Elastostatic methods

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1.General

-Analytical methods in linear elastostatics

2.Solving elastostatic problems

-Summary of basic problems

-Classification of problems

-Basic solving methods

-Uniqueness of solution

-**Saint-Venant's principle**

-Superposition principle

3.Solve space problems according to displacement

4.Solve space problems according to stress

Summary of basic equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{zx}}{\partial z} + \dfrac{\partial \tau_{yx}}{\partial y} + X = 0 \\ \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + \dfrac{\partial \tau_{xy}}{\partial x} + Y = 0 \\ \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} - Z = 0 \end{cases} \qquad (2.2)$$

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{cases} \qquad (3.4)$$

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \\ \varepsilon_y = \dfrac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)] \\ \varepsilon_z = \dfrac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{xz} = \dfrac{1}{G}\tau_{xz} \end{cases} \qquad (4.5)$$

-Classification of problems

- surface force boundary condition
- dispalcement boundary condition
- mixed boundary condition

-Basic solving methods

- stress method
- displacement method
- mixed method

-Saint-Venant's principle

Saint-Venant principle can be expressed as: a system of forces on a local area of surface of the object can be replaced by any system of forces which is statically equivalent to. Distribution of stress within the forces' area is significantly different than before. Stress distribution is almost the same at a considerable distance though.

