Variational formulation of elliptic problems

MA31-Numerical analysis of Partial Derivative Equations: Courses 09-10

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Outline

1. Introduction
2. Theoretical basis
   - Lax-Milgram theorem
   - Definitions of H^1 and H_0^1
3. Variational formulation
   - Variational formulation of the Laplace equation
   - Variational formulation of Neumann problem

Objective of the course

- to be able to transform an elliptic PDE problem with boundary condition into a variational problem,
- to know the underlying mathematical structures,
- to be prepared to understand the principle of Finite element method and inner energetic approximations.

Outline

Lax-Milgram theorem

Definition

a bilinear continuous form a on a Hilbert space $\mathcal{H}$ is said ***coercive*** *when* $\exists m > 0 \text{ such that } a(u,u) \geq m \parallel u \parallel^2$

Theorem

Let a be a bilinear continuous coercive form and L a linear continuous form on $\mathcal{H}$, then there exists a unique $u \in \mathcal{H}$ such that

$$\forall v \in \mathcal{H}, a(u,v) = L(v)$$

Moreover, if a is symmetric, u is the minmizer of the quadratic form $J(v) = \frac{1}{2}a(v,v) - L(v)$

Proof of Lax-Milgram theorem

- Since a is a bilinear continuous form, from the Riesz representation theorem it exists a unique linear continuous operator $A \in \mathcal{L}(H)$ such that

$$\forall (u, v) \in \mathcal{H} \times \mathcal{H}, a(u, v) = (Au \mid v)$$

- Moreover we get $\| Au \| \geq m \| u \|$ from coercivity and Cauchy-Schwarz inequality. This proves that A is invertible in $\mathcal{L}(H)$ (linear algebra proof).
- From the Riesz representation theorem there exist $b \in \mathcal{H}$ such that $\forall v \in \mathcal{H},\ (b \mid v) = L(v)$.
- Then solving $\forall v \in \mathcal{H}, a(u, v) = L(v)$ amounts to solve $Au = b$ which is solved by $u = A^{-1}b$
- We have $J(u + v) = J(u) + \frac{1}{2}a(v, v) + a(u, v) - L(v)$.

Outline

Definition of H^1

Definition

Let Ω be a 'regular domain' of $\mathbb{R}^n$ (with smooth boundary) and let $\mathcal{H}^1$ be the subspace of $u \in {}^2(\Omega, dx)$ such that $\vec{\nabla}(u) \in L^2(\Omega, dx)$ where the gradient is a weak derivative in the sense of distribution theory. $\mathcal{H}^1$ is called a ***Sobolev space***

Theorem

Define on $\mathcal{H}^1$ the scalar product

$$(u \mid v)_1 = (u \mid v) + (\vec{\nabla}(u) \mid \vec{\nabla}(v))$$

Then $\mathcal{H}^1$ is a Hilbert space for this scalar product.

Proof of hilbertian structure of $\mathcal{H}^1$

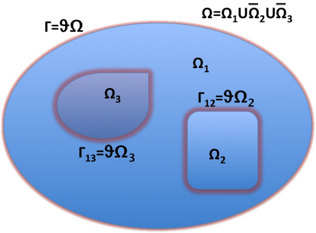
Proof

- Let (u_n) be a Cauchy sequence for $\| \,.\, \|_1$
- Then (u_n) and $(v_n = \vec{\nabla}(u_n))$ are Cauchy sequences in $L^2(\Omega, dx)$ and converges respectively towards u and v
- From the definition of weak derivative in distribution theory

$$\forall \phi \in \mathcal{D}(\Omega), \int_\Omega \vec{\nabla}(u_n)\phi(x)dx = -\int_\Omega u_n \vec{\nabla}(\phi(x))dx$$

- It comes that $v = \vec{\nabla}(u)$, then $u \in \mathcal{H}_1$ and $\| u_n - u \|_1 \to 0$

Piecewise $\mathcal{C}^1$ continuous functions



Definition

*Let Ω be a smooth bounded open domain of $\mathbb{R}^n$ and u a piecewise derivable function which is continuous on $\overline{\Omega}$. A **piecewise $\mathcal{C}^1$ function** is such that there is a finite partition of Ω defined by disjoint open sets with smooth boundaries Ω_i checking $\overline{\Omega} = \cup_i(\overline{\Omega_i})$ such that the restrictions u_i of u to $\overline{\Omega_i}$ are $\mathcal{C}^1(\overline{\Omega_i})$.*

Continuous piecewise C^1 functions are in H^1

Theorem

Let u be a continuous piecewise $\mathcal{C}^1$ function on a bounded support with smooth boundary $\overline{\Omega}$. then $u \in \mathcal{H}^1(\Omega)$.

