中国民航大学
Civil Aviation University of China

Structural Applications of Finite Elements

Chapter 10

Dynamic problems

2018-09-01

Outline

- **1D steady-state heat conduction**
- **2D steady-state heat conduction**
- **Torsion**

We define the Lagrangean by

$$L = T - \Pi$$

where T is the kinetic energy and Π is the potential energy.

$$I = \int_{t_1}^{t_2} L \, dt$$

If L can be expressed in terms of the generalized variables $(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ where $\dot{q}_i = dq_i/dt$, then the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1 \text{ to } n$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$\Pi = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

Using $L = T - \Pi$, we obtain the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

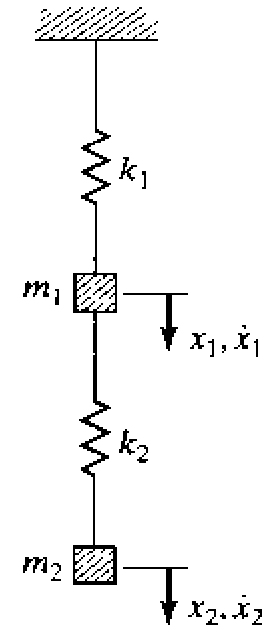
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

which can be written in the form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \mathbf{0}$$

which is of the form

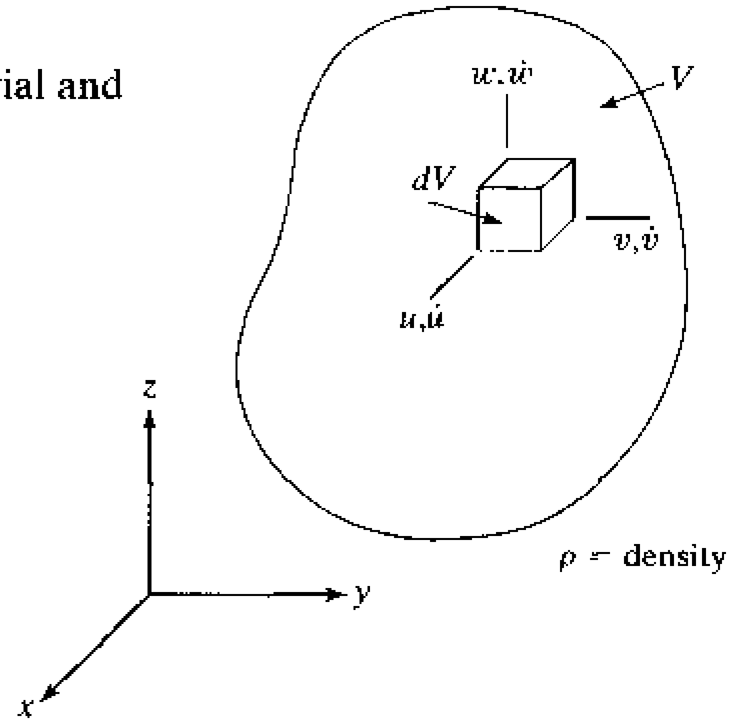
$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$



$$T = \frac{1}{2}\int_V \dot{\mathbf{u}}^{\mathrm{T}}\dot{\mathbf{u}}\rho\, dV$$

where ρ is the density (mass per unit volume) of the material and

$$\dot{\mathbf{u}} = [\dot{u}, \dot{v}, \dot{w}]^{\mathrm{T}}$$



$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$$\dot{\mathbf{u}} = \mathbf{N}\dot{\mathbf{q}}$$

$$T_e = \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_e \rho\mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV\right]\dot{\mathbf{q}} \qquad \mathbf{m}^e = \int_e \rho\mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV$$

$$T = \sum_e T_e = \sum_e \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{m}^e\dot{\mathbf{q}} = \tfrac{1}{2}\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{Q}}$$

$$\Pi = \tfrac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\mathbf{F} \qquad L = T - \Pi$$

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{F}$$

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = 0$$

$$\mathbf{Q} = \mathbf{U}\sin\omega t$$

$$\mathbf{K}\mathbf{U} = \omega^2\mathbf{M}\mathbf{U} \qquad \mathbf{K}\mathbf{U} = \lambda\mathbf{M}\mathbf{U}$$

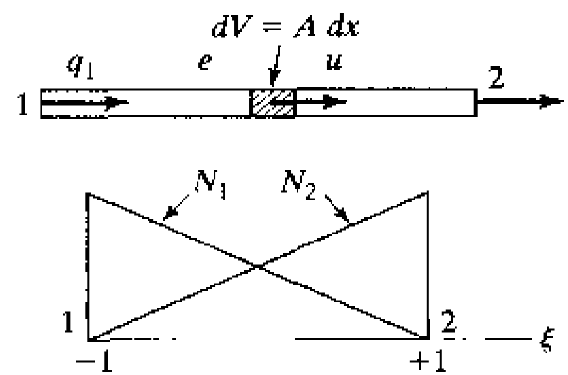
$$\mathbf{m}^e = \rho \int_e \mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV \qquad N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

$$\mathbf{q}^{\mathrm{T}} = [q_1 \quad q_2]$$

$$\mathbf{N} = [N_1 \quad N_2]$$

$$\mathbf{m}^e = \rho \int_e \mathbf{N}^{\mathrm{T}}\mathbf{N} A\, dx = \frac{\rho A_e \ell_e}{2} \int_{-1}^{+1} \mathbf{N}^{\mathrm{T}}\mathbf{N}\, d\xi$$

$$\mathbf{m}^e = \frac{\rho A_e \ell_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



$$N_1 = \frac{1-\xi}{2}$$

$$N_2 = \frac{1+\xi}{2}$$

$$dx = \frac{\ell_e}{2}\, d\xi$$

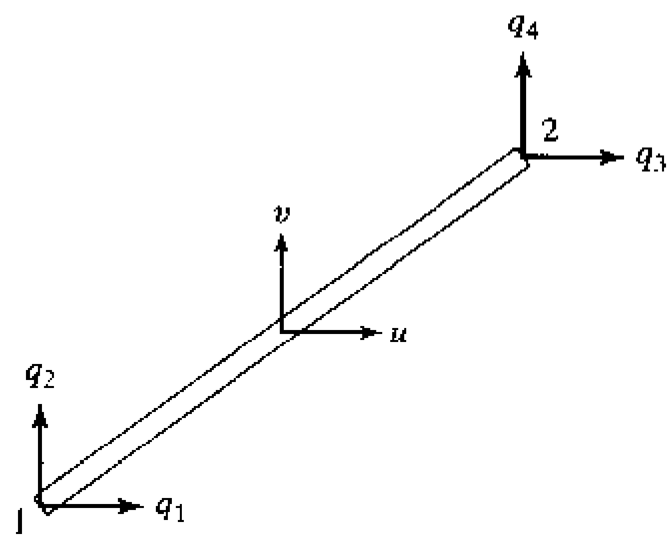
$$\mathbf{u}^{\mathrm{T}} = [u \quad v]$$

$$q^{\mathrm{T}} = [q_1 \quad q_2 \quad q_3 \quad q_4]$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

$$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

$$\mathbf{m}^e = \frac{\rho A_e \ell_e}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$



