Linear solid strain

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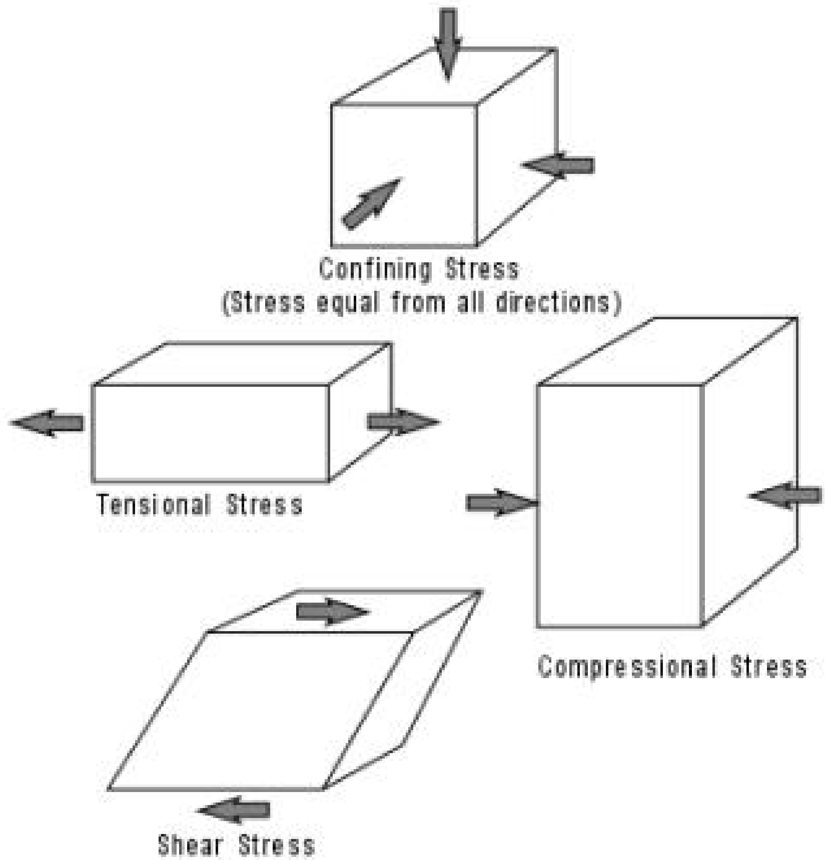

Figure 3.1

-Relation of displacement and strain --- Geometric Equation

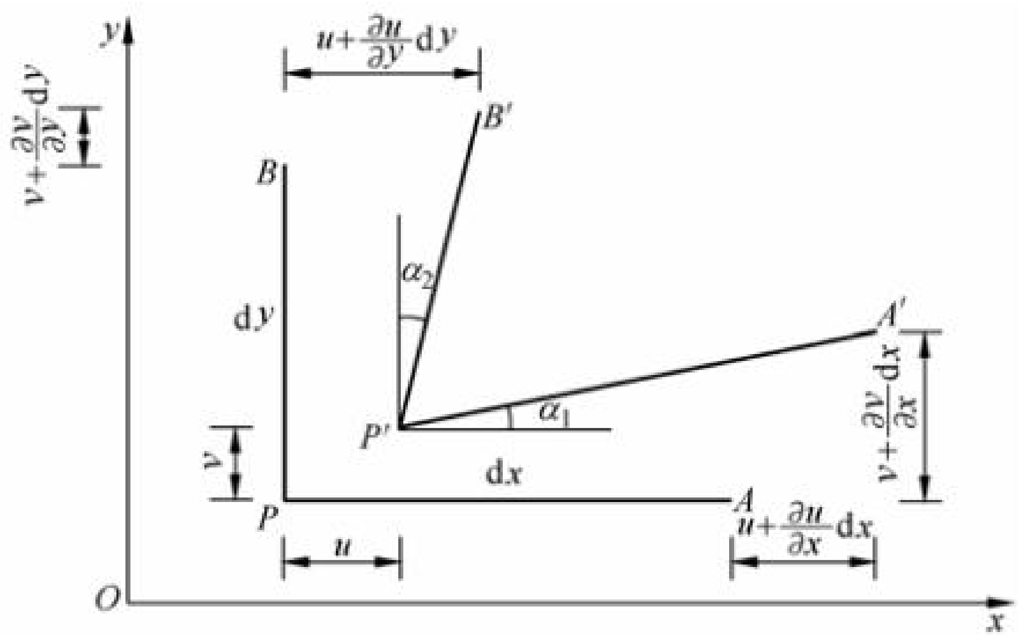

Figure 3.2

$$\varepsilon_x = \frac{\overline{P'A'} - \overline{PA}}{\overline{PA}} \approx \frac{u_A - u}{dx} = \frac{\partial u}{\partial x} \qquad (3.3a)$$

