

# Chapitre V

## Etude des régimes compressibles en Fluide Parfait

- Part 1 :      Introduction  
                 Description of compressible and inviscid flows  
                 Steady unidirectional compressible flows
- Part 2 :      From pressure waves to shock waves  
                 Normal and oblique shock waves  
                 Examples

*Avertissement : Ce chapitre est tiré du cours de Master donné dans le cadre du « Summer Program » associant les écoles aéronautiques du GEA. Seule la première partie sera traitée.*

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### Introduction (1)

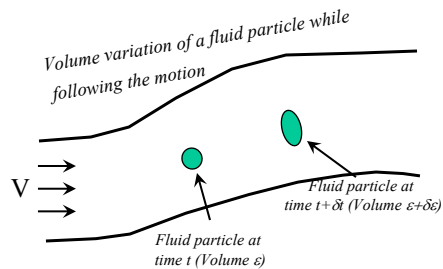
- Compressibility phenomena are associated with large velocities or large accelerations in a gas flow.
- The development of this field of fluid mechanics is linked with the evolution of aeronautics in the middle of the 20<sup>th</sup> century.
- In fact, it concerns a lot of applications :
  - ⇒ Pneumatic transport
  - ⇒ flows in Intake or exhaust ports of automotive engines
  - ⇒ .....

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→ Goals :

- ⇒ Physical description
- ⇒ Mathematical model
- ⇒ Definition and properties of stagnation quantities

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→ What are the consequences of fluid motion on the variation of the volume of an elementary fluid particle ?

→ Important parameter : Mach Number  $M = V/a$

→ First estimation (  $m = \rho \varepsilon$  constant mass of the fluid particle)

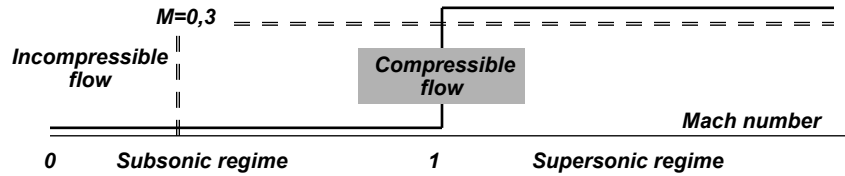
$$\bullet \left| \frac{\delta \varepsilon}{\varepsilon} \right| = \left| \frac{\delta \rho}{\rho} \right| = \frac{1}{a^2} \left| \frac{\delta p}{\rho} \right|$$

$$\bullet \text{ If } \rho \approx \text{constant} : \delta p \approx \rho \frac{V_\infty^2}{2}$$

$$\left| \frac{\delta \varepsilon}{\varepsilon} \right| \approx \frac{M_\infty^2}{2}$$

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→ Regimes of compressible flows



**Mach number :  $M=V/c$  ( $c$ : speed of sound)**

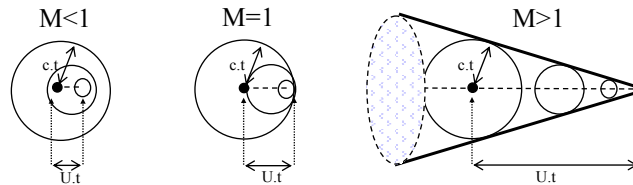
(For air in standard conditions of temperature and pressure,  
 $M=1$  corresponds to 330 m/s)

→ In some flows, for example on airfoils, both subsonic and supersonic regions can co-exist. We say that the flow regime is transsonic

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→ On both sides of Mach ONE !

→ SOURCE IN MOTION



⇒ Subsonic regime : Information arrives before the source

⇒ Supersonic regime : Information arrives after

• Mach cone of angle  $\alpha$  /  $\sin(\alpha) = 1/M$

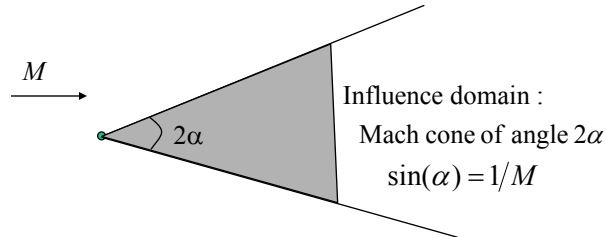
⇒ The properties of the equations of motion are changing :

\* Subsonic : System elliptic in space

\* Supersonic : System hyperbolic in space

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- On both sides of Mach ONE
- FIXED SOURCE in a Supersonic flow



- ⇒ In a supersonic flow, the fluid particle is not “informed” that there is an obstacle !!
  - This explains why we observe very sharp transitions
  - Explains the apparition of shock waves.

- Large Reynolds numbers
- Non buoyant fluid , No volumetric heat transfer
- We assume that the fluid is a perfect gaz.

\* Equation of state :  $p/\rho = rT$  where  $r = R/M$

\* Joule Laws :  $de = c_v dT$  ;  $dh = c_p dT$

$c_v$  and  $c_p$  supposed constant ;  $\gamma = c_p/c_v$

Meyer relation :  $r = c_p - c_v$

\* Entropy :  $s = c_v \log(p/\rho^\gamma)$

\* Velocity of sound :  $a^2 = \left. \frac{\partial p}{\partial \rho} \right|_s = \gamma rT = \gamma \frac{p}{\rho}$

*r is the material specific gaz constant.  
 $c_p$  and  $c_v$  are specific heats  
at constant pressure and volume*

- Tables and numerical integration must be used for more complex thermodynamics.

*Mathematical model for compressible and inviscid flows. (2)*

- Mass, Momentum and energy balance are written .
- If there are no irreversibilities and no volumetric heat flux,  
For a general unsteady flow :

$$\Rightarrow \frac{Ds}{Dt} = 0$$

| In a compressible and inviscid flow, the entropy is  
| constant along trajectories.

- BEWARE : This is only true « pieces by pieces »  
if there are shock waves.

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*Mathematical model for compressible and inviscid flows. (3)*

- Mass, Momentum and energy balance are written .
- If there are no irreversibilities and no volumetric heat flux  
Case of a permanent flow (which will be considered afterwards)

$$\Rightarrow \vec{V} \cdot \vec{grad}(s) = 0$$

$$\vec{V} \cdot \vec{grad}(h_i) = 0 \quad \text{where} \quad h_i = h + \frac{V^2}{2}$$

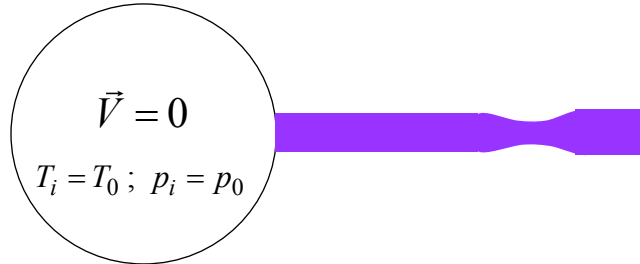
| In a permanent compressible and inviscid flow the entropy  
| and the stagnation enthalpy are constant along streamlines.