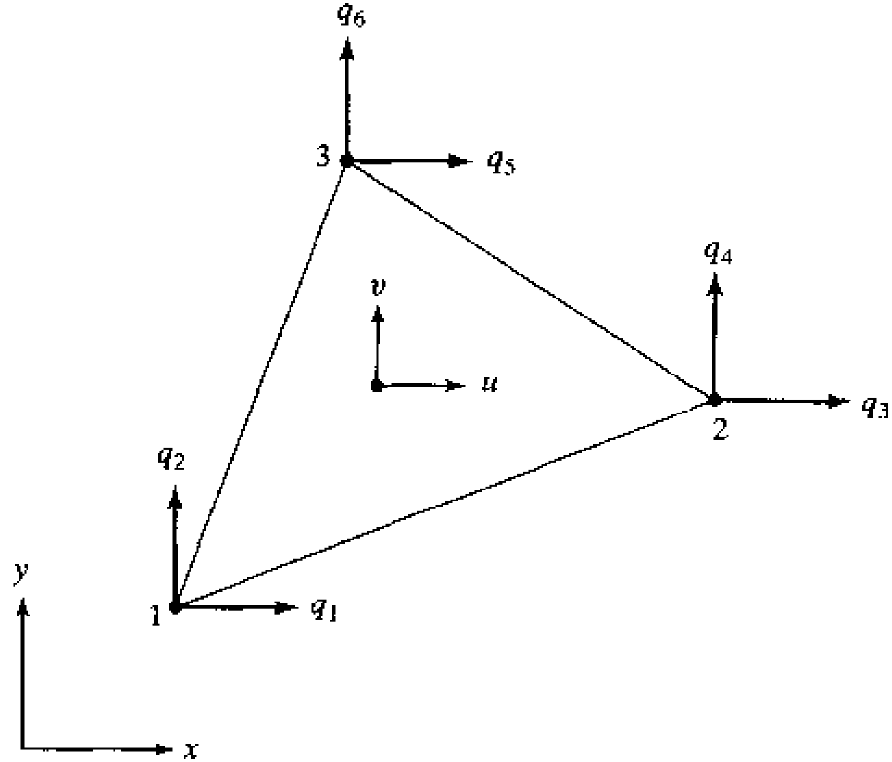
$$\mathbf{m}^e = \frac{\rho A_e \ell_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \frac{W}{2}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\boldsymbol{M}^e = \frac{W}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{u}^{\mathrm{T}} = [u \quad v]$$

$$\mathbf{q}^{\mathrm{T}} = [q_1 \quad q_2 \quad \cdots \quad q_6]$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$



$$\mathbf{m}^e = \rho t_e \int_e \mathbf{N}^{\mathrm{T}}\mathbf{N}\, dA$$

$$\mathbf{m}^e = \frac{\rho t_e A_e}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ & 2 & 0 & 1 & 0 & 1 \\ & & 2 & 0 & 1 & 0 \\ & & & 2 & 0 & 1 \\ \text{Symmetric} & & & & 2 & 0 \\ & & & & & 2 \end{bmatrix}$$

$$\frac{W}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u}^{\mathrm{T}} = [u \quad w]$$

$$\mathbf{m}^e = \int_e \rho \mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV = \int_e \rho \mathbf{N}^{\mathrm{T}}\mathbf{N} 2\pi r\, dA$$

$$\mathbf{m}^e = 2\pi\rho \int_e (N_1 r_1 + N_2 r_2 + N_3 r_3)\mathbf{N}^{\mathrm{T}}\mathbf{N}\, dA$$

$$\int_e N_1^3\, dA = \frac{2A_e}{20}, \int_e N_1^2 N_2\, dA = \frac{2A_e}{60}, \int_e N_1 N_2 N_3\, dA = \frac{2A_e}{120}, \text{etc.}$$

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

$$\mathbf{m}_e = \frac{\pi\rho A_e}{10}\begin{bmatrix} \frac{4}{3}r_1 + 2\bar{r} & 0 & 2\bar{r} - \frac{r_3}{3} & 0 & 2\bar{r} - \frac{r_2}{3} & 0 \\ & \frac{4}{3}r_1 + 2\bar{r} & 0 & 2\bar{r} - \frac{r_3}{3} & 0 & 2\bar{r} - \frac{r_2}{3} \\ & & \frac{4}{3}r_2 + 2\bar{r} & 0 & 2\bar{r} - \frac{r_1}{3} & 0 \\ & & & \frac{4}{3}r_2 + 2\bar{r} & 0 & 2\bar{r} - \frac{r_1}{3} \\ \text{Symmetric} & & & & \frac{4}{3}r_3 + 2\bar{r} & 0 \\ & & & & & \frac{4}{3}r_3 + 2\bar{r} \end{bmatrix}$$

$$\mathbf{u}^{\mathrm{T}} = [u \quad v]$$

$$\mathbf{q}^{\mathrm{T}} = [q_1 \quad q_2 \quad \cdots \quad q_8]$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$\mathbf{m}^e = \rho t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{N}^{\mathrm{T}} \mathbf{N} \det \mathbf{J} \, d\xi \, d\eta$$

$$\mathbf{u}^{\mathrm{T}} = [u \quad v \quad w]$$

$$\mathbf{N} = \left[\begin{matrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 \end{matrix} \right.$$

