Poisson process introduction to queueing theory

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Markov process with continuous time and discrete state

Recall that a Markov process $(N_t)_{t\in\mathbb{R}^+}$ with continuous time and discrete state checks

$$\begin{aligned}&\forall\ \ 0 \leq s_1 < s_2 < \cdots < s_k < t,\\ &\quad Pr(N_t = n|N_{s_1} = p_1, N_{s_2 1} = p_2, \ldots, N_{s_k} = p_k)\\ &= Pr(N_t = n|N_{s_k} = p_k)\end{aligned}$$

Moreover if the Markov dynamics is time-homogeneous, the transition is stationary i.e.

$$Pr(N_t = n|N_{s_k} = p_k) = Pr(N_{t-s_k} = n - p_k)$$

Counting Markov semi-groups

- Let us consider a stochastic process with **continuous positive time** and **integer values** where N_t is the number of random events that occurred between time 0 and time t.
- If we suppose that $\{N_t\}$ is a Markov process with homogeneous dynamics it means that

$$\forall \;\; 0 \leq s_1 < s_2 < \cdots < s_k < t,$$
$$Pr N_0 = p_0, N_{s_1} = p_1, N_{s_2} = p_2, \ldots, N_{s_k} = p_k)$$
$$= Pr(N_0 = p_0) Pr(N_{s_1} = p_1 - p_0) \ldots Pr(N_{s_k - s_{k-1}} = p_k - p_{k-1})$$

- If we take $N_0 = 0$ the Markov process is completely defined by the family of probability laws $P_t(N_t)$ of N_t for $t > 0$ checking

$$\sum_{k=0}^{n} P_s(k) P_t(n-k) = P_{t+s}(n) \Leftrightarrow P_s \star P_t = P_{t+s} \qquad (1)$$

Poisson process

Theorem

Let us suppose that for small time intervals, the occurrence of the event is instantaneous and infinitesimal which is modeled (with $\lambda > 0$)

$$\begin{cases} Pr(N_{\Delta t} = 0) = 1 - \lambda\Delta t - o(\Delta t) \\ Pr(N_{\Delta t} = 1) = \lambda\Delta t + o(\Delta t) \\ Pr(N_{\Delta t} > 1) = o(\Delta t) \end{cases}$$

Then

$$P_t(k) = Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

We say that $\{N_t\}$ is a Poisson process with intensity λ.

Remark

λ表示在一定时间（单位时间）内事件发生的平均次数。

Counting Markov semi-groups

- Set $f_k(t) = Pr(N_t = k)$.Then we have from (1) $P_{t+\Delta t} = P_t * P_{\Delta t}$ which amounts to

$$\begin{cases} f_0(t+\Delta t) = f_0(t) f_0(\Delta t) \\ f_1(t+\Delta t) = f_1(t) f_0(\Delta t) + f_0(t) f_1(\Delta t) \\ \dots\dots \\ f_{k+1}(t+\Delta t) = f_{k+1}(t) f_0(\Delta t) + f_k(t) f_1(\Delta t) \\ \qquad\qquad + \sum_{j=2}^{k+1} f_{k+1-j}(t) f_j(\Delta t) \end{cases}$$

- Now we use the transition assumptions when $\Delta t \to 0$

$$\begin{cases} f_0(t+\Delta t) = f_0(t)[1 - \lambda \Delta t - o(\Delta t)] \\ f_1(t+\Delta t) = f_1(t)[1 - \lambda \Delta t] + \lambda \Delta t f_0(t) + o(\Delta t) \\ \dots\dots \\ f_{k=1}(t+\Delta t) = f_{k+1}(t)[1 - \lambda \Delta t] + \lambda \Delta t f_0(t) f_k(t) + o(\Delta t) \end{cases}$$

Counting Markov semi-groups

- We get a linear stationary differential system:

$$\begin{cases} f_0'(t) = -\lambda f_0(t) \\ f_1'(t) = -\lambda f_1(t) + \lambda f_0(t) \\ \ldots\ldots \\ f_{k+1}'(t) = -\lambda f_{k+1}(t) + \lambda f_k(t) \end{cases}$$

- Taking into account the initial conditions

$$\begin{cases} f_0(t=0) = P(N_0 = 0) = 1 \\ f_k(t=0) = P(N_0 = k) = 0, \quad \forall \;\; k > 1 \end{cases}$$

we easily get by induction

$$P(N_t = k) = f_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Definition of the occurrence time

Definition

The occurrence time of the k-th event (ω) of an event process N_t is defined by

$$T_k(\omega) = \min\{t | N_t(\omega) = k\}$$

The occurrence time T_k is a stopping time in the sense of previous chapter:

$$(T_k \leq t) \in \mathcal{F}_t$$

where $\mathcal{F}_t$ is the σ-algebra of the past of processus N_t at time t
We shall prove the following theorem

