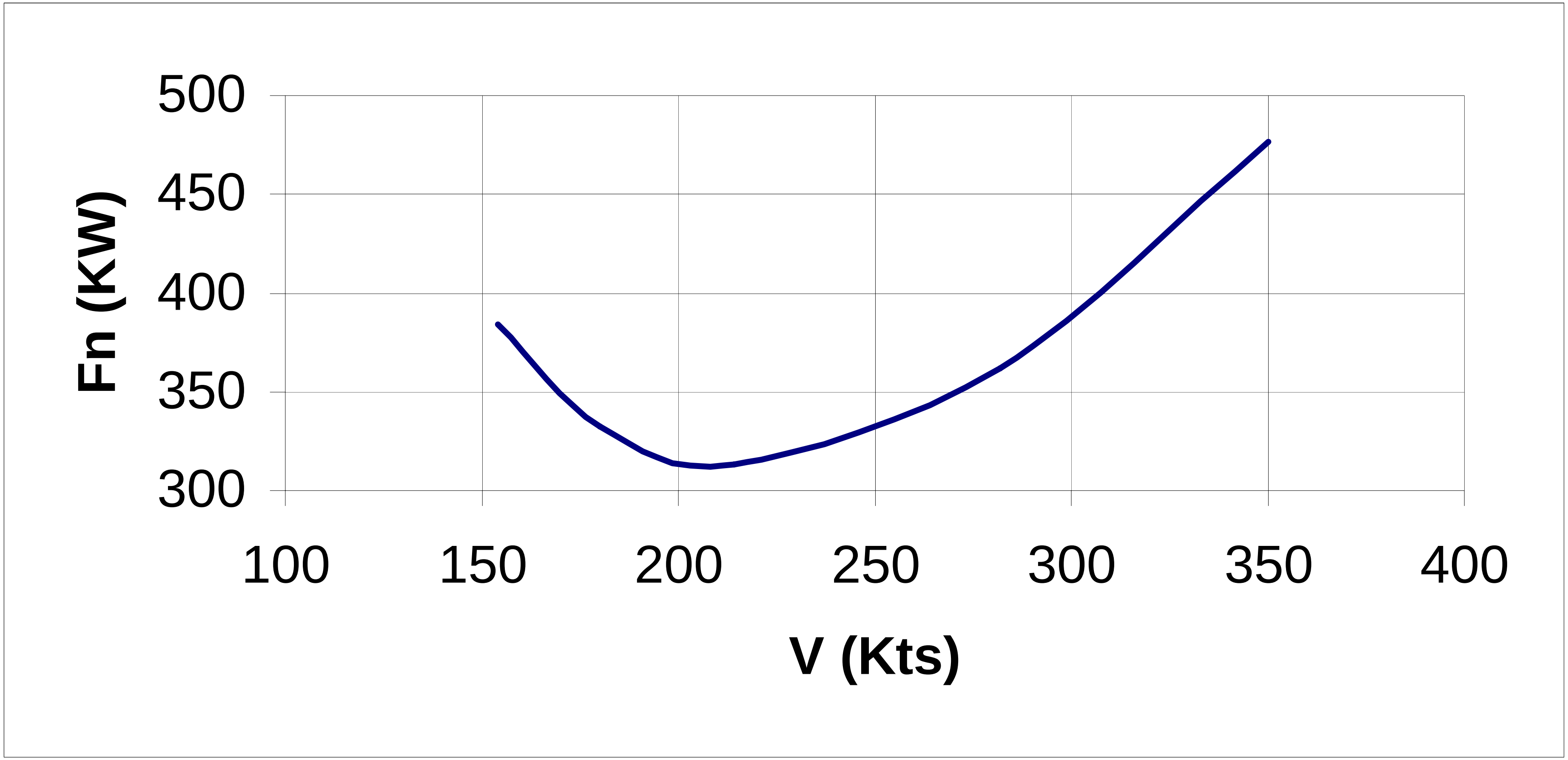
Annex



Flight mechanics

Correction of tutorial n°2
Low altitude cruise flight of a transonic twinjet

1 – Endurance and range

1.1 – Hourly consumption and range

- Hourly consumption C_h

It is the quantity of fuel consumed per unit of time (sometimes called *Fuel flow*) :

$$C_h = \frac{dQ}{dt} \qquad (1)$$

We have: $C_h = C_S F_u$, where C_S is the specific fuel consumption and F_u the available thrust (*i.e.* the thrust selected by the pilot).

In stabilized level flight, we have: $F_u = F_n(V) = \frac{mg}{f} = mg\frac{C_x}{C_z}$

Therefore : $$\boxed{C_h = C_S \frac{mg}{f} = C_S mg \frac{C_{x_0} + kC_z^2}{C_z}}$$ in kg/h

In the hypothesis where C_S is constant, C_h does not vary with altitude.

C_h is minimum when the lift-drag ratio is maximum.

Reminder: if we consider that the polar is similar to a parabola with equation $C_x = C_{x_0} + kC_z^2$, the point where the lift-drag ratio is maximum corresponds to $C_z = \sqrt{\frac{C_{x_0}}{k}}$ and $C_x = 2C_{x_0}$.

- Endurance τ

Endurance is the time during which it is possible to maintain level flight with a given quantity of fuel: $\tau = \int_{m_0}^{m_1} dt = \int_{m_0}^{m_1} \frac{1}{C_h} dQ$ according to equation (1).

As the aircraft burns fuel, its mass decreases : $dQ = -dm$

Hence the endurance $\tau = -\int_{m_0}^{m_1} \frac{1}{C_h} dm = -\int_{m_0}^{m_1} \frac{f}{C_S g}\frac{dm}{m}$

We made the hypothesis that specific fuel consumption was constant. Moreover, if we consider a level flight cruise with a constant C_z (*i.e.* with a constant angle of attack), the lift-drag ratio is constant, and we have: $\tau = -\frac{f}{C_S g}\int_{m_0}^{m_1} \frac{dm}{m} = \frac{f}{C_S g}\ln\frac{m_0}{m_1}$

Note: with $mg = 1/2\,\rho S V^2 C_z$, if $C_z = cst$, then the speed decreases.

Therefore : $\boxed{\tau = \dfrac{f}{C_S g} \ln \dfrac{m_0}{m_0 - Q}}$ in hours

1.2 – Specific range

Specific range is the distance traveled by unit of mass of consumed fuel.

$SR = \dfrac{dL}{dQ}$ where dL is the distance travelled during dt

$SR = \dfrac{V_k dt}{dQ} = \dfrac{V_k}{C_h} = \dfrac{1}{C_S} \dfrac{f}{mg} V_k$ where $V_k (ground\ \ speed) = V(true\ air\ \ speed) + W_e (wind)$

In the first place, we consider that there is no wind, so $V_k = V$

We can express V in m/s using the lift equation : $V = \sqrt{\dfrac{2mg}{\rho S C_z}}$

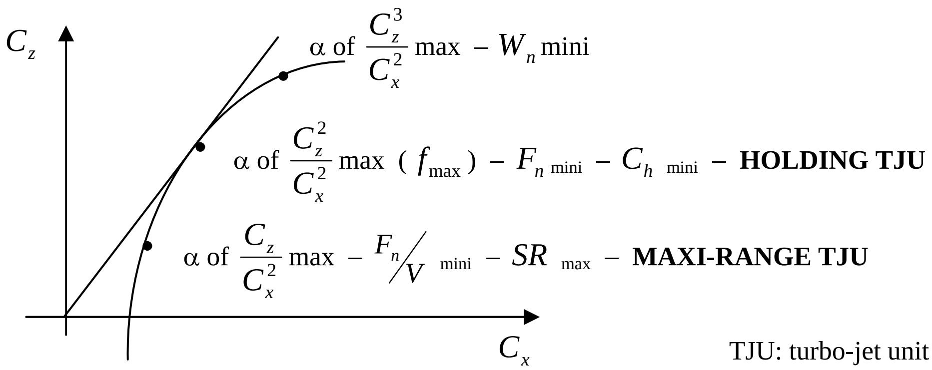
Consequently $V_{Kts} = \dfrac{3600}{1852} V_{m/s} = 1.944 * V_{m/s}$

$$SR = \frac{1}{C_S mg} \frac{C_z}{C_x} 1.944 \sqrt{\frac{2mg}{\rho S C_z}}$$

Therefore : $\boxed{SR = \dfrac{1.944}{C_S} \sqrt{\dfrac{2}{\rho S g}} \dfrac{1}{\sqrt{m}} \dfrac{\sqrt{C_z}}{C_x}}$ in Nm/kg

Note : *SR* is maximum if :

- ρ is as low as possible (high altitude)
- $\dfrac{C_x}{\sqrt{C_z}}$ is minimum, which corresponds to a minimum $\dfrac{F_n}{V}$



TJU: turbo-jet unit

Note: some numerical values

	Mass	Number of passengers	TAS	Mach number	Tank capacity	C_h	SR
Airbus A300	150 t	250	460 Kts	0.78	50 t	6 t/h	76.7 Nm/t
Boeing 747-100	330 t	400	490 Kts	0.84	150 t	12 t/h	41 Nm/t

- Maximum range D

$$D = -\int_{m_0}^{m_1} SRdm = -\int_{m_0}^{m_1} \frac{1.944}{C_S}\sqrt{\frac{2}{\rho Sg}}\frac{\sqrt{C_z}}{C_x}\frac{dm}{\sqrt{m}}$$

With $\int_{m_0}^{m_1} \frac{dm}{\sqrt{m}} = \left[2\sqrt{m}\right]_{m_0}^{m_1} = 2\left(\sqrt{m_1} - \sqrt{m_0}\right) = 2\left(\sqrt{m_0 - Q} - \sqrt{m_0}\right)$, we find that:

$$\boxed{D = 2\times\frac{1.944}{C_S}\sqrt{\frac{2}{\rho Sg}}\frac{\sqrt{C_z}}{C_x}\left(\sqrt{m_0} - \sqrt{m_0 - Q}\right)}$$

- Transport coefficient K

According to the above expression of D, for a given distance D, we have $\sqrt{m_0} - \sqrt{m_1} = \text{constant}$.