Proof We just have to prove that the weak derivative of u is the L^2 function which restriction on Ω_i is ∇u_i.

$$\int_\Omega u(x)\nabla\phi(x)dx = \sum_i \int_{\Omega_i} u_i(x)\nabla\phi(x)dx$$

$$\int_\Omega u(x)\nabla\phi(x)dx = -\sum_i \int_{\Omega_i} \nabla u_i(x)\phi(x)dx + \sum_{i,j} \int_{\Gamma_{ij}} u(x)\phi(x)\vec{n}dx$$

$$\int_\Omega u(x)\nabla\phi(x)dx = -\sum_i \int_{\Omega_i} \nabla u_i(x)\phi(x)dx$$

Definition of $\mathcal{H}_0^1$

Definition

Let $\mathcal{H}_0^1$ be the closure of $\mathcal{D}(\Omega)$ in $\mathcal{H}^1$.

Roughly speaking the elements of $\mathcal{H}_0^1$ are functions of $\mathcal{H}^1$ which are null on $\partial\Omega$. That cannot be a formal definition since functions of H_1 are defined only almost everywhere and $\partial\Omega$ is a smooth boundary of null Lebesgue measure. Thus we have

Theorem

Let $u \in \mathcal{H}^1 \cap \mathcal{C}^0(\overline{\Omega})$, then we have the equivaence

$$u \in \mathcal{H}_0^1 \Leftrightarrow \forall x \in \partial\Omega, u(x) = 0$$

Main properties of $\mathcal{H}_0^1$

Theorem

Any compactly supported function $u \in \mathcal{H}^1$ belongs to $\mathcal{H}_0^1$

Proof To prove this theorem, we use an approximate convolution unit $\phi_n \in \mathcal{D}(\Omega)$. Then $(\phi_n * u)$ is compactly supported for n large enough, smooth and converges towards u whe, $n \to \infty$. So $u \in \mathcal{H}_0^1$

Theorem

We have the following Green formula on $\mathcal{H}_0^1$

$$\forall (u,v) \in \mathcal{H}_0^1, \int_\Omega \vec{\nabla}(u) v(x) dx = -\int_\Omega u(x) \vec{\nabla}(v)(x) dx$$

Proof We use the density of $\mathcal{D}(\Omega)$ in H_0^1 and pass to the

The application trace γ

Theorem

*The linear continuous restriction operator γ of $\mathcal{C}^0(\overline{\Omega}) \cap \mathcal{H}^1$ into $\mathcal{C}^0(\overline{\Omega}) \cap \mathcal{H}^1$ into $\mathcal{C}(\partial\Omega))$ is prolongated into a linear continuous operator of $\mathcal{H}^1$ into $L^2(\Omega, ds)$ which is called the **trace operator**. The kernel of the trace operator γ is $\mathcal{H}_1^0$*

The trace operator allows to formulate the Green formula on a smooth bounded domain:

Theorem

For all $(u, v) \in \mathcal{H}^1 \times \mathcal{H}^1$, we have

$$\int_\Omega \vec{\nabla} u(x) v(x) dx = \int_{\partial\Omega} u(x) v(x) \vec{n}_x ds - \int_\Omega u(x) \vec{\nabla} v(x) dx$$

Poincaré inequality on $\mathcal{H}_D^1(\Omega)$

Theorem

Let a regular partition of the boundary of a smooth bounded domain Ω $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and let

$$\mathcal{H}_D^1 = \{v \in \mathcal{H}_D^1 \text{ such that } \gamma(v).1_D = 0\}$$