$$\varepsilon_y = \frac{\overline{P'B'} - \overline{PB}}{\overline{PB}} \approx \frac{v_B - v}{dy} = \frac{\partial v}{\partial y} \qquad (3.3b)$$

$$\begin{aligned} \gamma_{xy} &= \alpha_1 + \alpha_2 \\ &\approx \frac{v_A - v}{dx} + \frac{u_B - u}{dy} \\ &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (3.3c) \end{aligned}$$

-*Geometric Equation

$$
\begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} \\
\varepsilon_y = \dfrac{\partial v}{\partial y} \\
\varepsilon_z = \dfrac{\partial w}{\partial z} \\
\gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\
\gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\
\gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}
\end{cases}
\tag{3.4}
$$

Equation group (3.4) is called geometric equations, also known as Cauchy formulations.

-Special case-motion formulation of rigid

To rigid, there is no strain under any condition, we have

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \tag{3.5}$$

Substitute equation (3.5) into equation (3.4),

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \tag{3.6a}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{3.6b}$$

Integrate equation (3.6a)

$$u = f(y,z), v = g(x,z), w = h(x,y) \tag{3.7}$$

Substitute equation (3.7) into equation (3.6b),

$$\begin{cases} \dfrac{\partial g(x,z)}{\partial x} + \dfrac{\partial f(y,z)}{\partial y} = 0 \\ \dfrac{\partial h(x,y)}{\partial y} + \dfrac{\partial g(x,z)}{\partial z} = 0 \\ \dfrac{\partial f(y,z)}{\partial z} + \dfrac{\partial h(x,y)}{\partial x} = 0 \end{cases} \tag{3.8}$$

Compute partial derivative of x to first equation of (3.8), and of z to second equation of (3.8)

$$\begin{cases} \dfrac{\partial^2 g(x,z)}{\partial x^2} = 0 \\ \dfrac{\partial^2 g(x,z)}{\partial z^2} = 0 \end{cases} \tag{3.9}$$

A conclusion can be made that g(x,z) could only have constant term, x term, z term, and xz term. Function g(x,z) can be constructed as

$$g(x,z) = a + bx + cz + dxz \tag{3.10a}$$

Similarly, we have

$$f(y,z) = e + fy + gz + hyz \tag{3.10b}$$

$$h(x,y) = i + jx + ky + lxy \tag{3.10c}$$

Substitute equations (3.10) back into equation (3.8)

$$\begin{cases} k + c + (d + l)x = 0 \\ b + f + (d + h)z = 0 \\ g + j + (h + l)y = 0 \end{cases} \tag{3.11}$$

Solutions are

$$f = -b, c = -k, j = -g, d = h = l = 0$$

Thus, displacement functions of rigid are

$$\begin{cases} f(y,z) = e - by + gz \\ g(x,z) = a + bx - kz \\ h(x,y) = i - gx + ky \end{cases} \tag{3.12}$$

Or, in vector type

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} e \\ a \\ i \end{Bmatrix} - \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times \begin{Bmatrix} k \\ g \\ b \end{Bmatrix} \tag{3.13}$$

Equation (3.13) is obviously motion formulation of rigid.

-Displacement boundary condition

When rigid is researched, it has to feed the condition of zero-strain. According to different research object and constraints, displacement of points on the boundary is sometimes pre-decided to given constraints, which is called displacement boundary condition

$$\begin{cases} u_S = \overline{u} \\ v_S = \overline{v} \\ w_S = \overline{w} \end{cases} \tag{3.14}$$

S refers to boundary surface, $\vec{\mu}$, $\vec{v}$ and $\vec{w}$ are known values of points.