- BEWARE : This is again only true « pieces by pieces »  
if there are shock waves.

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Permanent regime: practical definitions of the stagnation quantities (1).

→ Practical point of view : In a tank :  $h_i = c_p T_i = c_p T_0$



→ Following theoretical results, along streamlines :

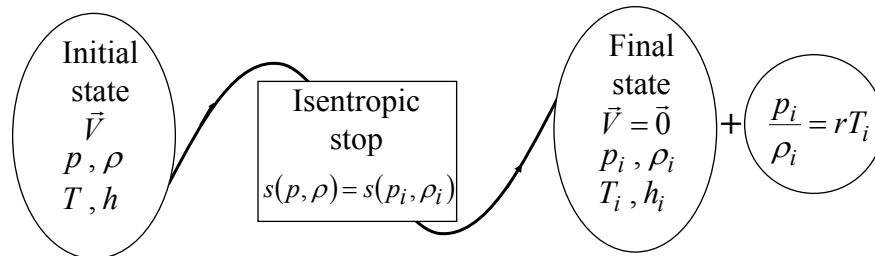
$$* h_i = h + \frac{V^2}{2} \text{ or } T_i = T + \frac{V^2}{2c_p} \text{ are constant quantities}$$

→ If then the properties of the gaz are uniform in the tank :

⇒  $h_i$  et  $T_i$  are constant EVERYWHERE.

→ Note that :  $h_i = \frac{a^2}{(\gamma-1)} + \frac{V^2}{2}$  and  $\frac{T_i}{T} = \left(1 + \frac{(\gamma-1)}{2} M^2\right)$  229

Permanent regime: practical definitions of the stagnation quantities (2).



→ Following theoretical results, along streamlines :

\*  $h_i(p_i, \rho_i) = c_p T_i$  AND  $s(p_i, \rho_i)$  are independant constant quantities.

⇒  $T_i$  ;  $p_i$  et  $\rho_i$  are constant quantities along streamlines

→ If then the properties of the gaz are uniform in the tank :

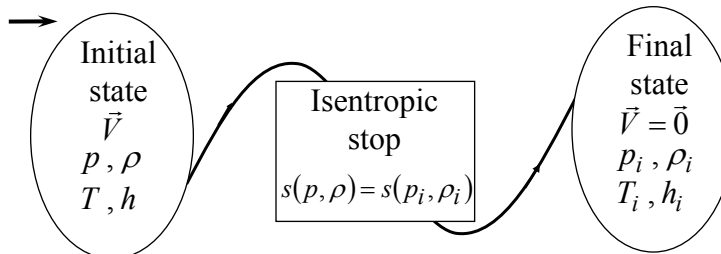
⇒  $T_i$  ;  $p_i$  et  $\rho_i$  are constant EVERYWHERE.

(Not valid if there are shock waves !!)

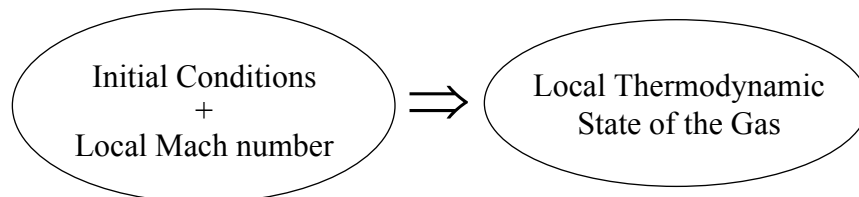
→ At a given Mach number  $M$ , the ratio between a quantity and the corresponding stagnation quantity are given by :

SAINT-VENANT RELATIONS

$$\begin{aligned} \frac{T}{T_i} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} & \frac{p}{p_i} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\gamma/(\gamma-1)} \\ \frac{\rho}{\rho_i} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1/(\gamma-1)} & \frac{a}{a_i} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1/2} \end{aligned}$$



→ In a permanent compressible and inviscid flow  
 If the boundary conditions are UNIFORM, and if there are NO shock waves :

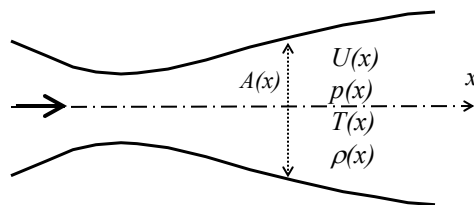


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Permanent compressible and inviscid  
Monodimensional flow  
-  
Laval Nozzle Flow

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*Permanent compressible and inviscid monodimensional flow: Laval nozzle flow (1)*

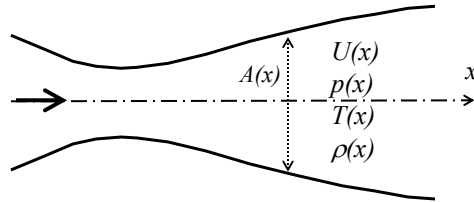


- \* Quasi-monodimensional flow along the x coordinate
- \* Slow variations of  $A(x)$  with  $dA/A \ll 1$
- \* Weak curvatures  $A/R^2 \ll 1$
- \* Uniform boundary conditions

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*Permanent compressible and inviscid monodimensional flow: Laval nozzle flow (2)*



\* Continuity :  $\rho U A = cste$

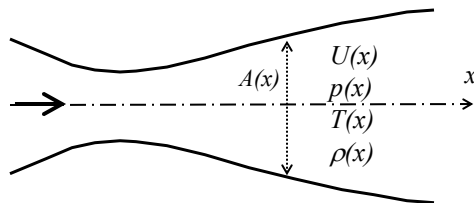
\* Momentum :  $\rho U \frac{dU}{dx} = -\frac{dp}{dx}$

\* Enthalpy :  $h_i = cste$

\* Entropy :  $s = cste \Rightarrow \frac{p}{\rho^\gamma} = cste$

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*Permanent compressible and inviscid monodimensional flow: Laval nozzle flow (3)*



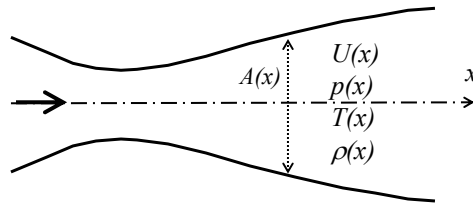
\* Continuity :  $\frac{d\rho}{\rho} + \frac{dU}{U} + \frac{dA}{A} = 0$

\* Momentum :  $\rho U \frac{dU}{dx} + \frac{dp}{dx} = 0$

\* Enthalpy :  $h_i = cste$

\* Entropy :  $\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \Rightarrow dp = a^2 d\rho$

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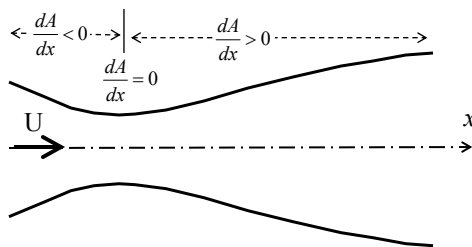


Hugoniot  
Theorem

\* Reducing variables : 
$$(M^2 - 1) \frac{dU}{U} = \frac{dA}{A}$$

- \* In a SUBSONIC FLOW, When section increases,  
The velocity decreases (and vice versa)
- \* In a SUPERSONIC FLOW, When section increases,  
The velocity increases (and vice versa)

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At the throat ??

$$(M^2 - 1) \frac{dU}{U} = \frac{dA}{A}$$

- \* If  $M=1$  , then  $dA=0$

If an isentropic monodimensional flow is sonic,  
then we are at a minimum of the cross-section.

