中国民航大学中欧航空工程师学院

Module SM55

Modélisation par Eléments Finis

Recueil d’exercices

(avec et sans solutions)

Elements finis

Exercise 1

Ritz's Method & Minimisation of the Total Potential Energy

A beam is clamped on its left side as shown on figure 1. A uniform mass is hung to the points A and B of the beam with 2 cables. The weight of the mass is :

$$Weight \quad \begin{cases} F_x = & 0 \\ F_y = & -2F \\ F_z = & 0 \end{cases}$$

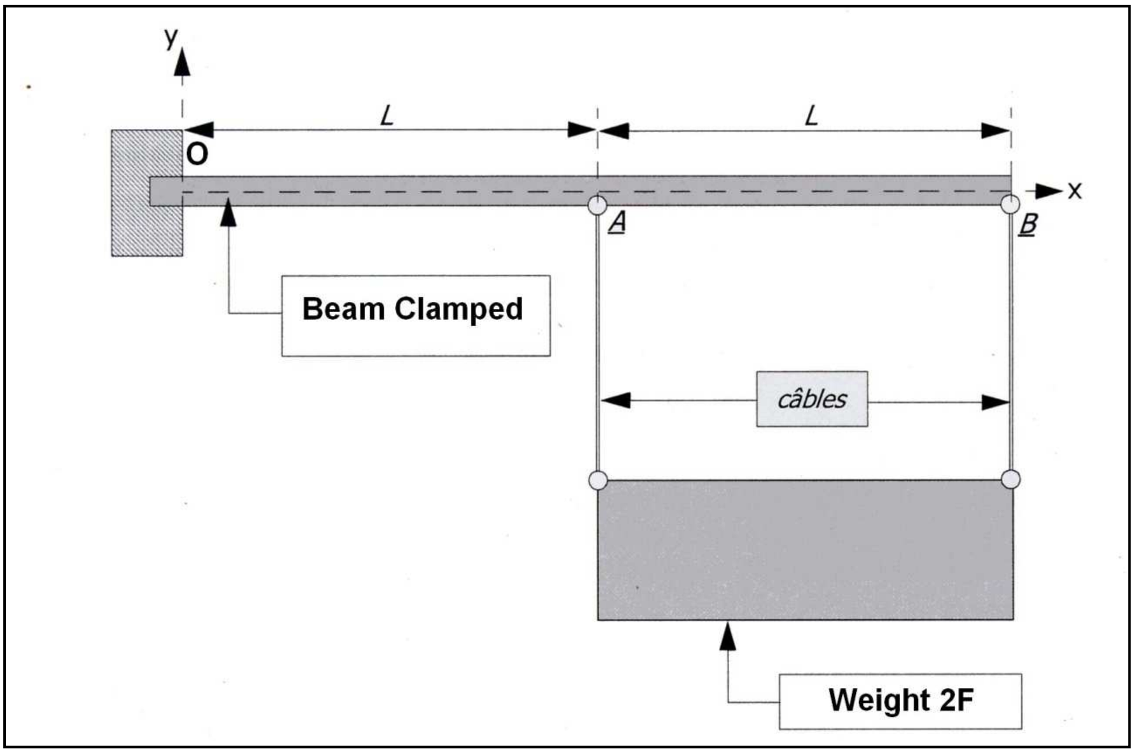


Figure 1 : Description of the structure

Questions :

1 – By writing the equilibrium of the mass, find the forces applied by the cables on the beam. Theses forces will be named $\overrightarrow{F_A}$ and $\overrightarrow{F_B}$

2 – According to the beam theory, write the bending moment for each cross section between O and A and then between A and B,

3 – Write the differential equation that gives a relation between the bending moment and the displacement of the middle line of the beam.

4 – Write the boundary conditions which allow to find the good solution after integration of this differential equation.

The results are :

$$\begin{cases} 0 < x < L & v(x) = \dfrac{F}{6EI_z}\left(2x^3 - 9Lx^2\right) \\ L < x < 2L & v(x) = \dfrac{F}{6EI_z}\left(x^3 - 6Lx^2 - 3L^2x + L^3\right) \end{cases}$$

5 – Using the Ritz method, and the minimum total potential energy theorem, find the vertical displacements of the two points A and B.

5.1 Propose a polynomial function which respect the boundary condition for the middle line of whole beam,

5.2 Compute the internal energy, according to the theory of beam, introduced the bending moment,

5.3 Write this internal energy under a matrix form like this :

$$W = \frac{1}{2}\begin{pmatrix} a & b \end{pmatrix}[K]\begin{pmatrix} a \\ b \end{pmatrix}$$

5.4 Compute the potential energy of the load applied to the beam,

5.5 Write this potential energy under a matrix form like this :

$$V = -\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} \alpha \cdot F_1 \\ \beta \cdot F_2 \end{pmatrix} \text{ with } \alpha \quad and \quad \beta \text{ to compute}$$

5.6 Minimise the total energy, in order to find an expression like this :

$$\begin{pmatrix} \alpha \cdot F_1 \\ \beta \cdot F_2 \end{pmatrix} = [K]\begin{pmatrix} a \\ b \end{pmatrix}$$

5.7 Write the equations which allow to find the different coefficients of the polynomial function of the displacement of the middle line. Solve them.

6 – Compare these displacements with the results given by using the beam theory.

7 – To improve the accuracy of the solution try a new shape function for the middle line of the beam by adding a polynomial term of degree 4.

7.1 Write under a matrix form the displacement and its first and second derivates,

7.2 Write under a matrix form the Internal energy,

7.3 Write under a matrix form the Potential energy,

7.4 Write the fundamental equation $(F) = [K](q)$

7.5 Solve the system to find $(q) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

7.6 Compare the news results with the previous in term of accuracy

Solution :

Question 0 : Forces applied by the cables in A and B.

The equilibrium of the plate allow us to find easily :

$$\overrightarrow{F_A} = \overrightarrow{F_B} = \begin{pmatrix} 0 \\ -F \\ 0 \end{pmatrix}$$

Question 1 : Bending moment M_z and Shear Force T_y

The bending moment M_z is the projection on the z axis of the resultant moment of the load applied on the right side of the cross section, calculated in the centre of gravity of the current cross section.
The shear force T_y is the projection on the y axis of the resultant force of the load applied on the right side of the cross section, calculated in the centre of gravity of the current cross section.

As we can define the position of the current cross section by the abscissa x, we obtain :

	From O t0 A	From A to B
	$0 < x < L$	$L < x < 2L$
M_z	$-F(2L-x)-F(L-x)$	$-F(2L-x)$
M_z	$2Fx-3FL$	$Fx-2FL$
T_y	$2F$	F

Question 2 : Differential equations to obtain the displacement of the neutral axis.

The beam theory teaches us that the deflexion of a point of the neutral axis (who joint the centre of gravity of each cross section) is done by the following relations :

$$\frac{d^2v}{dx^2} = \frac{M_z}{EI_z} \qquad \text{For the bending moment}$$

More often, the deflexion introduced by the shear force is very small in comparison with the one introduced by the bending moment. So we shall neglect the effect of the shear force. That is a usual practise. This assumption is very accurate when the following conditions are verified :

- the beam is made of an isotropic material like steel or light alloy,
- the length of the beam is great compared with the dimensions of the cross section.

The integration of the differential equations gives :

	From O to A	From A to B
	$0 < x < L$	$L < x < 2L$
M_z	$2Fx - 3FL$	$Fx - 2FL$
$EI_z \dfrac{d^2v}{dx^2} = M_z$	$2Fx - 3FL$	$Fx - 2FL$
$EI_z \dfrac{dv}{dx}$	$Fx^2 - 3FLx + C_1$	$F\dfrac{x^2}{2} - 2FLx + C_2$
$EI_z v$	$F\dfrac{x^3}{3} - \dfrac{3}{2}FLx^2 + C_1x + C_3$	$F\dfrac{x^3}{6} - FLx^2 + C_2x + C_4$
v	$v_{OA} = \dfrac{1}{EI_z}\left(F\dfrac{x^3}{3} - \dfrac{3}{2}FLx^2 + C_1x + C_3\right)$	$v_{AB} = \dfrac{1}{EI_z}\left(F\dfrac{x^3}{6} - FLx^2 + C_2x + C_4\right)$

Question 3 : Boundary conditions.

As we perform the integration in a first time between O and A, and in a second time between A and B, we shall obtain 4 different constants of integration.

We need 4 boundary conditions to define them.

They are :

For the left side of the cantilever who is clamped :

$$v(0) = 0$$
$$\frac{dv(0)}{dx} = 0$$

For the junction between the left side (OA) and the right side (AB) of the neutral axis of the beam we must have a continuous displacement and a continuous slope :

$$v_{OA}(L) = v_{AB}(L)$$
$$\frac{dv_{OA}(L)}{dx} = \frac{dv_{AB}(L)}{dx}$$

The results are :

$$\begin{cases} 0 < x < L & v(x) = \dfrac{F}{6EI_z}\left(2x^3 - 9Lx^2\right) \\ L < x < 2L & v(x) = \dfrac{F}{6EI_z}\left(x^3 - 6Lx^2 - 3L^2x + L^3\right) \end{cases}$$

We can find the displacement for the points A & B. They are :

Displacement of the point A	$-\dfrac{7}{6}\dfrac{FL^3}{EI_z}$
Displacement of the point B	$-\dfrac{7}{2}\dfrac{FL^3}{EI_z}$

Question 4 : Ritz method.

4.1 Approximation of the neutral axis equation :

To verify the boundary conditions of the clamped side of the cantilever we can choose a polynomial function with to two terms.

$$v(x) = ax^2 + bx^3 \qquad (1\text{-}1)$$

If the accuracy of the results are not enough great we could add a third term of degree five. This is the objective of the question 6.

From equation (1-1) we obtain

	Algebra writing	Matrix writing
$v(x) =$	$ax^2 + bx^3$	$\begin{pmatrix} x^2 & x^3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$
$\dfrac{dv(x)}{dx} =$	$2ax + 3bx^2$	$\begin{pmatrix} 2x & 3x^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$
$\dfrac{d^2v(x)}{dx^2} =$	$2a + 6bx$	$\begin{pmatrix} 2 & 6x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

4.2 Internal energy in the cantilever :

The internal energy is calculated with the results of the theory of beams (we neglect the internal energy introduced by the shear force) :

$$W = \int_0^{2L} EI_z \left(\frac{d^2 v}{dx^2} \right)^2 dx$$

4.3 Internal energy in the cantilever :

If we introduce inside this equation the results of the question 4.1, we obtain :

Analytical writing	Matrix writing
$W = \frac{1}{2}\int_0^{2L} EI_z \left(2a + 6bx\right)^2 dx$	$W = \frac{1}{2}\int_0^{2L} EI_z \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 2 \\ 6x \end{pmatrix} \begin{pmatrix} 2 & 6x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} dx$
$W = \frac{1}{2}\int_0^{2L} EI_z \left(4a^2 + +24abx + 36b^2x^2\right) dx$	$W = \frac{1}{2}\int_0^{2L} EI_z \begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} 4 & 12x \\ 12x & 36x^2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} dx$
$W = \frac{1}{2} EI_z \left(8La^2 + +48L^2ab + 96L^3b^2\right)$	$W = \frac{1}{2}\begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} 4LEI_z & 24L^2EI_z \\ 24L^2EI_z & 96L^3EI_z \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

4.4 Potential energy of the load applied to the beam,

The potential energy of each force can be defined by the relation $\overrightarrow{F_A} = -\overrightarrow{grad}\left(V_A\right)$.

This relation can be written like this.

$$\frac{\partial V_A}{\partial x} = -F_A^x = 0$$
$$\frac{\partial V_A}{\partial y} = -F_A^y = F$$
$$\frac{\partial V_A}{\partial z} = -F_A^z = 0$$

After integration :

$$V_A = Fy + V_0$$

We can define the level of the potential energy as we want, for instance we can choose $V_0 = 0$

And finally we have for the two forces the potential energy :

$$V = V_A + V_B = F \cdot y_A + F \cdot y_B$$

We have two forces applied on the cantilever :
- the force F at the point A with a displacement $v(L) = aL^2 + bL^3$
- the force F at the point B with a displacement $v(2L) = 4aL^2 + 8bL^3$

$$V = F \bullet v(L) + F \bullet v(2L) = \left(5aFL^2 + 9bFL^3\right) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5FL^2 \\ 9FL^3 \end{pmatrix}$$

4.5 Potential energy under a matrix form :

$$V = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5FL^2 \\ 9FL^3 \end{pmatrix}$$

4.6 Minimisation of the total energy.

The total potential energy inside the beam is the sum of the potential energy of the load more the internal energy in the beam.

We have :

$$EP_T = W + V = \frac{1}{2} EI_z \left(8La^2 + 48L^2ab + 96L^3b^2\right) + \left(5aFL^2 + 9bFL^3\right)$$

To minimise this energy we can write the two following equations :

$$\frac{\partial EP_T}{\partial a} = 0$$
$$\frac{\partial EP_T}{\partial b} = 0$$

That give :

$$\frac{\partial EP_T}{\partial a} = EI_z \left(8aL + 24bL^2\right) - \left(5FL^2\right)$$

$$\frac{\partial EP_T}{\partial b} = EI_z \left(24aL^2 + 96bL^3\right) - \left(9FL^3\right)$$

This result can be writing under a matrix form like this :

$$\begin{pmatrix} -5FL^2 \\ -9FL^3 \end{pmatrix} = \begin{bmatrix} 4LEI_z & 24L^2EI_z \\ 24L^2EI_z & 96L^3EI_z \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

4.7 Write the equations which allow to find the different coefficients of the polynomial function of the displacement of the neutral axis.

$$a+6b=-\frac{5FL}{4EI_z}$$
$$a+4b=-\frac{9FL}{24EI_z}$$

The solution is :

$$a=-\frac{11}{8}\frac{FL}{EI_z}$$
$$b=\frac{1}{4}\frac{F}{EI_z}$$

Displacement of the points A & B :

$$\boxed{v(x)=-\frac{11}{8}\frac{FL}{EI_z}x^2+\frac{1}{4}\frac{F}{EI_z}x^3}$$

5 – Comparison between the displacements calculated by the two methods.

	Theory of Beam	Ritz's method	Error
$v(L)$	$-\frac{7}{6}\frac{FL^3}{EI_z}$	$-\frac{9}{8}\frac{FL^3}{EI_z}$	3.7 %
$v(2L)$	$-\frac{7}{2}\frac{FL^3}{EI_z}$	$-\frac{7}{2}\frac{FL^3}{EI_z}$	0%

6 – Improvement of the shape function.

We add a term of degree four in the shape function of the displacement. Therefore the displacement and its two first derivates are :

$$v(x)=\begin{pmatrix}x^2 & x^3 & x^4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\frac{dv(x)}{dx}=\begin{pmatrix}2x & 3x^2 & 4x^3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$
$$\frac{d^2v(x)}{dx^2}=\begin{pmatrix}2 & 6x & 12x^2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

The stiffness matrix is easy to find according to the work already done :

$$W=\frac{1}{2}\int_0^{2L}\begin{pmatrix}a & b & c\end{pmatrix}EI_z\begin{pmatrix}2\\6x\\12x^2\end{pmatrix}\begin{pmatrix}2 & 6x & 12x^2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}dx$$

$$[K]=EI_z\begin{bmatrix}\int_0^{2L}4dx & \int_0^{2L}12xdx & \int_0^{2L}24x^2dx\\ \int_0^{2L}12xdx & \int_0^{2L}36x^2dx & \int_0^{2L}72x^3dx\\ \int_0^{2L}24x^2dx & \int_0^{2L}72x^3dx & \int_0^{2L}144x^4dx\end{bmatrix}$$

After integration the stiffness matrix is :

$$[K]=EI_z\begin{bmatrix}8L & 24L^2 & 64L^3\\ 24L^2 & 96L^3 & 288L^4\\ 64L^3 & 288L^4 & 921.6L^5\end{bmatrix}$$

The potential energy of the load is still :

$$V=F\bullet v(L)+F\bullet v(2L)=5aFL^2+9bFL^3+17cFL^3$$

$$V=\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}5FL^2\\9FL^3\\17FL^4\end{pmatrix}$$

The governing equation of the equilibrium is :

$$\begin{pmatrix}-5FL^2\\-9FL^3\\-17FL^4\end{pmatrix}=EI_z\begin{bmatrix}8L & 24L^2 & 64L^3\\ 24L^2 & 96L^3 & 288L^4\\ 64L^3 & 288L^4 & 921.6L^5\end{bmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

Resolution : The development of the matrix equation gives.

$$8a+24Lb+64L^2c=-\frac{5FL}{EI_z}$$

$$24a+96Lb+288L^2c=-\frac{9FL}{EI_z}$$

$$64a+288Lb+921.6L^2c=-\frac{17FL}{EI_z}$$

$$\begin{cases}a=-\dfrac{49}{32}\dfrac{F}{EI_z}L\\ b=\dfrac{13}{32}\dfrac{F}{EI_z}\\ c=-\dfrac{5}{128}\dfrac{F}{EI_z}\dfrac{1}{L}\end{cases}$$

$$\boxed{v(x)=\frac{FL^3}{128EI_z}\left(-196\left(\frac{x}{L}\right)^2+52\left(\frac{x}{L}\right)^3-5\left(\frac{x}{L}\right)^4\right)}$$

	Beam Theory	Ritz Method	Error
$v(L)$	$-\frac{7}{6}\frac{FL^3}{EI_z}$	$-\frac{149}{128}\frac{FL^3}{EI_z}$	0.2 %
$v(2L)$	$-\frac{7}{2}\frac{FL^3}{EI_z}$	$-3.5\frac{FL^3}{EI_z}$	0 %

Exercise 2

Finite Elements
Assembling of springs

Five (5) springs are assembling together as indicated in figure 1.

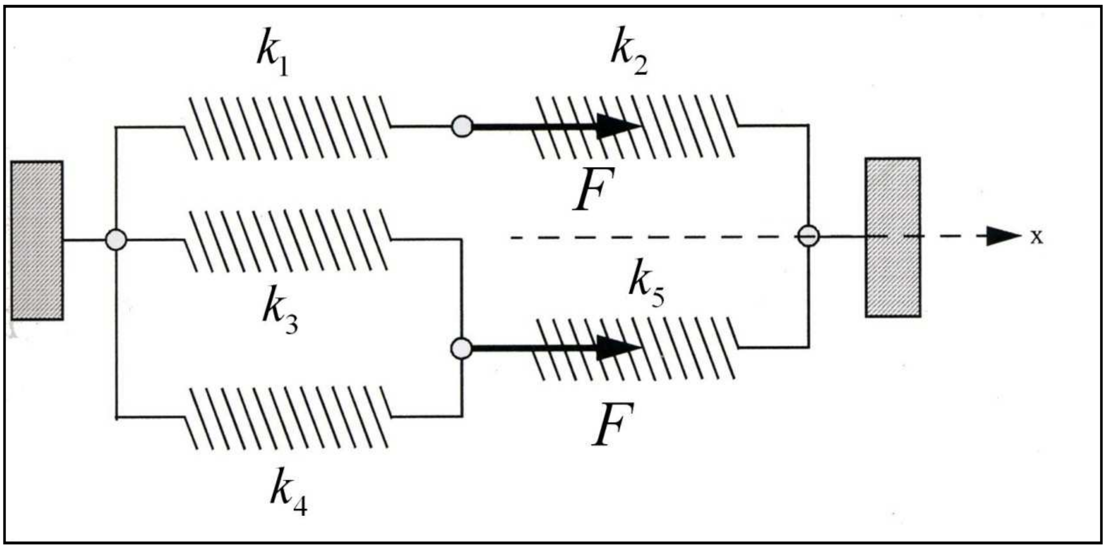


Figure 1 : Assembling of 5 springs

We want to study this assembling of springs by the finite elements method.

Questions :

0 – Pour un ressort ne pouvant travailler qu'en traction compression :

- **Donner la relation force allongement**
- **Tracer cette courbe**
- **Partant d'une force F, calculer le travail élémentaire pour passer à la force F+dF**
- **Représenter ce travail sur la courbe**
- **Calculer le travail lorsque l'on passe d'une force nulle à une force F**
- **Exprimer l'énergie interne dans le ressort en fonction de l'allongement et de la force.**
- **Quel est l'allongement du ressort en fonction des déplacements nodaux N1 et N2**
- **Calculer l'énergie dans le ressort sous la forme** $W = \frac{1}{2}\begin{pmatrix} u_1 & u_2 \end{pmatrix}[K]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$
- **En déduire la matrice de raideur d'un ressort de raideur k**
- **(partie ajoutée le 14 Octobre 2014 pour SIAE TianJin)**

1 – **Modelling :** Choose a set of nodes and give them a number.

2 – Give the expression of the internal energy inside **each** element in function of the nodal displacements. $\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}$

3 – Create the global stiffness matrix.

4 – Now, to simplify the calculation, we shall consider that the five springs have the same stiffness k. Is it possible to invert this stiffness matrix ?

- **Par example calculer le déterminant de la matrice de raideur.**

5 – Boundary conditions : Write the displacements in a column vector (q).

6 – Load : Define the load as the sum of two column vectors.

7 – Solve the equation $(F)=[K](q)$.

8 – Find (function of E,S and L) the stiffness of the springs if they are bars made of an elastic linear material define by :

- The elastic's modulus (or Young's Modulus) E
- The Poisson ratio ν
- The cross section S
- The length : L

9 – Find the stresses inside the different bars.

10 – Essayer de trouver ce qui est du domaine de Patran et du domaine de Nastran

Solution

1 – Choose a set of nodes and give them a number.

4 nodes allow to represent the displacement inside the system of springs, as indicated on the figure 2.

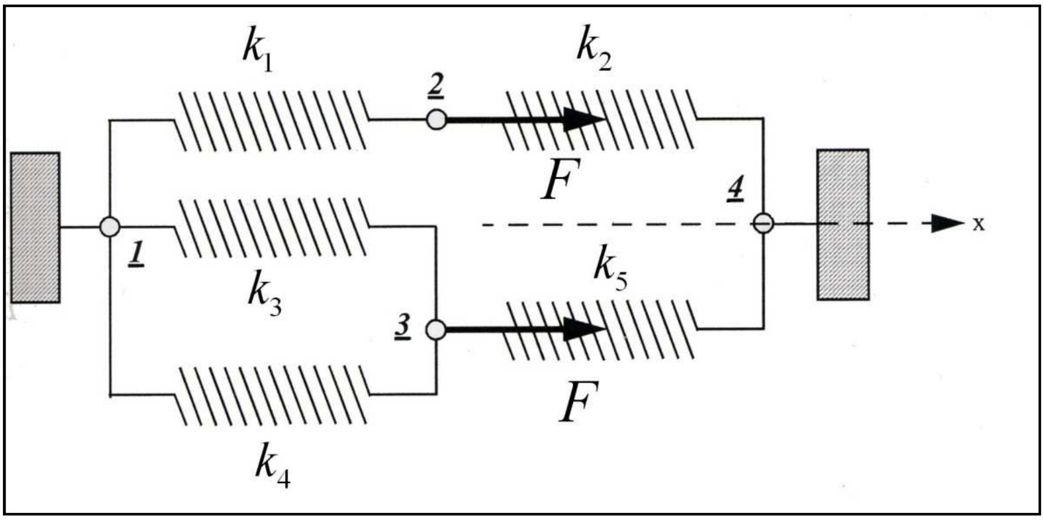


Figure 2 : Position and identification of the nodes

2 – Give the expression of the internal energy inside one element in function of the nodal displacements.

$$W = \frac{1}{2}k\left(u_i - u_j\right)^2 = \frac{1}{2}\begin{pmatrix} u_i & u_j \end{pmatrix}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} u_i \\ u_j \end{pmatrix}$$

3 – Create the global stiffness matrix.

$$[K] = \begin{bmatrix} k_1 + k_3 + k_4 & -k_1 & -k_3 - k_4 & 0 \\ -k_1 & k_1 + k_2 & 0 & -k_2 \\ -k_3 - k_4 & 0 & k_3 + k_4 + k_5 & -k_5 \\ 0 & -k_2 & -k_5 & k_2 + k_5 \end{bmatrix}$$

As all the spring have the same stiffness, this matrix becomes :

$$[K] = \begin{bmatrix} 3k & -k & -2k & 0 \\ -k & 2k & 0 & -k \\ -2k & 0 & 3k & -k \\ 0 & -k & -k & 2k \end{bmatrix}$$

4 – Is it possible to invert this stiffness matrix ?

No, because the determinant is equal to zero. To be able to solve the problem $(F) = [K](q)$ we have to remove the rigid body movements.

5 – Boundary conditions : Write the displacements in a column vector.

As the node 1 and the node 4 cannot move in the direction x, we have : $u_1 = 0 \quad u_2 = 0$

$$\boxed{(q) \quad = \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{pmatrix}}$$

7 – Load : Define the load as the sum of two column vectors.

The load applied to the system is the sum of two loads :

- the external load : $(F_{external}) = \begin{pmatrix} 0 \\ F \\ F \\ 0 \end{pmatrix}$

- the internal load : $(F_{internal}) = \begin{pmatrix} X_1 \\ 0 \\ 0 \\ X_4 \end{pmatrix}$

- The load vector is the sum of theses two loads :

$$\boxed{(F) = \begin{pmatrix} X_1 \\ F \\ F \\ X_4 \end{pmatrix}}$$

8 – Solve the equation $(F) = [K](q)$.

$$\boxed{\begin{pmatrix} X_1 \\ F \\ F \\ X_4 \end{pmatrix} = \begin{bmatrix} 3k & -k & -2k & 0 \\ -k & 2k & 0 & -k \\ -2k & 0 & 3k & -k \\ 0 & -k & -k & 2k \end{bmatrix} \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{pmatrix}} \quad (1)$$

This equation can be solved if we remove the lines 1 and 4 and the columns 1 and 4, corresponding to the displacements known. That give :

$$\boxed{\begin{pmatrix} F \\ F \end{pmatrix} = \begin{bmatrix} 2k & 0 \\ 0 & 3k \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \quad => \quad \begin{cases} u_2 = \dfrac{F}{2k} \\ u_3 = \dfrac{F}{3k} \end{cases}}$$

After that, to find the reactions we replace u_2 and u_3 in the equation (1). That give :

$$\boxed{\begin{pmatrix} X_1 \\ F \\ F \\ X_4 \end{pmatrix} = \begin{bmatrix} 3k & -k & -2k & 0 \\ -k & 2k & 0 & -k \\ -2k & 0 & 3k & -k \\ 0 & -k & -k & 2k \end{bmatrix} \begin{pmatrix} 0 \\ \dfrac{F}{2k} \\ \dfrac{F}{3k} \\ 0 \end{pmatrix} \quad => \quad \begin{cases} X_1 = -\dfrac{7F}{6} \\ X_4 = -\dfrac{5F}{6} \end{cases}}$$

9 – Stiffness of the springs

The normal stress in a beam submitted only to a normal force is : $\sigma_x = \dfrac{F}{S}$

The relation ship between stresses and strains gives : $\sigma_x = E\varepsilon_x$

And as we have : $\varepsilon_x = \dfrac{\Delta L}{L}$

We obtain : $\boxed{k = \dfrac{ES}{L}}$

10 – Find the stresses inside the most loaded beam.

As we know the displacement of each node, it's easy to find for each bar the strain :

$$\varepsilon_x = \frac{u_j - u_i}{L}$$

The relationship between strains and stresses gives the stress :

$$\sigma_x = E\varepsilon_x = E\frac{u_j - u_i}{L}$$

	u_i	u_j	σ_x
Spring 1	$u_1 = 0$	$u_2 = \dfrac{F}{2k} = \dfrac{FL}{2ES}$	$\sigma_x = \dfrac{F}{2S}$
Spring 2	$u_2 = \dfrac{F}{2k} = \dfrac{FL}{2ES}$	$u_4 = 0$	$\sigma_x = -\dfrac{F}{2S}$
Spring 3	$u_1 = 0$	$u_3 = \dfrac{F}{3k} = \dfrac{FL}{3ES}$	$\sigma_x = \dfrac{F}{3S}$
Spring 4	$u_1 = 0$	$u_3 = \dfrac{F}{3k} = \dfrac{FL}{3ES}$	$\sigma_x = \dfrac{F}{3S}$
Spring 5	$u_3 = \dfrac{F}{3k} = \dfrac{FL}{3ES}$	$u_4 = 0$	$\sigma_x = -\dfrac{F}{3S}$

Exercise 3

Finite Elements
Assembling of bars

A set of bars is defined by figure 1.

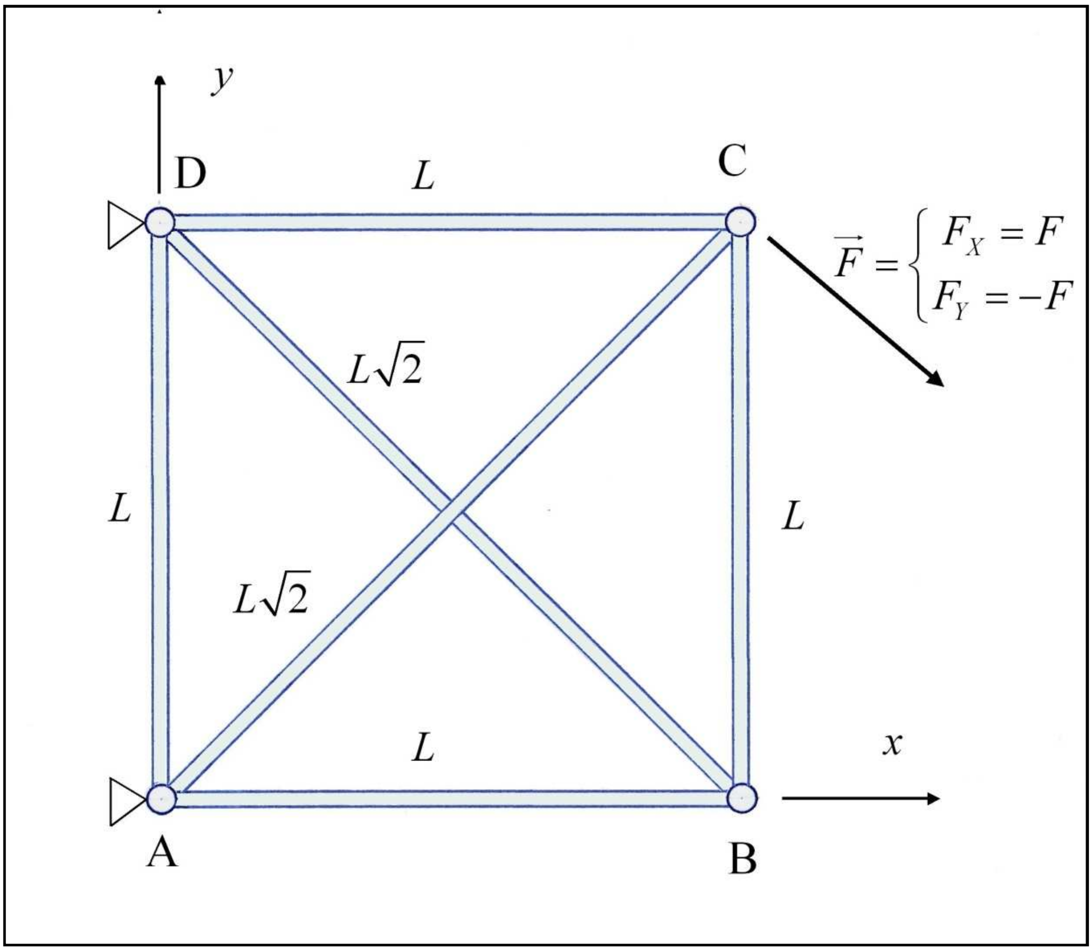


Figure 1 : Set of different bars

The 6 bars have the following mechanical characteristics :

- Young's modulus : $E = 200 \; GPa$
- Poisson's ratio : $\nu = 0.3$
- Cross section : $S = 400 \; mm$

The length for the bars AB, BC, CD, DA is : $L = 1000 \; mm$

The length for the bars AC, DB is : $L\sqrt{2} = 1414 \; mm$

Questions

1. Propose a modelling to solve this problem by the finite element method,
2. Choose a set of nodes and give them a number.
3. Write the stiffness matrix for each bar in its own axis system.
4. Write the stiffness matrix of each bar in the global axis system.
5. Create the global stiffness matrix for this structure,
6. Is it possible to invert this stiffness matrix ?
7. Boundary conditions : Write the displacements in a column vector (q).
8. Load : Define the load as the sum of two column vectors.
9. Solve the equation $(F)=[K](q)$.
10. Find the reaction in the points the displacement are imposed.
11. Indicate how the computer can find the normal stresses inside the bar.

Solution of the exercise 3

Question 1 : Modelling.

AB, BC, CD, DA, AC & DB will be modelled by 6 bar elements, because we can assume that there is only a constant normal stress in each beam of this structure. This hypothesis must be done by the user of the software finite element. This step is very important, because the computer will use this modelling to find the stiffness matrix of the structure. An error here and the problem cannot be solved by the computer with good results.

Question 2 : Nodes and DOF.

As we solve this problem with a 2D approach, each node has only two degrees of freedom. The first in the direction x, the second in the direction y.

- The displacement in the direction x is noted u
- The displacement in the direction x is noted v

The following figure shows the different notations used afterwards.

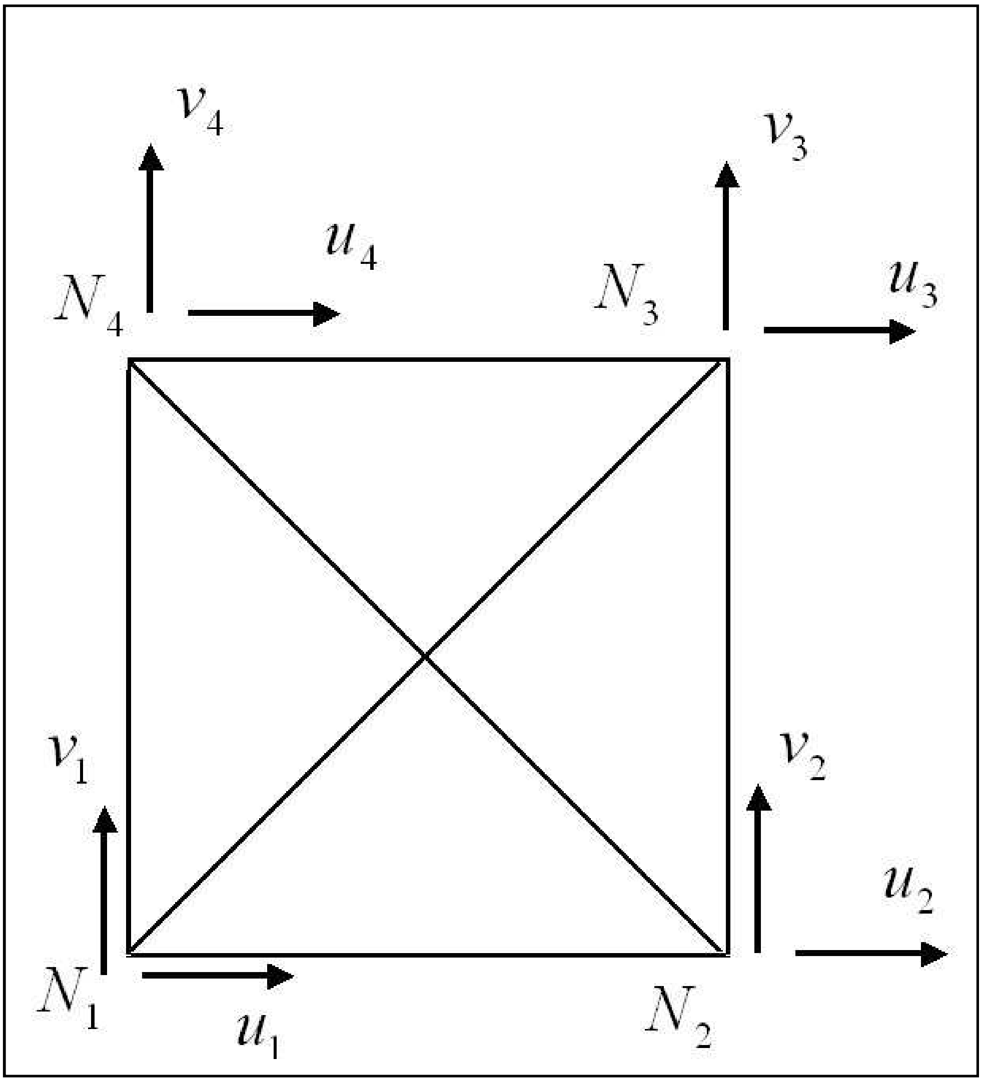


Figure 2 : Modelling Nodes & DOF

The displacement vector (q) can be written in a matrix column :

$$\boxed{(q)^T = (u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4)}$$

Question 3: Stiffness matrix of a bar in its own axis system

For instance if we consider the bar between the nodes N_1 and N_2, the local axis system is the same than the global axis system of the bar.

The internal energy in this bar is :

$$W_{12} = \frac{1}{2}(u_1 \quad v_1 \quad u_2 \quad v_2)\begin{bmatrix} \frac{ES}{L} & 0 & -\frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

Then the stiffness matrix is :

$$\boxed{[K]_{12} = \begin{bmatrix} \frac{ES}{L} & 0 & -\frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question 4: Stiffness matrix.

