Discrete Mechanical Vibrations SM32

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TWO DEGREES OF FREEDOM SYSTEMS

Undamped systems

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Non-conservative systems

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- Vibration absorber
- Suspended system
- Based supported system

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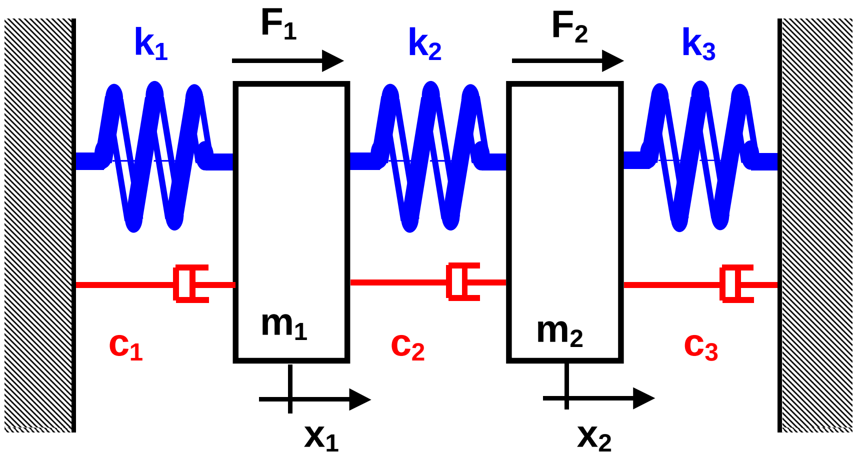
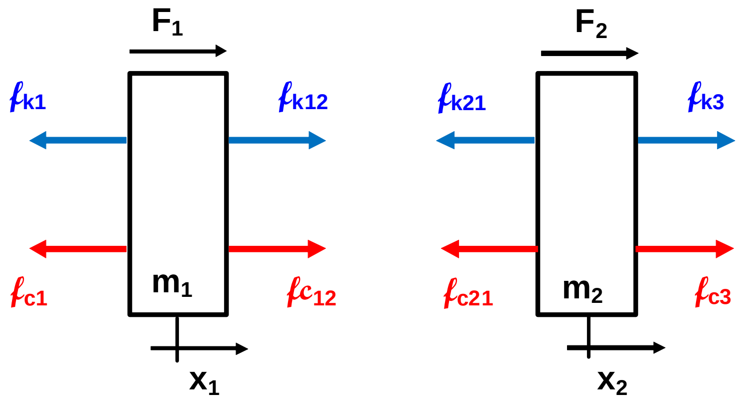
Two degree-of-freedom systems, even though included in ***N degree-of freedom*** systems, are treated separately. This is because their small size allows analytical solution, understanding of more general methods, and an introduction to the concept of coupling. In addition, they provide an explanation of useful applications such as the dynamic vibration absorber.

Certain properties of a vibrating system used here will not be proven until the next chapter and, as a preparation for the next chapter, the modal method is used even though direct calculations are simpler.



Isolated system:



For mass m_1 :

$$m_1\ddot{x}_1 = -\ell_{k1} + \ell_{k12} - \ell_{c1} + \ell_{c12} + F_1$$

For mass m_2 :

$$m_2\ddot{x}_2 = -\ell_{k21} + \ell_{k3} - \ell_{c21} + \ell_{c3} + F_2$$

with:

$$\ell_{k1} = k_1 x_1 \quad \ell_{k3} = -k_3 x_2 \quad \ell_{k12} = k_2\left(x_2 - x_1\right) \quad \ell_{k21} = k_2\left(x_2 - x_1\right)$$

$$\ell_{c1} = c_1 \dot{x}_1 \quad \ell_{c3} = -c_3 \dot{x}_2 \quad \ell_{c12} = c_2\left(\dot{x}_2 - \dot{x}_1\right) \quad \ell_{c21} = c_2\left(\dot{x}_2 - \dot{x}_1\right)$$

The two equations are:

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2\left(x_2 - x_1\right) - c_1\dot{x}_1 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + F_1$$

$$m_2\ddot{x}_1 = -k_2\left(x_2 - x_1\right) - k_3 x_2 - c_2\left(\dot{x}_2 - \dot{x}_1\right) - c_3\dot{x}_2 + F_2$$

With a matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

Or in the more compact form:

$$\boxed{M\ddot{x} + C\dot{x} + Kx = F}$$

Mass Matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Stiffness Matrix

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Damping Matrix

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

Displacement Vector

$$x = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

Forces Vector

$$F = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

As possible [, { or }] will be erased in the future relationships

Recall:

The Hamilton's Principle which states:

"*Of all the possible paths along which a dynamical system may more from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies*"

Without external forces:

$$\int_{t_1}^{t_2} (T - U)\, dt$$

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \qquad \text{and} \qquad \delta x_i(t_1) = \delta x_i(t_2) = 0$$

with:

T total Kinetic Energy of the system.

U total Potential Energy of the system.

This means that:

$$U = U(x_1, x_2) \quad \Longrightarrow \quad \delta U = \frac{\partial U}{\partial x_1}\delta x_1 + \frac{\partial U}{\partial x_2}\delta x_2$$

and

$$T = T(x_1, x_2, \dot{x}_1, \dot{x}_2) \quad \Longrightarrow \quad \begin{aligned} \delta T &= \frac{\partial T}{\partial \dot{x}_1}\delta \dot{x}_1 + \frac{\partial T}{\partial \dot{x}_2}\delta \dot{x}_2 + \frac{\partial T}{\partial x_1}\delta x_1 + \frac{\partial T}{\partial x_2}\delta x_2 \\ &= \frac{\partial T}{\partial \dot{x}_1}\frac{d}{dt}(\delta x_1) + \frac{\partial T}{\partial \dot{x}_2}\frac{d}{dt}(\delta x_2) + \frac{\partial T}{\partial x_1}\delta x_1 + \frac{\partial T}{\partial x_2}\delta x_2 \end{aligned}$$

Where terms such as $\delta x'_i$ will integrated by parts.

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{x}_1}\delta \dot{x}_1 \, dt = \left[\frac{\partial T}{\partial \dot{x}_1}\delta x_1\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right)\delta x_1 \, dt$$

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{x}_2}\delta \dot{x}_2 \, dt = \left[\frac{\partial T}{\partial \dot{x}_2}\delta x_2\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right)\delta x_2 \, dt$$

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Grouping terms of same kind, it follows:

$$\int_{t_1}^{t_2} \delta(T-U)\,dt = = \left[\frac{\partial T}{\partial \dot{x}_1}\delta x_1\right]_{t_1}^{t_2} + \left[\frac{\partial T}{\partial \dot{x}_2}\delta x_2\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\frac{\partial T}{\partial x_1}\delta x_1\,dt + \int_{t_1}^{t_2}\frac{\partial T}{\partial x_2}\delta x_2\,dt$$

$$-\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right)\delta x_1\,dt - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right)\delta x_2\,dt - \int_{t_1}^{t_2}\frac{\partial U}{\partial x_1}\delta x_1\,dt - \int_{t_1}^{t_2}\frac{\partial U}{\partial x_2}\delta x_2\,dt$$

$$\int_{t_1}^{t_2} \delta(T-U)\,dt = \left[\frac{\partial T}{\partial \dot{x}_1}\delta x_1\right]_{t_1}^{t_2} + \left[\frac{\partial T}{\partial \dot{x}_2}\delta x_2\right]_{t_1}^{t_2}$$

$$\left(-\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right)dt + \int_{t_1}^{t_2}\frac{\partial T}{\partial x_1}dt - \int_{t_1}^{t_2}\frac{\partial U}{\partial x_1}dt\right)\delta x_1$$

$$\left(-\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right)dt + \int_{t_1}^{t_2}\frac{\partial T}{\partial x_2}dt - \int_{t_1}^{t_2}\frac{\partial U}{\partial x_2}dt\right)\delta x_2$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0\\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0\end{aligned}}$$

For the previous 2 DOF system:

Kinetic Energy

$$T=\frac{1}{2}m_1\left(\frac{\partial x_1(t)}{\partial t}\right)^2+\frac{1}{2}m_2\left(\frac{\partial x_2(t)}{\partial t}\right)^2$$

Potential Energy or stress Energy

$$U=\frac{1}{2}k_1\left(x_1(t)\right)^2+\frac{1}{2}k_2\left(x_2(t)-x_1(t)\right)^2+\frac{1}{2}k_3\left(x_2(t)\right)^2$$

'Work done by dissipation forces' Rayleigh Function

$$W_d=\frac{1}{2}c_1\left(\frac{\partial x_1(t)}{\partial t}\right)^2+\frac{1}{2}c_2\left(\left(\frac{\partial x_2(t)}{\partial t}\right)-\left(\frac{\partial x_1(t)}{\partial t}\right)\right)^2+\frac{1}{2}c_3\left(\frac{\partial x_2(t)}{\partial t}\right)^2$$

see TD.

Remarks :

- for discret systems this leads to LAGRANGE's equations .
- it can also be used in static.

Kinetic Energy

$$\frac{\partial T}{\partial \dot{x}_1} = m_1\left(\frac{\partial x_1(t)}{\partial t}\right) \qquad \frac{\partial T}{\partial \dot{x}_2} = m_2\left(\frac{\partial x_2(t)}{\partial t}\right)$$

$$\Rightarrow$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) &= m_1\left(\frac{\partial^2 x_1(t)}{\partial t^2}\right)\\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) &= m_2\left(\frac{\partial^2 x_2(t)}{\partial t^2}\right)\end{aligned}}$$

Potential Energy (or stress Energy)

$$\frac{\partial U}{\partial x_1} \qquad \frac{\partial U}{\partial x_2}$$

$$\boxed{\begin{aligned}\frac{\partial U}{\partial x_1} &= k_1\left(x_1(t)\right) + k_2\left(x_1(t)\right) - k_2\left(x_2(t)\right)\\ \frac{\partial U}{\partial x_2} &= +k_2\left(x_2(t)\right) + k_3\left(x_2(t)\right) - k_2\left(x_1(t)\right)\end{aligned}}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0$$

$$\Rightarrow$$

$$\boxed{\begin{aligned}m_1\left(\frac{\partial x_1(t)}{\partial t}\right) + k_1\left(x_1(t)\right) + k_2\left(x_1(t)\right) - k_2\left(x_2(t)\right) &= 0\\ m_2\left(\frac{\partial x_2(t)}{\partial t}\right) + k_2\left(x_2(t)\right) + k_3\left(x_2(t)\right) - k_2\left(x_1(t)\right) &= 0\end{aligned}}$$

Note: $\partial T/\partial x_1 = \partial T/\partial x_2 = 0$

Free Vibrations

In order to clarify the presentation of the basic phenomena, the following simplifications are made:

$$m_1 = 3m \quad ; \quad m_2 = m \quad ; \quad k_1 = k_2 = k_3 = k$$

Free Vibrations of the mass-spring system:

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solutions are sought in the form

$$x_1 = X_1 e^{rt} \qquad \text{and} \qquad x_2 = X_2 e^{rt}$$

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} r^2 e^{rt} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{rt} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Free Vibrations

gives two homegeneous equations

$$\left(\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} r^2 + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{rt} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In matrix form:

$$\left[Mr^2 + K\right] X e^{rt} = 0$$

or with the more classical form

$$\begin{bmatrix} 3mr^2 + 2k & -k \\ -k & mr^2 + 2k \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{rt} = 0$$

! $\forall t$

Free Vibrations

Expansion of the determinant (without the trivial solution $X_1 = X_2 = 0$, which is out of interest)

Iff det = 0

$$\left(3mr^2 + 2k\right)\left(mr^2 + 2k\right) - k^2 = 0$$

If and **only if** det = 0

$$3m^2r^4 + 8mkr^2 + 3k^2 = 0$$

This second degree equation in r^2 admits these solutions

$$r_1^2 = -0.4514\frac{k}{m} \qquad \Rightarrow r_1 = \pm j\,0.6719\sqrt{\frac{k}{m}} = \pm j\,\omega_1$$

$$r_2^2 = -2.215\frac{k}{m} \qquad \Rightarrow r_2 = \pm j\,1.488\sqrt{\frac{k}{m}} = \pm j\,\omega_2$$

In a similar manner (as for the one DOF system), the frequencies of the TWO DOF system are defined as ω_1 and ω_2.

- **Mode shapes associated with ω_1**

Determinant must be equal to zero.

$$\begin{bmatrix} 3mr_1^2 + 2k & -k \\ -k & mr_1^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{r_1 t} = 0$$

for

$$r_1^2 = -0.4514 \frac{k}{m}$$

For the first line

$$\left[3m(-0.4514 \frac{k}{m}) + 2k \right] X_1 - k X_2 = 0$$

$$0.6458 X_1 - 1 X_2 = 0$$

Hence for ω_1

$X_1=1$ then $X_2=0.6458$

or/and also

$X_1=2$ then $X_2=1.2916$

- **Mode shapes associted with ω_1**

The two components X_1 and X_2 can therefore only be determined to within a multiplicative constant. The process of choosing this constant is called normalization. Then

$$\boxed{\phi_1 = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.6458 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{12} \end{Bmatrix}}$$

is called the first **mode shape of vibration** (associated with frequency ω_1).