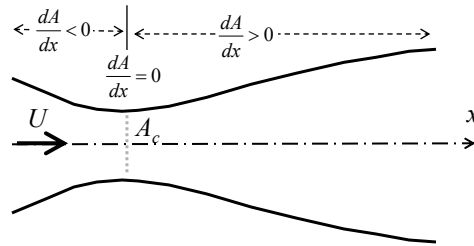
- \* Conversely, if  $dA=0$  :

$\Rightarrow$  Either  $dU=0$

$\Rightarrow$  Either  $M=1$ . In this case, we are at a MINIMUM of the section

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### Laval nozzle flow(6)



$$M = M_c = 1$$

$$\text{at } A = A_c$$

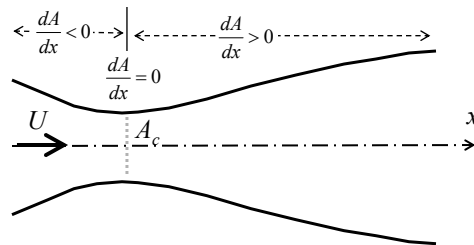
\* With a sonic flow at the throat, we obtain for the mass flux :

$$\Rightarrow Q_m = \sqrt{\frac{\gamma}{r}} \left( \frac{\gamma+1}{2} \right)^{-\frac{\gamma+1}{2(\gamma-1)}} \frac{p_i}{\sqrt{T_i}} A_c = (4.04 \cdot 10^{-2}) \frac{A_c p_i}{\sqrt{T_i}} \quad (\gamma = 1.4)$$

\* This relation has a lot of practical applications if one wants to regulate a mass flux just by controlling the initial stagnation pressure  $p_i$

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### Laval nozzle flow (7)



$$(M^2 - 1) \frac{dU}{U} = \frac{dA}{A}$$

\* By definition, the throat conditions are defined by :

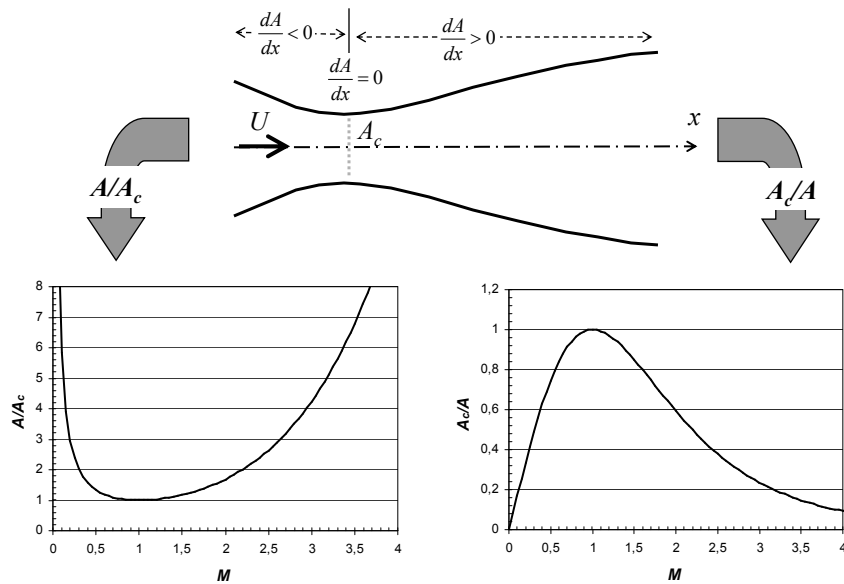
$$M = M_c = 1 \text{ at } A = A_c \quad (\text{This section can be virtual !!})$$

\* An important theoretical link between  $A/A_c$  and the local Mach number is :

$$\Rightarrow \frac{A}{A_c} = \frac{1}{M} \left[ \frac{2}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} M^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

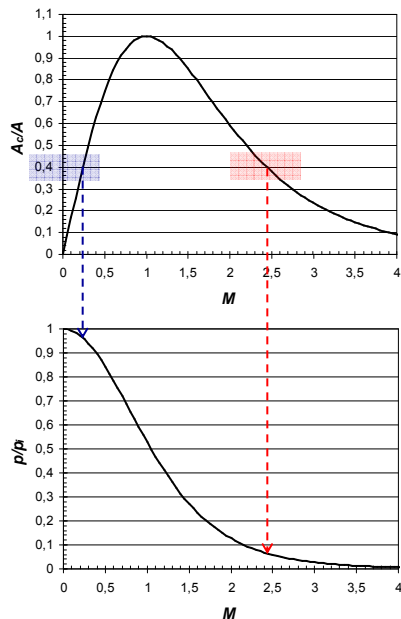
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### Laval nozzle flow (7)



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### Laval nozzle flow (8)



#### \* Applications :

- Atmospheric conditions at entrance
- Varying outlet pressure
- Outlet section :  $A_o/A_c = 2.5$

\* Subsonic flow and  $M=1$  at the minimum section.  $M_o, p_o$  ???

$$\Rightarrow M_o = 0.24 \text{ and } p_o/p_i = 0.96$$

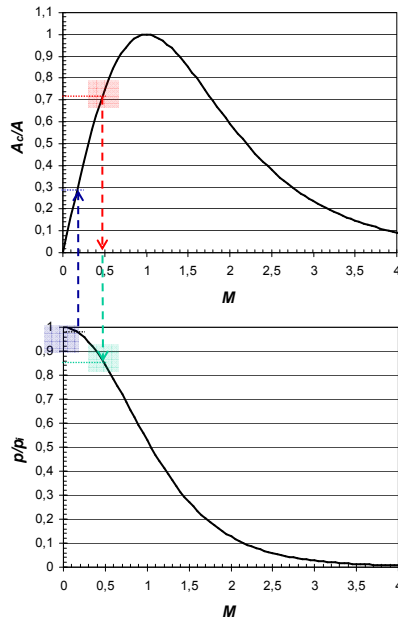
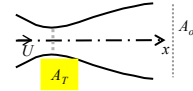
\* Supersonic flow and  $M=1$  at the minimum section.  $M_o, p_o$  ???

$$\Rightarrow M_o = 2.44 \text{ and } p_o/p_i = 0.064$$

AND FOR OTHER  
VALUES OF  $p_o/p_i$  ???

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### Laval nozzle flow (9)



#### \* Conditions :

- Atmospheric conditions at entrance
- Varying outlet pressure
- Outlet section :  $A_o/A_T = 2.5$

\*  $p_o/p_i = 0.98$

value of  $M_o, M_s$  ???

