[2nd]- Stresses in elastic solid [book1 7.1-7.3\book2 2.1-2.3\courseware reference CH2]

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1. Equilibrium of infinitesimal unit

-Direction rules of stress

As mentioned in 1st lesson section 2, stress is the expression of force density on a point. It always described in normal and tangential directions of researched plane. Normal projection of a stress vector is called normal stress, which has a relation with dimension change of a body. As shown in Figure 2.1, σ_x is a classic expression of normal stress. The below subscript x means the direction of this normal stress along x axis of the reference. Tangential projections of a stress vector are called shear stress or tangential stress. They have influence on shape of a body. τ_{yz} is used to express a shear stress. Its below subscript yz refers that it is located on a plane perpendicular to y axis with direction along z axis.

If a Descartes coordinate is used with axis along the sides of infinitesimal unit, rules of positive direction for stress are as following. A plane of infinitesimal unit is called positive plane if its direction of exterior normal is toward to positive direction of an axis. Stress value on positive plane is positive if its direction along positive direction of an axis. Otherwise, it is a negative value.

A plane is called negative plane if its direction of exterior normal is toward to negative direction of an axis. On the contrary, stress value on negative plane is positive, if its direction along negative direction of an axis. Otherwise, it is a negative value.

Signs of stresses shown in Figure 2.1 are all positive.

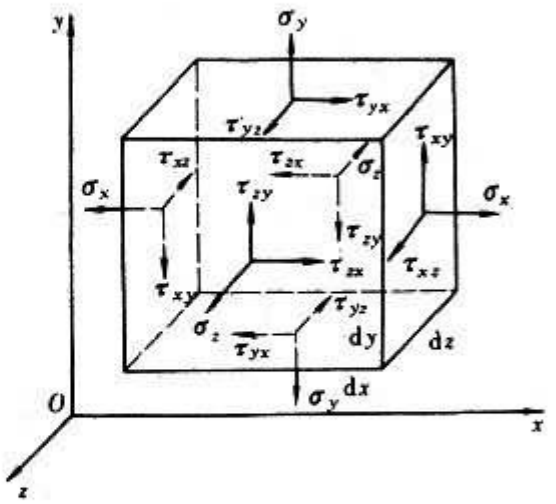

Figure2.1 An infinitesimal unit

-Balance Analysis

Any part of an object is under state of equilibrium, if its entirety is balanced. Under context of statics, any part's balance of a body can be described by a group of equations in a reference system, which is called equilibrium equations.

