[3rd]-- Linear solid strain [book1 7.4-7.5\book2 2.4-2.5\courseware reference CH3]
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1, Deformation and Strains
-Displacement of rigid
As we already knew, displacement refers position change of a body in space. Displacement of a particle or rigid is a sequence of motion. Motion of body is a result of external force. Any motion of a rigid can be regarded as combination of translation and rotation. As rigid, any to points inside rigid will keep distance between them under all conditions.
-Deformation of body
If an elastomer suffers external force, another effect calls interior effect—deformation. Deformation means change of shape or dimension of a body (see figure 3.1).

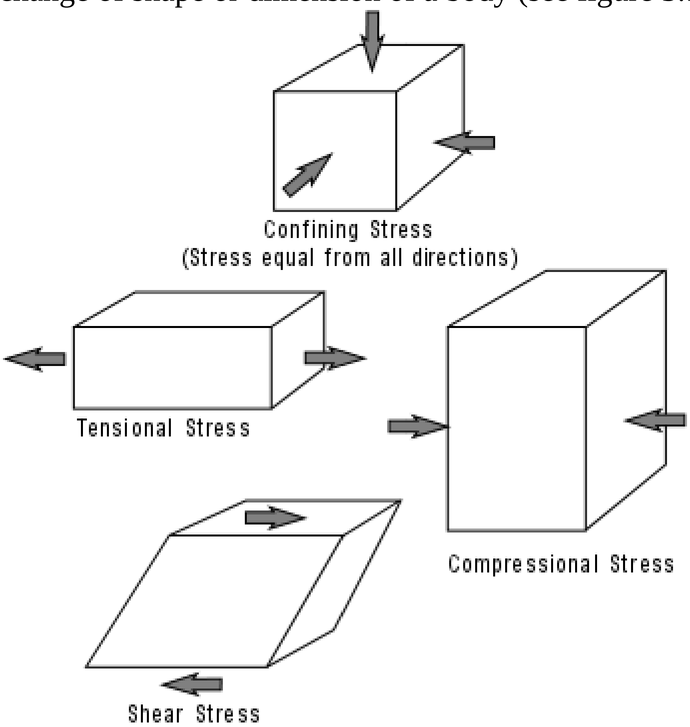

Figure 3.1
Obviously, distance of two points in an elastomer could change when there had a deformation.

-Definition of strain
To measure deformation of a point, normal strain and shear strain are defined. Normal strain

refers to relative change of differential segment of a direction by a point. It is positive if it is elongated. Shear strain is represented as angle change between two differential segments, also known as angular strain. It is positive if angle turns smaller. In Descartes coordinate, ε_x , ε_y , ε_z , is respectively to delegate normal strain of a point along axis; γ_{xy} , γ_{yz} , γ_{zx} is respectively shear strain of a point at three planes. Strain component is a dimensionless quantity.

Considering continuity, distribution of strains maybe used to express function of coordinate. And relationship between displacement and strain can be built to solve problems.

-Relation of displacement and strain- geometric equations
Displacement of elastomer relate to strain. Their relationship could be described by analyzing infinitesimal unit strains. Considering an infinitesimal-cuboid $dV = dxdydz$ from a random selected point $P(x, y, z)$, its displacements and strains can be researched in three projection planes respectively.

Firstly, we observe *xOy* plane. Projection sketch of two sides of infinitesimal cuboid are shown in Figure 3.2. Displacement component functions of studied elastomer are $u(x, y)$, $v(x, y)$. u and v are used to represent displacement component of point *P*.

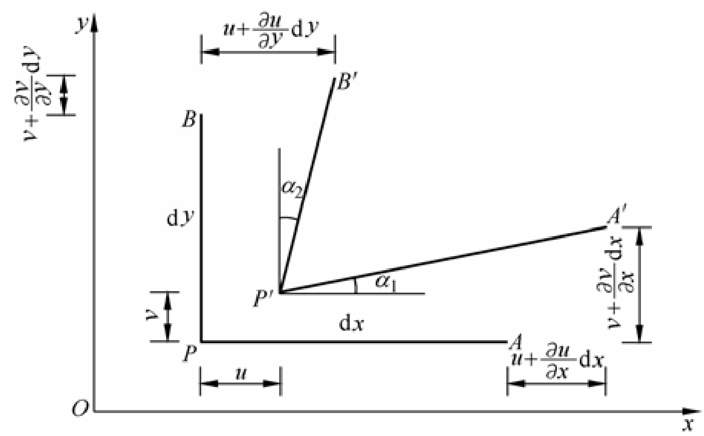

Figure 3.2

Since point *P*, *A*, and *B* had displacement respectively under external loads, displacement, shape and dimension of infinitesimal cuboid are all changed. We now formulate relationships between strains of infinitesimal cuboid and points' displacements.
Considering hypothesis of small deformation, displacements of point *A* can be written as two components

$$u_A = u + \frac{\partial u}{\partial x}dx \text{ , } v_A = v + \frac{\partial v}{\partial x}dx \tag{3.1}$$