$\Rightarrow M_o = 0.17$

$\Rightarrow A_c/A_o = 0.29$  \*  $A_o/A_T = 2.5$

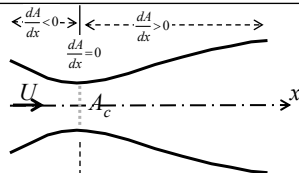
$\Rightarrow A_c/A_T = 0.72$

$\Rightarrow M_T = 0.48$

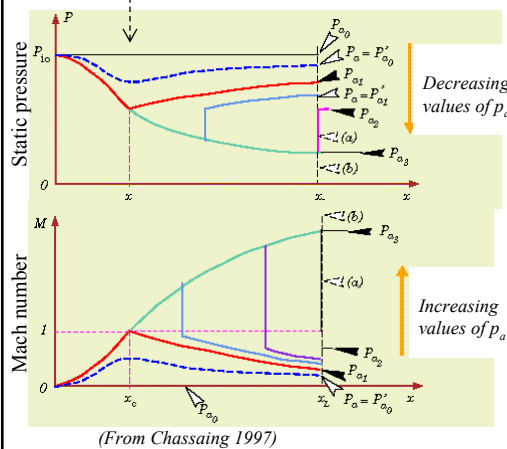
$\Rightarrow p_T/p_i = 0.854$

The section "C" is here virtual.

### Laval nozzle flow (10)



Hypothesis : Fixed upstream stagnation pressure  
 $p_o$  : Decreasing static pressure at outlet



\* For :  $p_{a1} < p_o$

- Isentropic subsonic flow

\* For :  $p_s = p_{a1}$

- Isentropic flow

$M=1$  at the throat

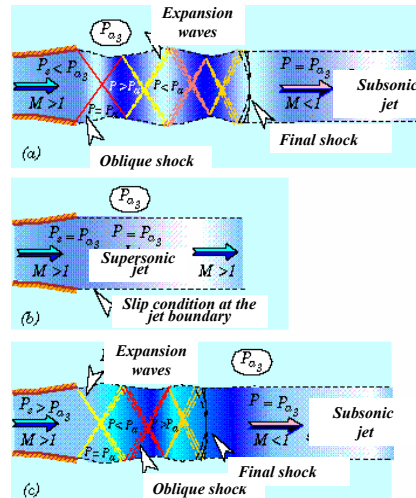
\* For :  $p_{a2} < p_s < p_{a1}$

- NON isentropic flow

- Formation of straight shock waves in the diverging section

(From Chassaing 1997)

### Laval nozzle flow (11)



(From Chassaing 1997)

\* For :  $p_{a3} < p_s < p_{a2}$   
- Compression at the outlet by oblique shock waves

\* For :  $p_s = p_{a3}$   
- Isentropic Supersonic flow

\* For :  $p_s < p_{a3}$   
- Expansion wave at the outlet reflecting on the boundary of the jet ( $p$  is a constant on this surface)

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### Synthesis of this first part

- Presentation of the mathematical model for compressible inviscid flows
- Presentation of physical properties of monodimensional and isentropic compressible inviscid flows
- The appearance of shock waves has been evidenced
- In the next part :
  - \* We explain what is a shock wave
  - \* We give some examples

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→ Sound wave : A very small perturbation of the pressure field

Examples of levels acceptable for a human ear ( $p_0 = 1 \text{ atm}$ ):

$$* p'/p_0 \in [10^{-9}, 10^{-5}]$$

\* Corresponds to  $u'/a_0$  et  $\rho'/\rho_0$  of the same order

→ Shock wave: finite amplitude jump

\* Irreversibility

\* The sound wave is a limiting case for asymptotically small pressure jumps.

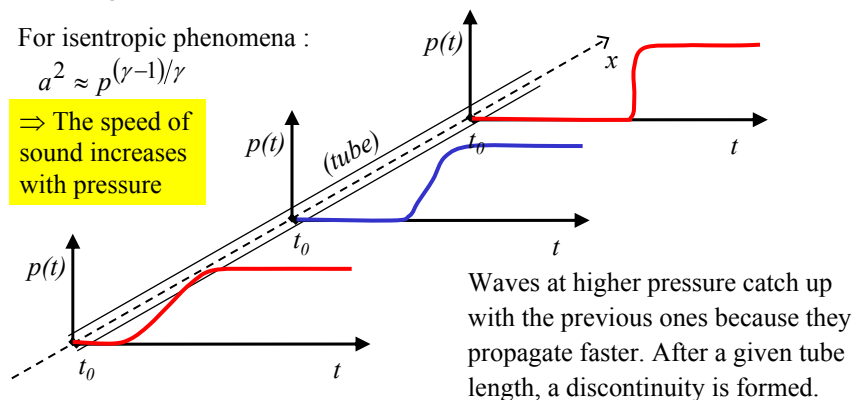
→ Shock waves are discontinuities. Why ?

Let's imagine that we increase the pressure level at a given location in a tube.

For isentropic phenomena :

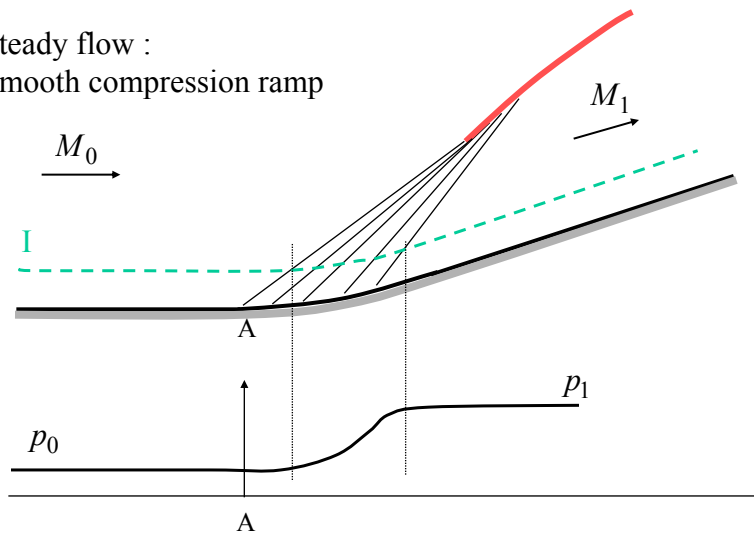
$$a^2 \approx p^{(\gamma-1)/\gamma}$$

⇒ The speed of sound increases with pressure



→ On the contrary, a smoothing of the pressure evolution occurs during an expansion.