If we differentiate : $\frac{1}{2}\left(\frac{dm_0}{\sqrt{m_0}} - \frac{dm_1}{\sqrt{m_1}}\right) = 0$

So: $\boxed{K = \frac{dm_0}{dm_1} = \sqrt{\frac{m_0}{m_1}}}$

Physical interpretation of the transport coefficient : it is the additional takeoff mass necessary to bring 1kg of extra load to destination (it generally varies with distance traveled and mass).
Two practical examples :

- fuel budget of my flight being done for a certain mass, I have to load additional freight dm_1 at the last minute; the transport coefficient enables to rapidly determine the takeoff mass variation dm_0, hence the additional fuel
- I take off from an airfield where fuel is cheap, whereas it is expensive at the next stop; it may be interesting to load additional fuel (hence an additional takeoff mass dm_0), but this fuel will cost me an extra consumption; thanks to the transport coefficient, I will be able to decide whether this operation is profitable or not.

For instance, the transport coefficient on a Paris - Rio flight is about 1.6.

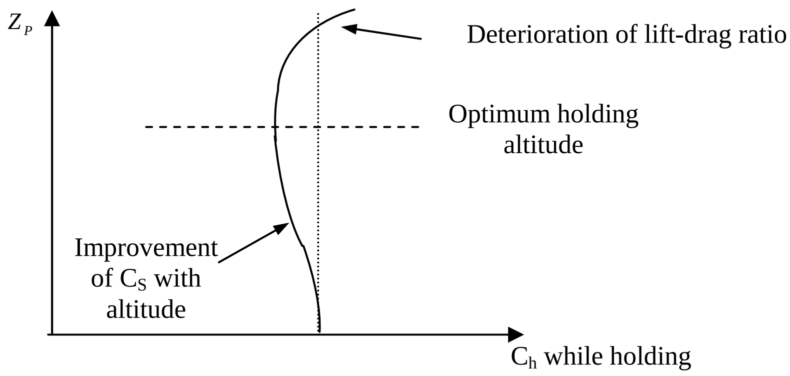
1.3 – True influence of altitude

Until now, we made the hypothesis that $C_S = cst$ and $C_x(C_z)$ is independent of Mach number (incompressible flow). We saw that in that case :

- the hourly consumption C_h is independent from altitude : for a turbojet, holding time doesn't change with altitude
- the specific range increases with altitude (for a turbojet), that is why aircraft adopt high cruise altitudes

If my angle of attack is set on Maxi Range value, when I burn fuel, mass decreases and I will be able to climb ($mg = 1/2\,\rho SV^2 C_z$) : optimum cruise, regarding consumption, is a continuous climbing cruise. For practical reasons, air traffic control does not authorize this kind of flight, so we perform step cruises.

In fact, C_S is actually getting better with altitude and lift-drag ratio decreases with Mach number. For jet engines, here is the evolution of C_h when holding:



2 – Influence of wind

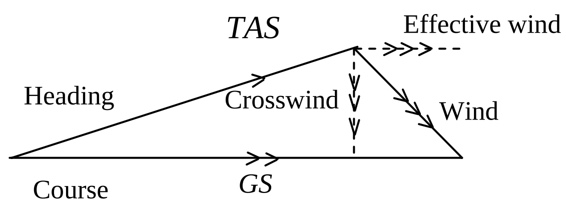
- Specific range with effective wind $SR(V,W_e)$

We have : $$SR(V,W_e)=\frac{V_k}{C_h}=\frac{V+W_e}{C_h}=\frac{V+W_e}{V}\frac{V}{C_h}=\left(1+\frac{W_e}{V}\right)\frac{V}{C_h}$$

We recognize specific range with no wind.

Therefore: $$\boxed{SR(V,W_e)=\left(1+\frac{W_e}{V}\right)SR(V,0)}$$

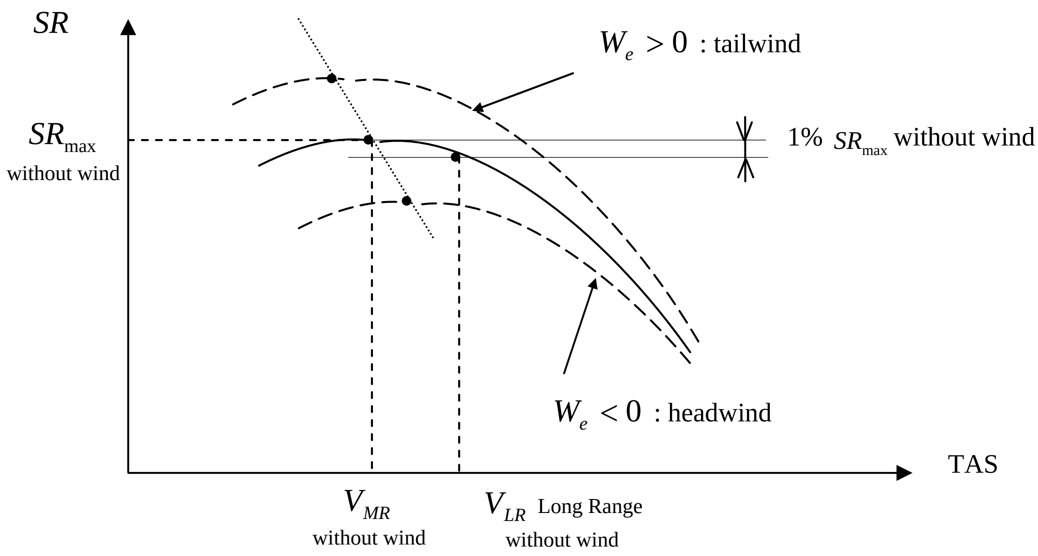
Note: the effective wind is the wind component that really affects the aircraft.



When there is tailwind ($W_e > 0$), the true air speed necessary to have the same ground speed decreases, while specific range increases ($SR(W_e)=(1+\varepsilon)SR(0)$ and $\varepsilon>0$). We can deduce the evolution of the curve $SR(TAS)$ when the effective wind varies (see figure below).

- Maxi-range speed V_{MR}

The maxi-range true air speed decreases with tailwind ($W_e > 0$) and increases with headwind ($W_e < 0$).



- Influence of wind on C_h and SR

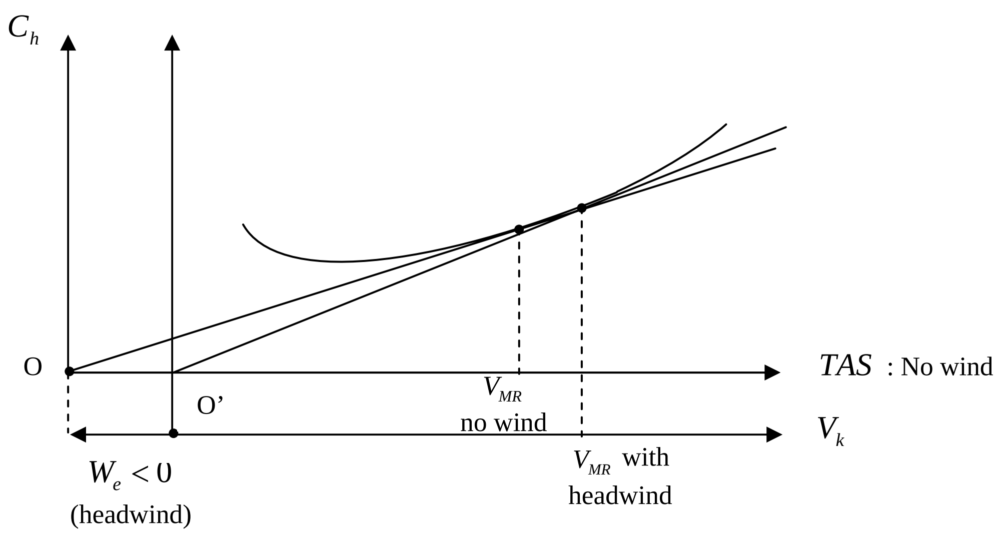
The maxi-range point corresponds to $\frac{V_k}{C_h}$ maxi.

Let us draw the curve $C_h = f(V) = C_S F_u$. Since C_S is constant, the shape of the curve is similar to that of $F_u = f(V)$, which is known.

Without wind, it is equivalent to drawing $C_h = f(TAS)$. We place the tangent to the curve from the origin; this indicates maxi-range.

With wind, we have $V_k = V + W_e$. For a given TAS, if we have for instance headwind $W_e < 0$, we have a smaller ground speed; it amounts to displacing the origin on the axis of speed, rightward in case of headwind.

We read a new maxi-range, with wind; with headwind, it corresponds to a higher speed and a smaller angle of attack.



- Interest of Long-Range

We place the point corresponding to long-range (99% of SR_{max} without wind) on the curve $SR = f(TAS)$.

The interest of Long-Range cruise is that you can fly faster, with a rather limited SR decrease (1%), hence more rotations; fixed charges are better paid off.

Note: nowadays, airlines try to take into account the actual fixed charges and to choose an adapted Maxi-Range / Long-Range ratio (instead of a standard 1% difference)

3 – Low altitude cruise

3.1 – Maxi-Range equivalent air speed

We saw that the lift equation gives : $V=\sqrt{\frac{2mg}{\rho SC_z}}$.

By definition, if $Z_P=0$, *i.e.* $\rho=\rho_0=1.225kg/m^3$: $EAS=\sqrt{\frac{2mg}{\rho_0 SC_z}}$.

We need to determine the value of C_z corresponding to Maxi-Range. We saw in §1.2 that it is the case if $\frac{\sqrt{C_z}}{C_x}$ is minimum or $\frac{C_x}{\sqrt{Cz}}$ is maximum.

With $\frac{C_x}{\sqrt{C_z}}=\frac{C_{x_0}}{\sqrt{C_z}}+C_z^{3/2}$, $\frac{C_x}{\sqrt{Cz}}$ is maximum when $\frac{\partial}{\partial C_z}\left(\frac{C_x}{\sqrt{C_z}}\right)=0$.

Once again we recognize one of the noticeable points of the polar curve : $C_z=\sqrt{\frac{C_{x_0}}{3k}}$

Numerical application: $C_{z_{MR}}=\sqrt{\frac{1.753}{3\times 3.881}}=0.388$

$$EAS=\sqrt{\frac{2\times 61\times 10^3\times 9.81}{1.225\times 122\times 0.388}}=\mathbf{143.66\,m/s=279.26Kts}$$

3.2 – $SR=f(\sigma)$

We saw in §1.1 that

$$\boxed{C_h=C_S\frac{mg}{f}=C_S mg\frac{C_{x_0}+kC_z^2}{C_z}}$$

Numerical application: with $C_{z_{MR}}=0.388$

$$C_h=0.065\times 61\times 10^3\times 9.81\times\frac{(1.753+3.881\times 0.388^2)10^{-2}}{0.388}=\mathbf{2.34t/h}$$

We saw in §1.2 that $SR=\frac{1.944}{C_S}\sqrt{\frac{2}{\rho Smg}}\frac{\sqrt{C_z}}{C_x}$.