Then $\exists \alpha > 0$ *such that* $\int_\Omega \mid u(x) \mid^2 dx \leq \alpha^2 \int_\Omega \| \vec{\nabla}(u) \|^2 dx$

The general proof is difficult. To give an idea of the proof consider $\Omega = [a, b]$ and $\partial\Omega_D = \{a\}$. Then,

$$\mid u(x) \mid = \mid \int_a^x u'(t)dt \mid \leq \sqrt{\int_a^b \mid u'(t) \mid^2 dt}$$

$$\int^b \mid u(x) \mid^2 dx \leq (b-a) \int^b \mid u'(t) \mid^2 dt$$

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Homogeneous Dirichlet boundary condition

Recall that from Stokes formula

$$\forall(u,v)\in\mathcal{H}_0^1(\Omega)\times\mathcal{H}_1^0(\Omega),\int_\Omega\vec{\nabla}(u)(x).\vec{\nabla}(v)(x)dx=-\int_\Omega\Delta u(x)v(x)dx$$

Theorem

Let Ω be an open bounded subset of $\mathbb{R}^n$ and $f\in L^2(\Omega)$. Then there exists a unique $u\in\mathcal{H}_0^1(\Omega)$ such that

$$\forall v\in\mathcal{H}_0^1(\Omega),\int_\Omega\vec{\nabla}(u)(x)\,.\,\vec{\nabla}(v)(x)dx=\int_\Omega f(x)v(x)dx$$

Proof We want to apply Lax-Milgram theorem with $a(u,v)=\int_\Omega\vec{\nabla}(u)(x)\,.\,\vec{\nabla}(v)(x)dx$ and $L(v)=\int_\Omega f(x)v(x)dx$

Proof of the solution of Laplace equation

- It is clear from Cauchy-Schwarz inequality that a and L are continuous.
- Poincaré inequality shows that the bilinear form a is coercive. since it gives

$$\| u \|_{\mathcal{H}_1}^2 = \int_\Omega \left[| u^2(x) | + \| \vec{\nabla}(u) \|^2 \right] dx \leq (1+\alpha^2) a(u,u)$$

Remark

Notice that the solution u minimizes the energy functional

$$J(v) = \frac{1}{2} \int_\Omega \| \vec{\nabla}(v)(x) \|^2 \, dx - \int_{\partial\Omega} f(x) v(x) ds$$

Actually, for elastic deformation, the variational formulation amounts to the principle of virtual work.

Outline

Elliptic equation with Neumann condition

Problem

Find $u \in \mathcal{H}^1(\Omega)$ such that

$$\begin{cases} \forall x \in \Omega, & -\Delta u(x) + a(x)u(x) = f(x) \\ \forall x \in \partial\Omega, & \frac{\partial u}{\partial n}(x) = g(x) \end{cases}$$

We suppose $f, g \in L^2(\Omega, dx)$ and $\forall x \in \Omega, 0 < m \leq a(x) \leq M$. Moreover we suppose that $\exists \hat{g} \in \mathcal{H}^1$ with $g = \gamma(\mathcal{H}_1)$. Then it exists a variational formulation of Neumann problem

Problem

Find $u \in \mathcal{H}_1(\Omega)$ such that $\forall v \in \mathcal{H}^1(\Omega)$,

$$\int_\Omega \left[\vec{\nabla}(u)(x) \,.\, \vec{\nabla}(v) + a(x)u(x)v(x)\right] dx = \int_\Omega f(x)v(x)dx + \int_{\partial\Omega} g(x)v(x)ds$$

Summary

- The Sobolev spaces provide the framework where PDE problems with boundary conditions are well-posed.
- For linear equation, Lax-Milgram theorem provides a powerful tool to solve variational formulations.
- The difficult point to apply Lax-Milgram theorem for elliptic equations is to check coercivity property from boundary conditions. Various inequalities which are difficult to prove where found to solve various problems.
- Poincare inequality is very useful for that purpose.

For Further Reading I

- Grégoire Allaire
  Analyse numérique et optimisation .
  Ecole Polytechnique 2005
- Bernard Larrroutourou, Pierre-Louis Lions
  Optimisation et analyse numérique .
  Ecole Polytechnique 1995