Theorem

The occurence time T_k form a random walk with exponential step of parameter λ

Computation of occurrence time distribution

Let us first compute the law of the first occurence time

$$Pr(T_1 \leq t) = Pr(N_t > 0) = 1 - e^{-\lambda t} \Rightarrow dPr_{T_1}(t) = \lambda e^{-\lambda t} dt$$

Let us now compute the law of the interval between consecutive arrivals

$$\begin{aligned} & Pr(T_{k+1} \leqslant t_k + t \,|T_k = t_k, T_1 = t_1, \ldots, T_{k-1} = t+_{k-1}) \\ = \quad & Pr(N_{t_k+t} > k \,|[N_{t_k} = k] \cap [T_k = t_k, T_1 = t_1, \ldots, T_{k-1} = k-1]) \\ = \quad & Pr(N_{t_k+t} > k \,|N_{t_k} = k) = 1 - e^{-\lambda t} \end{aligned}$$

We use the Markov homogeneous property of time transition in Poisson process by noticing the conditioning event is in $\mathcal{F}_{t_k}$

Definition of Poisson process from the occurrence time

Theorem

Let T_k a random walk on $\mathbb{R}^+$ with exponential step distribution with λ parameter. We define an event process $\{N_t\}$ by $N_t = \sum_{k=1}^{\infty} 1_{T_k \leq t}$. Then $\{N_t\}$ is a Poisson process of parameter λ

$$P_t(k) = Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

and (T_k) is the sequence of its event occurrence times

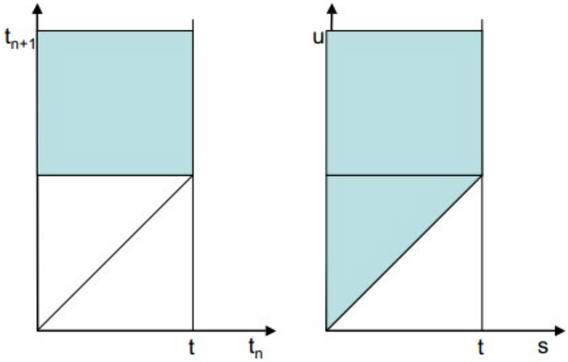
Proof   Since the proof is difficult, we shall prove two basic steps letting the completion to the reader of more advanced courses (see [2]). We shall prove the two following steps

- N_t is a Poisson law of parameter λ
- If $s < t$ then $Pr(N_t = p + n \,|N_s = p) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^n}{n!}$

Computation of N_t distribution

We have $Pr(N_t = n) = Pr(T_n \leq t, T_{n+1} > t)$

$$Pr(N_t = n) = \int_0^t \frac{\lambda e^{-\lambda s}(\lambda s)^{n-1}}{(n-1)!}\left[\int_{t-s}^{\infty} \lambda e^{-\lambda u} du\right] ds = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$



Computation of Markov transition

We want to show

$$Pr(N_s = p; N_t = p + n) = e^{-\lambda t} \frac{(\lambda s)^p}{p!} \frac{(\lambda(t - s))^n}{n!}$$

We apply the same method as previously (we suppose $n > 1$)

$$Pr(N_s = p; N_t = p + n) = Pr(T_p \leq s, T_{p+1} > s, T_{n+p} \leq t, T_{n+p+1} > t) =$$

$$\int_0^s \lambda e^{-\lambda u_1} \frac{(\lambda u_1)^{p-1}}{(p-1)!} \int_{s-u_1}^{t-u_1} \lambda e^{-\lambda u_2} \int_0^{t-u_1-u_2} \lambda e^{-\lambda u_3} \frac{(\lambda u_3)^{n-2}}{(n-2)!} \int_{t-u_1-u_2-u_3}^{\infty} \lambda e^{-\lambda t} du$$

$$= e^{-\lambda u_t} \int_0^s \frac{(\lambda u_1)^{p-1}}{(p-1)!} \int_{s-u_1}^{t-u_1} \lambda \int_0^{t-u_1-u_2} \frac{(\lambda u_3)^{n-2}}{(n-2)!} du_3 du_2 du_1$$

$$= e^{-\lambda u_t} \int_0^s \frac{(\lambda u_1)^{p-1}}{(p-1)!} \int_{s-u_1}^{t-u_1} \lambda \frac{[\lambda(t - u_1 - u_2)]^{n-1}}{(n-1)!} du_2 du_1$$

$$= e^{-\lambda u_t} [\int_0^s \frac{(\lambda u_1)^{p-1}}{(p-1)!} du_1] \frac{(\lambda(t - s))^n}{n!}$$