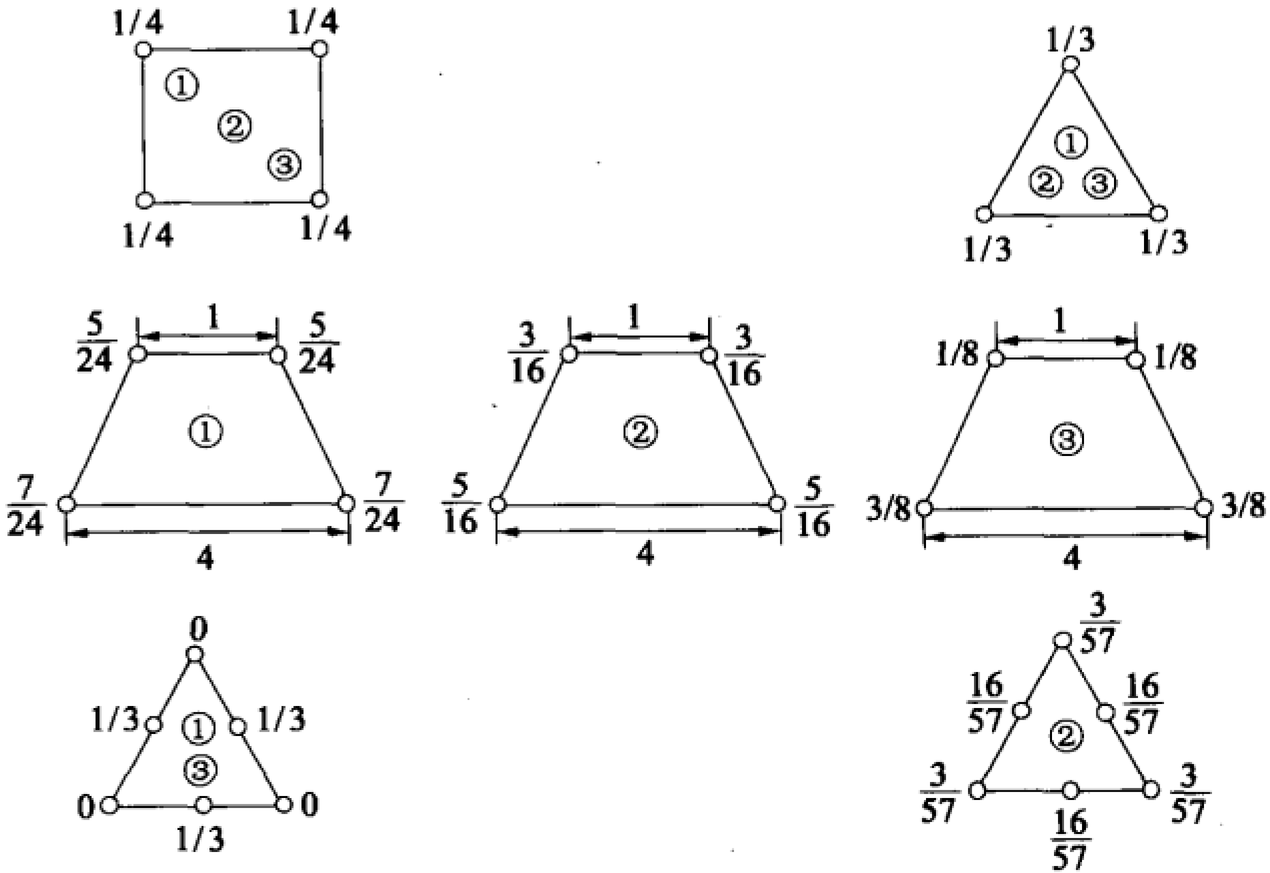
$$\mathbf{m}^e = \frac{\rho V_e}{20} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ & & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ & & & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ & \text{Symmetric} & & & & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ & & & & & & 2 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & & 2 & 0 & 0 & 1 & 0 \\ & & & & & & & & 2 & 0 & 0 & 1 \\ & & & & & & & & & 2 & 0 & 0 \\ & & & & & & & & & & 2 & 0 \\ & & & & & & & & & & & 2 \end{bmatrix}$$

$$\sum_r \tilde{M}_{rr} = \int_V \rho \mathrm{d}V$$

$$\tilde{M}_{rr} = \sum_s M_{rs}$$

$$\tilde{M}_{rr} = \alpha M_{rr}$$

将结点取为积分点



大型系统的特征值求解

- 向量迭代法
- 变换法
- 瑞利-里兹法
- 子空间迭代法
- Lanczos迭代法

Inverse iteration

$$\boldsymbol{K}\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{M}\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \frac{\bar{\boldsymbol{x}}_{k+1}}{(\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\boldsymbol{M}\bar{\boldsymbol{x}}_{k+1})^{\frac{1}{2}}}$$

为了提高计算效率，便于程序实现，引入向量

$$\boldsymbol{y}_k = \boldsymbol{M}\boldsymbol{x}_k$$

$$\bar{\boldsymbol{y}}_{k+1} = \boldsymbol{M}\bar{\boldsymbol{x}}_{k+1}$$

将式(3.71)和(3.72)代入式(3.55)和(3.56)中，并在式(3.56)两边左乘$\boldsymbol{M}$

$$\boldsymbol{K}\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{y}_k$$

$$\boldsymbol{y}_{k+1} = \frac{\bar{\boldsymbol{y}}_{k+1}}{(\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{y}}_{k+1})^{\frac{1}{2}}}$$

式(3.66)可改写为

$$\rho(\bar{\boldsymbol{x}}_{k+1}) = \frac{\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\boldsymbol{y}_k}{\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{y}}_{k+1}}$$

迭代过程为

1. 选取初始迭代向量 $\boldsymbol{x}_1$，利用式(3.71)计算 $\boldsymbol{y}_1$，并令$k=1$；
2. 对矩阵$\boldsymbol{K}$进行LDLT分解，即$\boldsymbol{K}=\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$；
3. 回代求解代数方程组(3.73)，得向量 $\bar{\boldsymbol{x}}_{k+1}$；
4. 由式(3.72)计算 $\bar{\boldsymbol{y}}_{k+1}$；
5. 由式(3.75)计算瑞利商 $\rho(\bar{\boldsymbol{x}}_{k+1})$；
6. 利用式(3.74)对 $\bar{\boldsymbol{x}}_{k+1}$正则化；
7. 检查$\dfrac{|\rho(\bar{\boldsymbol{x}}_{k+1})-\rho(\bar{\boldsymbol{x}}_k)|}{\rho(\bar{\boldsymbol{x}}_{k+1})}\leqslant \mathrm{tol}$?