With $\sigma=\frac{\rho}{\rho_0}$, we can express the specific range as a function of relative density:

$$\boxed{SR=\frac{1}{\sqrt{\sigma}}\frac{1.944}{C_S}\sqrt{\frac{2}{\rho_0 Smg}}\frac{\sqrt{C_z}}{C_x}}$$

3.3 – Limit of validity of the incompressible polar curve

For Maxi-Range speed, we have $C_z = 0.388$.

This C_z can be produced by the wings up to $M = 0.633$ (refer to graph giving the validity of the incompressible polar curve).

The lift equation using Mach number states that : $mg = \frac{\gamma}{2} PSC_z M^2$.

The maximum allowable Mach number corresponds to a minimum pressure $P_{\min} = \frac{2mg}{\gamma SC_z M_{\max}^2}$

In this specific case, $P_{\min} = \frac{2 \times 61000 \times 9.81}{1.4 \times 122 \times (0.633)^2 \times 0.388} = 450.7 hPa$

This corresponds to a maximum pressure altitude $Z_P =$ **20800ft** below which the incompressible polar curve can be used.

3.4 – Specific range at FL 100

Note: FL 100 is far too low for economical cruise; it is the level used in case of a depressurization.

We saw in §3.1 that $C_{z_{MR}} = \sqrt{\frac{C_{x_0}}{3k}} = 0.388$, which gives $C_{x_{MR}} = \frac{4}{3} C_{x_0} = 0.02337$

We read in the standard atmosphere table the relative density at flight level 100: $\sigma_{FL100} = 0.7385$

Therefore :

$$SR_{FL100} = \frac{1}{\sqrt{0.7385}} \frac{1.944}{0.065} \sqrt{\frac{2}{1.225 \times 122 \times 61000 \times 9.81}} \frac{\sqrt{0.388}}{0.02337} = \mathbf{0.1387 Nm/kg} = \mathbf{138.7 Nm/t}$$

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Flight mechanics

Correction of tutorial n°3
Performances of a twinjet

1 – Speed measurement at FL50

1.1 – True air speed and Mach number in ISA conditions

True air speed $TAS = EAS\sqrt{\frac{\rho_0}{\rho}} = EAS\sqrt{\frac{1}{\sigma}}$

Numerical application: $TAS = 250\sqrt{\frac{1}{0.8617}} = \mathbf{269.3Kts} = \mathbf{138.5m/s}$

Mach number : $M = \frac{TAS}{a}$

Numerical application: $M = \frac{269.3}{650.01} = \mathbf{0.414}$

1.2 – True air speed and Mach number in ISA+10 conditions

True air speed: $TAS_{(ISA+10)} = EAS\sqrt{\frac{1}{\sigma_{ISA+10}}}$

According to ideal gas law, P = ρrT = cst (it is constant because we are at the same pressure altitude). Therefore we can write : $\rho_{ISA}T_{ISA} = \rho_{ISA+10}T_{ISA+10}$

Hence : $\rho_{ISA+10} = \frac{\rho_{ISA}T_{ISA}}{T_{ISA+10}}$ and $\sigma_{ISA+10} = \frac{\sigma_{ISA}T_{ISA}}{T_{ISA+10}}$

Similarly : $TAS_{ISA+10} = TAS_{ISA}\sqrt{\frac{T_{ISA+10}}{T_{ISA}}}$

Numerical application: $\sigma_{ISA+10} = 0.8617 \times \frac{273+5.1}{273+15.1} = \mathbf{0.8318}$

$TAS_{ISA+10} = 250\sqrt{\frac{1}{0.8318}} = \mathbf{274.1Kts} = \mathbf{141m/s}$

Mach number: $M_{ISA+10} = \frac{TAS_{ISA+10}}{a_{ISA+10}}$

$a_{ISA+10} = \sqrt{\gamma r T_{ISA+10}} = a_{ISA}\sqrt{\frac{T_{ISA+10}}{T_{ISA}}}$ Hence: $\boxed{M_{ISA+10} = M_{ISA}}$

Note: this result can also be found using the following :
$1/2\rho_0 EAS^2 = 1/2\rho TAS^2 = 1/2\rho\, a^2 M^2 = 1/2\rho\, \gamma r T M^2 = \gamma/2\, P M^2$
At constant EAS and flight level, Mach number is independent from temperature.

2 – Thrust required for the flight – Acceleration performances

2.1 – Thrust required for the flight

Thrust required for level flight: $F_n = \dfrac{mg}{f} = mg\dfrac{C_{x_0} + kC_z^2}{C_z}$

We can determine the C_z required for a level flight at given EAS and mass:

$mg = 1/2\rho_0 EAS^2 SC_z$ Hence: $C_z = \dfrac{2mg}{\rho_0 EAS^2 S}$

Numerical application: $C_z = \dfrac{2\times 4400\times 3600^2}{1.225\times 250^2\times 1852^2} = \mathbf{0.43}$

Thrust required for level flight : $F_n = 53800\times 9.81\dfrac{(1.753+3.881\times 0.43^2)\times 10^{-2}}{0.43} = \mathbf{30.3kN}$

2.2 – Acceleration in level flight at maximum thrust

We write Newton's second law in level flight, along the Ox axis: $F_u - F_n = m\dfrac{dV}{dt}$, where F_u is the available thrust (here $F_u = F_{u\,max}$)

Hence: $\boxed{\dfrac{dV}{dt} = \dfrac{1}{m}\left(F_{u\max} - F_n\right)}$

Numerical application:

$F_{u\max} = \sigma F_{u0} = 0.8617\times 150 = \mathbf{129.3kN}$

$\dfrac{dV}{dt} = \dfrac{1}{53.8\times 10^3}(129.3-30.3)\times 10^3 = \mathbf{1.84ms^{-2}}$

3 – Turn with a bank angle of 30°

3.1 – Normal load factor

In a turn, normal load factor $n_z = \dfrac{1}{\cos\mu}$

Rate of turn Ω: $tg\mu = \dfrac{V^2}{Rg} = \dfrac{\Omega V}{g} = \dfrac{\Omega^2 R}{g}$

Hence: $\Omega = \dfrac{g\times tg\mu}{V}$ and $R = \dfrac{V^2}{g\times tg\mu}$

Numerical application:

$n_z = \dfrac{1}{\cos 30°} = \mathbf{1.15}$

$\Omega = \dfrac{9.81\times tg30°}{138.5} = \mathbf{4.09\times 10^{-2}\ rd/s = 2.34°/s}$

$R = \dfrac{138.5^2}{9.81\times tg30°} = \mathbf{3387m}$

3.2 – Thrust required for level flight

We can determine the C_z required for a banked level flight at given EAS and mass:

$n_z mg = 1/2\rho_0 EAS^2 SC_{z_{turn}}$ Hence: $C_{z_{turn}} = \dfrac{2n_z mg}{\rho_0 EAS^2 S}$

Numerical application: $C_{z_{turn}} = \dfrac{2\times 1.15\times 4400\times 3600^2}{1.225\times 250^2\times 1852^2} = \mathbf{0.50}$

We can determine the thrust F_n required to a banked level flight:

$$F_{n\,turn} = \frac{n_{z_{turn}} mg}{f_{turn}} = n_{z_{turn}} mg \frac{C_{x_0} + kC_{z_{turn}}^2}{C_{z_{turn}}}$$

Numerical application: $F_{n_{turn}} = 1.15 \times 53800 \times 9.81 \frac{(1.753 + 3.881 \times 0.50^2) \times 10^{-2}}{0.50} = \mathbf{33.06\,kN}$

4 – Avoidance maneuver with a g-load n_z = 1.2

We will make the hypothesis that this load factor is reached at the bottom point of the pull-up.

So: $n_z = 1 + \frac{\omega V}{g}$ Hence $\omega = \frac{(n_z - 1)g}{V}$

Numerical application: $\omega = \frac{0.2 \times 9.81}{138.5} = \mathbf{0.0142\,rd/s} = \mathbf{0.812}\ °/s$

Other hypothesis : speed is kept constant thanks to the autothrottle which manages thrust.

$\frac{Vz}{V} = \sin\gamma$, where γ is the climb gradient

We can calculate the time when we reach a rate of climb of 1500ft/mn.

$\sin\gamma = \frac{Vz}{V} = \frac{1500}{3.28 \times 60 \times 138.5} = 0.055$. This corresponds to γ = 3.15°

ω is the rate of climb variation $\omega = \frac{d\gamma}{dt}$. So the time you need to reach a rate of climb of 1500 ft/mn is:

$$dt = \frac{3.15}{0.812} = 3.88s \approx 4s$$

5 – Turn under maximum load factor

5.1 – Symmetrical turn with a load factor of 2.5

We can determine the maximum load factor that we can have while keeping EAS = 250 kt:

$$n_{\max} mg = \frac{1}{2} \rho_0 S\ EAS^2 C_{Z\max}$$

We saw in §2.1 that, in these conditions, the Mach number is M=0.414. So we use $C_{Z\max}$=1.1

Numerical application:

$$n_{\max} = \frac{0.5 \times 1.225 \times 250^2 \times 1852^2 \times 1.1}{3600^2 \times 4400} = 2.53$$

The symmetrical turn with a load factor of 2.5 is therefore possible.

We know that in a stabilized turn: $n_z = \frac{1}{\cos\mu}$ and $\tan\mu = \frac{V\omega}{g} = \frac{V^2}{Rg}$

Numerical application:

μ = **66.4°**

and $\omega = \frac{9.81 \times \tan(66.4°)}{138.5} = \mathbf{0.1623rd/s} = \mathbf{9.3}\ °/s$

$$R = \frac{(138.5)^2}{9.81 \times tg(66.4°)} = \mathbf{854m}$$

5.2 – Maintaining altitude and speed with a load factor of 2.5

We saw that we could fly in level flight with a load factor of 2.5 without stalling.
We now need to see if we can maintain the same speed. For this purpose, we calculate the thrust required to the banked flight with a load factor of 2.5 and we compare it to the maximum thrust available.
Available thrust at FL 50: $F_u = \sigma F_{u0}$

Numerical application:

$F_u = 0.8617 \times 150 = 129.255$ kN

Required thrust while turning :

$$F_{n\,turn} = \frac{n_z mg}{f_{turn}} \text{ with } f_{turn} = \frac{C_{z_{turn}}}{C_{x0} + kC_{z_{turn}}^{\ 2}} \text{ and with } C_{z_{turn}} = \frac{2n_z mg}{\rho_0 S * EAS^2}$$

Numerical application:

$$C_{z_{trn}} = \frac{2\times 2.5\times 4400}{1.225\times(128.61)^2} = 1.086$$

$$C_{x_{turn}} = 1.753\times 10^{-2} + 3.881\times 10^{-2}\times 1.086^2 = 0.0633$$

$$f_{turn} = \frac{1.086}{0.0633} = 17.16$$

Hence $F_{n\,turn} = \frac{2.5\times 53.8\times 10^3\times 9.81}{17.16} = \mathbf{76.9kN}$

We check that we have $F_{nturn} < F_u$.
Therefore, at this altitude, it is possible to maintain constant speed and altitude, while performing a turn with a load factor of 2.5.