Random telegraph signal process

Problem

Consider a stochastic process $(X_t) \in \{-1, 1\}$ such that $Pr(X_0 = -1) = Pr(X_0 = 1) = 0.5$. X_t changes its value at each occurrence of an intensity λ Poisson process $\{N_t\}$. Then $\{X_t\}$ is a Markov process with transition semi-group

$$(Pr(X_t = j|X_0 = i)_{ij}) = \begin{bmatrix} \frac{1}{2}(1 + e^{-2\lambda t}) & \frac{1}{2}(1 - e^{-2\lambda t}) \\ \frac{1}{2}(1 - e^{-2\lambda t}) & \frac{1}{2}(1 + e^{-2\lambda t}) \end{bmatrix}$$

Proof

- The fact that (X_t) is a Markov process comes from Markov property for $\{N_t\}$ since the transition state process for (X_t) is deterministic
- The computation of the transition matrix comes from the computation of

$$Pr(N_t \in 2\mathbb{N}) = \sum_{i=0}^{\infty} \frac{(\lambda t)^{2j}}{(2j)!} e^{-\lambda t} = \frac{1}{2} e^{-\lambda t} [e^{\lambda t} + e^{-\lambda t}]$$

Properties of the random telegraph signal process

- We notice that the Markov process has the uniform distribution as a unique stationary distribution and that the law of X_t converges to this law when $t \to \infty$. The process is ergodic.
- We can write the differential system in the following form

$$\begin{cases} P_{\Delta t}(1 \to 1) = 1 - \lambda\Delta t + o(\Delta t) \\ P_{\Delta t}(1 \to -1) = \lambda\Delta t + o(\Delta t) \\ P_{\Delta t}(-1 \to 1) = \lambda\Delta t + o(\Delta t) \\ P_{\Delta t}(-1 \to -1) = 1 - \lambda\Delta t + o(\Delta t) \end{cases}$$

In that form the transition matrix of Chapman-Kolmorov equations leads to a linear differential system with a transition rate matrix.

Markov chain with continuous time

- We shall describe now the (time homogeneous) Markov process with continuous time and discrete state space (Some times, they are called Markov chains with continuous time)
- Their transition dynamics is completely defined by the matrix of transition rates if we assume instantaneous and infinitesimal transition
- Let us take the matrix of transition rates Λ as a square matrix $\#\mathcal{E} \times \#\mathcal{E}$ with positive off-diagonal terms and equilibrium diagonal terms to get 0 as the sum along any line.
- Then we can write

$$P_{\Delta t} = I + \Lambda \Delta t + o(\Delta t)$$

Examples

- For the Poisson process, the matrix of transition rates Λ is infinite and defined by

$$\begin{bmatrix} \lambda_{00} = -\lambda, \lambda_{01} = \lambda & \ldots\ldots & \ldots, \lambda_{0,j} = 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \lambda_{k,j} = 0, & \lambda_{k,k} = -\lambda, \lambda_{k,k+1} = \lambda & \ldots, \lambda_{k,j} = 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

- For the random telegraph signal, the matrix of transition rates is

$$\Lambda = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

Description of the transition

Theorem

Take a continuous-time Markov chain (X_t) with transition-rate matrix Λ. Define the occurrence state transition rates $\lambda_i = -\lambda_{ii}$ and thestochastic state transition matrix $P = (p_{ij}) : p_{ij} = \lambda_i \lambda_{ij}$ if $i \neq j$ and $p_{ii} = 0$. Then the transition from $X_t = i$ may be considered in two successive steps
① Compute the transition time with an exponential law of parameter λ_i,
② When transition time occurs, choose next state according to the transition matrix i-th line $(p_{ij})_j$

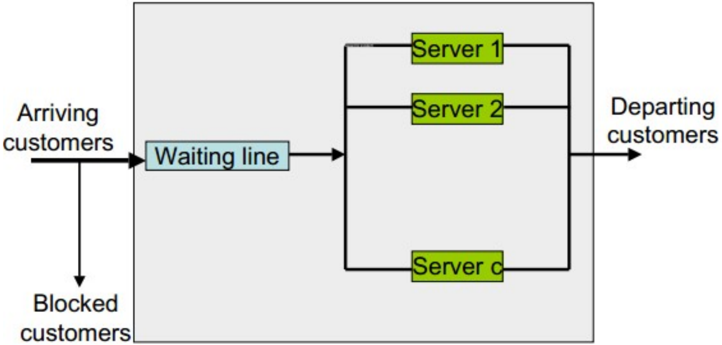
We can consider the inverse point of view and construct the Markov chain from the transition occurence rate vector $\lambda = (\lambda_i)$ and the state transition matrix $P = (p_{ij})$.