For instance, the bar between the nodes N_1 and N_3 is inclined of an angle $\alpha = \pi / 4$

The internal energy of this bar in the global axis system is defined by the following relation:

$$W_{13} = \frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix}^T \begin{bmatrix} [P] & [0] \\ [0] & [P] \end{bmatrix}\begin{bmatrix} \frac{ES}{L\sqrt{2}} & 0 & -\frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L\sqrt{2}} & 0 & \frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} [P]^T & [0] \\ [0] & [P]^T \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix}$$

With

$$[P]=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} \quad \& \quad [P]^T=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

For the bar between the nodes N_1 and N_3 we obtain with 45° as value for α :

$$W_{13}=\frac{1}{2}\begin{pmatrix}u_1\\v_1\\u_3\\v_3\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&-1&0&0\\1&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&0&0\\-1&1&0&0\\0&0&1&1\\0&0&-1&1\end{bmatrix}\begin{pmatrix}u_1\\v_1\\u_3\\v_3\end{pmatrix}$$

That gives :

$$W_{13}=\frac{1}{2}\begin{pmatrix}u_1\\v_1\\u_3\\v_3\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}\begin{pmatrix}u_1\\v_1\\u_3\\v_3\end{pmatrix}$$

For the bar between the nodes N_4 and N_2 we obtain we obtain with -45° as value for α :
 :

$$W_{24}=\frac{1}{2}\begin{pmatrix}u_4\\v_4\\u_2\\v_2\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&0&0\\-1&1&0&0\\0&0&1&1\\0&0&-1&1\end{bmatrix}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&-1&0&0\\1&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\begin{pmatrix}u_4\\v_4\\u_2\\v_2\end{pmatrix}$$

That gives :

$$W_{24}=\frac{1}{2}\begin{pmatrix}u_4\\v_4\\u_2\\v_2\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}\begin{pmatrix}u_4\\v_4\\u_2\\v_2\end{pmatrix}$$

Question 5 : Stiffness matrix of the system

The internal energy of the system is the sum of the energy in each bar. We have to do the assembling of the internal energy of each bar inside a great stiffness matrix according to the global displacement vector.

The result is :

$$W=\frac{1}{2}\begin{pmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\u_3\\v_4\end{pmatrix}^T\frac{ES}{L}\begin{bmatrix}1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0\\ \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1\\ -1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}\\ 0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0\\ 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}\end{bmatrix}\begin{pmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\u_3\\v_4\end{pmatrix}$$

So we can define the global stiffness matrix system like this :

$$[K]=\frac{ES}{L}\begin{bmatrix}1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0\\ \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1\\ -1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}\\ 0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0\\ 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}\end{bmatrix} \quad \text{(EQ1)}$$

Question 6 : It’s not possible to invert this matrix, because the system has rigid body movements. As it is a 2D system we need to remove 3 degrees of freedom. The two ball joints allow us to remove 4 degrees of freedom. It is more than we need.

Question 7 : Displacement vector

The boundary conditions of the set of bars allow us to write the displacement vector, as the sum of a well known displacement vector, and a unknown displacement vector :

$$(q)=\begin{pmatrix} u_1=0 \\ v_1=0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4=0 \\ v_4=0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ 0 \\ 0 \end{pmatrix}$$

Question 7 : Load vector

The load vector is the sum of the two nodal loads :
- External forces applied on the nodes,
- Internal reactions

$$(F)=\begin{pmatrix} X_1 \\ Y_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ X_4 \\ Y_4 \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F \\ -F \\ 0 \\ 0 \end{pmatrix}$$

Question 9 : Resolution of the problem

To solve the finite element problem, the computer must solve the equation $(F)=[K](q)$

Hence, the equation to solve is :

$$
\begin{pmatrix} X_1 \\ Y_1 \\ 0 \\ 0 \\ F \\ -F \\ X_4 \\ Y_4 \end{pmatrix} = \frac{ES}{L}
\begin{bmatrix}
1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 \\
-1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}
\end{bmatrix}
\begin{pmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ 0 \\ 0 \end{pmatrix}
$$

It's impossible to invert directly this global stiffness matrix. As we need, in a first time, to find the unknown displacements we can write this equation like the following after having removed the lines and columns corresponding to the displacement known. That gives :

$$
\begin{pmatrix} 0 \\ 0 \\ F \\ -F \end{pmatrix} = \frac{ES}{L}
\begin{pmatrix}
1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\
-\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & 0 & -1 \\
0 & 0 & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & -1 & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}
\end{pmatrix}
\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}
$$

For the computer it's very easy to solve this equation, because the determinant of the matrix is different of zero. We are sure of that because the boundary conditions have removed all the rigid body movements of the set of bars. Generally the computer use the Gauss's method to invert the small matrix (less than 15 000 x 15 000) or the gradient method for the biggest matrix. Generally, for the most common software, the user can choose the method he wants to invert the matrix.

After the inversion of the reduced stiffness matrix the computer knows the values of the displacement unknown before.

A numerical application gives :

$$\begin{pmatrix} 0 \\ 0 \\ F \\ -F \end{pmatrix} = \begin{pmatrix} 108284 & -28284 & 0 & 0 \\ -28284 & 108284 & 0 & -80000 \\ 0 & 0 & 108284 & 28284 \\ 0 & -80000 & 28284 & 108284 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

And then :

$$\begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = 10^{-4} \begin{pmatrix} \frac{27071}{244852} & \frac{17071}{244852} & \frac{-7071}{489704} & \frac{27071}{489704} \\ \frac{17071}{244852} & \frac{462129041}{1731348492} & \frac{-27071}{244852} & \frac{732839041}{3462696984} \\ \frac{-7071}{489704} & \frac{-27071}{244852} & \frac{27071}{244852} & \frac{-17071}{244852} \\ \frac{27071}{489704} & \frac{732839041}{3462696984} & \frac{-17071}{244852} & \frac{462129041}{1731348492} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ F \\ -F \end{pmatrix}$$

The displacements are after inverting the stiffness matrix :

$$\boxed{\begin{array}{l} u_2 = -6.97 \quad 10^{-6} \quad \text{F} \quad , \quad v_2 = -2.67 \quad 10^{-5} \quad \text{F} \\ u_3 = -1.80 \quad 10^{-5} \quad \text{F} \quad , \quad v_3 = -3.37 \quad 10^{-5} \quad \text{F} \end{array}}$$

Question 10 : Values of the reactions

Using the equation (EQ1) $(F) = [K](q)$, the computer can find easily the reaction that appears when nodes have displacement imposed. For instance:

$$\boxed{\begin{array}{l} X_1 = -u_2 - \frac{\sqrt{2}}{4}u_3 - \frac{\sqrt{2}}{4}v_3 \quad , \quad Y_1 = -\frac{\sqrt{2}}{4}u_3 - \frac{\sqrt{2}}{4}v_3 \\ X_4 = -\frac{\sqrt{2}}{4}u_2 + \frac{\sqrt{2}}{4}u_3 - u_3 \quad , \quad Y_4 = \frac{\sqrt{2}}{4}u_2 - \frac{\sqrt{2}}{4}u_3 \end{array}}$$

Question 11 : Stresses in the different bars.

The computer knows the initial position for each node :

$$N_i = \begin{pmatrix} x_i & y_i \end{pmatrix}$$

It also knows the displacement vector for each node. Hence it can find the final position for each node :

$$N_i = \begin{pmatrix} x_i + u_i & y_i + v_i \end{pmatrix}$$

The computer can define the strain ε_{ij} in each bar who joint the node N_i to the node N_j :

$$\varepsilon_{ij} = \frac{\sqrt{\left(x_j + u_j - x_i - u_i\right)^2 + \left(y_j + v_j - y_i - v_i\right)^2} - \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}}{\sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}}$$

And finally the stresses σ_{ij} in each bar who joint the node N_i to the node N_j is :

$$\boxed{\sigma_{ij} = E\varepsilon_{ij}}$$

Exercise 4

Finite Elements
Beam defined by 3 nodes.

In a finite element software, you can choose (see page 3), among a lot of different elements, a beam element defined by 3 nodes. The objective of this exercise is to find the stiffness matrix of such a finite element.

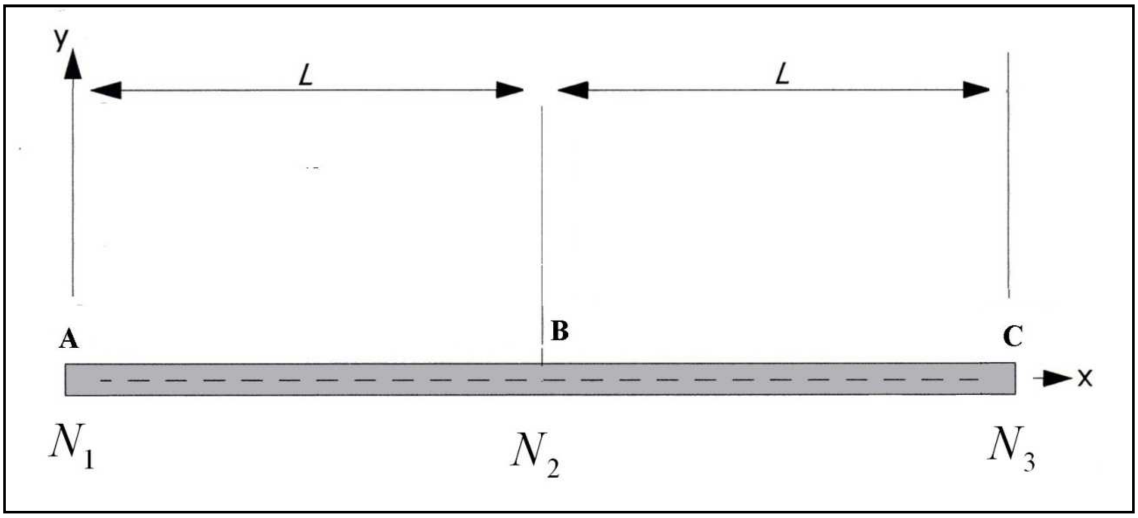


Figure 1 : Beam Finite Element defined by 3 nodes

We start from a beam defined by the figure 1 with three nodes $\left(N_1 \quad N_2 \quad N_3\right)$.

The mechanical characteristic of the material and of the beam are supposed to be known :

- Young's modulus : E
- Poisson's ratio : ν
- Shear modulus : G
- Cross section : S
- Quadratic moments : I_y & I_z
- Coefficient of reduced cross section : k_y & k_z
- Twisting stiffness : J

Questions :

Remark : For the question 1 to 3, you will take in account the normal force N_x, the shear force T_y and T_z the twisting moment M_t and the bending moments M_y and M_z.

Remark : For the question 4 to 8, you will take in account only the bending moment M_z.

1. Defined the most general displacement vector for this beam.

2. Give the analytical internal energy, function of the internal load ($N_x \quad T_y \quad T_z \quad M_t \quad M_y \quad M_z$)

3. Give the internal energy, function of the displacement of the middle line of the beam, by using relations as :

$$\frac{d^2v(x)}{dx^2} = \frac{M_z}{EI_z}$$

4. We shall reduce the study to the internal energy introduced by the bending moment M_z. Propose a field of displacement for the middle line of the beam, with a polynomial function as $v(x) = a + bx + cx^2 + dx^3 + \dots\dots\dots\dots\dots$

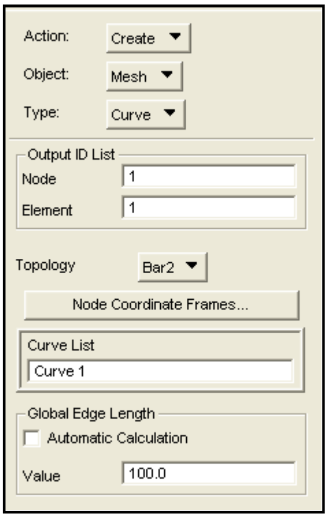
5. Write the boundary conditions this field displacement must verify,

6. Find the coefficients of the polynomial as function of the nodal displacements,

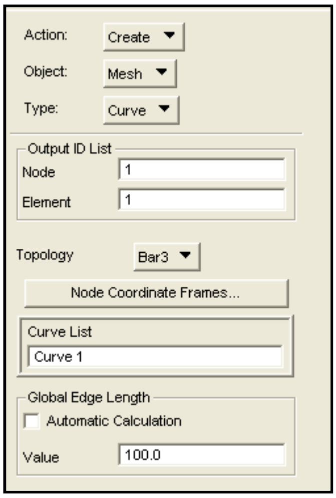
7. Write the internal energy as a function of the nodal displacements,

8. Find the coefficient of the stiffness matrix of this element.

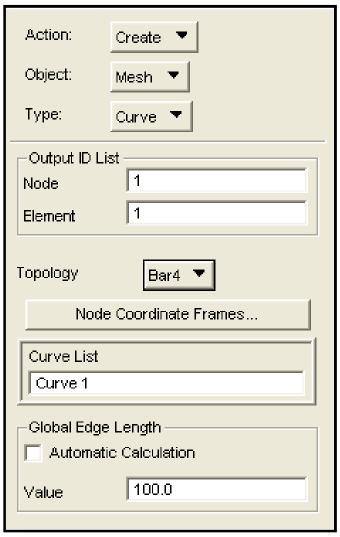
Example of a finite element of beams defined by several nodes in Nastra-Patran



Beam defined by two nodes



Beam defined by three nodes



Beam defined by four nodes

Solution of the Exercise 4

Question 1 : The beam theory approximate the field of displacement of each point of the cross section as the result of the three translations of the centre of gravity and the three rotations around the principal axis of the cross section.

As the modelling of the beam is made with 3 nodes, the most general displacement vector for this element is a 18 components one :

$$(q)^T = \left(u_1 \quad \theta_1^x \quad v_1 \quad \theta_1^y \quad w_1 \quad \theta_1^z \quad .... \quad \theta_3^y \quad w_3 \quad \theta_3^z\right)$$

Question 2 : Analytical internal energy,

$$\frac{dW}{dx} = \frac{1}{2}\frac{N_X^2}{ES} + + \frac{1}{2}\frac{T_Y^2}{Gk_Y S} + \frac{1}{2}\frac{T_Z^2}{Gk_Z S} + \frac{1}{2}\frac{M_t^2}{GJ} + \frac{1}{2}\frac{M_Y^2}{EI_Y} + \frac{1}{2}\frac{M_Z^2}{EI_Z} \qquad \text{(EQ 1)}$$

Question 3 :

The theory of beams gives us the following relation between the displacement and the internal load :

$$\begin{cases} \dfrac{du}{dx} = \dfrac{N_x}{ES} \\ \dfrac{dv}{dx} = \dfrac{T_y}{Gk_Y S} \\ \dfrac{dw}{dx} = \dfrac{T_z}{Gk_Z S} \end{cases} , \quad \begin{cases} \dfrac{d\theta_x}{dx} = \dfrac{M_t}{GJ} \\ \dfrac{d\theta_y}{dx} = \dfrac{M_y}{EI_Y} \\ \dfrac{d\theta_z}{dx} = \dfrac{M_z}{EI_Z} \end{cases} \qquad \text{(EQ 2)}$$

Hence the equations (EQ 1) and (EQ 2) give :

$$W = \frac{1}{2}\int_0^{2L} ES\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} Gk_Y S\left(\frac{dv}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} Gk_Z S\left(\frac{dw}{dx}\right)^2 dx +$$
$$+ \frac{1}{2}\int_0^{2L} GJ\left(\frac{d\theta_X}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} EI_Y\left(\frac{d\theta_Y}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} EI_Z\left(\frac{d\theta_Z}{dx}\right)^2 dx$$

If the angles are small, we have the following relations :

$$\begin{cases} \theta_y = \dfrac{dw}{dx} \\ \theta_z = \dfrac{dv}{dx} \end{cases}$$

And then the internal energy becomes :

$$\boxed{\begin{aligned} W = {} & \frac{1}{2}\int_0^{2L} ES\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} Gk_Y S\left(\frac{dv}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} Gk_Z S\left(\frac{dw}{dx}\right)^2 dx + \\ & + \frac{1}{2}\int_0^{2L} GJ\left(\frac{d\theta_X}{dx}\right)^2 dx + \frac{1}{2}\int_0^{2L} EI_Y\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^{2L} EI_Z\left(\frac{d^2v}{dx^2}\right)^2 dx \end{aligned}}$$

Question 4 :

If we consider only the internal energy of the bending moment M_z, we just need an approximation of the field of displacement v(x).

$$W = \frac{1}{2}\int_0^{2L} EI_Z\left(\frac{d\theta_Z}{dx}\right)^2 dx = \frac{1}{2}\int_0^{2L} EI_Z\left(\frac{d^2v}{dx^2}\right)^2 dx$$

As the beam is defined by 3 nodes, and as each node has 2 degrees of freedom, (the translation of the centre of gravity of the cross section in the direction y, and the rotation of the same cross section around a z axis), we can define an interpolated field of displacement with 6 coefficients. A polynomial function alloys that for instance :

$$\boxed{v(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5} \qquad (1)$$

Question 5 :

The boundary conditions can be written like this :

$x=0$	$v(o)=v_1$	$\dfrac{dv(0)}{dx}=\theta_1$
$x=L$	$v(o)=v_2$	$\dfrac{dv(L)}{dx}=\theta_2$
$x=2L$	$v(o)=v_3$	$\dfrac{dv(2L)}{dx}=\theta_3$

These boundary conditions can be written like this, with respect of the interpolated function for the field of displacement v(x)

$$v(0) = a = v_1$$

$$\frac{dv}{dx}(0) = b = \theta_1$$

$$v(L) = a + bL + cL^2 + dM^3 + eL^4 + fL^5 = v_2$$

$$\frac{dv}{dx}(L) = b + 2cL + 3dL^2 + 4eL^3 + 5fL^4 = \theta_2$$

$$v(2L) = a + b(2L) + c(2L)^2 + d(2L)^3 + e(2L)^4 + f(2L)^5 = v_3$$

$$\frac{dv}{dx}(2L) = b + 2c(2L) + 3d(2L)^2 + 4e(2L)^3 + 5f(2L)^4 = \theta_3$$

Question 6 : This set of 6 equations with 6 unknown coefficient have only one solution.

$$a=\frac{1}{4}\begin{pmatrix}4 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$b=\frac{1}{4}\begin{pmatrix}0 & 4 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$c=\frac{1}{4}\begin{pmatrix}\frac{-23}{L^2} & \frac{-12}{L} & \frac{16}{L^2} & \frac{-16}{L} & \frac{7}{L^2} & \frac{-2}{L}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$d=\frac{1}{4}\begin{pmatrix}\frac{33}{L^3} & \frac{13}{L^2} & \frac{-16}{L^3} & \frac{32}{L^2} & \frac{-17}{L^3} & \frac{5}{L^2}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$e=\frac{1}{4}\begin{pmatrix}\frac{-17}{L^4} & \frac{-6}{L^3} & \frac{4}{L^4} & \frac{-20}{L^3} & \frac{13}{L^4} & \frac{-4}{L^3}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$f=\frac{1}{4}\begin{pmatrix}\frac{3}{L^5} & \frac{1}{L^4} & 0 & \frac{4}{L^4} & \frac{-3}{L^5} & \frac{1}{L^4}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

If we replace the different coefficients inside the interpolated function (1), we find

$$v(x)=\begin{pmatrix}\varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix} \quad , \quad \frac{dv(x)}{dx}=\begin{pmatrix}\frac{d\varphi_1}{dx} & \frac{d\varphi_2}{dx} & \frac{d\varphi_3}{dx} & \frac{d\varphi_4}{dx} & \frac{d\varphi_5}{dx} & \frac{d\varphi_6}{dx}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

$$\frac{d^2v(x)}{dx^2}=\begin{pmatrix}\frac{d^2\varphi_1}{dx^2} & \frac{d^2\varphi_2}{dx^2} & \frac{d^2\varphi_3}{dx^2} & \frac{d^2\varphi_4}{dx^2} & \frac{d^2\varphi_5}{dx^2} & \frac{d^2\varphi_6}{dx^2}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}$$

With the fonction defined $\begin{pmatrix}\varphi_i & \frac{d\varphi_i}{dx} & \frac{d^2\varphi_i}{dx^2}\end{pmatrix}$ on the following page :

$$\varphi_1=\frac{1}{4}\left(4-23\frac{x^2}{L^2}+33\frac{x^3}{L^3}-17\frac{x^4}{L^4}+3\frac{x^5}{L^5}\right)\qquad \frac{d\varphi_1}{dx}=\frac{1}{4}\left(46\frac{x}{L^2}+99\frac{x^2}{L^3}-68\frac{x^3}{L^4}+15\frac{x^4}{L^5}\right)\qquad \frac{d^2\varphi_1}{dx^2}=\frac{1}{4}\left(\frac{46}{L^2}+\frac{198x}{L^3}-\frac{204x^2}{L^4}+\frac{60x^3}{L^5}\right)$$

$$\varphi_2=\frac{1}{4}\left(4x-12\frac{x^2}{L}+13\frac{x^3}{L^2}-6\frac{x^4}{L^3}+\frac{x^5}{L^4}\right)\qquad \frac{d\varphi_2}{dx}=\frac{1}{4}\left(4-24\frac{x}{L}+39\frac{x^2}{L^2}-24\frac{x^3}{L^3}+5\frac{x^4}{L^4}\right)\qquad \frac{d^2\varphi_2}{dx^2}=\frac{1}{4}\left(-\frac{24}{L}+\frac{78x}{L^2}-\frac{72x^2}{L^3}+\frac{20x^3}{L^4}\right)$$

$$\varphi_3=\frac{1}{4}\left(16\frac{x^2}{L^2}-16\frac{x^3}{L^3}+4\frac{x^4}{L^4}\right)\qquad \frac{d\varphi_3}{dx}=\frac{1}{4}\left(32\frac{x}{L^2}-48\frac{x^2}{L^3}+16\frac{x^3}{L^4}\right)\qquad \frac{d^2\varphi_3}{dx^2}=\frac{1}{4}\left(32\frac{1}{L^2}-96\frac{x}{L^3}+48\frac{x^2}{L^4}\right)$$

$$\varphi_4=\frac{1}{4}\left(-16\frac{x^2}{L}+32\frac{x^3}{L^2}-20\frac{x^4}{L^3}+4\frac{x^5}{L^4}\right)\qquad \frac{d\varphi_4}{dx}=\frac{1}{4}\left(-32\frac{x}{L}+96\frac{x^2}{L^2}-80\frac{x^3}{L^3}+20\frac{x^4}{L^4}\right)\qquad \frac{d^2\varphi_4}{dx^2}=\frac{1}{4}\left(-32\frac{1}{L}+192\frac{x}{L^2}-240\frac{x^2}{L^3}+80\frac{x^3}{L^4}\right)$$

$$\varphi_5=\frac{1}{4}\left(7\frac{x^2}{L^2}-17\frac{x^3}{L^3}+13\frac{x^4}{L^4}-3\frac{x^5}{L^5}\right)\qquad \frac{d\varphi_5}{dx}=\frac{1}{4}\left(14\frac{x}{L^2}-51\frac{x^2}{L^3}+52\frac{x^3}{L^4}-15\frac{x^4}{L^5}\right)\qquad \frac{d^2\varphi_5}{dx^2}=\frac{1}{4}\left(14\frac{1}{L^2}-102\frac{x}{L^3}+156\frac{x^4}{L^4}-60\frac{x^3}{L^5}\right)$$

$$\varphi_6=\frac{1}{4}\left(-2\frac{x^2}{L}+5\frac{x^3}{L^2}-4\frac{x^4}{L^3}+\frac{x^5}{L^4}\right)\qquad \frac{d\varphi_6}{dx}=\frac{1}{4}\left(-4\frac{x}{L}+15\frac{x^2}{L^2}-16\frac{x^3}{L^3}+5\frac{x^4}{L^4}\right)\qquad \frac{d^2\varphi_6}{dx^2}=\frac{1}{4}\left(-4\frac{1}{L}+30\frac{x}{L^2}-48\frac{x^2}{L^3}+20\frac{x^3}{L^4}\right)$$

Question 7 : Internal energy in the element of beam

$$W=\frac{1}{2}\int_0^{2l} EI_z\left(\frac{d^2v}{dx^2}\right)^2 dl=\frac{1}{2}\int_0^{2l} EI_z\left(\frac{d^2v}{dx^2}\right)^T\left(\frac{d^2v}{dx^2}\right)dl$$

$$\boxed{W=\frac{1}{2}\int_0^{2l}\begin{pmatrix}v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3\end{pmatrix} EI_z \begin{pmatrix}\frac{d^2\varphi_1}{dx^2}\\ \frac{d^2\varphi_2}{dx^2}\\ \frac{d^2\varphi_3}{dx^2}\\ \frac{d^2\varphi_4}{dx^2}\\ \frac{d^2\varphi_5}{dx^2}\\ \frac{d^2\varphi_6}{dx^2}\end{pmatrix}\begin{pmatrix}\frac{d^2\varphi_1}{dx^2} & \frac{d^2\varphi_2}{dx^2} & \frac{d^2\varphi_3}{dx^2} & \frac{d^2\varphi_4}{dx^2} & \frac{d^2\varphi_5}{dx^2} & \frac{d^2\varphi_6}{dx^2}\end{pmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{pmatrix}dx}$$

$$\boxed{W=\frac{1}{2}\int_0^{2l}(q)^T\, EI_z\left[\sum_{i=1}^{6}\sum_{j=1}^{6}\int_0^{2L}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx\right](q)}$$

Question 8 : Stiffness matrix

The general term of the stiffness matrix is :

$$\boxed{K_{ij}=EI_z\int_0^{2L}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx}$$

For instance :

$$K_{22}=\frac{166}{35}\frac{EI_z}{L}$$

Exercise 5

Finite Elements
Assembling of beams and bars

We want to study the system defined by the figure 1 with the finite elements method.

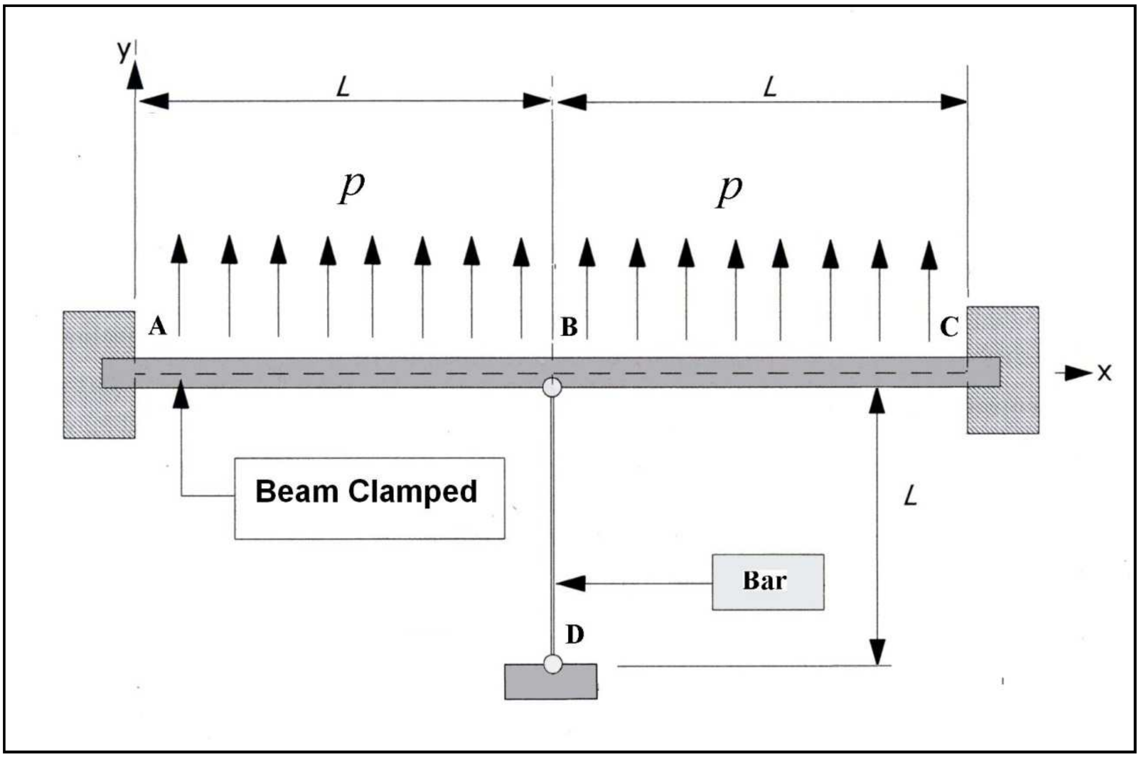


Figure 1 : Case study

All the mechanical characteristics are supposed known, as for instance :

- the area of the cross section of the bar $S^* = 400 \quad mm^2 = S/4$
- the area of the cross section of the beam $S = 1600 \quad mm^2$
- the quadratic moment around the z axis for the beam, $I_z = 2563333 \quad mm^4$
- the Young's modulus of the material $E = 200 \quad GPa$
- the half length of the beam $L = 1000 \quad mm$
- the length of the bar $L = 1000 \quad mm$

The following figure gives the details of the cross section of the beam as defined by the software PATRAN

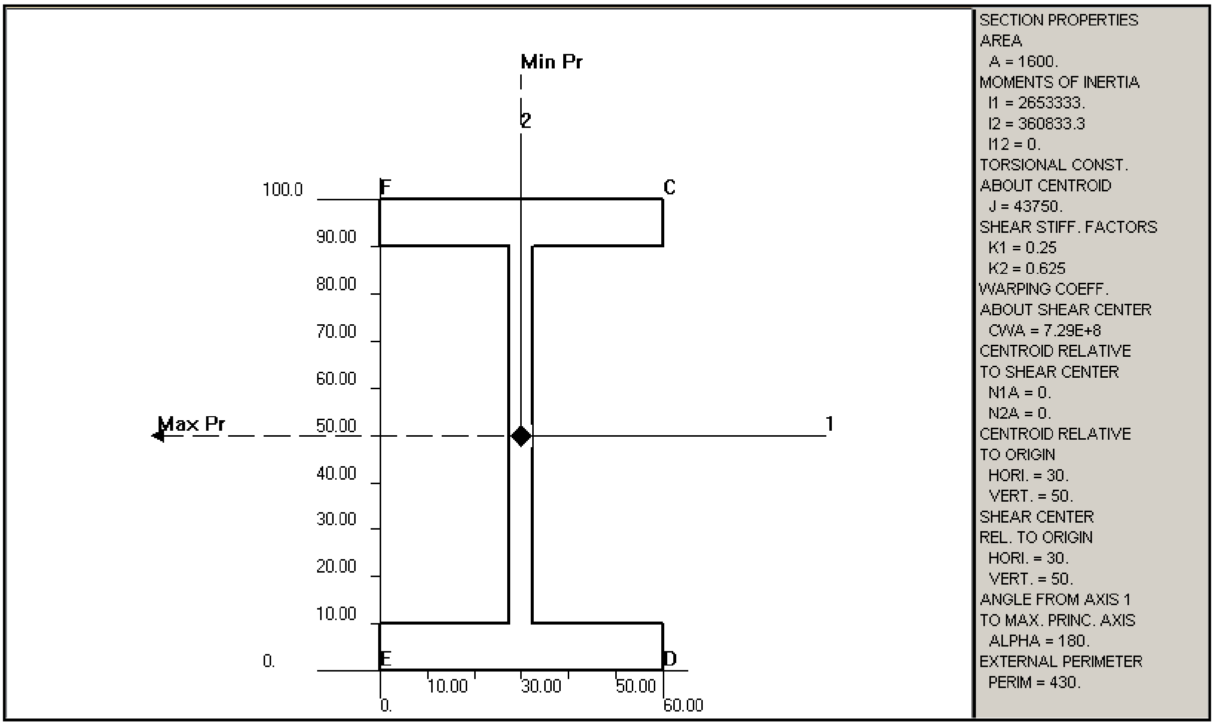


Figure 2 : Cross section defined by a library in PATRAN

Questions :

1 Propose a modelling to solve this problem with the finite element method,

2 Choose a set of nodes and give them a number.

3 Write the stiffness matrix of a beam, in its own axis system, with only the internal energy created by the bending moment M_z

4 Write the stiffness matrix of the bar element with the internal energy created by the normal force N_x

5 Create the global stiffness matrix of this structure,

6 Is it possible to invert this stiffness matrix ?

7 Boundary conditions : Write the displacements in a column vector (q).

8 Reaction Load : Write the reaction load in a column vector.

9 Replace the real distributed load p by an equivalent nodal load. For that you have to use the following shape functions for the interpolated displacement of a beam defined by two nodes.

$$v(x)=\left(\left\{1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}\right\} \quad L\left\{\frac{x}{L}-2\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\} \quad \left\{3\frac{x^2}{L^2}-2\frac{x^3}{L^3}\right\} \quad L\left\{-\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\right)\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} \text{ (Eq 0)}$$

10 External load : Write the external load in a column vector,

11 Global Load : Write the global load in a column vector,

12 Solve the equation $\left(F\right)=\left[K\right]\left(q\right)$,

13 Indicate how the computer can find the values of the unknown reactions. Compute theses reactions,

14 Indicate how the computer can find the normal stresses inside the beam.

15 If the internal energy of the beam would have been calculated with respect of the shear force, indicate how the computer could have found this internal energy.

16 Explain how you can solve this problem with the analytic theory of beams by studying only the half part of the structure.

17 Find the stresses in the point A and B with the beam theory.

18 Find the value of the normal force in the bar BD.

19 Compare the results issued of the beam theory with the ones issued of the finite element method.

Solution

Question 1 :Modelling.

AC will be modelled by 2 beam elements, because the bending moment and the shear force inside this part of the structure create normal and shear stresses.
BD will be modelled by a bar element, because we know that a beam with 2 ball joints can only works in traction or compression with a constant normal stress.

Question 2 : Nodes and DOF.

We can reduce the size of the displacement vector because we are sure that there is no normal force inside the beam elements. Hence the displacement vector can be reduced to 7 components :

$$(q)^T = \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \end{pmatrix}$$

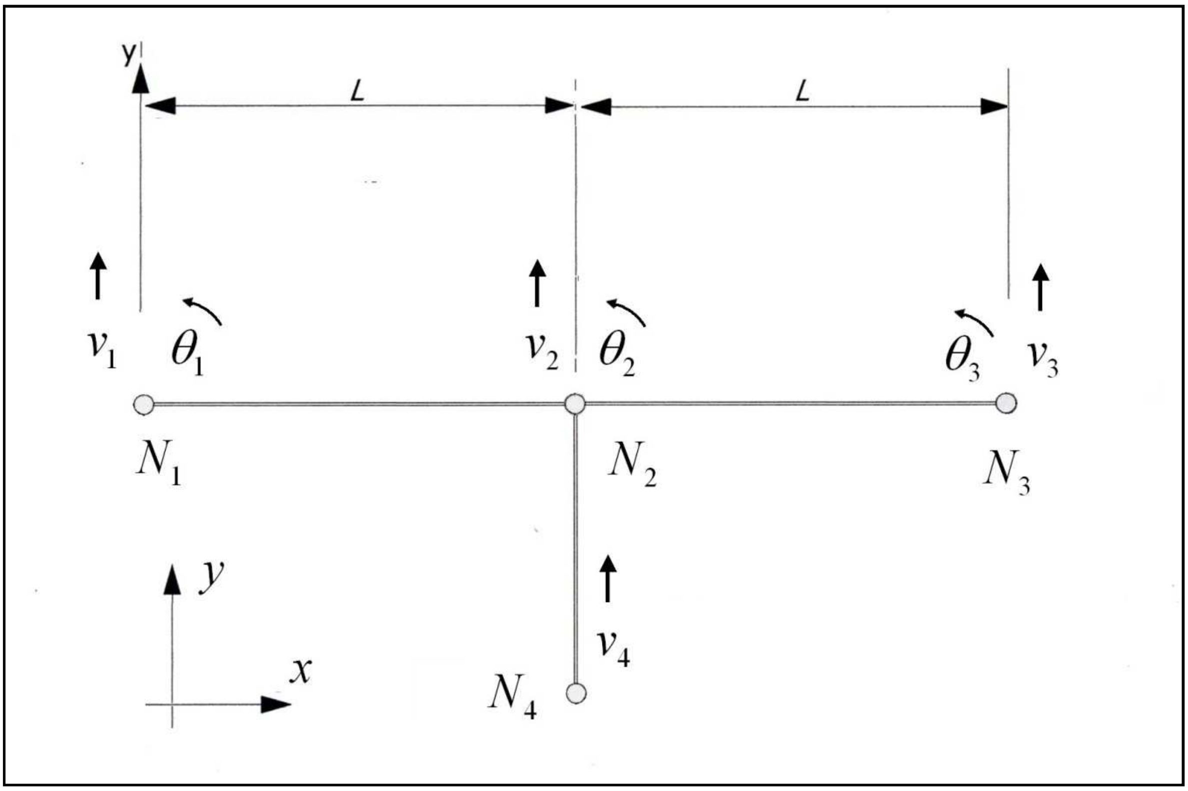


Figure 2 : Position of nodes and DOF for each node

Question 3 : Stiffness matrix of a beam

The internal energy created by the bending moment M_z in the beam between the nodes N_1 and N_2 is equal to :

$$W=\frac{1}{2}\begin{pmatrix}v_1 & \theta_1 & v_2 & \theta_2\end{pmatrix}\frac{EI_z}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{pmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{pmatrix}$$

Hence the stiffness matrix is :

$$[K]=\frac{EI_z}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}$$

Question 4 : Stiffness matrix of a bar

Consider the beam between the nodes N_2 and N_4. The internal energy is only created in the field of displacement v :

$$W=\frac{1}{2}\begin{pmatrix}v_1 & v_4\end{pmatrix}\begin{bmatrix}\frac{ES^*}{L} & -\frac{ES^*}{L}\\ -\frac{ES^*}{L} & \frac{ES^*}{L}\end{bmatrix}\begin{pmatrix}v_1\\ v_4\end{pmatrix}$$

Hence the stiffness matrix of the vertical bar is :

$$[K]=\begin{bmatrix}\frac{ES^*}{L} & -\frac{ES^*}{L}\\ -\frac{ES^*}{L} & \frac{ES^*}{L}\end{bmatrix}=\frac{ES^*}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Question 5 : Global stiffness matrix,

To simplify the writing of the global stiffness matrix we can use the following gimmick :

$$\frac{ES^*}{L}=\frac{SL^2}{4I_z}\frac{EI_z}{L^3}=\alpha\frac{EI_z}{L^3}\qquad \text{with } \alpha=\frac{SL^2}{4I_z}$$

The assembling of the stiffness matrix is easy to do by adding the different energy inside each element in a great matrix associated to the full displacement vector.