• Mode shapes associted with ω_2

Determinant must be equal to zero.

$$\begin{bmatrix} 3mr_2^2 + 2k & -k \\ -k & mr_2^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{r_2 t} = 0$$

for

$$r_2^2 = -2.215 \frac{k}{m}$$

$$-kX_1 + \left[m(-2.215)\frac{k}{m} + 2k \right] X_2 = 0$$

$$-1X_1 - 0.215 X_2 = 0$$

Hence for ω_2

$X_1 = 1$ then $X_2 =$ -4.646

or/and also

$X_1 = 2$ then $X_2 =$ -9.392

- **Mode shapes associted with ω_2**

This relation between the two components (in fact the eigenvector) is defined with a constant coefficient. Then the second **mode shape of vibration** (associated with frequency ω_2) is:

$$\boxed{\phi_2 = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -4.646 \end{Bmatrix} = \begin{Bmatrix} \phi_{21} \\ \phi_{22} \end{Bmatrix}}$$

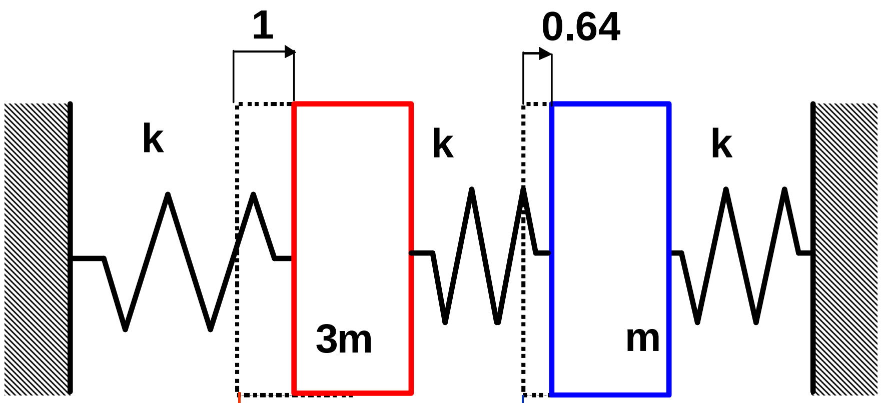
Free Motion

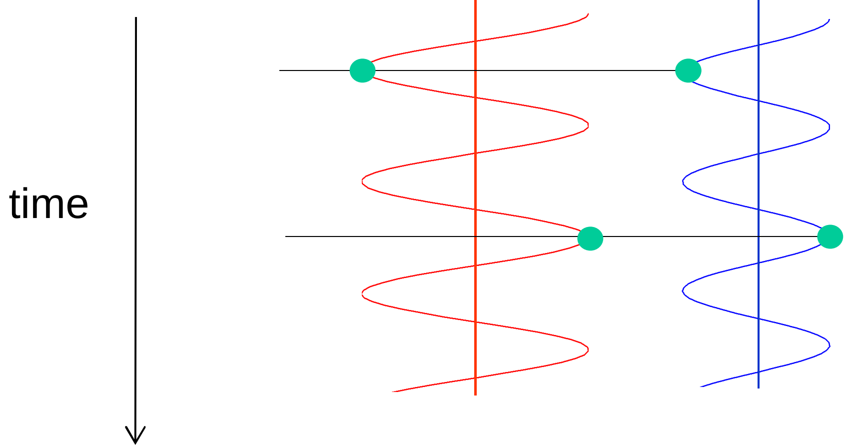
Mode shape associated with ω_1

First Mode of Vibration : **In phase**

$$\omega_1 = 0.6719\sqrt{\frac{k}{m}}$$

$$\begin{Bmatrix} \phi_{11} \\ \phi_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.6458 \end{Bmatrix}$$





The masses reache Maxi (and Min) at the same instants.

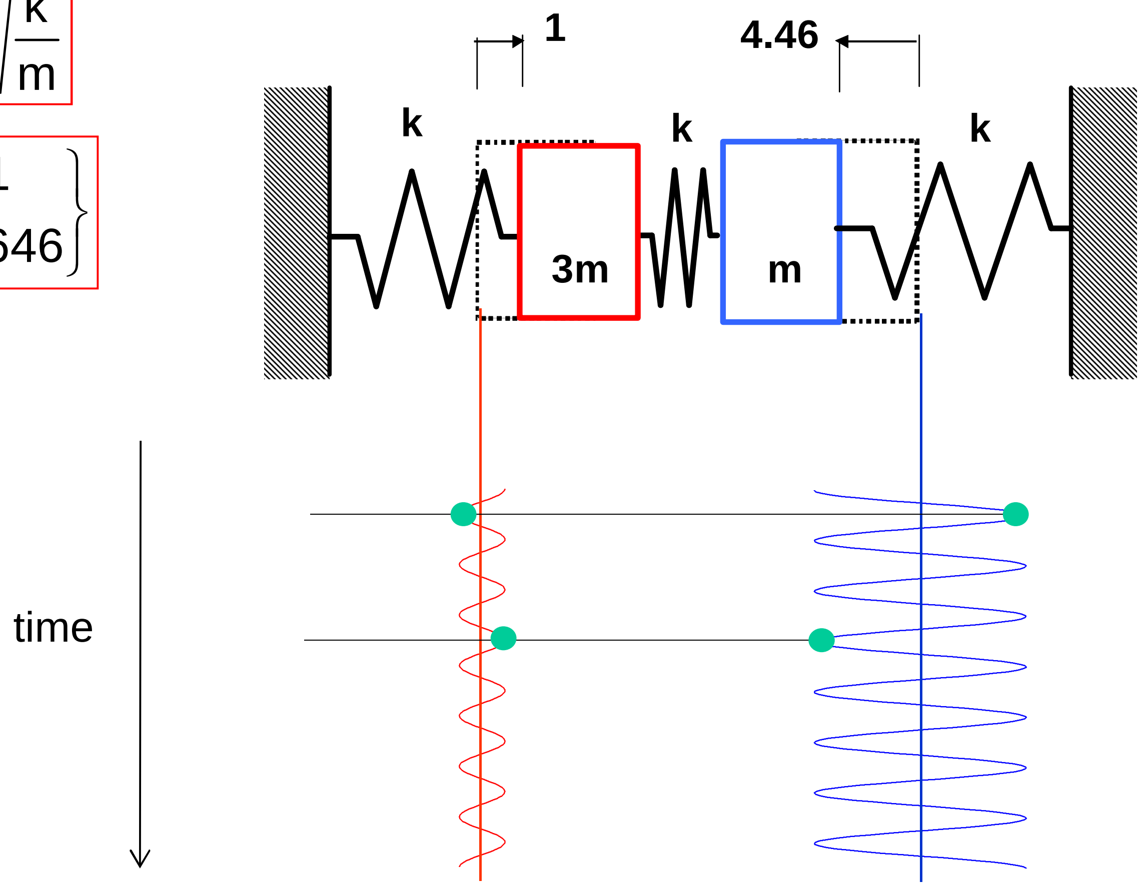
Free Motion

Mode shape associated with ω_2

Second Mode of Vibration : **In opposition of phase**

$$\omega_2 = 1.488\sqrt{\frac{k}{m}}$$

$$\begin{Bmatrix} \phi_{21} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -4.646 \end{Bmatrix}$$



The two masses are in an opposite motion.

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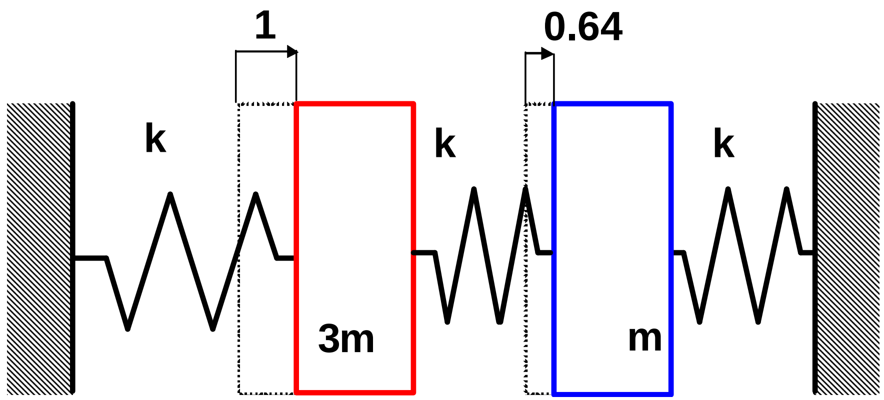
Free Motion

Mode shape associated with ω_1

First Mode of Vibration In phase

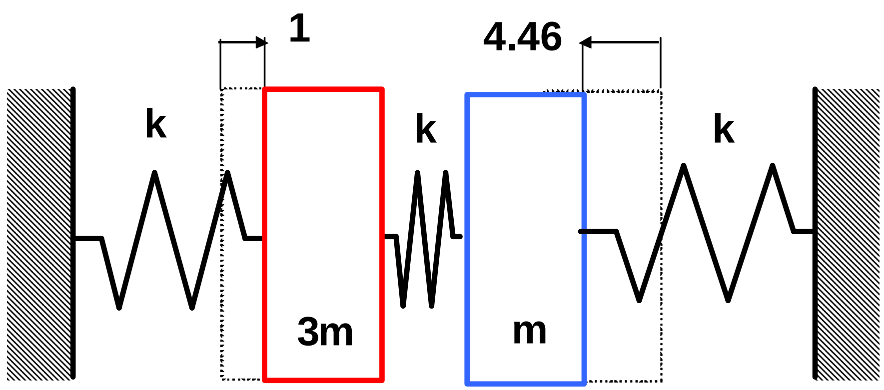
$$\begin{Bmatrix} \phi_{11} \\ \phi_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.6458 \end{Bmatrix}$$



Mode shape associated with ω_2

Second Mode of Vibration In opposition of phase

$$\begin{Bmatrix} \phi_{21} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -4.646 \end{Bmatrix}$$



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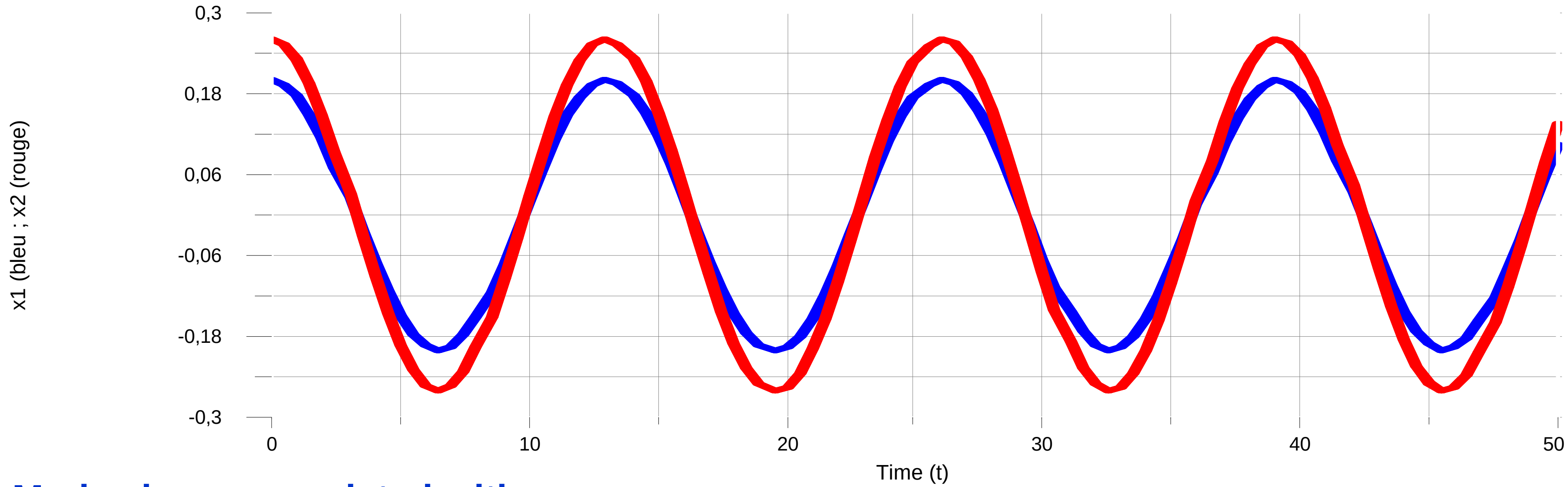
Free Motion