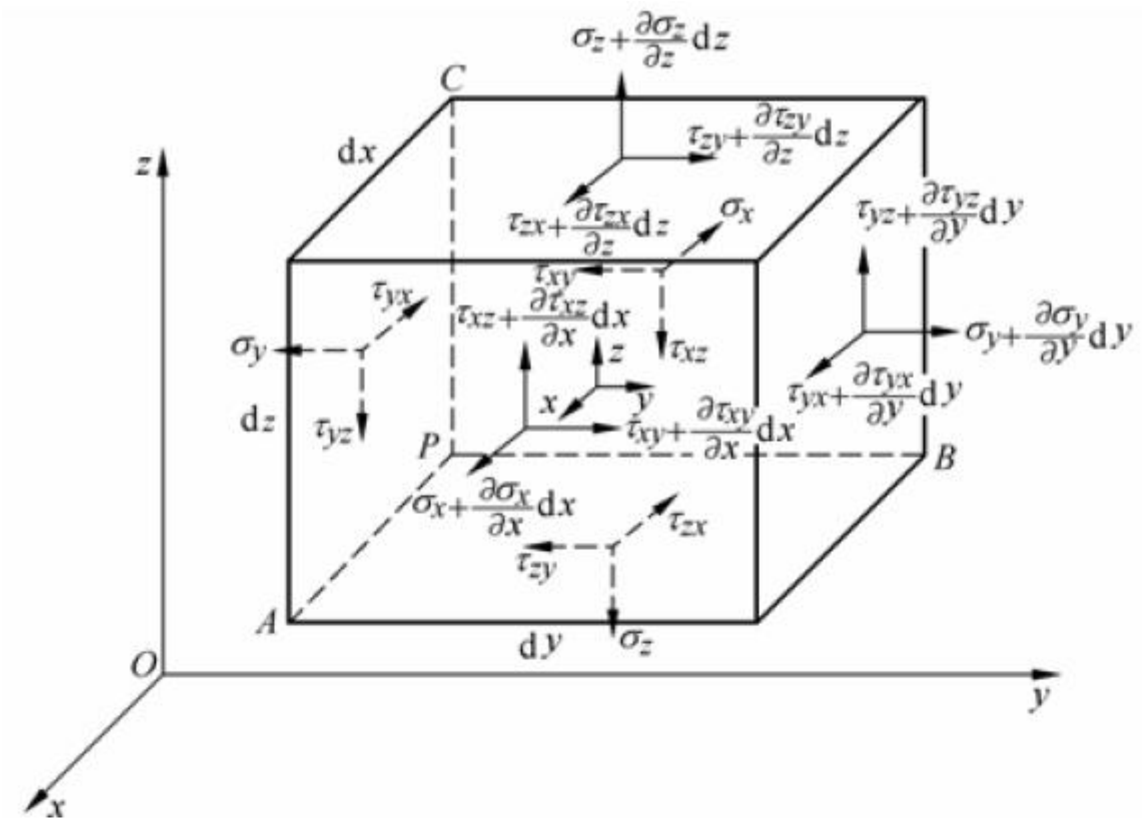

Figure2.2 Balance of infinitesimal unit

Considering an infinitesimal unit on a point P of an elastomer, a Descartes coordinate is built up to describe it. Sides' lengths of this infinitesimal cuboid are dx, dy, and dz. According to continuity, stress value of this elastomer at any point can be expressed as a continuous function like σ_x. If stress values on three planes which near point P are noted as types shown in Figure 2.2, then stress values on other three planes have little differences. Stress values on two opposite planes have a difference of increment. For example, σ_y is used to express average normal stress value on infinitesimal plane AC (which is actual value of stress on point P). Then, the normal stress value on the plane opposite to AC can be express as

$$\sigma_y + \frac{\partial \sigma_y}{\partial y} dy .$$

Other stress values are similar. Considering existence of volume force, X, Y, and Z are used to express projections of volume force at this cuboid.

According to balance condition, equilibrium equations can be written down, if this unit is under state of balance.

Take geometric centre of unit as reference point, equations are

$$\begin{cases} \sum F_x = 0 & \sum M_x = 0 \\ \sum F_y = 0 & \sum M_y = 0 \\ \sum F_z = 0 & \sum M_z = 0 \end{cases}$$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dydz - \sigma_x dydz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) dxdy - \tau_{zx} dxdy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dzdx - \tau_{yx} dzdx + Xdxdydz = 0 \tag{2.1a}$$

$$\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy \right) dxdz - \sigma_y dxdz + \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz \right) dxdy - \tau_{zy} dxdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) dzdy - \tau_{xy} dzdy + Ydxdydz = 0 \tag{2.1b}$$

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz \right) dxdy - \sigma_z dxdy + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx \right) dzdy - \tau_{xz} dzdy + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy \right) dzdx - \tau_{yz} dzdx + Zdxdydz = 0 \tag{2.1c}$$

$$\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy \right) dzdx \cdot \frac{dy}{2} + \tau_{yz} dzdx \cdot \frac{dy}{2} - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz \right) dxdy \cdot \frac{dz}{2} - \tau_{zy} dxdy \cdot \frac{dz}{2} = 0 \tag{2.1d}$$

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) dxdy \cdot \frac{dz}{2} + \tau_{zx} dxdy \cdot \frac{dz}{2} - \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx \right) dzdy \cdot \frac{dx}{2} - \tau_{xz} dzdy \cdot \frac{dx}{2} = 0 \tag{2.1e}$$

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) dzdy \cdot \frac{dx}{2} + \tau_{xy} dzdy \cdot \frac{dx}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dzdx \cdot \frac{dy}{2} - \tau_{yx} dzdx \cdot \frac{dy}{2} = 0 \tag{2.1f}$$

-Reciprocal theorem of shear stress

Considering equation (2.1d), it is simplified into following type with ignoring infinitesimal of higher order.