The colour code used in the next page can help you to understand this assembling

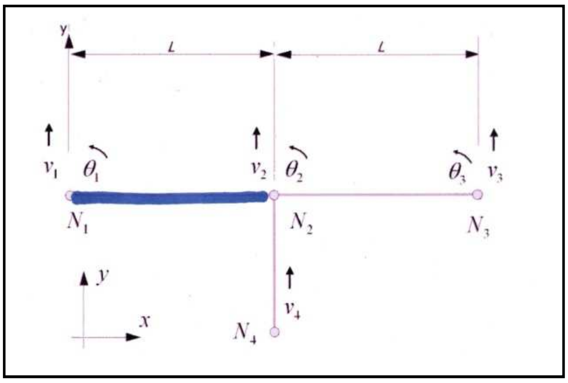


Figure 3 : First Beam Element

$\frac{EI_z}{L^3}$

	v_1	θ_1	v_2	θ_2	v_3	θ_3	v_4
v_1	12	$6L$	-12	$6L$			
θ_1	$6L$	$4L^2$	$-6L$	$2L^2$			
v_2	-12	$-6L$	12	$-6L$			
θ_2	$6L$	$2L^2$	$-6L$	$4L^2$			
v_3							
θ_3							
v_4							

Figure 4 : Terms of stiffness for the first Beam Element

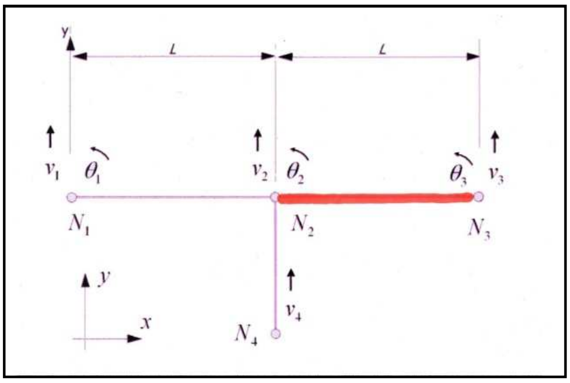


Figure 5 : Second Beam Element

$\frac{EI_z}{L^3}$

	v_1	θ_1	v_2	θ_2	v_3	θ_3	v_4
v_1							
θ_1							
v_2			12	$6L$	-12	$6L$	
θ_2			$6L$	$4L^2$	$-6L$	$2L^2$	
v_3			-12	$-6L$	12	$-6L$	
θ_3			$6L$	$2L^2$	$-6L$	$4L^2$	
v_4							

Figure 6 : Terms of stiffness for the Second Beam Element

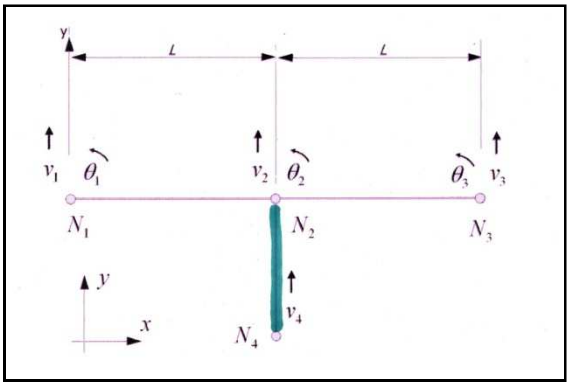


Figure 7 : Bar Element

$\frac{EI_z}{L^3}$

	v_1	θ_1	v_2	θ_2	v_3	θ_3	v_4
v_1							
θ_1							
v_2			α				$-\alpha$
θ_2							
v_3							
θ_3							
v_4			$-\alpha$				α

Figure 8 : Terms of stiffness for the Bar Element

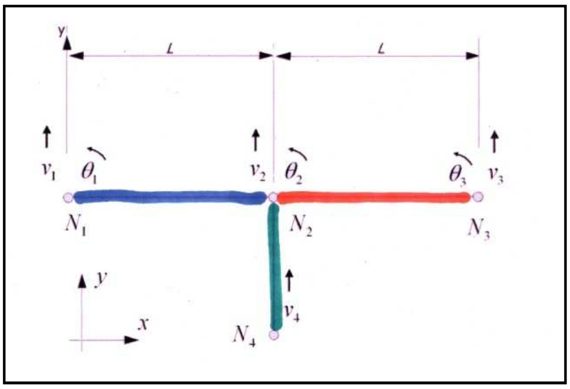


Figure 9 : Structure designed

$\frac{EI_z}{L^3}$

	v_1	θ_1	v_2	θ_2	v_3	θ_3	v_4
v_1	12	$6L$	-12	$6L$	0	0	0
θ_1	$6L$	$4L^2$	$-6L$	$2L^2$	0	0	0
v_2	-12	$-6L$	$24+\alpha$	0	-12	$6L$	$-\alpha$
θ_2	$6L$	$2L^2$	0	$8L^2$	$-6L$	$2L^2$	0
v_3	0	0	-12	$-6L$	12	$-6L$	0
θ_3	0	0	$6L$	$2L^2$	$-6L$	$4L^2$	0
v_4	0	0	$-\alpha$	0	0	0	α

Figure 10 : Stiffness matrix of the structure

Finally we obtain for the global internal energy

$$W = \frac{1}{2}\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{pmatrix}^T \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 12+12+\alpha & -6L+6L & -12 & 6L & -\alpha \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6l & 2L^2 & 0 \\ 0 & 0 & -12 & 6l & 12 & -6l & 0 \\ 0 & 0 & 6l & 2L^2 & -6l & 4L^2 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & \alpha \end{bmatrix}\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{pmatrix}$$

Then the stiffness matrix is :

$$\boxed{[K] = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24+\alpha & 0 & -12 & 6L & -\alpha \\ 6L & 2L^2 & 0 & 8L^2 & -6l & 2L^2 & 0 \\ 0 & 0 & -12 & 6l & 12 & -6l & 0 \\ 0 & 0 & 6l & 2L^2 & -6l & 4L^2 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & \alpha \end{bmatrix}}$$

Question 6 : Is it possible to invert this stiffness matrix ?

We cannot invert this matrix, because it has 3 eigen values equal to zero because there are three rigid body movements. For instance :

- a translation in the x direction,
- a translation in the y direction,
- a rotation around and axis parallel to z.

Question7 : Boundary conditions : Nodal displacements vector (q).

The physicals joins impose physical displacements.
For instance, the cross section of the beam in the point A is clamped. Therefore the translations and the rotation of this cross section are equal to zero.

According to that we can write the displacements of the different nodes concerned :

$$(q) = \begin{pmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 \\ \theta_2 \\ v_3 = 0 \\ \theta_3 = 0 \\ v_4 = 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Question 8 : Reaction Load

When a degree of freedom (DOF) is removed or imposed by a join an internal reaction appears :

- this reaction is a force when the DOF is a translation,
- this reaction is a moment when the DOF is a rotation.

For this study we have :

$$(F_{INT}) = \begin{pmatrix} Y_1 \\ M_1 \\ 0 \\ 0 \\ Y_3 \\ M_3 \\ Y_4 \end{pmatrix}$$

Question 9 : Equivalent nodal load

The external load is introduced by a distributed load. The computer needs, to solve the problem, an equivalent nodal load. We must replace the distributed load by a nodal equivalent load.

Let us consider a beam defined by two nodes as shown on the following figure :

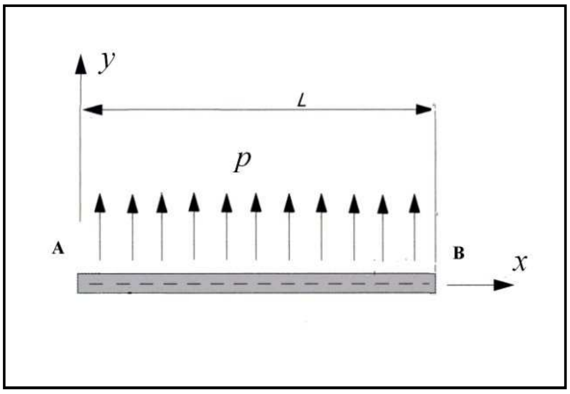


Figure 11 : Distributed load for a generic beam element

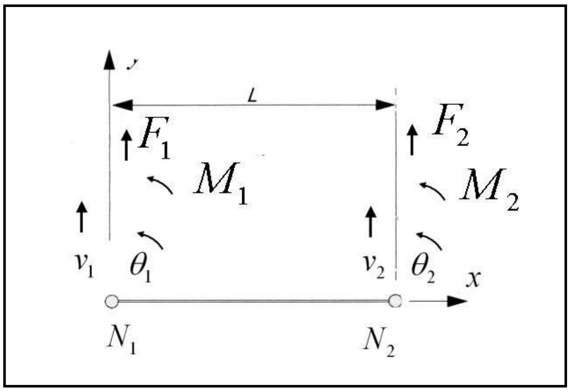


Figure 12 : Equivalent Nodal Load

To find the equivalent nodal load we generally say that the work of the nodal load in the nodal displacement is equal to the work of the real load in the interpolated field of displacement.

This can be written like this :

$$\boxed{v_1 F_1 + \theta_1 M_1 + v_2 F_2 + +\theta_2 M_2 = \int_0^L v(x) p(x) dx} \qquad \text{Eq 3}$$

If we remember that the shape functions are :

$$v(x) = \left(\left\{ 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \right\} \quad L\left\{ \frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3} \right\} \quad \left\{ 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \right\} \quad L\left\{ -\frac{x^2}{L^2} + \frac{x^3}{L^3} \right\} \right) \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

We can write the equation (Eq3) like this :

$$\begin{pmatrix} F_1 & M_1 & F_2 & M_2 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}_1 = \int_0^L \left(\{\Phi_1\} \quad \{\Phi_2\} \quad \{\Phi_3\} \quad \{\Phi_4\} \right) \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} p(x) dx$$

With

$$\Phi_1 = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

$$\Phi_2 = L\left(\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3} \right)$$

$$\Phi_3 = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$

$$\Phi_4 = L\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3} \right)$$

This relation give the following results :

$$F_1 = \int_0^L \Phi_1 \times p(x)\,dx$$
$$M_1 = \int_0^L \Phi_2 \times p(x)\,dx$$
$$F_2 = \int_0^L \Phi_3 \times p(x)\,dx$$
$$M_2 = \int_0^L \Phi_4 \times p(x)\,dx$$

Finally we have the nodal equivalent load :

$$F_1 = \int_0^L p\left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right)dx$$
$$M_1 = \int_0^L pL\left(\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)dx$$
$$F_2 = \int_0^L p\left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)dx$$
$$M_2 = \int_0^L pL\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)dx$$

After computation we find :

$$\boxed{\begin{aligned} F_1 &= \frac{pL}{2} \\ M_1 &= \frac{pL^2}{12} \\ F_2 &= \frac{pL}{2} \\ M_2 &= -\frac{pL^2}{12} \end{aligned}} \quad \text{(Eq 4)}$$

Question 10 : External Load

The Equation (EQ 4) allows us to define the equivalent nodal load for the structure like this :

$$
(F_{EXT}) = \begin{pmatrix} \frac{pL}{2} \\ \frac{pL^2}{12} \\ \frac{pL}{2} \\ -\frac{pL^2}{12} \\ \frac{0}{} \\ \frac{0}{} \\ \frac{0}{} \end{pmatrix} + \begin{pmatrix} \frac{0}{} \\ \frac{0}{} \\ \frac{pL}{2} \\ \frac{pL^2}{12} \\ \frac{pL}{2} \\ -\frac{pL^2}{12} \\ \frac{0}{} \end{pmatrix} = \begin{pmatrix} \frac{pL}{2} \\ \frac{pL^2}{12} \\ \frac{2pL}{2} \\ \frac{0}{} \\ \frac{pL}{2} \\ -\frac{pL^2}{12} \\ \frac{0}{} \end{pmatrix}
$$

Question 11 : Global load

The global load is the sum of the reaction load (unknown) and the external load (well known)

$$
(F) = \begin{pmatrix} \frac{pL}{2} \\ \frac{pL^2}{12} \\ \frac{2pL}{2} \\ 0 \\ \frac{pL}{2} \\ -\frac{pL^2}{12} \\ 0 \end{pmatrix} + \begin{pmatrix} Y_1 \\ M_1 \\ 0 \\ 0 \\ Y_3 \\ M_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} Y_1 + \frac{pL}{2} \\ M_1 + \frac{pL^2}{12} \\ pL \\ 0 \\ Y_3 + \frac{pL}{2} \\ M_3 - \frac{pL^2}{12} \\ Y_4 \end{pmatrix}
$$

Question 12 : Solve the equation $(F) = [K](q)$.

The fundamental equation of the finite element theory $(F) = [K](q)$ can be written like this according to the real boundary conditions of the structure studied :

$$\boxed{\begin{pmatrix} Y_1+\frac{pL}{2} \\ M_1+\frac{pL^2}{12} \\ pL \\ 0 \\ Y_3+\frac{pL}{2} \\ M_3-\frac{pL^2}{12} \\ Y_4 \end{pmatrix} = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24+\alpha & 0 & -12 & 6L & -\alpha \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & 6L & 12 & -6L & 0 \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & \alpha \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}} \text{(Eq 5)}$$

If we remove the lines and columns for which the displacements are well known, we obtain :

$$\begin{pmatrix} pL \\ 0 \end{pmatrix} = \frac{EI_z}{L^3}\begin{bmatrix} 24+\alpha & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix}$$

This equation gives the unknown displacement :

$$\boxed{\begin{array}{l} v_2 = \dfrac{pL^4}{EI_z\left(24+\alpha\right)} = \dfrac{4pL^4}{E\left(96I_z+SL^2\right)} \\ \theta_2 = 0 \end{array}}$$

Numerical application :

$$\begin{array}{l} L = 1 \quad m \\ S = 1600 \quad mm^2 \\ E = 200000 \quad MPa \\ I_z = 2653333 \quad mm^4 \\ p = 100 \quad N/mm \end{array}$$

$$\alpha = \frac{SL^2}{I_z} = 150.753788$$

$$v_2 = 3.05 \quad mm$$

Question 13 : Reactions

The equation (Eq 5) allows the computer to find the unknown reactions :

$$Y_1 = -\frac{pL}{2} - \frac{12EI_z}{L^3} v_2$$
$$M_1 = -\frac{pL^2}{12} - \frac{6EI_z}{L^2} v_2$$
$$Y_3 = -\frac{pL}{2} - \frac{12EI_z}{L^3} v_2$$
$$M_3 = \frac{pL^2}{12} + \frac{6EI_z}{L^2} v_2$$
$$Y_4 = -\frac{ES}{L} v_2$$

Numerical application :

$Y_1 = Y_3 =$ N
$M_1 = M_3 =$ mN
$Y_4 =$ N

Question 14 : Normal stresses.

The Normal stress is given by the beam theory. We have :

$$\sigma_x = -\frac{M_z}{I_z} y$$

but we have another relation :

$$\frac{d^2 v}{dx^2} = \frac{M_z}{EI_z}$$

Finally we find :

$$\boxed{\sigma_x = -E \frac{d^2 v}{dx^2} y} \qquad \text{EQ 6}$$

We can extract the normal stress from the first beam element or from the second beam element.

Extraction from the first beam element :

The first beam element is defined between the nodes 1 and 2 and for it the value of x will be taken equal to L to find the different values at the point B,

- The second beam element is defined between the nodes 2 and 3 and to find the equivalent values we have to take x as equal to 0,

The equation (Eq 0) fives after derivation for the first element :

$$\frac{d^2v(x)}{dx^2} = \left(\left\{ -\frac{6}{L^2} + \frac{12x}{L^3} \right\} \quad \left\{ -\frac{6}{L} + \frac{6x}{L^2} \right\} \quad \left\{ \frac{6}{L^2} - \frac{12x}{L^3} \right\} \quad \left\{ -\frac{2}{L} + \frac{6x}{L^2} \right\} \right) \begin{pmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 \\ \theta_2 = 0 \end{pmatrix}$$

As only is different of zero this equation becomes :

$$\frac{d^2v(x)}{dx^2} = \left(\frac{6}{L^2} - \frac{12x}{L^3} \right) v_2$$

If we introduce the value of v2 we have :

$$\frac{d^2v(x)}{dx^2} = \left(\frac{6}{L^2} - \frac{12x}{L^3} \right) \frac{pL^4}{E\left(24I_z + SL^2\right)}$$

Then the equation EQ 6 gives :

$$\sigma_x = -\frac{6pL\left(L-2x\right)}{\left(24I_z + SL^2\right)} y$$

For $x = L$ we find the value of the normal stress for the points of the cross section C :
:

$$\boxed{\sigma_x = \frac{6pL^2}{\left(24I_z + SL^2\right)} y}$$

Extraction from the second beam element :

The second beam element is defined between the nodes 2 and 3 and to find the equivalent values we have to take x as equal to 0,

The equation (Eq 0) fives after derivation for the first element :

$$\frac{d^2v(x)}{dx^2} = \left(\left\{ -\frac{6}{L^2} + \frac{12x}{L^3} \right\} \quad \left\{ -\frac{6}{L} + \frac{6x}{L^2} \right\} \quad \left\{ \frac{6}{L^2} - \frac{12x}{L^3} \right\} \quad \left\{ -\frac{2}{L} + \frac{6x}{L^2} \right\} \right) \begin{pmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 \\ \theta_2 = 0 \end{pmatrix}$$

If we use the same computation that for the beam element we find :

$$\boxed{\sigma_x\left(0\right) = \frac{6pL^2}{24I_z + SL^2} y}$$

Remark : We find the same value for the normal stress in each of the two elements.

Numerical application :

The cross section has been selected inside a library. The computer knows the value of the coordinate y for the 4 specific different points of the cross section named C,D,E,F (see the figure 1 and the following table).

Point	Y	Z
C	50 mm	-30 mm
D	-50 mm	-30 mm
E	-50 mm	30 mm
F	50 mm	30 mm

The computer can compute the value of the normal stress for theses different points.

The results are :

Point	σ_x	Unit
C	-64.7	Mpa
D	64.7	Mpa
E	64.7	Mpa
F	-64.7	Mpa

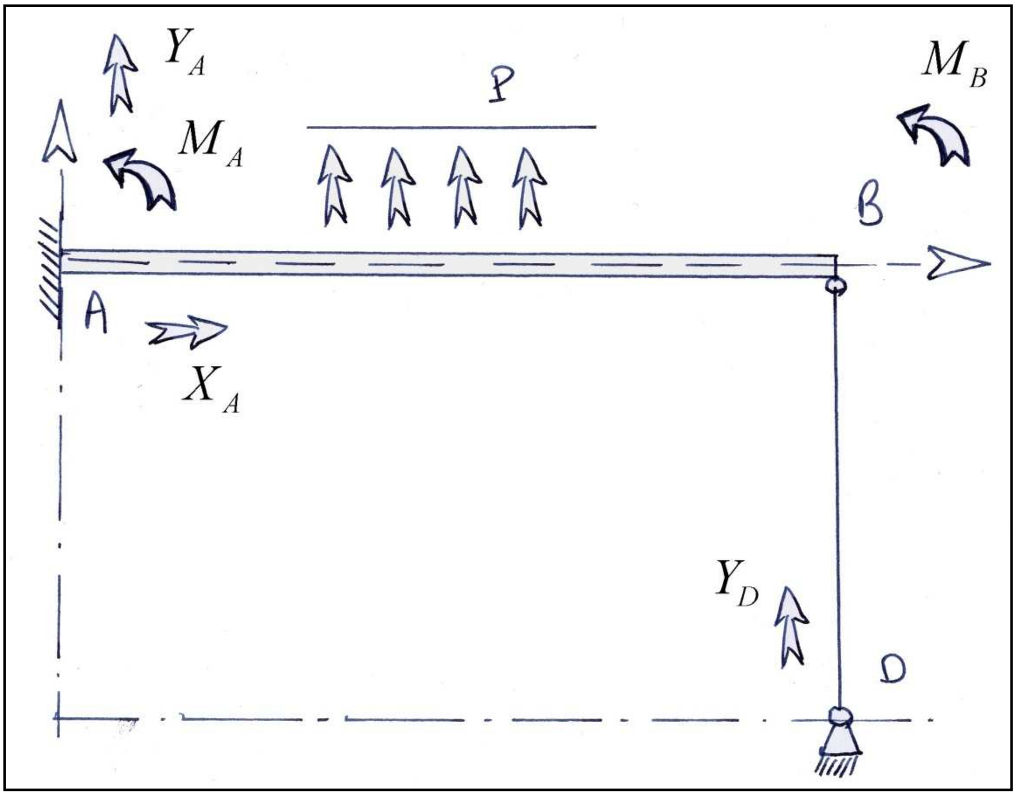
Question 15 : Shear stresses.

The displacement is calculated with respect of the shear force, but as the compute knows only the mechanical coefficients of the beam; it's impossible for it to compute the shear stress.

Question 18 : Analytical solution.

As there is a plane of symmetry for the geometry and for the load we can reduce the study to the half part of the structure. Take care that the cross section of the bar is now $S/2$.

The following figure gives the notations used :



Half study of the structure

Equilibrium of the whole structure :

$$\begin{cases} Y_A + pl + Y_D = 0 \\ X_A + X_B = 0 \\ M_A + P\dfrac{L^2}{2} + M_B + Y_D L = 0 \end{cases}$$

Bending moment in the beam:

$$M_z = p\frac{(L-x)^2}{2} + Y_D(L-x) + M_B$$

Shear Force in the horizontal beam :

$$T_Y = p(L-x) + Y_D$$

Normal Force in the beam :

$$N_X = X_B$$

Internal Energy :

$$W = \int_0^L \frac{1}{2EIz}\left[p\frac{(L-x)^2}{2} + Y_D(L-x) + M_B \right]^2 dx + \int_0^L \frac{1}{2GK_yS}\left[p(L-x) + Y_D \right]^2 dx + \frac{1}{2}\frac{N_X^2 L}{ES^*}$$

If we neglect the internal energy introduce by the shear force we have :

$$W = \int_0^L \frac{1}{2EIz}\left[p\frac{(L-x)^2}{2} + Y_D(L-x) + M_B \right]^2 dx + \frac{1}{2}\frac{N_X^2 L}{ES^*}$$

Castigliano’s Theorem :

$$\boxed{\begin{aligned} \theta_B^z &= \frac{\partial W}{\partial M_B} = 0 \\ v_D &= \frac{\partial W}{\partial Y_D} = 0 \\ u_B &= \frac{\partial W}{\partial X_B} = 0 \end{aligned}}$$

The first of these three equations give :

$$\frac{\partial W}{\partial M_B} = \int_0^L \frac{1}{EIz}\left[p\frac{(L-x)^2}{2} + Y_D(L-x) + M_B \right] dx = 0$$

$$\left[-p\frac{(L-x)^3}{6} - Y_D\frac{(L-x)^2}{2} + M_B x \right]_0^L = 0$$

$$\boxed{pL^2 + 3LY_D + 6M_B = 0} \qquad \text{(Eq 7)}$$

The second equation gives :

$$\frac{\partial W}{\partial Y_D} = \int_0^L \frac{1}{EIz}\left[p\frac{(L-x)^3}{2} + Y_D(L-x)^2 + M_B(L-x) \right] dx = 0$$

$$\left[-p\frac{(L-x)^4}{8} - Y_D\frac{(L-x)^3}{3} - \frac{(L-x)^2}{2}M_B \right]_0^L = 0$$

$$\boxed{3pL^2 + 8Y_D L + 12M_B = 0} \qquad \text{(Eq 8)}$$

The third equation give :

$$\frac{\partial W}{\partial X_B} = \frac{N_B L}{ES} = 0$$

$$\boxed{N_B = 0}$$

Results :

If we introduce the numerical values in the equation 7 and 8, we obtain a set of 2 equations with 2 unknown quantities.

$$\begin{cases} -2pL^2 - 6LY_D - 12M_B = 0 \\ 3pL^2 + 8Y_D L + 12M_B = 0 \end{cases}$$

$$3pL^2 + 8Y_D L + 12M_B = 0$$

Question 19 : Stresses in the point A and B with the beam theory.

$$\boxed{\sigma_x = -\frac{M_Z}{I_Z} y}$$

Question 20 : Value of the normal force in the bar BD.

Question 21 : Comparison

Exercise 7

Finite Elements
Assembling of a set of symmetric bars and beams

We consider the structure described by the figure 1. We shall do the following assumption to create a modelling with the finite element method :

- the part DC will be considered as a beam element,
- the part BD, BA and BC will be considered as bar elements.

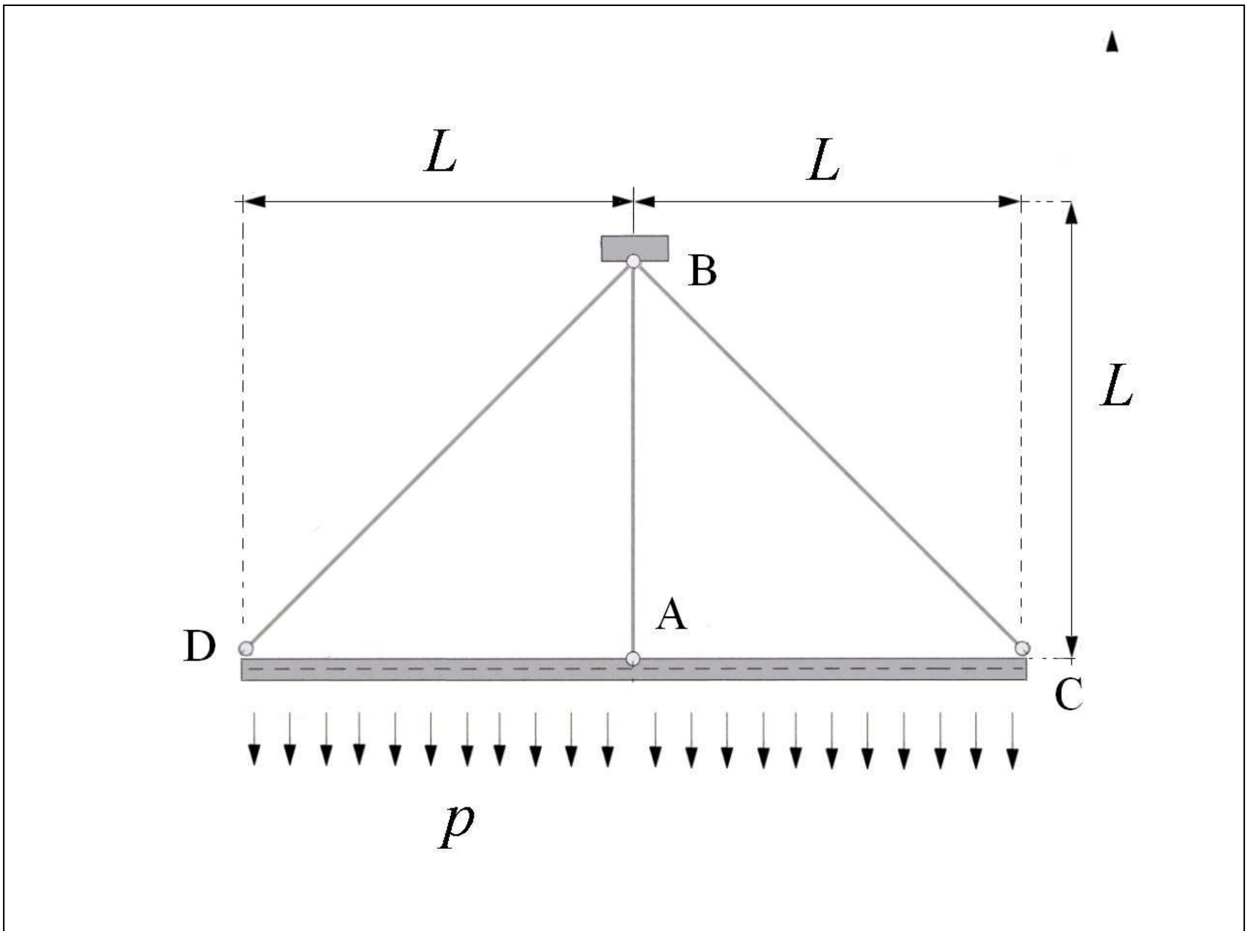


Figure 1 : Set of different bars

The different beams and bars are made of an elastic material whose properties are :
- Young's modulus : E
- Poisson's ratio : ν

The area of the cross section of bars and beam is the same :
- Area of cross section of beam and bars : S
- The length of the bar AB is : L
- The length of the bars DB and DC is : $L\sqrt{2}$
- The length of the beam is : $2L$
- The quadratic moments of the beam : I_Z *and* I_y

The different joins are :

- at the point B : a join ball between all the bars,
- at the points D, A and C : a join ball between each bar and the beam,

The load apply consist only in a distributed load which can represents the weight of the beam for instance. The weight of the bars can be neglected in this study.

To simplify the problem, we shall neglect, in the beam, the effect of the shear force T_y.

Questions

20 Simplify the study with consideration on symmetry.
21 Write the boundary conditions in term of displacements for the nodes inside the plane of symmetry,
22 If you study only the half part of the structure, what is the cross section to take in account for the bar AB
23 Choose a set of nodes and represent them with the corresponding DOF.
24 Create the global stiffness matrix for this structure,
25 Boundary conditions : Write the displacements in a column vector (q).
26 Define the load vector as the sum of the external and internal load.
27 Write the fundamental equation of the finite element method.
28 Write the equation, without solving it, to find the displacement unknown.
29 Write the equation to find the unknown reactions.
30 Find the different displacements for this problem with the following numerical values:

$$\begin{aligned} E &= 78 \quad GPa \\ L &= 1200 \quad mm \\ I_z &= 1{,}2.10^7 \quad mm^4 \\ S &= 130 \quad mm^2 \\ P &= 2000 \quad N/m \end{aligned}$$

Solution of the exercise 7

Question 1 : Half study of the structure.

As the structure studied have a plane of symmetry for the load and the geometry we can study only the half part of it.

Question 2 : Boundary conditions in the plane of symmetry.

The plane of symmetry cannot have a translation in a normal direction to it, and rotations are not allowed for any axis in the plane of symmetry.

$$node \quad N_1 \quad \begin{cases} u_1 = 0 \\ \theta_1 = 0 \end{cases}$$

$$node \quad N_2 \quad \{u_2 = 0$$

Question 3 : Cross section of the vertical bar.

The cross section of the bar in the plane of symmetry must be changed in $S/2$ to respect the symmetry of the problem

Question 4 : Nodes and DOF.

As we solve this problem with a 2D approach, the degree of freedom will be the following :

- for each node of a beam 3 DOF : 2 translations and 1 rotation,
- for each node of a bar 2 DOF : 2 translations

The following figure shows the different notations used afterwards.

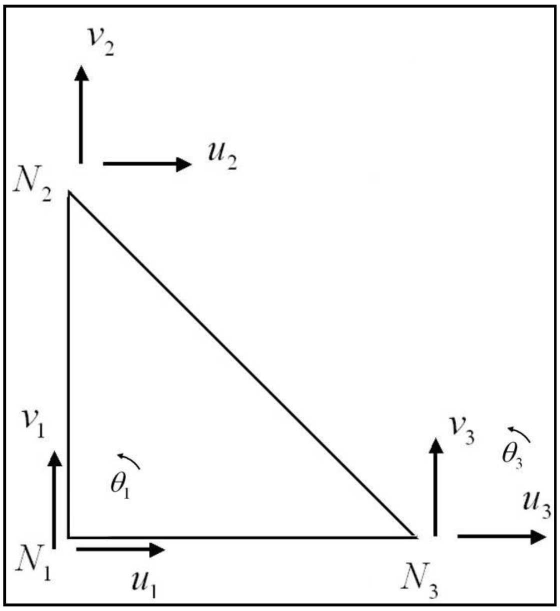


Figure 2 : Modelling - Nodes & DOF

The displacement vector (q) can be written in a matrix column, with :

$$\boxed{(q)^T = (u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad v_3 \quad u_3 \quad \theta_3)}$$

Question 5.1: Stiffness matrix of the vertical bar, between the nodes N_1 and N_2.

For this bar the internal energy is created in the field of displacement v_1 and v_2. We have to remember that the cross section of the vertical bar must be divided by 2.

$$[K]_{12} = \begin{bmatrix} \frac{ES}{2L} & -\frac{ES}{2L} \\ -\frac{ES}{2L} & \frac{ES}{2L} \end{bmatrix} \qquad \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix}$$

Question 5.2 : Stiffness matrix of the bar between the nodes N_2 and N_3 inclined with a negative angle with $\alpha = -\pi/4$.

This bar has a length equal to $L\sqrt{2}$.

The stiffness matrix of this bar in the global axis system is given by the following relation:

$$[K]_{23} = \begin{bmatrix} [P] & [0] \\ [0] & [P] \end{bmatrix} \begin{bmatrix} \frac{ES}{L\sqrt{2}} & 0 & -\frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L\sqrt{2}} & 0 & \frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [P]^T & [0] \\ [0] & [P]^T \end{bmatrix}$$

With

$$[P] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$[P]^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We have :

$$[K]_{23} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \frac{ES}{L\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

That gives :

$$[K]_{23} = \frac{ES\sqrt{2}}{4L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Question 5.3 : Stiffness matrix of the beam.

As we neglect the effect of the shear force, we have just to take in account the internal energy of the bending moment and of the normal force.

$$[K]_{13} = \begin{bmatrix} \frac{ES}{L} & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\ 0 & 12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\ -\frac{ES}{L} & 0 & 0 & \frac{ES}{L} & 0 & 0 \\ 0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & 0 & 12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}$$

Question 5.4 : Stiffness matrix of the structure.

$$[K] = [K]_{12} + [K]_{23} + [K]_{13}$$

$$[K]=\begin{bmatrix}
\frac{ES}{L} & 0 & 0 & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\
0 & 12\frac{EI_z}{L^3}\frac{ES}{2L} & 6\frac{EI_z}{L^2} & 0 & -\frac{ES}{2L} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\
0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\
0 & 0 & 0 & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & 0 \\
0 & -\frac{ES}{2L} & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES}{2L}+\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\
-\frac{ES}{L} & 0 & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\
0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 12\frac{EI_z}{L^3}+\frac{ES\sqrt{2}}{4L} & -6\frac{EI_z}{L^2} \\
0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L}
\end{bmatrix}$$

Question 6 : Displacement vector

The boundary conditions are :
- a join ball, at the node N_2 , which impose the two translation to be equal to zero,
- the symmetry, at the node N_1, with respect of the vertical plane which impose :
  - a rotation for the node N_1 equal to zero,
  - a translation in the x direction for the node N_1 to be equal to zero.

$$(q)=\begin{pmatrix} u_1=0 \\ v_1 \\ \theta_1=0 \\ u_2=0 \\ v_2=0 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}=\begin{pmatrix} 0 \\ v_1 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}$$

Question 7 : Load vector

The load vector is the sum of the two nodal loads :
- External forces applied on the nodes,
- Internal reactions.

$$(F)=\begin{pmatrix} X_1 \\ 0 \\ M_1 \\ X_2 \\ Y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -pL/2 \\ -pL^2/12 \\ 0 \\ 0 \\ 0 \\ -pL/2 \\ pL^2/12 \end{pmatrix}$$

Question 9 : Fundamental equation of the finite element

$$\begin{pmatrix} X_1 \\ -pL/2 \\ M_1-pL^2/12 \\ X_2 \\ Y_2 \\ 0 \\ -pL/2 \\ pL^2/12 \end{pmatrix}=\begin{bmatrix} \frac{ES}{L} & 0 & 0 & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\ 0 & 12\frac{EI_z}{L^3}\frac{ES}{L} & 6\frac{EI_z}{L^2} & 0 & -\frac{ES}{L} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\ 0 & 0 & 0 & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & 0 \\ 0 & -\frac{ES}{L} & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\ -\frac{ES}{L} & 0 & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\ 0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 12\frac{EI_z}{L^3}+\frac{ES\sqrt{2}}{4L} & -6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix}\begin{pmatrix} 0 \\ v_1 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}$$

Question 9 : Equations to solve the problem.

If we remove the lines and columns for which the displacements is imposed we can reduce the problem to a matrix equation with a solution :

$$\begin{pmatrix} -pL/2 \\ -pL^2/12 \\ -pL/2 \\ pL^2/12 \end{pmatrix}=\begin{bmatrix} 12\frac{EI_z}{L^3}+\frac{ES}{2L} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\ -12\frac{EI_z}{L^3} & -\frac{ES\sqrt{2}}{4L} & 12\frac{EI_z}{L^3}+\frac{ES\sqrt{2}}{4L} & -6\frac{EI_z}{L^2} \\ 6\frac{EI_z}{L^2} & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix}\begin{pmatrix} v_1 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}$$

Question 10 : Values of the reactions.