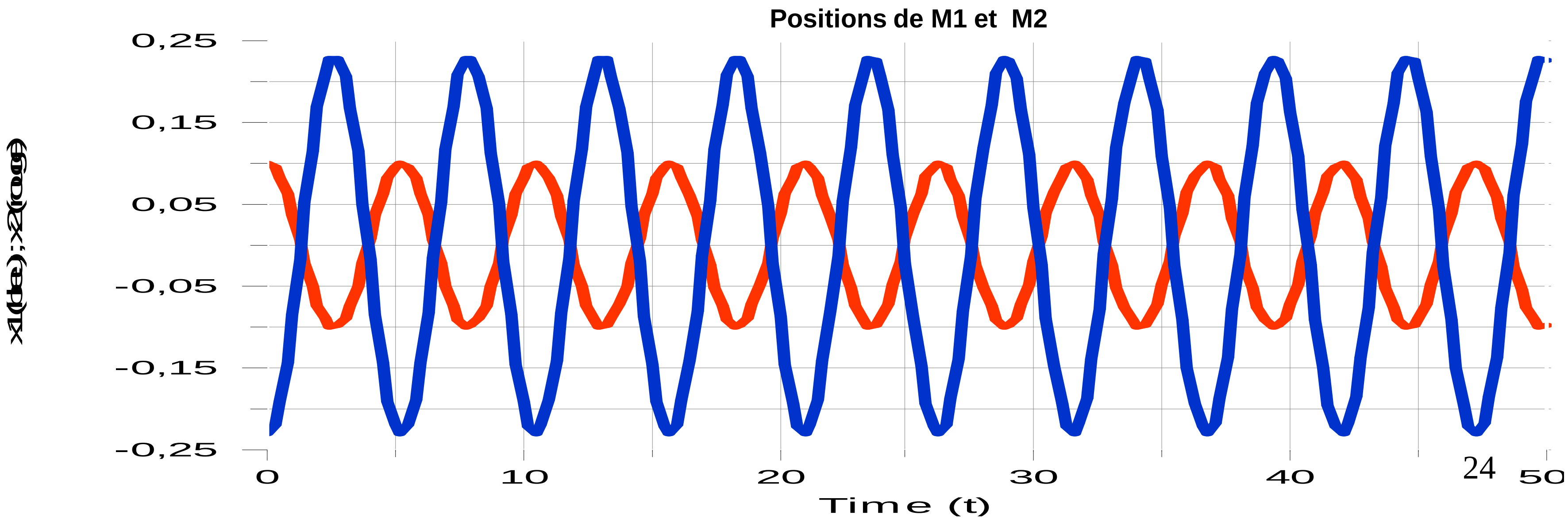
Mode shape associated with ω_1

Mode 1 _ In phase



Mode shape associated with ω_2

Mode 2 _ In opposition ofphase



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Mode Matrix:

Using a compact form:

$$\begin{array}{cc} \phi_{1i} & \phi_{2i} \\ \Downarrow & \Downarrow \end{array}$$

$$\phi = [\phi_1, \phi_2] = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.6458 & -4.646 \end{bmatrix}$$

Remark:

The relationship between the two components is defined with a multiplicative constant. There is no unique way to proceed for the normalization . For systems with only a few degrees of freedom several choices are possible, for example: the second relation was taken to 1 (it is the simplest form).

Other possible 'norms' are:

- Euclidian norm of the vector = 1 spécific $\sqrt{x_1^2 + x_2^2} = 1$

-' *norm*' associated with the modal mass usefull for computation in FE

$$[\phi]^t [M][\phi] = [I]$$

- ...

Approach using Modal basis

The basic idea is to express the two coupled equations with particular coordinates which make the new system un-coupled. The two new coordinates are called coordinates in modal basis. The system in expressed in **a normal form.**

Assuming that frequencies ω_i and mode shapes ϕ_i are defined, it can be stated

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \Longrightarrow x = \phi\, q$$

and also

$$\dot{x} = \phi\, \dot{q} \quad \text{and} \quad \ddot{x} = \phi\, \ddot{q}$$

$$M\ddot{x} + Kx = 0$$

$$\Longrightarrow \quad M\phi\, \ddot{q} + K\phi\, q = 0$$

Multiplication (on the left) with the transpose matrix ϕ leads:

$$\Longrightarrow \quad \boxed{\phi^t M\phi\, \ddot{q} + \phi^t K\phi\, q = 0}$$

Approach using Modal basis

It becomes for the previous example,

$$\boxed{\begin{aligned}&\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}^t \begin{bmatrix}3m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix}\\ &\qquad+\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}^t \begin{bmatrix}2k & -k\\ -k & 2k\end{bmatrix}\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}\end{aligned}}$$

recalling

$$\phi=[\phi_1,\phi_2]=\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0.6458 & -4.646\end{bmatrix}$$

and

$$\boxed{\varphi^t=\begin{bmatrix}\phi_{11} & \phi_{21}\\ \phi_{12} & \phi_{22}\end{bmatrix}^t=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}}$$

hence

$$\begin{bmatrix} 3.417m & 0 \\ 0 & 24.58m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 1.542k & 0 \\ 0 & 54.46k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\boxed{\mathscr{M}\, \ddot{q} + \mathscr{K}\, q = 0}$$

Mass modal Matrix :

$$[\mathscr{M}] = \phi^t M \phi = \begin{bmatrix} 3.417m & 0 \\ 0 & 24.58m \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Stiffness Modal Matrix

$$[\mathscr{K}] = \phi^t K \phi = \begin{bmatrix} 1.542k & 0 \\ 0 & 54.46k \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

Now, equations are expressed in a **normal form**. Modal matrices are diagonal and the two equations which are presented in a new basis q_1 and q_2 are un-coupled.

$$\boxed{\begin{aligned} m_1\ddot{q}_1 + k_1 q_1 &= 0 \\ m_2\ddot{q}_2 + k_2 q_2 &= 0 \end{aligned}}$$

It is possible to consider the problem as a problem of two equations of one DOF system in q_1 and q_2.

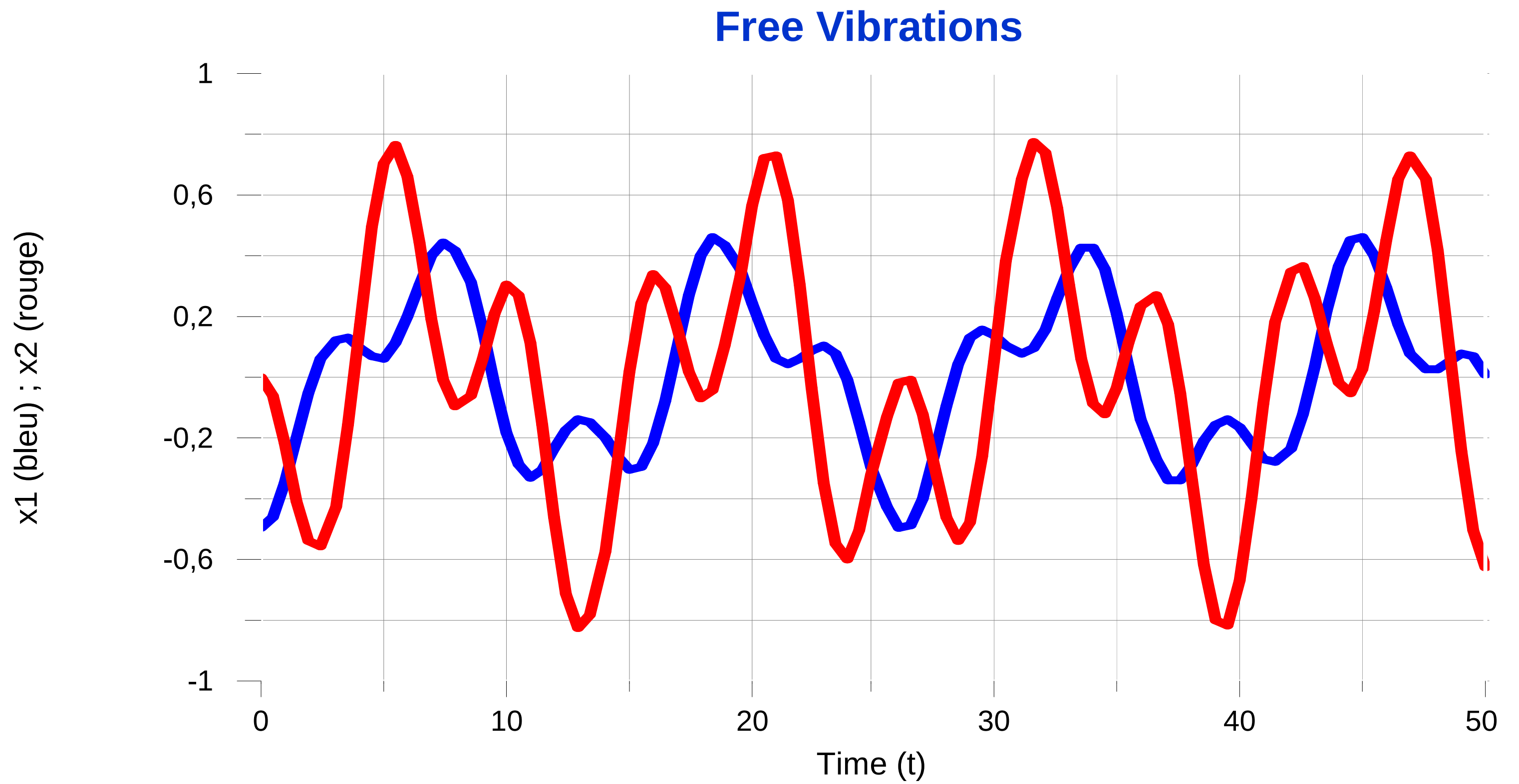
The independent solutions are sought in the form (see chapter for **ONE dof** system)

$$q_1 = A_1 \sin\omega_1 t + B_1 \cos\omega_1 t$$
$$q_2 = A_2 \sin\omega_2 t + B_2 \cos\omega_2 t$$

The 4 constantes A_1, A_2, B_1, B_2 are obtained with initial conditions (2 initial displacements, 2 initial velocities).

From a mathematical viewpoint, any other solutions can be obtained with a linear combination of two independant solutions, so that if q_1 and q_2 are defined, results are:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \Longrightarrow \begin{aligned} x_1 &= \phi_{11}\, q_1 + \phi_{21}\, q_2 \\ x_2 &= \phi_{12}\, q_1 + \phi_{22}\, q_2 \end{aligned}$$



Any displacement is a linear combinaison of two independent solutions.

So,

<u>Any displacement is a linear combinaison of the two mode shapes ϕ_1 and ϕ_2</u>

Forced Harmonic Vibration

A forcing term is added:

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \sin \Omega t \\ 0 \end{Bmatrix}$$

As for single undamped system, the total solution of the equations system with forcing term will be composed of a homogeneous solution (transient) and of a particular solution.

It is important to note that this system is an academic system because damping is supposed to be zero but Engineering Systems always possess damping (even if small).

Two ways are presented.

Direct method

Modal basis method

Harmonic excitation: direct method

Solutions are sought in the form:

$$x_1 = A_1 \sin\Omega t + B_1 \cos\Omega t$$
$$x_2 = A_2 \sin\Omega t + B_2 \cos\Omega t$$

Substituting into EOM and considering that it must hold for all time, each equation gives two relations corresponding to the vanishing of the coefficients of **sinΩt** *and* **cosΩt** .

The results are

$$-\Omega^2 [M] \begin{Bmatrix} A_1 \sin\Omega t + B_1 \cos\Omega t \\ A_2 \sin\Omega t + B_2 \cos\Omega t \end{Bmatrix} + [K] \begin{Bmatrix} A_1 \sin\Omega t + B_1 \cos\Omega t \\ A_2 \sin\Omega t + B_2 \cos\Omega t \end{Bmatrix} = \begin{Bmatrix} F \sin\Omega t \\ 0 \end{Bmatrix}$$

$$\boxed{\left[K - M\Omega^2\right] \begin{Bmatrix} A_1 \sin\Omega t + B_1 \cos\Omega t \\ A_2 \sin\Omega t + B_2 \cos\Omega t \end{Bmatrix} = \begin{Bmatrix} F \sin\Omega t \\ 0 \end{Bmatrix}}$$

Harmonic excitation: direct method

$$\begin{bmatrix} 2k-3m\Omega^2 & -k \\ -k & 2k-m\Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \sin\Omega t + B_1 \cos\Omega t \\ A_2 \sin\Omega t + B_2 \cos\Omega t \end{Bmatrix} = \begin{Bmatrix} F\sin\Omega t \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2k-3m\Omega^2 & -k \\ -k & 2k-m\Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \sin\Omega t \\ A_2 \sin\Omega t \end{Bmatrix} = \begin{Bmatrix} F\sin\Omega t \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2k-3m\Omega^2 & -k \\ -k & 2k-m\Omega^2 \end{bmatrix} \begin{Bmatrix} B_1 \cos\Omega t \\ B_2 \cos\Omega t \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

for any time

$$\begin{bmatrix} 2k-3m\Omega^2 & -k \\ -k & 2k-m\Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

and

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Harmonic excitation: direct method

$$A_1 = \frac{F(2k - m\Omega^2)}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2}$$

$$A_2 = \frac{kF}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2}$$

or

$$\boxed{x_1(t) = \frac{F(2k - m\Omega^2)}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2}\sin\Omega t}$$

$$\boxed{x_2(t) = \frac{kF}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2}\sin\Omega t}$$

Harmonic excitation: direct method

The values of Ω for which the denominators of $x_1(t)$ and $x_2(t)$:

$$(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2$$

vanishes correspond to the frequencies $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$ found for free vibrations.

