Modeling functional framework of NAPDE

MA31-Numerical analysis of PDE: Courses 07-08

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Outline

Modeling tools for numerical analysis

- In numerical computation with computers, the objective changes: the "exact solution" is no more searched. Once its existence is ensured, an efficient approximation is the objective of the computation.
- The solution of the problem is no more individual. What is important is to identify a space of solutions and to move in this space towards "the exact solution" with a compromise between accuracy of approximation and computational resources (time and information size).
- Algorithms are the objective of research. Mathematical models analysis allows to use algorithms in other application domains: (example: finite element analysis was first developed for structure computation and then carried in other fields that are based on variational formulations.

Objectives of the course

- MA31 is basically a course of numerical analysis for partial differential equations.
- These equations are present in every field of aeronautical engineering
- A lot of software are already existing, other are currently produced by searchers and engineers in industry .
- It is also important to combine various software in "platforms".
- This course gathers mathematical tools that will be used further in MA31.
- Functional spaces and Fourier analysis have been already introduced in MA11. Optimization have been reviewed in MA12. Distributions are quickly introduced for further use.

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Definition of Banach spaces

Definition

*A **Banach space** $(\mathcal{B}, \| \, . \, \|)$ is a complete normed vector space (i.e. every Cauchy sequence is convergent)*

- Every real or complex finite-dimensional vector space is complete. Therefore it is a Banach space.
- The space $\mathbb{R}^{(\mathbb{N})}$ of real sequences null except a finite number of terms is not complete for norms $\| \, . \, \|_1$ and $\| \, . \, \|_\infty$
- The space of the differentiable functions on $[a, b]$ which is noted by $\mathcal{C}^1([a, b])$ with the norm of uniform convergence $\| f \|_\infty = \sup_{x \in [a,b]} | f(x) |$ is not complete.

Banach space of continuous functions

Theorem

Let (f_n) be a Cauchy sequence of continuous function on a compact K for the norm $\| \, . \, \|_\infty$ of uniform convergence. Its limit is continuous.

- The space $\mathcal{C}(K)$ of continuous functions on $[a, b]$, more generally on a compact subsect K of $\mathbb{R}^d$, endowed with the norm of uniform convergence is A Banach space.
- The space $\mathcal{C}_0(\mathbb{R}^d)$ of continuous functions going to 0 at ∞ endowed with $\| \, . \, \|_\infty$ is a Banach space.
- The space $\mathcal{C}_k(\mathbb{R}^d)$ of continuous functions with compact support is dense for $\| \, . \, \|_\infty$ in $\mathcal{C}_0(\mathbb{R}^d)$. It is not a Banach space.

Banach space of bounded measures

Definition

A bounded measure μ on $\mathbb{R}^d$ is defined by a module 1 measurable function $\arg(\mu)$ *and a positive bounded measure* $\mid \mu \mid$. *A bounded measure μ is used*

- *to define the measure of any measurable set:* $\mu(A) = \int_A \arg(\mu)(x) d \mid \mu \mid (x)$
- *to define the integral of any measurable bounded function:* $\int f(x) d\mu(x) = \int f(x) \arg(\mu)(x) d \mid \mu \mid (x)$

- The space $\mathcal{M}^1(\mathbb{R}^d)$ of bounded measures on $\mathbb{R}^d$ endowed with the norm $\| \mu \|_1 = \mid \mu \mid (\mathbb{R}^d)$ is a Banach space.
- The space of integrable functions $L^1(\mathbb{R}^d, dx)$ is a Banach subspace of $\mathcal{M}^1(\mathbb{R}^d)$ (measures with density)

Linear mappings and duality

- A linear mapping A of a normed space $\mathcal{E}$ into a Banach space $\mathcal{F}$ is continuous iff

$$\exists k > 0, \forall x \in \mathcal{E}, \| Ax \|_{\mathcal{F}} \leq k \| x \|_{F}$$