Once the displacement are well known by the resolution of the equation (Eq3), we can find the reactions by using the complete equation $(F)=[K](q)$:

$$\begin{pmatrix} X_1 \\ -pL/2 \\ M_1 - pL^2/12 \\ X_2 \\ Y_2 \\ 0 \\ -pL/2 \\ pL^2/12 \end{pmatrix} = \begin{bmatrix} \frac{ES}{L} & 0 & 0 & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\ 0 & 12\frac{EI_z}{L^3}\frac{ES}{L} & 6\frac{EI_z}{L^2} & 0 & -\frac{ES}{L} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\ 0 & 0 & 0 & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & 0 \\ 0 & -\frac{ES}{L} & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\ -\frac{ES}{L} & 0 & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\ 0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 12\frac{EI_z}{L^3}+\frac{ES\sqrt{2}}{4L} & -6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix} \begin{pmatrix} 0 \\ v_1 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix}$$

$$\boxed{\begin{aligned} &X_1 = -\frac{ES}{L}u_3 \\ &M_1 = \frac{pL^2}{12} + 6\frac{EI_z}{L^2}(v_1 - v_3) + 2\frac{EI_z}{L}\theta_3 \\ &X_2 = \frac{ES\sqrt{2}}{4L}(v_3 - u_3) \\ &Y_2 = -\frac{ES}{L}v_1 + \frac{ES\sqrt{2}}{4L}(u_3 - v_3) \end{aligned}}$$

Exercise 9

Finite Elements
Bars with constant distributed load

Part 1

A column represented on figure 1, is made of light alloy with E as Young's Modulus.

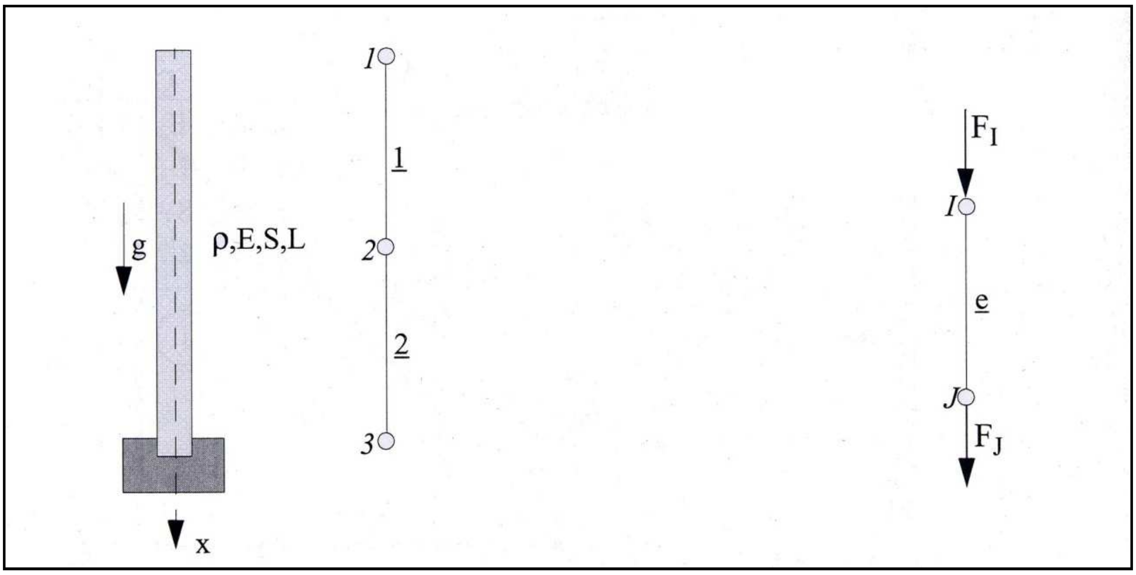


Figure 1 : Beam under its own weight

The mechanical properties of this column are :

The length	$2L = 2 \quad m$
The cross section	$S = 25 \quad cm^2$
The density	$\rho = 7.8 \quad Kg / dm^3$

This column is subjected to its own weight.

Questions :

1 – Propose a modelling with 3 nodes for this structure, as indicated on figure 1.
2 – Find the nodal load F_I and F_J equivalent to the distributed load created by the weight of the beam,
3 – Write under a matrix form, the unknown displacement vector,
4 – Write under a matrix form, the load vector,
5 – Solve the finite element problem and find the nodal displacement,
6 – Compare the results with the analytical solution given by the beam theory, in terms of reaction, displacement and stresses,
7 – Propose a method to increase the accuracy of the results.

Exercise 9

Finite Elements
Bars with non constant distributed load

A blade in rotation represented on figure 1, is made of light alloy with E as Young's Modulus. This blade is submitted to an inertial load. We shall neglect in this study the other effects like, the bending moment and the shear force.

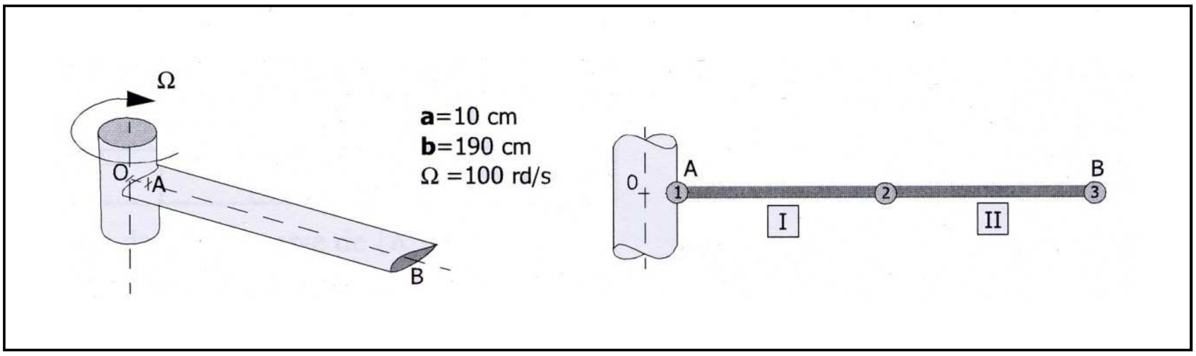


Figure 1 : Blade submitted to inertial load

The mechanical properties of the blade are :

The length	$2L \; = \; 1.8 \quad m$
The cross section	$S \; = \; 300 \quad cm^2$
The density	$\rho \; = \; 7.8 \quad Kg/dm^3$
The Young's modulus	$E \; = \; 200000 \quad MPa$

We propose a modelling with 3 nodes for this structure, and two bar elements of the same length, as indicated on figure 1.

Questions :

1 – Solve this problem with an analytical solution by using the beam theory. For that you will consider only the effect of the normal force N_x,

1 – 1– Find the reaction, on the side clamped,
1 – 2– Find the weight of the blade,
1 – 3– Compare this reaction with the weight of the blade,
1 – 4– Find the displacement of the different points of the beam,
1 – 5– Find the stresses for each point,
1 – 6– Plot the normal stress when x ranges from a to b,

1 – 7– Give the numerical values of the displacements, the strains and the normal stresses for the points defined by $\begin{cases} r = \; 0.1 \quad m \\ r = \; 1.0 \quad m \\ r = \; 1.9 \quad m \end{cases}$

2 – Write the form functions for the displacement inside a bar element,
3 – Find the nodal load F_I and F_J equivalent to the inertial load created by the high rotation speed of the blade. For that use the notations of the following figure :

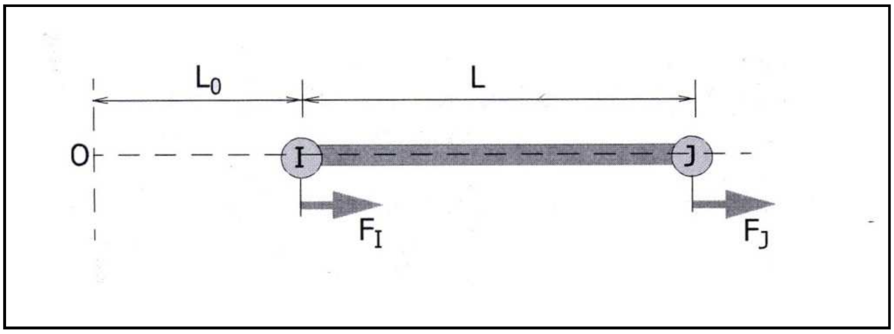


Figure 2 : Element of bar submitted to an inertial load

4 – Write the stiffness matrix of this structure,
5 – Write under a matrix form, the unknown displacement vector,
6 – Write under a matrix form, the load vector,
7 – Solve the finite element problem and find the nodal displacement u_2 and u_3,
8 – Find the numerical values for the displacement u_2 and u_3,
9 – Compare these values with the ones given by the analytic solution,
10 – Compare these values with the following ones given by the software NASTRAN,

$$\begin{cases} u_1 & = & 0.000000 \quad 10^{0} \\ u_2 & = & 6.160054 \quad 10^{-1} \\ u_3 & = & 9.161106 \quad 10^{-1} \end{cases}$$

11 – Find the numerical values for the reaction on the clamped side,
12 – Compare this value with the one given by the analytic solution,
13 – Compare this value with the following one given by the software NASTRAN,

$$\begin{cases} X_1 & = & 2.1060012 \quad 10^{6} \end{cases}$$

14 – Find the numerical values for the normal stresses in the elements,
15 – Compare these values with the ones given by the analytic solution,
16 – Compare theses values with the following ones given by the software NASTRAN,

$$\begin{cases} \sigma_x & = & \qquad MPa \\ \sigma_x & = & \qquad MPa \end{cases}$$

17 – Propose a method to increase the accuracy of these results.
18 – You will find below, the values of displacements, strains, nodal load and stresses obtained with NASTRAN and with a modelling using 18 bars element. Compare the analytical and finite element results.

Solution of the exercise 9

Question 1 : Analytical solution

1 – 1– Reaction, on the clamped side :

The reaction is given by the following equation of equilibrium:

$$X_1 + \int_a^b \rho S \omega^2 r dr = 0$$

That give :

$$\boxed{X_1 = -\rho S \omega^2 \frac{b^2 - a^2}{2} = -4.21 \quad 10^6 \quad N}$$

1 – 2– Find the weight of the blade,

1 – 3– Compare this reaction with the weight of the blade,

1 – 4– Find the displacement of the different points of the beam,
1 – 5– Find the stresses for each points,
1 – 6– Plot the normal stress when x ranges from a to b,

1 – 7– Give the numerical values of the displacements, the strains and the normal stresses for the points defined by $\begin{cases} r = & 0.1 \quad m \\ r = & 1.0 \quad m \\ r = & 1.9 \quad m \end{cases}$

Question 1 : Form functions

The form functions describe the field of displacement inside a bar. It's a linear function of x.

$$u(X) = \frac{X}{L}\left(u_J - u_I\right) + u_I$$

We can verify that this formula allows us to verify the boundary conditions :

- for $x = 0$ we have $x(o) = u_I$
- for $x = L$ we have $x(L) = u_J$

This form function can be written under a matrix form as following :

$$\boxed{u(X) = \begin{pmatrix} \frac{L-X}{L} & , & \frac{X}{L} \end{pmatrix} \begin{pmatrix} u_I \\ u_J \end{pmatrix}} \qquad (1)$$

Question 2 : Equivalent nodal load

To find the equivalent nodal load, we will write that the work of the distributed load inside the interpolated displacement, is equal to the work of the nodal load inside the nodal displacement.

The equation (1) gives the form functions of the bar, inside the axis system of the bar.

The distributed load is an inertial load. To find it, we can consider a small slice of the bar, with dx as thickness. This slice is submitted to an inertial load. This inertial load is equal to :

$$dF = dm\omega^2 r = \rho S\omega^2 r dr$$

The work or this inertial load inside the interpolated displacement of the bar is :

$$\boxed{W = \int_{L_0}^{L_0+L} u(r)dF = \int_{L_0}^{L_0+L} u(r)\rho S\omega^2 r dr} \quad (2)$$

As we have the relation $r = L_0 + X$ the equation (2) becomes :

$$\boxed{W = \int_{L_0}^{L_0+L} \rho S\omega^2 r \begin{pmatrix} \frac{L-X}{L} & , & \frac{X}{L} \end{pmatrix} \begin{pmatrix} u_I \\ u_J \end{pmatrix} dr}$$

$$\boxed{W = \int_{L_0}^{L_0+L} \rho S\omega^2 r \begin{pmatrix} \frac{L-r+L_0}{L} & , & \frac{r-L_0}{L} \end{pmatrix} \begin{pmatrix} u_I \\ u_J \end{pmatrix} dr} \qquad (3)$$

The work of the nodal load inside the nodal displacement is equal to :

$$\boxed{W = F_I.u_I + F_J.u_J = \begin{pmatrix} F_I & , & F_J \end{pmatrix} \begin{pmatrix} u_I \\ u_J \end{pmatrix}} \qquad (4)$$

If we identify (3) and (4) we find, the equivalent nodal load.

$$F_I = \int_{L_0}^{L_0+L} \rho S \omega^2 r \frac{L - r + L_0}{L} dr$$

$$F_J = \int_{L_0}^{L_0+L} \rho S \omega^2 r \frac{r - L_0}{L} dr$$

The integration gives

$$\boxed{\begin{aligned} F_I &= \rho S \omega^2 L \left(\frac{Lo}{2} + \frac{L}{6} \right) \\ F_J &= \rho S \omega^2 L \left(\frac{Lo}{2} + \frac{L}{3} \right) \end{aligned}} \qquad (5)$$

Question 3 :Stiffness matrix

$$K = \frac{ES}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Question 4 : Displacement vector

$$(q) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix}$$

Question 5 : Load Vector

$$(F) = \rho S \omega^2 L \begin{pmatrix} \frac{5a+b}{12} \\ \frac{4a+2b}{12} \\ 0 \end{pmatrix} + \rho S \omega^2 L \begin{pmatrix} 0 \\ \frac{2a+4b}{12} \\ \frac{a+5b}{12} \end{pmatrix} + \begin{pmatrix} X_1 \\ 0 \\ 0 \end{pmatrix}$$

The load vector is :

$$(F)=\begin{pmatrix}\rho S\omega^2 L\dfrac{5a+b}{12}+X_1\\ \rho S\omega^2 L\dfrac{6a+6b}{12}\\ \rho S\omega^2 L\dfrac{a+5b}{12}\end{pmatrix}$$

A numerical application gives :

$$(F)=\begin{pmatrix}421200\\2106000\\1684800\end{pmatrix}+\begin{pmatrix}X_1\\0\\0\end{pmatrix}$$

1.053001E+05
2.106001E+06
2.000701E+06

0.0
6.160054E-01
9.161106E-01

Question 6 : Resolution

$$\begin{pmatrix}\rho S\omega^2 L\dfrac{5a+b}{12}+X_1\\ \rho S\omega^2 L\dfrac{6a+6b}{12}\\ \rho S\omega^2 L\dfrac{a+5b}{12}\end{pmatrix}=\frac{ES}{L}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{pmatrix}0\\u_2\\u_3\end{pmatrix}\qquad \textbf{(6)}$$

If we remove the line and column for the well known displacement we have :

$$\frac{\rho S\omega^2 L}{12}\begin{pmatrix}6a+6b\\a+5b\end{pmatrix}=\frac{ES}{L}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{pmatrix}u_2\\u_3\end{pmatrix}$$

This equation can be written like this :

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=\frac{\rho\omega^2 L^2}{12E}\begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{pmatrix}6a+6b\\a+5b\end{pmatrix}$$

The literal values are :

$$\boxed{\begin{aligned} u_2 &= \frac{\rho\omega^2 L^2}{12E}(7a + 11b) \\ u_3 &= \frac{\rho\omega^2 L^2}{12E}(8a + 16b) \end{aligned}}$$

The numerical values are :

$$\boxed{\begin{aligned} u_2 &= 0.00 \quad 10^{-4} \quad m \\ u_2 &= 5.69 \quad 10^{-4} \quad m \\ u_3 &= 8.21 \quad 10^{-4} \quad m \end{aligned}}$$

Question 7 : Analytical solution with the beam theory

The relation ship between stresses and strains allows to find the normal stresses inside each bar :

For the first bar, between nodes 1 and 2, we have :

$$\sigma_x = \frac{(u_2 - u_1)}{L} E = 126.4 \quad MPa$$

For the second bar, between nodes 2 and 3, we have :

$$\sigma_x = \frac{(u_3 - u_2)}{L} E = 56.2 \quad MPa$$

The equation (6) give the reaction :

$$X_1 = -\left(\rho S \omega^2 L \frac{5a + b}{12} + \frac{ES}{L} u_2 \right)$$

$$\boxed{X_1 = -\rho S \omega^2 L (a + b)}$$

$$\boxed{X_1 = -4.21 \quad 10^6 \quad N}$$

Question 8 : Analytical solution with the beam theory

The normal force on a cross section defined by its abscissa r is :

$$N_x = \int_r^b \rho S \omega^2 r dr = \rho S \omega^2 \frac{b^2 - r^2}{2}$$

The normal stress is :

$$\boxed{N_x = \rho \omega^2 \frac{b^2 - r^2}{2}}$$

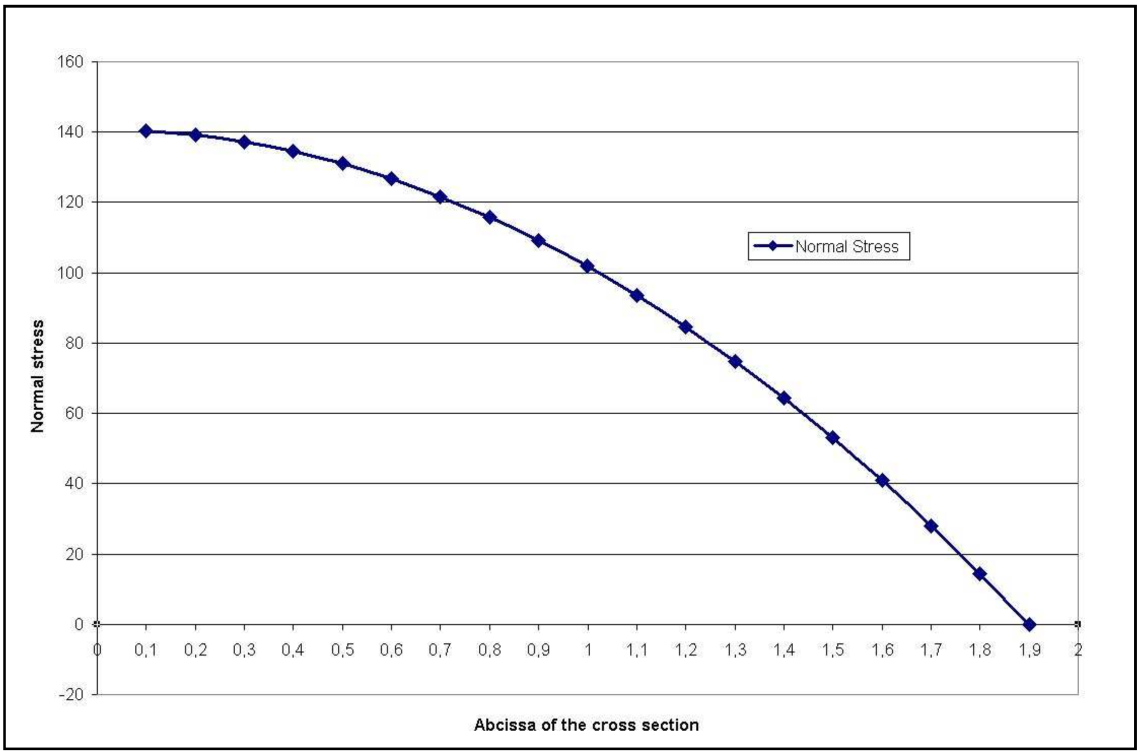


Figure 5 : Evolution of the normal stress in span

Question 9 : Weight of the blade

The weight of the blade is :

$$Weight = \rho S (b - a) g$$
$$Weight = 4132 \quad N$$

The ratio between the inertial load and the weight is bigger than 1000. The modelling with bars is a good modelling.

Exercise 8

Finite Elements

Study of a plate with different meshing of CST Constant Strain Triangle

We consider a squarred plate whose dimensions are :

- L for the length of each side,
- e for the thickness of the plate.

This plate is clampted on the left side. Two different modelling of this plate with 2 triangular membrane elements are proposed. We want to compare the results in term of displacement.

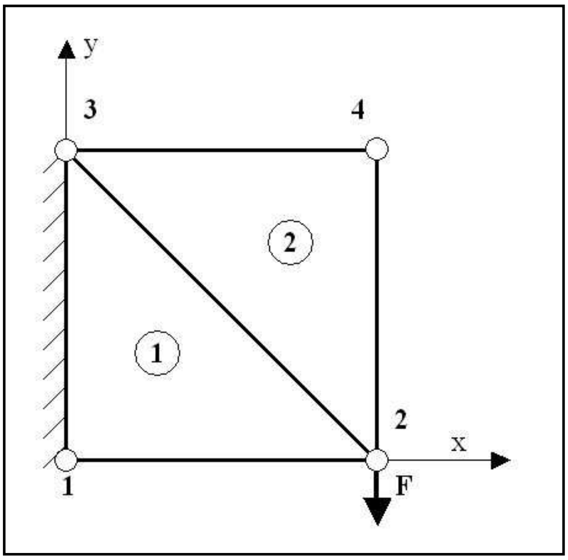


Figure 1 : First study case

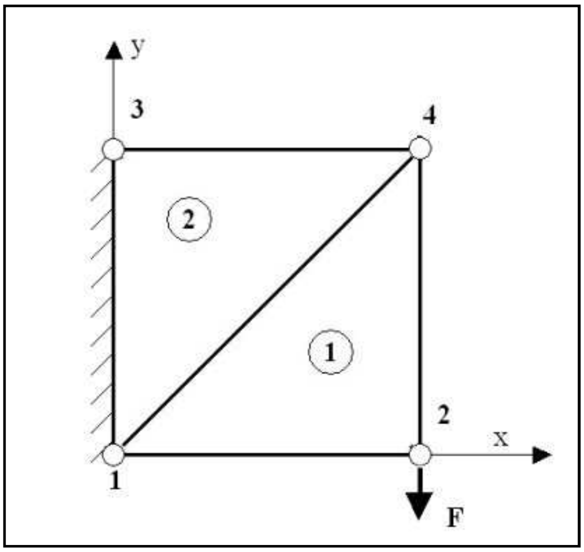


Figure 2 : Second study case

The plate is made of an elastic material whose properties are :

- Young's modulus : E
- Poisson's ratio : $\nu = 1/3$

The triangular finite element chosen are defined each one by 3 nodes. The strains and the stresses are constant inside such elements. The are named **CST** for Constant Strain Triangle.

Questions

0 –Justify the choose of membrane element.

1 – To simplify the calculation of the stiffness matrix, show that the second study case can be replaced by the following. Conclude that the stiffness matrix of the two study cases is the same.

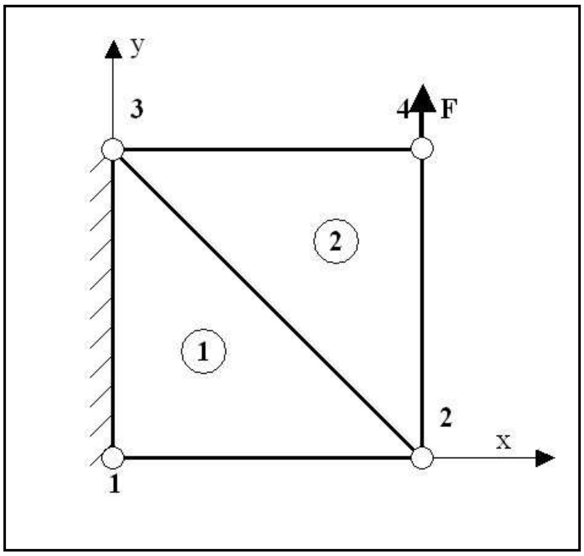


Figure 3 : New second study case

2 – Give the detailed degree of freedom (DOF) of this problem.

3 – Write the stiffness matrix for each of the two elements (use for that the results of the lectures detailed in page 4 & 5).

4 – Write the stiffness matrix of the whole plate.

5 – Write the two different loads, each in a column vector.

6 – Write (without solving them) the two fundamental equations of the finite element method.

7 – Write (without solving them) the two systems of equation that allow to solve the problem.

8 – Write (without solving them) the two systems of equation that allow to find the unknown reactions.

9 – The displacement obtained after resolution are the following

- case study n°1 : $$\begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = \frac{\left(1-\nu^2\right)F}{34eE} \begin{Bmatrix} -12 \\ -66 \\ +9 \\ -57 \end{Bmatrix}$$

- case study n°2 : $$\begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = \frac{\left(1-\nu^2\right)F}{34eE}\begin{Bmatrix} +15 \\ +57 \\ -24 \\ +84 \end{Bmatrix}$$

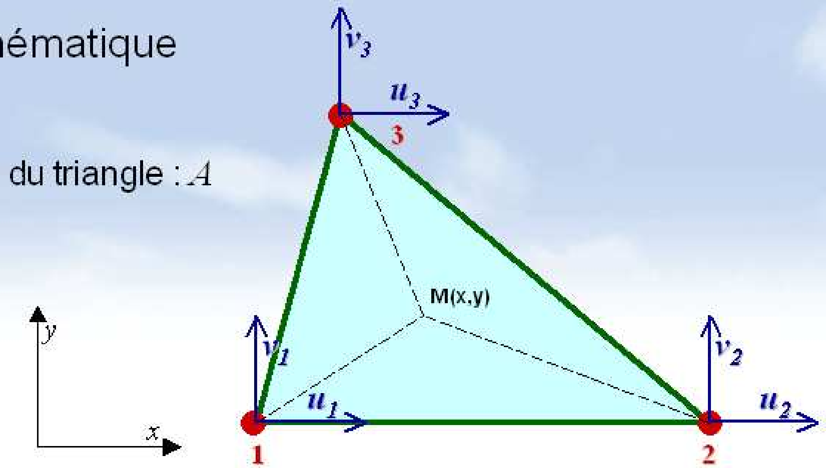
Compare the two results.

10 – How could you increase the accuracy of the solution given by the finite element method.

Elément Triangle - Membrane (CST = Constant Strain Triangle)

- Cinématique

Aire du triangle : A



Permutation circulaire des indices (i,j,k)

i	j	k
1	2	3
2	3	1
3	1	2

- Interpolation des déformations $\varepsilon = Bq$

$$\begin{Bmatrix} \varepsilon_{xx} = \dfrac{\partial u}{\partial x} \\ \varepsilon_{yy} = \dfrac{\partial v}{\partial y} \\ 2\varepsilon_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & b_1 & b_2 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

Elément Triangle - Membrane (CST = Constant Strain Triangle)

- Loi de comportement $\sigma = C\varepsilon$
  cas d'un matériau élastique, linéaire, isotrope

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

- Matrice de rigidité

$$K = \int_V B^T C B dv$$

Solution of the exercise 8

Question 0 : We can consider that the plate will not have deflexion in the z direction and that the stresses do not depend of the direction z. This hypothesis is false near the clamped side because of the Poisson's ratio effect.

By using the St Venant's assumption, we can assume that the normal stress σ_x is a bi-linear stress, in the direction x and in the direction y. An approximate value of the stresses far from the clamped side and far for the point the load F is introduced is something like this according to the theory of beam :

$$\boxed{\begin{aligned} &\sigma_x = \pm 12\frac{F(L-x)}{eL^3}y \\ &\sigma_y = 0 \\ &\sigma_z = 0 \\ &\tau_{xy} = \frac{3}{2}\frac{F}{eL^3}\left(4y^2 - L^2\right) \end{aligned}}$$

Remark : The theory of beam is not very accurate for this study case because of the dimmension of the structure, but can give an idea of the field of stresses inside it.

Question 1 : A rotation of 180 ° around the x axis give the same geometry and the same load. The two study cases are the same.

Question 2 : Nodes and Degree Of Freedom (DOF)

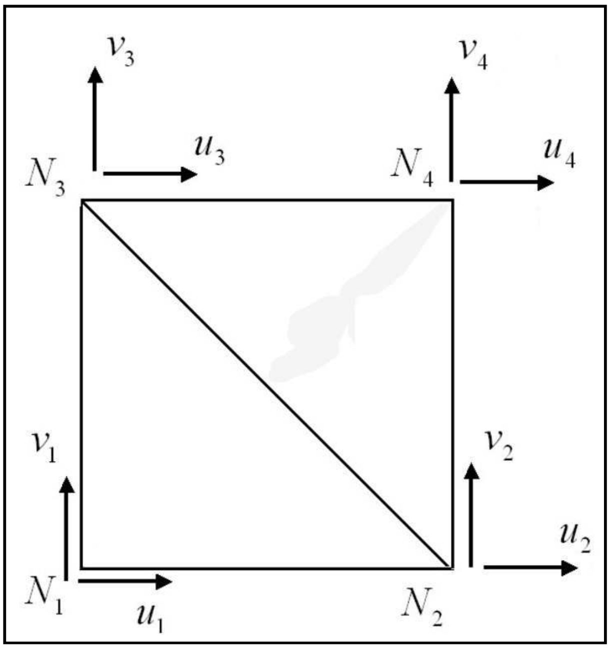


Figure 4 :Nodes and DOF

The boundary conditions give : $\begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{cases}$

Question 3.1 : Stiffness matrix of the first element.

We know that for a triangular finite element defined by three nodes the shape functions for the displacements are : $\begin{cases} u(x,y) = a + bx + cy \\ v(x,y) = A + Bx + Cy \end{cases}$

The coordinates of the nodes are :

$$Node \quad N_1 \quad \begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases}$$

$$Node \quad N_2 \quad \begin{cases} x_2 = L \\ y_2 = 0 \end{cases}$$

$$Node \quad N_3 \quad \begin{cases} x_3 = 0 \\ y_3 = L \end{cases}$$

The stiffness matrix can be obtained with ther following formula :

$$[K]_{El} = \frac{1}{2A} \iiint_{Volume} \frac{E}{1-\nu^2} \begin{pmatrix} b_i & 0 & c_i \\ b_j & 0 & c_j \\ b_k & 0 & c_k \\ 0 & c_i & b_i \\ 0 & c_j & b_j \\ 0 & c_k & b_k \end{pmatrix} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} b_i & b_j & b_k & 0 & 0 & 0 \\ 0 & 0 & 0 & c_i & c_j & c_k \\ c_i & c_j & c_k & b_i & b_j & b_k \end{pmatrix} dv \quad \text{(Eq 1)}$$

Remark 1 : The different functions b_i , c_j are given by the following relations

$\begin{cases} b_i = y_j - y_k \\ c_i = x_k - x_j \end{cases}$ With

i	j	k
1	2	3
2	3	1
3	1	2

$$\begin{cases} b_1 = y_2 - y_3 \\ b_2 = y_3 - y_1 \\ b_3 = y_1 - y_2 \end{cases} \qquad \begin{cases} b_1 = L \\ b_2 = -L \\ b_3 = 0 \end{cases}$$

$$\begin{cases} c_1 = x_3 - x_2 \\ c_2 = x_1 - x_3 \\ c_3 = x_2 - x_1 \end{cases} \qquad \begin{cases} c_1 = -L \\ c_2 = 0 \\ c_3 = L \end{cases}$$

Remark 2 : A represent the area of the triangular finite element. For the first element defined by the Node 1 to the Node3 we can compute the area like the computer will do it :

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & L & 0 \\ 1 & 0 & L \end{bmatrix} = \frac{L^2}{2} \qquad \text{(Eq 2)}$$

- | -

As the terms of the matrix are constants the integration is easy to do.The equation (Eq1) becomes :

$$[K]_{El_1} = \frac{e}{L^2}\frac{E}{1-\nu^2}\begin{pmatrix} -L & 0 & -L \\ L & 0 & 0 \\ 0 & 0 & L \\ 0 & -L & -L \\ 0 & 0 & L \\ 0 & L & 0 \end{pmatrix}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}\begin{pmatrix} -L & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L & 0 & L \\ -L & 0 & L & L & -L & 0 \end{pmatrix}$$

As the Poisson’s ratio is equal to 1/3.

$$[K]_{El_1} = \frac{e}{L^2}\frac{E}{1-\nu^2}L^2\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & L & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$$

Finally the Stiffness matrix for the first element is :

$$[K]_{El_1} = \frac{eE}{3(1-\nu^2)} \begin{bmatrix} 4 & -3 & -1 & 2 & -1 & -1 \\ -3 & 3 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 2 & -1 & -1 & 4 & -1 & -3 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -3 & 0 & 3 \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Question 2.2 : Stiffness matrix of the second element.

The coordinates of the nodes are :

$$Node \quad N_2 \quad \begin{cases} x_2 = L \\ y_2 = 0 \end{cases}$$

$$Node \quad N_4 \quad \begin{cases} x_4 = L \\ y_4 = L \end{cases}$$

$$Node \quad N_3 \quad \begin{cases} x_3 = 0 \\ y_3 = L \end{cases}$$

Remark 3 : The nodes N_1, N_4 and N_3 are in the anti counter clock, then the area given by equation 2 is a positive value.

Now the different functions b_i , c_j are :

$$\begin{cases} b_i = y_j - y_k \\ c_i = x_k - x_j \end{cases}$$

With

i	j	k
2	4	3
4	3	2
3	2	4

$$\begin{cases} b_2 = y_4 - y_3 \\ b_4 = y_3 - y_2 \\ b_3 = y_2 - y_4 \end{cases} \qquad \begin{cases} b_2 = 0 \\ b_4 = L \\ b_3 = -L \end{cases}$$

$$\begin{cases} c_2 = x_3 - x_4 \\ c_4 = x_2 - x_3 \\ c_3 = x_4 - x_2 \end{cases} \qquad \begin{cases} c_2 = -L \\ c_4 = L \\ c_3 = 0 \end{cases}$$

$$[K]_{El_1} = \frac{e}{L^2}\frac{E}{1-\nu^2}\begin{pmatrix} 0 & 0 & -L \\ L & 0 & L \\ -L & 0 & 0 \\ 0 & -L & 0 \\ 0 & L & L \\ 0 & L & -L \end{pmatrix}\frac{1}{3}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & L & -L & 0 & 0 & 0 \\ 0 & 0 & 0 & -L & L & 0 \\ -L & L & 0 & 0 & L & -L \end{pmatrix}$$

Finally the Stiffness matrix for the second element is :

$$[K]_{El_2} = \frac{eE}{3\left(1-\nu^2\right)}\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 4 & -3 & -1 & 2 & -1 \\ 0 & -3 & 3 & 1 & -1 & 0 \\ 0 & -1 & 1 & 3 & -3 & 0 \\ -1 & 2 & -1 & -3 & 4 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} u_2 \\ u_4 \\ u_3 \\ v_2 \\ v_4 \\ v_3 \end{pmatrix}$$

Question 4 : Stiffness matrix of the whole plate.

