Exercises for Fourier Analysis with solutions

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Contents

1 Basics on integral Fourier transform (C09)

1.1 Computation of Fourier transforms

For the following basic functions:

1. **step function:** $f(t) = 1$ **iff** $-0.5 \leq t \leq 0.5$**,** $f(t) = 0$ **elsewhere**
2. **hat function:** $f(t) = (1- \mid t \mid)^+$
3. **pairwise decreasing exponential:**$f(t) = \exp(- \mid t \mid)$
4. **Gaussian probability density function:** $f(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})$

Investigate the following

- **in time domain: check** $f \in L^1$**. Is the function continuous? Compute its norm** 1 **and** ∞**.**
- **in frequency domain: compute its Fourier transform** $\hat{f}$**. Is the Fourier transform in** $\mathcal{C}_0$ **? Is it integrable ?**
- **Compute** : $norm\hat{f}_\infty$**. Check the continuity and the contraction property of Fourier transform from** L^1 **to** $\mathcal{C}_0$

Solution

1. Step function: $f(t) = 1$ iff $-0.5 \leq t \leq 0.5$, $f(x) = 0$ elsewhere
   The function f is in L^1. It is not continuous, but it is bounded and
   $$\| f \|_1 = \| f \|_\infty = 1$$
   In the frequency domain, $\hat{f}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega} = \text{sinc}(\pi\omega)$. This function is continuous and null to infinity but it is not integrable.

   We have $\| \hat{f} \|_1 = \infty$, $\| \hat{f} \|_\infty = 1$.

   The contraction property $\| \hat{f} \|_\infty \leq \| f \|_1$ is checked.
2. Hat function: $f(t) = (1- \mid t \mid)^+$
   The function f is in L^1. It is continuous, bounded though it is not differentiable. We have
   $$\| f \|_1 = \| f \|_\infty = 1$$

In the frequency domain, from the pairwise type of the function, we get

$$\begin{array}{ccc}
\hat{f}(\omega) & = & 2\int_0^1 \cos(2\pi\omega t)(1-t)dt \\
\int_0^1 \cos(2\pi\omega t)dt & = & \frac{\sin(2\pi\omega}{2\pi\omega} \\
\int_0^1 t\cos(2\pi\omega t)dt & = & \frac{1}{4\pi^2\omega^2}\int_0^{2\pi\omega} t\cos t dt \\
\int_0^1 t\cos(2\pi\omega t)dt & = & \frac{1}{4\pi^2\omega^2}[2\pi\omega\sin(2\pi\omega)+\cos(2\pi\omega)-1] \\
\hat{f}(\omega) & = & \frac{\sin^2(\pi\omega)}{\pi^2\omega^2}
\end{array}$$

This function is continuous, null to infinity and integrable.

We have $\| \hat{f} \|_\infty = 1$ and the contraction property is checked.

3. Pairwise decreasing exponential: $f(x) = \exp(- \mid x \mid)$
The function f is in L^1. It is continuous, bounded though it is not differentiable. We have

$$\| f \|_1 = 2, \| f \|_\infty = 1$$

In the frequency domain, from the pairwise type of the function, we get

$$\hat{f}(\omega) = 2\Re\left[\int_0^\infty \exp(-2i\pi\omega t)dt\right] = 2\Re\frac{1}{1+2i\pi\omega} = \frac{2}{1+4\pi^2\omega^2}$$

This function is continuous, null to infinity and integrable.

We have $\| \hat{f} \|_\infty = 2$ and the contraction property is checked.

4. Gaussian probability density function: $f(t) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})$

The function f is in L^1. It is continuous, bounded and infinitely differentiable. We have

$$\| f \|_1 = 1, \| f \|_\infty = \frac{1}{\sqrt{2\pi}}$$

In the frequency domain, we get

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(\frac{1}{2}(-t^2 - 4i\pi\omega t\right)dt = \exp(-2\pi^2\omega^2)$$

This function is continuously differentiable, decrasing quickly to infinity and so integrable.

We have $\| \hat{f} \|_\infty = 1$ and the contraction property is checked.

*Notice that in that exercise we perform integration in the complex plane without any specific problem because we integrate analytic functions in the relevant domain. Complex integral has to be fully taken into account to compute Fourier integrals if the complex function to be integrated presents some pole.

1.2 Elementary properties of Fourier transforms

Prove rigorously the basic properties of linearity and scaling for integrable functions and bounded measures.