   (1) 成立，则已收敛，且有$\lambda_1=\rho(\bar{\boldsymbol{x}}_{k+1})$，$\boldsymbol{\phi}_1=\boldsymbol{x}_{k+1}=\dfrac{\bar{\boldsymbol{x}}_{k+1}}{(\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{y}}_{k+1})^{\frac{1}{2}}}$

   (2) 不成立，则令$k=k+1$，转向3继续迭代。

例 3-1 用向量迭代法求广义特征值问题 $\boldsymbol{K}\boldsymbol{\phi}=\lambda\boldsymbol{M}\boldsymbol{\phi}$ 的第一阶特征对$(\lambda_1,\boldsymbol{\phi}_1)$，其中

$$\boldsymbol{K}=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},\qquad \boldsymbol{M}=\begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

并取误差范数 $\mathrm{tol}=10^{-6}$。本问题的精确解为

$$\lambda_1=\frac{1}{2}-\frac{\sqrt{2}}{4},\quad \boldsymbol{\phi}_1=\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1+\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2=\frac{1}{2}+\frac{\sqrt{2}}{4},\quad \boldsymbol{\phi}_2=\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{-1+\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_3=\infty,\quad \boldsymbol{\phi}_3=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_4=\infty,\quad \boldsymbol{\phi}_4=\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

解： 取初始向量为

$$\boldsymbol{x}_1 = [\ 1 \quad 1 \quad 1 \quad 1\ ]^{\mathrm{T}}$$

对$k=1$，有

$$\boldsymbol{y}_1 = [\ 0 \quad 2 \quad 0 \quad 1\ ]^{\mathrm{T}}$$

$$\bar{\boldsymbol{x}}_2 = [\ 3 \quad 6 \quad 7 \quad 8\ ]^{\mathrm{T}}$$

$$\bar{\boldsymbol{y}}_2 = [\ 0 \quad 12 \quad 0 \quad 8\ ]$$

$$\rho(\bar{\boldsymbol{x}}_2) = 0.1470588$$

$$\boldsymbol{y}_2 = [\ 0 \quad 1.02899 \quad 0 \quad 0.68599\ ]$$

k	$\bar{\boldsymbol{x}}_{k+1}$	$\bar{\boldsymbol{y}}_{k+1}$	$\rho(\bar{\boldsymbol{x}}_{k+1})$	$\dfrac{\lvert\lambda_1^{(k+1)}-\lambda_1^{(k)}\rvert}{\lambda_1^{(k+1)}}$	$\boldsymbol{y}_k$
1	$\begin{bmatrix}3\\6\\7\\8\end{bmatrix}$	$\begin{bmatrix}0\\12\\0\\8\end{bmatrix}$	0.1470588	—	$\begin{bmatrix}0\\1.02899\\0\\0.68599\end{bmatrix}$
2	$\begin{bmatrix}1.71499\\3.42997\\4.11597\\4.10896\end{bmatrix}$	$\begin{bmatrix}0\\6.85994\\0\\4.80196\end{bmatrix}$	0.1464646	4.057×10^{-3}	$\begin{bmatrix}0\\1.00504\\0\\0.70353\end{bmatrix}$
3	$\begin{bmatrix}1.70856\\3.41713\\4.12066\\4.82418\end{bmatrix}$	$\begin{bmatrix}0\\6.83426\\0\\4.82418\end{bmatrix}$	0.1464471	1.195×10^{-4}	$\begin{bmatrix}0\\1.00087\\0\\0.70649\end{bmatrix}$
4	$\begin{bmatrix}1.70736\\3.41472\\4.12121\\4.82771\end{bmatrix}$	$\begin{bmatrix}0\\6.82944\\0\\4.82771\end{bmatrix}$	0.1464466	3.519×10^{-6}	$\begin{bmatrix}0\\1.00015\\0\\0.70700\end{bmatrix}$
5	$\begin{bmatrix}1.70715\\3.41430\\4.12130\\4.82830\end{bmatrix}$	$\begin{bmatrix}0\\6.82860\\0\\4.82830\end{bmatrix}$	0.1464466	1.03589×10^{-7}	$\begin{bmatrix}0\\1.00003\\0\\0.70709\end{bmatrix}$

$$\boldsymbol{M}\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{K}\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \frac{\bar{\boldsymbol{x}}_{k+1}}{(\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\boldsymbol{M}\bar{\boldsymbol{x}}_{k+1})^{\frac{1}{2}}}$$

$$(\boldsymbol{K} - \alpha\boldsymbol{M})\boldsymbol{\phi} = \hat{\lambda}\boldsymbol{M}\boldsymbol{\phi}$$

Rayleigh quotient iteration

1. 选取初始迭代向量 $\boldsymbol{x}_1$ 和初始移轴量 $\rho(\bar{\boldsymbol{x}}_1)$，计算 $\boldsymbol{y}_1 = \boldsymbol{M}\boldsymbol{x}_1$，并令 $k = 1$；
2. 求解代数方程组 $[\boldsymbol{K} - \rho(\bar{\boldsymbol{x}}_k)\boldsymbol{M}]\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{y}_k$；
3. 计算 $\bar{\boldsymbol{y}}_{k+1} = \boldsymbol{M}\bar{\boldsymbol{x}}_{k+1}$；
4. 计算瑞利商 $\rho(\bar{\boldsymbol{x}}_{k+1}) = \dfrac{\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\boldsymbol{y}_k}{\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{y}}_{k+1}} + \rho(\bar{\boldsymbol{x}}_k)$；
5. 对$\bar{\boldsymbol{y}}_{k+1}$正则化$\boldsymbol{y}_{k+1} = \dfrac{\bar{\boldsymbol{y}}_{k+1}}{(\bar{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{y}}_{k+1})^{1/2}}$；

1. 选取$n\times q$阶初始迭代矩阵$\boldsymbol{X}_1=[\boldsymbol{x}_1^{(1)} \ \ \boldsymbol{x}_2^{(1)} \ \ \cdots \ \ \boldsymbol{x}_q^{(1)}]$，计算$\boldsymbol{Y}_1=\boldsymbol{M}\boldsymbol{X}_1$，并令$k=1$。