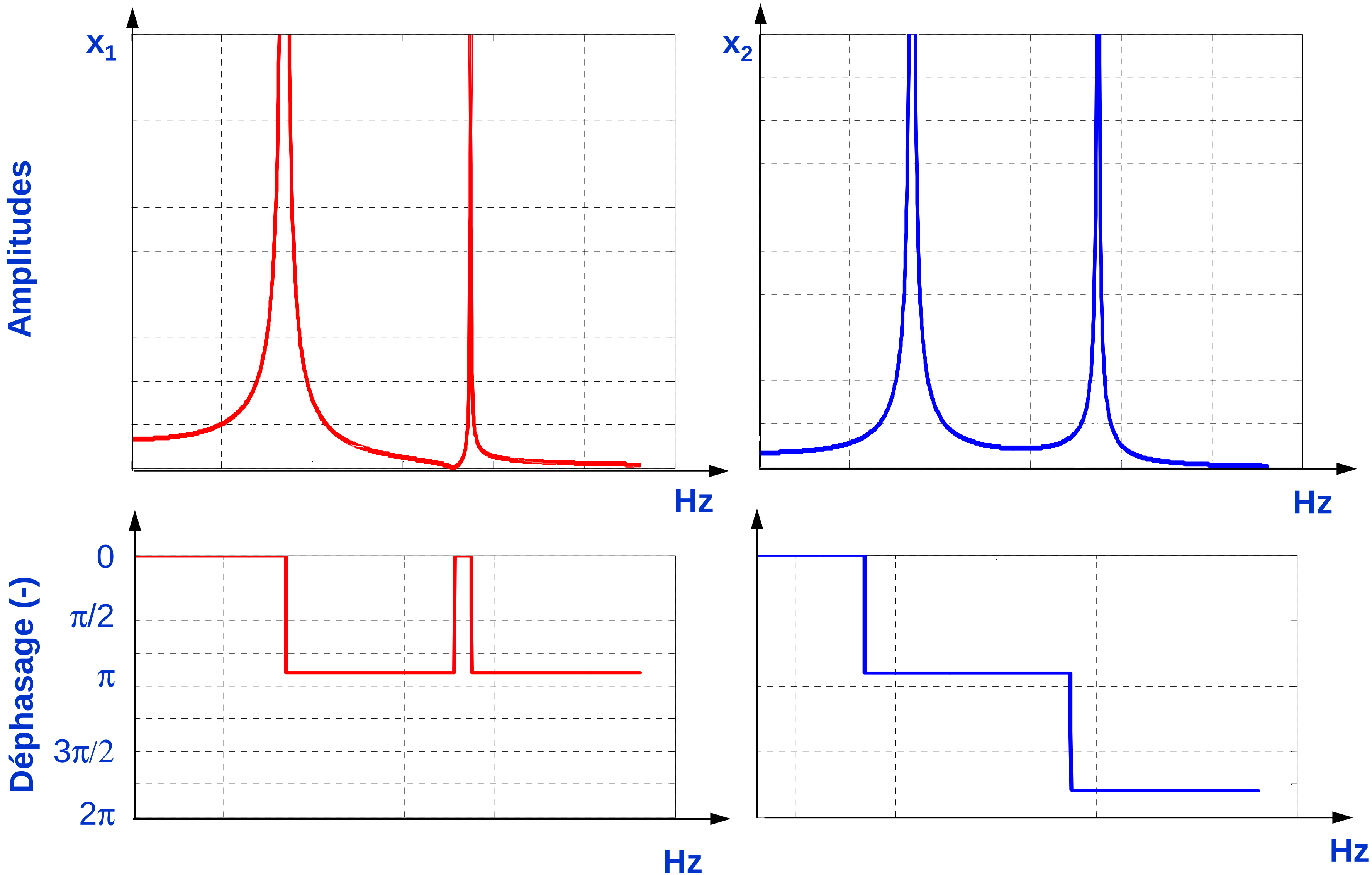
If the system is subjected to a harmonic force whose frequency is equal to ω_1 or ω_2 , the amplitude of the response will approach infinity.

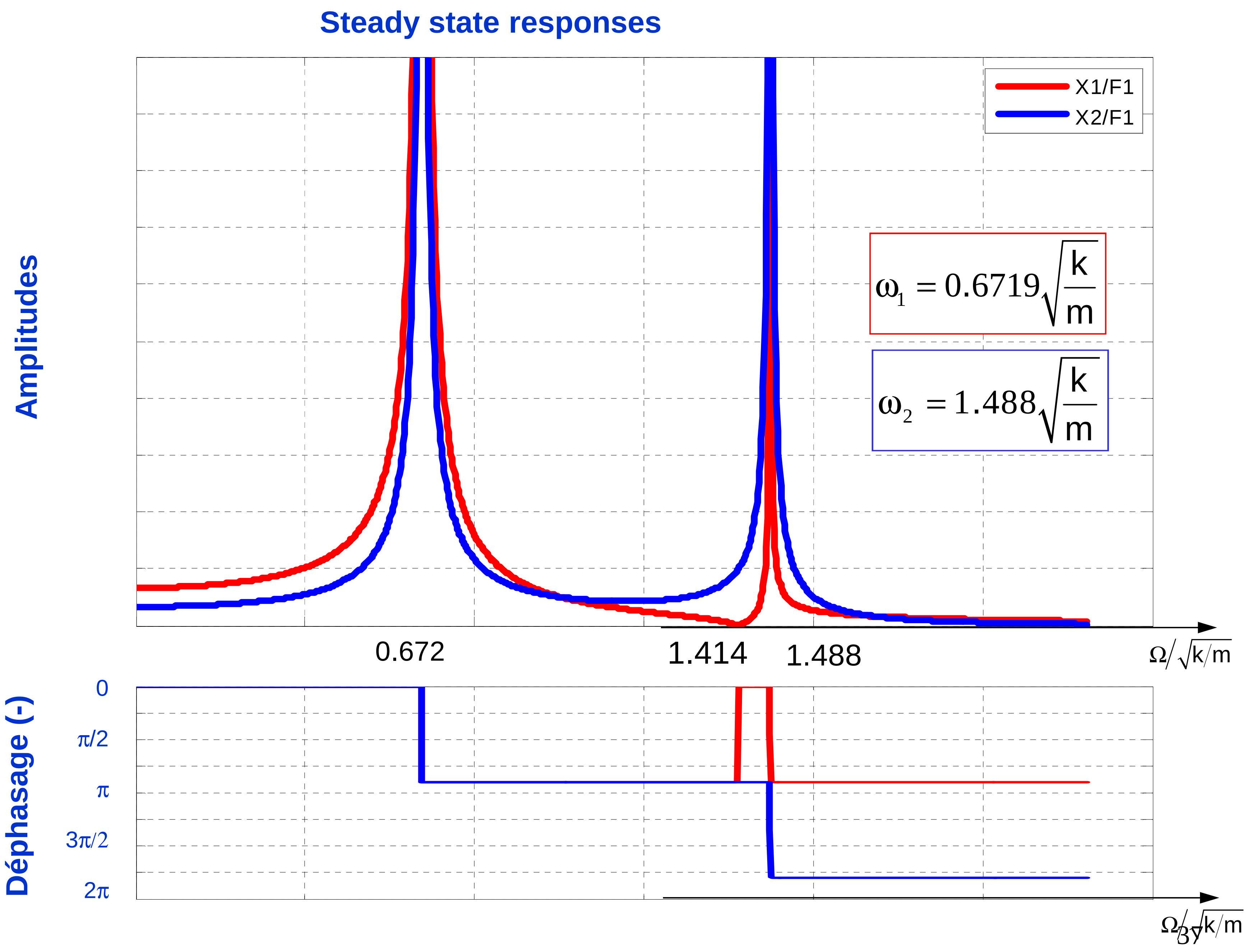
$$\Omega = \omega_1 \qquad \text{and} \qquad \Omega = \omega_2$$

Also, if $\Omega = \sqrt{2k/m}$ A_1 (and x_1) is nil, but not A_2 (and x_2) , it is the **anti-resonance phenomenon.**

However, if some damping is included, the constants B_1 and B_2 will be nonzero, and the amplitude of response at resonance will be finite.

Réponses en fréquences régime permanent





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Approach using Modal basis

With

$$x = \phi q$$

and

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.6458 & -4.646 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

It becomes (same as for the previous example plus excitation term):

$$\boxed{\phi^t M \phi \, \ddot{q} + \phi^t K \phi \, q = \phi^t F}$$

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Approach using Modal basis

Then multiplication (on the left) with the transpose matrix ϕ, it becomes:

$$
\begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix}^t \begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}
+ \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix}^t \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}
= \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix}^t \begin{Bmatrix} F \sin \Omega t \\ 0 \end{Bmatrix}
$$

It becomes (same as for the previous example) plus excitation term:

$$
\boxed{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} F \sin \Omega t \\ \phi_{21} F \sin \Omega t \end{Bmatrix}}
$$

It is important to note that matrices are still **diagonal**.

Approach using Modal basis

The two systems are still un-coupled and solutions in q_1 and q_2 can be obtained with the results of the previous chapter.

$$\boxed{\begin{aligned} m_1\ddot{q}_1 + k_1 q_1 &= \phi_{11} F \sin\Omega t \\ m_2\ddot{q}_2 + k_2 q_2 &= \phi_{21} F \sin\Omega t \end{aligned}}$$

So, in **steady state motions**:

$$q_1 = \frac{\phi_{11} F \sin\Omega t}{k_1 - m_1\Omega^2} \quad \text{and} \quad q_2 = \frac{\phi_{21} F \sin\Omega t}{k_2 - m_2\Omega^2}$$

hence

$$\boxed{\begin{aligned} x_1 &= \phi_{11} q_1 + \phi_{21} q_2 = \phi_{11} \frac{\phi_{11} F \sin\Omega t}{k_1 - m_1\Omega^2} + \phi_{21} \frac{\phi_{21} F \sin\Omega t}{k_2 - m_2\Omega^2} \\ x_2 &= \phi_{12} q_1 + \phi_{22} q_2 = \phi_{12} \frac{\phi_{11} F \sin\Omega t}{k_1 - m_1\Omega^2} + \phi_{22} \frac{\phi_{21} F \sin\Omega t}{k_2 - m_2\Omega^2} \end{aligned}}$$

Approach using Modal basis

Or by re-arranging:

$$x_1 = \left[\phi_{11}\frac{\phi_{11}}{k_1 - m_1\Omega^2} + \phi_{21}\frac{\phi_{21}}{k_2 - m_2\Omega^2}\right] F\sin\Omega t = \Lambda_1 \sin\Omega t$$

$$x_2 = \left[\phi_{12}\frac{\phi_{11}}{k_1 - m_1\Omega^2} + \phi_{22}\frac{\phi_{21}}{k_2 - m_2\Omega^2}\right] F\sin\Omega t = \Lambda_2 \sin\Omega t$$

Remarks :

If forcing frequency $\boldsymbol{\Omega}$ is close by the first natural frequency ω_1 this means that:

$$\Omega \approx \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

then:

$$x_1 = \left[\phi_{11}\frac{\phi_{11}}{\text{small}} + \phi_{21}\frac{\phi_{21}}{\text{fini}}\right] F\sin\Omega t \Rightarrow \boxed{\cong \phi_{11}\frac{\phi_{11} F\sin\Omega t}{k_1 - m_1\Omega^2}}$$

$$x_2 = \left[\phi_{12}\frac{\phi_{11}}{\text{small}} + \phi_{22}\frac{\phi_{21}}{\text{fini}}\right] F\sin\Omega t \Rightarrow \boxed{\cong \phi_{12}\frac{\phi_{11} F\sin\Omega t}{k_1 - m_1\Omega^2}}$$

Approach using Modal basis

Remarks cont'ed :

If forcing frequency Ω is close by the second natural frequency ω_2 , this means that:

$$\Omega \approx \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

then

$$x_1 = \left[\phi_{11} \frac{\phi_{11}}{\text{fini}} + \phi_{21} \frac{\phi_{21}}{\text{petit}} \right] F \sin \Omega t \Rightarrow \boxed{\cong \phi_{21} \frac{\phi_{21} F \sin \Omega t}{k_2 - m_2 \Omega^2}}$$

$$x_2 = \left[\phi_{12} \frac{\phi_{11}}{\text{fini}} + \phi_{22} \frac{\phi_{21}}{\text{petit}} \right] F \sin \Omega t \Rightarrow \boxed{\cong \phi_{22} \frac{\phi_{21} F \sin \Omega t}{k_2 - m_2 \Omega^2}}$$