Similarly, displacements of point *B* are

$$u_B = u + \frac{\partial u}{\partial y}dy \text{ , } v_B = v + \frac{\partial v}{\partial y}dy \tag{3.2}$$

According to definition of strain, normal strains of point *P* are length changes of infinitesimal segment $\overline{PA}$ and $\overline{PB}$, namely

$$\varepsilon_x = \frac{\overline{P'A'} - \overline{PA}}{\overline{PA}} \approx \frac{u_A - u}{dx} = \frac{\partial u}{\partial x} \tag{3.3a}$$

$$\varepsilon_y = \frac{\overline{P'B'} - \overline{PB}}{\overline{PB}} \approx \frac{v_B - v}{dy} = \frac{\partial v}{\partial y} \tag{3.3b}$$

Shear strain of point *P*, which is change of intersection angle of $\overline{PA}$ and $\overline{PB}$, can be expressed as

$$\begin{aligned} \gamma_{xy} &= \alpha_1 + \alpha_2 \\ &\approx \frac{v_A - v}{dx} + \frac{u_B - u}{dy} \\ &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned} \tag{3.3c}$$

On projection plane *yOz* and *zOx*, similar results are achieved. We have a group of equations

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{cases} \tag{3.4}$$

Equation group (3.4) is called geometric equations, also known as Cauchy formulations.

-special case—motion formulation of rigid

Cauchy formulations are more universal than motion formulation of rigid. Motion formulation of rigid can be educed from Cauchy formulations.

To rigid, there is no strain under any condition, we have

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \tag{3.5}$$

Substitute equation (3.5) into equation (3.4),

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \tag{3.6a}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{3.6b}$$

Integrate equation (3.6a)

$$u = f(y,z), v = g(x,z), w = h(x,y) \tag{3.7}$$

Substitute equation (3.7) into equation (3.6b),

$$\begin{cases} \dfrac{\partial g(x,z)}{\partial x} + \dfrac{\partial f(y,z)}{\partial y} = 0 \\ \dfrac{\partial h(x,y)}{\partial y} + \dfrac{\partial g(x,z)}{\partial z} = 0 \\ \dfrac{\partial f(y,z)}{\partial z} + \dfrac{\partial h(x,y)}{\partial x} = 0 \end{cases} \tag{3.8}$$

Compute partial derivative of x to first equation of (3.8), and of z to second equation of (3.8)

$$\begin{cases} \dfrac{\partial^2 g(x,z)}{\partial x^2} = 0 \\ \dfrac{\partial^2 g(x,z)}{\partial z^2} = 0 \end{cases} \tag{3.9}$$

A conclusion can be made that $g(x,z)$ could only have constant term, x term, z term, and xz term. Function $g(x,z)$ can be constructed as

$$g(x,z) = a + bx + cz + dxz \tag{3.10a}$$

Similarly, we have

$$f(y,z) = e + fy + gz + hyz \tag{3.10b}$$

$$h(x,y) = i + jx + ky + lxy \tag{3.10c}$$

Substitute equations (3.10) back into equation (3.8)

$$\begin{cases} k + c + (d+l)x = 0 \\ b + f + (d+h)z = 0 \\ g + j + (h+l)y = 0 \end{cases} \tag{3.11}$$

Solutions are

$$f = -b, c = -k, j = -g, d = h = l = 0$$

Thus, displacement functions of rigid are

$$\begin{cases} f(y,z) = e - by + gz \\ g(x,z) = a + bx - kz \\ h(x,y) = i - gx + ky \end{cases} \tag{3.12}$$

Or, in vector type

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} e \\ a \\ i \end{Bmatrix} - \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \times \begin{Bmatrix} k \\ g \\ b \end{Bmatrix} \tag{3.13}$$

Equation (3.13) is obviously motion formulation of rigid.

-Displacement boundary condition

When rigid is researched, it has to feed the condition of zero-strain. According to different research object and constraints, displacement of points on the boundary is sometimes pre-decided to given constraints, which is called displacement boundary condition

$$\begin{cases} u_S = \bar{u} \\ v_S = \bar{v} \\ w_S = \bar{w} \end{cases} \tag{3.14}$$

S refers to boundary surface, $\bar{u}$, $\bar{v}$, and $\bar{w}$ are known values of points.

In elasticity, there are three boundary conditions. Equation (2.10) is called stress boundary condition, which supply surface force situation of boundary. The third is called mixed boundary condition. It gives out displacements of part of boundary surface and surface force of another part. Or, it can be aware of some part of boundary for both displacement and surface force.

-Strain compatibility equations

Through the geometric equations (3.4), the relationship between elastic strain and displacement are connected. In view of mathematics, strain components of elastomer should have relations to keep a continuous deformation instead of being destroyed. This

mathematical relationship is known as the strain compatibility equation (or compatible equation).
Such compatible equations fall into two categories: the first is strain compatibility equation inside the plane; the second category is about equations between planes. Strain compatibility equation is derived from the geometric equation.

