[5th]-Elastostatic methods [book1 7.7-8.1/8.6 \book2 2.7, 3.1-3.3\courseware reference CH5]

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1. General

-Analytical methods in linear elastostatics

Purpose to analyze an elastostatics problem is mainly to grasp the distributions of stress and strain of an object under specific load environment. Or on the opposite way, to design a part for an application. According to different boundary conditions, there are displacement formulation, stress formulation, and mixed formulation.

To a spacial problem, there are fifteen variable functions: three displacement components, six stress components, and six strain components. Also, there are fifteen independent equations: three differential equilibriam equations (2.2), six geometric equations (3.4), and six physical equations (4.5). Plus, we have boundary condition to ensure uncertain coefficients. Theoretically, all elastostatical problems are solvable. However, with limited ability to partial differential equation handling, many problems have not analytical solution. Thus, two constructive methods: inverse method and semi-inverse method are developed. Tentative calulation is necessary to use these methods.

2. Solving elastostatic problems

-Summary of basic equations

Until now, we have Navier equation (2.2)

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{zx}}{\partial z} + \dfrac{\partial \tau_{yx}}{\partial y} + X = 0 \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + \dfrac{\partial \tau_{xy}}{\partial x} + Y = 0 \\ \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} - Z = 0 \end{cases} \tag{2.2}$$

Which can link stresses with volume forces (loads).

Also, we have Cauchy formulations (3.4)

$$
\begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} \\
\varepsilon_y = \dfrac{\partial v}{\partial y} \\
\varepsilon_z = \dfrac{\partial w}{\partial z} \\
\gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\
\gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\
\gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}
\end{cases}
\tag{3.4}
$$

That associates strains with displacement of point.

And, we introduced the constitutive equation (4.5) or (4.8)

$$
\begin{cases}
\varepsilon_x = \dfrac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \\
\varepsilon_y = \dfrac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)] \\
\varepsilon_z = \dfrac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] \\
\gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\
\gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\
\gamma_{xz} = \dfrac{1}{G}\tau_{xz}
\end{cases}
\tag{4.5}
$$

$$
\begin{cases}
\sigma_x = \lambda\theta + 2v\varepsilon_x \\
\sigma_y = \lambda\theta + 2v\varepsilon_y \\
\sigma_z = \lambda\theta + 2v\varepsilon_z \\
\tau_{xy} = v\gamma_{xy} \\
\tau_{yz} = v\gamma_{yz} \\
\tau_{xz} = v\gamma_{xz}
\end{cases}
\tag{4.8}
$$

They descirbed relationship between stresses and strains in isotropic materials. Sometimes, we need consider strain compatibility equations (3.15) to keep a continuous deformation.

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} + \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \dfrac{\partial^2 \varepsilon_y}{\partial z^2} + \dfrac{\partial^2 \varepsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \dfrac{\partial^2 \varepsilon_z}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{zx}}{\partial z \partial x} \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{xy}}{\partial z} + \dfrac{\partial \gamma_{yz}}{\partial x} - \dfrac{\partial \gamma_{zx}}{\partial y}\right) = 2\dfrac{\partial^2 \varepsilon_y}{\partial x \partial z} \\ \dfrac{\partial}{\partial z}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{zx}}{\partial y} - \dfrac{\partial \gamma_{xy}}{\partial z}\right) = 2\dfrac{\partial^2 \varepsilon_z}{\partial y \partial x} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial \gamma_{zx}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z} - \dfrac{\partial \gamma_{yz}}{\partial x}\right) = 2\dfrac{\partial^2 \varepsilon_x}{\partial z \partial y} \end{cases} \tag{3.15}$$

According to these equations, a relations' chain from force to displacement is built up. These four groups of equations are called basic equations.

-Classification of problems

Mission of elastostatics is to calculate fifteen variables by solving fifteen basic equations. Generally, computation of all of fifteen variable is not necessary. To lower difficulty of work, a simplicity is executived by selecting some basic variables.

Mathematically, it calls boundary-value problem to solve partial differential equation group with given boundary condition.

According to different boundary conditions, elastostaitics problems are classified into three categories.

The first boundary-value problem refers to all external forces including volume force andsurface forces with components F_{sx},F_{sy},and F_{sz} are given. It calls surface force boundary condition.

The second kind is known as displacement boundary condition. Which means volume force of elastomer and surface displacement components are predecided.

The third kind problem covers elastomers with known volume components, part of displacement components and other part of surface force components. This is called mixed boundary condition. These three kind of problems delegate some simplified practical engineering problems.