- The vector space $\mathcal{L}(\mathcal{E}, \mathcal{F})$ of continuous linear mappings of $\mathcal{E}$ into $\mathcal{F}$ endowed with the norm $\| A \| = \sup_{\|x\|_{E} \leq 1} \| Ax \|_{F}$ is a Banach space.
- Notably, if $\mathcal{F}$ is the basic field ($\mathbb{R}$ ou $\mathbb{C}$), the Banach space $\mathcal{L}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^*$ is called the dual space of $\mathcal{E}$.
- For instance, $M^1(\mathbb{R}^d)$ is the dual of $\mathcal{C}_0(\mathbb{R}^d)$.
- $L^p(\mathbb{R}^d, dx)$ is the dual space of $L^q(\mathbb{R}^d, dx)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Banach fixed point theorem

Why is completeness of functional space necessary ?

Banach fixed point theorem

Let T be a contraction mapping of a Banach space. Then it admits one and only one fixed point solution of $Tx = x$

Proof: It is shown that the sequence of iterates $T^n(x)$ is Cauchy.

Cauchy-Lipschitz theorem

The initial value ODE $y'(t) = f(t, y);\ y(t_0) = y_0$ admits a local unique solution provided f is Lipschitz continuous.

Proof:The fixed point theorem is applied in $\mathcal{C}([t_0 - \epsilon, t_0 + \epsilon])$

$$T(\phi)(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

Compactness

Compactness is a key concept in analysis.

- One can extract from any sequence in a compact a convergent subsequence.
- The continuous image of a compact set is compact
- So, a continuous real function is bounded on a compact and reaches its bounds

Unfortunately, the Riesz theorem prevents a straightforward transition from finite-dimensional analysis to functional analysis:

Riesz theorem

The unit ball of a normed space $\mathcal{E}$ is compact (for normed topology) if and only if $\mathcal{E}$ is finite-dimensional. In that case, the compact sets are the bounded closed subsets of $\mathcal{E}$.

Weak topology

Definition of weak topology

Let $\mathcal{E}$ a Banach space and $\mathcal{E}^*$ its dual space.
The weak topology $\sigma(E, E^*)$ on E is defined by :

$$x_n \rightarrow_w x \Leftrightarrow \forall f \in \mathcal{E}^*, < f, x_n > \rightarrow < f, x >$$

The weak topology $\sigma(E^*, E)$ on $\mathcal{E}^*$ is defined by*

$$f_n \rightarrow_{w^*} f \Leftrightarrow \forall x \in \mathcal{E}, < f_n, x > \rightarrow < f, x >$$

For instance, the topology of law convergence in probability is a weak* topology. The major fact is the Banach-Alaoglu theorem which ensures that **the unit ball of the dual space is weakly* compact**

Dirac approximation and density theorems

Recall that in Fourier analysis, we obtained approximation theorems through weak convergence and convolution.

Definition

Let h be a positive function of the Schwarz space $\mathcal{S}$ such that $\int h(x)dx = 1$. Then we define the sequence h_n in $\mathcal{S}$ by

$$h_n(x) = nh(nx)$$

(h_n) is called an approximating sequence (AS) with base h.

Proposition

h_n is converging towards the Dirac measure δ_0 for the weak topology $\sigma(\mathcal{M}^1, \mathcal{C}_0)$

Outline

Definition and main properties

Definition of Hilbert spaces

*A **Hilbert space** is a Banach space where the norm comes from a scalar product:* $\| \xi \| = \sqrt{< \xi \mid \xi >}^{\frac{1}{2}}$

The main properties of a Hilbert space $\mathcal{H}$ are the orthogonal projection and the isomorphism with dual:

- Let $\mathcal{K} \subset \mathcal{H}$ be a sub-Hilbert space, then for any $x \in \mathcal{H}$, there exists $Px \in \mathcal{K}$ such that $x - Px \in \mathcal{K}^{\perp}$. The linear mapping P is called the **orthogonal projection (projector)** onto $\mathcal{K}$.
- For all $x \in \mathcal{H}$ we define the linear form x^t on $\mathcal{H}$ by $x^t(\xi) =< x \mid \xi >$. The application $x \to x^t$ is an isometric isomorphism of $\mathcal{H}$ onto its dual.