5.3 – Level flight with a load factor of 2.5 and an equivalent air speed of 210 Kts

We can calculate the required C_z with an EAS of 210 Kts and a load factor of 2.5:

$$C_{z_{turn}} = \frac{2n * mg}{\rho_0 S * EAS^2}$$

Numerical application:

$$C_{z_{turn}} = \frac{2\times 2.5\times 4400\times 3600^2}{1.225\times 210^2\times 1852^2} = 1.54$$

We now calculate the corresponding Mach number with an EAS of 210 Kts at FL50:

$$TAS = EAS\sqrt{\frac{\rho_0}{\rho}} = EAS\sqrt{\frac{1}{\sigma}} = 210\times\sqrt{\frac{1}{0.8617}} = 226.23Kts$$

Hence $M = \frac{TAS}{a} = \frac{226.23}{650.01} = 0.348$

We remark that at this Mach number, the available $C_{z\ max}$ is necessarily less than 1.45 ($C_{z\ max}$ available for M=0.26), because we know that $C_{z\ max}$ decreases when Mach number increases.

Therefore we won't be able to perform a turn while maintaining altitude at EAS=210Kts with a load factor of 2.5, because we would need C_z=1.54.

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Flight mechanics

Correction of tutorial n°4
High altitude cruise of a transonic twinjet

We saw in tutorial n°2 that the higher a twin-engine jet flies, the more distance it can travel. The choice of the angle of attack also permits to optimize what we want: a maximum lift-drag ratio minimizes hourly consumption (holding); maximum C_z/C_x^2 maximizes specific range (Maxi-range).
In fact, this reasoning is valid if all parameters are free to vary. In reality, we often fly at a given Mach number: optimization is then different.
Moreover, we generally can't perform climb cruises, because we are given a flight level (actually, we perform step climb cruises).
This is what we will be dealing with in this tutorial.

1 – Aerodynamic optimum

1.1 – Specific range SR

We already saw (tutorial n°2) that : $SR = \dfrac{V_k}{C_S F_u}$

Without wind, $V_k = V$ and in stabilized cruise, $F_u = F_n = \dfrac{mg}{f}$

Therefore : $$\boxed{SR = \frac{Vf}{C_S mg} = \frac{MaC_z}{C_S mgC_x}}$$

1.2 – Optimum specific range

Consider that C_S is constant in the flight envelope, SR is then maximum when Maf is maximum.

$a = \sqrt{\gamma r T}$ is a function of temperature; above 11Km, because temperature in standard atmosphere is constant, a is constant.

The cruise flight level for a turbo-jet unit being more than 11Km, maximum SR corresponds to the maximum Mf product.

If the Mach number is forced, *i.e.* if the pilot flies at a constant Mach number, SR will be maximum at the maximum lift-drag ratio for this specific Mach number.

In addition, we must satisfy the lift equation: $mg = \gamma/2\, PSC_z M^2$

M and $C_z = C_z(f_{\max})$ being fixed, this imposes a pressure, and consequently an altitude.

To summarize, for each Mach number, SR is maximum at the altitude where the angle of attack for stabilized level flight is similar to the one for maximum lift-drag ratio for this Mach number.

Note: in the case where all parameters are free to vary (Mach number not imposed), we saw that SR is maximum for the angle of attack of maximum $\dfrac{C_z}{C_x^2}$.

1.3 – Aerodynamic optimum (Mach,C_z)

The best couple (Mach, C_z) is the one that gives the best $Mf_{\max}$ product.

$Mach$	0.45	0.50	0.60	0.70	0.76	0.78	0.79	0.80	0.82
$f_{\max}$	19.17	18.98	18.90	18.13	17.54	17.12	16.59	16.07	13.81
$M \cdot f_{\max}$	8.63	9.49	11.34	12.69	13.33	13.35	13.11	12.86	11.32

The optimum couple is therefore : $\boxed{M = 0.78 \text{ and } C_z = 0.567}$

Remark: Note that it is different to fly at the best lift-drag ratio (Mach number not imposed, optimum: M=0.78 and $f_{\max} = 17.12$) and to fly at a given Mach number, at the maximum lift-drag ratio angle of attack for this Mach number (if for instance M=0.7, $f_{\max} = 18.13$ and the angle of attack is different from the one in the first case).

2 – Cruise flight at Mach 0.78

2.1 – Optimum flight level as a function of mass.

We know that $mg = 0.7PSC_zM^2$.

Mach number being given, and C_z being equal to the optimum value 0.567, each pressure altitude corresponds to an optimum mass :

$$m_{opt} = \frac{0.7PSC_zM^2}{g} = \frac{0.7\times 122\times (0.78)^2\times 0.567}{9.81}P$$

FL	350	370	390	410
P (Pa)	23841.9	21662.4	19677.0	17873.5
m (tons)	71.6	65.1	59.1	53.7

See curve in annex.

Note : reciprocally, each mass corresponds to an optimum flight level; in order to remain in optimum conditions, you need to climb as you burn fuel, hence the notion of climb cruise.

2.2 – Climb cruise

Hypotheses: $M = 0.78$ $a = cst$ $C_S = cst$

We saw (§1.1) that $SR = \dfrac{aMf}{C_S mg}$

But $SR = \dfrac{dD}{dQ} = \dfrac{dD}{-dm}$ where dD is the distance traveled during dt with fuel dQ

If we consider that the climb cruise is performed with an angle of attack such that $f = f_{\max} = cst$, we have:

$$\boxed{D = -\int_{m_0}^{m_1} \frac{aMf}{C_S g}\cdot\frac{dm}{m} = \frac{aMf}{C_S g}\ln\frac{m_0}{m_1}}$$

With $m_0 = 65t$ and $m_1 = 55t$, the initial altitude is close to 37 000 ft and the final altitude close to 41 000 ft. We are still above 11 Km, so we can consider that speed of sound is constant. In standard atmosphere, temperature is –56.5°C, hence:

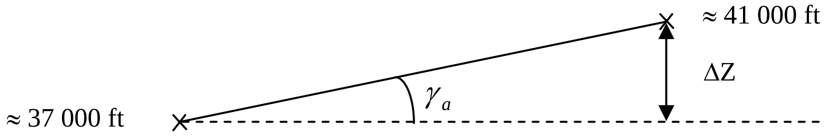
$$a = 20.05\sqrt{T} = 20.05\sqrt{273-56.5} = 295.01 m/s = 573.45 Kts$$

In the above formula, C_S is expressed in Kg/N/h; if we express a in Kts, we obtain D in Nautical miles (Nm).

Numerical application: $D = \dfrac{573.45 \times 0.78 \times 17.12}{0.065 \times 9.81} \ln \dfrac{65}{55} = 2006 Nm$

We disregarded climb gradient γ_a, so that we could say that $F_u = F_n = \dfrac{mg}{f}$ to calculate SR and D.

Actually :



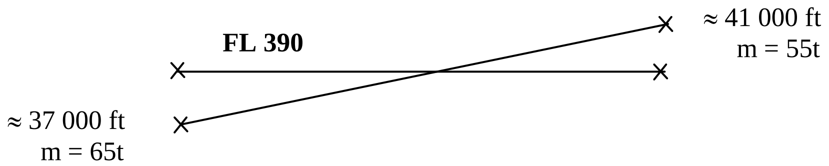
$$\Delta Z \approx 4000 ft = \frac{4000}{3.28} = 1220 m$$

$$\gamma_a = \frac{\Delta Z_{(m)}}{D_{(m)}} = \frac{1220}{2006 \times 1852} = 3 \cdot 10^{-4} rd = 0.02°$$

It is therefore legitimate to disregard γ_a.

2.3 – Level cruise flight – Odd flight levels

This flight is made in RVSM airspace (Reduced Vertical Separation Minimum). This means that in the upper controlled airspace (from FL 290 to FL 410 – Europe, oceanic airspace), vertical separation between 2 aircraft flying on reciprocal headings is 1000 ft.
If we take a look at the curve showing optimum level as a function of mass, we notice that if m = 65t, the optimum level is about FL 370, and if m = 55t, about FL 410.
The only usable odd level between those two levels is FL 390.



In the previous paragraph, we made the hypothesis that lift-drag ratio was constant and maximum ($f_{\max}$).
In the case where FL = FL390 = cst, f is optimum only when m = 59.1t (see §2.1). For the rest of the flight, f varies and is not at its optimum value.
Each value of mass corresponds to a value of C_z :

$$C_z = \frac{mg}{0.7 PSM^2} = \frac{9.81}{0.7 \times 122 \times 19677 \times (0.78)^2} m = 9.6 \times 10^{-3} m(t)$$

Each value of C_z corresponds to a value of lift-drag ratio (see curve $f = f(C_z)$ given in the text for M=0.78).
f being known, we can calculate SR :

$$SR = \frac{aMf}{C_S gm} = \frac{573.45 \times 0.78}{0.065 \times 9.81} \frac{f}{m(t)} = 701.47 \frac{f}{m(t)}$$

Hence the following results:

$m_{(t)}$	65	63	61	59	57	55
C_z	0.624	0605	0.586	0.566	0.547	0.528
f (curve)	17.05	17.09	17.11	17.12	17.10	17.04
$SR_{(Nm/t)}$	184.00	190.29	196.76	203.54	210.44	217.33

To calculate distance traveled D', we consider that SR is constant on each 2t interval, equal to the mean of the 2 bounds of the interval. Hence:

$$D' = 2 \times 187.15 + 2 \times 193.53 + 2 \times 200.15 + 2 \times 206.99 + 2 \times 213.88$$

$$\boxed{D' = 2003.4 Nm}$$

The difference with optimum climb cruise is 3Nm, which can be disregarded.

3 – Pressurization failure

At FL 390, with m = 61 t, we have $C_z = 9.6 \times 10^{-3} m(t) = 0.586$ and $f = 17.11$

Therefore : $SR = 701.47 \dfrac{f}{m(t)} = 196.8 Nm/t$

At FL 100, with m = 61 t, we found (end of tutorial n°2) $SR = 138.7 Nm/t$.