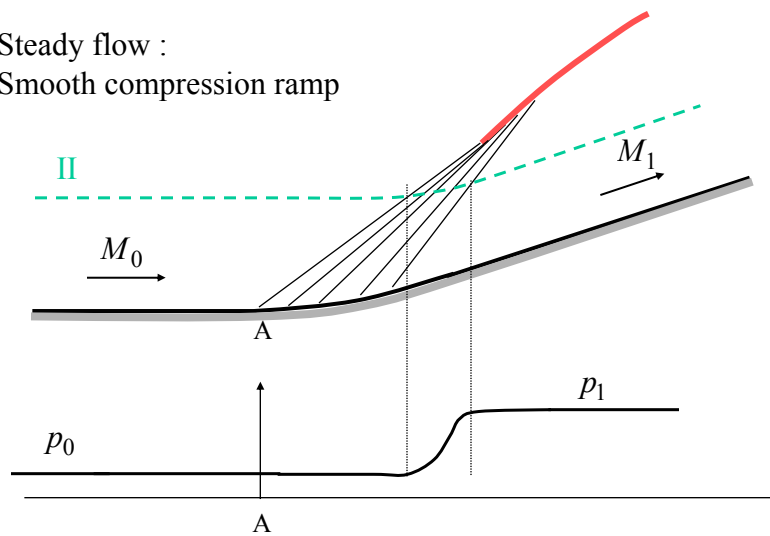
→ Steady flow :  
Smooth compression ramp



→ Evolution of the pressure along trajectory I

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→ Steady flow :  
Smooth compression ramp

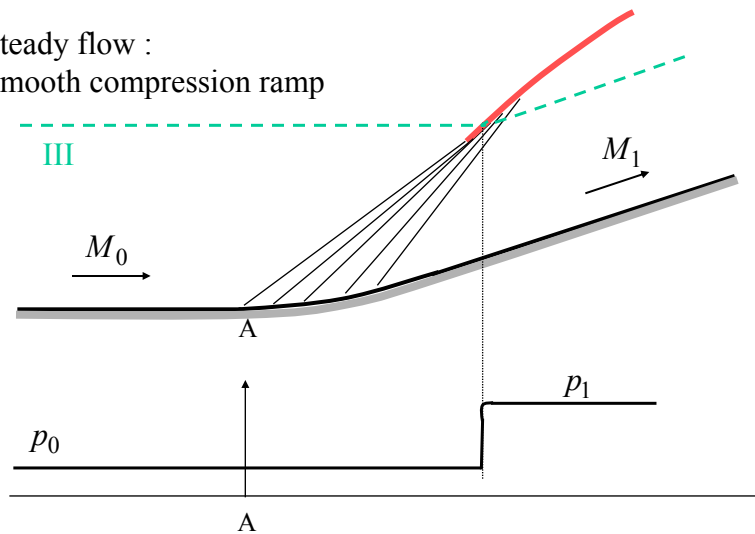


→ Evolution of the pressure along trajectory II

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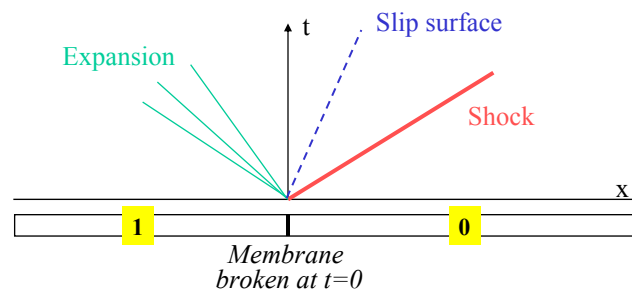
→ Steady flow :  
Smooth compression ramp



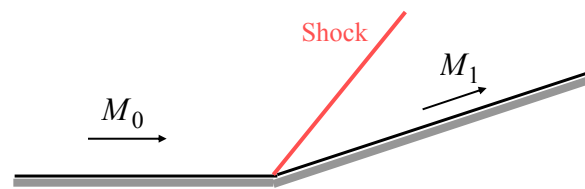
→ Evolution of the pressure along trajectory III

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→ Unsteady regime : Shock tube  $p_1 > p_0$

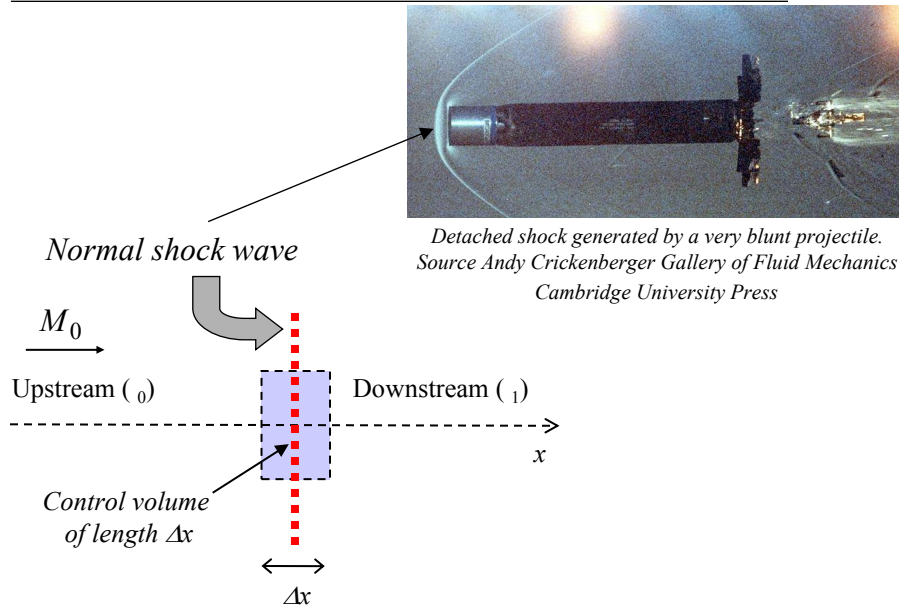


→ Steady regime : Sudden deviation of the wall



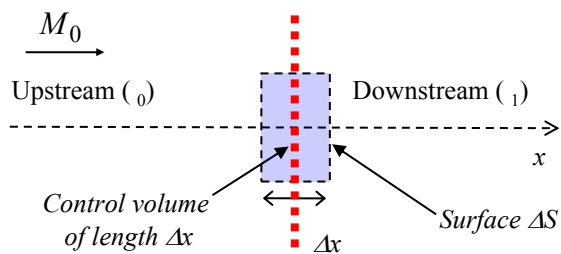
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## Normal shock waves



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## Normal shock waves

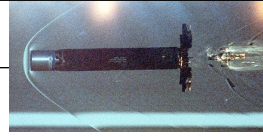


\* If  $\Delta x \rightarrow 0$ , balances are used to link fluxes through  $\Delta S$

- Mass	: $\rho_0 U_0 = \rho_1 U_1$
- Momentum	: $p_0 + \rho_0 U_0^2 = p_1 + \rho_1 U_1^2$
- Enthalpy	: $h_0 + \frac{U_0^2}{2} = h_1 + \frac{U_1^2}{2}$

• In steady flows :  $h_i$  (and then  $T_i$ ) are constant across the shock wave. *Natural consequence of first principle in thermodynamics.*