2. 解方程

$$\boldsymbol{K}\bar{\boldsymbol{X}}_{k+1}=\boldsymbol{Y}_k \tag{3.189}$$

3. 计算

$$\bar{\boldsymbol{Y}}_{k+1}=\boldsymbol{M}\bar{\boldsymbol{X}}_{k+1} \tag{3.190}$$

4. 以$\bar{\boldsymbol{X}}_{k+1}$为里兹基向量，计算缩减自由度数后的刚度矩阵$\bar{\boldsymbol{K}}$和质量矩阵$\bar{\boldsymbol{M}}$，即

$$\bar{\boldsymbol{K}}=\bar{\boldsymbol{X}}_{k+1}^{\mathrm{T}}\boldsymbol{Y}_k \tag{3.191}$$

$$\bar{\boldsymbol{M}}=\bar{\boldsymbol{X}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{Y}}_{k+1} \tag{3.192}$$

5. 求解广义特征值问题

$$\bar{\boldsymbol{K}}\bar{\boldsymbol{\Phi}}=\bar{\boldsymbol{M}}\bar{\boldsymbol{\Phi}}\bar{\boldsymbol{\Lambda}} \tag{3.193}$$

得到全部q个特征值$\bar{\lambda}_i(i=1,2,\cdots,q)$和相应的特征向量$\bar{\boldsymbol{\phi}}_i$，即

$$\bar{\boldsymbol{\Lambda}}=\mathrm{diag}(\bar{\lambda}_1,\bar{\lambda}_2,\cdots,\bar{\lambda}_q) \tag{3.194}$$

$$\bar{\boldsymbol{\Phi}}=[\bar{\boldsymbol{\phi}}_1, \ \bar{\boldsymbol{\phi}}_2, \ \cdots, \ \bar{\boldsymbol{\phi}}_q] \tag{3.195}$$

6. 如果各特征值已满足精度要求$\dfrac{\left|\bar{\lambda}_i^{(k+1)}-\bar{\lambda}_i^{(k)}\right|}{\bar{\lambda}_i^{(k+1)}}<\mathrm{tol}$，$i=1,2,\cdots,p$，用Sturm序列检查是否已求得了全部待求的特征对，并取

$$\boldsymbol{\Phi}=\boldsymbol{X}_{k+1}=\bar{\boldsymbol{X}}_{k+1}\bar{\boldsymbol{\Phi}} \tag{3.196}$$

$$\boldsymbol{\Lambda}=[\bar{\lambda}_1,\bar{\lambda}_2,\cdots,\bar{\lambda}_q] \tag{3.197}$$

否则计算

$$\boldsymbol{Y}_{k+1}=\bar{\boldsymbol{Y}}_{k+1}\bar{\boldsymbol{\Phi}}$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}; \quad M = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \lambda_1 = 2,\ \lambda_2 = 4,\ \lambda_3 = 6$$

解： 初始迭代矩阵$\boldsymbol{X}_1$取为

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}; \quad Y_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

由式(3.189) ~ (3.198)得

$$\bar{\boldsymbol{X}}_2 = \boldsymbol{K}^{-1}\boldsymbol{Y}_1 = \begin{bmatrix} 0.5 & 0.29167 \\ 0.5 & 0.08333 \\ 0.5 & 0.04167 \end{bmatrix}$$

$$\bar{\boldsymbol{Y}}_2 = \boldsymbol{M}\bar{\boldsymbol{X}}_2 = \begin{bmatrix} 0.25 & 0.14583 \\ 0.25 & 0.08333 \\ 0.25 & 0.02083 \end{bmatrix}$$

$$\bar{\boldsymbol{K}} = \bar{\boldsymbol{X}}_2^{\mathrm{T}}\bar{\boldsymbol{Y}}_1 = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.14583 \end{bmatrix}$$

$$\bar{\boldsymbol{M}} = \bar{\boldsymbol{X}}_2^{\mathrm{T}}\bar{\boldsymbol{Y}}_2 = \begin{bmatrix} 0.5 & 0.125 \\ 0.125 & 0.05035 \end{bmatrix}$$

解广义特征值问题$\bar{K}\bar{\Phi}=\bar{M}\bar{\Phi}\bar{\Lambda}$，得

$$\bar{\Lambda}=\begin{bmatrix}2 & 0\\ 0 & 4.3636\end{bmatrix},\quad \bar{\Phi}=\begin{bmatrix}-1 & 0.2425350\\ 0 & -0.9701425\end{bmatrix}$$

按类似的计算过程，继续迭代，经过16次迭代最终收敛($\mathrm{tol}=10^{-6}$)于：

$$\Lambda=\begin{bmatrix}2.0 & 0\\ 0 & 4.0000023\end{bmatrix},\quad \Phi=\begin{bmatrix}-0.7071 & -1.0008\\ -0.7071 & 0.0008\\ -0.7071 & 0.9992\end{bmatrix}$$

如果取$q=3$，即

$$X_1=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$$

则只需一次迭代即得到了精确解：

$$\Lambda=\begin{bmatrix}2 & & \\ & 4 & \\ & & 6\end{bmatrix},\quad \Phi=\begin{bmatrix}-0.7071 & -1.0 & 0.7071\\ -0.7071 & 0.0 & 0.7071\\ -0.7071 & 1.0 & 0.7071\end{bmatrix}$$

运动方程的求解

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_t + \boldsymbol{C}\dot{\boldsymbol{a}}_t + \boldsymbol{K}\boldsymbol{a}_t = \boldsymbol{Q}_t$$

- 振型叠加法
- Central difference method
- Houbolt法
- Newmark法
- Wilson θ 法
- 广义α方法
- 精细积分法
- 其他