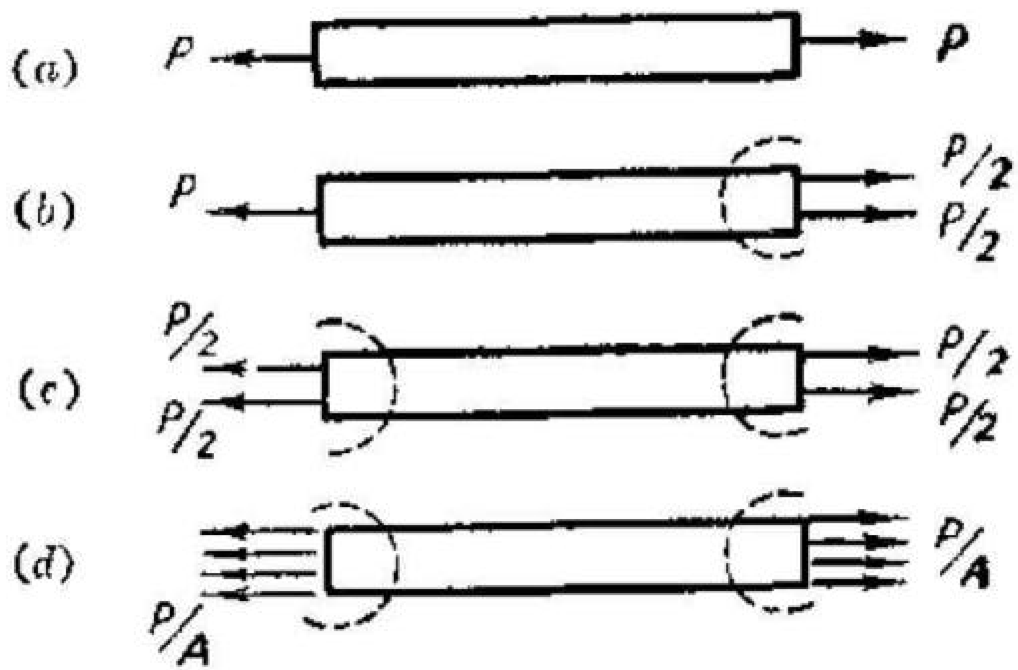

Figure 5.2. Saint-Venant's principle

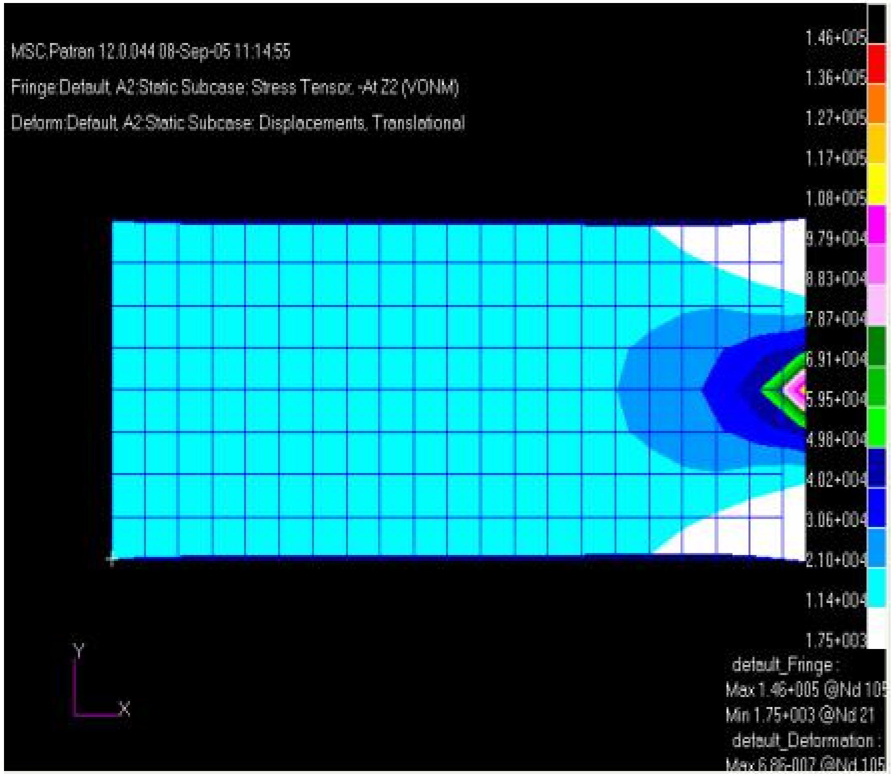

Figure 5.3 A simulation of Saint-Venant's principle

-Superposition principle

The superposition principle, also known as superposition property, states that, for all linear systems, the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually.
So that if input A produces response X and input B produces response Y then input (A + B) produces response (X + Y).
Mathematically, for all linear systems F(x) = y, where x is some sort of stimulus (input) and yis some sort of response (output), the superposition (i.e., sum) of stimuli yields a superposition of the respective responses:

$$F(x_1 + x_2) = F(x_1) + F(x_2)$$

Under hypothesis of small deformation, total effect of an object caused by several group of loads equal to sum of effects caused by each group of load individually.

3.Solve space problems according to displacement

Displacement method take displacement components u, v, and w as basic variables. Solution is achieved by expressing stress and strain with displacement. Its procedure is:

a.Utilizing constitutive equations and Cauchy formulations, stress components can be described by displacements components by eliminating strain components. Combinate Navier equations with new achieved equations, three equilibrium equations including three uncertain displacement components are formulated

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$
$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$
$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0 \qquad (5.1)$$

where $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, which is another expression of equation (3.22). Where

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as Laplace operator.

b.In the process of solving group (5.1), undetermined coefficients will appear because of integral operation. These coefficients can be computed with boundary condition.

Boundary condition (2.10) should be expressed according to displacement

$$\bar{X} = \left(\lambda e + 2G\frac{\partial u}{\partial x}\right)l + G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)m + G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)n$$

$$\bar{Y} = \left(\lambda e + 2G\frac{\partial v}{\partial y}\right)m + G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)n + G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)l$$

$$\bar{Z} = \left(\lambda e + 2G\frac{\partial w}{\partial z}\right)n + G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)l + G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)m \qquad (5.2)$$

c.After getting displacement component functions, strain components can be obtained by partial differentiating geometric equations (3.4).

d.Similarly, stress components can be gained with physical equations (4.8).

a. 位移解法

以位移作为未知数

↓

几何方程求应变

↓

物理方程求应力

平衡方程

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+X=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+Y=0$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+Z=0$$

几何方程方程

$$\varepsilon_x=\frac{\partial u}{\partial x} \qquad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\varepsilon_y=\frac{\partial v}{\partial y} \qquad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}$$

$$\varepsilon_z=\frac{\partial w}{\partial z} \qquad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