-Basic solving methods

According to classification of elastostatics problems, there are three methods: displacement method, stress method, and mixed method. It is called displacement method if displacement component functions u, v, and w are taken as basic variables. It is stress method to utilize σ_x,σ_y,σ_z,τ_{xy},τ_{yz} and τ_{zx} as basic variables. Mixed method, stress components functions takes part of stress components and part of displacement components as basic variables.

-Uniqueness of solution

Without considering rigid displacement, the three types of boundary value problems are unique. Commonly, elastomer is effected by knowen volume forces. On the boundary of object, surface force or displacement, or part of the surface force and other part of the displacement will be known. The stress and strain at any point of elastomer are unique only if the body is balanced. To the latter two cases, displacement is also unique.

-Saint-Venant's principle

Solving elasticity problems, the stress components, strain components and displacement components are easily expressed via basic equations. However, distribution of stress on the boundary is unknown or very hard to be satisfied in some cases. The only information could be resultant force and resultant moment. To handle this kind of situation, we need rely on Saint-Venant's principle which was firstly identified and generalized by an academician of French Academy of Sciences, Saint-Venant. The Saint-Venant's principle allows elasticians to replace complicated stress distributions or weak boundary conditions into ones that are easier to solve, as long as that boundary is geometrically short.

Figure 5.1. Jean Claude Saint-Venant

Saint-Venant was a student at the École Polytechnique, entering the school in 1813 when he was sixteen years old. He graduated in 1816 and spent the next 27 years as a civil engineer.

Saint-Venant principle can be expressed as: a system of forces on a local area of surface of the object can be replaced by any system of forces which is statically equivalent to. Distribution of stress within the forces' area is significantly different than before. Stress distribution is almost the same at a considerable distance though.

For example, a cylinder part is effected by two equal forces P in opposite directions along the axis on geometric centers of two end sections, like shown in Figure 5.2 (a). If an equivalent force system is used to replace force P on one end or both ends (see figure 5.2 (b) or 5.2 (c)), stress distribution of part will be changed. According to Saint-Venant's principle, stress distribution in region arround the force point within dashed line will has a significant difference while other regions' have slight changes. Also, a distributed force with a intensity P/A can be use to substitute the concentrated force P, where A is area of section (see figure 5.2(d)). Still, regions with prominent influence only occur near to two ends. All these four cases have very similar stress distribution in regions at a big distance to ends.

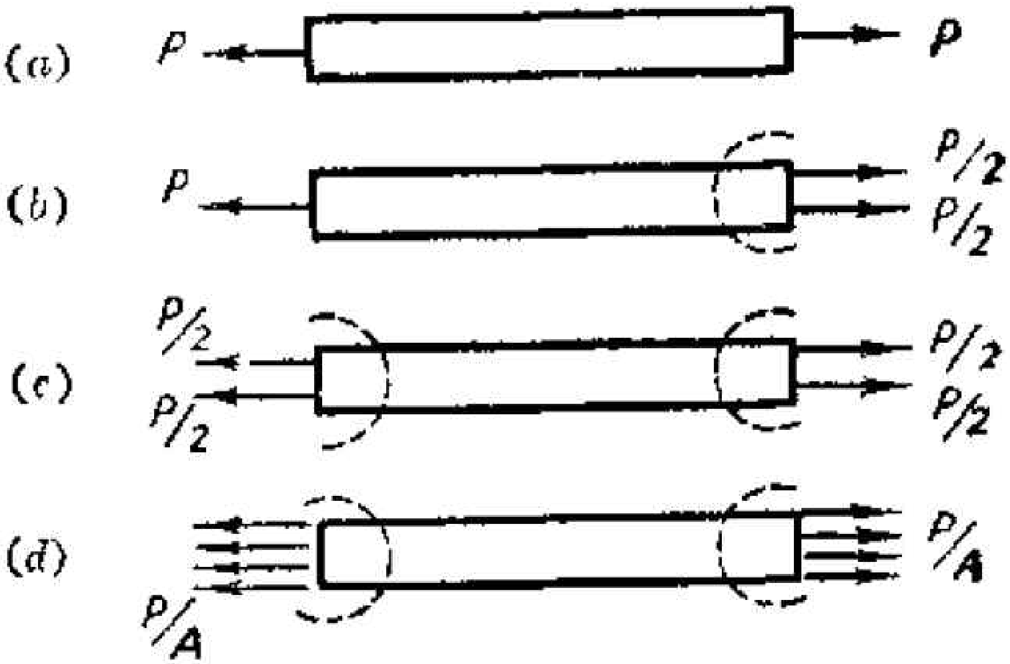

Figure 5.2. Saint-Venant's principle

A simple boundary condition can result in a simple and easy achieved answer. In figure 5.2, distribution of surface forces at (a), (b), and (c) are not continuous and difficult to be described. Using mode (d) to analyze stress distribution at regions far away from ends, answer is quite easy. This answer can be regarded as answers of other three cases with limited errors. The result is already testified by experiment and theoretical analysis, see figure 5.3.