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Approach using Modal basis

Finally:

if

$$\Omega \cong \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

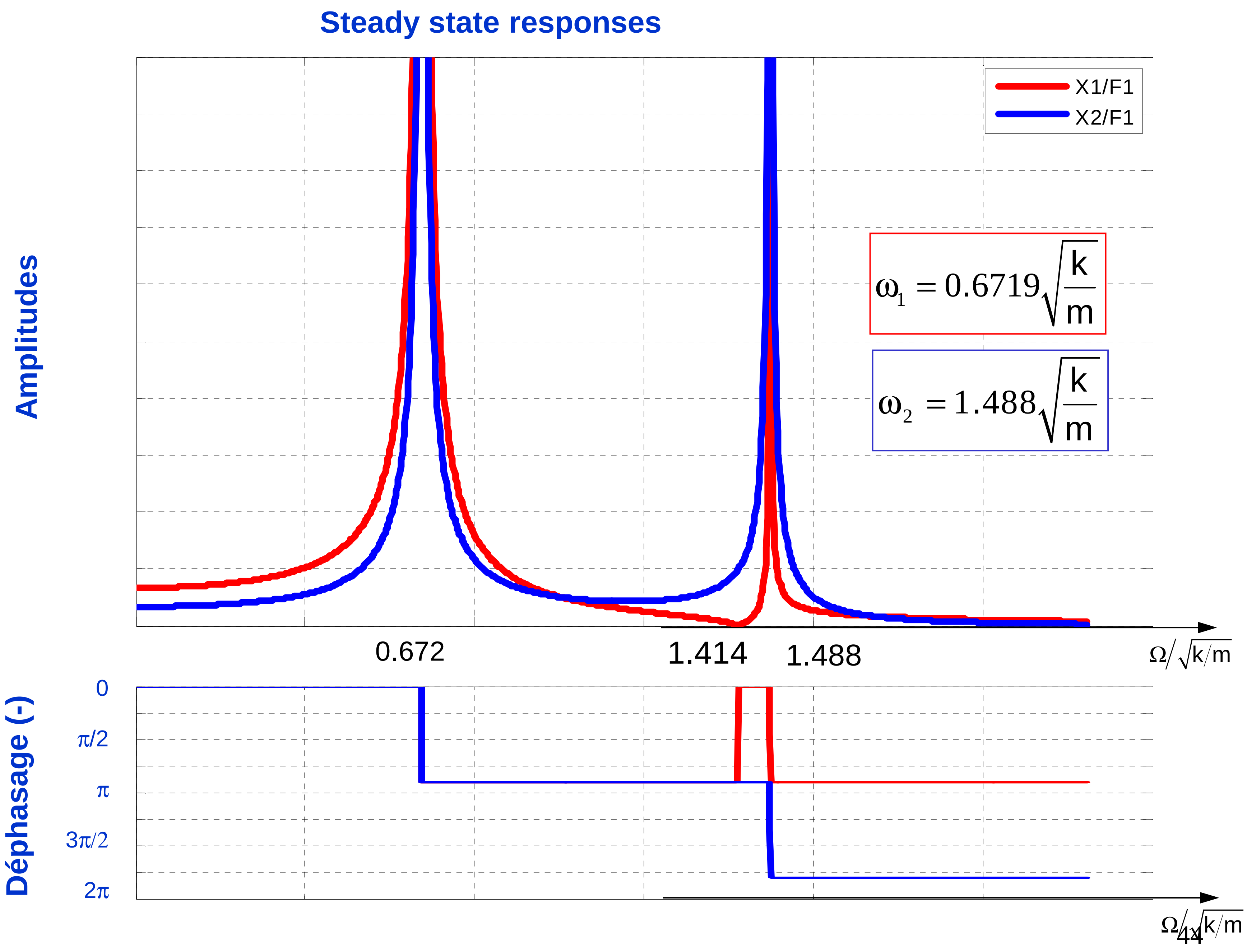
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{12} \end{Bmatrix} q_1(t) = \{\phi_1\} q_1(t)$$

And if

$$\Omega \cong \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{21} \\ \phi_{22} \end{Bmatrix} q_2(t) = \{\phi_2\} q_2(t)$$

It can be concluded that near a natural frequency (i.e. ω_1 and ω_2) the dynamic behavior of the TWO-DOF system is equivalent to whose of ONE-DOF system.



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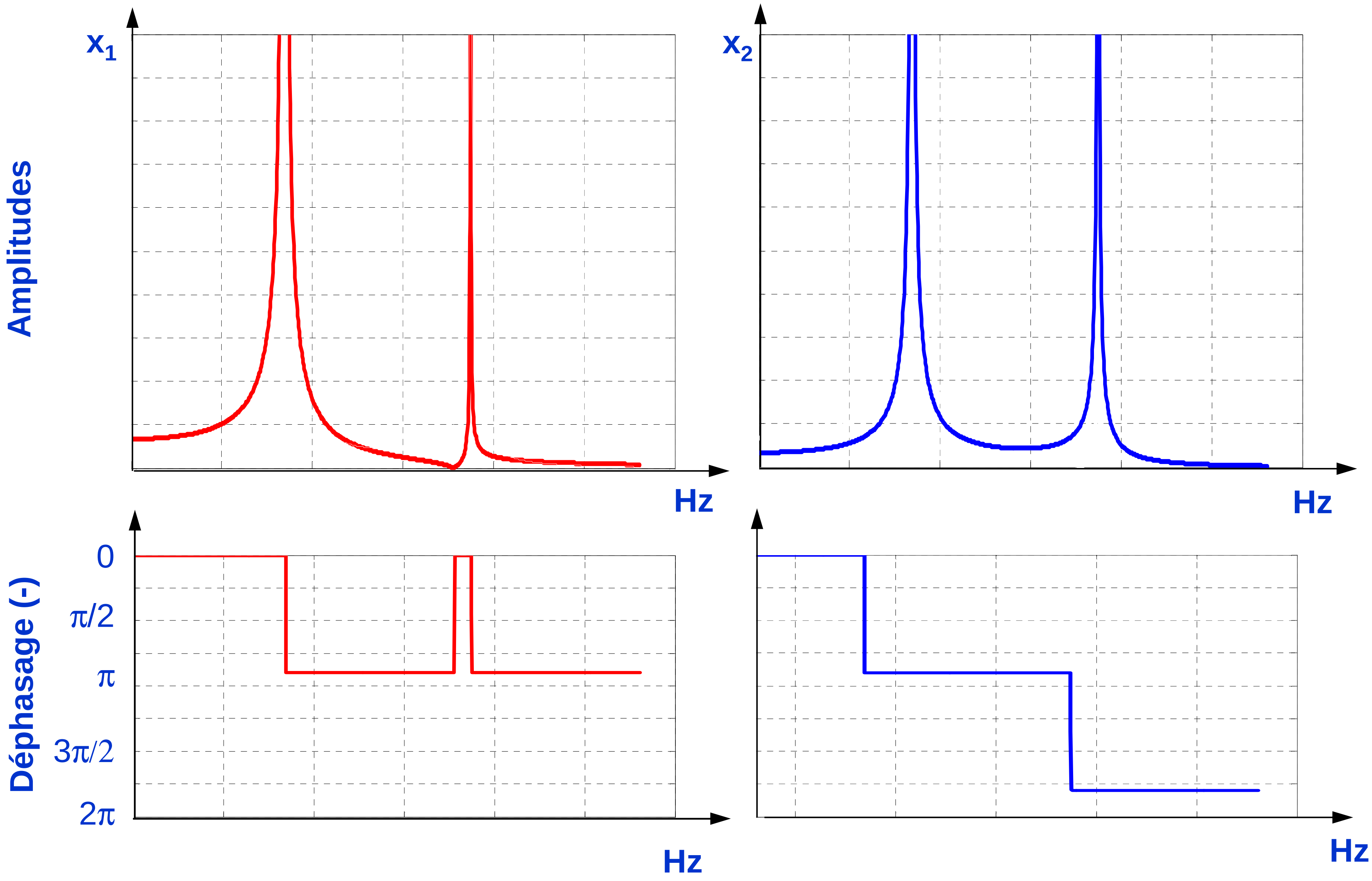
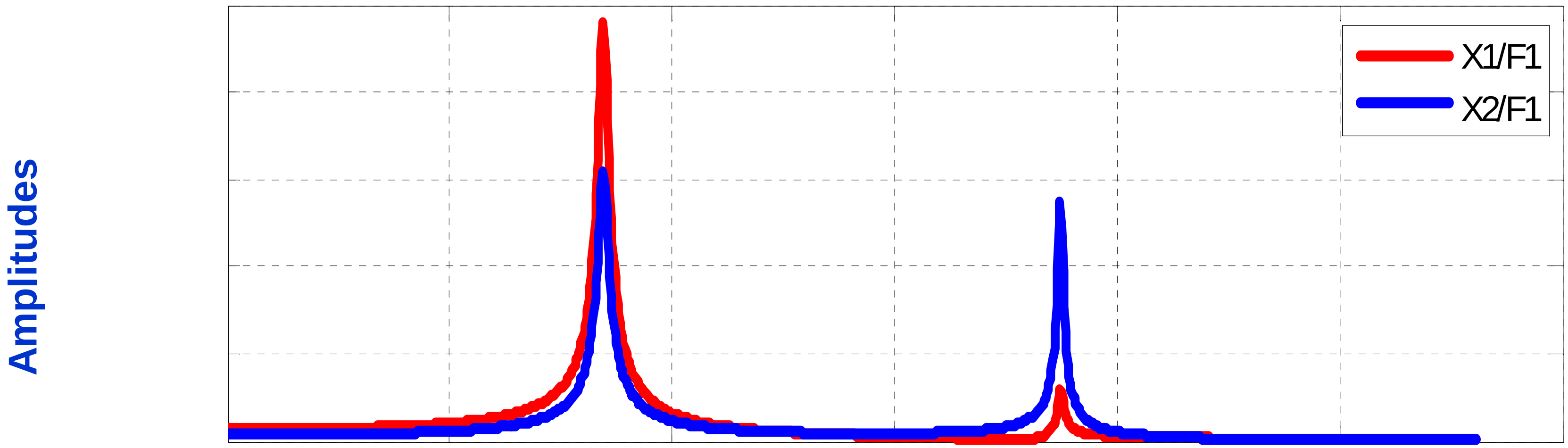
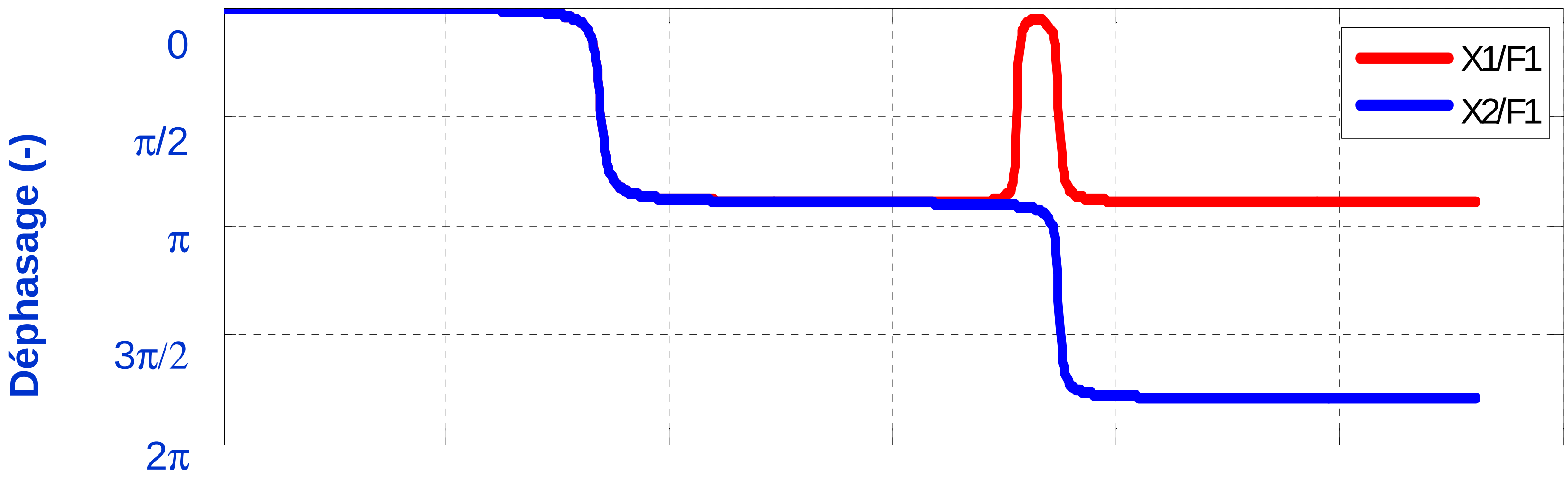
Réponses en fréquences régime permanent