$$(\tau_{yz}\,\mathrm{dzdxdy} - \tau_{zy}\,\mathrm{dxdydz}) \cdot \frac{1}{2} = 0$$

Divided by volume of unit dxdydz,

$$\Rightarrow \tau_{yz} = \tau_{zy} \qquad (2.2)$$

Similar results can be produced from equation (2.1e) and equation (2.1f)

$$\tau_{zx} = \tau_{xy}\ ,\ \ \tau_{xy} = \tau_{yx}$$

In two mutually perpendicular planes, shear stress component of one plane with direction vertical to equals one of another plane only if these tow shear stress components are both vertical to intersection of planes. They both point to the intersection line or be away from it at the same moment. This is called reciprocal theorem of shear stress.

-Equations of Differential Equilibrium

The first three equations of (2.1) are practical to solving variables. The following type can be achieved with a slight simplification.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{yx}}{\partial y} + X = 0 \quad (2.2a)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \quad (2.2b)$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - Z = 0 \quad (2.2c)$$

Equations (2.2) are equations of differential equilibrium, or can be called Navier equations.

2. Stress state at a point

-Usages of stress state analysis

Stress state refers to stress distribution of a point at all directions. Solutions of equations (2.2) just give out stress distribution along directions of given coordinate axis, which have not directly relations with destroying or yield of material. Stress state analysis may be helpful to justify if material would be destroyed under given loads and which direction will this destruction occurs along with.

-Stress on an inclined plane

According to equations (2.2), there have only six individual stress components at an infinitesimal unit. If solutions of equations (2.2) are achieved, namely six individual stress components at three mutually perpendicular planes are known. Stresses at an inclined plane through the same point can be formulated by balance condition.

Figure 2.2 shows the infinitesimal-tetrahedron by the elastic body by a point P. It is consist of three orthogonal and one tilted planes. △ ABC indicate a plane in an arbitrary direction. The external normal unit vector is

$$\vec{n} = l\vec{i} + m\vec{j} + n\vec{k} \qquad (2.3)$$

Suppose dS is area of ΔABC, areas of three orthogonal planes are

ΔBPC = ldS, ΔAPC = mdS, ΔAPB = ndS

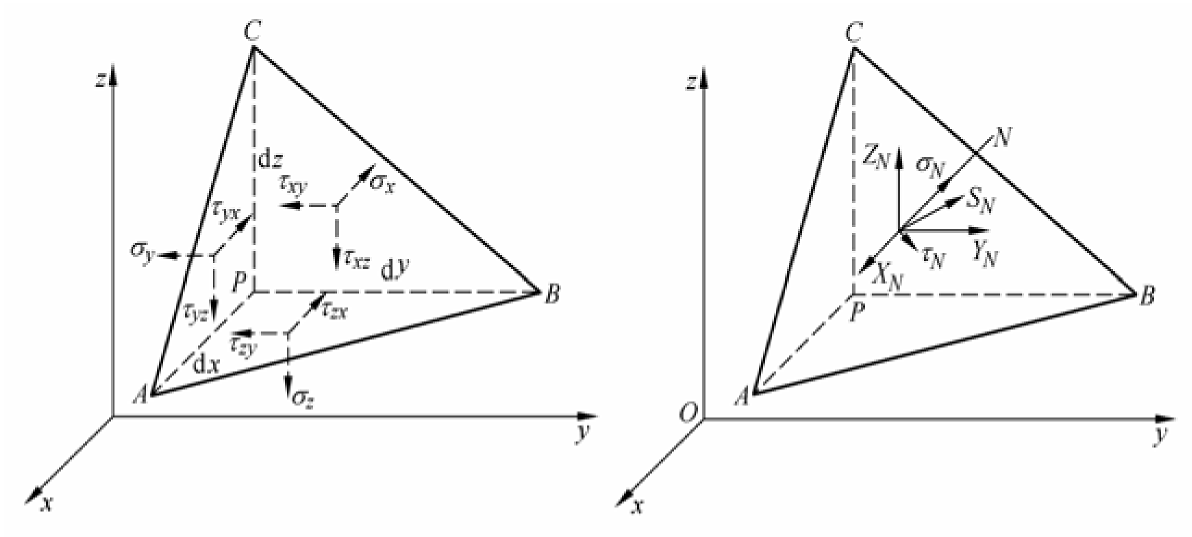

Figure 2.2

The volume of tetrahedron is

$$dV = \frac{1}{6} dxdydz$$

Stress vector of tilted plane can be express as three stress components along directions of axis (Seen in Figure 2.2). It can also be described along normal and tangential directions though. According to balance condition of tetrahedron, we have