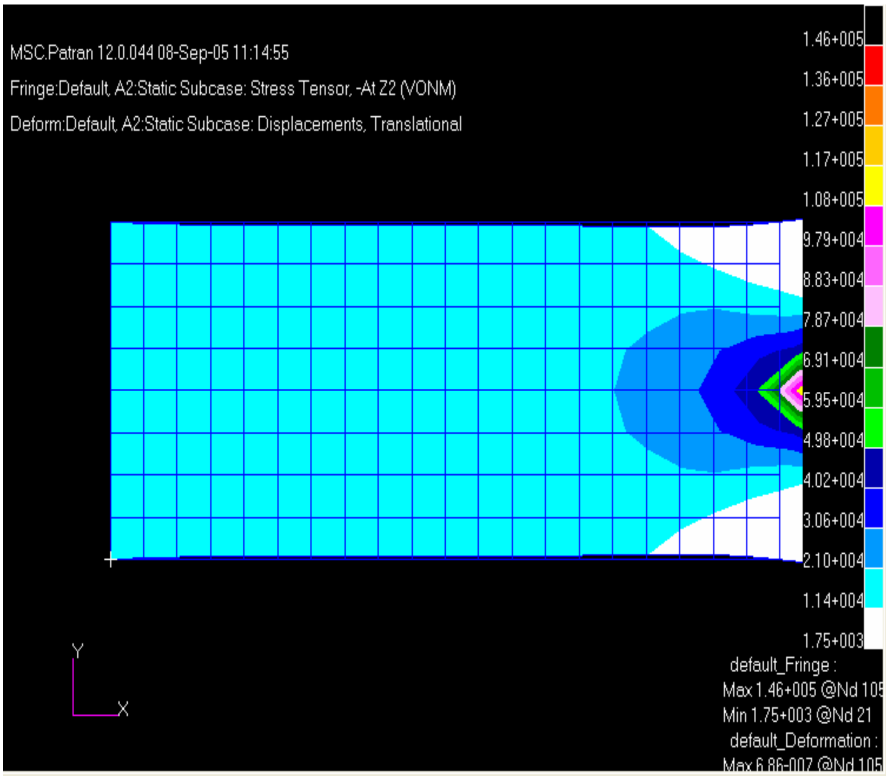

Figure 5.3 A simulation of Saint-Venant's principle

Two points among Saint-Venant's principle should be paid more attention: range of researched system of forces is local; and, the replaced system of forces is statically equivalent to original one. So, it also calls principle of locality.

It can be expressed in another way: If A small part of object is effected by a balanced system of force, stress distribution caused by this system of force only exist in region near the system of force. The influence of this system of force sharply reduces in regions far away.

-Superposition principle

The superposition principle, also known as superposition property, states that, for all linear systems, the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually.

So that if input A produces response X and input B produces response Y then input (A + B) produces response (X + Y).

Mathematically, for all linear systems F(x) = y, where x is some sort of stimulus (input) and yis some sort of response (output), the superposition (i.e., sum) of stimuli yields a superposition of the respective responses:

$$F(x_1 + x_2) = F(x_1) + F(x_2)$$

Under hypothesis of small deformation, total effect of an object caused by several group of loads equal to sum of effects caused by each group of load individually.

3. Solve space problems according to displacement

Displacement method take displacement components u, v, and w as basic variables. Solution is achieved by expressing stress and strain with displacement. Its procedure is:

a.Utilizing constitutive equations and Cauchy formulations, stress components can be described by displacements components by eliminating strain components. Combinate Navier equations with new achieved equations, three equilibrium equations including three uncertain displacement components are formulated

$$\begin{aligned}
&(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0 \\
&(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0 \\
&(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0
\end{aligned} \tag{5.1}$$

where $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, which is another expression of equation (3.22). Where

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as Laplace operator.