The decrease of specific range due to a lower flight level is 58.1 Nm/t, which is very important.

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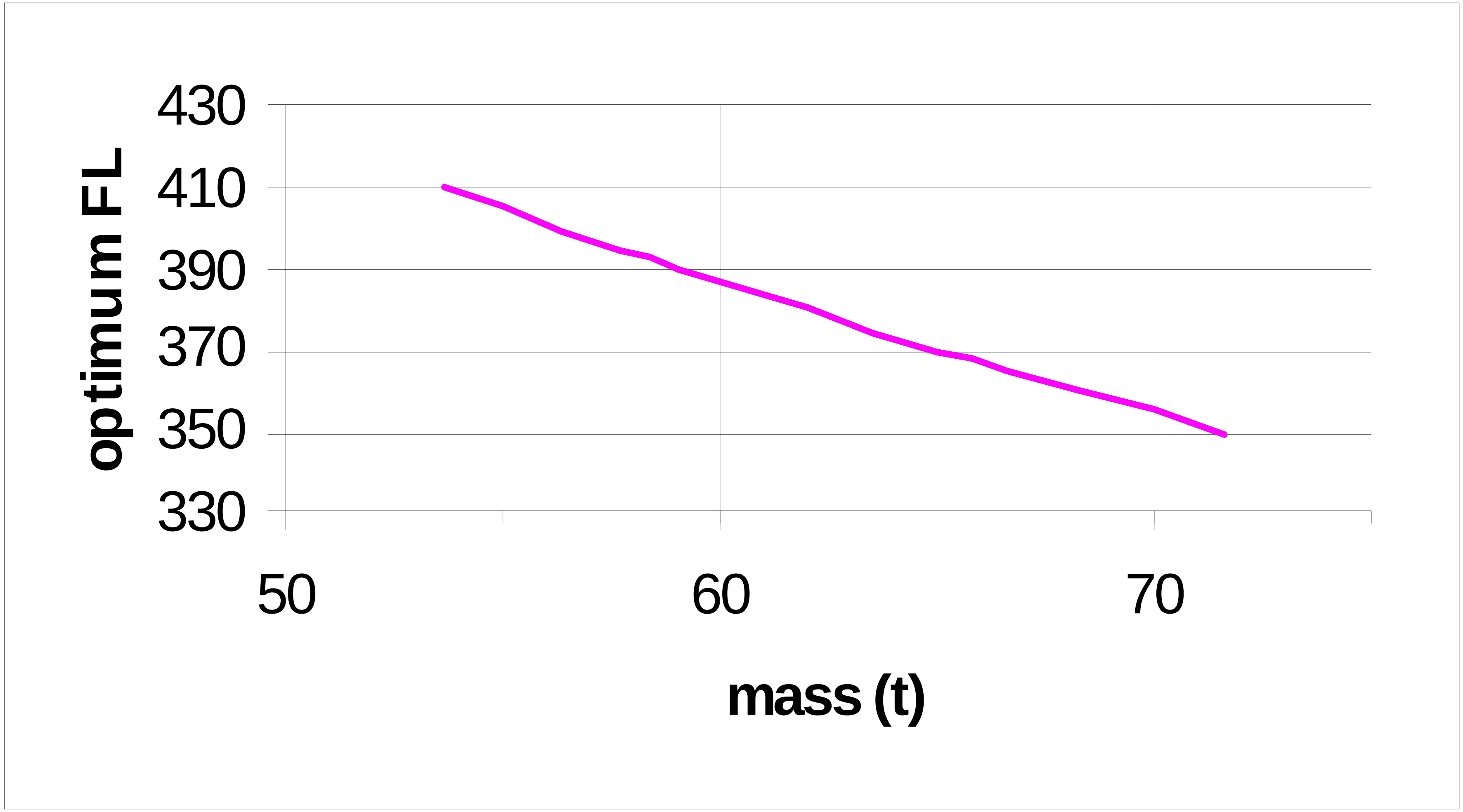
Annex



Flight mechanics

Correction of tutorial n°5
High altitude performances of a transonic twinjet

This tutorial deals with the determination of the flight envelope of the twinjet. In particular, we will draw the buffet onset chart of the aircraft.

1 – Aerodynamic ceiling

For an aircraft in horizontal stabilized flight: $n_z mg = 0.7 PSC_z M^2$.

The aerodynamic ceiling is the maximum altitude (*i.e.* minimum pressure) where you can have enough lift to satisfy this equation.

For each Mach number value, there is a ceiling, reached for $C_z = C_{z_{max}}$.

If we maximize ($C_{z_{max}} M^2$), we obtain the absolute aerodynamic ceiling for a particular Mach number.

Thanks to the table that you find in the subject, we notice that ($C_{z_{max}} M^2$) is maximum when **M= 0.78**. It corresponds to the absolute aerodynamic ceiling.

Just before stalling: $n_z mg = 0.7 PSC_{z\,max} M^2$.

By definition, the ceiling corresponds to $n_z = 1$. When you reach the ceiling, you cannot perform any avoiding maneuver (if n_z increases, you have to descend, otherwise you stall).

We now consider that $n_z = 1.3$. Operational ceiling is a lower altitude than the previous one, which allows a margin in order to perform maneuvers ($n_z = 1.3$ corresponds to a turn with a banking angle of 40°).

$$P_{1g} = \frac{mg}{0.7S(C_{z_{max}} M^2)} = \frac{550 \times 9.81}{0.7 \times 0.5135} = 15010 Pa$$

By interpolating the standard atmosphere table, we can say that the aerodynamic ceiling is at a pressure altitude of **44 600 ft**.

1.3g buffet limited altitude is reached for a pressure $P_{1.3g} = 1.3 P_{1g} = 19513 Pa$, which corresponds to a pressure altitude of **39 200 ft**.

<u>Note 1:</u> instead of using standard atmosphere tables, convenient but little precise, we could use a rigorous formula, obtained by integrating the equation: $dP = -\rho g dh = -\rho g_0 dZ$:

$\frac{P_2}{P_1} = e^{-\frac{g_0}{RT}(Z_2 - Z_1)}$ So $P = 22632.056 e^{-0.157688 \times 10^{-3}(H - 11000)}$ with H in meters (this formula is valid between 11 and 20 km)

For instance, for the aerodynamic ceiling, we find 44 622 ft.

Note 2: thanks to this formula, we can demonstrate that the difference between aerodynamic ceiling and 1.3g buffet limited altitude is constant and about 5 000 ft. We just have to write that: $\frac{P_2}{P_1} = e^{-\frac{g_0}{RT}(Z_2 - Z_1)}$, Hence $Z_2 - Z_1 = \frac{RT}{g_0} Log \frac{P_2}{P_1}$

In this case, $\frac{P_{1,3g}}{P_{1g}} = 1.3$ so we find: $\Delta H_{(m)} = \frac{10^3}{0.157688} Log(1.3) = 1663.8m = 5459 ft$

2 – Flight envelope at FL 390

2.1 – In stabilized flight level, $n_z = 1$ and we can write that $mg = 0.7 PSC_z M^2$

Hence: $C_z M^2 = \frac{9.81 \times m}{0.7 \times 122 \times P}$

In stabilized level flight, the values of mass and altitude determine the value of the $C_z M^2$ product.

Numerical application: m = 67.1t Z_P = FL390

$$C_z M^2 = \frac{9.81 \times 67.1 \times 10^3}{0.7 \times 122 \times 19677} = \mathbf{0.3917}$$

This value is necessary to maintain stabilized level flight. But, according to the table $C_{z\max} M^2 = f(M)$, for M=0.78, wings can give a $C_{z\max} M^2$ of 0.5135. The difference can therefore be used to perform an instantaneous maneuver with a load factor equal to:

$$n_{z_{\max}} \text{ instantaneous} = \frac{C_{z_{\max}} M^2}{C_z M^2 (stabilized\ level\ flight)} = \frac{0.5135}{0.3917} = \mathbf{1.31}$$

2.2 – Buffet onset chart

Thanks to the table which is in the subject, we draw the curve $C_{z\max} M^2 = f(M)$ (see annex 1).

In the flight conditions of §2.1, with $C_z M^2 = 0.3917$, we read on the curve the limits of the flight envelope in level flight:

low speed stall Mach number : 0.658
high speed stall Mach number : 0.831

3 – Maximum cruise thrust limited altitude at Mach 0.78

The difference between required thrust F_n and available thrust F_u determines climbing capacities. We will study the evolution of these capacities when pressure altitude Z_P varies, *i.e.* the evolution of the 2 curves $F_n = f(V)$ and $F_u = f(V)$.

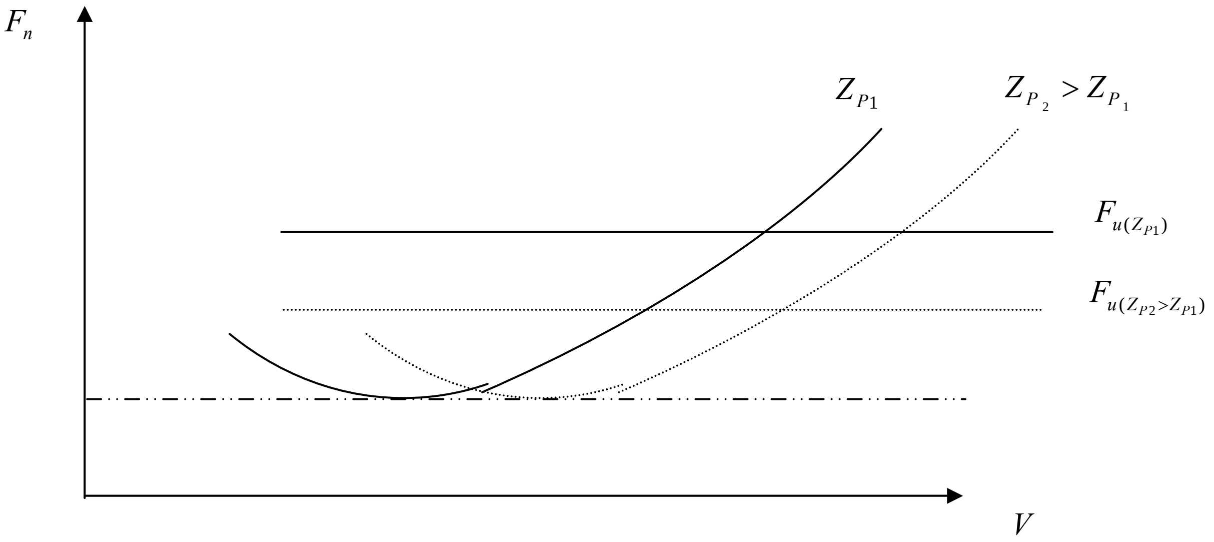
- $F_n = \frac{mg}{f}$

  Is the angle of attack is kept constant :
  - f does not depend on altitude, so neither does F_n
  - V depends on Z_P : $V_{Z_P} = V_{Z_P=0} \sqrt{\frac{\rho_0}{\rho}}$, *i.e.* V increases with altitude

When Z_P increases, the different curves $F_n = f(V)$ are obtained from the curve at $Z_P = 0$ by an affine transformation along the speed axis.