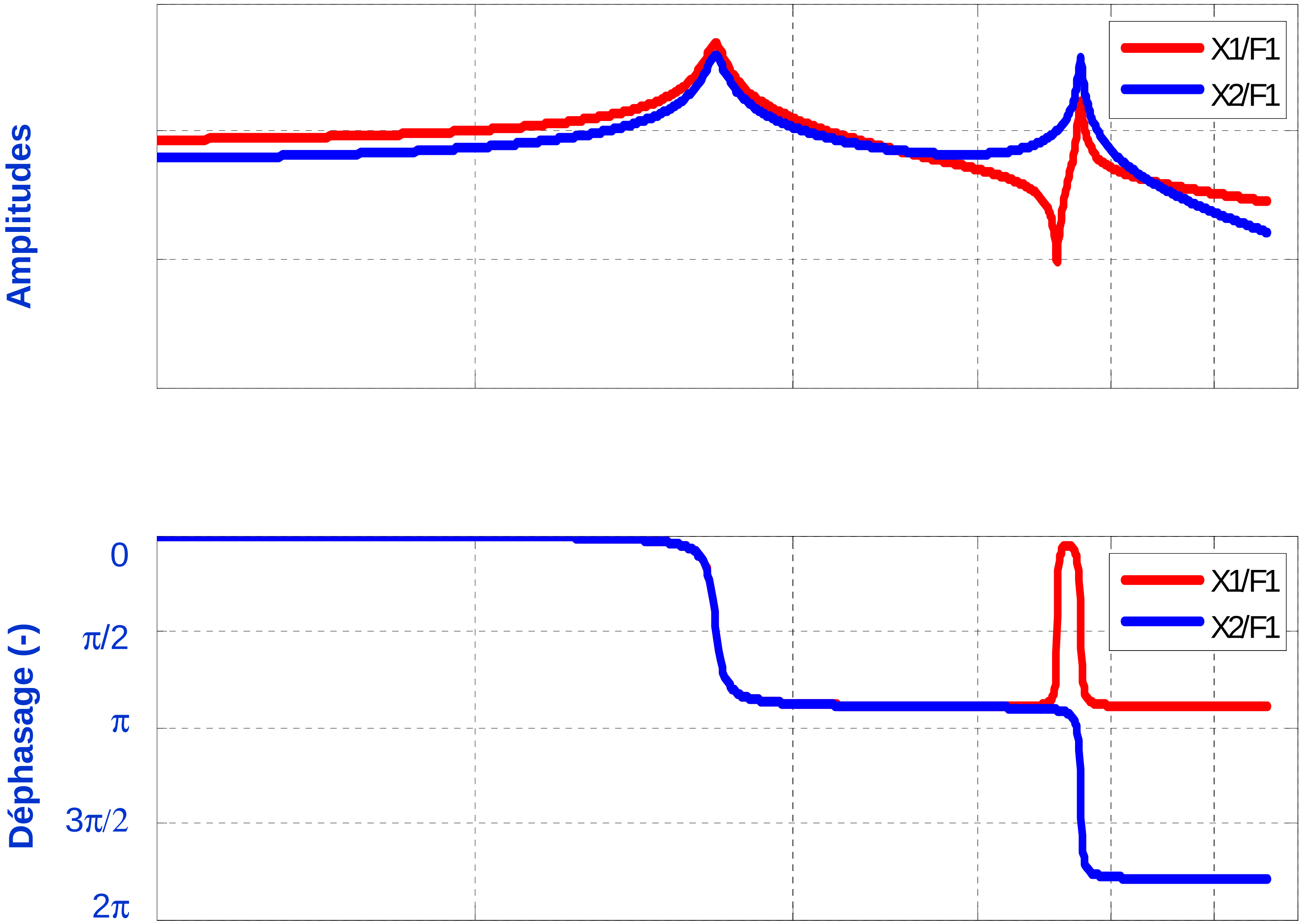


Steady state responses (other representations)





Steady state responses (other representations)



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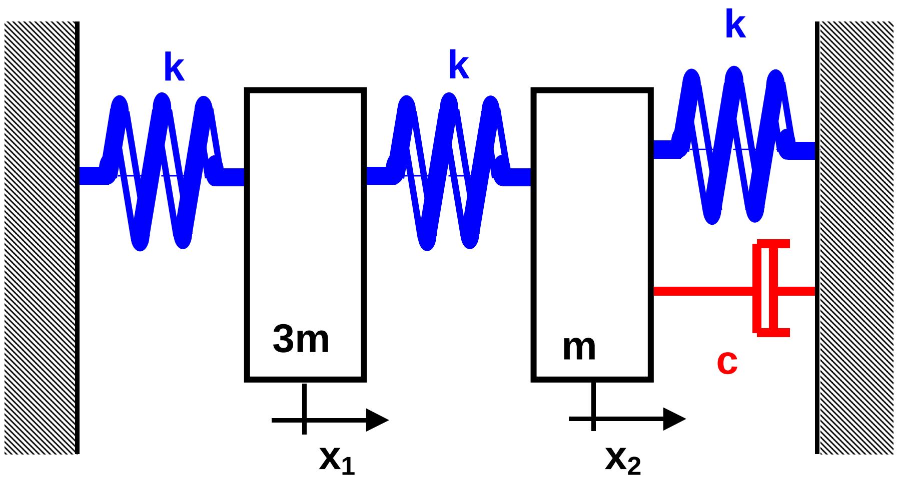
Non-conservative systems

Free Vibrations

Linear viscous damping



$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In matrix form:

$$\boxed{M\ddot{x} + C\dot{x} + Kx = 0}$$

Frequencies and mode shapes

For the compact system: [2x2]

$$M\ddot{x} + C\dot{x} + Kx = 0$$

Solutions are sought in the form

$$x_1 = X_1 e^{rt} \quad \text{and} \quad x_2 = X_2 e^{rt}$$

$$\begin{bmatrix} 3mr^2 + 2k & -k \\ -k & mr^2 + cr + 2k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [0]$$

For any time **t ,** solutions are for $\mathbf{X_i}$:

$$r_1 = \alpha_1 + j\beta_1 \quad \text{and} \quad r_2 = \alpha_1 - j\beta_1$$

$$r_3 = \alpha_2 + j\beta_2 \quad \text{and} \quad r_4 = \alpha_2 - j\beta_2$$

The quantities α_1 and a α_2 characterize the damping and the quantities β_1 and β_2 the frequencies.

General solutions are:

$$x_1(t) = e^{-\alpha_1 t}(a_{11}\cos(\omega_1 t) + b_{11}\cos(\omega_1 t)) + e^{-\alpha_2 t}(a_{21}\cos(\omega_2 t) + b_{21}\cos(\omega_2 t))$$

$$x_2(t) = e^{-\alpha_1 t}(a_{12}\cos(\omega_1 t) + b_{12}\cos(\omega_1 t)) + e^{-\alpha_2 t}(a_{22}\cos(\omega_2 t) + b_{22}\cos(\omega_2 t))$$

Stability is deduced from real parts of solutions:

Negative parts correspond to damping and positive parts leads to instability.

Imaginary parts correspond ot frequencies.

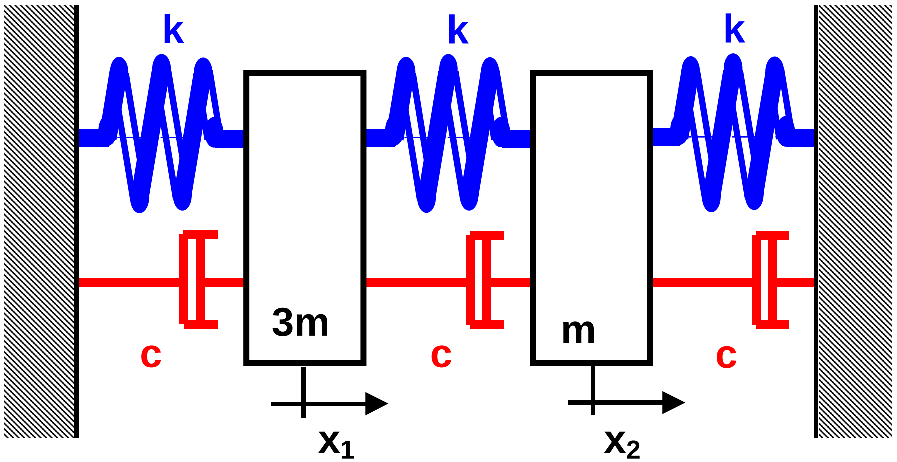
Initial conditions (2 displacements and 2 velocities) serve as the determination of the four constants a_{ij}.

Proportionnal viscous damping

Supposed a proportionnal damping (according to Basile, Caughey or Rayleigh)

$$[C] = \alpha[M] + \beta[K]$$

In order to clarify the presentation **C** is taken as **C**= α **. M +**β **.K** but $\alpha = 0$

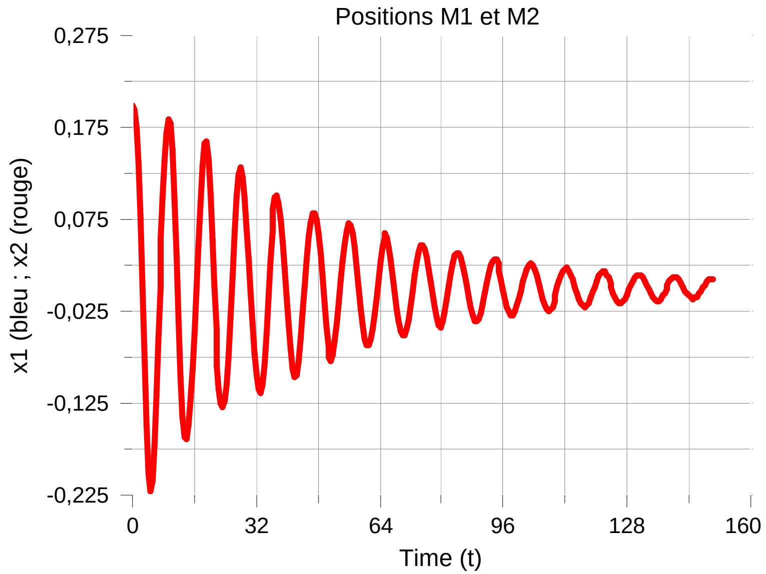


$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

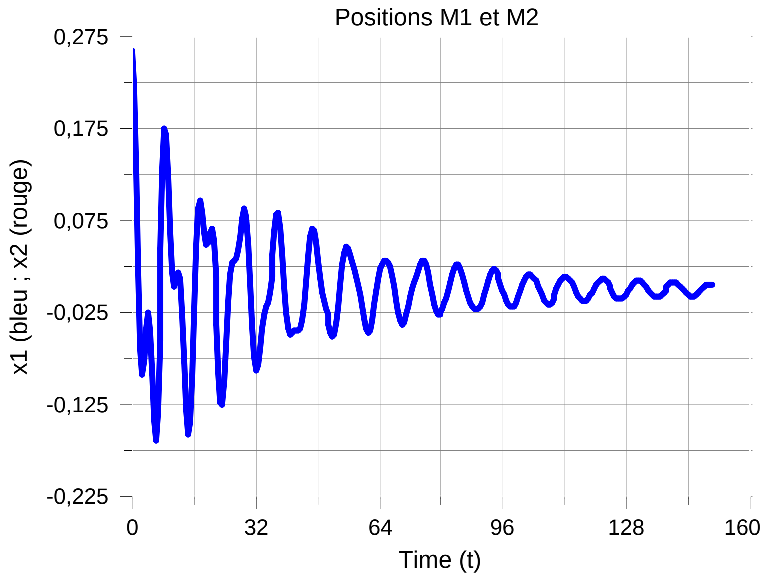
$$M\ddot{x} + C\dot{x} + Kx = 0$$

Proportionnal Viscous Damping

Displacement of x_1



Displacement of x_2



Harmonic Excitation

Proportionnal Viscous Damping

$$[C] = \alpha[M] + \beta[K]$$

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \sin \Omega t \\ 0 \end{Bmatrix}$$

- **Direct Method**

  not suitable

- **Modal Method**

  more preferable

Harmonic Excitation

Modal Method

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \sin \Omega t \\ 0 \end{Bmatrix}$$

Assuming that frequencies ω_i and mode shapes ϕ_i are defined it can be stated and using:

$$x = \phi\, q$$

$$[c] = [\phi]^t [C][\phi] = [\phi]^t \left(\alpha[M] + \beta[K]\right)[\phi]$$

$$[c] = [\phi]^t \left(\beta[K]\right)[\phi]$$

$$= \beta[\phi]^t [K][\phi]$$

[c] is a diagonal matrix.

$$\boxed{[c] = \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix}^t \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}}$$

Modal Method

Hence equations are un-coupled.

$$\boxed{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} F \sin \Omega t \\ \phi_{21} F \sin \Omega t \end{Bmatrix}}$$

$$m_1 \ddot{q}_1 + c_1 \dot{q}_1 + k_1 q_1 = F_1(t)$$

$$m_2 \ddot{q}_2 + c_2 \dot{q}_2 + k_2 q_2 = F_2(t)$$

$$\boxed{m\,\ddot{q} + c\,\dot{q} + k\,q = F(t)}$$

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As for the case of ONE DOF system, solutions in steady state motions are:

$$q_1(t) = \frac{\phi_{11} F \sin(\Omega t - \psi_1)}{\sqrt{(k_1 - m_1\Omega^2)^2 + c_1{}^2\Omega^2}}$$

and

$$q_2(t) = \frac{\phi_{21} F \sin(\Omega t - \psi_2)}{\sqrt{(k_2 - m_2\Omega^2)^2 + c_2{}^2\Omega^2}}$$

$$x_1(t) = \phi_{11} q_1 + \phi_{21} q_2$$
$$= \phi_{11}\frac{\phi_{11} F \sin(\Omega t - \psi_1)}{\sqrt{(k_1 - m_1\Omega^2)^2 + c_1{}^2\Omega^2}} + \phi_{21}\frac{\phi_{21} F \sin(\Omega t - \psi_2)}{\sqrt{(k_2 - m_2\Omega^2)^2 + c_2{}^2\Omega^2}}$$

$$x_2(t) = \phi_{12} q_1 + \phi_{22} q_2$$
$$= \phi_{12}\frac{\phi_{11} F \sin(\Omega t - \psi_1)}{\sqrt{(k_1 - m_1\Omega^2)^2 + c_1{}^2\Omega^2}} + \phi_{22}\frac{\phi_{21} F \sin(\Omega t - \psi_2)}{\sqrt{(k_2 - m_2\Omega^2)^2 + c_2{}^2\Omega^2}}$$

In the same manner, it can be concluded that near a natural frequency (i.e. ω_1 and ω_2) the dynamic behavior of the TWO-DOF system is governed by the equivalent ONE DOF system.

