Stresses in Elastic Solid

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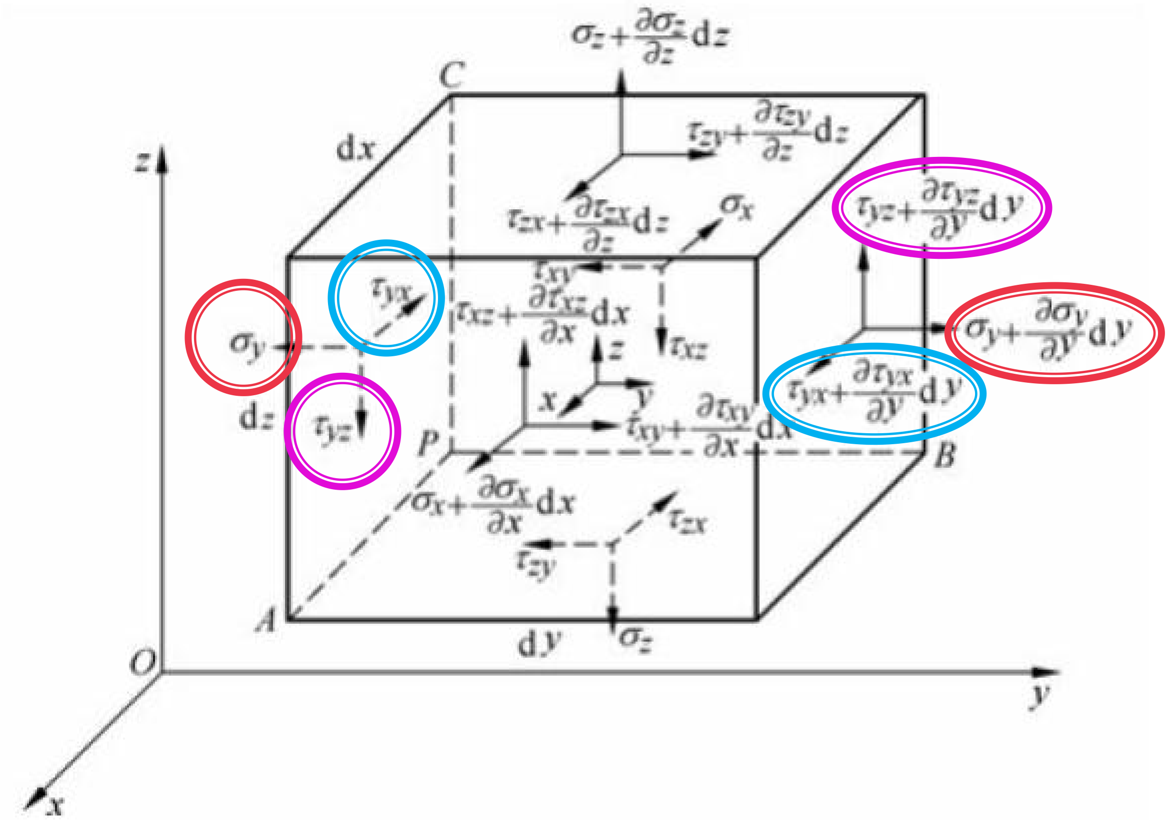
1.Equilibrium of infinitesimal unit



1.Equilibrium of infinitesimal unit

Take geometric centre of unit as reference point, equations are

$$\begin{cases} \sum F_x = 0 & \sum M_x = 0 \\ \sum F_y = 0 & \sum M_y = 0 \\ \sum F_z = 0 & \sum M_z = 0 \end{cases}$$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dydz - \sigma_x dydz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy - \tau_{zx} dxdy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dzdx - \tau_{yx} dzdx + Xdxdydz = 0 \tag{2.1a}$$

$$\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy\right) dxdz - \sigma_y dxdz + \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz\right) dxdy - \tau_{zy} dxdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dzdy - \tau_{xy} dzdy + Ydxdydz = 0 \tag{2.1b}$$

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz\right) dxdy - \sigma_z dxdy + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx\right) dzdy - \tau_{xz} dzdy + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) dzdx - \tau_{yz} dzdx + Zdxdydz = 0 \tag{2.1c}$$

$$\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) dzdx \cdot \frac{dy}{2} + \tau_{yz} dzdx \cdot \frac{dy}{2} - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz\right) dxdy \cdot \frac{dz}{2} - \tau_{zy} dxdy \cdot \frac{dz}{2} = 0 \tag{2.1d}$$

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy \cdot \frac{dz}{2} + \tau_{zx} dxdy \cdot \frac{dz}{2} - \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx\right) dzdy \cdot \frac{dx}{2} - \tau_{xz} dzdy \cdot \frac{dx}{2} = 0 \tag{2.1e}$$