We can create the global stiffness matrix as the sum of the two elementary stiffness matrix.
For that we add the two elementary stiffness matrix inside one, and we obtain :

$$[K]_{El_1} = \frac{eE}{3\left(1-\nu^2\right)}\begin{bmatrix} 4 & -3 & -1 & 0 & 2 & -1 & -1 & 0 \\ -3 & 3 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 4 & -1 & -3 & 0 \\ -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{eE}{3\left(1-\nu^2\right)}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -3 & 0 & 1 & 0 & -1 \\ 0 & -1 & -3 & 4 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 3 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 & 0 & -3 & -1 & 4 \end{bmatrix}$$

Finally the global stiffness matrix is :

$$[K]=\frac{eE}{3\left(1-\nu^2\right)}\begin{bmatrix} 4 & -3 & -1 & 0 & 2 & -1 & -1 & 0 \\ -3 & 4 & 0 & -1 & -1 & 0 & 2 & -1 \\ -1 & 0 & 4 & -3 & -1 & 2 & 0 & -1 \\ 0 & -1 & -3 & 4 & 0 & -1 & -1 & 2 \\ 2 & -1 & -1 & 0 & 4 & -1 & -3 & 0 \\ -1 & 0 & 2 & -1 & -1 & 4 & 0 & -3 \\ -1 & 2 & 0 & -1 & -3 & 0 & 4 & -1 \\ 0 & -1 & -1 & 2 & 0 & -3 & -1 & 4 \end{bmatrix}$$

Question 5.1 : Load for the case study 1.

$$Load_1=\begin{pmatrix} X_1 \\ 0 \\ X_3 \\ 0 \\ Y_1 \\ -F \\ Y_3 \\ 0 \end{pmatrix}$$

Question 5.2 : Load for the case study 2.

$$Load_2=\begin{pmatrix} X_1 \\ 0 \\ X_3 \\ 0 \\ Y_1 \\ 0 \\ Y_3 \\ F \end{pmatrix}$$

Question 6.1: Fundamental equation of the finite element for the first load case

$$\begin{pmatrix} X_1 \\ 0 \\ X_3 \\ 0 \\ Y_1 \\ -F \\ Y_3 \\ 0 \end{pmatrix} = \frac{eE}{3\left(1-\nu^2\right)} \begin{bmatrix} 4 & -3 & -1 & 0 & 2 & -1 & -1 & 0 \\ -3 & 4 & 0 & -1 & -1 & 0 & 2 & -1 \\ -1 & 0 & 4 & -3 & -1 & 2 & 0 & -1 \\ 0 & -1 & -3 & 4 & 0 & -1 & -1 & 2 \\ 2 & -1 & -1 & 0 & 4 & -1 & -3 & 0 \\ -1 & 0 & 2 & -1 & -1 & 4 & 0 & -3 \\ -1 & 2 & 0 & -1 & -3 & 0 & 4 & -1 \\ 0 & -1 & -1 & 2 & 0 & -3 & -1 & 4 \end{bmatrix} \begin{pmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ v_2 \\ 0 \\ v_4 \end{pmatrix} \quad \text{(Eq 3)}$$

Question 6.2: Fundamental equation of the finite element for the second load case

$$: \begin{pmatrix} X_1 \\ 0 \\ X_3 \\ 0 \\ Y_1 \\ 0 \\ Y_3 \\ F \end{pmatrix} = \frac{eE}{3\left(1-\nu^2\right)} \begin{bmatrix} 4 & -3 & -1 & 0 & 2 & -1 & -1 & 0 \\ -3 & 4 & 0 & -1 & -1 & 0 & 2 & -1 \\ -1 & 0 & 4 & -3 & -1 & 2 & 0 & -1 \\ 0 & -1 & -3 & 4 & 0 & -1 & -1 & 2 \\ 2 & -1 & -1 & 0 & 4 & -1 & -3 & 0 \\ -1 & 0 & 2 & -1 & -1 & 4 & 0 & -3 \\ -1 & 2 & 0 & -1 & -3 & 0 & 4 & -1 \\ 0 & -1 & -1 & 2 & 0 & -3 & -1 & 4 \end{bmatrix} \begin{pmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ v_2 \\ 0 \\ v_4 \end{pmatrix} \quad \text{(Eq 4)}$$

Question 7.1 : Field of displacement for the first load case

$$\begin{pmatrix} 0 \\ 0 \\ -F \\ 0 \end{pmatrix} = \frac{eE}{3\left(1-\nu^2\right)} \begin{bmatrix} 4 & 0 & -1 & -1 \\ 0 & 8 & -4 & 2 \\ -1 & -4 & 6 & -3 \\ -1 & 2 & -3 & 6 \end{bmatrix} \begin{pmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{pmatrix}$$

These relation can be inverted :

$$\begin{pmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{pmatrix} = \frac{3\left(1-\nu^2\right)}{34eE} \begin{bmatrix} 10 & 1 & 4 & 5 \\ 1 & 12 & -3 & -8 \\ 4 & -3 & 22 & 19 \\ 5 & -8 & 19 & 28 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -F \\ 0 \end{pmatrix}$$

The solution is :

$$\boxed{\begin{aligned} u_2 &= -\frac{12\left(1-\nu^2\right)F}{34eE} \\ u_4 &= \frac{9\left(1-\nu^2\right)F}{34eE} \\ v_2 &= -\frac{66\left(1-\nu^2\right)F}{34eE} \\ v_4 &= -\frac{57\left(1-\nu^2\right)F}{34eE} \end{aligned}}$$

Question 7.2 : Field of displacement for the second load case

$$\boxed{\begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix} = \frac{eE}{3\left(1-\nu^2\right)} \begin{bmatrix} 4 & 0 & -1 & -1 \\ 0 & 8 & -4 & 2 \\ -1 & -4 & 6 & -3 \\ -1 & 2 & -3 & 6 \end{bmatrix} \begin{pmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{pmatrix}}$$

These relation can be inverted :

$$\boxed{\begin{pmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{pmatrix} = \frac{3\left(1-\nu^2\right)}{34eE} \begin{bmatrix} 10 & 1 & 4 & 5 \\ 1 & 12 & -3 & -8 \\ 4 & -3 & 22 & 19 \\ 5 & -8 & 19 & 28 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix}}$$

The solution is :

$$\boxed{\begin{aligned} u_2 &= \frac{15\left(1-\nu^2\right)F}{34eE} \\ u_4 &= -\frac{24\left(1-\nu^2\right)F}{34eE} \\ v_2 &= \frac{57\left(1-\nu^2\right)F}{34eE} \\ v_4 &= \frac{84\left(1-\nu^2\right)F}{34eE} \end{aligned}}$$

Question 9 :

As we know the displacement of the free nodes, we can compute the reactions with (Eq3) and (Eq4)

Question 9 :

The results are differents. The triangular elements defined with 3 nodes, with a great size are not enough accurate because in such elements the stresses and the strains are constant.

Question 10 :

We can change triangular finite element by quadrangular finite element. We can also decrease the size of the triangular elements.

Exercise 11

Finite Elements
Beam element with an offset

We want to study the system define by figure 1 with the finite elements method.
The load is applied on the two opposite sides of the plate. To avoid the buckling of the plate, a few stiffeners are bonded on the plate with rivets.

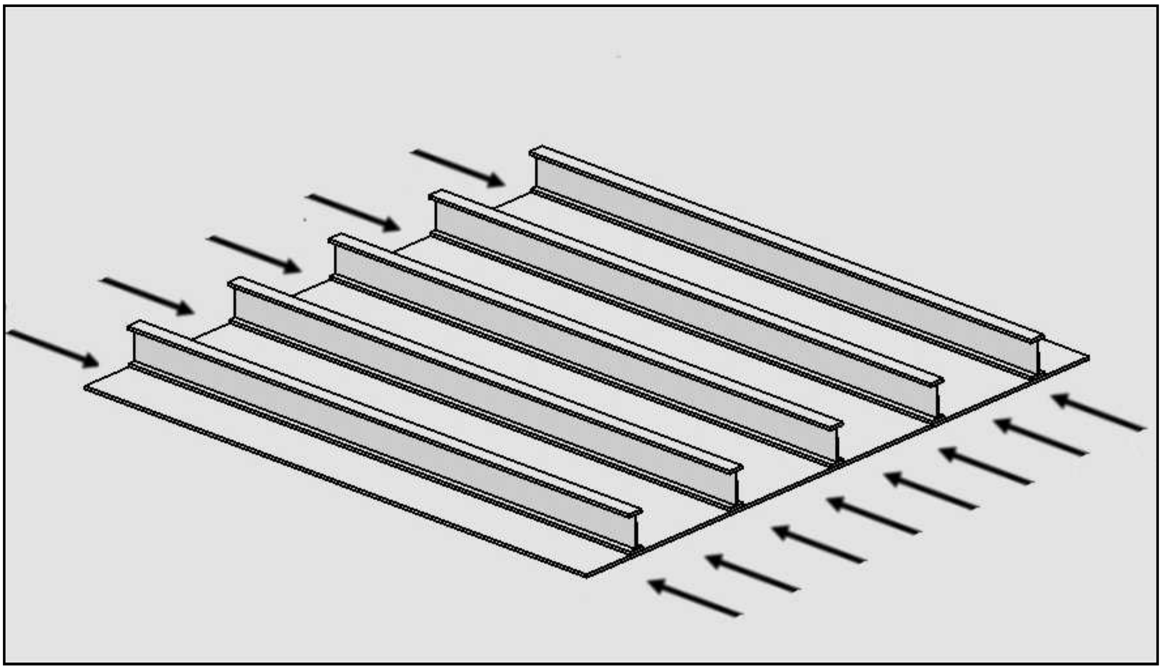


Figure 1 : Case study

Propose different modelling to solve this problem by the finite element method.

One method consists in modelling the beam with an offset. The purpose of this exercise is to find the stiffness matrix of a beam element defined by two nodes when the neutral axis of the beam is parallel to the line of the nodes. The position of the neutral axis is defined by the offset d as represented on figure 2.

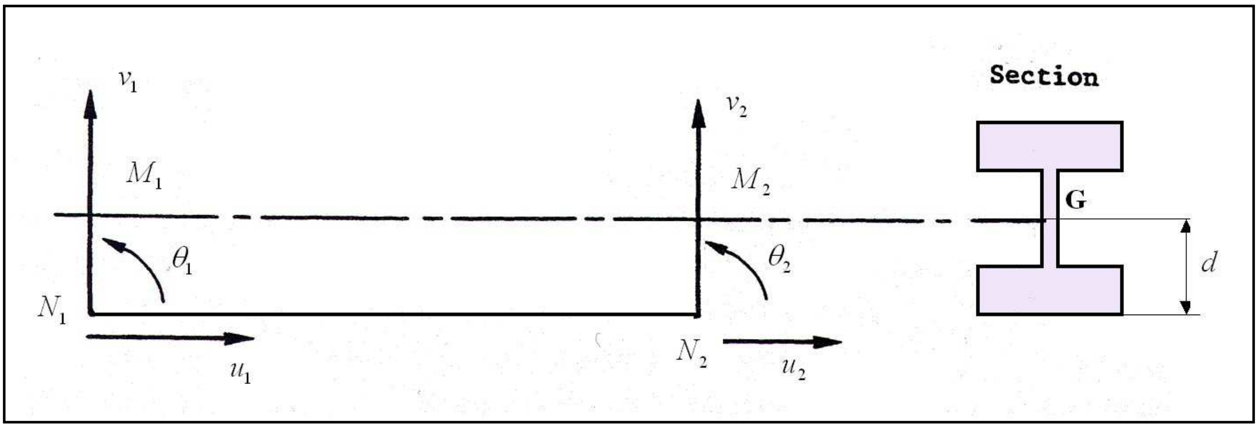


Figure 2 : Beam defined by two nodes with an offset

Remarks :

- All the mechanical characteristics are supposed known.
- To simplify the study the internal energy introduces by the shear force T_y inside the beam will be neglected.

Questions :

1 What are the different terms of the internal energy in the beam you need,

2 Find the strain ε_x in the beam in function of $u(x,0)$ and $v(x,0)$.

3 Find that the internal energy inside the beam can be written under the following form,

$$\boxed{W = \int_0^L \left(\iint_S \frac{E\varepsilon_x^2}{2} dS \right) dx} \qquad (1)$$

4 By using the relation (1) find the internal energy inside the beam as the sum of :
   a. The internal energy introduce by the normal force N_x,
   b. The internal energy introduce by the bending moment M_z,
   c. A coupling term between the normal force N_x and the bending moment M_z.

5 Give the shape functions for the displacement in the beam, function of the nodal displacement.
   a. Displacement $u(x)$
   b. Displacement $v(x)$

6 Find the different terms of the stiffness matrix of a beam with an offset.

Solution

Exercise 12

Finite Elements
Assembling of bars

A set of bars is defined by figure 1.

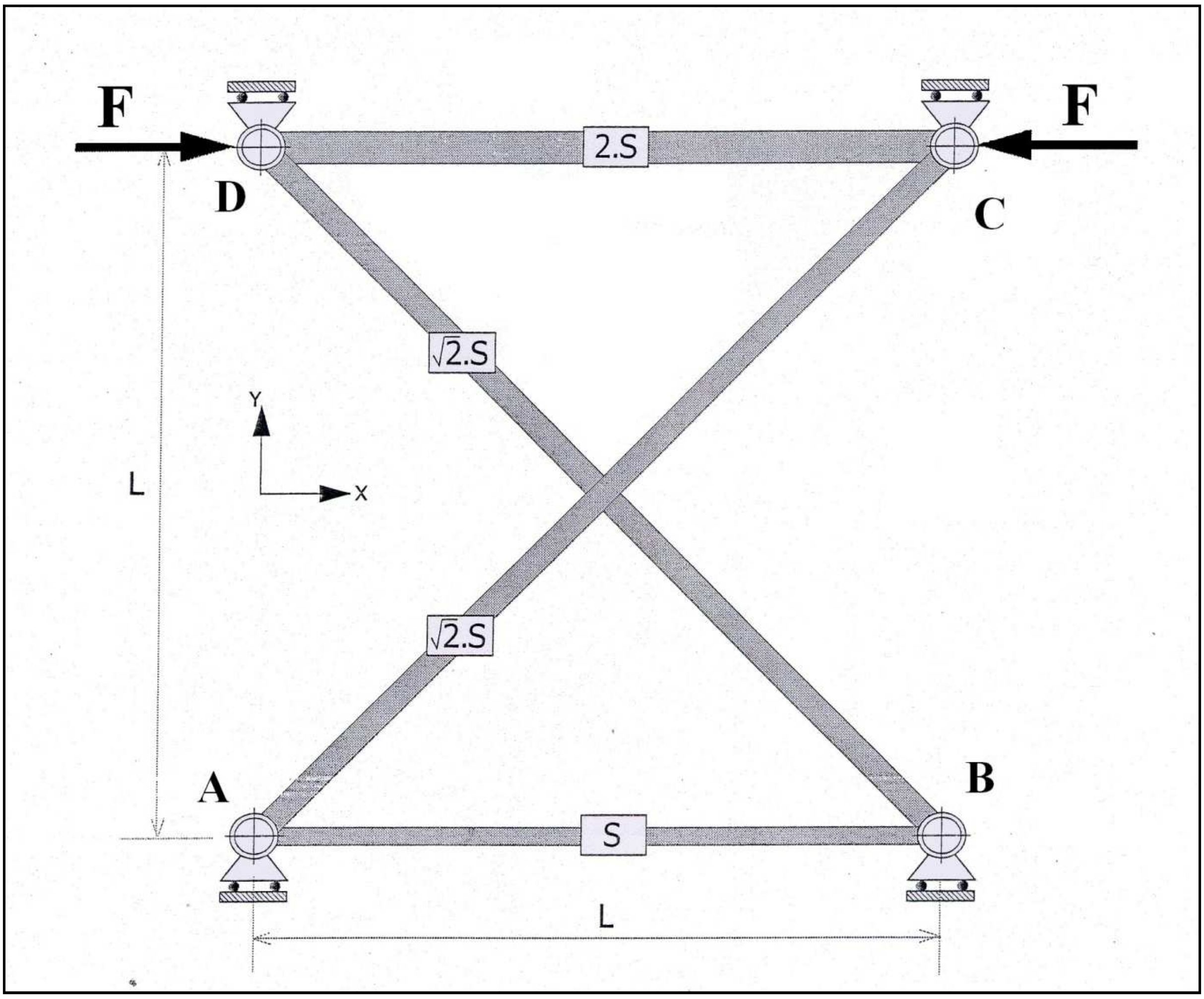


Figure 1 : Set of different bars

The 4 bars are made of a material with the following mechanical characteristics :
- Young's modulus : $E = 210 \quad GPa$
- Poisson's ratio : $\nu = 0.3$

The cross sections of the different bars are :

- Bar AB $S = 200 \quad mm^2$
- Bar CD $2S$
- Bar AC and BD $S\sqrt{2}$

The lengths of the different bars are :

- Bar AB and CD $L = 1 \quad m$
- Bar AC and BD $L\sqrt{2}$

The load consists in two opposite forces on the point B and C with :

$F = 10000 \quad N$

Questions

7 Propose a modelling to solve this problem by the finite element method,
8 Choose a set of nodes and give them a number.
9 Write the stiffness matrix of a bar in its own axis system.
10 Write the stiffness matrix of a bar in the global axis system.
11 Create the global stiffness matrix for this structure,
12 Boundary conditions : Write the displacements in a column vector (q).
13 Load : Write the load in a column vector $[F]$
14 Is it possible to solve the equation $(F)=[K](q)$. Find a method to solve.
15 Solve the equation $(F)=[K](q)$.
16 Find the numerical values of the different displacements.
17 Find the normal stress inside the bar the more loaded.
18 Find the reaction in A,B,C and D

Solution of the exercise 12

Question 1 : Modelling.

As the joints in A,B,C, and D are ball joints, the beams AB, BC, AC & DB will be modelled by 4 bar elements

Question 2 : Nodes and DOF.

As we solve this problem with a 2D approach, we can consider to simplify the writing that each node has only two degrees of freedom. The first in the direction x, the second in the direction y.

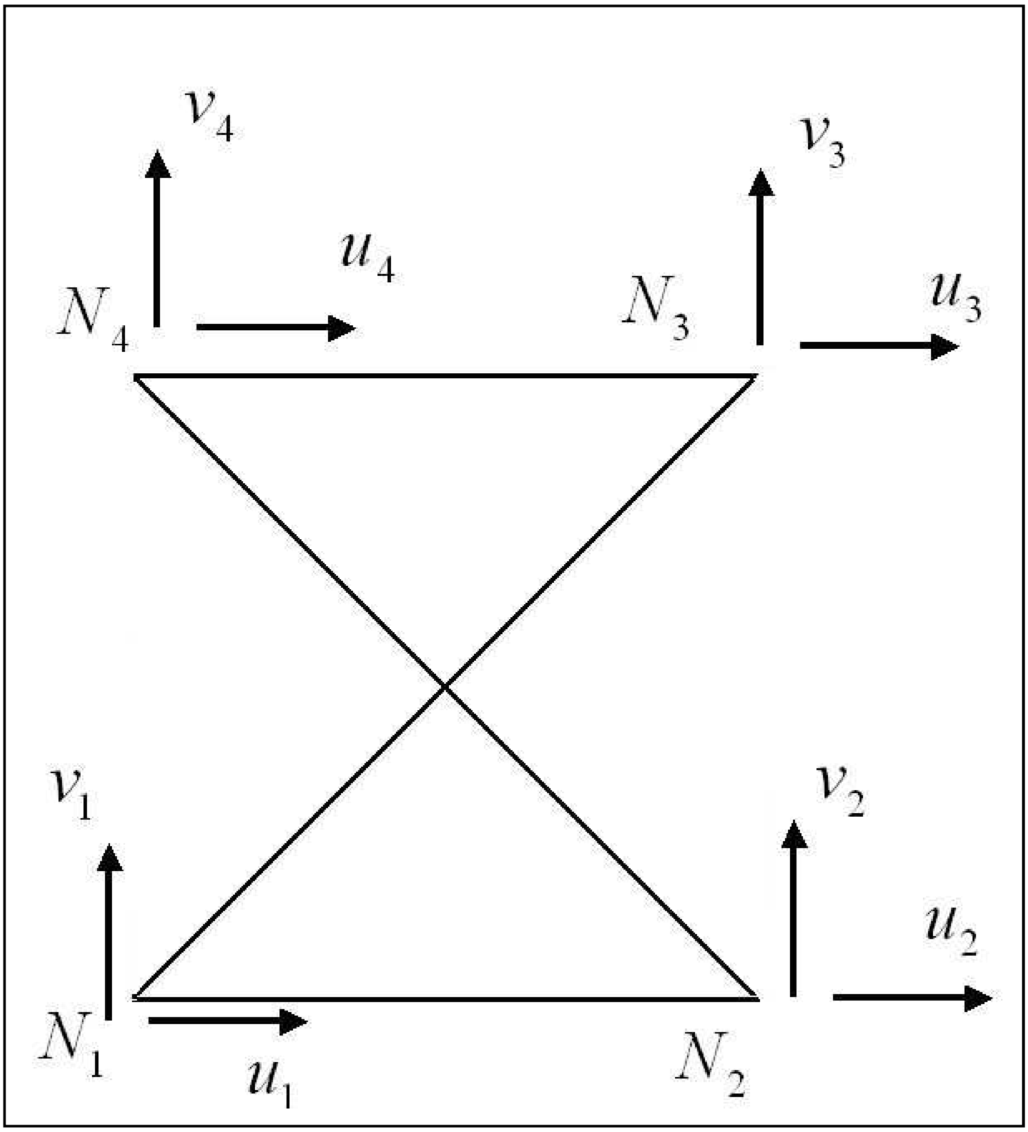


Figure 2 : Modelling Nodes & DOF

The displacement vector (q) can be written in a matrix column such as we have :

$$\boxed{(q)^T = (u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4)}$$

Question 3: Stiffness matrix of a bar in its own axis system

For instance if we consider the bar between the nodes N_1 and N_2, the local axis system is the same than the global axis system of the bar.

The internal energy in this bar is :

$$W=\frac{1}{2}\begin{pmatrix}u_1 & v_1 & u_2 & v_2\end{pmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}u_1\\ v_1\\ u_2\\ v_2\end{pmatrix}$$

Then the stiffness matrix of the bar AB is :

$$\boxed{[K]_{12}=\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}$$

For the bar between the nodes N_3 and N_4 we have, as the cross section is $2S$

$$\boxed{[K]_{34}=\begin{bmatrix}\frac{2ES}{L} & 0 & -\frac{2ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{2ES}{L} & 0 & \frac{2ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}$$

Question 4: Stiffness matrix of a bar two nodes N_i and N_j, inclined of an angle α such as $\left(\vec{x} \quad \overrightarrow{X}\right)=\alpha$ with

- $\vec{x}$ the global axis
- $\overrightarrow{X}$ the local axis of the bar parallel to the neutral axis of the bar
- L the length of the bar
- S the cross section of the bar
- E the Young's modulus of the material whose bar is made

$$[K]=\begin{bmatrix}[P] & [0]\\ [0] & [P]\end{bmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}[P]^T & [0]\\ [0] & [P]^T\end{bmatrix} \qquad (1)$$

$$\text{with } [P]=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} \quad \& \quad [P]^T=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

If we develop the equation (1) we obtain :

$$[K]=\frac{ES}{L}\begin{bmatrix}Cos(\alpha)Cos(\alpha) & Cos(\alpha)Sin(\alpha) & -Cos(\alpha)Cos(\alpha) & -Cos(\alpha)Sin(\alpha)\\ Cos(\alpha)Sin(\alpha) & Sin(\alpha)Sin(\alpha) & -Cos(\alpha)Sin(\alpha) & -Sin(\alpha)Sin(\alpha)\\ -Cos(\alpha)Cos(\alpha) & -Cos(\alpha)Sin(\alpha) & Cos(\alpha)Cos(\alpha) & Cos(\alpha)Sin(\alpha)\\ -Cos(\alpha)Sin(\alpha) & -Sin(\alpha)Sin(\alpha) & Cos(\alpha)Sin(\alpha) & Sin(\alpha)Sin(\alpha)\end{bmatrix} \qquad (2)$$

- I -

For instance, the bar between the nodes N_1 and N_3 is inclined of an angle $\alpha=\pi/4$
As the length is $L\sqrt{2}$ and the cross section $S\sqrt{2}$

The internal energy of this bar in the global axis system is :

$$W=\frac{1}{2}\begin{pmatrix}u_1\\ v_1\\ u_3\\ v_3\end{pmatrix}^T\begin{bmatrix}[P] & [0]\\ [0] & [P]\end{bmatrix}\begin{bmatrix}\frac{ES}{L} & 0 & -\frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\\ -\frac{ES}{L} & 0 & \frac{ES}{L} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}[P]^T & [0]\\ [0] & [P]^T\end{bmatrix}\begin{pmatrix}u_1\\ v_1\\ u_3\\ v_3\end{pmatrix}$$

With

$$[P]=\begin{bmatrix}Cos\left(\frac{\pi}{4}\right) & -Sin\left(\frac{\pi}{4}\right)\\ Sin\left(\frac{\pi}{4}\right) & Cos\left(\frac{\pi}{4}\right)\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Finally, we can apply (1) or (2) to compute the stiffness matrix of the bar between the nodes N_1 and N_3 .

We obtain : $$[K]_{13} = \frac{ES}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

For the bar between the nodes N_4 and N_2 we have $\alpha = -\pi/4$:

$$[K]_{24} = \frac{ES}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Question 5 : Stiffness matrix of the system

As the internal energy of the system is the sum of the energy in each bar., we obtain the stiffness matrix by assembling the stiffness matrix of each bar inside a large matrix.

$$[K] = \frac{ES}{2L}\begin{bmatrix} 3 & 1 & -2 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 3 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 5 & 1 & -4 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -4 & 0 & 5 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Question 6 : Displacement vector

The boundary conditions of the set of bars allows us to write the displacement vector, as the sum of a well known displacement vector, and a unknown displacement vector :

$$(q) = \begin{pmatrix} u_1 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 = 0 \\ u_4 \\ v_4 = 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ 0 \end{pmatrix}$$

Question 7 : Load vector

The load vector is the sum of the two nodal loads :

- External forces applied on the nodes,
- Internal reactions

$$(F)=\begin{pmatrix}0\\Y_1\\0\\Y_2\\0\\Y_3\\0\\Y_4\end{pmatrix}+\begin{pmatrix}0\\0\\0\\0\\-F\\0\\F\\0\end{pmatrix}$$

Question 8 : Resolution of the problem

To solve the finite element problem, the computer must solve the equation $(F)=[K](q)$.

$$\boxed{\begin{pmatrix}0\\Y_1\\0\\Y_2\\-F\\Y_3\\F\\Y_4\end{pmatrix}=\frac{ES}{2L}\begin{bmatrix}3&1&-2&0&-1&-1&0&0\\1&1&0&0&-1&-1&0&0\\-2&0&3&-1&0&0&-1&1\\0&0&-1&1&0&0&1&-1\\-1&-1&0&0&5&1&-4&0\\-1&-1&0&0&1&1&0&0\\0&0&-1&1&-4&0&5&-1\\0&0&1&-1&0&0&-1&1\end{bmatrix}\begin{pmatrix}u_1\\0\\u_2\\0\\u_3\\0\\u_4\\0\end{pmatrix}} \qquad (3)$$

For that it remove the lines and the column whose degree of freedom are known and reactions unknown. That gives :

$$\begin{pmatrix}0\\0\\-F\\F\end{pmatrix}=\frac{ES}{2L}\begin{pmatrix}3&-2&-1&0\\-2&3&0&-1\\-1&0&5&-4\\0&-1&-4&5\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} \qquad (4)$$

You can verify that the determinant of this stiffness matrix of the equation (4) is equal to zero. It is because there is a rigid body movement in a direction parallel to x. We must kill a degree of freedom in the x direction to remove this rigid body movement.

For instance, we can impose a the following displacement for the node N_1 : $u_1=0$

That will introduce a reaction X_1

Remark : The same displacement in the direction x , for each node, will not increase the internal energy of the system. This displacement is free, because there is no join to avoid it.

Question 8 : Resolution of the problem

The new equation $(F)=[K](q)$ gives :

$$F\begin{pmatrix}0\\-1\\1\end{pmatrix}=\frac{EL}{2L}\begin{bmatrix}3&0&-1\\0&5&-4\\-1&-4&5\end{bmatrix}\begin{pmatrix}u_2\\u_3\\u_4\end{pmatrix}$$

To solve this equation by hand, we can use the Cramer formulas.
Remark : The computer never use this method. It prefer the gauss Method or the Gradient method.

$$\begin{cases} u_2=\dfrac{2FL}{ES}\dfrac{\begin{bmatrix}0&0&-1\\-1&5&-4\\1&-4&5\end{bmatrix}}{\begin{bmatrix}3&0&-1\\0&5&-4\\-1&-4&5\end{bmatrix}}=\dfrac{2FL}{ES}\dfrac{1}{22}=\dfrac{FL}{11ES} \\ u_3=\dfrac{2FL}{ES}\dfrac{\begin{bmatrix}3&0&-1\\0&-1&-4\\-1&1&5\end{bmatrix}}{\begin{bmatrix}3&0&-1\\0&5&-4\\-1&-4&5\end{bmatrix}}=\dfrac{2FL}{ES}\dfrac{-2}{22}=-\dfrac{2FL}{11ES} \\ u_4=\dfrac{2FL}{ES}\dfrac{\begin{bmatrix}3&0&0\\0&5&-1\\-1&-4&1\end{bmatrix}}{\begin{bmatrix}3&0&-1\\0&5&-4\\-1&-4&5\end{bmatrix}}=\dfrac{2FL}{ES}\dfrac{3}{22}=\dfrac{3FL}{11ES} \end{cases}$$

Question 10 : Displacements

A numerical application gives :

$$\begin{cases} u_2 &= & 2.16 & 10^{-5} & m \\ u_3 &= & -4.32 & 10^{-5} & m \\ u_4 &= & 6.49 & 10^{-5} & m \end{cases}$$

Question 11 : The bar the more loaded is the bar CD

$$\boxed{\sigma_{34} = E\varepsilon_x = E\frac{(u_3 - u_4)}{L} = -\frac{5}{11}\frac{F}{S} = 22.7 \quad MPa}$$

Question 12 : Reactions

By using the initial equation $(F) = [K](q)$ (eq 3), we can find directly the reactions, now than all the displacements are well known.

$$\begin{pmatrix} X_1 \\ Y_1 \\ 0 \\ Y_2 \\ -F \\ Y_3 \\ F \\ Y_4 \end{pmatrix} = \frac{ES}{2L}\begin{bmatrix} 3 & 1 & -2 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 3 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 5 & 1 & -4 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -4 & 0 & 5 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ 0 \end{pmatrix}$$

We find

$$\boxed{\begin{aligned} X_1 &= (-2u_2 - u_3)\frac{ES}{2L} = 0 \quad N \\ Y_1 &= -u_3\frac{ES}{2L} = \frac{F}{11} = 909 \quad N \\ Y_2 &= (-u_2 + u_4)\frac{ES}{2L} = \frac{F}{11} = 909 \quad N \\ Y_3 &= u_3\frac{ES}{2L} = -\frac{F}{11} = -909 \quad N \\ Y_4 &= (u_2 - u_4)\frac{ES}{2L} = -\frac{F}{11} = -909 \quad N \end{aligned}}$$

Exercise 13

Finite Elements
Beam with a distributed load

A Beam loaded by a distributed force p is described by figure 1.

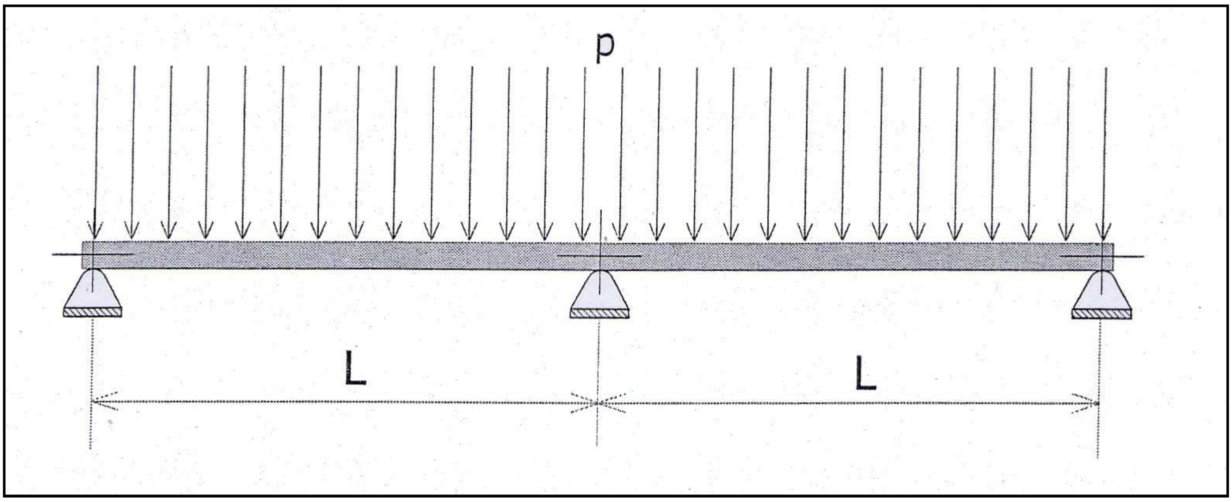


Figure 1 : Beam with a distributed load

The beam is made of a material with the following mechanical characteristics :
- Young's modulus : $E = 210 \quad GPa$
- Poisson's ratio : $\nu = 0.3$

The cross sections is a rectangular cross section whose dimensions are :
Thickness $e = 5 \quad mm$
Height $h = 20 \quad mm$

The quadratic moment is I_z

The length of the beam is $2L = 4 \quad m$

The load consists in the weight of the beam. The distributed load created by the weight will be named p

Questions

19 Indicate why you can study the half part of the structure only with the finite element method.
20 Give the boundary condition, in term of displacement in the plane of symmetry,

- | -

The modelling of the beam will be made with 2 beams elements as described on figure 2. Three nodes define the two beam elements

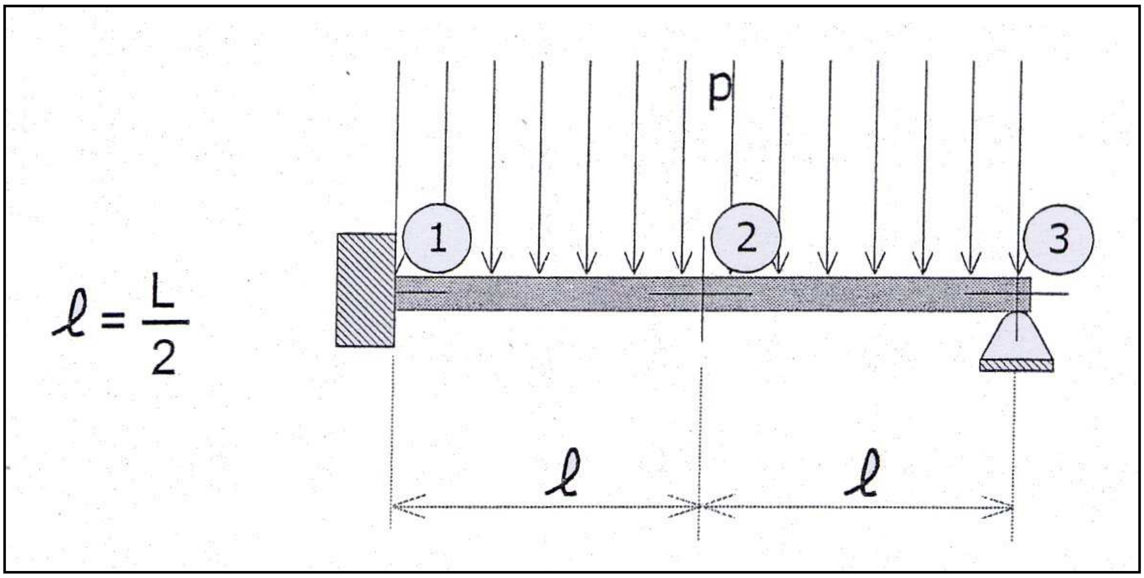


Figure 2 : Half beam with 2 beam elements and three nodes

- | -

21 Explain why the stiffness matrix of this finite element problem can be reduced to a 6x6 matrix
22 Write the stiffness matrix with the terms of bending energy,
23 Write the load as a column vector
24 Write the displacement as a column vector
25 The resolution of $(F)=[K](q)$ gives the following results :

$$\boxed{\begin{aligned} v_2 &= -\frac{2pl^4}{24EI_z} \\ \theta_2 &= -\frac{pl^3}{24EI_z} \\ \theta_3 &= \frac{4pl^3}{24EI_z} \end{aligned}} \qquad (1)$$

26 Explain how the computer can find from these results the displacement in the different elements, for instance :

- Element 1 from node N_1 to node N_2:

$$v(x)=\frac{pl^4}{24EI_z}\left(-5\left(\frac{x}{l}\right)^2+3\left(\frac{x}{l}\right)^3\right) \quad for \quad 0<x<l$$

- Element 1 from node N_2 to node N_3:

- $$v(x)=\frac{pl^4}{24EI_z}\left(-2\left(\frac{x}{l}\right)+4\left(\frac{x}{l}\right)^2-3\left(\frac{x}{l}\right)^3\right) \quad for \quad 0<x<l$$

27 Compute $\frac{d^2v(x)}{dx^2}$:

28 Find from that the bending moment obtained with the finite element method.

29 Compare these results with the ones obtained by the beam theory,

30 Is it possible to do a modelling of the beam with 2 dimensional finite elements,

31 What type of element could you choose ?

32 How can you do if you an accurate value for the stress between two nodes near the one from the other ?

Solution of exercise 14

Question 1 : Study of the half part of the structure.

The structure studied has a plane of symmetry for the geometry and for the load. We can limit the study to the half part of the structure. The results on the other part will be deduced directly:

Question 2 : Boundary conditions, in term of displacement in the plane of symmetry,

The symmetry impose

$$\begin{cases} u(2L) = 0 \\ \dfrac{dv(2L)}{dx} = 0 \end{cases}$$

As the joint in the plane of symmetry remove the displacement in the direction y, wue have :

$$v(2L) = 0$$

Hence the joint in the plane of symmetry is a clamped joint. We can replace the study of the whole structure by the half part of the structure defines by figure 3.