-Strain compatibility equations

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} + \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \dfrac{\partial^2 \varepsilon_y}{\partial z^2} + \dfrac{\partial^2 \varepsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \dfrac{\partial^2 \varepsilon_z}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases} \tag{3.15a}$$

To obtain strain compatibility equation between planes, other partial derivative computation are done to equation (3.4)

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z}$$
$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 v}{\partial z \partial x}$$
$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 w}{\partial x \partial y}$$

Subtract the third equation with summation of first two equations,

$$\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} = 2\frac{\partial^2 v}{\partial x \partial z}$$

Differentiate the above equation by y

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z}$$

Similarly, other two equations are achieved

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{xy}}{\partial z} + \dfrac{\partial \gamma_{yz}}{\partial x} - \dfrac{\partial \gamma_{zx}}{\partial y}\right) = 2\dfrac{\partial^2 \varepsilon_y}{\partial x \partial z} \\ \dfrac{\partial}{\partial z}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{zx}}{\partial y} - \dfrac{\partial \gamma_{xy}}{\partial z}\right) = 2\dfrac{\partial^2 \varepsilon_z}{\partial y \partial x} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial \gamma_{zx}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z} - \dfrac{\partial \gamma_{yz}}{\partial x}\right) = 2\dfrac{\partial^2 \varepsilon_x}{\partial z \partial y} \end{cases} \qquad (3.15b)$$

2.Strain state at a point of body

-Analysis of strain state

Strain state is stain situation of a point inside an elastomer in all directions. As same as stress components, six strain components in any points of an object change their values as rotation of coordinate system. There are also three elastic strain principal directions perpendicular to each other. After deformation, differential lines along these three directions only have changes of lengths and remain right angles among them, which means zero shear strain. These three principal strains are noted as ε_1, ε_2, and ε_3. Considering direction normal (2.3) and hypothesis of small deformation, we have

$$\varepsilon_N = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{yx} ml + \gamma_{zx} nl + \gamma_{zy} nm \qquad (3.16)$$

-Strain invariants

When principal strains are wanted, equations can be built up suppose direction cosine {l m n},

$$\begin{cases}(\varepsilon_x-\varepsilon)l+\frac{1}{2}\gamma_{yx}m+\frac{1}{2}\gamma_{zx}n=0\\ \frac{1}{2}\gamma_{xy}l+(\varepsilon_y-\varepsilon)m+\frac{1}{2}\gamma_{zy}n=0\\ \frac{1}{2}\gamma_{xz}l+\frac{1}{2}\gamma_{yz}m+(\varepsilon_z-\varepsilon)n=0\end{cases} \qquad (3.17)$$

Unknown variables l, m, and n can not be all zeros, hence

$$\begin{vmatrix}(\varepsilon_x-\varepsilon) & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx}\\ \frac{1}{2}\gamma_{xy} & (\varepsilon_y-\varepsilon) & \frac{1}{2}\gamma_{zy}\\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & (\varepsilon_z-\varepsilon)\end{vmatrix}=0 \qquad (3.18)$$

Expanding determinant (3.18), a strain state characteristic formulation is achieved

$$\varepsilon^3-J_1\varepsilon^2+J_2\varepsilon-J_3=0 \qquad (3.19)$$

Where J_1, J_2, J_3 are respectively the first, second, and third strain invariants

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\gamma_{xy}^2 - \frac{1}{4}\gamma_{yz}^2 - \frac{1}{4}\gamma_{xz}^2 \qquad (3.20)$$

$$J_3 = \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix}$$

Equation (3.19) have three real number roots. They are principal strains ε_1, ε_2 and ε_3. The following types of strain invariants are expressed by principal strains

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$J_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \qquad (3.21)$$

$$J_3 = \varepsilon_1\varepsilon_2\varepsilon_3$$

-Principal strains and principal axis of strains

Respectively take three principal strains back into equation group (3.17), direction normal vectors of these principal strains can be solved. If the studied material are isotropic, principal axis of stresses and principal axis of strains are coincident.

-Volumetric strain

Volumetric strain refers to unit volume change of elastomer, noted as θ. After deformation of body, infinitesimal cuboid have new sides' length $(1+\varepsilon_x)dx$, $(1+\varepsilon_y)dy$ and $(1+\varepsilon_z)dz$.

Considering hypothesis of small deformation, volumetric change is

$$V^{'} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)dxdydz$$

Volumetric change is

$$\theta = \frac{dV^{'}}{V} = \frac{(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)dxdydz - dxdydz}{dxdydz}$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x + \varepsilon_x\varepsilon_y\varepsilon_z$$

Ignoring higher infinitesimal, volumetric strain is

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad (3.22)$$

According to geometric formulation (3.4)

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (3.23)$$