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dzdy \cdot \frac{dx}{2} + \tau_{xy} dzdy \cdot \frac{dx}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dzdx \cdot \frac{dy}{2} - \tau_{yx} dzdx \cdot \frac{dy}{2} = 0 \tag{2.1f}$$

-*Equations of Differential Equilibrium

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{zx}}{\partial z}+\frac{\partial \tau_{yx}}{\partial y}+X=0 \quad (2.2a)$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\frac{\partial \tau_{xy}}{\partial x}+Y=0 \quad (2.2b)$$

$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}-Z=0 \quad (2.2c)$$

-*Reciprocal theorem of shear stress

$$\tau_{yz}=\tau_{zy}$$

$$\tau_{zx}=\tau_{xy}$$

$$\tau_{xy}=\tau_{yx}$$

-*Cauchy equations

$$X_N = \sigma_x l + \tau_{yx} m + \tau_{zx} n \quad (2.5a)$$
$$Y_N = \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad (2.5b)$$
$$Z_N = \tau_{xz} l + \tau_{yz} m + \sigma_z n \quad (2.5c)$$

Value of tilted section stress can be calculated as:

$$S = \sqrt{X_N^2 + Y_N^2 + Z_N^2} \qquad (2.6)$$

$$\sigma_N = \vec{S} \cdot \vec{n} = X_N l + Y_N m + Z_N n \quad (2.7)$$

$$\Rightarrow \begin{aligned} \sigma_N &= (\sigma_x l + \tau_{yx} m + \tau_{zx} n) l + (\tau_{xy} l + \sigma_y m + \tau_{zy} n) m + (\tau_{xz} l + \tau_{yz} m + \sigma_z n) n \\ &= \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{yx} ml + 2\tau_{zx} nl + 2\tau_{zy} nm \end{aligned} \qquad (2.8)$$

Then, according to equation (2.6)

$$\tau_N = \sqrt{S^2 - \sigma_N^2} = \sqrt{X_N^2 + Y_N^2 + Z_N^2 - \sigma_N^2} \quad (2.9)$$

-*Stress boundary conditions

$$\bar{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n \quad (2.10a)$$

$$\bar{Y} = \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad (2.10b)$$

$$\bar{Z} = \tau_{xz} l + \tau_{yz} m + \sigma_z n \quad (2.10c)$$

3.Principal stress, maximum and minimum stress of a point

According to description of principal stress plane, we have

$$\tau_N = 0, \quad \sigma_N = \sigma$$

According to equation (2.9)

$$\sigma_N = S$$

Considering equation (2.3)

$$X_N = l\sigma, \quad Y_N = m\sigma, \quad Z_N = n\sigma \qquad (2.11)$$

Take equation (2.11) into equation (2.5), we have

$$\begin{cases} (\sigma_x - \sigma)l + \tau_{yx}m + \tau_{zx}n = 0 \\ \tau_{xy}l + (\sigma_y - \sigma)m + \tau_{zy}n = 0 \\ \tau_{xz}l + \tau_{yz}m + (\sigma_z - \sigma)n = 0 \end{cases} \qquad (2.12)$$

As direction normal, we also know

$$l^2 + m^2 + n^2 = 1 \qquad (2.13)$$

As homogeneous linear equations with non-zero solution vector, coefficient matrix of equation (2.12) has to feed the following condition.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \qquad (2.14)$$

Considering reciprocal theorem of shear stress, a cubic equation about σ is achieved

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad (2.15)$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

-Maximum and minimum stress of a point

$$\sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3, \tau_{zx} = \tau_{xz} = 0, \tau_{yx} = \tau_{xy} = 0, \tau_{zy} = \tau_{yz} = 0$$

According to equation (2.8), normal stress on a tilted plane is

$$\sigma_N = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 \qquad (2.16)$$

Maximum and minimum values of shear stress can be worked out. In view of equation (2.15) and (2.16), equation (2.9) is rewritten as

$$\tau_N{}^2 = \sigma_1{}^2 l^2 + \sigma_2{}^2 m^2 + \sigma_3{}^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2 \qquad (2.17)$$

Table 2.1 Extreme values of stresses

l	m	n	τ_N^2	σ_N
± 1	0	0	0	σ_1
0	± 1	0	0	σ_2
0	0	± 1	0	σ_3
0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\left(\frac{\sigma_2-\sigma_3}{2}\right)^2$	$\frac{\sigma_2+\sigma_3}{2}$
$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	$\left(\frac{\sigma_1-\sigma_2}{2}\right)^2$	$\frac{\sigma_1+\sigma_2}{2}$
$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	$\left(\frac{\sigma_1-\sigma_3}{2}\right)^2$	$\frac{\sigma_1+\sigma_3}{2}$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \qquad (2.18)$$