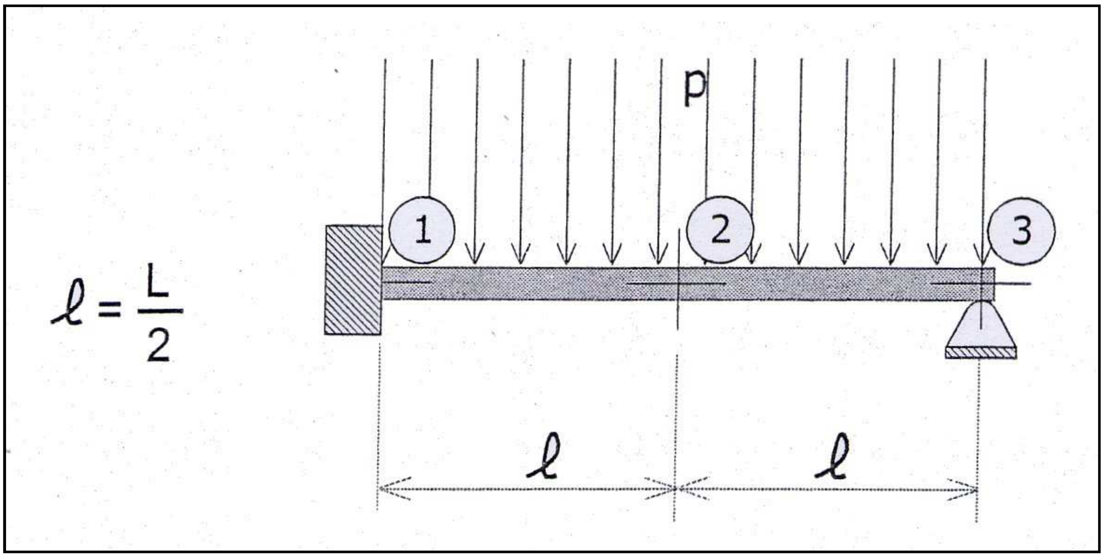


Figure 3 : Half beam with 2 beam elements and three nodes

Question 3 : The stiffness matrix can be reduced to a 6x6 matrix

We know that there is no normal force N_x inside the structure studied. So the internal energy in the field of displacement $u(x)$ is equal to zero. We can reduce the study at the following displacement vector.

$$^{T}(q) = \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{pmatrix}$$

Question 4 : Stiffness matrix

If we are good students who are listened their professor, we remember that the stiffness matrix of a beam inside its own axis is :

$$[K] = \frac{EI_z}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

It's very easy to find the stiffness matrix of the structure studied, because there are only two beam elements.

$$\boxed{[K] = \frac{EI_z}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6L & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}}$$

Question 5 : Write the load as a column vector

$$\boxed{[F] = \begin{pmatrix} Y_1 \\ M_1 \\ 0 \\ 0 \\ Y_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{Pl}{2} \\ -\frac{Pl^2}{12} \\ -pl \\ 0 \\ -\frac{Pl}{2} \\ \frac{Pl^2}{12} \end{pmatrix}}$$

Question 6 : Write the displacement as a column vector

$$[q]=\begin{pmatrix}0\\0\\v_2\\\theta_2\\0\\\theta_3\end{pmatrix}$$

Question 7 : The resolution of $(F)=[K](q)$ gives the following results :

$$\boxed{\begin{pmatrix}Y_1-\dfrac{Pl}{2}\\M_1-\dfrac{Pl^2}{12}\\-pl\\0\\Y-\dfrac{Pl}{2}\\\dfrac{Pl^2}{12}\end{pmatrix}=\frac{EI_z}{l^3}\begin{bmatrix}12&6l&-12&6l&0&0\\6L&4l^2&-6l&2l^2&0&0\\-12&-6l&24&0&-12&6l\\6l&2l^2&0&8l^2&-6l&2l^2\\0&0&-12&-6l&12&-6l\\0&0&6l&2l^2&-6l&4l^2\end{bmatrix}\begin{pmatrix}0\\0\\v_2\\\theta_2\\0\\\theta_3\end{pmatrix}}$$

The resolution of this system after removing lines and columns for which displacement are known gives :

$$\boxed{\begin{aligned}v_2&=-\frac{2pl^4}{24EI_z}\\\theta_2&=-\frac{pl^3}{24EI_z}\\\theta_3&=\frac{4pl^3}{24EI_z}\end{aligned}}\qquad(2)$$

Question 8 :

We are seen during the lectures that the shape function for a beam defines by two nodes is inside an axis system dedicated to the beam :

$$v(x)=\left(\left\{1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}\right\}\quad L\left\{\frac{x}{L}-2\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\quad\left\{3\frac{x^2}{L^2}-2\frac{x^3}{L^3}\right\}\quad L\left\{-\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\right)\begin{pmatrix}v_1\\\theta_1\\v_2\\\theta_2\end{pmatrix}\qquad(3)$$

If we introduce the displacements (2) inside the equation (3) and if we remember that $\begin{cases}v_1=0\\\theta_1=0\\v_3=0\end{cases}$

We obtain for the first element between nodes N_1 and N_2:

$$v(x)=\left(\left\{1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}\right\}\quad L\left\{\frac{x}{L}-2\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\quad \left\{3\frac{x^2}{L^2}-2\frac{x^3}{L^3}\right\}\quad L\left\{-\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\right)\begin{pmatrix}0\\0\\-\dfrac{2pl^4}{24EI_z}\\-\dfrac{pl^3}{24EI_z}\end{pmatrix}$$

$$v(x)=\frac{pl^4}{24EI_z}\left(-5\left(\frac{x}{l}\right)^2+3\left(\frac{x}{l}\right)^3\right)\qquad for\quad 0<x<l$$

We obtain for the second element between nodes N_2 and N_3:

$$v(x)=\left(\left\{1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3}\right\}\quad L\left\{\frac{x}{L}-2\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\quad \left\{3\frac{x^2}{L^2}-2\frac{x^3}{L^3}\right\}\quad L\left\{-\frac{x^2}{L^2}+\frac{x^3}{L^3}\right\}\right)\begin{pmatrix}-\dfrac{2pl^4}{24EI_z}\\-\dfrac{pl^3}{24EI_z}\\0\\\dfrac{4pl^3}{24EI_z}\end{pmatrix}$$

$$v(x)=\frac{pl^4}{24EI_z}\left(-2-\left(\frac{x}{l}\right)+4\left(\frac{x}{l}\right)^2-\left(\frac{x}{l}\right)^3\right)\qquad for\quad 0<x<l$$

Question 9 : $\dfrac{d^2v(x)}{dx^2}$

Directly from the question 8, we have :

For the first element between node N_1 to node N_2:

$$\boxed{\frac{d^2v(x)}{dx^2}=\frac{pl}{24EI_z}\left(-10l+18x\right)\qquad for\quad 0<x<l}$$

For the second element between node N_2 to node N_3:

$$\boxed{\frac{d^2v(x)}{dx^2} = \frac{pl}{24EI_z}(8l-6x) \qquad for \quad 0<x<l}$$

Question 10 :

We remember that : $\frac{d^2v(x)}{dx^2} = \frac{M_z}{EI_z}$

So we have :

Between node N_1 to node N_2 : $\boxed{M_z = \frac{pl}{12}(-5l+9x) \qquad for \quad 0<x<l}$

Between node N_2 to node N_3 : $\boxed{M_z = \frac{pl}{12}(4l-3x) \qquad for \quad 0<x<l}$

Question 11 : Comparison

The results from the beam theory are drawn on the following figures :

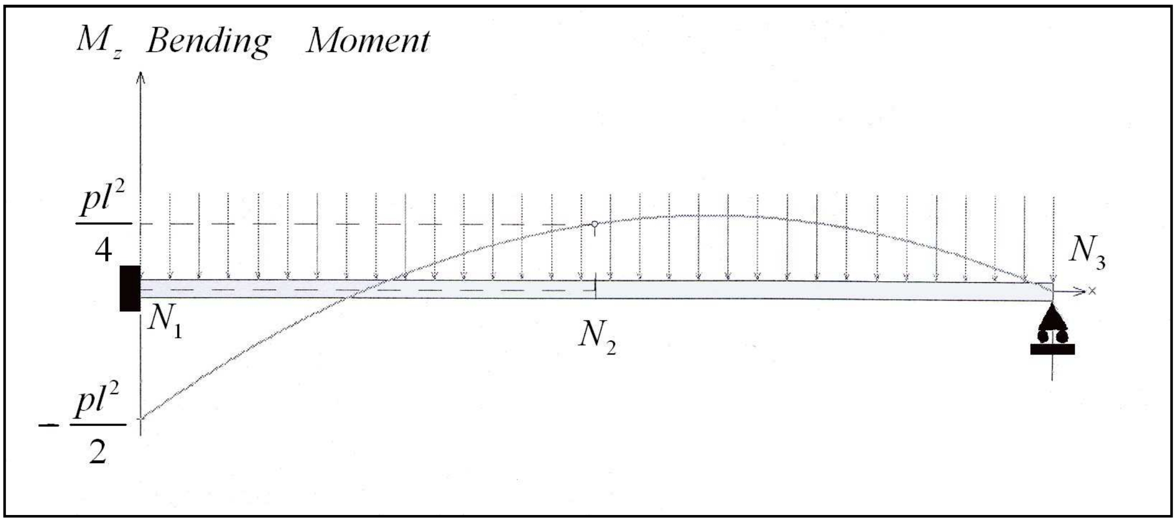


Figure 4 : Diagram of the bending moment issued of the beam theory

- | -

Question 12 : Modelling of the beam with 2 dimensional finite elements

According to the beam theory, the state of stress inside the structure doesn't depend of the direction z. There are only normal stress σ_x and shear stress τ_{xy} . We can use 2 dimensional finite elements to study this structure.

Question 13 : Type of element

Membrane elements for instance, with the nodes in the planes xy

Question 14 : Accurate value for the stress between two nodes

To find an accurate value for the normal stress between two nodes, you can add between these two nodes a bar element with a very very small surface for its cross section (for instance $S = 0.0001mm^2$), but made of the same material than the original structure.

This bar will not affect the global stiffness of the structure, so that will not change the result given by the finite element software. In fact the behaviour of the bar will be given by the behaviour of the structure. After that you can extract the value of the normal stress inside the bar to have an accurate value for it between the two nodes.

It's the same effect than a stain gage.

Exercise 15

Finite Elements
Dynamic study of a set of symmetric bars and beams

We consider the structure described by the figure 1. We shall do the following assumption to create a modelling with the finite element method :

- the part DC will be considered as a beam element,
- the part BD, BA and BC will be considered as bar elements.

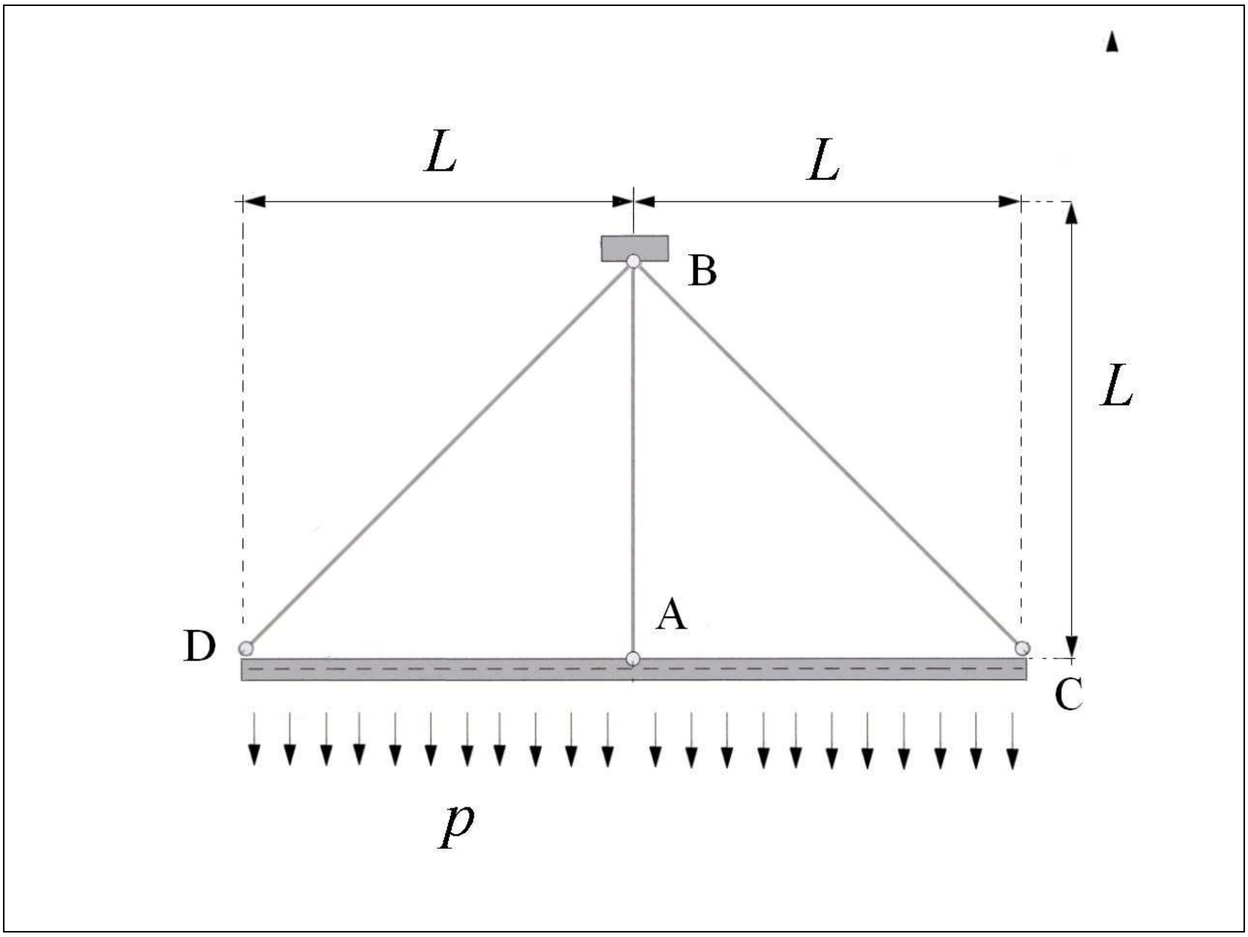


Figure 1 : Set of different bars

The different beams and bars are made of an elastic material whose properties are :
- Young's modulus : E
- Poisson's ratio : ν

The area of the cross section of bars and beam is the same :
- Area of cross section of beam and bars : S
- The length of the bar AB is : L
- The length of the bars DB and DC are : $L\sqrt{2}$
- The length of the beam DC is : $2L$
- The quadratic moments of the beam are : I_Z *and* I_y

- The mass of theAB bar is : m
- The mass of the bar db and DC are : $m\sqrt{2}$
- The mass of the beam DC is is : $2m$

The different joins are :

- at the point B : a join ball between all the bars,
- at the points D, A and C : a join ball between each bar and the beam,

The load apply consist only in a distributed load which can represents the weight of the beam for instance. The weight of the bars can be neglected in this study.

To simplify the problem, we shall neglect, in the beam, the effect of the shear force T_y.

This problem as been already studied (exo 7) from a static point of vieuw with the finite element method. The stiffness matrix found is :

$$[K]=\begin{bmatrix}
\frac{ES}{L} & 0 & 0 & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\
0 & 12\frac{EI_z}{L^3}+\frac{ES}{2L} & 6\frac{EI_z}{L^2} & 0 & -\frac{ES}{2L} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\
0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\
0 & 0 & 0 & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & 0 \\
0 & -\frac{ES}{2L} & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES}{2L}+\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\
-\frac{ES}{L} & 0 & 0 & -\frac{ES\sqrt{2}}{4L} & \frac{ES\sqrt{2}}{4L} & \frac{ES}{L}+\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 0 \\
0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & \frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & -\frac{ES\sqrt{2}}{4L} & 12\frac{EI_z}{L^3}+\frac{ES\sqrt{2}}{4L} & -6\frac{EI_z}{L^2} \\
0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & 0 & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L}
\end{bmatrix}$$

Questions

33 Simplify the study with consideration on symmetry.
34 Write the boundary conditions in term of displacements for the nodes inside the plane of symmetry,
35 If you study only the half part of the structure, what is the cross section to take in account for the bar AB
36 Choose a set of nodes and represent them with the corresponding DOF.
37 Create the mass matrix of the differents bar,
38 Create the mass matrix of the beam,
39 Create the global mass matrix of this problem,
40 Boundary conditions : Write the displacements in a column vector (q).
41 In order to find the shape and frequencies, write the fundamental equation of the finite element method.
42 Indicate how the computer can find the differents shapes and frequencies :

$$E = 78 \quad GPa$$
$$L = 1200 \quad mm$$
$$I_z = 1,2.10^7 \quad mm^4$$
$$S = 130 \quad mm^2$$
$$P = 2000 \quad N/m$$

Solution of the exercise 15

Question 1 : Half study of the structure.

As the structure studied has a plane of symmetry for the load and for the geometry we can study only the half part of it.

Question 2 : Boundary conditions in the plane of symmetry.

The plane of symmetry cannot have a translation in a normal direction to it, and rotations are not allowed for any axis in the plane of symmetry.

$$node \quad N_1 \quad \begin{cases} u_1 = 0 \\ \theta_1 = 0 \end{cases}$$

$$node \quad N_2 \quad \{ u_2 = 0$$

Question 3 : Cross section of the vertical bar.

The cross section of the bar in the plane of symmetry must be changed in $S/2$ to respect the symmetry of the problem

Question 4 : Nodes and DOF.

As we solve this problem with a 2D approach, the degree of freedom will be the following :

- for each node of a beam 3 DOF : 2 translations and 1 rotation,
- for each node of a bar 2 DOF : 2 translations

The following figure shows the different notations used afterwards.

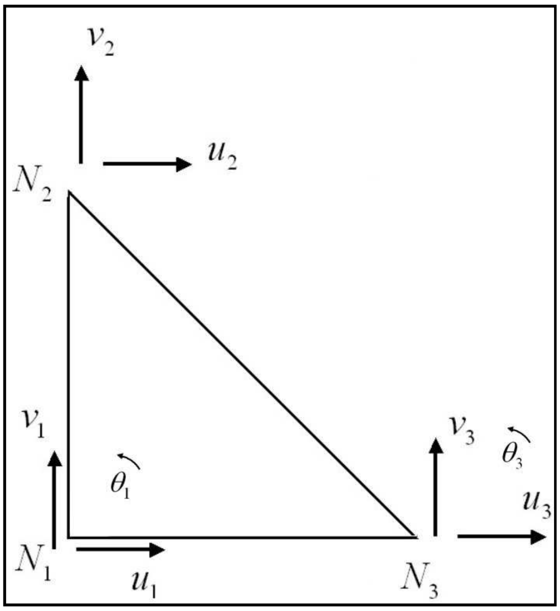


Figure 2 : Modelling - Nodes & DOF

The displacement vector (q) can be written in a matrix column, with :

$$\boxed{(q)^T = (u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad v_3 \quad u_3 \quad \theta_3)}$$

43 Mass matrix of the bar AB,

As the initial mass of the bar AB is m and as we have reduced the study to the half part of the system, the mass to take in account for the bar AB must be divided by 2.

In its local axis system we have

$$[M]_{AB} = \frac{m}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

In the global axis system the mass matrix :

$$[M]_{AB} = \frac{m}{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

44 Mass matrix of the bar CB,

As the mass of the bar CB is $m\sqrt{2}$, the mass matrix of this bar in its local axis system is :

$$[M]_{BC} = \frac{m\sqrt{2}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The angle $(\vec{x} \quad , \quad BC) = -45°$ so we can compute the mass matrix by using the classical process :

$$[M]_{BC} = \begin{bmatrix} [P] & [0] \\ [0] & [P] \end{bmatrix} \frac{m\sqrt{2}}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} [P]^T & [0] \\ [0] & [P]^T \end{bmatrix}$$

With

$$[P] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The result is :

$$[M]_{BC} = \frac{m\sqrt{2}}{12}\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

45 Mass matrix of the beam,

As the mass of the beam DC is 2m and as we have reduced the study to the half part of the system, the mass to take in account for the half beam is m. The local and the global axis system are the same for the beam.

$$[M]_{AC} = \frac{m}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & 22L & 4L^2 \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

46 global mass matrix of this problem,

$$[M]_{AC} = \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & 22L & 4L^2 \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

$$[M] = \frac{m}{420}\begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 70 & 0 & 0 \\ 0 & 226 & 22L & 35 & 0 & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 0 & 0 & 13L & -3L^2 \\ 0 & 0 & 0 & 70\sqrt{2} & 35\sqrt{2} & -70\sqrt{2} & -35\sqrt{2} & 0 \\ 0 & 35 & 0 & 70+35\sqrt{2} & 70\sqrt{2} & -35\sqrt{2} & -70\sqrt{2} & 0 \\ 70 & 0 & 0 & -70\sqrt{2} & -35\sqrt{2} & 70+70\sqrt{2} & 35\sqrt{2} & 0 \\ 0 & 54 & 13L & -35\sqrt{2} & -70\sqrt{2} & 35\sqrt{\ } & 156+70\sqrt{2} & -22L \\ 0 & -13L & -3L^2 & 0 & 0 & 0 & 22L & 4L^2 \end{bmatrix}$$

47 Boundary conditions : Write the displacements in a column vector (q).

48 In order to find the shape and frequencies, write the fundamental equation of the finite element method.

49 Indicate how the computer can find the differents shapes and frequencies :

$$E = 78 \quad GPa$$
$$L = 1200 \quad mm$$
$$I_z = 1{,}2.10^7 \quad mm^4$$
$$S = 130 \quad mm^2$$
$$P = 2000 \quad N/m$$

Mass matrix of a beam in its own axis system

Considerering a finite element of beam defined by two nodes and submitted to a bending moment Mz, as defined by the following figure :

According to the theory of beam, we know that a slice of beam whose thickness if dx will have a motion we can decomposed in :

- a translation of components v(x)
- a rotation around the neutral axis of an angle $\theta(x)$.

This motion is a function of the time, so we can define the kinetic energy of this slice of beam as the sum of the kinetic energy of translation and the kinetic energy of rotation.

Hence we have :

$$KineticEnergy = \frac{1}{2} dm \left(\frac{d}{dt}\big(v(x)\big) \right)^2 + \frac{1}{2} dJ \left(\frac{d}{dt}\big(\theta(x)\big) \right)^2$$

For the smaller shape modes, the second term of this equation can be neglected in front of the first term.

$$KineticEnergy = \frac{1}{2} \rho S \int_0^L \left(\frac{d}{dt}\big(v(x)\big) \right)^2 dx = \frac{1}{2} \rho S \int_0^L \left(\frac{d}{dt}\big(v(x)\big) \right)^T \left(\frac{d}{dt}\big(v(x)\big) \right) dx$$

If we remember that the displacement of the neutral axis is defined by shape functions like this :

$$v(x) = \left(\left\{ 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \right\} \quad L\left\{ \frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3} \right\} \quad \left\{ 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \right\} \quad L\left\{ -\frac{x^2}{L^2} + \frac{x^3}{L^3} \right\} \right) \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} \quad \text{(Eq 0)}$$

If we use these notations :

$$\Phi_1 = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$

$$\Phi_2 = L\left(\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3} \right)$$

$$\Phi_3 = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$

$$\Phi_4 = L\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3} \right)$$

The equation (0) becomes :

$$v(x) = \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

(Eq 0)

We can write :

$$\frac{dv(x)}{dt} = \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \end{pmatrix} \begin{pmatrix} \frac{dv_1}{dt} \\ \frac{d\theta_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{d\theta_2}{dt} \end{pmatrix}$$

To compute the total kinetic energy of the whole beam, we must do an integration when x ranges from o to L. Hence we have :

The general term Kik of the Mass matrix is $M = \int_0^L \Phi_i \quad \Phi_j \quad dx$

With

Φ_i

The result is :

If we take in account the kinetic energy in the motion in the direction u, we have :

Exercise 16

Finite Elements

Static and Dynamic study of a set of bars

We shall study a set of bars with the finite element method. The first part will be the static study and the second part will be the dynamic study.

Description of the structure

A set of 6 bars is defined by the figure 1.

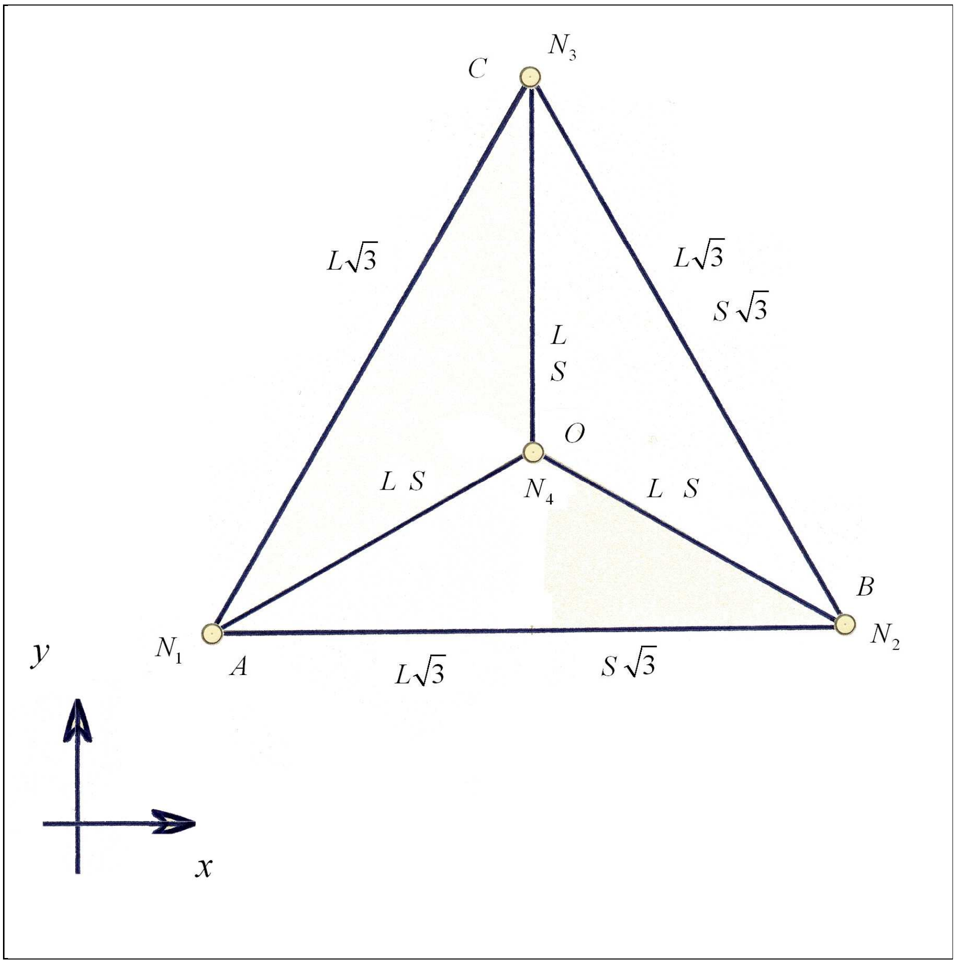


Figure 1 : Structure

The bars AB, BC, CA, respectively b_1 , b_2 , b_3 on the external edges of the triangle have a length equal to $L\sqrt{3}$ and a cross section equal to $S\sqrt{3}$

The bars AO, BO, CO, respectively b_4 , b_5 , b_6, inside the triangle have a length equal to L and a cross section equal to S

The different bars are made of the same elastic material defined by its Young's modulus E

Static Study

Question 1 : Write the stiffness matrix of each bar b_i in its own axis system,

Question 2 : Write the stiffness matrix of each bar b_i in the global axis system,

Question 3 : Write the stiffness matrix of this structure in the global axis system,
Question 4 : Is it possible to invert this matrix whose dimension is 8x8 ? Propose an example of the minimum joints that allow inverting this matrix.
Question 5 : Now, we shall choose the following joins, defined by the figure 2, and we shall apply a load on the node N4 whose components are $\vec{F} = (F_X , F_Y)$

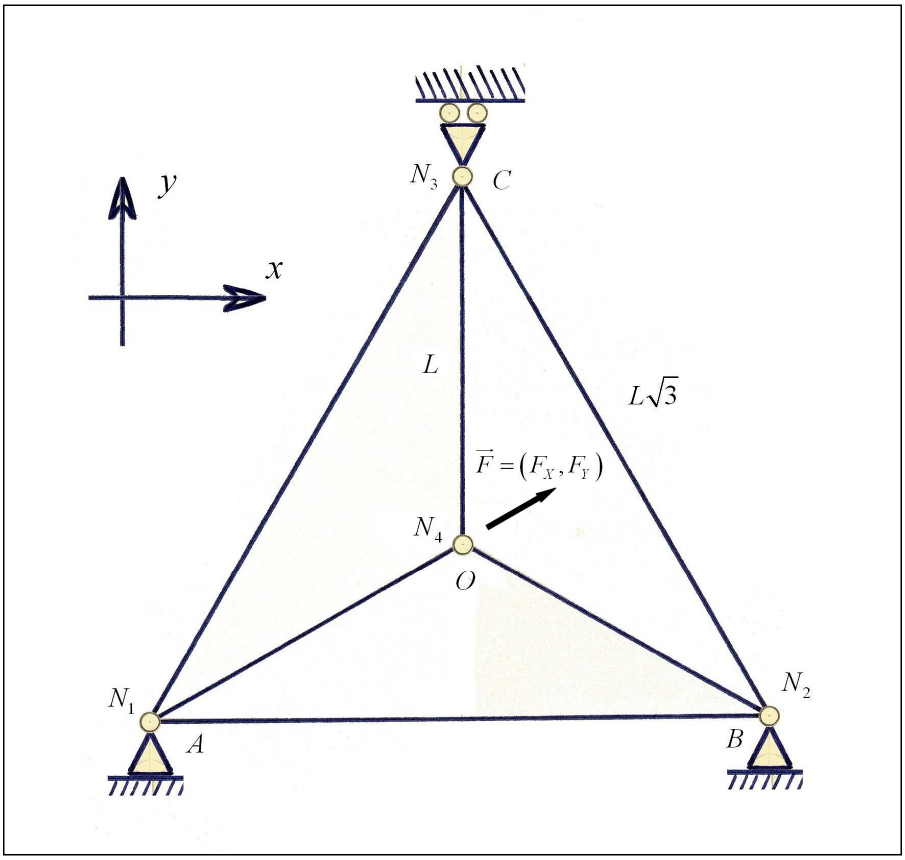


Figure 2 : Joins and Load

Write the values of the different displacements imposed by the joins.

Question 6 : Write the equation the computer has to solve under the form $(F)=[K](q)$

Write this equation with respect of boundary conditions.

Question 7 : Solve this equation to find the unknown displacement.

Question 8 : Find the values of the reactions introduced by the joins.
Question 9 : Do you think the solution given by the finite element method is as accurate as the solution you could find by applying the theory of bars and beams ?
Question 10 : Give the value of the normal stress in the bars b_1 (AB) , b_4 (BO) , b_6 (CO)

Dynamic Study

Question 11 : Write the mass matrix of each bar b_i in its own axis system,

Question 12 : Write the mass matrix of each bar b_i in the global axis system,

Question 13 : Write the mass matrix of this structure in the global axis system,

Verify that this mass matrix is equal to :

$$[M]=\frac{\rho LS}{24}\begin{bmatrix} 36 & 8\sqrt{3} & 12 & 0 & 3 & 3\sqrt{3} & 3 & \sqrt{3} \\ 8\sqrt{3} & 20 & 0 & 0 & 3\sqrt{3} & 9 & \sqrt{3} & 1 \\ 12 & 0 & 36 & -8\sqrt{3} & 3 & -3\sqrt{3} & 3 & -\sqrt{3} \\ 0 & 0 & -8\sqrt{3} & 20 & -3\sqrt{3} & 9 & -\sqrt{3} & 1 \\ 3 & 3\sqrt{3} & 3 & -3\sqrt{3} & 12 & 0 & 0 & 0 \\ 3\sqrt{3} & 9 & -3\sqrt{3} & 9 & 0 & 32 & 0 & 4 \\ 3 & \sqrt{3} & 3 & -\sqrt{3} & 0 & 0 & 12 & 0 \\ \sqrt{3} & 1 & -\sqrt{3} & 1 & 0 & 4 & 0 & 12 \end{bmatrix}$$

Question 14 : Write the fundamental equation of dynamic without external dynamic load,

Question 15 : Choose a solution as $q(x,t)=Q(x) \quad \times \quad A\sin\omega t$ in order to solve this equation. Explain what is $Q(x)$.

Question 16 : Write the system the computer has to solve, under the form $\omega^2=[M]^{-1}[K]$, with respect of boundary conditions

Question 17 : Solve this system and find the find the natural frequencies of this system.

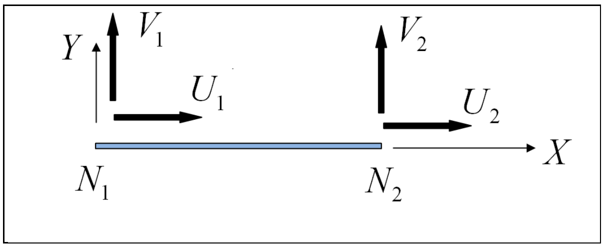
Question 18 : Explain to shat correspond these natural fraguencies.

Solution of the exercise 16

Question 1 : Stiffness matrix of each bar b_i in its own axis system

The stiffness matrix is the same for the all the bars when it is written in the local axis system. Take care to the length and the cross section that are different.

Bar number 1 between the nodes N_1 and N_2

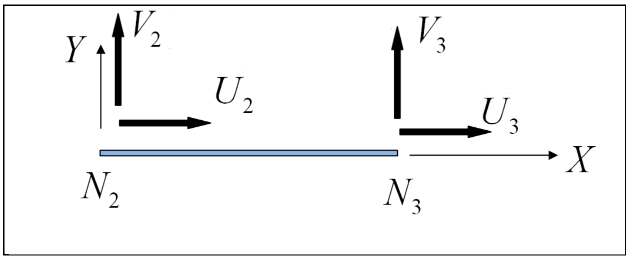


Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

$$[K]_{12} = \frac{ES\sqrt{3}}{L\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 2 between the nodes N_2 and N_3

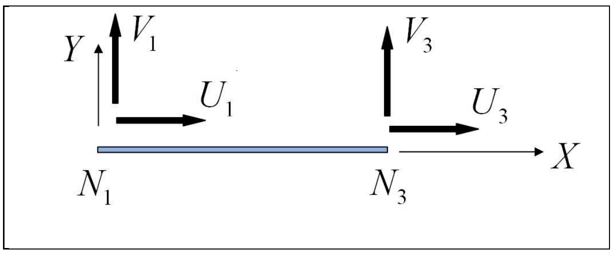


Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

$$[K]_{23} = \frac{ES\sqrt{3}}{L\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 3 between the nodes N_1 and N_3

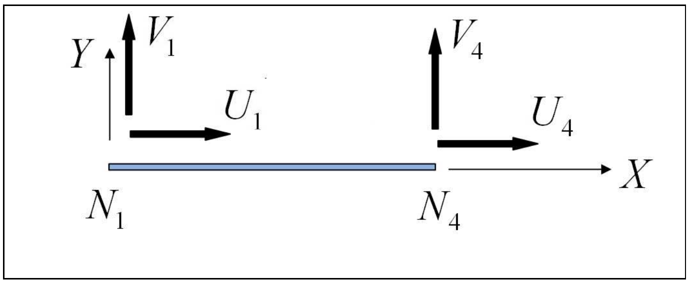


Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

$$[K]_{13} = \frac{ES\sqrt{3}}{L\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

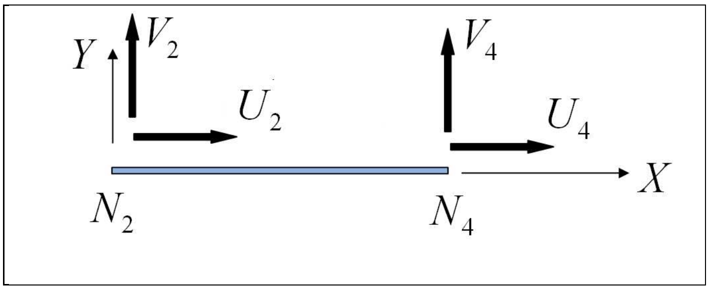
Bar number 4 between the nodes N_1 and N_4



Length of the bar : L
Cross section : S

$$[K]_{14} = \frac{ES}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

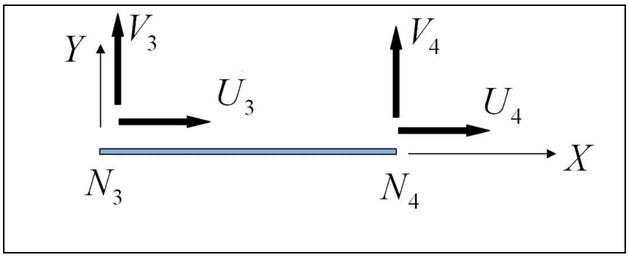
Bar number 5 between the nodes N_2 and N_4



Length of the bar : L
Cross section : S

$$[K]_{24} = \frac{ES}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 6 between the nodes N_3 and N_4



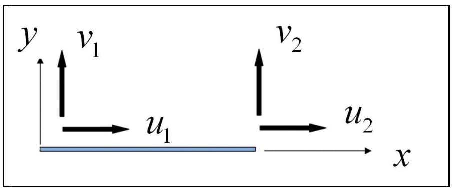
Length of the bar : L
Cross section : S

$$[K]_{34} = \frac{ES}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 2 : Stiffness matrix of each bar b_i in the global axis system

We can use the relation
$$[K]_{b_i} = \begin{bmatrix} C^2 & SC & -C^2 & -SC \\ SC & S^2 & -SC & -S^2 \\ -C^2 & -SC & C^2 & SC \\ -SC & -S^2 & SC & S^2 \end{bmatrix} \frac{E_{b_i} S_{b_i}}{L_{b_i}}$$

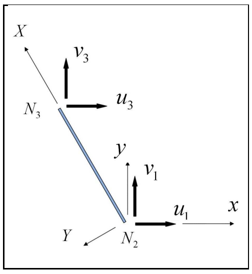
Bar number 1 between the nodes N_1 and N_2



Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = 0$

$$[K]_{12} = \frac{ES\sqrt{3}}{4L\sqrt{3}} \begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{ES}{4L} \begin{bmatrix} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

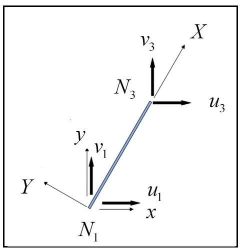
Bar number 2 between the nodes N_2 and N_3



Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = 2\pi/3$

$$[K]_{23} = \frac{ES\sqrt{3}}{4L\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix} = \frac{ES}{4L} \begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}$$

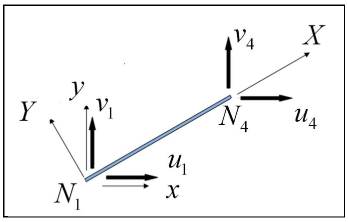
Bar number 3 between the nodes N_1 and N_3



Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = \pi/3$

$$[K]_{13} = \frac{ES\sqrt{3}}{4L\sqrt{3}} \begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix} = \frac{ES}{4L} \begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

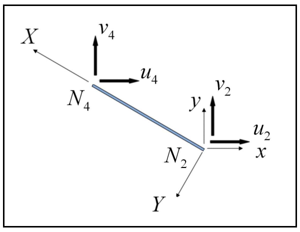
Bar number 4 between the nodes N_1 and N_4



Length of the bar : L
Cross section : S
Angle : $\alpha \;=\; \pi/6$

$$[K]_{14}=\frac{ES}{4L}\begin{bmatrix} 3 & \sqrt{3} & -3 & -\sqrt{3} \\ \sqrt{3} & 1 & -\sqrt{3} & -1 \\ -3 & -\sqrt{3} & 3 & \sqrt{3} \\ -\sqrt{3} & -1 & \sqrt{3} & 1 \end{bmatrix}$$

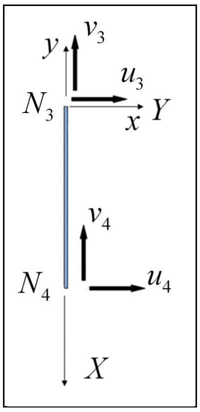
Bar number 5 between the nodes N_2 and N_4



Length of the bar : L
Cross section : S
Angle : $\alpha \;=\; 5\pi/6$

$$[K]_{24}=\frac{ES}{4L}\begin{bmatrix} 3 & -\sqrt{3} & -3 & \sqrt{3} \\ -\sqrt{3} & 1 & \sqrt{3} & -1 \\ -3 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & 1 \end{bmatrix}$$

Bar number 6 between the nodes N_3 and N_4



Length of the bar : L
Cross section : S
Angle : $\alpha \;=\; -\pi/2$

$$[K]_{34}=\frac{ES}{4L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{bmatrix}$$

Question 3 : Stiffness matrix of this structure

Assembling the global stiffness matrix with the stiffness matrix of each bar can be done very easily like this :

$$[K]=\frac{ES}{4L}\begin{bmatrix} 8 & 2\sqrt{3} & -4 & 0 & -1 & -\sqrt{3} & -3 & -\sqrt{3} \\ 2\sqrt{3} & 4 & 0 & 0 & -\sqrt{3} & -3 & -\sqrt{3} & -1 \\ -4 & 0 & 8 & -2\sqrt{3} & -1 & \sqrt{3} & -3 & \sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 4 & \sqrt{3} & -3 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 & 0 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 10 & 0 & -4 \\ -3 & -\sqrt{3} & -3 & \sqrt{3} & 0 & 0 & 6 & 0 \\ -\sqrt{3} & -1 & \sqrt{3} & -1 & 0 & -4 & 0 & 6 \end{bmatrix}$$

Question 4 : Example of the minimum joints that allow to invert the reduced stiffness matrix.