Solution

Linearity is obvious. Set $A > 0, f_a(t) = f(at)$ Then

$$\hat{f_a}(t) = \int \exp(-2i\pi\omega t) f(at)dt = \frac{1}{a}\hat{f}(\frac{\omega}{a})$$

1.3 2-dimensional Fourier transform and derivation

One considers the 2-dim Fourier transform which is defined by

$$\hat{f}(\lambda, \mu) = \int\int f(x, y) \exp[-2i\pi(\lambda x + \mu y)]dxdy$$

Let us suppose that $f \in \mathcal{S}$. Compute the Fourier transform of the laplacian Δf of f in function of the Fourier transform of f.

Solution

For one variable, we know that $\hat{f'}(\omega) = 2i\pi\omega\hat{f}(\omega)$ Accordingly, we have

$$(\frac{\partial f}{\partial x})\hat{}(\lambda, \mu) = 2i\pi\lambda\hat{f}(\lambda, \mu)$$

$$(\Delta f)\hat{}(\lambda, \mu) = -4\pi^2(\lambda^2 + \mu^2)\hat{f}(\lambda, \mu)$$

1.4 Norm of a bounded measure

Check the computation of the norms of linear forms defined on $\mathcal{C}^b$ by integral with respect to

1. **a point measure:** $\mu = \sum_{j=1}^{\infty} \alpha_i \delta_{x_j}$ **(**$\| \mu \|_1 = \sum_{j=1}^{\infty} \mid \alpha_j \mid$**)**
2. **a density measure:** $d\mu(t) = f(t)dt$ **(**$\| \mu \|_1 = \int \mid f(t) \mid dt$**)**

Solution

The majorations of the norm are obvious from simple inequalities. We have to check that the max is approached

1. point measure: $\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}$.
   Take as a test function sequence continuous functions h_n such that $\forall t, \mid h_n(t) \mid \leq 1$ and for $i \leq n, h_n(x_i) = \frac{a_i^\star}{|a_i|}$

2. density measure: $d\mu(t) = f(t)dt$.
   If f is continuous, it is enough to take $h(t) = \frac{f(t)^\star}{|f(t)|}$. If it is not, first approach f by a a function sequence (f_n) using the density of $\mathcal{C}_k$ in L^1 and then set $h_n(t) = \frac{f_n(t)^\star}{|f_n(t)|}$

1.5 Fourier transform of a bounded measure

The Fourier transform of a bounded measure μ is the function $\hat{\mu}$ which is defined by

$$\hat{\mu}(\omega) = \int \exp(-2i\pi\omega t)d\mu(t)$$

1. **Prove that $\| \hat{\mu} \|_\infty = \max_\omega | \hat{\mu}(\omega) | \leq \| \mu \|_1$**
2. **Consider first that μ has a bounded support and prove in that case the uniform continuity of $\hat{\mu}$.**
3. **Extend the proof of continuity to the general case using that the real line is an increasing countable union of bounded measurable sets (for instance intervals $[-n, n]$.**

<u>*Solution*</u>

1. The contraction principle is obvious from integral inequality.
2. We have $| \hat{\mu}(\omega) - \hat{\mu}(\varpi) | \leq \int | \exp(-2i\pi\omega t) - \exp(-2i\pi\varpi t) | \, d | \mu | (t)$
   Then if the support of μ is $[-A, A]$, it exists $\eta > 0$ such that

   $$| t | \leq A, | \omega - \varpi | \leq \eta \Rightarrow | \exp(-2i\pi\omega t) - \exp(-2i\pi\varpi t) | \leq \frac{\epsilon}{\| \mu \|_1}$$

   So $| t | \leq A, | \omega - \varpi | \leq \eta \Rightarrow | \hat{\mu}(\omega) - \hat{\mu}(\varpi) | \leq \epsilon$
3. We approach in norm 1, μ by a sequence (μ_n) of compactly supported bounded measures.

   Then $\hat{\mu}$ is the uniform limit (from the Fourier contraction principle) of the sequence $\hat{\mu_n}$ of bounded uniformly continuous functions. So it is uniformly continuous.