Modal Damping

A pratical used of modal damping is (finite elements models).

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{Bmatrix} \text{dampings} \\ ? \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \sin \Omega t \\ 0 \end{Bmatrix}$$

Using

$$x = \phi\, q$$

and left multiplication by ϕ^t for M and K only

It is required to have:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} ? & 0 \\ 0 & ? \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} F \sin \Omega t \\ \phi_{21} F \sin \Omega t \end{Bmatrix}$$

Modal Damping

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} F \sin \Omega t \\ \phi_{21} F \sin \Omega t \end{Bmatrix}$$

Using results for one dof system, it is stated that:

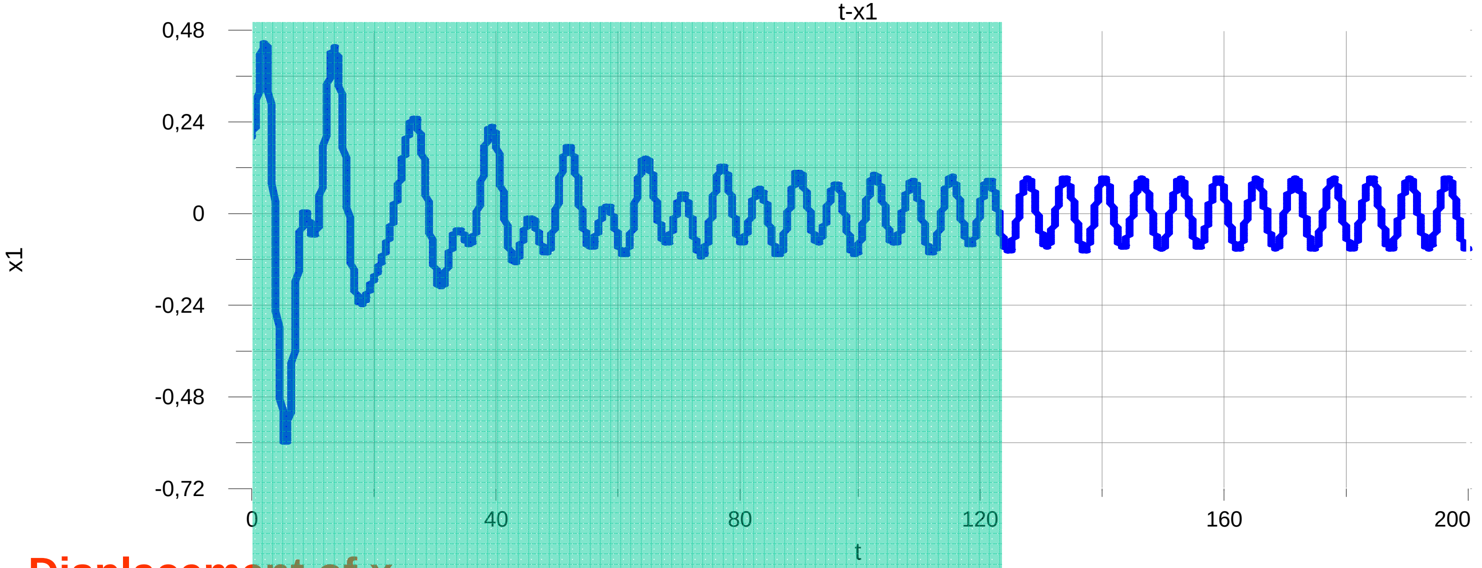
$$\alpha_i = \frac{c_i}{c_{ci}} = \frac{c_i}{2\sqrt{k_i m_i}} = \frac{\text{current damping}_i}{\text{critical damping}_i}$$

$$\boxed{\alpha_1 = \frac{c_1}{2\sqrt{k_1 m_1}}} \qquad \boxed{\alpha_2 = \frac{c_2}{2\sqrt{k_2 m_2}}}$$

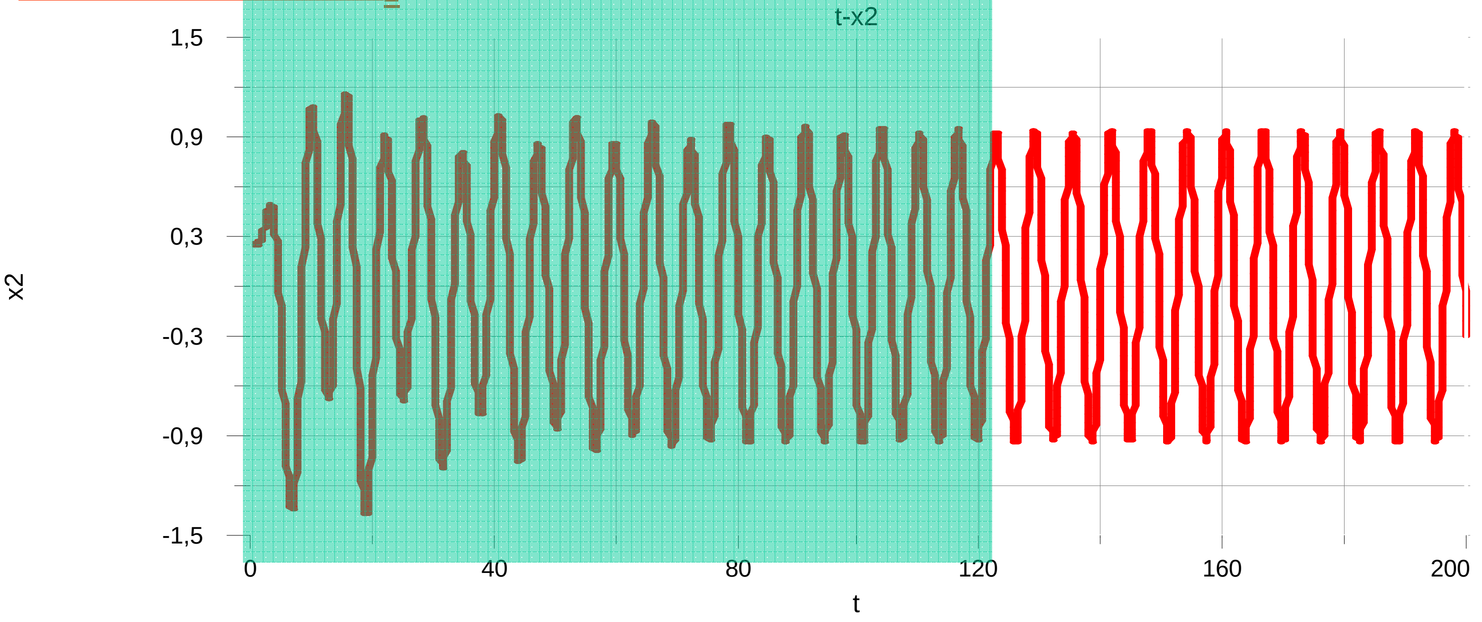
where α_1 and α_2 may be obtained with Half-Power Bandwith.

Displacement of x_1

Transient motion. **Steady state motion**



Displacement of x_2



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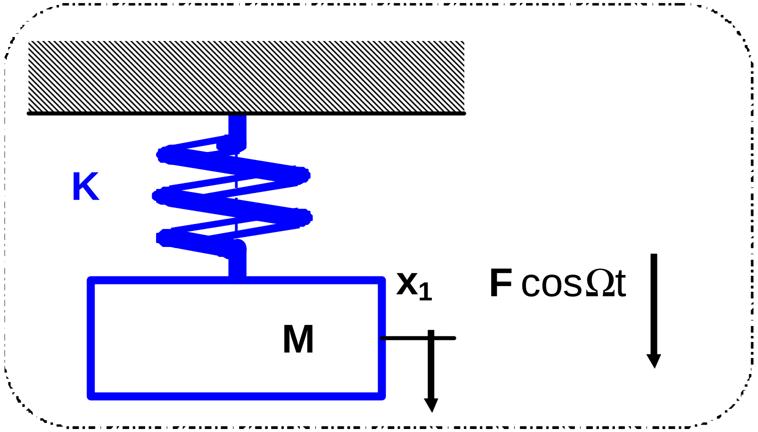
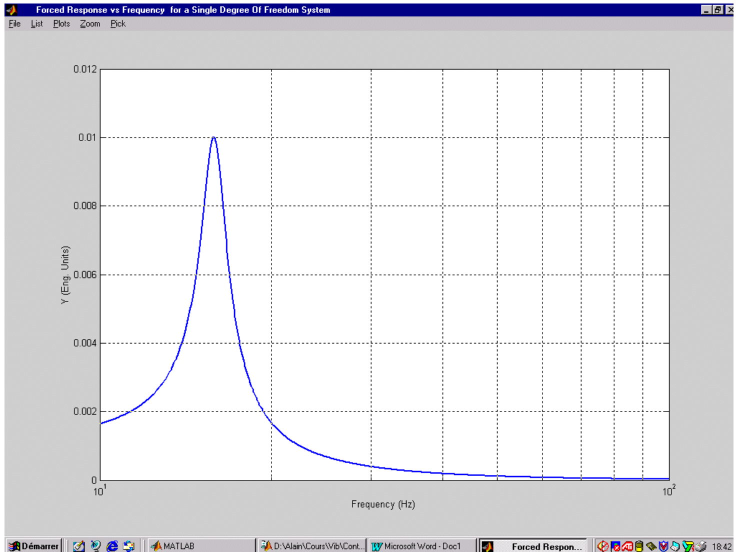
APPLICATIONS

Dynamic absorber



The principle of a vibration absorber is simple and this device is frequently used to reduce the amplitude of a vibrating system.

Let a single degree-of freedom system (K,M) be subjected to a force **F cos Ωt**. In steady-state motion,



$$x_1(t) = \frac{F}{K - M\Omega^2} \cos \Omega t$$

and,

$$\omega_1 = \sqrt{\frac{K}{M}}$$

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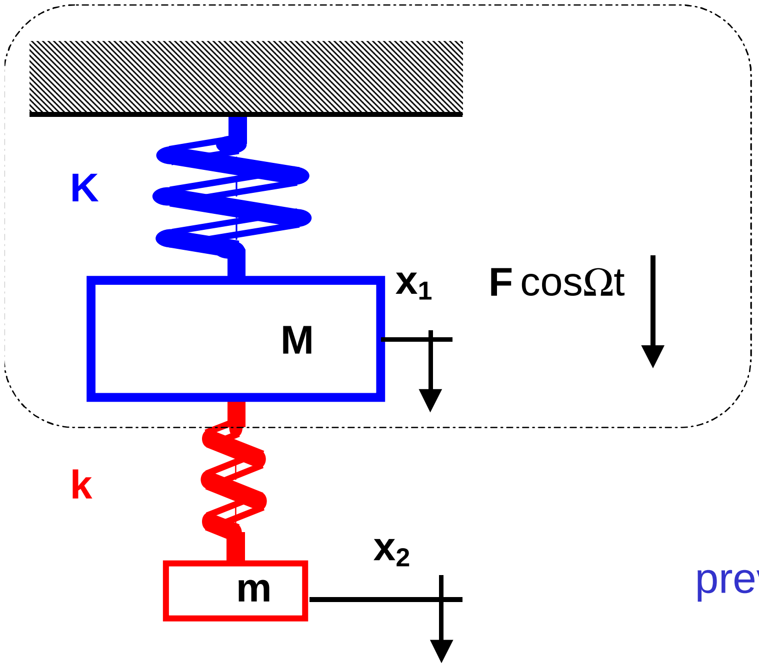
Dynamic absorber



Suppose now that one adds to the original system a second spring-mass system (k, m) Then:

$$x_1(t) = X_1 \cos \Omega t$$

$$x_2(t) = X_2 \cos \Omega t$$

The equations of this combined system are (see previous chapter):

$$X_1 = \frac{F(k - m\Omega^2)}{(K + k - M\Omega^2)(k - m\Omega^2) - k^2}$$

$$X_2 = \frac{Fk}{(K + k - M\Omega^2)(k - m\Omega^2) - k^2}$$

APPLICATIONS

Dynamic absorber

It must be noted that for:

$$\Omega = \sqrt{\frac{k}{m}} \qquad X_1 = 0 \qquad X_2 = -\frac{F}{k}$$

the motion of the original spring-mass system is completely suppressed. This is the principle of the **vibration absorber**.