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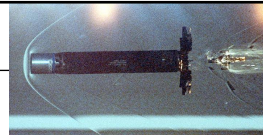
\* Static pressure and density jumps are only dependant on the upstream Mach number : *(After a tedious work with equations !)*

$$\frac{p_1}{p_0} = 1 + \frac{2\gamma}{\gamma+1} (M_0^2 - 1) \quad \text{and} \quad \frac{\rho_1}{\rho_0} = \frac{(\gamma+1)M_0^2}{(\gamma-1)M_0^2 + 2}$$

\* After eliminating  $M_0$ , we obtain an important relation due to Hugoniot :

$$\frac{\rho_1}{\rho_0} = \frac{1 + \frac{\gamma+1}{\gamma-1} \frac{p_1}{p_0}}{\frac{\gamma+1}{\gamma-1} + \frac{p_1}{p_0}} \quad \text{or} \quad \left( \frac{\rho_1}{\rho_0} - 1 \right) = \frac{2}{\gamma-1} \frac{\left( \frac{p_1}{p_0} - 1 \right)}{\frac{\gamma+1}{\gamma-1} + \frac{p_1}{p_0}}$$

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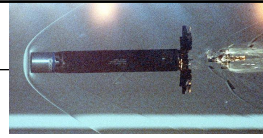
\* What means Hugoniot relation for  $p_1/p_0 \approx 1$  ???

$$\begin{aligned} p_1/p_0 = 1 + \varepsilon \quad (\varepsilon \ll 1) &\Rightarrow \left( \frac{\rho_1}{\rho_0} - 1 \right) \approx \frac{2}{\gamma-1} \frac{(\varepsilon)}{\frac{\gamma+1}{\gamma-1} + 1} \approx \frac{1}{\gamma} \left( \frac{p_1}{p_0} - 1 \right) \\ &\Rightarrow \text{Log} \left( \frac{\rho_1}{\rho_0} \right) \approx \frac{1}{\gamma} \text{Log} \left( \frac{p_1}{p_0} \right) \quad \text{ou} \quad \left( \frac{p_1}{\rho_1^\gamma} \right) \approx \left( \frac{p_0}{\rho_0^\gamma} \right) \end{aligned}$$

\* For an asymptotically weak shock wave, we thus tend toward an isentropic evolution

\* In the general case, the evolution through the shock wave **IS NOT** isentropic. Irreversibility occurs.

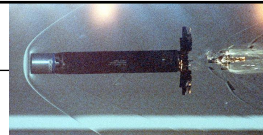
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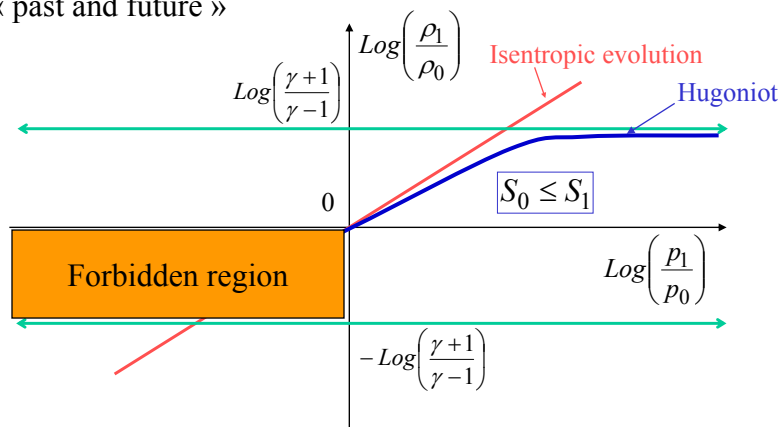
\* Second principle of thermodynamics

$$S_0 \leq S_1 \Leftrightarrow \text{Log}\left(\frac{p_0}{\rho_0^\gamma}\right) \leq \text{Log}\left(\frac{p_1}{\rho_1^\gamma}\right) \Leftrightarrow \text{Log}\left(\frac{\rho_1}{\rho_0}\right) \leq \frac{1}{\gamma} \text{Log}\left(\frac{p_1}{p_0}\right)$$

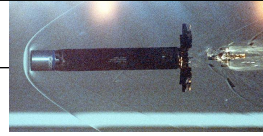
\* Knowing that  $p_1/p_0 \in [0, +\infty[$ , we just compare the evolution of the pressure obtained by Hugoniot to an isentropic evolution.



\* The second principle enables to distinguish « past and future »



\* Thus  $S_0 \leq S_1 \Rightarrow \underline{p_1/p_0 \geq 1 \text{ et } \rho_1/\rho_0 \geq 1}$



\* Evolution of the Mach number :

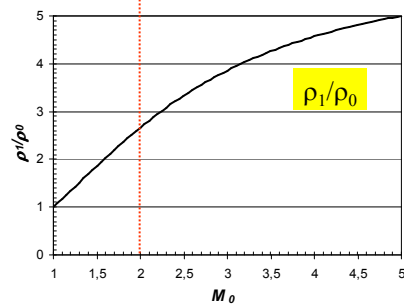
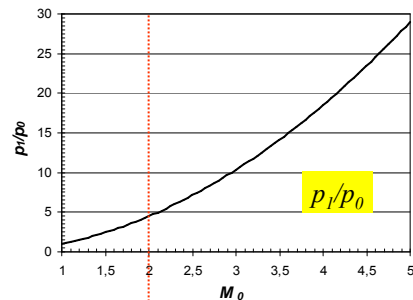
$$\text{Upstream : } \frac{p_1}{p_0} - 1 = \frac{2\gamma}{\gamma+1} (M_0^2 - 1) \Rightarrow \underline{M_0^2 \geq 1}$$

$$\text{Downstream : } M_1^2 - 1 = (\gamma+1) \frac{(1 - M_0^2)}{2\gamma M_0^2 - (\gamma-1)} \Rightarrow \underline{M_1^2 \leq 1}$$

\* If a normal shock wave is found in a steady flow, then :

- The upstream Mach number is greater than one (the flow has to be supersonic !)
- The downstream Mach number is lower than one.

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• The pressure and density jumps across a normal shock wave are large.

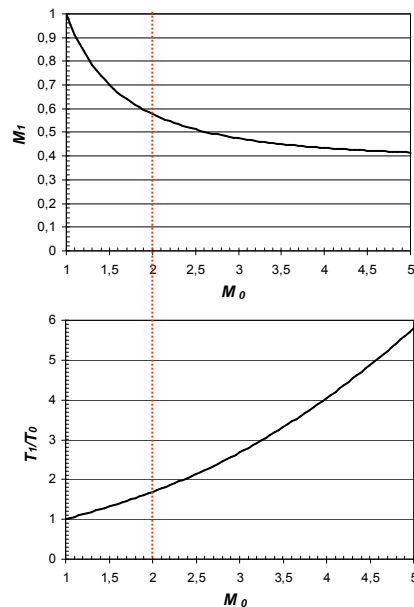
• For example :

-  $M_0=2$  implies  $p_1/p_0 = 4.5$  and  $\rho_1/\rho_0 = 2.67$

- As a comparison, we remind that a sound wave of rms fluctuating pressure level of 2 Pa ( $2 \cdot 10^{-5}$  bar) corresponds to 100 dB and that pain occurs at 120 dB (20 Pa rms)

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### Normal shock waves



- Decrease of the Mach number and increase of the static temperature.