2 Convolution (C10)

2.1 Straightforward computation

Compute the convolution $g = f \star f$ of f by itself in the case where $f(t) = 1$ iff $-0.5 \leq t \leq 0.5$, $f(t) = 0$ elsewhere. Check the value of the Fourier transform $\hat{g}$

<u>*Solution*</u>

- If $t \leq 0, g(t) =\mid [t-0.5, t+0.5] \cap [-0.5, 0.5] \mid = (t+1)^+$
- If $0 \leq t, g(t) =\mid [t-0.5, t+0.5] \cap [-0.5, 0.5] \mid = (t-1)^+$

We computed $\hat{g}$ in exercise 1.1 . It is the square of $\hat{f}$, since g is he convolution square of f.

2.2 Gauss pdf convolution

We denote $g_{a,\sigma}$ the Gauss probability density function (pdf)defined by

$$g_{a,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{-(x-a)^2}{2\sigma^2}$$

We want to compute $g_{a,\sigma} \star g_{b,\lambda}$. The straightforward computation is tedious. So use the Fourier transform (we shall see it is invertible):

- **Compute the Fourier transform of $g_{a,\sigma}$**
- **Check with the Fourier transform that the Gauss family $\{g_{a,\sigma}\}$ is invariant by convolution product and compute the resulting parameters.**

<u>*Solution*</u>

We saw that $\hat{g_{0,1}}(\omega) = \exp(-2\pi^2\omega^2)$
Then from the scaling principle $\hat{g_{0,\sigma}}(\omega) = \exp(-2\pi^2\sigma^2\omega^2)$
Then from the Fourier transform of the shift, $\hat{g_{a,\sigma}}(\omega) = \exp(2i\pi a\omega - 2\pi^2\sigma^2\omega^2)$
So $(g_{a,\sigma} * g_{b,\tau}\hat{)}(\omega) = \exp(2i\pi(a+b)\omega - 2\pi^2(\sigma^2+\tau^2)\omega^2) = (g_{a+b,\sqrt{\sigma^2+\tau^2}}\hat{)}(\omega)$
* This result is obtained easily in probability theory where a is the expectation and σ is the standard deviation of the Gaussian probability law

2.3 Stability of $\mathcal{S}$ under convolution and density in L^1

1. **Show that if $\mu \in \mathcal{M}_1$ and $f \in \mathcal{S}$, then $\mu \star f \in \mathcal{C}^\infty$. Compute its derivatives.**

2. **Suppose that $\mu \in \mathcal{M}_1$ with bounded support: $\exists A, \mu\{x :\mid x \mid\geq A\} = 0$. Show that if $f \in \mathcal{S}$, then $\mu \star f \in \mathcal{S}$.**

3. **Set $f(x) = \frac{1}{\pi(1+x^2)}$. It is called the <u>Cauchy probability density</u> and a complex variable integration shows that $\hat{f}(\omega) = \exp(-2\pi \mid \omega \mid)$. Take $g = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$ for the Gauss function. Prove that the density probability $f \star g$ does not belong to $\mathcal{S}$ by showing its Fourier transform is not derivable.**

4. **Let f in L^1. Cut first the tail of f to produce a compactly supported function f_A and then use the convergence of the convolution operator by the Gauss AS (h_n) towards identity to check that $\mathcal{S}$ is dense in L^1.**

<u>*Solution*</u>

1. From Lebesgue derivation of integrals of differntiable functions with respect to a parameter we get easily

$$\frac{d}{dx}(\mu * f)(x) = \frac{d}{dx}\left[\int f(x-y)d\mu(y)\right] = \int f'(x-y)d\mu(y) = \mu * f'(x)$$

since f and its derivative f' are bounded and therefore uniformly μ-integrable. And from the same argument, we have

$$\frac{d^n}{dx^n}(\mu * f)(x) = \mu * f^{(n)})(x)$$

2. We have $\mid (\mu \star f)(x) \mid \leq \parallel \mu \parallel_1 \max_{x-A \leq y \leq x+A} \mid f(y) \mid$
Then if $f \in \mathcal{S}$, when $x \to \infty$,

$$\mid x^n(\mu \star f)(x) \mid \leq \parallel \mu \parallel_1 \frac{(x+A)^n}{(x-A)^n} \max_{x-A \leq y \leq x+A} \mid y^n f(y) \mid \to 0$$

The same is true for any derivative, so $\mu \star f \in \mathcal{S}$.

3. The Fourier transform of $f * g$ is $\hat{f}.\hat{g}$. It is not derivable in 0.

Therefore, since $\mathcal{S}$ is stable by Fourier transform, $f * g \notin \mathcal{S}$.