$$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0$$

⇒

$$X_N dS - \sigma_x ldS - \tau_{yx} mdS - \tau_{zx} ndS + XdV = 0 \; (2.4a)$$
$$Y_N dS - \tau_{xy} ldS - \sigma_y mdS - \tau_{zy} ndS + YdV = 0 \; (2.4b)$$
$$Z_N dS - \tau_{xz} ldS - \tau_{yz} mdS - \sigma_z ndS + ZdV = 0 \; (2.4c)$$

In equations (2.4), the last item in each equation can be ignored because dV is an infinitesimal higher than dS. Hence, solutions of equations (2.4) are

$$X_N = \sigma_x l + \tau_{yx} m + \tau_{zx} n \quad (2.5a)$$

$$Y_N = \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad (2.5b)$$

$$Z_N = \tau_{xz} l + \tau_{yz} m + \sigma_z n \quad (2.5c)$$

Equations (2.5) are tilted section stress equations, also called Cauchy equations. Value of tilted section stress can be calculated as

$$S = \sqrt{X_N^{\ 2} + Y_N^{\ 2} + Z_N^{\ 2}} \qquad (2.6)$$

Using equations (2.5) and equation (2.6), stress vector of tilted section can be achieved. Also, stress vector can be express as normal and tangential stresses.

According to equation (2.3) and (2.5),

$$\sigma_N = \bar{S} \cdot \vec{n} = X_N l + Y_N m + Z_N n \quad (2.7)$$

$$\Rightarrow \begin{aligned} \sigma_N &= (\sigma_x l + \tau_{yx} m + \tau_{zx} n) l + (\tau_{xy} l + \sigma_y m + \tau_{zy} n) m + (\tau_{xz} l + \tau_{yz} m + \sigma_z n) n \\ &= \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{yx} ml + 2\tau_{zx} nl + 2\tau_{zy} nm \end{aligned} \qquad (2.8)$$

Then, according to equation (2.6)

$$\tau_N = \sqrt{S^2 - \sigma_N^{\ 2}} = \sqrt{X_N^{\ 2} + Y_N^{\ 2} + Z_N^{\ 2} - \sigma_N^{\ 2}} \quad (2.9)$$

As we can see, any section at a point with direction $\vec{n}$ has stress can be described by equations (2.8) and (2.9), only if six individual stress components are known.

-Stress boundary conditions

Another usage of Cauchy equation is to achieve a stress boundary condition. To points which are at the edges of elastomer, infinitesimal unit is chosen as infinitesimal-tetrahedron type with the tilted section exactly along the boundary. Since balance condition is still works, equation (2.5) can be used. Substituting stress components with surface force components, we have

$$\bar{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n \quad (2.10a)$$

$$\bar{Y} = \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad (2.10b)$$

$$\bar{Z} = \tau_{xz} l + \tau_{yz} m + \sigma_z n \quad (2.10c)$$

3, Applications of stress state analysis

-Principal stress and stress invariants

According to Cauchy equations (2.5), values stress components on a tilted section are depended on direction normal{l m n}. Considering equation (2.8) and (2.9), normal stress and tangential stress of a section at a point are also variable. In fact, there are special sections, on which, shear stresses are zeros. These sections are principal stress planes at a point.