Central difference method

$$a_{t+\Delta t} = a_t + \dot{a}_t \Delta t + \frac{1}{2}\ddot{a}_t \Delta t^2 + \frac{1}{6}\dddot{a}_t \Delta t^3 + O(\Delta t^4)$$

$$a_{t-\Delta t} = a_t - \dot{a}_t \Delta t + \frac{1}{2}\ddot{a}_t \Delta t^2 - \frac{1}{6}\dddot{a}_t \Delta t^3 + O(\Delta t^4)$$

$$\dot{a}_t \Delta t = \frac{1}{2}(a_{t+\Delta t} - a_{t-\Delta t}) + O(\Delta t^3)$$

$$\ddot{a}_t \Delta t^2 = (a_{t+\Delta t} - 2a_t + a_{t-\Delta t}) + O(\Delta t^4)$$

$$\dot{a}_t = \frac{1}{2\Delta t}(a_{t+\Delta t} - a_{t-\Delta t})$$

$$\ddot{a}_t = \frac{1}{\Delta t^2}(a_{t+\Delta t} - 2a_t + a_{t-\Delta t})$$

$$M\ddot{a}_t + C\dot{a}_t + Ka_t = Q_t$$

$$\hat{M} a_{t+\Delta t} = \hat{Q}_t$$

Explicit

$$\hat{M} = \left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right)$$

$$\hat{Q}_t = Q_t - \left(K - \frac{2}{\Delta t^2}M\right)a_t - \left(\frac{1}{\Delta t^2}M - \frac{1}{2\Delta t}C\right)a_{t-\Delta t}$$

$$a_{-\Delta t} = a_0 - \dot{a}_0 \Delta t + \frac{1}{2}\ddot{a}_0 \Delta t^2$$

1. 初始计算

(1) 形成刚度矩阵$\boldsymbol{K}$，质量阵$\boldsymbol{M}$和阻尼阵$\boldsymbol{C}$；

(2) 给定$\boldsymbol{a}_0$、$\dot{\boldsymbol{a}}_0$，并求解$\ddot{\boldsymbol{a}}_0$；

(3) 选择时间步长Δt，并计算积分常数$c_0 = 1/\Delta t^2$、$c_1 = 1/2\Delta t$、$c_2 = 2c_0$、$c_3 = 1/c_2$；

(4) 计算$\boldsymbol{a}_{-\Delta t} = \boldsymbol{a}_0 - \Delta t\dot{\boldsymbol{a}}_0 + c_3\ddot{\boldsymbol{a}}_0$；

(5) 形成有效质量矩阵(effective mass matrix)：$\hat{\boldsymbol{M}} = c_0\boldsymbol{M} + c_1\boldsymbol{C}$；

(6) 对$\hat{\boldsymbol{M}}$进行三角分解：$\hat{\boldsymbol{M}} = \boldsymbol{LDL}^{\mathrm{T}}$。

2. 对于每一时间步

(1) 计算时刻t的有效载荷$\hat{\boldsymbol{Q}}_t$：$\hat{\boldsymbol{Q}}_t = \boldsymbol{Q}_t - (\boldsymbol{K} - c_2\boldsymbol{M})\boldsymbol{a}_t - (c_0\boldsymbol{M} - c_1\boldsymbol{C})\boldsymbol{a}_{t-\Delta t}$；

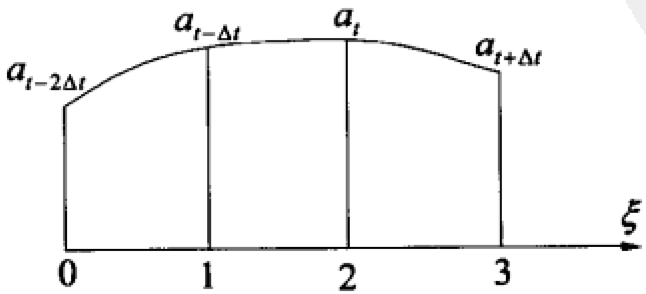
(2) 求解时刻$t+\Delta t$的位移$\boldsymbol{LDL}^{\mathrm{T}}\boldsymbol{a}_{t+\Delta t} = \hat{\boldsymbol{Q}}_t$；

(3) 如果需要，计算时刻t的加速度和速度

$$\ddot{\boldsymbol{a}}_t = c_0(\boldsymbol{a}_{t-\Delta t} - 2\boldsymbol{a}_t + \boldsymbol{a}_{t+\Delta t})$$

$$\dot{\boldsymbol{a}}_t = c_1(-\boldsymbol{a}_{t-\Delta t} + \boldsymbol{a}_{t+\Delta t})$$

Houbolt method



$$N_i = \prod_{\substack{j=1 \\ j\neq i}}^{4} \frac{\xi - \xi_j}{\xi_i - \xi_j}, \quad \xi = \frac{\tau - t + 2\Delta t}{\Delta t} \quad (0 \leqslant \xi \leqslant 3)$$

$$\boldsymbol{a}(\xi) = \sum_{i=1}^{4} N_i \boldsymbol{a}_i = N_1 \boldsymbol{a}_{t+\Delta t} + N_2 \boldsymbol{a}_t + N_3 \boldsymbol{a}_{t-\Delta t} + N_4 \boldsymbol{a}_{t-2\Delta t}$$

$$\begin{aligned} \boldsymbol{a}(\xi) = & \frac{1}{6}\xi(\xi-1)(\xi-2)\boldsymbol{a}_{t+\Delta t} - \frac{1}{2}\xi(\xi-1)(\xi-3)\boldsymbol{a}_t + \\ & \frac{1}{2}\xi(\xi-2)(\xi-3)\boldsymbol{a}_{t-\Delta t} - \frac{1}{6}(\xi-1)(\xi-2)(\xi-3)\boldsymbol{a}_{t-2\Delta t} \end{aligned}$$

$$\dot{\boldsymbol{a}}_{t+\Delta t} = \frac{1}{6\Delta t}(11\boldsymbol{a}_{t+\Delta t} - 18\boldsymbol{a}_t + 9\boldsymbol{a}_{t-\Delta t} - 2\boldsymbol{a}_{t-2\Delta t}) \qquad \xi = 3$$

$$\ddot{\boldsymbol{a}}_{t+\Delta t} = \frac{1}{\Delta t^2}(2\boldsymbol{a}_{t+\Delta t} - 5\boldsymbol{a}_t + 4\boldsymbol{a}_{t-\Delta t} - \boldsymbol{a}_{t-2\Delta t})$$

$$\dot{\boldsymbol{a}}_{t+\Delta t} = \frac{1}{6\Delta t}(11\boldsymbol{a}_{t+\Delta t} - 18\boldsymbol{a}_t + 9\boldsymbol{a}_{t-\Delta t} - 2\boldsymbol{a}_{t-2\Delta t})$$

$$\ddot{\boldsymbol{a}}_{t+\Delta t} = \frac{1}{\Delta t^2}(2\boldsymbol{a}_{t+\Delta t} - 5\boldsymbol{a}_t + 4\boldsymbol{a}_{t-\Delta t} - \boldsymbol{a}_{t-2\Delta t})$$