4. We have $\forall \epsilon, \exists A, f_A(x) = f(x).1_{[-A,A]}(x), \parallel f - f_A \parallel_1 \leq \frac{\epsilon}{2}$.
Then $\exists N, n > N, \parallel g_N * f_A - f_A \parallel_1 \leq \frac{\epsilon}{2} \Rightarrow \parallel f - g_n * f_A \parallel_1 \leq \epsilon$

3 Fourier inversion (C11)

3.1 Riemann lemma

Prove Riemann Lemma: If $f \in L^1, \hat{f} \in \mathcal{C}_0$

1. **Take the Gauss AS (h_n), and show Riemann lemma for $h_n \star f$.**
2. **Let $n \to \infty$ and conclude that $\hat{f} \in \mathcal{C}_0$ as the $\| \,.\, \|_\infty$ limit of $\hat{h_n}\hat{f}$.**

<u>*Solution*</u>

1. $(h_n \star f)\hat{} = \hat{h_n}.\hat{f}$ is the product of a function of $\mathcal{S}$ by a function of $\mathcal{C}_b$, so it belongs to $\mathcal{C}_0$
2. Since $\mathcal{C}_0$ is a Banach space for the norm $\|\|\|_\infty$ of uniform convergence, and since $\| \hat{f} - \hat{f}.\hat{h_n}) \|_\infty \to 0$ from the AS property then $\hat{f} \in \mathcal{C}_0$.

3.2 The hat-based AS

We recall from exercise 2.1 that the hat function g is the convolution square of the step function f. What are the properties of the sequence g_n defined by $g_n(t) = ng(nt)$ and its Fourier transform $\hat{g_n}$?

<u>*Solution*</u>

The sequence (g_n) is in $\mathcal{C}_k$. It has a compact support. The sequence of convolution operators converges to identity weakly and in norm 1. But it is not smooth. So its Fourier transform is not fast decreasing for high frequency.

It cannot be used to prove inversion formula since the Fourier inverse transform of $\hat{g} = \text{sinc}^2$ is difficult to compute explicitly.

3.3 Smooth AS with bounded support

Set $k(x) = \exp\left[\frac{-1}{1-x^2}\right], h(x) = \frac{k(x)}{\int k(y)dy}$. We build $h_n(x) = nh(nx)$. Show this AS has all the good properties to prove Fourier inversion formula. Let $\mathcal{D}$ the space of infinitely differentiable functions with bounded support. Use the AS (h_n) to prove the density of $\mathcal{D}$ in L^1 and in $\mathcal{C}_0$.

<u>*Solution*</u>

Since $(h_n) \in \mathcal{S}$, it is a good AS. To prove the density we have to check that $h_n * f \in \mathcal{D}$. But it is not true generally.

If $g \in \mathcal{C}_k$, the space of continuous functions with compact support, then $h_n * g$ has

a compact support and therefore $h_n * f \in \mathcal{D}$.
So first use the density of $\mathcal{C}_k$ both in L^1 and in $\mathcal{C}_0$:

$$\forall f \in L^1, \forall \epsilon > 0, \exists f_\epsilon \in \mathcal{C}_k \parallel f - f_\epsilon \parallel_1 \leq \frac{\epsilon}{2}$$

$$\exists N, n \geq N, \parallel f_\epsilon - h_n * f_\epsilon \parallel_1 \leq \frac{\epsilon}{2}$$

The density of $\mathcal{D}$ in L^1 is proven. The density in $\mathcal{C}_\prime$ is obtained following he same proof.

4 Fourier transform on L^2 and Energy (C12)

4.1 Convolution in L^2

Show that the convolution product may be defined on $L^2 \times L^2$ with values in L^∞

Solution

$(f * f)(x) = \int f(y)f(x-y)dy$ is well defined and from Schwarz inequality in Hilbert space L^2, we get $| f * f(x) | \leq \| f \|_2^2$

4.2 Convolution by sinc

Show that the convolution $f \in L^2 \to \mathrm{sinc}(\pi.) \star f$ is well defined as a bounded operator from L^2 to L^2.

Solution

We saw in last exercise since $\mathrm{sinc}(\pi.) \in L^2$ that the convolution integral $\mathrm{sinc}(\pi.) \star f$ is well defined but generally convolution is not an operation in L^2.