Orthonormal basis

A practical property of Hilbert spaces is the existence of orthonormal basis noted (ξ_n). One has:

- Each $x \in \mathcal{H}$ may be developped as $x = \sum_n < \xi_n \mid x > \xi_n$. This sum converges in norm and is subject to truncation in numerical approximation.
- Then if $A \in \mathcal{L}(\mathcal{H})$ is a linear continuous operator of $\mathcal{H}$,

$$\forall x \in \mathcal{H}, Ax = \sum_n < \xi_n \mid x > A\xi_n = \sum_{n,p} < \xi_n \mid x >< \xi_p \mid A\xi_n > \xi_p$$

- In numerical analysis it is crucial to choose relevant basis to minimize the number of terms in the approximation. The approximation has to be adapted to the problem.
- The idea is to project the problem onto a finite dimensional space

Self-adjoint operators

Definition

Let $\mathcal{H}$ a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$, then the ***adjoint*** *A^* of A is the linear operator which is defined by*

$$\forall (x, y) \in \mathcal{H} \times \mathcal{H}, < Ax \mid y >=< x \mid A^* y >$$

A is said ***self-adjoint*** *iff $A = A^*$, i.e.*

$$\forall (x, y) \in \mathcal{H} \times \mathcal{H}, < Ax \mid y >=< x \mid Ay >$$

The set of $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible is called the ***spectrum*** *of A. It is a non-void compact subset of $\mathbb{C}$ bounded by $\| A \|$ and it is noted $\mathrm{Sp}(A)$. The spectrum of a self-adjoint operator is real.*

Spectral theory

Spectral theory is a generalization of matrix diagonalization. If an orthonormal basis (λ_n, ξ_n) of eigenvalues-eigenvectors if found for a normal operator A, one gets

$$Ax = \sum_n \lambda_n < \xi_n \mid x > \xi_n$$

This expression is easy for further processing as inversion ...
The general spectral theorem for normal operators is less practical since there may be **continuous spectrum**:

Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ a normal bounded linear operator. There exists $(\Omega, \mathcal{F}, m)$ such that $\mathcal{H}$ is isomorphic to $L^2(\Omega, \mathcal{F}, m)$ and A is transformed by this isomorphism into a multiplication operator T_f by a bounded measurable function f.

Case of Hilbert-Schmidt operators

Definition

Let $\mathcal{H} = L^2(\Omega, m)$, let $k \in L^2(\Omega \times \Omega, m \otimes m)$, then the linear bounded operator defined by

$$\forall \xi \in \mathcal{H}, K\xi(s) = \int k(s,t)\xi(t)dm(t)$$

*is said a **Hilbert-Schmidt operator** or an **integral operator** of **kernel** k.*

- K is a **compact operator** which transforms bounded weakly convergent sequences into norm-convergent sequences.
- The spectrum of K consists into $\{0\}$ and a collection of eigenvalues (λ_n) which accumulates only on 0.

Case of unbounded operators

Definition

- *Let $\mathcal{D} \in \mathcal{H}$ be a dense vector subspace of $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{D}, \mathcal{H})$. $(A, \mathcal{D})$ is called an* ***operator with domaine*** *$\mathcal{D}$.*
- *Set $\mathcal{D}^* = \{\xi \in \mathcal{H}, such\ that\ \eta \in \mathcal{D} \rightarrow (\xi \mid A\eta) is\ continuous\}$ and for $\xi \in \mathcal{D}^*$, define $A^*\xi$ by $(A^*\xi \mid \eta) = (\xi \mid A\eta)$. The operator $(\mathcal{D}^*, A^*)$ is called the* ***adjoint*** *of $(\mathcal{D}, A)$*

- The concept of self-adjoint and normal operators, the definition of spectrum, the spectral decomposition can be extended to unbounded operators.
- Unbounded operators are represented in spectral decomposition by the multiplication by an unbounded function.