Remarks :

In using these results for designing vibration absorbers it is necessary to fulfil some requirements:

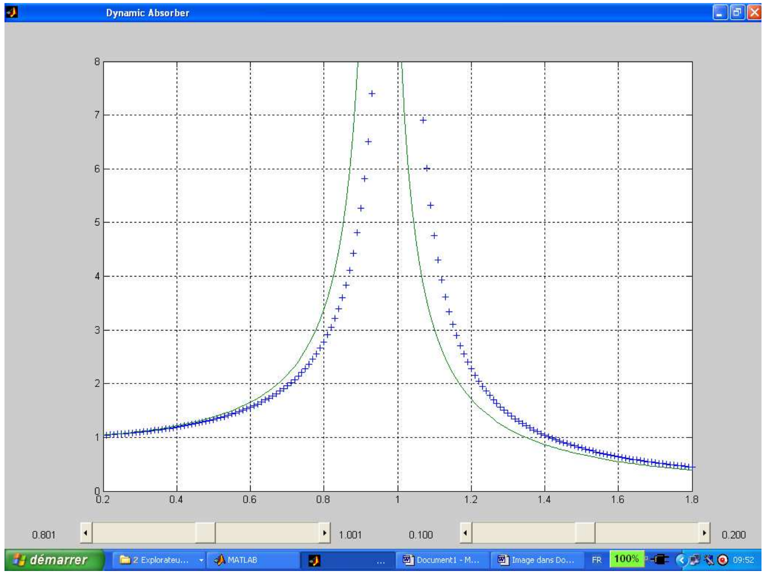
-The frequency Ω must be constant or varying only over a small range because the attachment of the absorber splits ω_1 into two resonant frequencies, one on either side of edt. Thus, if Ω is too far above or below its design value of ω_1 one will get resonance instead of absorption of its motion.

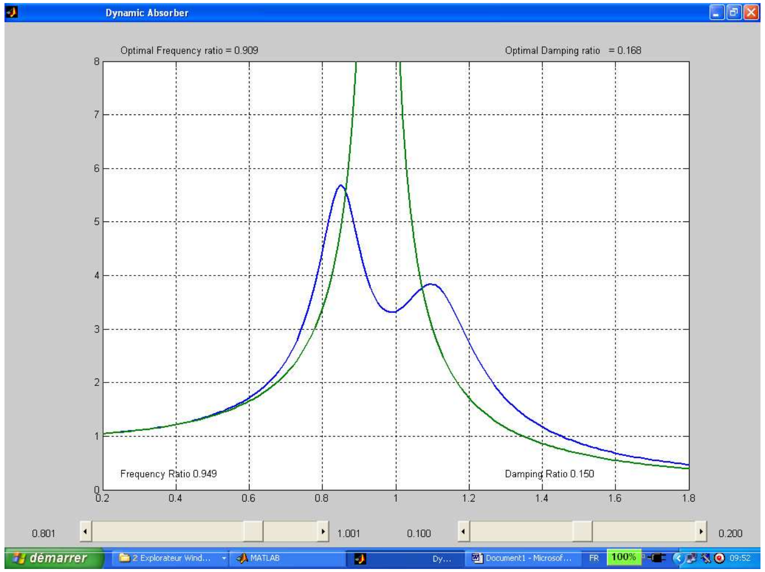
-The addition of an auxiliary system to the original system must be technically feasible.

A viscous damper if often added with the secondary mass.

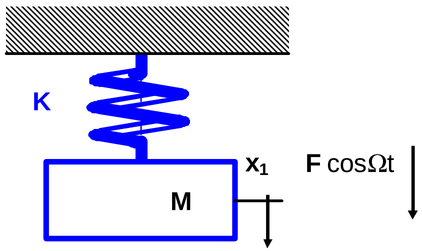
APPLICATIONS

Dynamic absorber

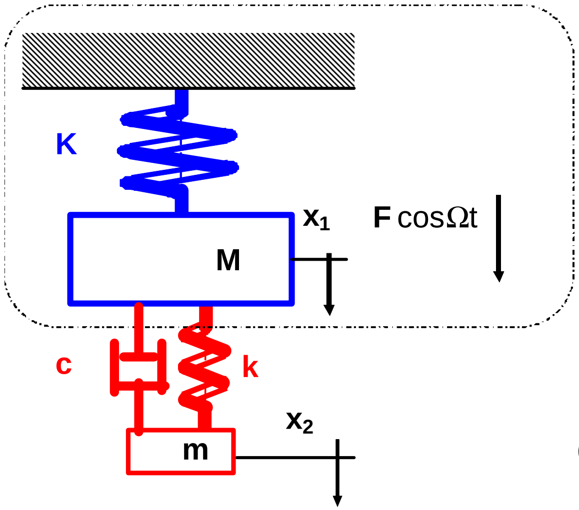




One dof system



Two dof system



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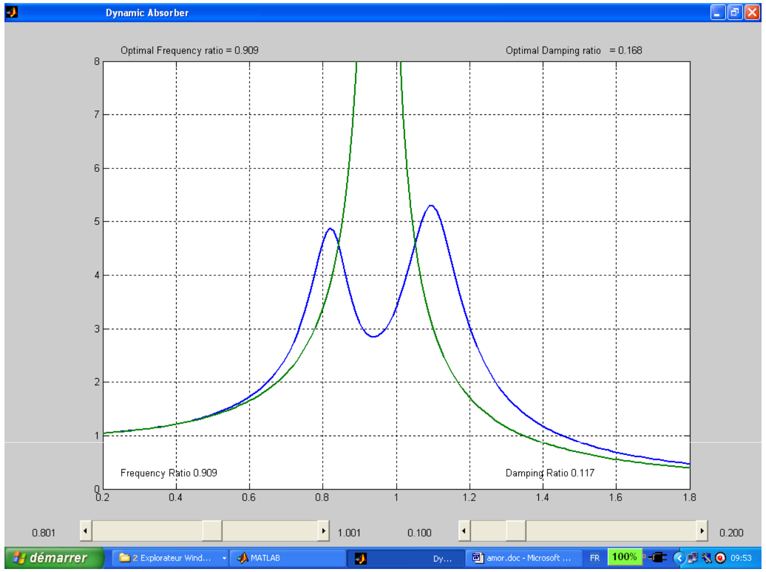
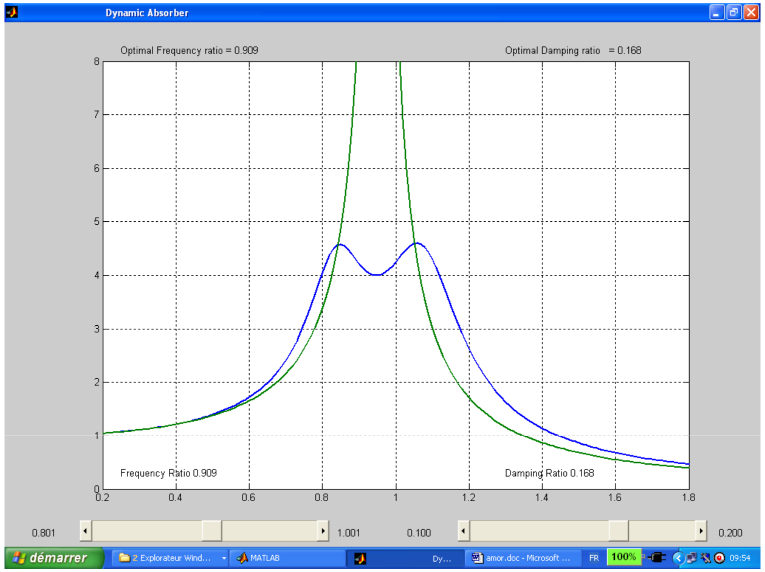
APPLICATIONS

Dynamic absorber



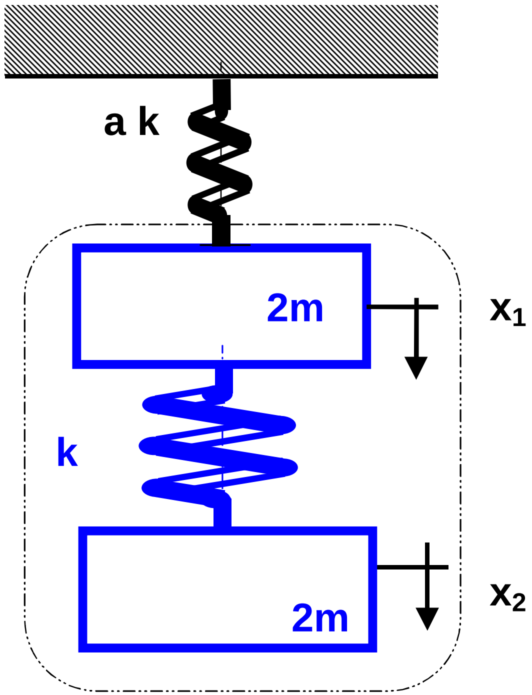
Two dof system tuned in frequency



Two dof system tuned in amplitude

APPLICATIONS

Suspended system



A two degree-of-freedom system (2m, k, 2m) is mounted on a rigid support by a spring of stiffness a.k (where k is a parameter to be determined).

$$\omega = \sqrt{\frac{k}{m}}$$

Initial system has two frequencies and the the nonzero frequency of the initial system is noted ω .The two frequencies of the complete system are ω_1 and ω_2.

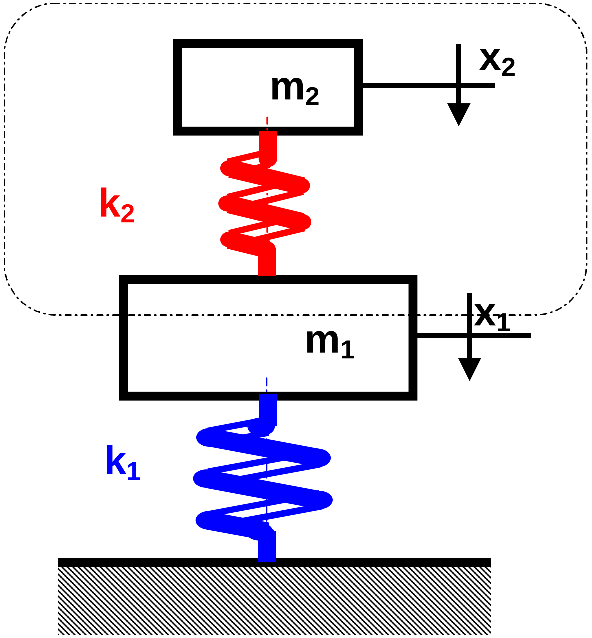
Quelle raideur du sandow choisir pour que la mesure soit proche de celle du système non suspendu ?

Ratio/frequencies according to parameter **a:**

a	**1**	**0.5**	**0.2**	**0.1**	**0.05**	**0.02**	**0.01**
ω_1/ω	0.44	0.33	0.22	0.16	0.11	0.07	0.05
ω_2/ω	1.14	1.068	1.026	1.013	1.006	1.003	1.001

APPLICATIONS

Based supported system



The initial system is a single degree-of-freedom system with frequency $\omega_2 = (k_2/m_2)^{½}$ is mounted on another single degree-of-freedom system having frequency $\omega_1 = (k_1/m_1)^{½}$

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

A new system with two degree-of-freedom is obtained with two associated frequencies ω^*_1 and ω^*_2.

Frequencies of the final setup (ω^*_1 and ω^*_2) can be measured and are well defined. The question is: How to define initial system (composed of k_1 and m_1) for a good determination of frequency of the main system (composed of k_2 and m_2)

*

ω_2/ω_1	1		3		10		100	
m_2/m_1	e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2
0.001	0.984	0.016	0.333	0.0006	0.100	0.0005	0.010	**0.0005**
0.003	0.973	0.027	0.333	0.0017	0.100	0.0015	0.010	0.0015
0.01	0.951	0.051	0.331	0.006	0.099	0.005	0.010	0.005
0.03	0.917	0.090	0.280	0.016	0.098	0.015	0.009	0.015
0.1	0.854	0.117	0.316	0.054	0.095	0.049	0.009	0.049
0.3	0.763	0.311	0.289	0.152	0.087	0.141	0.008	0.140
1	0.618	0.618	0.232	0.434	0.071	0.416	0.007	0.414

Where for simplicity

$$e_1 = \frac{\omega_1^* - 0}{\omega_2} \quad \text{and} \quad \boxed{e_2 = \frac{\omega_2^* - \omega_2}{\omega_2}}$$

Values of ω_2/ω_1 and m_2/m_1 (or values of k_1, m_1 for k_2 , m_2 considered as constant) which are in the upper right-hand comer of this table result $\omega^*_2 \approx \omega_2$.

This implies that with judiciously chosen values of k_1 and m_1, it is possible to 'soft' mount a system k_2 , m_2 and still measure its 'hard' mounted, or fixed-base, frequency.