- For example :

$$-M_o=2 \text{ implies } T_i/T_o = 1.69 \text{ and } M_i=0.58$$

What about stagnation quantities ??

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### Normal shock waves

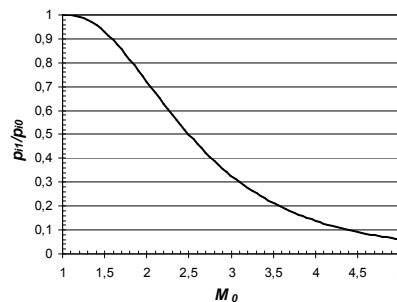
- \* For steady flow :  $h_i$  (and thus  $T_i$ ) remain constant across the shock.

- \* After a little algebra, one finds :  $S_1 - S_0 = c_v(1-\gamma) \text{Log} \left( \frac{p_{i1}}{p_{i0}} \right)$

$$\Rightarrow \frac{p_{i1}}{p_{i0}} \leq 1 \text{ et } \frac{\rho_{i1}}{\rho_{i0}} \leq 1$$

- \* The stagnation pressure decreases across the shock

- \* The shock induces logically a loss due to dissipative phenomena.



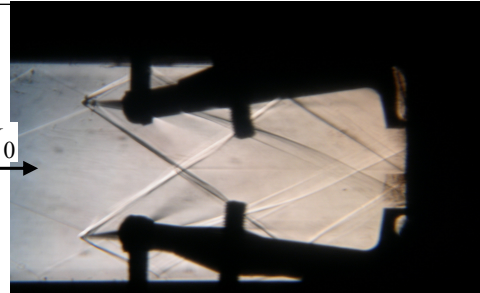
Example :  $M_o=2$  implies  $p_{i1}/p_{i0} = 0.72$

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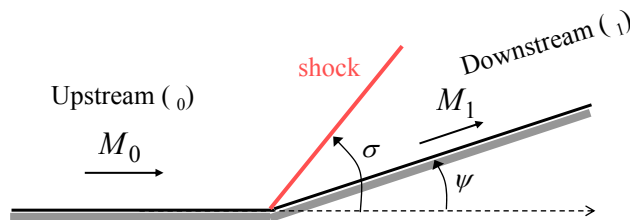
## Oblique shock waves

### Oblique shock wave

$M_0$

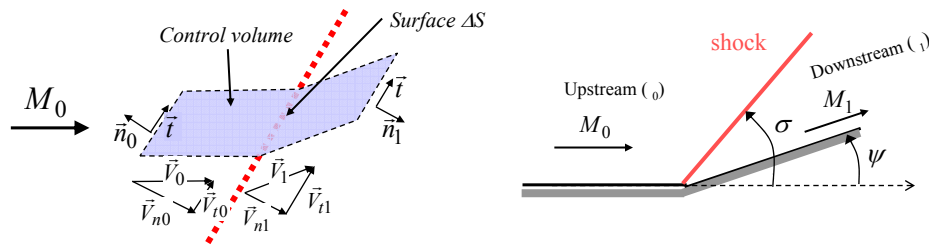


Oblique shock waves generated by two "knives" arranged in a symmetrical way in a supersonic wind tunnel at ENSMA. Courtesy of E. Collin.



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## Oblique shock waves



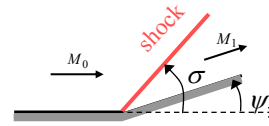
### \* Flux relations across $\Delta S$

- Mass :  $\rho_0 U_{n0} = \rho_1 U_{n1}$
- Momentum /  $\vec{n}$  :  $p_0 + \rho_0 U_{n0}^2 = p_1 + \rho_1 U_{n1}^2$
- Momentum /  $\vec{t}$  :  $U_{t0} = U_{t1}$
- Enthalpy:  $h_0 + \frac{1}{2}(U_{t0}^2 + U_{n0}^2) = h_1 + \frac{1}{2}(U_{t1}^2 + U_{n1}^2)$

\* In steady flow,  $h_i$  (and  $T_i$ ) are constant across the shock wave

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### Oblique shock waves



\* By eliminating  $U_{t0} = U_{t1}$  in the enthalpy equation :

$$\begin{aligned} \text{- Mass :} & \quad \rho_0 U_{n0} = \rho_1 U_{n1} \\ \text{- Momentum / } \vec{n} : & \quad p_0 + \rho_0 U_{n0}^2 = p_1 + \rho_1 U_{n1}^2 \\ \text{- Enthalpy :} & \quad h_0 + \frac{1}{2} U_{n0}^2 = h_1 + \frac{1}{2} U_{n1}^2 \end{aligned}$$

\* These are STRICTLY the same equations than the one found for normal shock waves.

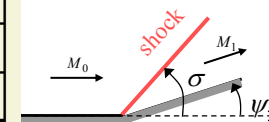
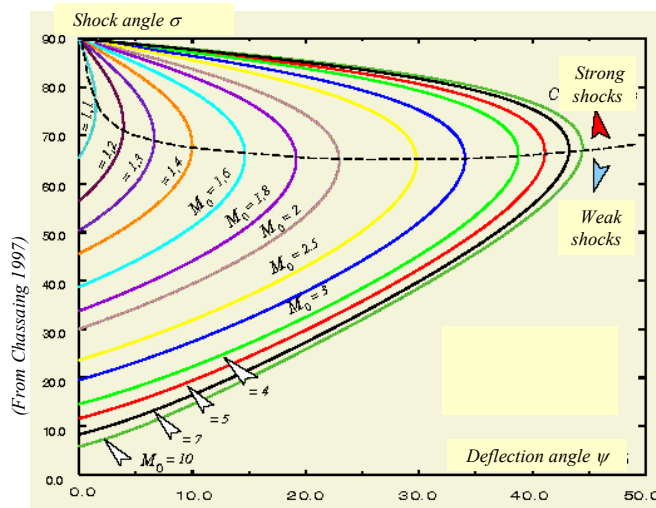
\* Jump relations are therefore EXACTLY THE SAME if written here as a function of  $M_{n0} = U_{n0}/a_0$  (see slides 43 and 44)

\* In particular, the steady oblique shock wave exists if :  
 $M_{n0} = M_0 \sin(\varepsilon) > 1$

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### Oblique shock waves : about angles