Outline

Analytic solution and characteristics

The following PDE is current in the transport problem, notably in fluid mechanics:

$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = f \\ u(x,0) = u_0(x) \end{cases}$$

- We consider here the case where c is constant and $f = 0$.
- The change of variables: $v(z,t) = u(x - ct, t)$ amounts to equation $\frac{\partial v}{\partial t} = 0 \Rightarrow v(z,t) = v(z) \Rightarrow u(x,t) = u_0(x - ct)$
- The only solution is constant along characteristics $x = ct$.
- If u_0 is not differentiable, it cannot be a classical solution, still it is a physical solution.
- Classical derivation needs to be generalized.

Non-smooth solution

- Take as initial data the step function: $u(0,x) = Y(x) = 1_{x\geq 0}$, it has no derivative.
- Notice that $Y(x) = \int_{-\infty}^{x} d\delta(x)$ where δ is the Dirac measure at 0. Can δ_0 be the derivative of Y ?
- To generalize derivation, notice that if f is $C^1(\mathbb{R})$,

$$\forall\phi \in \mathcal{D}(\mathbb{R}), \int f'(x)\phi(x)dx = -\int f(x)\phi'(x)dx$$

- and for a step function

$$\forall\phi \in \S(\mathbb{R}), \phi(0) = \int \phi(x)d\delta_0(x) = -\int Y(x)\phi'(x)dx = \phi(0)$$

This shows how to derive an element of the dual of the space of differentiable functions which goes to 0 at ∞

Outline

Test-function space

Definition

Ω is an open set of $\mathbb{R}^d$. Let $\mathcal{D}(\Omega)$ be the space of infinitely derivable functions on Ω with compact support. The topology in $\mathcal{D}$ is defined as follows:

$$\phi_n \to_{\mathcal{D}} \phi \Leftrightarrow$$

1. *All the ϕ_n are supported by a common compact set*
2. *$\forall p, \phi_n^{(p)}$ converges uniformly towards $\phi^{(p)}$*

$\mathcal{D}$ is the space of test functions. It is as small as possible (smaller than $\mathcal{C}_0$ for instance) in order for its dual to be as big as possible (larger than $\mathcal{M}^1$).

Definition of a distribution

Definition

A distribution T on Ω is a continous linear form on $\mathcal{D}(\Omega)$ i.e.

$$\phi_n \to_{\mathcal{D}} \phi \Leftarrow < T, \phi_n > \to < T, \phi >$$

The vector space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$.

Definition

Let $T \in \mathcal{D}'(\Omega)$. Then the linear form DT defined on $\mathcal{D}(\Omega)$ by $< DT, \phi > = - < T, \phi' >$ is a distribution. It is called the ***derivative*** *of T.*

These definitions are written for $\Omega \subset \mathbb{R}$. They may be thoroughly extended to several variables and partial derivatives.

Main examples

- Any continuous function is a distribution. Any derivable function is a distribution and its classical derivative is its derivative according to the distribution derivation definition.
- Any bounded measure, any measure that ensures finiteness for compact sets (as Lebesgue measure) is a distribution since $\mathcal{D}(\mathbb{R})$ is embedded continuously in $\mathcal{C}_0(\mathbb{R})$, i.e. $\mathcal{D}$-convergence is stronger than uniform convergence.
- $x \in \mathbb{R}^{+*} \to \log(x)$ is continuous on $\mathbb{R}^{+*}$ thus it defines a distribution on $\mathbb{R}^{+*}$. But $x \to \log(x)1_{x>0}$ is not continuous on $\mathbb{R}$. But this function is a density of a finite measure on all intervals $[a, b]$, thus $\log \in \mathcal{D}(\mathbb{R})$

Derivations of piecewise continuous functions

Definition

A function f on $\mathbb{R}$ is said piecewise $\mathcal{C}^k$ if there exists a partition of $\mathbb{R}$ into intervals $(]a_K, a_{k+1}])$ such that f is C^k on each open interval and that f and its k derivatives admits right and left limits on each a_k which are denoted $f(a_k + 0)$ and $f(a_k - 0)$

Proposition

Let f be a piecewise $\mathcal{C}^1$ function with classical derivative f'. Then its derivative in the distribution sense is given by

$$Df = f' + \sum_k [f(a_k + 0) - f(a_k - 0)]\delta_{a_k}$$

The proof is obtained by a part integration on a test-function ϕ.