Normal stress on these planes is called principal stresses. Their values and directions can be acquired as following process.
According to description of principal stress plane, we have

$$\tau_N = 0, \quad \sigma_N = \sigma$$

According to equation (2.9)

$$\sigma_N = S$$

Considering equation (2.3)

$$X_N = l\sigma, \quad Y_N = m\sigma, \quad Z_N = n\sigma \qquad (2.11)$$

Take equation (2.11) into equation (2.5), we have

$$\begin{cases} (\sigma_x - \sigma)l + \tau_{yx}m + \tau_{zx}n = 0 \\ \tau_{xy}l + (\sigma_y - \sigma)m + \tau_{zy}n = 0 \\ \tau_{xz}l + \tau_{yz}m + (\sigma_z - \sigma)n = 0 \end{cases} \qquad (2.12)$$

As direction normal, we also know

$$l^2 + m^2 + n^2 = 1 \qquad (2.13)$$

As homogeneous linear equations with non-zero solution vector, coefficient matrix of equation (2.12) has to feed the following condition.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \qquad (2.14)$$

Considering reciprocal theorem of shear stress, a cubic equation about σ is achieved

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad (2.15)$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

To ensure uniqueness of solutions in a certain stress state, coefficients I_1, I_2, and I_3 must be invariant no matter which coordinate we use. I_1, I_2, and I_3 are respectively called the first, second, and third stress invariant. Equation (2.15) is called stress state characteristic formulation.

Three roots of equation (2.15) are principal stresses of a point. Big to small, they are labelled σ_1,σ_2 and σ_3. Considering equation (2.12) and (2.13), corresponding direction normal as$_{123}$ values can be solved with principal stresses, known as principal stresses' direction.

-Properties of stress invariants

a. Invariance

As mentioned before, stress invariants (associate with principal stresses and their directions) are decided by loads, structure of body and constraint conditions. Configuration of reference has no effect to them.

b. Reality

Three roots of stress state characteristic equation (2.15), or principal stresses, are all realties.

c. Orthogonality

Directions of principal stresses are orthogonal. In other words, three principal stress planes are perpendicular to each other.

Orthogonality of stress invariants is easy to comprehend if account reciprocal theorem of shear stress.

-Maximum and minimum stress of a point

According to orthogonality of principal stresses, a coordinate can be built along the principal stress directions. Selected infinitesimal cuboid is consisted of principal stress planes. Vaules of stress components of cuboid in predesigned reference are

$$\sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3, \tau_{zx} = \tau_{xz} = 0, \tau_{yx} = \tau_{xy} = 0, \tau_{zy} = \tau_{yz} = 0$$

According to equation (2.8), normal stress on a tilted plane is

$$\sigma_N = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 \qquad (2.16)$$

Considering equation (2.13), Lagrange extreme value theorem can be used to research equation (2.16). Three extreme values of equation (2.16)are σ_1,σ_2 and σ_3. Thus, σ_1 is normal stress maximum of point, σ_3 is normal stress minimum of point. They both appear on normal stress of point. Particularly, stresses on all directions are same as principal stress if values of three principal stresses are equivalent.

Maximum and minimum values of shear stress can be worked out. In view of equation (2.15) and (2.16), equation (2.9) is rewritten as

$$\tau_N{}^2 = \sigma_1{}^2 l^2 + \sigma_2{}^2 m^2 + \sigma_3{}^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2 \qquad (2.17)$$

Utilizing equation (2.13), Lagrange extreme value theorem can be used to research equation (2.17). All extreme values of stress state of a point and their direction normal components are recorded in Table 2.1.

Table 2.1 Extreme values of stresses

l	m	n	τ_N^2	σ_N
± 1	0	0	0	σ_1
0	± 1	0	0	σ_2
0	0	± 1	0	σ_3
0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\left(\frac{\sigma_2-\sigma_3}{2}\right)^2$	$\frac{\sigma_2+\sigma_3}{2}$
$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	$\left(\frac{\sigma_1-\sigma_2}{2}\right)^2$	$\frac{\sigma_1+\sigma_2}{2}$
$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	$\left(\frac{\sigma_1-\sigma_3}{2}\right)^2$	$\frac{\sigma_1+\sigma_3}{2}$

As shown in Table 2.1, the first three extreme values are located on principal stress planes. On these planes, normal stresses are principal stresses, and shear stress is zero (minimum value).All extreme values of shear stress occur on directions 45 degrees from principal planes. Maximum value of shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \tag{2.18}$$

It is located at plane 45 degrees from planes with maximal and minimal principal stress, parallel with plane with principal stress σ_2.