No because there are three rigid body movement. We must remove them to allow the computer to solve the problem by inverting a reduced stiffness matrix. We must remove the two translations in the direction x and y for instance, and the rotation around a z axis.

The joins on the following figure allows to remove the rigid body movements. There are many other solutions for that.

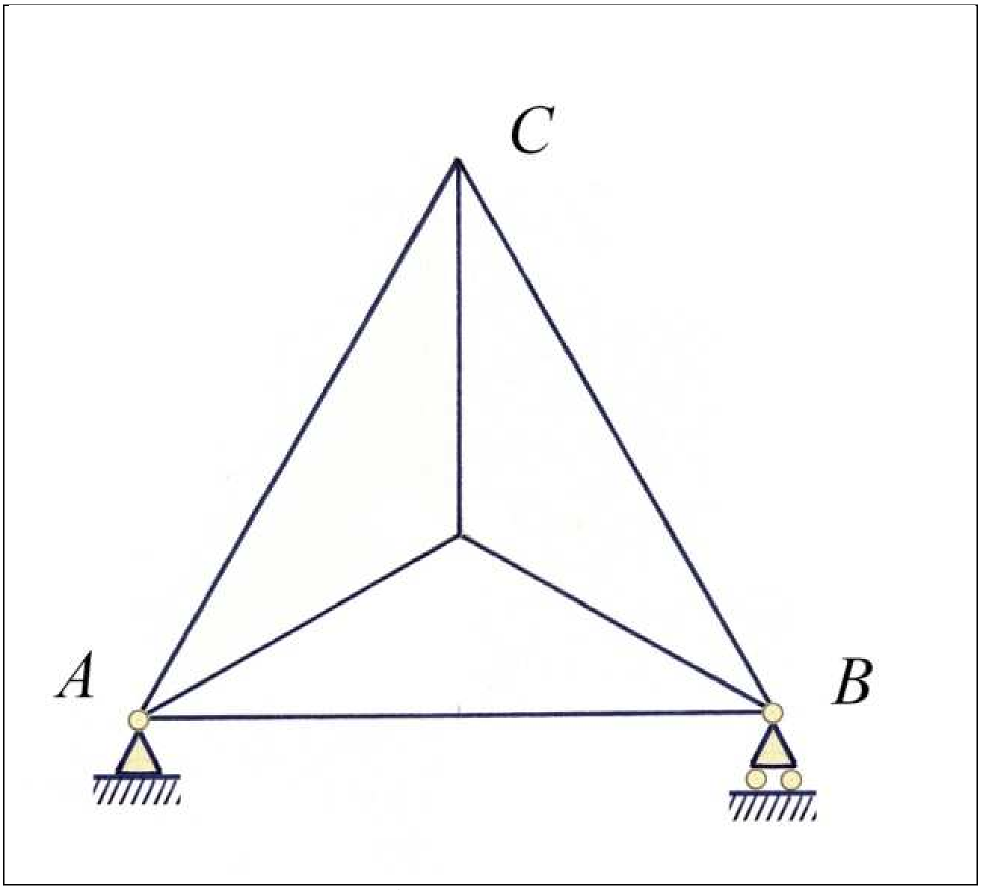


Figure 3 : Example of joins to remove the rigid body movement

Question 5 : Displacements imposed by the joins.

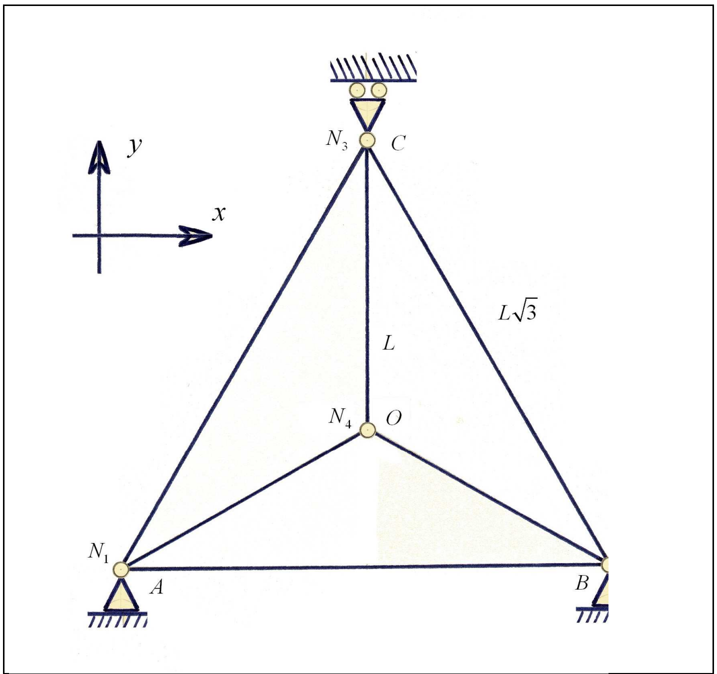


Figure 4 : Joins

$$\begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = 0 \\ v_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} u_3 = Free \\ u_4 = Free \\ v_4 = Free \end{cases}$$

Question 6 : The computer has to solve $(F) = [K](q)$

The fundamental equation of finite element for static is :

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ 0 \\ Y_3 \\ F_X \\ F_Y \end{bmatrix} = \frac{ES}{4L} \begin{bmatrix} 8 & 2\sqrt{3} & -4 & 0 & -1 & -\sqrt{3} & -3 & -\sqrt{3} \\ 2\sqrt{3} & 4 & 0 & 0 & -\sqrt{3} & -3 & -\sqrt{3} & -1 \\ -4 & 0 & 8 & -2\sqrt{3} & -1 & \sqrt{3} & -3 & \sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 4 & \sqrt{3} & -3 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 & 0 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 10 & 0 & -4 \\ -3 & -\sqrt{3} & -3 & \sqrt{3} & 0 & 0 & 6 & 0 \\ -\sqrt{3} & -1 & \sqrt{3} & -1 & 0 & -4 & 0 & 6 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ v_4 \end{pmatrix} \end{bmatrix}$$

After removing the lines and the columns 1,2,3,4,6, corresponding to the displacement known (imposed by the joins), we obtain the reduced equation :

$$\begin{bmatrix} 0 \\ F_X \\ F_Y \end{bmatrix} = \frac{ES}{2L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Question 7 : Unknown displacements.

This relation can be inverted very easily. Hence we obtain :

$$\begin{bmatrix} u_3 \\ u_4 \\ v_4 \end{bmatrix} = \frac{2L}{ES} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ F_X \\ F_Y \end{bmatrix}$$

And finally the displacements unknown are :

$$\boxed{\begin{aligned} & u_3 = 0 \\ & u_4 = \frac{2L}{3ES} F_X \\ & v_4 = \frac{2L}{3ES} F_Y \end{aligned}}$$

Question 8 : Reactions introduced by the joins.

As the displacements are known, we can use the equation of the question 6 to write

$$\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ 0 \\ Y_3 \\ F_X \\ F_Y \end{bmatrix} = \frac{ES}{4L} \begin{bmatrix} 8 & 2\sqrt{3} & -4 & 0 & -1 & -\sqrt{3} & -3 & -\sqrt{3} \\ 2\sqrt{3} & 4 & 0 & 0 & -\sqrt{3} & -3 & -\sqrt{3} & -1 \\ -4 & 0 & 8 & -2\sqrt{3} & -1 & \sqrt{3} & -3 & \sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 4 & \sqrt{3} & -3 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 & 0 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 10 & 0 & -4 \\ -3 & -\sqrt{3} & -3 & \sqrt{3} & 0 & 0 & 6 & 0 \\ -\sqrt{3} & -1 & \sqrt{3} & -1 & 0 & -4 & 0 & 6 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ v_4 \end{pmatrix} \end{bmatrix}$$

We obtain :

$$\boxed{\begin{aligned} & X_1 = \frac{ES}{4L}\left(-3u_4 - \sqrt{3}v_4\right) = -\frac{1}{2}F_X - \frac{\sqrt{3}}{6}F_Y \\ & Y_1 = \frac{ES}{2L}\left(-\sqrt{3}u_4 - v_4\right) = -\frac{\sqrt{3}}{6}F_X - \frac{1}{6}F_Y \\ & X_2 = \frac{ES}{2L}\left(-3u_4 + \sqrt{3}v_4\right) = -\frac{1}{2}F_X + \frac{\sqrt{3}}{6}F_Y \\ & Y_2 = \frac{ES}{2L}\left(\sqrt{3}u_4 - v_4\right) = \frac{\sqrt{3}}{6}F_X - \frac{1}{6}F_Y \\ & Y_3 = \frac{ES}{2L}\left(-4v_4\right) = -\frac{2}{3}F_Y \end{aligned}}$$

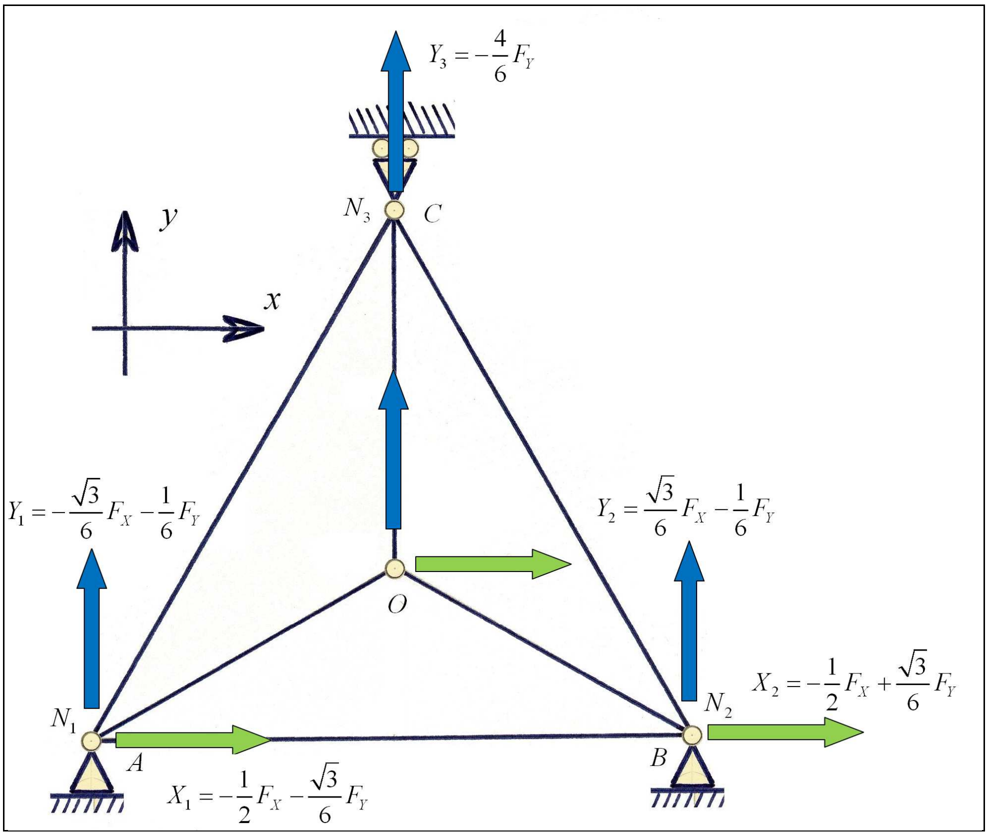


Figure 4 : Reactions

Question 9 : Comparison between finite element solution and analytical solution.

The solution given by the finite element method is the same than the solution found with the theory of bars, because for the bars the hypothesis are the same for these two theories.

Question 10 : Normal stress in the bars $b_1 \quad (AB)$ *,* $b_2 \quad (BC)$ *,* $b_6 \quad (CO)$

The displacements of the nodes 1, 2 and 3 are equal to zero. Therefore the variation of length of the each bar b1, b2 and b3 is equal to zero. Hence we have:

$$\sigma_x(b_1) = \sigma_x(b_2) = \sigma_x(b_3) = 0$$

The displacements of the nodes 1 and 3 have a component only in the direction Y according to the local axis of the bar b6. Then the stress is equal to zero.

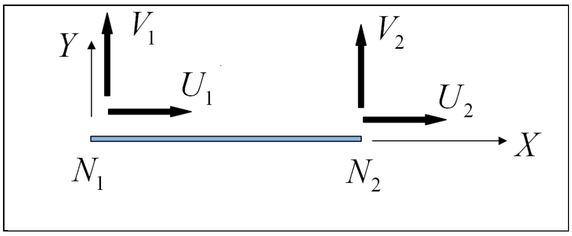
$$\sigma_x(b_6) = 0$$

Dynamic study

Question 11 : Mass matrix of each bar b_i in its own axis system

The mass matrix is the same for all the bars when it is written in the local axis system of the bar. Take care to the length and the cross section that are different.

Bar number 1 between the nodes N_1 and N_2



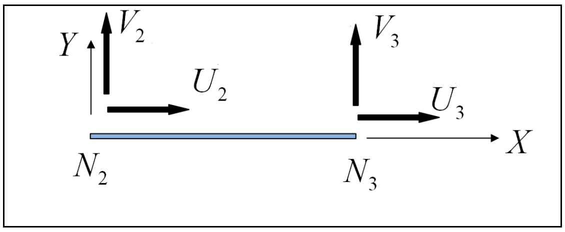
Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

Density : ρ

$$[M]_{12} = \frac{3\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 2 between the nodes N_2 and N_3



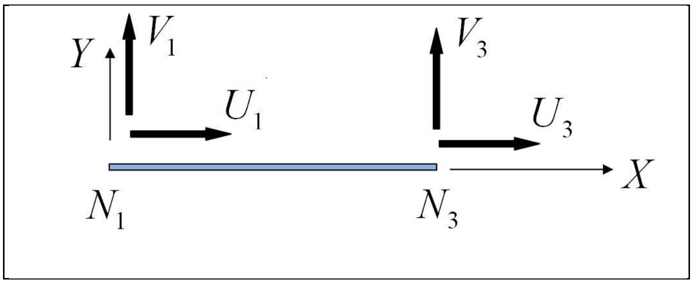
Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

Density : ρ

$$[M]_{23} = \frac{3\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 3 between the nodes N_1 and N_3



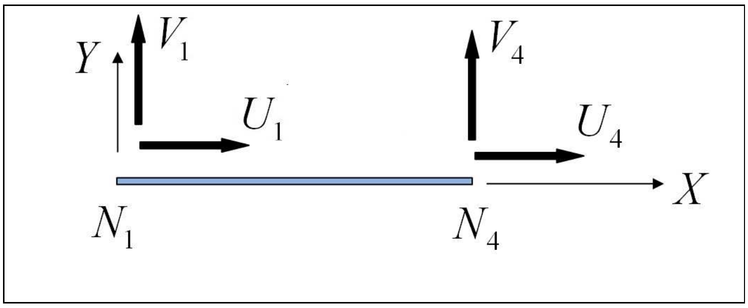
Length of the bar : $L\sqrt{3}$

Cross section : $S\sqrt{3}$

Density : ρ

$$[M]_{13} = \frac{3\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

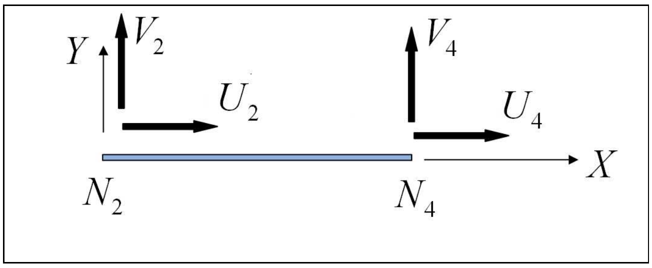
Bar number 4 between the nodes N_1 and N_4



Length of the bar : L
Cross section : S
Density : ρ

$$[M]_{14} = \frac{\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

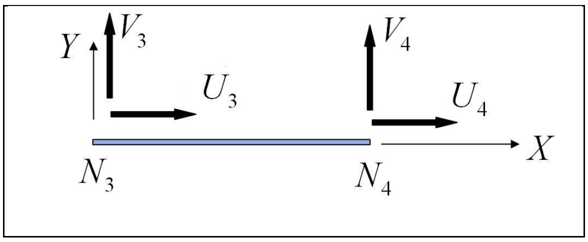
Bar number 5 between the nodes N_2 and N_4



Length of the bar : L
Cross section : S
Density : ρ

$$[M]_{24} = \frac{\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 6 between the nodes N_3 and N_4



Length of the bar : L
Cross section : S
Density : ρ

$$[M]_{34} = \frac{\rho LS}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 12 : Mass matrix of each bar b_i in the global axis system

We can use the relation $$[M]_{b_i} = \frac{M_{b_i}}{6}\begin{bmatrix} C^2 & SC & -C^2 & -SC \\ SC & S^2 & -SC & -S^2 \\ -C^2 & -SC & C^2 & SC \\ -SC & -S^2 & SC & S^2 \end{bmatrix}$$

Bar number 1 between the nodes N_1 and N_2

Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = 0$

$$[M]_{12} = \frac{3\rho LS}{24}\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bar number 2 between the nodes N_2 and N_3

Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = 2\pi/3$

$$[M]_{23} = \frac{3\rho LS}{24}\begin{bmatrix} 2 & -2\sqrt{3} & 1 & -\sqrt{3} \\ -2\sqrt{3} & 6 & -\sqrt{3} & 3 \\ 1 & -\sqrt{3} & 2 & -2\sqrt{3} \\ -\sqrt{3} & 3 & -2\sqrt{3} & 6 \end{bmatrix}$$

Bar number 3 between the nodes N_1 and N_3

Length of the bar : $L\sqrt{3}$
Cross section : $S\sqrt{3}$
Angle : $\alpha = \pi/3$

$$[M]_{13} = \frac{3\rho LS}{24}\begin{bmatrix} 2 & 2\sqrt{3} & 1 & \sqrt{3} \\ 2\sqrt{3} & 6 & \sqrt{3} & 3 \\ 1 & \sqrt{3} & 2 & 2\sqrt{3} \\ \sqrt{3} & 3 & 2\sqrt{3} & 6 \end{bmatrix}$$

Bar number 4 between the nodes N_1 and N_4

Length of the bar : L
Cross section : S
Angle : $\alpha = \pi/6$

$$[M]_{14} = \frac{\rho LS}{24}\begin{bmatrix} 6 & 2\sqrt{3} & 3 & \sqrt{3} \\ 2\sqrt{3} & 2 & \sqrt{3} & 1 \\ 3 & \sqrt{3} & 6 & 2\sqrt{3} \\ \sqrt{3} & 1 & 2\sqrt{3} & 2 \end{bmatrix}$$

Bar number 5 between the nodes N_2 and N_4

Length of the bar : L
Cross section : S
Angle : $\alpha = 5\pi/6$

$$[M]_{24} = \frac{\rho LS}{24}\begin{bmatrix} 6 & -2\sqrt{3} & 3 & -\sqrt{3} \\ -2\sqrt{3} & 2 & -\sqrt{3} & 1 \\ 3 & -\sqrt{3} & 6 & -2\sqrt{3} \\ -\sqrt{3} & 1 & -2\sqrt{3} & 2 \end{bmatrix}$$

Bar number 6 between the nodes N_3 and N_4

Length of the bar : L
Cross section : S
Angle : $\alpha = -\pi/2$

$$[M]_{24} = \frac{\rho LS}{24}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$$

Question 13 : Mass matrix of the set of bars

Assembling the mass stiffness matrix with the mass matrix of each bar can be done very easily like this :

$$[M]=\frac{\rho LS}{24}\begin{bmatrix} 36 & 8\sqrt{3} & 12 & 0 & 3 & 3\sqrt{3} & 3 & \sqrt{3} \\ 8\sqrt{3} & 20 & 0 & 0 & 3\sqrt{3} & 9 & \sqrt{3} & 1 \\ 12 & 0 & 36 & -8\sqrt{3} & 3 & -3\sqrt{3} & 3 & -\sqrt{3} \\ 0 & 0 & -8\sqrt{3} & 20 & -3\sqrt{3} & 9 & -\sqrt{3} & 1 \\ 3 & 3\sqrt{3} & 3 & -3\sqrt{3} & 12 & 0 & 0 & 0 \\ 3\sqrt{3} & 9 & -3\sqrt{3} & 9 & 0 & 32 & 0 & 4 \\ 3 & \sqrt{3} & 3 & -\sqrt{3} & 0 & 0 & 12 & 0 \\ \sqrt{3} & 1 & -\sqrt{3} & 1 & 0 & 4 & 0 & 12 \end{bmatrix}$$

Question 14 : Fundamental equation of dynamic without external dynamic load

$$\boxed{[M]\frac{d^2q}{dt^2}+[K]q=0}$$

Question 14 : $q(x,t)=Q(x) \quad \times \quad A\sin\omega t$

$Q(x)$***. Represent the displacement. This displacement is represented with the shape function used in the static study.***

$$\boxed{\begin{aligned} &[M]\frac{d^2q(x,t)}{dt^2}+[K]q(x,t)=0 \\ &A\sin(\omega t)\left(-\omega^2[M]+[K]\right)q(x)=0 \\ &\omega^2[M]=[K] \\ &\omega^2=[M]^{-1}[K] \end{aligned}}$$

Question 16 : Write the system the computer has to solve, under the form $\omega^2=[M]^{-1}[K]$, with respect of boundary conditions

The joins give the values of the displacement. If we remove the lines and column for which the displacement is equal to zero we obtain.

$$\omega^2 = \begin{bmatrix} \frac{\rho LS}{2} & 0 & 0 \\ 0 & \frac{\rho LS}{2} & 0 \\ 0 & 0 & \frac{\rho LS}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{ES}{2L} & 0 & 0 \\ 0 & \frac{3ES}{2L} & 0 \\ 0 & 0 & \frac{3ES}{2L} \end{bmatrix}$$

To invert the mass matrix is easy because it is a diagonal matrix.

$$\begin{bmatrix} \frac{\rho LS}{2} & 0 & 0 \\ 0 & \frac{\rho LS}{2} & 0 \\ 0 & 0 & \frac{\rho LS}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{\rho LS} & 0 & 0 \\ 0 & \frac{2}{\rho LS} & 0 \\ 0 & 0 & \frac{2}{\rho LS} \end{bmatrix}$$

And finally the result is :

$$\omega^2 = \begin{bmatrix} \frac{E}{\rho L^2} & 0 & 0 \\ 0 & \frac{3E}{\rho L^2} & 0 \\ 0 & 0 & \frac{3E}{\rho L^2} \end{bmatrix}$$

Question 17 : Solve this system and find the find the natural frequencies of this system.

The joins give the values of the displacement. If we remove the lines and column for which the displacement is equal to zero we obtain.

$$\omega^2 = \begin{bmatrix} \frac{\rho LS}{2} & 0 & 0 \\ 0 & \frac{\rho LS}{2} & 0 \\ 0 & 0 & \frac{\rho LS}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{ES}{2L} & 0 & 0 \\ 0 & \frac{3ES}{2L} & 0 \\ 0 & 0 & \frac{3ES}{2L} \end{bmatrix}$$

To invert the mass matrix is easy because it is a diagonal matrix.

$$\begin{bmatrix} \frac{\rho LS}{2} & 0 & 0 \\ 0 & \frac{\rho LS}{2} & 0 \\ 0 & 0 & \frac{\rho LS}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{\rho LS} & 0 & 0 \\ 0 & \frac{2}{\rho LS} & 0 \\ 0 & 0 & \frac{2}{\rho LS} \end{bmatrix}$$

And finally the result is :

$$\omega^2 = \begin{bmatrix} \frac{E}{\rho L^2} & 0 & 0 \\ 0 & \frac{3E}{\rho L^2} & 0 \\ 0 & 0 & \frac{3E}{\rho L^2} \end{bmatrix}$$

We have immediately the three eigen values :

$$\boxed{\begin{array}{l} \omega_1 = \frac{1}{L}\sqrt{\frac{E}{\rho}} \\ \omega_2 = \frac{\sqrt{3}}{L}\sqrt{\frac{E}{\rho}} \\ \omega_3 = \frac{\sqrt{3}}{L}\sqrt{\frac{E}{\rho}} \end{array}}$$

Exercise 18

Finite Elements : Assembling of beams and bars
Static and dynamic study

We want to study the plane system defined by the figure 1 with the finite elements method.

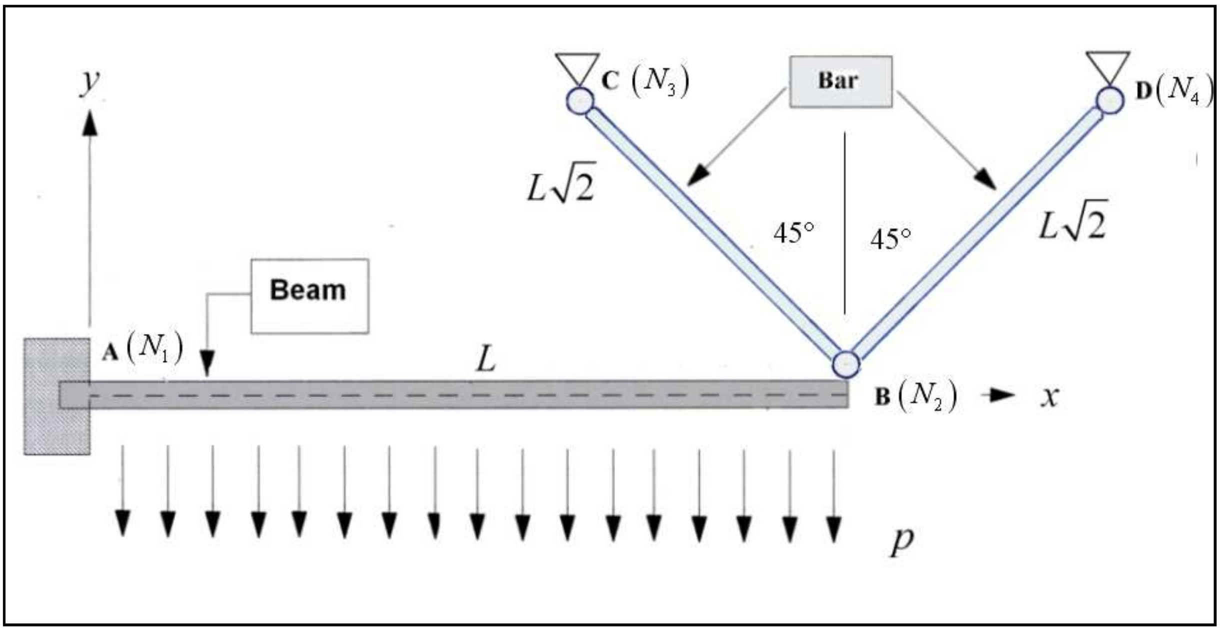


Figure 1 : Case study

The system is made of a beam (AB) and two bars (BC and BD) inclined of an angle of –45° for BC and 45° for BD.

All the mechanical characteristics are supposed known, as for instance :

- the same cross section for the bars and for the beam $S = 10000 \quad mm^2$
- the quadratic moment around the z axis for the beam, $I_z = 10^{-5} \quad m^4$
- Young's modulus of material bars and beam are made of : $E = 200 \quad GPa$
- the length of the beam $L = 1 \quad m$
- the length of each bar $L\sqrt{2} = 1.414 \quad m$
- the distributed load applied to the beam $p = 100 \quad N/mm$
- the weight of the bar is neglicted
- the mass of the beam is $M = 80 \quad Kg$
- the mass of each bar is $m = \dfrac{M}{10} = 8 \quad Kg$

The beam will be modelized by one beam element between the nodes N_1 (A) and N_2 (B)
Each bar will be modelized by one bar element between the nodes N_2 (B) and N_3 (C) for the bar BC and the nodes N_2 (B) and N_4 (D) for the bar BD.

The beam is clamped on the left side (Cross section A)
In C and D the join is a ball join.

Notation 1 : To simplyfy the writing of the stiffnes matrix we shall use the coefficient α defined like this :

$$\frac{ES}{L} = \alpha \frac{EI_z}{L^3} \qquad \text{with} \qquad \alpha = \frac{L^2 S}{I_z}$$

Notation 2 : The displacements of the different nodes, in the global axis system,will be noted :

$$\begin{cases} u_1 \\ v_1 \\ \theta_1 \end{cases} for \quad N_1 \qquad \begin{cases} u_2 \\ v_2 \\ \theta_2 \end{cases} for \quad N_2 \qquad \begin{cases} u_3 \\ v_3 \end{cases} for \quad N_3 \qquad \begin{cases} u_4 \\ v_4 \end{cases} for \quad N_4$$

Remark : For the beam the terms of the stiffness matrix introduced by the shear force will be neglicted.

Questions :

Static study :

1 Write the stiffness matrix of a beam in its own axis system just submetted to the normal force N_x and the bending moment M_z ,
2 Write the stiffness matrix of a bar in its own axis system,
3 Write the stiffness matrix of a beam AB in the global axis system,
4 Write the stiffness matrix of a bar BC in the global axis system,
5 Write the stiffness matrix of a bar BD in the global axis system,
6 Write the stiffness matrix of the system in the global axis system,
7 Write the nodal equivalent load,
8 After removing the displacement well known, write (without solving) the equation that allow the computer to solve the finite element problem.
9 Find the numerical values of the unknown displacements.

Dynamic study :

10 Write the masss matrix of a beam in its own axis system,
11 Write the mass matrix of a bar in its own axis system,
12 Write the mass matrix of a beam AB in the global axis system,
13 Write the mass matrix of a bar BC in the global axis system,
14 Write the mass matrix of a bar BD in the global axis system,
15 Write the mass matrix of the system in the global axis system,
16 Write the fundamental equation of finite element for the dynamic problem.
17 Write the polynomial equation that allow to find the different angular frequencies.
18 Compute the numerical value for the first angular frequency.