Other derivations

- The derivative of the Dirac is δ' defined by $< \delta', \phi >= -\phi'(0)$
- Let us try to derive $1_{\mathbb{R}^+}(x)\log(x)$ on $\mathbb{R}$. It comes

$$
\begin{aligned}
< D\log, \phi >= & \quad -\int_0^\infty \log(x)\phi'(x)dx \\
< D\log, \phi >= & \quad \lim_{\epsilon\to 0}\left[-\log\epsilon\phi(\epsilon) + \int_\epsilon^\infty \frac{\phi(x)}{x}dx\right]
\end{aligned}
$$

This distribution is called $\mathrm{fp}(\frac{1_{\mathbb{R}^+}(x)}{x})$ (Hadamard's finite part)

Outline

Definition of the Schwarz space.

Definition

The Schwarz space $\mathcal{S}$ is the space of functions which are infinitely differentiable and fastly decreasing to infinity, such that $\forall n, p, \lim_{|x|\to\infty} x^n f^{(p)}(x) = 0$. This functional space is embedded in all the classical Banach spaces $L^1, L^2, \mathcal{C}_0, \mathcal{C}_0^{(p)}$.

- Obviously $\mathcal{S}$ is stable by translation, derivation and multiplication by the variable.
- $\mathcal{S}$ is an excellent space where we can look for solution of differentiable or partial derivative equations.
- We proved last year (MA11) that $\mathcal{S}$ is dense in all these functional spaces

Fourier transform and the Schwarz space.

Theorem

$\mathcal{S}$ is stable by Fourier transform and functions of $\mathcal{S}$ check the Fourier inversion formula

Proof

Let $f \in \mathcal{S}$ and define $q_{m,p}(\hat{f}) = \sup_\omega (1+\omega^2)^m \mid \hat{f}^{(p)}(\omega) \mid$ From the exchange of derivation and multiplication through Fourier transform, we get

$$(1+\omega^2)\hat{f}(\omega) = (f - \frac{1}{4\pi^2} f'')\hat{}(\omega) = (Pf)\hat{}(\omega)$$

and more generally

$$(1+\omega^2)^m \mid \hat{f}^{(p)}(\omega) \mid = (P^m g)\hat{}(\omega)$$

where $g(x) = \left[\frac{-x}{2i\pi}\right]^p f(x)$ such that $\hat{g}(\omega) = \hat{f}^{(p)}(\omega)$

Tempered distributions

A topology can be defined on $\mathcal{S}$ such that

$$\phi_n \to_\phi \Leftrightarrow lim_{|x| \to \infty} x^n [\phi_n^{(p)}(x) - \phi^{(p)}(x)] = 0$$

$\mathcal{D}$ is embedded into $\mathcal{S}$ by a continuous inclusion.

Definition

*A **tempered distribution** is a distribution which is continuous for $\mathcal{S}$-topology. So it can be extended to define a continuous form on $\mathcal{S}$. Let $\mathcal{S}'$ be the space of tempered distributions.*

- Distributions with compact support are tempered.
- Bounded measures and square-integrable functions are tempered distributions.
- Continuous functions with polynomial increasing are tempered distributions.

Derivation and multiplication

Definition of product function.distribution

If $f \in \mathcal{C}^\infty$ is polynomially increasing and if $T \in \mathcal{S}'$, define fT by $\forall \phi \in \mathcal{S}, < fT, \phi >=< T, f\phi >$, then $fT \in \mathcal{S}'$

Proof: $f\phi \in \mathcal{S}$ and $f \in \mathcal{S} \to f\phi \in \mathcal{S}$ is continuous

Proposition

If $f \in \mathcal{C}^\infty$ is polynomially increasing and if $T \in \mathcal{S}'$, we have $D(fT) = f'.T + f.DT$