- Curves $F_u = f(V)$ are horizontal, F_u decreases with altitude.

There comes a time when $F_u = F_n$ for one single value of V; at this altitude, you can still maintain level flight, but it is impossible to climb : it is the maximum cruise thrust limited altitude.



3.1 – Determination of rate of climb V_Z at constant Mach number

- $C_Z = \dfrac{mg}{0.7PSM^2} = \dfrac{550 \times 9.81}{0.7 \times (0.78)^2} \times \dfrac{1}{P}$
- We read the lift-drag ratio on the curve $f = f(C_Z)$ given for M=0.78
- $F_n = \dfrac{mg}{f} = 67100 \times 9.81 \times \dfrac{1}{f}$
- For one engine : $F_u = 22000 \times \dfrac{\rho}{0.3164}$
- Climb gradient at constant true air speed : $\gamma_a = \dfrac{F_u - F_n}{mg}$

  Note : above 11 Km, a climb at constant Mach number is similar to a climb at constant true air speed.

- Vertical speed : $V_Z \approx V \cdot \gamma_a = a \cdot M \cdot \gamma_a$, with $a \cdot M = 295.07 \times 0.78 = 230.15 m/s$

  Hence the following table :

FL	$P_{(Pa)}$	C_Z	f	$F_{n(N)}$	$F_{u(N)}$	$\gamma_{a(rd)}$	$V_{Z(m/s)}$
390	19677	0.644	17.00	38721	44000	$8.02\ 10^{-3}$	1.846
410	17874	0.709	16.74	39322	39967	$0.98\ 10^{-3}$	0.226

3.2 – Maximum cruise thrust limited altitude

By extrapolating the standard atmosphere tables, we deduce the maximum cruise thrust limited altitude at M=0.78 ($V_Z = 0$) : **41 300 ft**

The service ceiling is the one that allows a rate of climb $V_Z = 1.5 m/s$. By interpolation, we can say that it is approximately **39 400 ft**.

4 – Wind gradient at FL 390

Let us consider a wind gradient Wxx (horizontal wind $W_X = W_{X0} + W_{XX}.x$).

When there is horizontal wind in cruise flight, ground speed V_k, true air speed V and wind W_X are collinear. We have: $V_k = V + W_X = aM + W_X$

Therefore : $\frac{dV_k}{dt} = a\frac{dM}{dt} + M\frac{da}{dt} + \frac{dW_X}{dt}$

In this case temperature is constant (above FL360), therefore the speed of sound is constant. Moreover, we have made the assumption we are cruising at constant Mach number.

Therefore : $\frac{dV_k}{dt} = \frac{dW_X}{dt}$

Since $\frac{dW_X}{dt} = W_{XX}\frac{dx}{dt} = W_{XX} \cdot V_k$ then $\frac{dV_k}{dt} = W_{XX}(aM + W_X)$

In stabilized level flight, the thrust equation is : $m\frac{dV_k}{dt} = F - F_n$

where F is thrust selected by the pilot and F_n is the required thrust.

FL 390 is the service ceiling, *i.e.* the altitude that allows a vertical speed $V_z = 1{,}5 m/s$.

We know that : $V_Z = V\gamma = V\frac{F - F_n}{mg} = aM\frac{F - F_n}{mg}$

We deduce : $\frac{dV_k}{dt} = \frac{gV_Z}{aM}$ or $W_{XX}(aM + W_X) = \frac{gV_Z}{aM}$

If we consider that there is initially no wind : $W_X = 0$ at $t = 0$

Hence : $\boxed{W_{XX} = \frac{gV_Z}{(aM)^2}}$

Numerical application:

$$W_{XX} = \frac{9.81 \times 1.5}{(295.07 \times 0.78)^2} = 2.78 \times 10^{-4} s^{-1} = 3600 \times 2.78 \times 10^{-4} Kt/Nm$$

Finally: $\mathbf{W_{XX} = 1\ Kt/Nm}$

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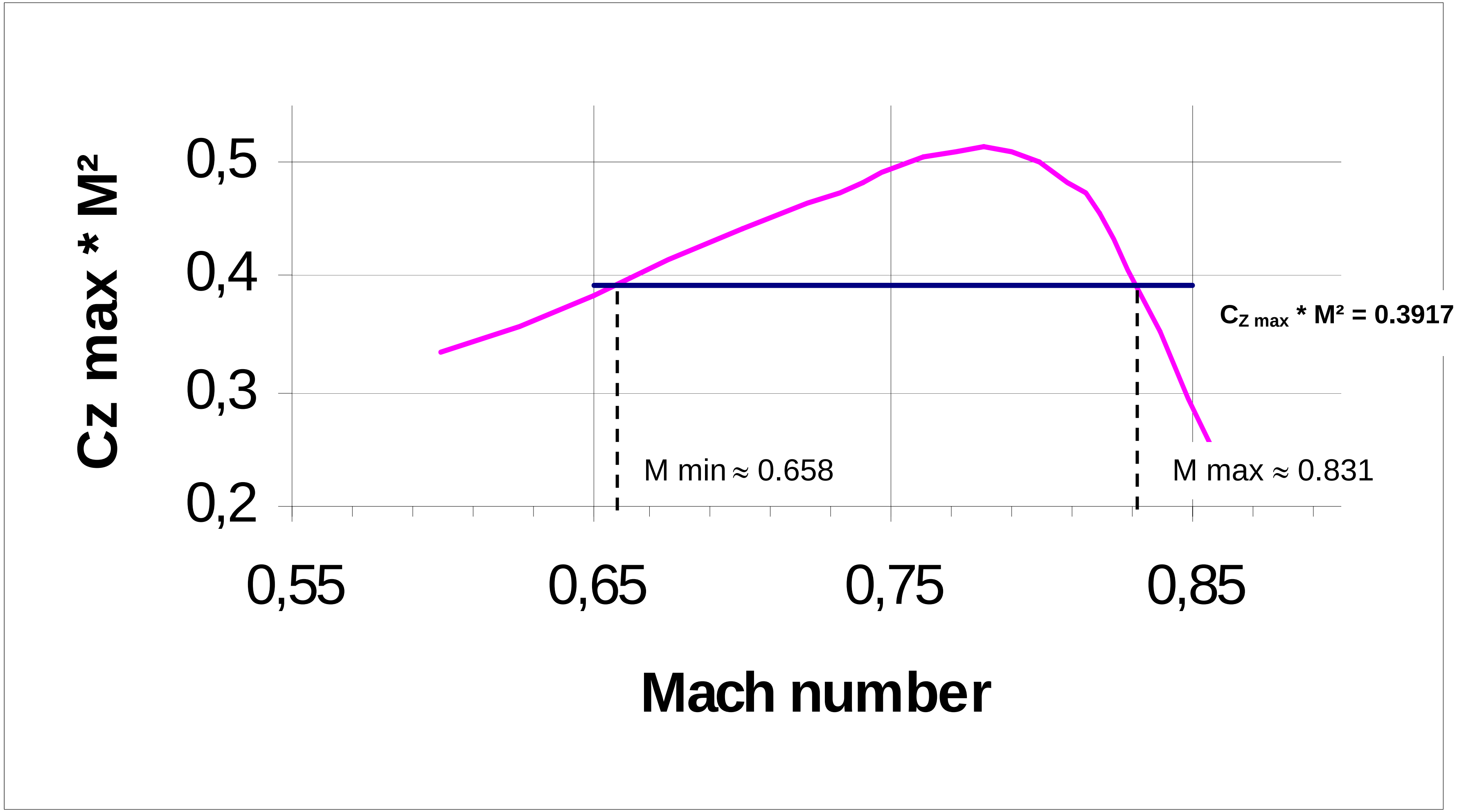
ANNEX 1 : BUFFET ONSET CHART



Flight mechanics

Correction of tutorial n°6
Power diagram of a twin-turboprop

1 – Power diagram

1.1 – Lift equation under its general form : $nmg = 1/2\,\rho S V^2 C_z$

At sea level ($Z_P = 0$): $\rho_0 = 1.225\,kg/m^3$

We want the stall speed (V_{stall}) for n=1 (« 1g »).

This speed is obtained when $C_z = C_{z\max}$

So $V_{S1g} = \sqrt{\dfrac{2mg}{\rho_0 S C_{z\max}}} = \sqrt{\dfrac{2 \times 5600 \times 9.81}{1.225 \times 28 \times 1.55}} = \mathbf{45.46\,m/s}$

$$V_{S_{Kts}} = \frac{3600}{1852} \times V_{S_{m/s}} \approx \mathbf{88Kts}$$

1.2 – Motive power limitation at sea level (relative density $\sigma = \dfrac{\rho}{\rho_0} = 1$) can be due to:

- either a thermodynamic limitation : $W_{m_1} = 910 \times \sigma = 910Kw$
- or mechanical limitations : $W_{m_2} = N_{\max} \times \Gamma_{\max} = 209.2 \times 3.024 = 632.6Kw$

$$W_{m_{\max}} = \min(W_{m_1}, W_{m_2}) = \mathbf{632.6Kw}$$

In this case, mechanics is the limiting factor.

1.3 – Available power diagram.

If η_H is the propeller efficiency, the available thrust for this twin-turboprop is:

$W_u(V) = 2 \times \eta_H \times W_{m_{\max}} = 2 \times 632.6 \times \eta_H$

The advance ratio γ is the ratio of the aircraft speed to the propeller rotation speed by π or so:

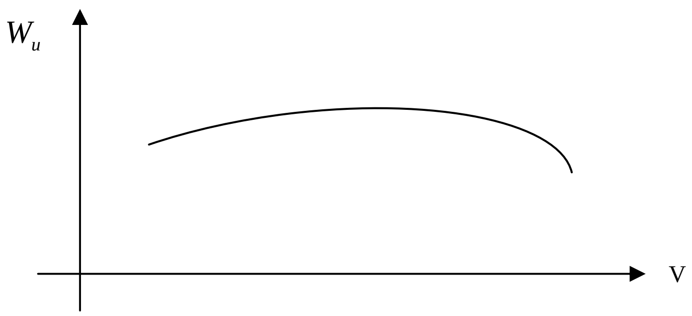
$\gamma = \dfrac{V}{ND}$

Hence $V_{m/s} = \gamma \times N_{tr/s} \times D_m = \gamma \times 33.3 \times 2.5$

From the table $\eta_H = f(\gamma)$ given in the text, we can calculate the corresponding values of V and W_u .