Solution of the exercice 18

Static study :

Question 1 : Stiffness matrix of a beam in its own axis system just submitted to the normal force N_x and the bending moment M_z,

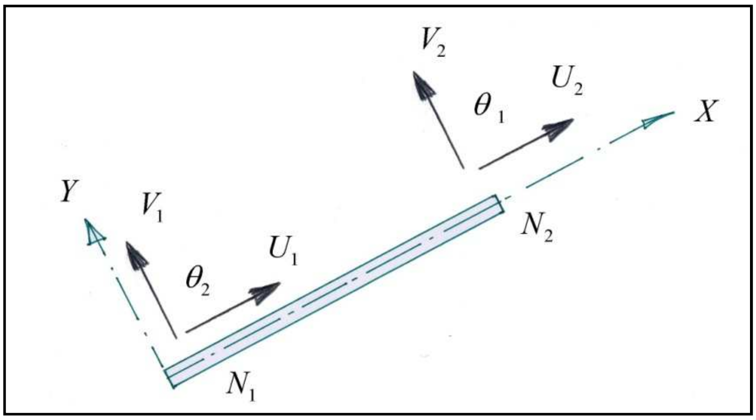


Figure 2 : Beam element in its axis system (XY)

With the notation $\alpha = \dfrac{L^2 S}{I_z}$ we have :

$$[K]_{BEAM} = \frac{EI_Z}{L^3}\begin{bmatrix} \alpha & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \qquad \begin{pmatrix} U_1 \\ V_1 \\ \theta_1 \\ U_2 \\ V_2 \\ \theta_2 \end{pmatrix}$$

Question 2 : Stiffness matrix of a bar in its own axis system,

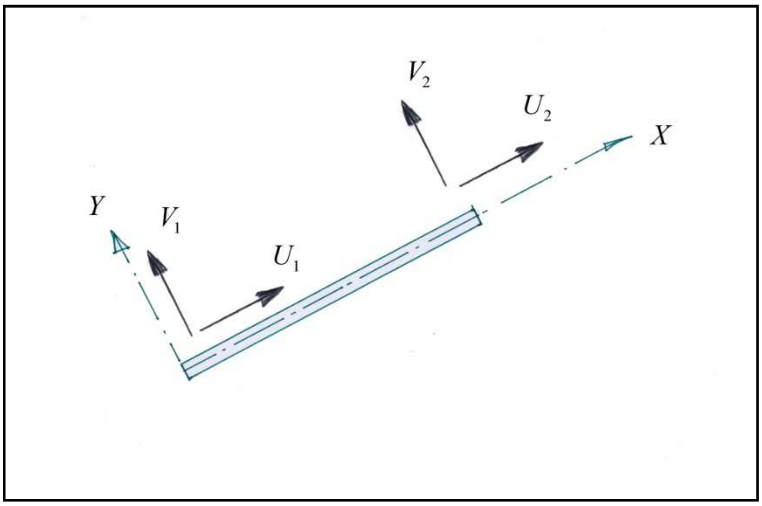


Figure 3 : Bar element in its axis system (XY)

$$[K]_{BAR} = \frac{ES}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}$$

Question 3 : Stiffness matrix of a beam AB in the global axis system,

As the global axis system and the local axis system are the same we have :

$$[K]_{AB} = \frac{EI_Z}{L^3}\begin{bmatrix} \alpha & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix}$$

Question 4 : Stiffness matrix of a bar BC in the global axis system,

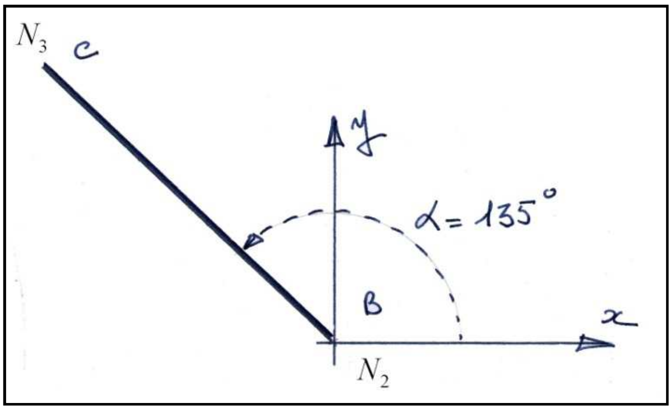


Figure 4 : BC bar element

This bar has a length equal to $L\sqrt{2}$ and the angle $\alpha = 135°$
The stiffness matrix of this bar in the global axis system is given by the following relation:

$$[K]_{BC} = \begin{bmatrix} [P] & [0] \\ [0] & [P] \end{bmatrix} \begin{bmatrix} \frac{ES}{L\sqrt{2}} & 0 & -\frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L\sqrt{2}} & 0 & \frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [P]^T & [0] \\ [0] & [P]^T \end{bmatrix}$$

With

$$[P]=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \quad \& \quad [P]^T=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

We have :

$$[K]_{BC}=\frac{ES}{L\sqrt{2}}\frac{\sqrt{2}}{2}\begin{bmatrix}-1 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & -1 & -1\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix}-1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -1\end{bmatrix}$$

$$[K]_{BC}=\frac{ES}{L}\frac{\sqrt{2}}{4}\begin{bmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix} \qquad \begin{pmatrix}u_2\\ v_2\\ u_3\\ v_3\end{pmatrix}$$

Question 5 : Stiffness matrix of a bar BD in the global axis system,

This bar has a length equal to $L\sqrt{2}$ and the angle $\alpha = 45°$

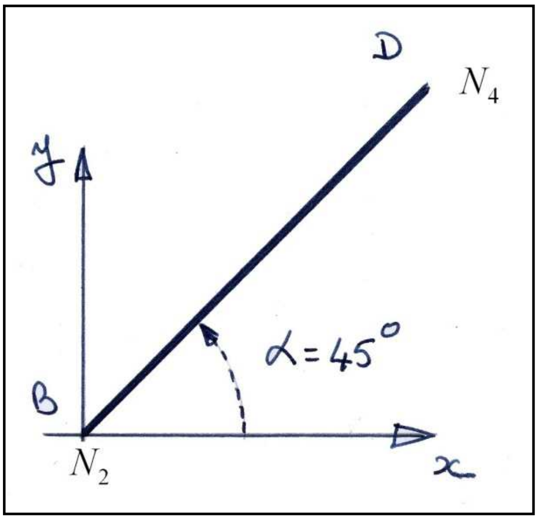


Figure 5 : BC bar element

$$[K]_{BD}=\frac{ES}{L\sqrt{2}}\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$$[K]_{BD} = \frac{ES}{L}\frac{\sqrt{2}}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix}$$

Question 6 : Stiffness matrix of the system in the global axis system,

$$[K] = \frac{EI_Z}{L^3}\begin{bmatrix} \alpha & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L & 0 & 0 & 0 & 0 \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & \alpha\left(1+\frac{\sqrt{2}}{2}\right) & 0 & 0 & -\alpha\frac{\sqrt{2}}{4} & \alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} \\ 0 & -12 & -6L & 0 & 12+\alpha\frac{\sqrt{2}}{2} & -6L & \alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha\frac{\sqrt{2}}{4} & \alpha\frac{\sqrt{2}}{4} & 0 & \alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & 0 & \alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & 0 & -\alpha\frac{\sqrt{2}}{4} & \alpha\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & 0 & -\alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & 0 & 0 & 0 & \alpha\frac{\sqrt{2}}{4} & \alpha\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & -\alpha\frac{\sqrt{2}}{4} & -\alpha\frac{\sqrt{2}}{4} & 0 & 0 & 0 & \alpha\frac{\sqrt{2}}{4} & \alpha\frac{\sqrt{2}}{4} \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

Question 7 : Nodal equivalent load,

$$[F]_{EXT} = \begin{pmatrix} 0 \\ -pL/2 \\ -pL^2/12 \\ 0 \\ -pL/2 \\ pL^2/12 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad [F]_{REACTION} = \begin{pmatrix} X_1 \\ Y_1 \\ M_1 \\ 0 \\ 0 \\ 0 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{pmatrix}$$

Question 8 : After removing the displacement well known, the equation that allow the computer to solve the finite element problem is :

$$\begin{pmatrix} 0 \\ \dfrac{-pL}{2} \\ \dfrac{pL^2}{12} \end{pmatrix} = \frac{EI_Z}{L^3}\begin{bmatrix} \alpha\left(1+\dfrac{\sqrt{2}}{2}\right) & 0 & 0 \\ 0 & 12+\alpha\dfrac{\sqrt{2}}{2} & -6L \\ 0 & -6L & 4L^2 \end{bmatrix}\begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \end{pmatrix}$$

Question 9 : Numerical values of the unknown displacements.

$$\frac{EI_Z}{L^3} = 2000 \quad N/mm \quad , \quad \alpha = \frac{SL^2}{I_Z} = 1000 \quad , \quad \frac{pL}{2} = 50000 \quad N \quad , \quad \frac{pL}{12} = 8333 \quad N$$

The matrix equation is equivalent to :

$$\begin{cases} u_2 = 0 \\ 719v_2 - 6000\theta_2 = -25 \\ -6v_2 + 4000\theta_2 = 4.167 \end{cases}$$

The resolution gives :

$$\begin{cases} u_2 = 0 & mm \\ v_2 = & mm \\ \theta_2 = & rad \end{cases}$$

Dynamic study :

Question 10 : Mass matrix of a beam in its own axis system,

$$[M]_{BEAM} = \frac{M}{420}\begin{bmatrix} 70 & 0 & 0 & 35 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 35 & 0 & 0 & 70 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}$$

Question 11 : Mass matrix of a bar in its own axis system,

$$[M]_{BAR} = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Question 12 : Mass matrix of a beam AB in the global axis system,

As for the stiffness matrix, the global axis system and the local axis system are the same. Hence we have :

$$[M]_{BEAM} = \frac{M}{420}\begin{bmatrix} 70 & 0 & 0 & 35 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 35 & 0 & 0 & 70 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix}$$

Question 13 : Mass matrix of a bar BC in the global axis system,

$$[M]_{BC} = \frac{m}{6}\frac{\sqrt{2}}{2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$[M]_{BC} = \frac{m}{12}\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix} \qquad \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Question 14 : Mass matrix of a bar BD in the global axis system,

$$[M]_{BD} = \frac{m}{6}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$[M]_{BD} = \frac{m}{12}\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \qquad \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix}$$

Question 15 : Mass matrix of the system in the global axis system,

$$[M]=\frac{M}{420}\begin{bmatrix} 70 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22L & 0 & -54 & -13L & 0 & 0 & 0 & 0 \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 & 0 & 0 & 0 & 0 \\ 35 & 0 & 0 & 84 & 0 & 0 & 3.5 & -3.5 & 3.5 & 3.5 \\ 0 & -54 & 13L & 0 & 170 & -22L & -3.5 & 3.5 & 3.5 & 3.5 \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.5 & -3.5 & 0 & 7 & -7 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 3.5 & 0 & -7 & 7 & 0 & 0 \\ 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 7 & 7 \end{bmatrix} \qquad \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

Question 15 : Fundamental equation of finite element for the dynamic problem.

$$[M]\frac{d^2q(x,t)}{dt^2}+[K]q(x,t)=0$$

We try to find solution of the form : $q(x,t)=A\sin\omega t.q(x)$

Hence we have :

$$q(x,t)=A\sin\omega t.q(x)$$
$$\frac{dq(x,t)}{dt}=-A\omega\cos\omega t.q(x)$$
$$\frac{d^2q(x,t)}{dt2}=-A\omega^2\sin\omega t.q(x)$$

And finally the fundamental equation of the dynamic becomes :

$$[K]q(x)=\omega^2[M]q(x)$$

Question 16 : Polynomial equation that allow to find the different angular frequencies.

To find the natural frequency and the shape mode associated we have to solve the equation of the question 15. The natural frequencies are the squared roots of the eigens values of the matrix $[M]^{-1}[K]$

$$\boxed{\omega^2=[M]^{-1}[K]}$$

Question 17 : Numerical values for the differents angular frequencies.

Stiffness matrix of a beam, whose length is L, in its own axis system. The shear force T_y is neglected.

$$[K]_{Beam} = \begin{bmatrix} \frac{ES}{L} & 0 & 0 & -\frac{ES}{L} & 0 & 0 \\ 0 & 12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\ -\frac{ES}{L} & 0 & 0 & \frac{ES}{L} & 0 & 0 \\ 0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & 0 & 12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix}$$

With the condensed notations we obtain:

$$[K]_{Beam} = \frac{EI_z}{L^3}\begin{bmatrix} \alpha & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

1- Stiffness matrix of a bar, whose lenght is L, in its own axis system

$$[K]_{Bar} = \begin{bmatrix} \frac{ES}{L} & -\frac{ES}{L} \\ -\frac{ES}{L} & \frac{ES}{L} \end{bmatrix} = \frac{ES}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EI_z}{L^3}\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix}$$

2- Stiffness matrix of the bar BC in the global axis system,

$$[K]_{BC}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{bmatrix}\frac{ES}{L\sqrt{2}}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}$$

That gives :

$$[K]_{BC}=\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{bmatrix}$$

$$[K]_{BC}=\frac{EI_z}{L^3}\begin{bmatrix}\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}\end{bmatrix}$$

Exercise 19

Finite Elements
Assembling of two bars

We want to study the system defined by the figure 1 with the finite elements method.

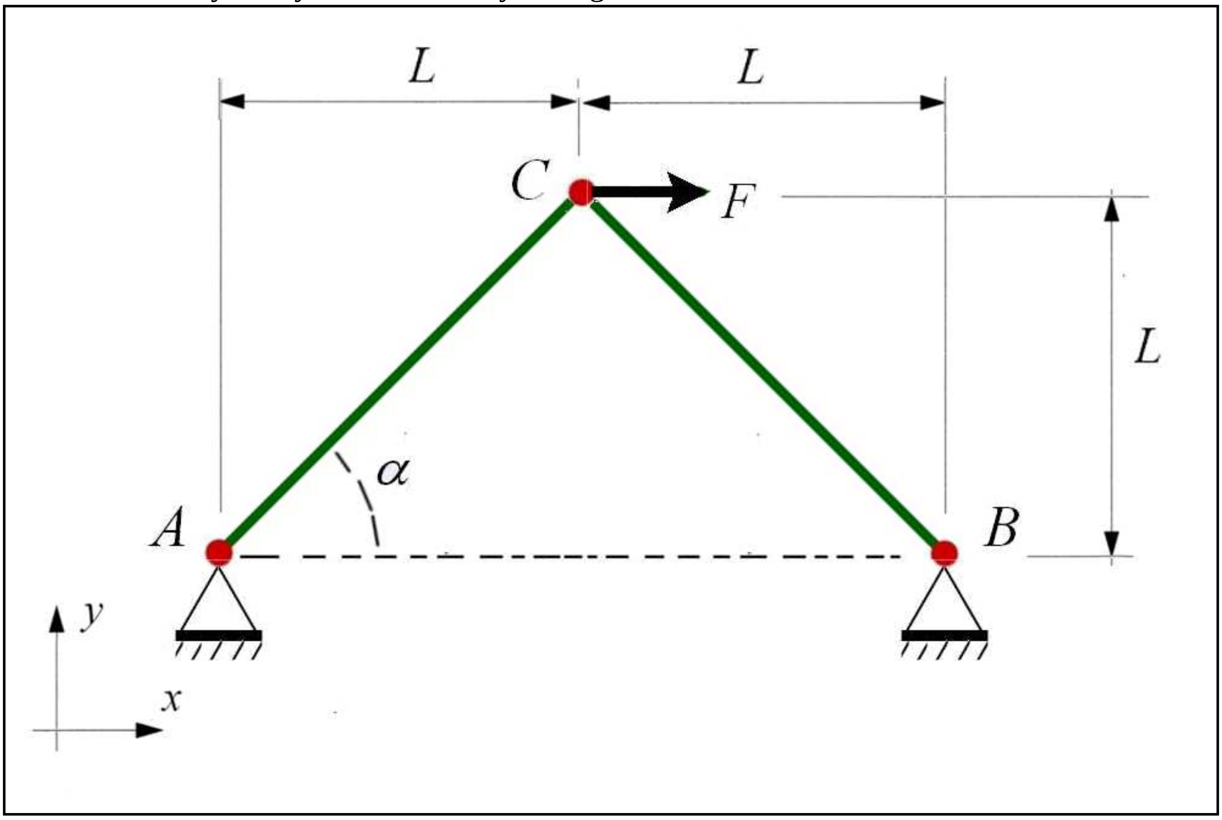


Figure 1 : Case study

All the mechanical characteristics are supposed known, as for instance :

- the cross section of the bar S
- the quadratic moment around the z axis for the beam, I_z
- the young's modulus of the material E

Questions :

Question 1 : Propose a modelling for this structure.

Question 2 : Choose a set of nodes and define for each of them the DOF you need.

Question 3 : Compute the stiffness matrix of the left bar

Question 4 : Compute the stiffness matrix of the right bar

Question 5 : Create by assembling the global stiffness matrix,

Question 6 : Is it possible to invert this stiffness matrix ?

Question 7 : Write the boundary conditions and define the displacements vector (q).

Question 8 : Write the external load

Question 9 : Write the internal load

Question 10 : Write the global load

Question 11: Write the equation $(F)=[K](q)$.

Question 12 : Find the displacement unknown

Question 13 : Find the reactions

Question 14 : Verify of the equilibrium.

Question 15 : Compute the normal stresses in the two bars

Solution

Question 1 :Modelling.

As we can assume that the two beams can only works with a constant state of uniform normal stress, the best modelling is to use two bar elements.

Question 2 : Nodes and DOF.

$$(q)^T = \begin{pmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{pmatrix}$$

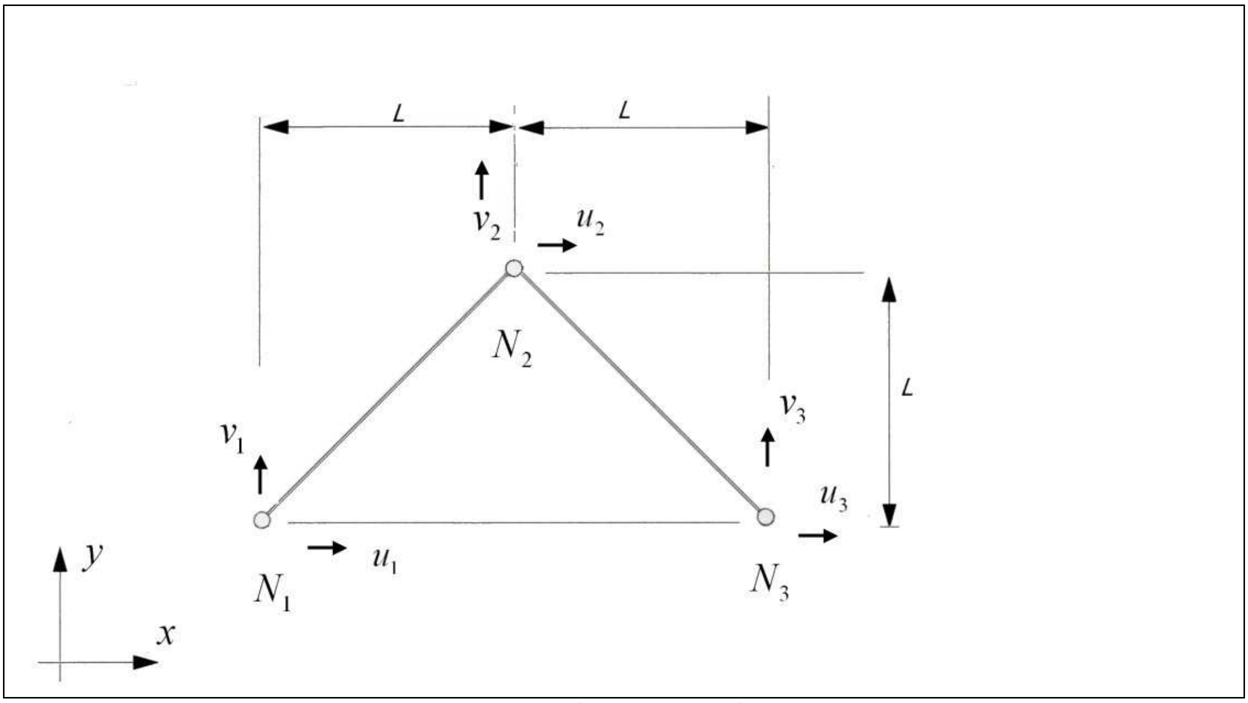


Figure 2 : Position of nodes and DOF for each node

Question 3 : Stiffness matrix of the left bar

For the bar between the nodes N_1 and N_2 we obtain with 45° as value for α :

$$W_{12} = \frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_3 \end{pmatrix}^T \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{ES}{L\sqrt{2}} & 0 & -\frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ES}{L\sqrt{2}} & 0 & \frac{ES}{L\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

That gives :

$$W_{12}=\frac{1}{2}\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix}$$

And finally the stiffness matrix of this bas is :

$$K_{12}=\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}$$

Question 4 : Stiffness matrix of the right bar

For the bar between the nodes N_2 and N_3 we obtain we obtain with -45° as value for α : :

$$W_{24}=\frac{1}{2}\begin{pmatrix}u_2\\v_2\\u_3\\v_3\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&0&0\\-1&1&0&0\\0&0&1&1\\0&0&-1&1\end{bmatrix}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&-1&0&0\\1&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\begin{pmatrix}u_2\\v_2\\u_3\\v_3\end{pmatrix}$$

That gives :

$$W_{23}=\frac{1}{2}\begin{pmatrix}u_2\\v_2\\u_3\\v_3\end{pmatrix}^T\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}\begin{pmatrix}u_2\\v_2\\u_3\\v_3\end{pmatrix}$$

And finally the stiffness matrix of this baris :

$$K_{23}=\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}$$

Question 5 : Global stiffness matrix,

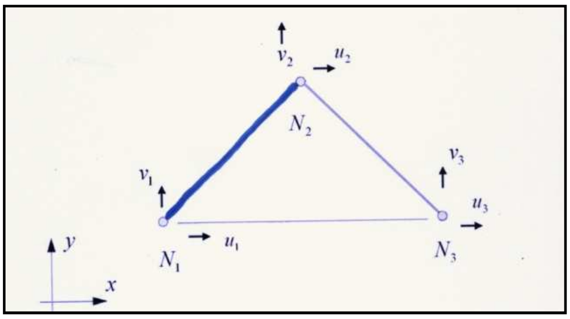


Figure 3 : First Bar Element

$\frac{ES\sqrt{2}}{4L}$

	u_1	v_1	u_2	v_2	u_3	v_3
u_1	1	1	-1	-1		
v_1	1	1	-1	-1		
u_2	-1	-1	1	1		
v_2	-1	-1	1	1		
u_3						
v_3						

Figure 4 : Terms of stiffness for the first Beam Element

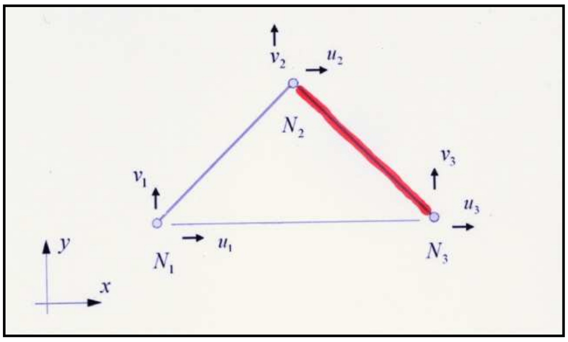


Figure 5 : Second Beam Element

$\frac{ES\sqrt{2}}{4L}$

	u_1	v_1	u_2	v_2	u_3	v_3
u_1						
v_1						
u_2			1	-1	-1	1
v_2			-1	1	1	-1
u_3			-1	1	1	-1
v_3			1	-1	-1	1

Figure 6 : Terms of stiffness for the Second Beam Element

Finally we obtain for the global internal energy

$$W = \frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}^T \frac{ES\sqrt{2}}{4L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Then the stiffness matrix is :

$$[K]=\frac{ES\sqrt{2}}{4L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Finally we obtain for the global internal energy

$$W=\frac{1}{2}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}^T \frac{ES\sqrt{2}}{4L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

Then the stiffness matrix is :

$$[K]=\frac{ES\sqrt{2}}{4L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Question 6 : Is it possible to invert this stiffness matrix ?

We can’t invert this matrix, because there are rigid body movements.

Question7 : Boundary conditions : Nodal displacements vector (q).

The physical joins impose physical displacements.

According to that we can write the displacements of the different nodes concerned :

$$
(q) = \begin{pmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 = 0 \\ v_3 = 0 \end{pmatrix}
$$

Question 8 : Reaction Load

When a degree of freedom (DOF) is removed or imposed by a join an internal reaction appears :

- this reaction is a force when the DOF is a translation,
- this reaction is a moment when the DOF is a rotation.

For this study we have :

$$
(R_{eactions}) = \begin{pmatrix} X_1 \\ Y_1 \\ 0 \\ 0 \\ X_3 \\ Y_3 \end{pmatrix}
$$

Question 9 : External load

The external load is introduced by a punctual force.

$$
(F_{applied}) = \begin{pmatrix} 0 \\ 0 \\ F_c \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Question 10 : Internal Load

When a degree of freedom (DOF) is removed or imposed by a join an internal reaction appears :

- this reaction is a force when the DOF is a translation,
- this reaction is a moment when the DOF is a rotation.

For this study we have :

$$\left(R_{eactions}\right)=\begin{pmatrix}X_1\\Y_1\\0\\0\\X_3\\Y_3\end{pmatrix}$$

Question 11 : Global load

The global load is the sum of the reaction load (unknown) and the external load (well known)

$$\left(F\right)=\begin{pmatrix}X_1\\Y_1\\F_C\\0\\X_3\\Y_3\end{pmatrix}$$

Question 12 : Equation $\left(F\right)=\left[K\right]\left(q\right)$.

The fundamental equation of the finite element theory $\left(F\right)=\left[K\right]\left(q\right)$ can be written like this according to the real boundary conditions of the structure studied :

$$\begin{pmatrix}X_1\\Y_1\\F_C\\0\\X_3\\Y_3\end{pmatrix}=\frac{ES\sqrt{2}}{4L}\begin{bmatrix}1&1&-1&-1&0&0\\1&1&-1&-1&0&0\\-1&-1&2&0&-1&1\\-1&-1&0&2&1&-1\\0&0&-1&1&1&-1\\0&0&1&-1&-1&1\end{bmatrix}\begin{pmatrix}0\\0\\u_2\\v_2\\0\\0\end{pmatrix}\qquad \text{EQ 21-1}$$

From this equation we extract a new equation that gives us the unknown displacements:

Question 13 : Displacements

If we remove from the EQ 21-1the lines and columns for which the displacements are known, we obtain the displacements:

$$\begin{pmatrix} F_C \\ 0 \end{pmatrix} = \frac{ES\sqrt{2}}{4L} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \quad => \quad \begin{cases} u_2 = \dfrac{L\sqrt{2}}{ES} F_C \\ v_2 = 0 \end{cases}$$

Question 14 : Reactions

The equation EQ 21-1 allows the computer to find the unknown reactions :

$$X_1 = Y_1 = -\frac{F_C}{2}$$
$$-X_3 = Y_3 = \frac{F_C}{2}$$

Question 15 : Verification of the equilibrium.

We can verify easily that the reactions and the load applied are a system of forces in equilibrium:

$$X_1 + X_3 + F_C = 0$$
$$Y_1 + Y_3 = 0$$
$$Y_3 2L - F_C L = 0$$

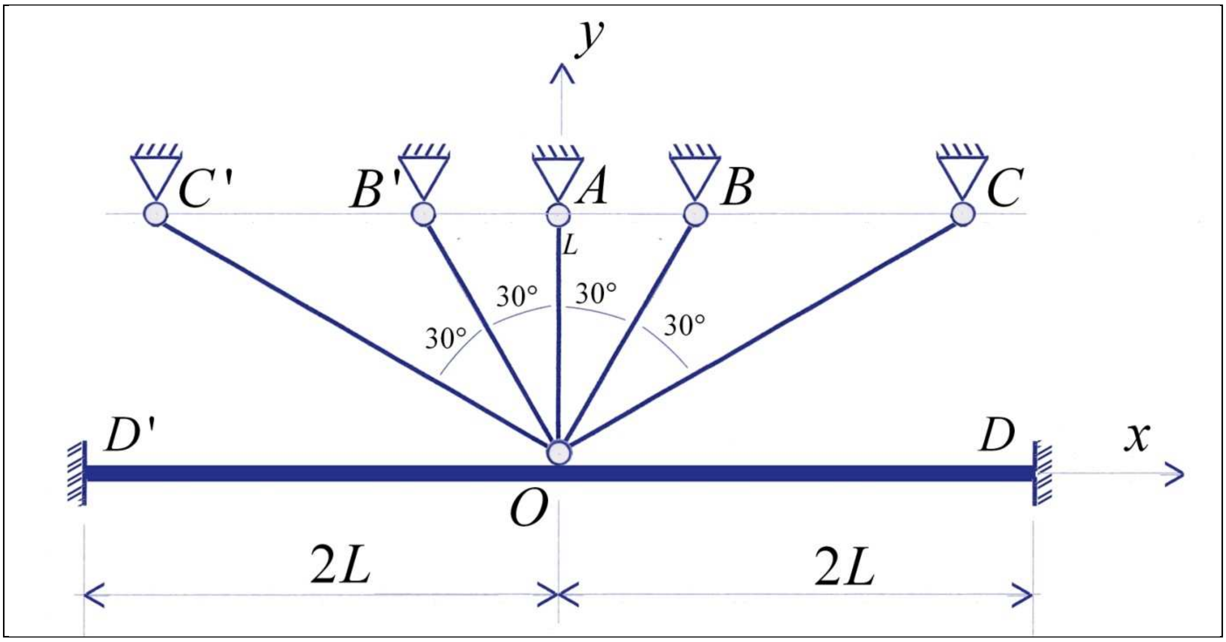
Question 14 : Normal stresses.

This question has been solved in a previous exercice.

Exercise 21

Finite Elements
Assembling of bars and beam : Static and Dynamic

On se propose de calculer par la méthode des éléments finis le système représenté par la figure suivante.



Ce système se compose :

- D'une poutre de longueur 4L, encastrée à ses deux extrémités.
- De 5 barres reliées entre elles et à la poutre au point O.

Caractéristiques des barres

	Longueur	Masse volumique	Section droite	Masse	Module Young
Barre OA	L	ρ	$2S$	2m	E
Barre OB et OB'	A calculer	ρ	$\frac{S\sqrt{3}}{2}$	A calculer	E
Barre OC et OC'	A calculer	ρ	$\frac{S}{2}$	A calculer	E

L'angle entre deux barres consécutives est de 30°.

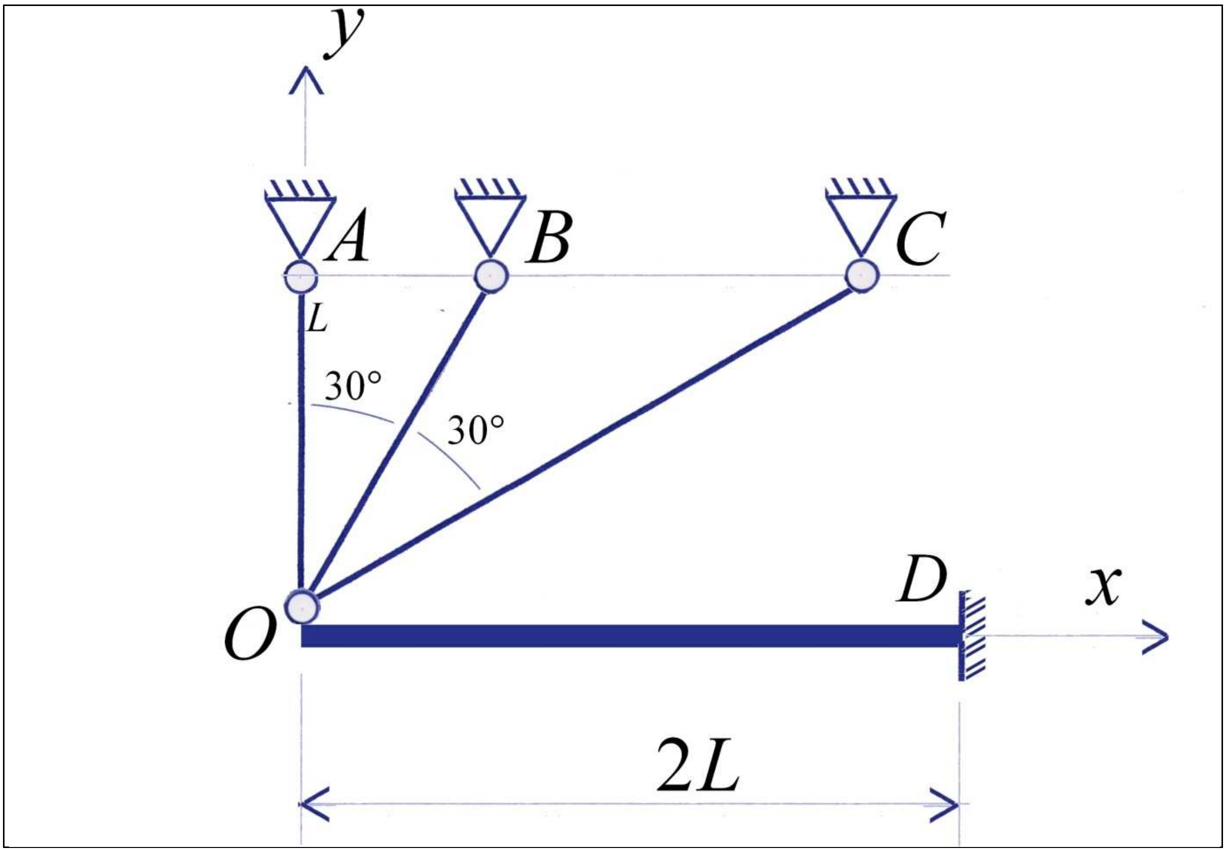
Caractéristiques de la poutre :

	Longueur	Masse volumique	Section droite	Masse	Module Young	Moment quadratique
Poutre DD’	4L	ρ	$100S$	400m	E	I_z

Ce système est soumis au poids de la poutre qui s’exerce selon la direction $\vec{g}\begin{pmatrix}0 & -g & 0\end{pmatrix}$.
On négligera le poids des barres pour la partie statique mais pas leur masse pour la partie dynamique.

Remarque liminaire :

La structure possède un plan de symétrie. Il est donc intéressant d’en modéliser uniquement la moitié, celle représentée sur la figure suivante par exemple.



Questions !

Question 1 : Discrétisation de la demi-structure. Par combien d’éléments, et de quels types, peut-on modéliser cette structure. Combien de nœuds sont nécessaires.

Question 2 : Quelles sont les conditions aux limites que l’on doit imposer en O et en A pour tenir compte de la symétrie. On traitera le problème en 2D dans le plan xoy.

Question 3 : Quelle section et quelle masse doit on prendre en compte pour la barre OA

Partie Statique

Question 4 : Quels sont les efforts appliqués sur le système (poids de la poutre)

Question 5 : Ecrire la matrice de rigidité des barres OA, OB et OC dans le système d'axes des barres

Question 6 : Ecrire la matrice de rigidité des barres OA, OB et OC dans le système d'axes global

Question 7 : Ecrire la matrice de rigidité de la poutre OD dans le système d'axes local

Question 8 : Ecrire la matrice de rigidité de la poutre OD dans le système d'axes global

Question 9 : Assembler la matrice de rigidité globale dans le tableau joint en annexe.

Question 10 : Ecrire le vecteur déplacement en tenant compte des conditions aux limites et des conditions de symétrie

Question 11 : Ecrire le vecteur force en le décomposant en forces de liaisons (réaction) et forces extérieures.

Question 12 : Résoudre ce système et trouver les déplacements inconnus.

Question 13 : Calculer l'effort normal dans la barre OA.

Partie Dynamique

Question 14 : Ecrire la matrice de masse des barres OA, OB et OC dans le système d'axes des barres

Question 15 : Ecrire la matrice de masse des barres OA, OB et OC dans le système d'axes global

Question 16 : Ecrire la matrice de masse de la poutre OD dans le système d'axes local

Question 17 : Ecrire la matrice de masse de la poutre OD dans le système d'axes global

Question 18 : Assembler la matrice de masse globale dans le tableau joint en annexe.

Question 19 : A partir des matrices de masses et de rigidité on peut écrire l'équation de la dynamique sous la forme suivante. $[M]\left(\frac{d^2q}{dt^2}\right)+[K][q]=0$

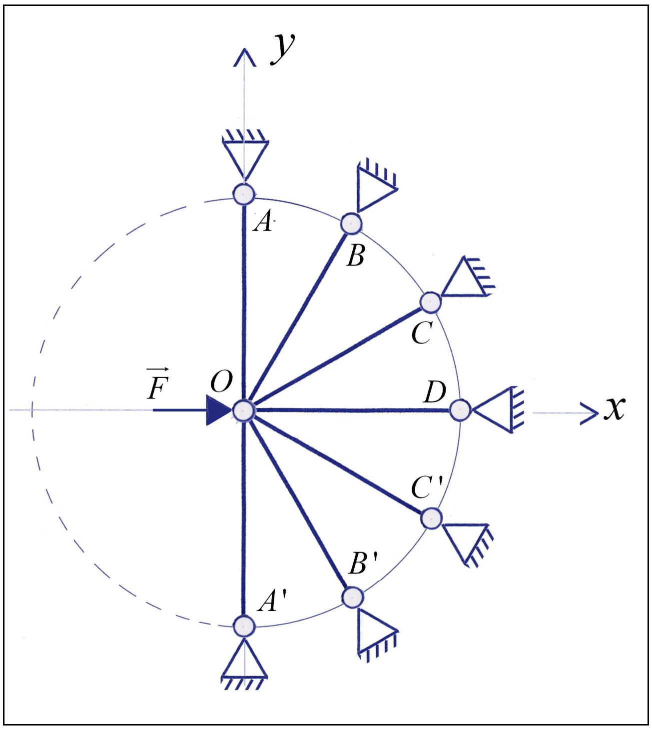
En supposant qu'il existe une solution du type $q(x,y,z,t)=q_i(x,y,z)[A\cos\omega_i t+B\sin\omega_i t]$ écrire en quelques lignes le processus qui permet de déterminer les fréquences de résonnance et les modes propres de cette structure.

Exercise 22

Finite Elements

Assembling of bars: Static and Dynamic

Partie statique :
On se propose de calculer les contraintes et les déplacements de la structure représentée ci-dessous.

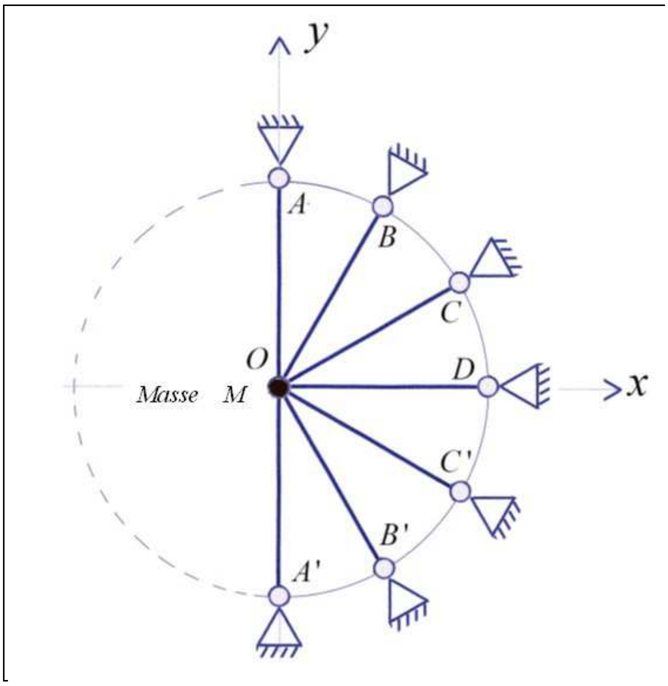


Les sept barres sont identiques, de longueur L, de section droite S, réalisé dans le même matériau de module de Young E. L'angle entre deux barres consécutives est de 30 °. Le problème sera traité en 2D dans le plan xoy.

Questions :

1- Indiquer le plan de symétrie pour cette structure.
2- Comment peut-on réduire l'étude de cette structure en prenant en compte cette symétrie. Dessiner le nouveau système en indiquant les conditions aux limites à prendre en compte dans le plan de symétrie
3- Modéliser cette structure avec le nombre minimal de nœuds.
4- Ecrire la matrice de rigidité d'une barre dans son propre repère,
5- Ecrire la matrice de rigidité de chaque barre dans le repère global,
6- Assembler la matrice de rigidité globale dans le tableau joint en annexe,
7- Conditionner ce problème sous la forme F=K.q, en indiquant le chargement (forces intérieures et forces extérieures) et les déplacements.
8- Résoudre pour trouver les déplacements inconnus,
9- Calculer l'effort normal et la contrainte normale dans la barre OC.

Partie dynamique :
On se propose de calculer les fréquences de résonnance du système précédent représenté sur la figure suivante:



La géométrie est la même que pour la question précédente. La force n'existe plus. Par contre on a rajouté une masse ponctuelle au point O. Cette masse à pour valeur M.
La masse m de chacune des barres est prise en compte pour l'étude dynamique.

Questions :

1- Indiquer pourquoi on ne peut pas tenir compte de la symétrie pour l'étude de cette structure.
2- On décide néanmoins de réduire l'étude de cette structure en prenant en compte cette symétrie. Ecrire la matrice de masse d'une barre dans son propre repère,
3- Ecrire la matrice de masse de chaque barre de la structure étudiée dans le repère global.
4- Assembler la matrice de masse globale dans le tableau joint en annexe.
5- A partir des matrices de masses et de rigidité on peut écrire l'équation de la dynamique sous la forme suivante. $[M]\left(\dfrac{d^2q}{dt^2}\right)+[K][q]=0$

En supposant qu'il existe une solution du type $\left[q(x,y,t)\right]=\left[q(x,y)\right]\left[A\cos\omega_i t+B\sin\omega_i t\right]$ écrire en quelques lignes le processus qui permet de déterminer les fréquences de résonnance et les modes propres de cette structure.

6- Calculer la fréquence de résonnance du premier mode