Strain compatibility equation inside the plane can be formulated as following according to partial derivative computation to geometric (3.4)

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} + \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \dfrac{\partial^2 \varepsilon_y}{\partial z^2} + \dfrac{\partial^2 \varepsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \dfrac{\partial^2 \varepsilon_z}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases} \tag{3.15a}$$

To obtain strain compatibility equation between planes, other partial derivative computation are done to equation (3.4)

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 v}{\partial z \partial x}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 w}{\partial x \partial y}$$

Subtract the third equation with summation of first two equations,

$$\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} = 2\frac{\partial^2 v}{\partial x \partial z}$$

Differentiate the above equation by *y*

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z}$$

Similarly, other two equations are achieved

$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{xy}}{\partial z} + \dfrac{\partial \gamma_{yz}}{\partial x} - \dfrac{\partial \gamma_{zx}}{\partial y}\right) = 2\dfrac{\partial^2 \varepsilon_y}{\partial x \partial z} \\ \dfrac{\partial}{\partial z}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{zx}}{\partial y} - \dfrac{\partial \gamma_{xy}}{\partial z}\right) = 2\dfrac{\partial^2 \varepsilon_z}{\partial y \partial x} \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial \gamma_{zx}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z} - \dfrac{\partial \gamma_{yz}}{\partial x}\right) = 2\dfrac{\partial^2 \varepsilon_x}{\partial z \partial y} \end{cases} \tag{3.15b}$$

Equation (3.15a) and (3.15b) are compatibility equations, also called Saint-Venant equations. Displacement of elastomer can not be continuous single-valued function, if equations (3.15) are not satisfied by stress components. And, this means that continuous structure of an elastomer is destroyed with cracks or overlapping. It is proved that satisfaction of Saint-Venant equations is a necessary and sufficient condition to continuum.

2, Strain state at a point of body

-Analysis of strain state

Strain state is stain situation of a point inside an elastomer in all directions. As same as stress components, six strain components in any points of an object change their values as rotation of coordinate system. There are also three elastic strain principal directions perpendicular to each other. After deformation, differential lines along these three directions only have changes of lengths and remain right angles among them, which means zero shear strain. These three principal strains are noted as ε_1, ε_2, and ε_3. Considering direction normal (2.3) and hypothesis of small deformation, we have

$$\varepsilon_N = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{yx} ml + \gamma_{zx} nl + \gamma_{zy} nm \qquad (3.16)$$

-Strain invariants

When principal strains are wanted, equations can be built up suppose direction cosine $\{l \quad m \quad n\}$,

$$\begin{cases} (\varepsilon_x - \varepsilon)l + \frac{1}{2}\gamma_{yx} m + \frac{1}{2}\gamma_{zx} n = 0 \\ \frac{1}{2}\gamma_{xy} l + (\varepsilon_y - \varepsilon)m + \frac{1}{2}\gamma_{zy} n = 0 \\ \frac{1}{2}\gamma_{xz} l + \frac{1}{2}\gamma_{yz} m + (\varepsilon_z - \varepsilon)n = 0 \end{cases} \qquad (3.17)$$

Unknown variables *l*, *m*, and *n* can not be all zeros, hence

$$\begin{vmatrix} (\varepsilon_x - \varepsilon) & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & (\varepsilon_y - \varepsilon) & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & (\varepsilon_z - \varepsilon) \end{vmatrix} = 0 \qquad (3.18)$$

Expanding determinant (3.18), a strain state characteristic formulation is achieved

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0 \qquad (3.19)$$

Where J_1, J_2, and J_3 are respectively the first, second, and third strain invariants

$$\begin{aligned} J_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\ J_2 &= \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\gamma_{xy}^2 - \frac{1}{4}\gamma_{yz}^2 - \frac{1}{4}\gamma_{xz}^2 \\ J_3 &= \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix} \end{aligned} \qquad (3.20)$$

Equation (3.19) have three real number roots. They are principal strains ε_1, ε_2, and ε_3. The following types of strain invariants are expressed by principal strains

$$\begin{aligned} J_1 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ J_2 &= \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \end{aligned} \qquad (3.21)$$

$$J_3 = \varepsilon_1 \varepsilon_2 \varepsilon_3$$

-Principal strains and principal axis of strains

Respectively take three principal strains back into equation group (3.17), direction normal vectors of these principal strains can be solved. If the studied material are isotropic, principal axis of stresses and principal axis of strains are coincident.

-Volumetric strain

Volumetric strain refers to unit volume change of elastomer, noted as θ. After deformation of body, infinitesimal cuboid have new sides' length $(1+\varepsilon_x)dx$, $(1+\varepsilon_y)dy$, and $(1+\varepsilon_z)dz$. Considering hypothesis of small deformation, volumetric change is

$$V^{'} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)dxdydz$$

Volumeric strain is

$$\theta = \frac{dV^{'}}{V} = \frac{(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)dxdydz - dxdydz}{dxdydz}$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x + \varepsilon_x\varepsilon_y\varepsilon_z$$

Ignoring higher infinitesimal, volumetric strain is

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{3.22}$$

According to geometric formulation (3.4)

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{3.23}$$

Equation (3.23) indicate the simple differential relations between volumetric strain and displacement components.