γ	0.5	0.625	0.75	1.00	1.25	1.5	1.7
η_H	0.65	0.74	0.79	0.84	0.86	0.84	0.78
$V_{(Kts)}$	80.9	101.1	121.4	161.8	202.3	242.7	275.1
$W_{u(Kw)}$	822.4	936.2	999.5	1062.8	1088.1	1062.8	986.9

Hence the shape of the curve $W_u = f(V)$:



See curve in annex 1.

1.4 – Necessary power for a level flight W_n .

$W_n = F_n \times V$

In stabilized level flight, we know that $F_n = \dfrac{mg}{f}$

According to the lift equation: $mg = 1/2\,\rho_0 S V^2 C_z$, therefore $C_z = \dfrac{2mg}{\rho_0 S V^2}$

For different values of V, we can calculate $C_z, C_x\,(C_x = C_{x_0} + kC_z^2 = 0.027 + 0.06C_z^2)$ and W_n .

$$C_z = \frac{2 \times 5600 \times 9.81}{1.225 \times 28 \times V^2} \quad \text{and} \quad W_n = 5600 \times 9.81 \times \frac{(2.7 + 6C_z^2) \times 10^{-2}}{C_z} \times V$$

- Power when stalling : we just need to apply the above formulae with $C_z = C_{z\max} = 1.55$ and $V = V_{S1g} = \mathbf{45.46m/s} = \mathbf{88.4Kts}$ (calculated in 1.1).

  We obtain $W_n = \mathbf{275.8Kw}$

- Minimum required power for a level flight :

  $$V = \sqrt{\frac{2mg}{\rho_0 S C_z}}$$

  This permits to express W_n as a function of C_z :

  $$W_n = \frac{mg}{f} V = mg\left(\frac{C_{x_0} + kC_z^2}{C_z}\right) \times \sqrt{\frac{2mg}{\rho_0 S C_z}}$$

  W_n is minimum if $\left(\dfrac{C_{x_0} + kC_z^2}{C_z^{3/2}}\right)$ is minimum, *i.e.* $C_z = \sqrt{\dfrac{3C_{x_0}}{k}} = \sqrt{\dfrac{3 \times 2.7}{6}} = 1.162$

  Finally: $V = \mathbf{52.5m/s} = \mathbf{102.1Kts}$ and $W_n = \mathbf{268.1Kw}$

- Tangent at the origin = power so that $\frac{W_n}{V}$ is minimum

  $\frac{W_n}{V} = \frac{mg}{f}$.

  Therefore, $\frac{W_n}{V}$ $_{\text{mini}}$ corresponds to $f_{\max}$.

  with $f = \frac{C_z}{C_x} = \frac{C_z}{C_{x_0} + kC_z^2}$, $C_{z_{f\max}} = \sqrt{\frac{C_{x_0}}{k}} = 0.671$

  So : $V = \mathbf{69.1m/s} = \mathbf{134.3Kts}$ and $W_n = \mathbf{305.6Kw}$

- Power at maximum operating speed VMO :
  VMO is given and equals 270Kts (138.9m/s).

We have the following table, for some speed values:

	V_S	W_n min			$\frac{W_n}{V}$ min					VMO
V_{Kts}	88.4	102.1	115	130	134.3	170	200	230	260	270
$V_{m/s}$	45.46	52.50	59.16	66.88	69.09	87.46	102.89	118.32	133.76	138.90
$C_z = \frac{2mg}{\rho SV^2}$	1.55	1.16	0.91	0.72	0.67	0.42	0.30	0.23	0.18	0.17
$W_{n_{Kw}} = mg\frac{C_x}{C_z}V$	275.8	268.1	274.3	296.4	305.6	430.4	610.5	855.7	1187.3	1317.1

1.5 – Required power diagram $W_n(V)$

See curve in annex 1.

Thanks to the curves, we can determine the maximum speed in level flight $V_{\max} = 249Kts$, intersection of $W_u(V)$ and $W_n(V)$.

2 – Excess power

2.1 – We measure the difference $(W_u - W_n)$ on the diagram and we copy it out on the ordinate axis.

2.2 –

- Maximum speed in level flight (see above): $V_{\max} = \mathbf{249Kts}$

- Speed corresponding to maximum ΔW :

  We read $V_{\Delta W\max} = \mathbf{127Kts}$

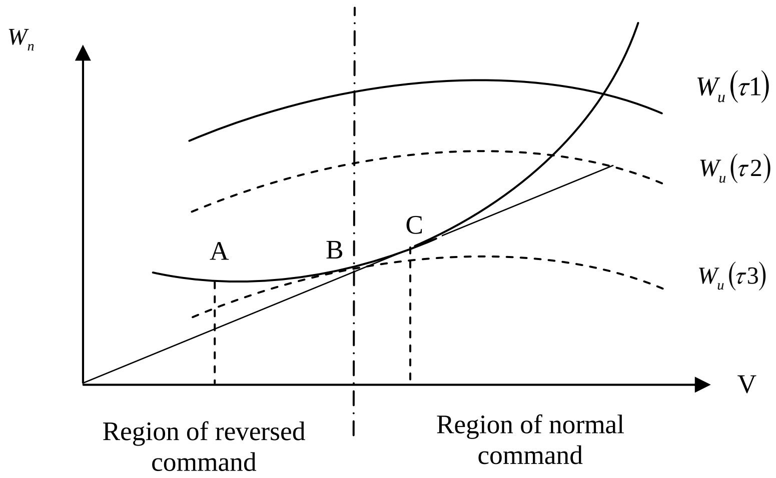
  During a climb at constant true air speed, rate of climb $Vz = \frac{\Delta W}{mg}$, so $\Delta W_{\max}$ corresponds to Vz_{max} when climbing.

- Speed corresponding to maximum $\Delta W/V$:

  We obtain maximum $\Delta W/V$ point by drawing the curve coming from the origin that is tangent to ΔW with the maximum slope. It is almost stalling speed V_S = **88Kts**.

  During a climb at constant true air speed, climb gradient $\gamma = \frac{\Delta F}{mg} = \frac{\Delta W}{V} \times \frac{1}{mg}$, therefore $\Delta W/V_{\max}$ corresponds to $\gamma_{\max}$.

Note 1: when position of the throttle lever τ varies, the curve $W_u(V)$ moves along a vertical translation direction. The point separating region of normal and reversed command is given by the tangency point between $W_n(V)$ and $W_u(\tau, V)$.



A : $W_{n\ \text{mini}}$ C : $\frac{W_n}{V}_{\text{mini}}$ or maximum lift-drag ratio

We notice that the maximum of ΔW (point B), boundary between normal and reversed command, is closer to the maximum lift-drag ratio (point C) than to $W_{n\ \text{mini}}$ (point A).

Note 2: the curve $W_n(V)$ is very flat (W_n is almost constant between stalling speed and 115 Kts) ; $\Delta W/V_{\max}$ ($\gamma_{\max}$ when climbing) therefore corresponds to V_S .

In fact, $C_x = C_{x_0} + kC_z^2$ is not a good approximation when we are close to stalling speed, with large angles of attack, we should add a ΔC_x term due to boundary layer separation.

3 – Single engine ceiling

3.1 – We saw in §1.3 that the available engine power is $W_u = 2 \times \eta_H \times W_{m_{\max}}$.

In case of engine failure, with a constant efficiency of 0.85, $W_{u\max} = 0.85 \times W_{m_{\max}}$.

Maximum motive power (see §1.2) is the minimum of:

- thermodynamic limitation : $W_{m_1} = 910 \times \sigma \quad (\sigma \neq 1)$
- mechanical limitation : $W_{m_2} = N_{\max} \times \Gamma_{\max} = 209.2 \times 3.024 = 632.6 Kw$

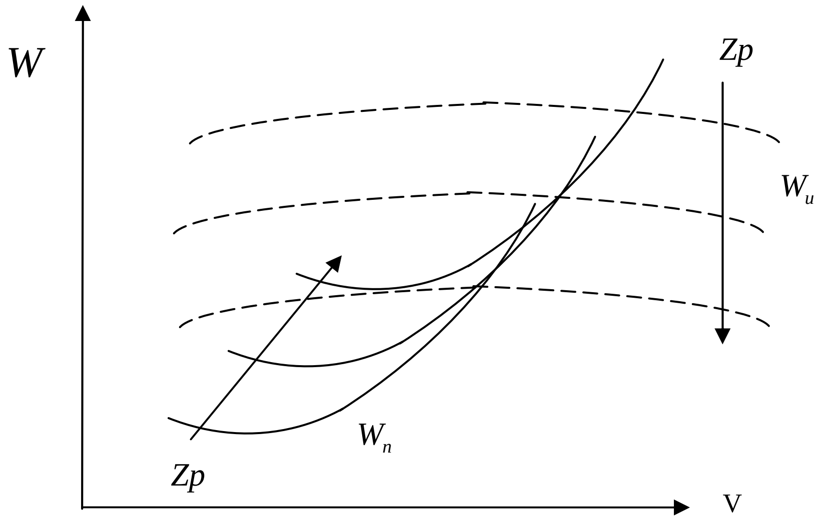
We can calculate, for several relative density values, the maximum available power W_u with one engine (see following table). We remark that mechanics is the limiting factor when relative density is high (low altitude); when it decreases, the limit is thermodynamic.
We also remark that the power delivered by the engine decreases with altitude.

Required power for a level flight at pressure altitude Zp is :

$$W_n(\sigma) = mg\frac{C_x}{C_z}V_{Zp} = mg\frac{C_x}{C_z}\times\frac{1}{\sqrt{\sigma}}V_{Zp=0} = \frac{1}{\sqrt{\sigma}}W_n(\sigma=1)$$

When altitude increases, σ decreases and the required power increases.

Here are the evolutions of $W_u(V)$ and $W_n(V)$ as a function of altitude :



When reaching a certain altitude, the available power is not enough to maintain level flight; this limit is the maximum cruise thrust limited altitude.

In the particular case where the efficiency is considered constant, W_u does not depend on V (the curve $W_u(V)$ is horizontal).