Here if we take $f \in \mathcal{S}$, it is clear that $\mathrm{sinc}(\pi.) * f$ has a Fourier transform which is $h.\hat{f}$ where h is the step function which is the Fourier-Plancherel transform of $\mathrm{sinc}(\pi.)$

Since the multiplication operator $\hat{f} \in L^2 \to h.\hat{f} \in L^2$ is a bounded linear operator its Fourier inverse representation $f \in L^2 \to \mathrm{sinc}(\pi.) * f \in L^2$ is also an equivalent bounded linear operator (same norm, same spectrum).

Note: The convolution by $\mathrm{sinc}(\pi.)$ is the ideal low band filter of engineers

5 Application to signal theory (C13)

5.1 Energy localization

1. **Show that the time-frequency localization product $(\Delta t)_f.(\Delta\omega)_f$ is invariant by time shift and scaling of the signal f. It only depends on the shape of the signal.**

2. **Compute the energy localization of a Gaussian signal and check it is optimum in term of the product $\Delta t.\Delta\omega$**

3. **Compute the energy localization of the ideal low-pass filter impulse response and compare its value to the same term for the Gaussian.**

4. **Compute the same for the RC impulse response $t \to 1_{\mathcal{R}^+}(t)\exp(-\frac{t}{RC})$.**

<u>*Solution*</u>

1. Time shift in time domain change by a translation time-localization of energy so it does not change Δt. In frequency domain, it induces phase shift of frequency representation so it does not change frequency-localization of energy.
   Scaling $f_a(t) = f(at)$ changes the time-localization of energy: $(\Delta t)_{f_a} = \frac{1}{a}(\Delta t)_f$. In frequency domain, $\hat{f}(\omega) = \frac{1}{a}\hat{f}(\frac{\omega}{a})$. So $(\Delta\omega)_{f_a} = a(\Delta\omega)_f$. To conclude $(\Delta t)_{f_a}.(\Delta\omega)_{f_a} = (\Delta t)_f.(\Delta\omega)_f$

2. For Gaussian signal $h(t) = \frac{1}{\sqrt{2\pi}}\exp(\frac{-t^2}{2}), \Delta t = 1$. The frequency localization is given by $\hat{h}(\omega) = \exp(-2\pi^2\omega^2) = \exp(\frac{-4\pi^2\omega^2}{2})$. So $\Delta\omega = \frac{1}{2\pi}$

3. The step function has a frequency representation of sinc type which has an infinite standard deviation (too much energy distributed upon large frequencies)

6 Basics on Laplace transform (C14)

6.1 Computation of Laplace transforms

Compute some of the Laplace transforms of the table We have $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp -t^2 dt$

Original (t)	Laplace transform (p)
$Y(t) = 1_{R^+}(t)$	$\frac{1}{p}$
$Y(t)\exp(-at)$	$\frac{1}{p+a}$
$Y(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\exp(-t)$	$\frac{1}{p^\alpha}$
$Y(t)\frac{\sin at}{a}$	$\frac{1}{p^2+a^2}$
$Y(t)\cos at$	$\frac{p}{p^2+a^2}$
$Y(t)\frac{exp(\frac{-t^2}{4})}{\sqrt{\pi}}$	$\exp(p^2)[1-\mathrm{erf}(p)]$
$Y(t)\mathrm{erf}(t)$	$\exp(\frac{p^2}{4})[1-\mathrm{erf}(\frac{p}{2})]$

6.2 Laplace final value theorem

Let $f \in L^+$. Show that if $\lim_{t\to\infty} f(t) = \eta$ then $\alpha_f \leq 0$ and $\eta = \lim_{p\to 0} p\mathcal{L}f(p)$
<u>Solution</u>

6.3 Symbolic calculus

Show that the solution of the constant coefficient differential equation $y^{(n)} - a_1 y^{(n-1)} - ... - a_n y = u$ with null initial conditions for $y^{(n-1)}(0) = ... = y(0) = 0$ is $y = (Y.e^{\lambda_1 t}) \star ... \star (Y.e^{\lambda_n t})$ where $\{\lambda_j\}_j$ are the roots of the characteristic polynomial $\lambda^n - a_1\lambda^{n-1} - ... - a_n = 0$.

<u>Solution</u>

7 Spectral theory of operators(C15)

7.1 Norm and spectral radius

Show that if $A \in \mathcal{L}(\mathcal{H})$, then $\mid \lambda \mid > \parallel A \parallel \Rightarrow (A - \lambda I)$ is invertible.