Differential equation group (5.1) is a synthesis of statics, geometry, and physics, which is basic equation group to solve elastostatics space problems according to displacement. When Lamé firstly achieved this eqaution group, he did not even know the usage of it. After many years, its purpose was finally found. So, it is called Lamé formulations too.

b.In the process of solving group (5.1), undetermined coefficients will appear because of integral operation. These coefficients can be computed with boundary condition.

Boundary condition (2.10) should be expressed according to displacement

$$\begin{aligned}
\overline{X} &= \left(\lambda e + 2G\frac{\partial u}{\partial x} \right) l + G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) m + G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) n \\
\overline{Y} &= \left(\lambda e + 2G\frac{\partial v}{\partial y} \right) m + G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) n + G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) l \\
\overline{Z} &= \left(\lambda e + 2G\frac{\partial w}{\partial z} \right) n + G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) l + G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) m
\end{aligned} \tag{5.2}$$

c.After getting displacement component functions, strain components can be obtained by partial differentiating geometric equations (3.4).

d.Similarly, stress components can be gained with physical equations (4.8).

4. Solve space problems according to stress

Stress method regard stress components as basic variables. Firstly, Navier equations and stress boundary condition should be satisfied. To decide six stress components, three more complementary equations are needed. Usually, compatibility equations (3.15) are accounted. Physically, strain compatibility equations expressed by stress components considering physical equations (4.5) can make sure that stress components satisfy not only balance condition but also continuity of body.

a.Stress components could be gained through equations (2.2), (4.5) and (3.15). Using physical equations (4.5), strain components can be eliminated from equations (3.15).

$$
\begin{aligned}
&\frac{\partial^2\Theta}{\partial x^2}+(1+\mu)\nabla^2\sigma_x=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial X}{\partial x}+\mu\frac{\partial Y}{\partial y}+\mu\frac{\partial Z}{\partial z}\right]\\
&\frac{\partial^2\Theta}{\partial y^2}+(1+\mu)\nabla^2\sigma_y=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial Y}{\partial y}+\mu\frac{\partial X}{\partial x}+\mu\frac{\partial Z}{\partial z}\right]\\
&\frac{\partial^2\Theta}{\partial z^2}+(1+\mu)\nabla^2\sigma_z=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial Z}{\partial z}+\mu\frac{\partial Y}{\partial y}+\mu\frac{\partial X}{\partial x}\right]\\
&\frac{\partial^2\Theta}{\partial y\partial x}+(1+\mu)\nabla^2\tau_{xy}=-(1+\mu)\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right)\\
&\frac{\partial^2\Theta}{\partial y\partial z}+(1+\mu)\nabla^2\tau_{yz}=-(1+\mu)\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)\\
&\frac{\partial^2\Theta}{\partial z\partial x}+(1+\mu)\nabla^2\tau_{zx}=-(1+\mu)\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)
\end{aligned} \tag{5.3}
$$

Equations (5.3) is compatibility equations in stress type, which is formulated by Beltrami-Michell at 1899.

Particularly, Italian scientist Beltrami gave another type of (5.3) in case of a zero or other constants volume force.

$$\frac{\partial^2 \Theta}{\partial x^2} + (1+\mu)\nabla^2 \sigma_x = 0$$
$$\frac{\partial^2 \Theta}{\partial y^2} + (1+\mu)\nabla^2 \sigma_y = 0$$
$$\frac{\partial^2 \Theta}{\partial z^2} + (1+\mu)\nabla^2 \sigma_z = 0$$
$$\frac{\partial^2 \Theta}{\partial y \partial x} + (1+\mu)\nabla^2 \tau_{xy} = 0$$
$$\frac{\partial^2 \Theta}{\partial y \partial z} + (1+\mu)\nabla^2 \tau_{yz} = 0$$
$$\frac{\partial^2 \Theta}{\partial z \partial x} + (1+\mu)\nabla^2 \tau_{zx} = 0 \tag{5.4}$$

b.Integrating stress compatibility equaitons (5.3) and Navier equaitons (2.2), stress component functions will be obtained with help of boundary condition.

c.Using physical equations (4.5), strain components are achieved.

d.To get displacement components, geometric equations (3.4) need to be integrated. Constraint conditions can be used to decide undetermined coefficients.

Displacement method, stress method, and mixed method (according to different boundary conditions) are categoried as directly methods. Although the building of these methods have great theoretical meaning. They are hardly used in their original types because of complicity and difficulty of mathematical analysis. Many numerical methods such as finite difference method and finite element method are more popular now. Still, analytical methods are important foundation of masion of elasticity.