本构方程

$$\sigma_x=2G\varepsilon_x+\lambda\theta \qquad \tau_{xy}=G\gamma_{xy}$$

$$\sigma_y=2G\varepsilon_y+\lambda\theta \qquad \tau_{yz}=G\gamma_{yz}$$

$$\sigma_z=2G\varepsilon_z+\lambda\theta \qquad \tau_{zx}=G\gamma_{zx}$$

$$\theta=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

由位移表示的平衡微分方程

$$
\begin{aligned}
&(\lambda+G)\frac{\partial e}{\partial x}+G\nabla^2 u+X=0\\
&(\lambda+G)\frac{\partial e}{\partial y}+G\nabla^2 v+Y=0\\
&(\lambda+G)\frac{\partial e}{\partial z}+G\nabla^2 w+Z=0
\end{aligned}
\tag{5.1}
$$

应力边界条件：用位移表示

$$
\begin{aligned}
\bar{X}&=\left(\lambda e+2G\frac{\partial u}{\partial x}\right)l+G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)m+G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)n\\
\bar{Y}&=\left(\lambda e+2G\frac{\partial v}{\partial y}\right)m+G\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)n+G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)l\\
\bar{Z}&=\left(\lambda e+2G\frac{\partial w}{\partial z}\right)n+G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)l+G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)m
\end{aligned}
\tag{5.2}
$$

4.Solve space problems according to stress

a.Stress components could be gained through equations (2.2), (4.5) and (3.15).

b.Integrating stress compatibility equaitons (5.3) and Navier equaitons (2.2), stress component functions will be obtained with help of boundary condition.

c.Using physical equations (4.5), strain components are achieved.

d.To get displacement components, geometric equations (3.4) need to be integrated. Constraint conditions can be used to decide undetermined coefficients.

4.Solve space problems according to stress

- 满足平衡微分方程、应力协调方程和应力边界条件

$$
\begin{aligned}
&\frac{\partial^2\Theta}{\partial x^2}+(1+\mu)\nabla^2\sigma_x=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial X}{\partial x}+\mu\frac{\partial Y}{\partial y}+\mu\frac{\partial Z}{\partial z}\right]\\
&\frac{\partial^2\Theta}{\partial y^2}+(1+\mu)\nabla^2\sigma_y=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial Y}{\partial y}+\mu\frac{\partial X}{\partial x}+\mu\frac{\partial Z}{\partial z}\right]\\
&\frac{\partial^2\Theta}{\partial z^2}+(1+\mu)\nabla^2\sigma_z=-\frac{1+\mu}{1-\mu}\left[(2-\mu)\frac{\partial Z}{\partial z}+\mu\frac{\partial Y}{\partial y}+\mu\frac{\partial X}{\partial x}\right]\\
&\frac{\partial^2\Theta}{\partial y\partial x}+(1+\mu)\nabla^2\tau_{xy}=-(1+\mu)\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right)\\
&\frac{\partial^2\Theta}{\partial y\partial z}+(1+\mu)\nabla^2\tau_{yz}=-(1+\mu)\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)\\
&\frac{\partial^2\Theta}{\partial z\partial x}+(1+\mu)\nabla^2\tau_{zx}=-(1+\mu)\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)
\end{aligned}
\tag{5.3}
$$

此式为Beltrami-Michell于1899年导出的应力形式表示的协调方程。

在体力为零或常量时，早在1892年就被意大利科学家贝尔特拉密所导出：

$$
\begin{aligned}
&\frac{\partial^2 \Theta}{\partial x^2} + (1+\mu)\nabla^2 \sigma_x = 0 \\
&\frac{\partial^2 \Theta}{\partial y^2} + (1+\mu)\nabla^2 \sigma_y = 0 \\
&\frac{\partial^2 \Theta}{\partial z^2} + (1+\mu)\nabla^2 \sigma_z = 0 \\
&\frac{\partial^2 \Theta}{\partial y \partial x} + (1+\mu)\nabla^2 \tau_{xy} = 0 \\
&\frac{\partial^2 \Theta}{\partial y \partial z} + (1+\mu)\nabla^2 \tau_{yz} = 0 \\
&\frac{\partial^2 \Theta}{\partial z \partial x} + (1+\mu)\nabla^2 \tau_{zx} = 0
\end{aligned}
\tag{5.4}
$$

6 个应力分量满足平衡微分方程，满足应力相容方程，并在边界上满足应力边界方程。