Elements of a queueing system

- Customers arrive at random arrival times. Generally the arrival times are a process with iid independent increments and the process of arriving customers is a Poisson process with arrival rate λ.
- There are c servers. If all the servers are busy. The customers wait in a waiting line or "queue". The service time τ of any customer are i.i.d. in basic models. They may obey to various specifications: exponential with rate $\mu(M)$, deterministic (D), more general law: uniform,... (G).
- In some models (telephonic queue), the waiting line is bounded and the customers are outed (blocked customers)

Elements of a queueing system

These elements are presented in the following figure ([1])



The two points of view

- According to the nature of the service, the discipline of the service may be different. The more classical one is "FIFO" (first in, first out). But for some computer networks the ""pile" discipline may be considered "LIFO" (Last in, First out). In transportation networks and others, the discipline may be more specific with priority regimes.
- From the point of view of the customer i, the important quantity is the total delay $T_i = W_i + \tau_i$, where W_i is his waiting time and τ_i is his service time. If there is also a blocking discipline, the rate of blocked customers have to be considered.
- The allocation of human resources consider the rate of unemployed servers and compare it to the rate of blocked customers.

Little's formula

- Little's formula is a general and intuitive formula which connects
  - $< N >$ is the average number of customers in the system,
  - λ is the average arrival rate,
  - $< T >$ is the average service time, it is the sum of the average waiting time and the average service time

  Little's formula states that $< N >= \lambda < T >$
- The average may be empirical average over finite time when the time is going to infinity or mathematical expectation in steady regime. They are equal from ergodic theorem under stability assumptions.
- It is shown that the current waiting times of the customers who are still in the system vanishes in the average over long time.

Description of the M/M/1 queue

- Let $\{N_t\}$ the arrival process. It is a Poisson process of arrival rate λ
- We note seemingly (D_t) the departure process which depends also on μ, the rate of service
- The number of customers in the system at time t is $L_t = N_t - D_t$. We are interested in this Markov chain with continuous time and we want to answer the following questions
  1. At which conditions does it exist a stationary regime?
  2. What is this stationary regime and the average number of customers in the system?
  3. Interpret in that simple case the Little's formula.

Chapman-Kolmogorov equation for (L_t)

It is quite easy to write the (infinite) transition rate matrix for (L_t). It is equal to

$$\begin{bmatrix} -\lambda & \lambda & \dots & \dots & \dots & \dots & \dots & \dots \\ \mu & -\lambda-\mu & -\lambda & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \mu & -\lambda-\mu & -\lambda & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

It gives rise to the following transition matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

About M/M/1 stationary regime

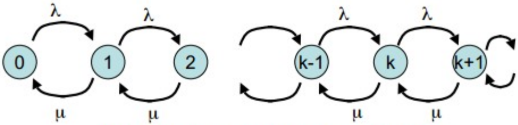
Let π be the stationary probability if it exists. We shall have

$$\begin{cases} \lambda\pi(0) = \mu\pi(1) \\ (\lambda+\mu)\pi(1) = \lambda\pi(0) + \mu\pi(2) \\ \dots \\ (\lambda+\mu)\pi(k) = \lambda\pi(k-1) + \mu\pi(k+1) \\ \dots \end{cases}$$

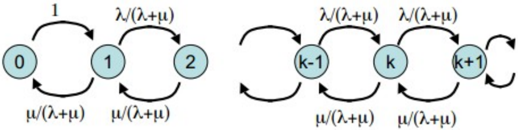
It gives $\quad [\pi(k+1) - \pi(k)] = \frac{\lambda}{\mu}[\pi(k) - \pi(k-1)]$
This recurrence converges if $\lambda < \mu$. In that case the chain is recurrent positive and its stationary probability is given by

$$\pi(k) = (\frac{\lambda}{\mu})^k \frac{\mu - \lambda}{\mu}$$

Transition graphs of M/M/1



(a) Rate transition graph of the MM1 queue



(b) Discrete probability transition graph of the MM1 queue

Practical conclusions on M/M/1

- Thus the number of customers L_t in the system in the stationary regime follows a geometric law with expectation $\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$
- We can apply Little's formula to find the average time spent in the system: $< T >= \mathbb{E}(L_t)\frac{1}{\lambda} = \frac{1}{\lambda-\mu}$
- The average length of the queue and the average time expectation go to infinity when $\lambda \to \mu$. The service rate with one server has to be larger that the arrival rate.
- Last we can check that the departure process is a Poisson process with intensity λ which confirms the stability of the system, (left to exercise)

Practical: to compute continuous-time Markov chains

Problem

① Get a precise model by writing Chapman-Kolmogorov equations.

② Get a concrete idea of what is going on by drawing the rate transition graph.

③ Compute the stationary state if any by putting the derivatives to 0.

④ Precise the nature of the chain (ergodic, recurrent, transitory with absorbing states...) by trying to solve the Chapman-Kolmogorov differential system.