The maximum altitude is reached when $W_u(Z_{P\max}) = W_n(Z_{P\max})$, when these two curves are tangent, *i.e.* when the speed corresponds to $W_{n\,\text{mini}}$.

We saw in §1.4 that $W_{n\,\text{mini}}$ at sea level was obtained for V=102.1Kts and was equal to 268kW.

Therefore : $\boxed{W_{n_{\text{mini}}}(\sigma) = \frac{268}{\sqrt{\sigma}}}$ minimum required power at altitude Zp (relative density σ)

For several relative density values :

σ	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
$W_{u\max}$ (kW) single engine	538.1	538.1	538.1	538.1	464.1	386.8	309.4	232.1	154.7
$W_{n\,\text{mini}}$ (kW)	268.0	282.5	299.6	320.3	346.0	379.0	423.7	489.3	599.3

We draw the curves $W_u(\sigma)$ and $W_{n\,\text{mini}}(\sigma)$ (see annex 2).

Gross single engine ceiling (or maximum thrust limited altitude) is obtained when $W_n = W_{a\,\text{mini}}$.

We read on the curve $\sigma = 0.49$, which approximately corresponds to a pressure altitude **Zp=22 500 ft**.

The maximum thrust limited altitude is reached for a speed of 102Kts at Zp=0. This is the definition of equivalent air speed: **EAS = 102 Kts**.

True air speed at Zp=22 500ft is $TAS = \frac{EAS}{\sqrt{\sigma}} = \frac{102}{\sqrt{0.49}}$ so **TAS = 145.7 Kts**

3.2 – Same question in ISA+10 conditions

The intersection of the curves $W_u(\sigma)$ and $W_{n\ \mathrm{mini}}(\sigma)$ happens when $\sigma = 0.49$.

This relative density value corresponds to Zp = 22 500 ft in standard atmosphere. If the atmosphere was hotter, we would have this relative density at a lower altitude.

Ideal gas law: $P = \rho RT$

For the same relative density : $\frac{P}{T} = Cst$ therefore $\frac{P_{ISA+10}}{T_{ISA+10}} = \frac{P_{ISA}}{T_{ISA}}$

Yet we have $P_{ISA} = 41900Pa$ with $\sigma = 0.49$, and $T_{ISA} = 245°K$

Hence : $P_{ISA+10} = \frac{P_{ISA}}{T_{ISA}}(T_{ISA} + 10) = \frac{41900}{245} \times 255 = 43600Pa$, which corresponds to a pressure altitude **Zp = 21 500 ft**

Relative density has not changed, therefore the corresponding EAS and TAS remain the same.

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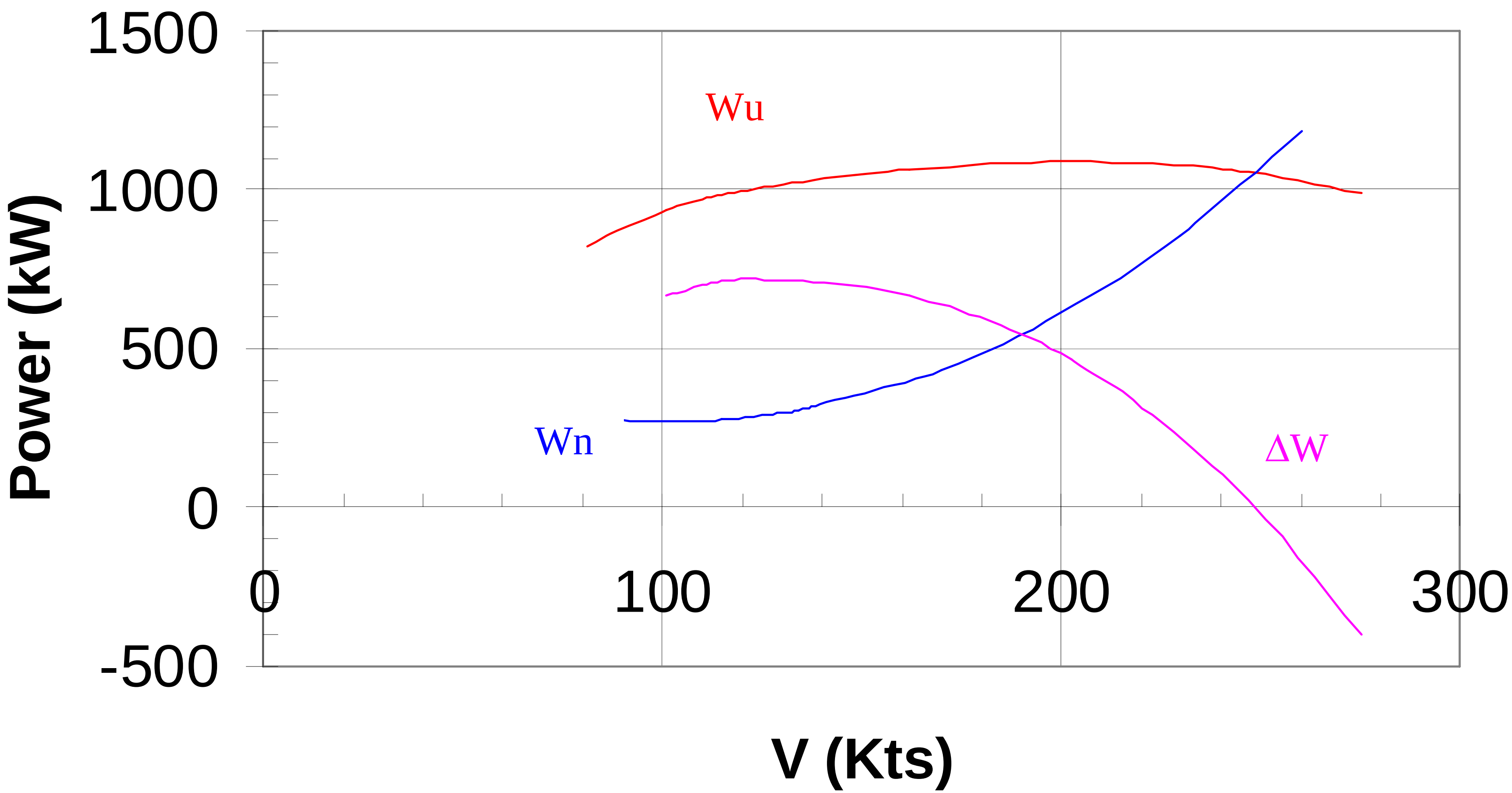
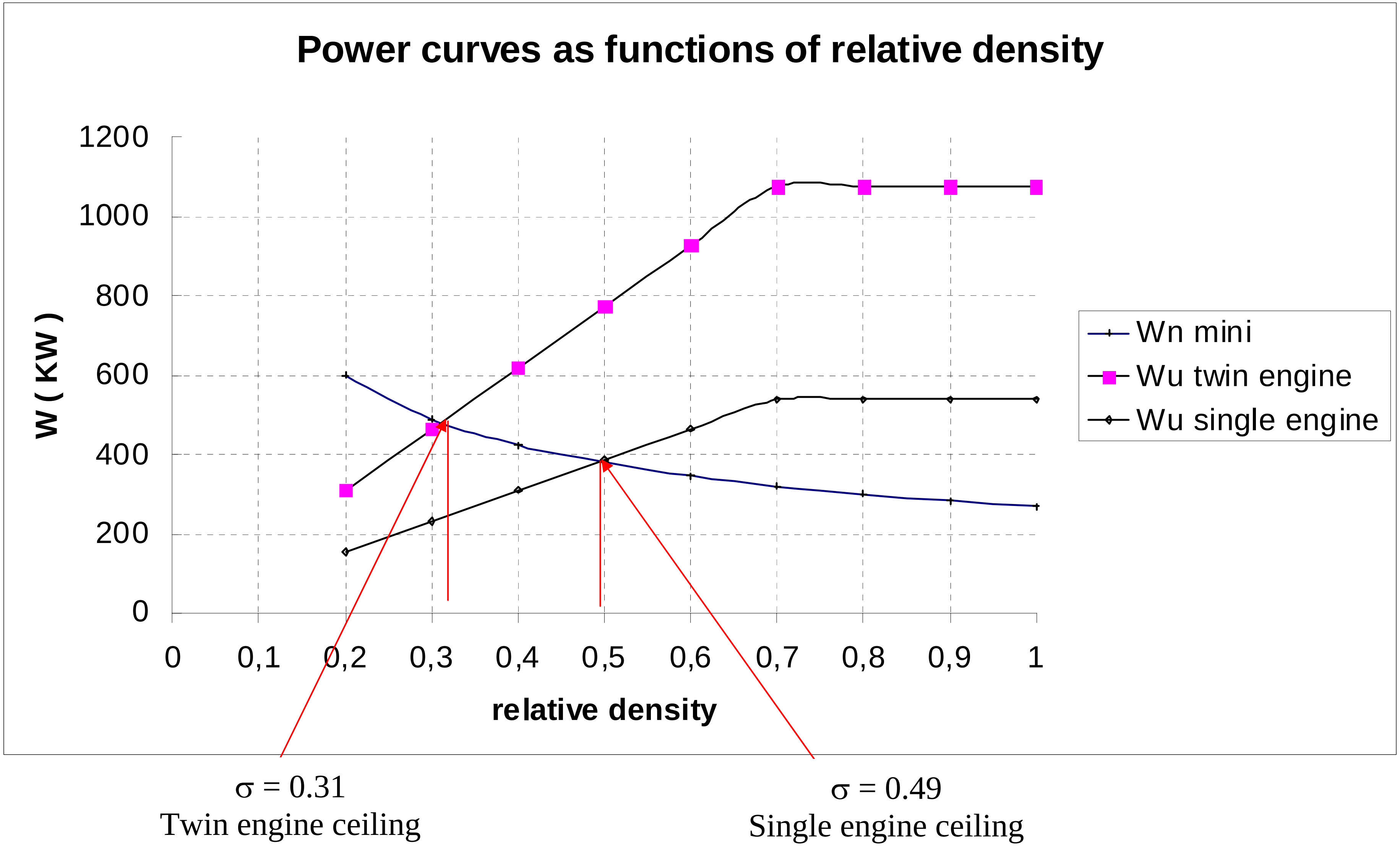
Annex 1



Annex 2


Power curves as functions of relative density
W (KW)
1200
1000
800
600
400
200
0
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1
relative density
Wn mini
Wu twin engine
Wu single engine
σ = 0.31
Twin engine ceiling
σ = 0.49
Single engine ceiling