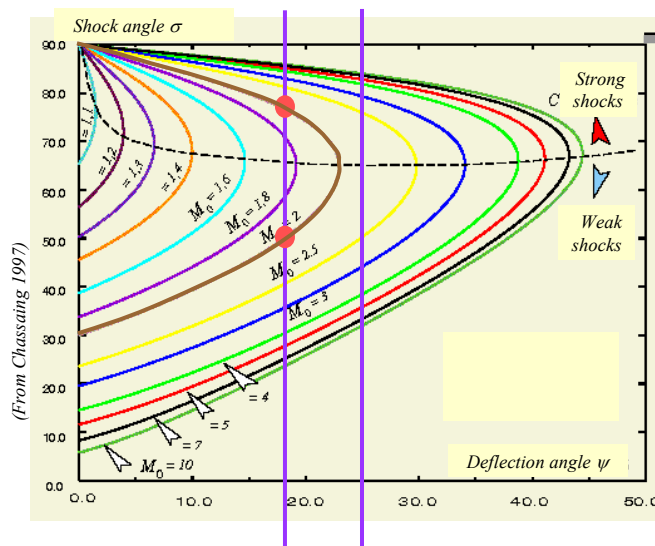
$$\text{* Mass balance + jump relation } \rho_1/\rho_0 \Rightarrow \frac{\tan(\sigma - \psi)}{\tan(\sigma)} = \frac{2}{(\gamma + 1) M_0^2 \sin^2(\sigma)} + \frac{(\gamma - 1)}{(\gamma + 1)}$$



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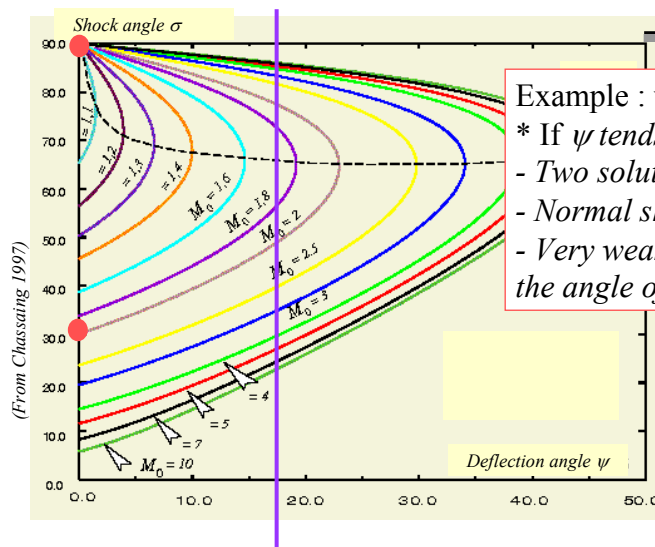


### Oblique shock waves : about angles



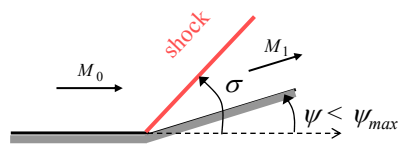
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### Oblique shock waves : about angles

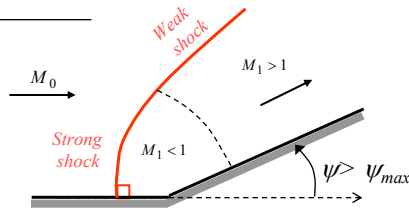


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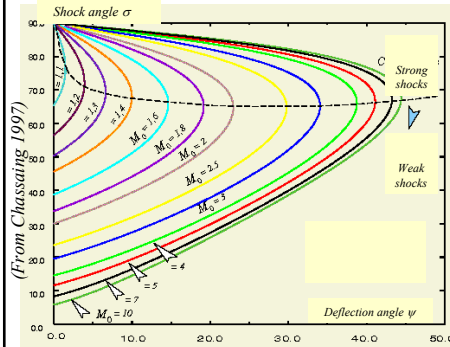
### Oblique shock waves : about angles



Attached shock wave :  
analytical solution



Detached shock wave :  
NO analytical solution



Detached shock waves are complex.

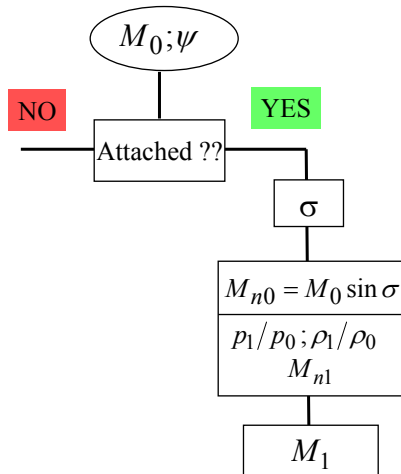
- The shock has to be normal along the wall (no deviation of the stream line)
- Therefore, there is a subsonic region and information can move upstream the corner.
- The shock is curved and the downstream region is NOT uniform.

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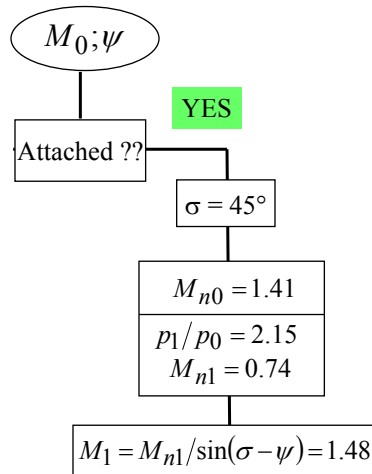
### Oblique shock waves : An example of computation

\* Algorithm :

If  $M_0$  and  $\psi$  are known :



\* Example :  $M_0 = 2$  and  $\psi = 15^\circ$ :

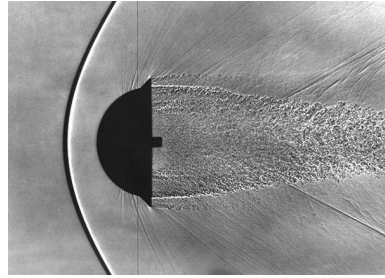


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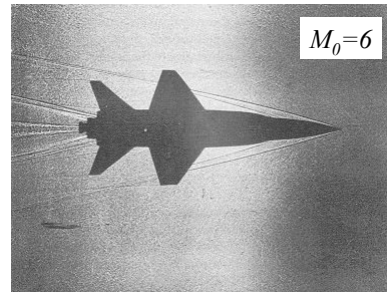
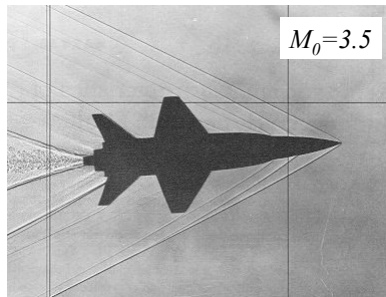
*Shock waves : Some examples*

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**Blunt Body  
Shock Waves**  
(Source NASA)



\* Free supersonic flight of an X15 model in the 50's and 60's !



*Conclusions*

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- Shock waves result from the progressive or rapid focussing of pressure waves.
- Shock waves are associated with large jumps in pressure, density, ... and are dissipative phenomena..
- Normal or oblique shocks can be simply computed using tables.
- Detached shock waves are complex. Numerical simulations are needed.
- These basic informations can be completed by the study of :
  - \* Expansion and compression waves using the method of characteristics for 2D or axisymmetric steady supersonic flows
  - \* Monodimensional unsteady compressible flows
  - \* ...