<u>*Solution*</u>

We have $\parallel \frac{1}{\lambda} A^n \parallel \leq \frac{\|A\|^n}{|\lambda|^n}$

So the power series $\Sigma_n \frac{A^n}{\lambda^n}$ converges in operator norm towards $(I - \frac{A}{\lambda})^{-1}$. That means that $A - \lambda I) = \lambda(\frac{A}{\lambda} - I)$ is invertible and therefore $\lambda \notin \mathrm{Sp}(A)$.

7.2 Spectral properties of normal operators

We consider a continuous normal operator $A \in \mathcal{L}(\mathcal{H})$.

1. **If for $\lambda \in C$, there exists (x_n) with $\parallel x_n \parallel = 1$ and $\parallel Ax_n - \lambda x_n \parallel \rightarrow 0$ then we say that λ is an approached eigenvalue of A. Prove that in this case λ lies in the spectrum of A , i.e. $A - \lambda I$ is not invertible.**

2. **Show that A is invertible if and only if it exists c such that**

$$\forall x \in \mathcal{H}, \parallel Ax \parallel \geq c \parallel x \parallel$$

3. **Show that the spectrum of A is the set of approached eigenvalues.**

4. **Show that if A is self-adjoint, its spectrum consists in real values.**

<u>*Solution*</u>

1. We have
$$1 = \parallel x_n \parallel \leq \parallel A - \lambda I)^{-1} \parallel . \parallel (A - \lambda I)x_n \parallel$$
So if $(A - \lambda I)$ is invertible, λ cannot be an approached eigenvalue, i.e. the set of approached eigenvalues is included in the spectrum of the operator.

2. If A is invertible then
$$\parallel x \parallel \leq \parallel A^{-1} \parallel \parallel Ax \parallel \Rightarrow \parallel Ax \parallel \geq \frac{1}{\parallel A^{-1} \parallel}. \parallel Ax \parallel$$
Inversely if $\parallel Ax \parallel \geq c \parallel x \parallel$ then A is injective. As A is normal, $\parallel Ax \parallel = \parallel A * x \parallel$, so
$$\mathrm{Ker}(A) = \mathrm{Ker}(A^*) \Rightarrow \overline{\mathrm{Im}(A)} = \overline{\mathrm{Im}(A^*)} = \mathrm{Ker}(A)^{\perp} = 0$$
So $\mathrm{Im}(A)$ is dense.

Now it is clear that $\mathrm{Im}(A)$ is closed. Take $Ax_n \to y$, then (Ax_n) is a Cauchy sequence, so from the assumed inequality, (x_n) is a Cauchy sequence too. Then $x_n \to x$ and $y = Ax$.

So A is bijective, we can take the inverse linear mapping A^{-1}, it is a bounded operator and $\| A^{-1} \| \leq c$.

3. We can apply the previous results to $A - \lambda I$. It shows that the spectrum is identical to the set of approached eigenvalues because

   - either λ is an approached eigenvalue of A,
   - or $\exists c = \min_{\|x\|=1} \frac{\|Ax-\lambda x\|}{\|x\|} > 0$ et donc $\forall x, \| Ax - \lambda x \| \geq c \| x \|$

4. A self-adjoint operator A cannot have no-real approached eigenvalue.

   Indeed,

$$\forall \lambda \in \mathcal{R}, \| A - \lambda ix \|^2 = \| Ax \|^2 + | \lambda |^2 \| x \|^2 \geq | \lambda |^2 \| x \|^2$$

   So from last question a self-adjoint operator cannot have non-real elements in his spectrum

7.3 Inverse of differential operator

1. **Show that for periodic $\mathcal{C}^\infty$ functions, the differential operator is not invertible. kernel operator which inverts differentiation is an Hilbert-Schmidt operator.**

2. **Show that the differential operator operator $(\mathcal{S}, D)$ is invertible on $\mathcal{S}$and that its inverse is a kernel operator which is not integrable.**

Solution

1.

2. The inverse of the differential operator if it exists

8 Application to partial derivative equations (C16)

8.1 Application of convolution to heat conduction

For $t \geq 0$, set $f_t = \frac{1}{2\sqrt{\pi t}} \exp -\frac{x^2}{4t}$. Show that for any $g \in L^1$, $u(t,x) = f_t \star g$ is the solution of the heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x)$$

such that the initial condition is $u(0,x) = \lim_{t \to 0^+} u(t,x) = g(x)$

- Apply the theorem about derivative of a convolution product to check the partial derivative equation.
- Show the convergence of the limit of integral by splitting the integration domain.
- Check the solution in the frequency domain