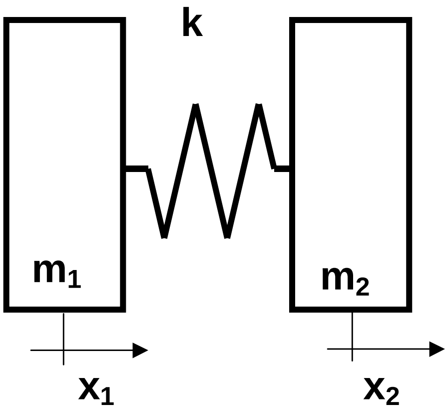
SYSTEME A DEUX DEGRES DE LIBERTE



1 – *Equations de Lagrange*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i}\right) + \left(\frac{\partial U}{\partial q_i}\right) = 0$$

avec ici

$$q_i \quad = \quad x_1 \quad \text{et} \quad x_2$$

1 – 1 Energie Cinétique

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

1 – 2 Energie de déformation

$$U = \frac{1}{2} k (x_1 - x_2)^2$$

Il y a deux équations :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)-\frac{\partial T}{\partial x_1}+\frac{\partial U}{\partial x_1}=0$$

et

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right)-\frac{\partial T}{\partial x_2}+\frac{\partial U}{\partial x_2}=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)=?$$

$$\frac{\partial T}{\partial \dot{x}_1}=m_1\dot{x}_1$$

$$\frac{d}{dt}\left(-\right)=m_1\ddot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2}=m_2\dot{x}_2$$

$$\frac{d}{dt}\left(-\right)=m_2\ddot{x}_2$$

$$\frac{\partial T}{\partial q_i} = ?$$

$$\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial U}{\partial q_i} = ?$$

$$\frac{\partial U}{\partial x_1} = ?$$

$$\frac{\partial U}{\partial x_1} = k\left(x_1 - x_2\right)$$

$$\frac{\partial U}{\partial x_2} = ?$$

$$\frac{\partial U}{\partial x_2} = k\left(x_2 - x_1\right)$$

d'ou :

$$m_1\ddot{x}_1 + k\left(x_1 - x_2\right) = 0$$

$$m_2\ddot{x}_2 + k\left(x_2 - x_1\right) = 0$$

Matrice de raideur

$$K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Matrice de masse

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

2 - Equations du mouvement :

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

$$M\ddot{x} + Kx = 0$$

$x_i = X_i e^{rt}$

$$\begin{bmatrix} m_1 r^2 + k & -k \\ -k & m_2 r^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\}$$

$$m_1 m_2 r^4 + r^2 (m_1 + m_2) k = 0$$

$$r^2 \left(r^2 m_1 m_2 + k(m_1 + m_2) \right) = 0$$

$$r_1^2 = 0 \qquad \text{et} \qquad r_2^2 = -\frac{k(m_1 + m_2)}{m_1 m_2}$$

<u>Rappel</u>

$$\omega_1 = 0 \qquad \text{et} \qquad \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

donc pour

$r_1^2 = 0$

$$\begin{bmatrix} m_1 0 + k & -k \\ -k & m_2 0 + k \end{bmatrix} \begin{Bmatrix} X_{11} \\ X_{12} \end{Bmatrix} = \{0\}$$

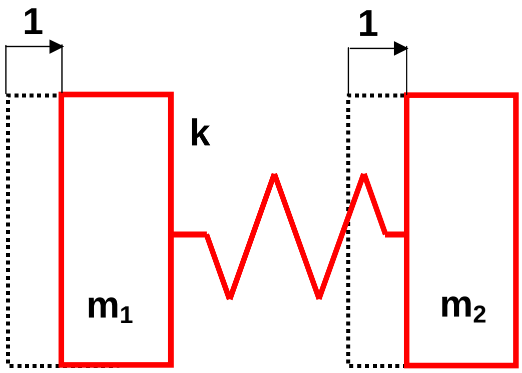
$$\phi_1 = \begin{Bmatrix} X_{11} \\ X_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

et pour

$r_2^2 = -\dfrac{k(m_1 + m_2)}{m_1 m_2}$

$$\begin{bmatrix} -\dfrac{k(m_1 + m_2)}{m_2} + k & -k \\ -k & -\dfrac{k(m_1 + m_2)}{m_1} + k \end{bmatrix} \begin{Bmatrix} X_{21} \\ X_{22} \end{Bmatrix} = \{0\}$$

$$\phi_2 = \begin{Bmatrix} X_{21} \\ X_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -m_1 / m_2 \end{Bmatrix}$$

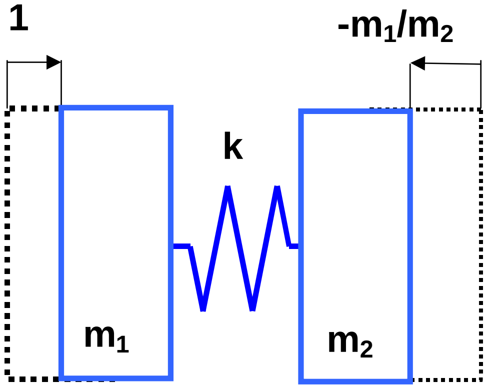


Premier mode (mode du corps rigide)

Rappel :

Pour ce mode, l’ énergie de déformation vaut

$$U = \frac{1}{2}k(\Delta x)^2 = 0$$



Second Mode

<u>Matrice des Modes</u>

$$\{x\} = [\Phi]\{q\}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \dfrac{-m_1}{m_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$[\phi_1 \quad \phi_2]^t \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} [\phi_1 \quad \phi_2] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} +$$

$$[\phi_1 \quad \phi_2]^t \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} [\phi_1 \quad \phi_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{0\}$$

$$\begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_1\left(1+\dfrac{m_1}{m_2}\right) \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k\left(1+\dfrac{m_1}{m_2}\right)^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{0\}$$

$$M\ddot{q} + Kq = 0$$

Les équation sont découplées

$$\sqrt{\frac{K_{11}}{M_{11}}} = 0 = \omega_1$$

et

$$\sqrt{\frac{K_{22}}{M_{22}}} = \sqrt{\frac{k\left(1+\dfrac{m_1}{m_2}\right)^2}{m_1\left(1+\dfrac{m_1}{m_2}\right)}} = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}} = \omega_2$$