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_{t+\Delta t} + \boldsymbol{C}\dot{\boldsymbol{a}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{a}_{t+\Delta t} = \boldsymbol{Q}_{t+\Delta t}$$

$$\hat{\boldsymbol{K}}\boldsymbol{a}_{t+\Delta t} = \hat{\boldsymbol{Q}}_{t+\Delta t}$$

Implicit

$$\hat{\boldsymbol{K}} = \left(\frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{C} + \boldsymbol{K}\right)$$

$$\hat{\boldsymbol{Q}}_{t+\Delta t} = \boldsymbol{Q}_{t+\Delta t} + \left(\frac{5}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{C}\right)\boldsymbol{a}_t - \left(\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{3}{2\Delta t}\boldsymbol{C}\right)\boldsymbol{a}_{t-\Delta t} + \left(\frac{1}{\Delta t^2}\boldsymbol{M} + \frac{1}{3\Delta t}\boldsymbol{C}\right)\boldsymbol{a}_{t-2\Delta t}$$

1. 初始计算

(1) 形成刚度矩阵$\boldsymbol{K}$、质量阵$\boldsymbol{M}$和阻尼阵$\boldsymbol{C}$；

(2) 给定$\boldsymbol{a}_0$、$\dot{\boldsymbol{a}}_0$，并计算$\ddot{\boldsymbol{a}}_0$；

(3) 选择时间步长并计算积分常数

$$c_0 = \frac{2}{\Delta t^2}, \quad c_1 = \frac{11}{6\Delta t}, \quad c_2 = \frac{5}{\Delta t^2}, \quad c_3 = \frac{3}{\Delta t},$$
$$c_4 = -2c_0, \quad c_5 = -\frac{c_3}{2}, \quad c_6 = \frac{c_0}{2}, \quad c_7 = \frac{c_3}{9}$$

(4) 使用其他起步算法计算$a_{\Delta t}$和$a_{2\Delta t}$；

(5) 计算有效刚度(effective stiffness matrix) $\hat{\boldsymbol{K}} = \boldsymbol{K} + c_0\boldsymbol{M} + c_1\boldsymbol{C}$；

(6) 对$\hat{\boldsymbol{K}}$进行三角化：$\hat{\boldsymbol{K}} = \boldsymbol{LDL}^{\mathrm{T}}$。

2. 对每一个时间步

(1) 计算$t+\Delta t$时刻的有效载荷(effective load)

$$\hat{\boldsymbol{Q}}_{t+\Delta t} = \boldsymbol{Q}_{t+\Delta t} + \boldsymbol{M}(c_2\boldsymbol{a}_t + c_4\boldsymbol{a}_{t-\Delta t} + c_6\boldsymbol{a}_{t-2\Delta t}) + \quad (4.$$
$$\boldsymbol{C}(c_3\boldsymbol{a}_t + c_5\boldsymbol{a}_{t-\Delta t} + c_7\boldsymbol{a}_{t-2\Delta t}) \quad (4.$$

(2) 求解$t+\Delta t$时刻的位移$\boldsymbol{a}_{t+\Delta t}$

$$\boldsymbol{LDL}^{\mathrm{T}}\boldsymbol{a}_{t+\Delta t} = \hat{\boldsymbol{Q}}_{t+\Delta t}$$

(3) 根据需要计算$t+\Delta t$时刻的加速度和速度

$$\ddot{\boldsymbol{a}}_{t+\Delta t} = c_0\boldsymbol{a}_{t+\Delta t} - c_2\boldsymbol{a}_t - c_4\boldsymbol{a}_{t-\Delta t} - c_6\boldsymbol{a}_{t-2\Delta t}$$
$$\dot{\boldsymbol{a}}_{t+\Delta t} = c_1\boldsymbol{a}_{t+\Delta t} - c_3\boldsymbol{a}_t - c_5\boldsymbol{a}_{t-\Delta t} - c_7\boldsymbol{a}_{t-2\Delta t}$$

Newmark method

$$a_{t+\Delta t} = a_t + \dot{a}_t \Delta t$$
$$\dot{a}_{t+\Delta t} = \dot{a}_t + \ddot{a}_t \Delta t$$

$$M\ddot{a}_t + C\dot{a}_t + Ka_t = Q_t$$ 欧拉法 self-starting

$$a_{t+\Delta t} = a_t + \frac{1}{2}(\dot{a}_t + \dot{a}_{t+\Delta t})\Delta t$$
$$\dot{a}_{t+\Delta t} = \dot{a}_t + \frac{1}{2}(\ddot{a}_t + \ddot{a}_{t+\Delta t})\Delta t$$

$$a_{t+\Delta t} = a_t + \dot{a}_t \Delta t + \frac{1}{4}(\ddot{a}_t + \ddot{a}_{t+\Delta t})\Delta t^2$$ 平均加速度法

$$\dot{a}_{t+\Delta t} = \dot{a}_t + (1-\gamma)\ddot{a}_t \Delta t + \gamma \ddot{a}_{t+\Delta t}\Delta t$$
$$a_{t+\Delta t} = a_t + \dot{a}_t \Delta t + \left(\frac{1}{2} - \beta\right)\ddot{a}_t \Delta t^2 + \beta \ddot{a}_{t+\Delta t}\Delta t^2$$

$\gamma = 1/2,\ \beta = 1/4 \quad (\ddot{a}_{t+\tau} = (\ddot{a}_t + \ddot{a}_{t+\Delta t})/2)$ 平均加速度法

$\gamma = 1/2,\ \beta = 0$ 中心差分法

$\gamma = 1/2,\ \beta = 1/6 \quad (\ddot{a}_{t+\tau} = \ddot{a}_t + (\ddot{a}_{t+\Delta t} - \ddot{a}_t)\tau/\Delta t)$