Proof: We have

$$< D(fT), \phi >= - < fT, \phi' >= - < T, f\phi' >$$

$< f'.T, \phi >=< T, f'\phi >$ and

$$< f.DT, \phi >=< DT, f\phi >= - < T, f'\phi > - < T, f\phi' >$$

Convolution

Lemma

If $S \in \mathcal{S}'$ is compactly supported and if $\phi \in \mathcal{S}$, the function $\tilde{S} * \phi$ defined by $(\tilde{S} * \phi(x)) =< S_y, \phi(x+y) >$ is in $\mathcal{S}$, the mapping $\phi \to \tilde{S} * \phi$ is a continuous linear operator of $\mathcal{S}$ and $(\tilde{S} * \phi)^{(p)} = \tilde{S} * \phi^{(p)}$

Definition

If $S \in \mathcal{S}'$ is compactly supported and if $T \in \mathcal{S}'$ we define the convolution product $S * T \in \mathcal{S}'$ by the formula $< S * T, \phi >=< T, \tilde{S} * \phi >$

We have $DT = \delta_0' * T$ and so $D(S * T) = DS * T = S * DT$. More generally, linear differential operators with constant coefficients may be represented by convolution products.

Fourier transform of tempered distributions

Recall that Fourier inversion formula induces that if ϕ and ψ are functions of $\mathcal{S}(\mathbb{R})$, $\int \phi(x)\hat{\psi}(x)dx = \int \psi(\omega)\hat{\phi}(\omega)d\omega$. In the same way, if $\mu \in \mathcal{M}^1$ and if $\phi \in \mathcal{S}$, $\int \hat{\phi}(x)d\mu(x) = \int \phi(\omega)\hat{\mu}(\omega)d\omega$. So,

Definition of Fourier transforms for tempered distributions

Let $T \in \mathcal{S}'$, we define $\mathcal{F}(T) = \hat{T} \in \mathcal{S}$ by $< \hat{T}, \phi > = < T, \hat{\phi} >$

Theorem

- *Fourier transform exchanges convolution and multiplication. More precisely if S is compactly supported, $\hat{S}$ is a C^∞ function; if it is polynomially increasing, we have: $\mathcal{F}(S * T) = \hat{S}.\mathcal{F}(T)$*
- *We have $\mathcal{F}(DT) = 2i\pi\omega.\mathcal{F}(T)$ and $\mathcal{F}(-2i\pi t.T) = D\mathcal{F}(T)$.*

Periodic distributions and Fourier transforms

Distribution theory allows to put in the same frame Fourier integrals and Fourier series. The Fourier transform of a periodic function is a discrete measure supported by the spectrum.

Poisson sum formula

*Let $\mathfrak{d}_T = \sum_{n\in\mathbb{Z}} \delta_{nT}$ be the **Dirac comb** with convergence in $\mathcal{S}'$. Its Fourier transform is a Dirac comb in the frequency domain. Then we have: $\mathcal{F}(\mathfrak{d}_T) = \frac{1}{T}\mathfrak{d}_{\frac{1}{T}}$*

Proof Let $\phi \in \mathcal{S}$, then the function $f(t) = \sum_n \phi(t - nT)$ is periodic and C^∞. So it is the sum of its Fourier series which is converging fast: $\sum_n \phi(t - nT) = \sum_k a_k(f)e^{2i\pi\frac{kt}{T}}$ with $a_k(f) = \frac{1}{T}\int e^{-2i\pi\frac{ks}{T}}\phi(s)ds$. So $\mathfrak{d}_T * \phi = \frac{1}{T}\sum_k e^{2i\pi\frac{k.}{T}} * \phi$
Eventually, $\mathfrak{d}_T = \frac{1}{T}\sum_k e^{2i\pi\frac{k.}{T}} = \frac{1}{T}\mathcal{F}^{-1}(\mathfrak{d}_{\frac{1}{T}})$

Summary

- Banach space structure is recalled, two basic applcation theorems show how it is used. weak topology is introduced.
- Hilbert spaces are recalled with specific focus as orthogonal projection and spectral theory of operators
- Distributions are introduced to generalize derivation and PDE to non smooth objects
- Distributions are used to extend Fourier transform and convolution in order to include derivation and PDE in its application scope as it was originally done for heat equation by Fourier.

For Further Reading I

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