线性加速度法

$\gamma = 1/2,\ \beta = 1/8$ $\quad \ddot{a}_{t+\tau} = \begin{cases} \ddot{a}_t & \tau \leqslant \dfrac{\Delta t}{2} \\ \ddot{a}_{t+\Delta t} & \dfrac{\Delta t}{2} \leqslant \tau \leqslant \Delta t \end{cases}$

$$\dot{a}_{t+\Delta t} = \dot{a}_t + (1-\gamma)\ddot{a}_t \Delta t + \gamma \ddot{a}_{t+\Delta t}\Delta t$$

$$a_{t+\Delta t} = a_t + \dot{a}_t \Delta t + \left(\frac{1}{2} - \beta\right)\ddot{a}_t \Delta t^2 + \beta \ddot{a}_{t+\Delta t}\Delta t^2$$

$$M\ddot{a}_{t+\Delta t} + C\dot{a}_{t+\Delta t} + K a_{t+\Delta t} = Q_{t+\Delta t}$$

$$\ddot{a}_{t+\Delta t} = \frac{1}{\beta\Delta t^2}(a_{t+\Delta t} - a_t) - \frac{1}{\beta\Delta t}\dot{a}_t - \left(\frac{1}{2\beta} - 1\right)\ddot{a}_t$$

$$\dot{a}_{t+\Delta t} = \frac{\gamma}{\beta\Delta t}(a_{t+\Delta t} - a_t) + \left(1 - \frac{\gamma}{\beta}\right)\dot{a}_t + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t \ddot{a}_t$$

$$\hat{K} a_{t+\Delta t} = \hat{Q}_{t+\Delta t}$$

Implicit

$$\hat{K} = K + \frac{1}{\beta\Delta t^2}M + \frac{\gamma}{\beta\Delta t}C$$

$$\hat{Q}_{t+\Delta t} = Q_{t+\Delta t} + M\left[\frac{1}{\beta\Delta t^2}a_t + \frac{1}{\beta\Delta t}\dot{a}_t + \left(\frac{1}{2\beta} - 1\right)\ddot{a}_t\right] + C\left[\frac{\gamma}{\beta\Delta t}a_t + \left(\frac{\gamma}{\beta} - 1\right)\dot{a}_t + \left(\frac{\gamma}{2\beta} - 1\right)\Delta t \ddot{a}_t\right]$$

用Newmark法求解运动方程的步骤可归纳为

1. 初始计算

(1) 形成刚度矩阵$\boldsymbol{K}$、质量矩阵$\boldsymbol{M}$和阻尼矩阵$\boldsymbol{C}$；

(2) 给定$\boldsymbol{a}_0$、$\dot{\boldsymbol{a}}_0$并计算$\ddot{\boldsymbol{a}}_0$；

(3) 选择时间步长Δt、参数β和γ，并计算积分常数

$$c_0 = \frac{1}{\beta \Delta t^2}, \quad c_1 = \frac{\gamma}{\beta \Delta t}, \quad c_2 = \frac{1}{\beta \Delta t}, \quad c_3 = \frac{1}{2\beta} - 1,$$

$$c_4 = \frac{\gamma}{\beta} - 1, \quad c_5 = \Delta t\left(\frac{\gamma}{2\beta} - 1\right), \quad c_6 = \Delta t(1-\gamma), \quad c_7 = \gamma \Delta t$$

(4) 形成有效刚度矩阵

$$\hat{\boldsymbol{K}} = \boldsymbol{K} + c_0 \boldsymbol{M} + c_1 \boldsymbol{C}$$

(5) 三角分解$\hat{\boldsymbol{K}} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$。

2. 对于每一时间步

(1) 计算时间$t+\Delta t$的有效载荷

$$\hat{\boldsymbol{Q}}_{t+\Delta t} = \boldsymbol{Q}_{t+\Delta t} + \boldsymbol{M}(c_0 \boldsymbol{a}_t + c_2 \dot{\boldsymbol{a}}_t + c_3 \ddot{\boldsymbol{a}}_t) + \boldsymbol{C}(c_1 \boldsymbol{a}_t + c_4 \dot{\boldsymbol{a}}_t + c_5 \ddot{\boldsymbol{a}}_t)$$

(2) 求解时刻$t+\Delta t$的位移

$$\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}} \boldsymbol{a}_{t+\Delta t} = \hat{\boldsymbol{Q}}_{t+\Delta t}$$

(3) 计算时间$t+\Delta t$的加速度和速度

$$\ddot{\boldsymbol{a}}_{t+\Delta t} = c_0(\boldsymbol{a}_{t+\Delta t} - \boldsymbol{a}_t) - c_2 \dot{\boldsymbol{a}}_t - c_3 \ddot{\boldsymbol{a}}_t$$

$$\dot{\boldsymbol{a}}_{t+\Delta t} = \dot{\boldsymbol{a}}_t + c_6 \ddot{\boldsymbol{a}}_t + c_7 \ddot{\boldsymbol{a}}_{t+\Delta t}$$

$$\gamma \geqslant \frac{1}{2}, \qquad \beta \geqslant \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2$$

$$\Delta t_{\mathrm{cr}} = \frac{T}{\pi} \frac{1}{\sqrt{\left(\gamma + \frac{1}{2}\right)^2 - 4\beta}}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

$\Delta t = T_2/10 = 0.28$; $\gamma = 0.5$, $\beta = 0.25$ $\quad \boldsymbol{a}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $\dot{\boldsymbol{a}}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $\ddot{\boldsymbol{a}}_0 = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$

$c_0 = 51.0$, $c_1 = 7.14$, $c_2 = 14.3$, $c_3 = 1.00$, $c_4 = 1.00$,

$c_5 = 0.00$, $c_6 = 0.14$, $c_7 = 0.14$

$$\hat{\boldsymbol{K}} = \begin{bmatrix} 108 & -2 \\ -2 & 55 \end{bmatrix} \qquad \hat{\boldsymbol{Q}}_{t+\Delta t} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} (51 a_t + 14.3 \dot{a}_t + 1.0 \ddot{a}_t)$$

时刻	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$	$11\Delta t$	$12\Delta t$
$\boldsymbol{a}(t)$	0.00673	0.0505	0.189	0.485	0.961	1.58	2.23	2.76	3.00	2.85	2.28	1.40
	0.364	1.35	2.68	4.00	4.95	5.34	5.13	4.48	3.64	2.90	2.44	2.31

$\Delta t = 10 T_2 = 28$.

时刻	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$	$11\Delta t$	$12\Delta t$
$\boldsymbol{a}(t)$	1.99	0.028	1.94	0.112	1.83	0.248	1.67	0.429	1.47	0.648	1.23	0.894
	5.99	0.045	5.90	0.177	5.72	0.393	5.47	0.685	5.14	1.04	4.76	1.45