Structural Applications of Finite Elements

Chapter 2

Generalization of Finite Element Concepts

2018-09-01

Outline

- **Principle of minimum potential energy**
- **Rayleigh-Ritz method**
- **Galerkin's method**
- **Structural element and structural system**

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

$$\Pi = \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV - \int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV - \int_S \mathbf{u}^{\mathrm{T}}\mathbf{T}\, dS - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

Example

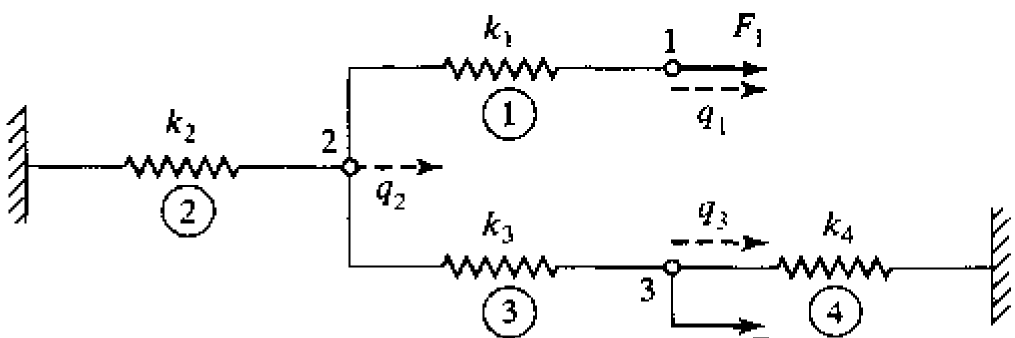
$$\Pi = \tfrac{1}{2}k_1\delta_1^2 + \tfrac{1}{2}k_2\delta_2^2 + \tfrac{1}{2}k_3\delta_3^2 + \tfrac{1}{2}k_4\delta_4^2 - F_1q_1 - F_3q_3$$

where δ_1, δ_2, δ_3, and δ_4 are extensions of the four springs. Since $\delta_1 = q_1 - q_2$, $\delta_2 = q_2$, $\delta_3 = q_3 - q_2$, and $\delta_4 = -q_3$, we have

$$\Pi = \tfrac{1}{2}k_1(q_1 - q_2)^2 + \tfrac{1}{2}k_2q_2^2 + \tfrac{1}{2}k_3(q_3 - q_2)^2 + \tfrac{1}{2}k_4q_3^2 - F_1q_1 - F_3q_3$$

where q_1, q_2, and q_3 are the displacements of nodes 1, 2, and 3, respectively.

$$\frac{\partial \Pi}{\partial q_i} = 0 \qquad i = 1, 2, 3$$



(a)

$$\frac{\partial \Pi}{\partial q_1} = k_1(q_1 - q_2) - F_1 = 0$$

$$\frac{\partial \Pi}{\partial q_2} = -k_1(q_1 - q_2) + k_2q_2 - k_3(q_3 - q_2) = 0$$

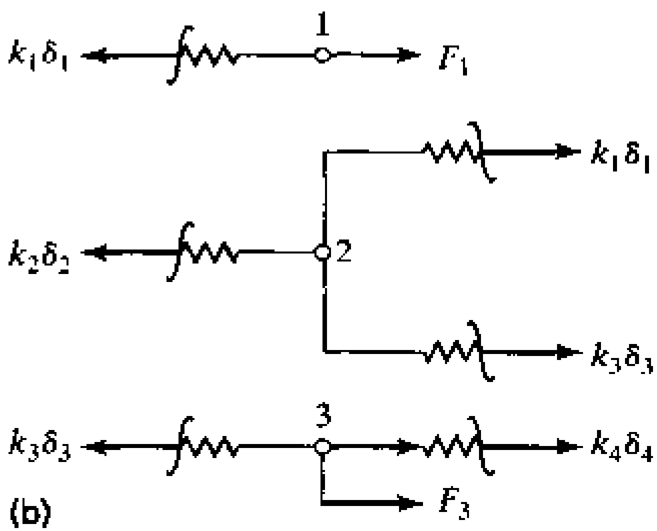
$$\frac{\partial \Pi}{\partial q_3} = k_3(q_3 - q_2) + k_4q_3 - F_3 = 0$$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

$$k_1\delta_1 = F_1$$
$$k_2\delta_2 - k_1\delta_1 - k_3\delta_3 = 0$$
$$k_3\delta_3 - k_4\delta_4 = F_3$$

$\delta_1 = q_1 - q_2$, $\delta_2 = q_2$,

$\delta_3 = q_3 - q_2$, and $\delta_4 = -q_3$



(b)

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

Rayleigh-Ritz method

Boundary conditions

Stress-strain relation

Strain-displacement relation

$$u = \sum a_i \phi_i(x, y, z) \qquad i = 1 \text{ to } \ell$$

$$v = \sum a_j \phi_j(x, y, z) \qquad j = \ell + 1 \text{ to } m$$

$$w = \sum a_k \phi_k(x, y, z) \qquad k = m + 1 \text{ to } n$$

$$n > m > \ell$$

$$\Pi = \frac{1}{2} \int_V \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon} \, dV - \int_V \mathbf{u}^{\mathrm{T}} \mathbf{f} \, dV - \int_S \mathbf{u}^{\mathrm{T}} \mathbf{T} \, dS - \sum_i \mathbf{u}_i^{\mathrm{T}} \mathbf{P}_i$$

$$\Pi = \Pi(a_1, a_2, \ldots, a_r)$$

$$\frac{\partial \Pi}{\partial a_i} = 0 \qquad i = 1, 2, \ldots, r$$

Example

The potential energy for the linear elastic one-dimensional rod

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

where $u_1 = u(x = 1)$.

Let us consider a polynomial function

$$u = a_1 + a_2 x + a_3 x^2$$

This must satisfy $u = 0$ at $x = 0$ and $u = 0$ at $x = 2$. Thus,

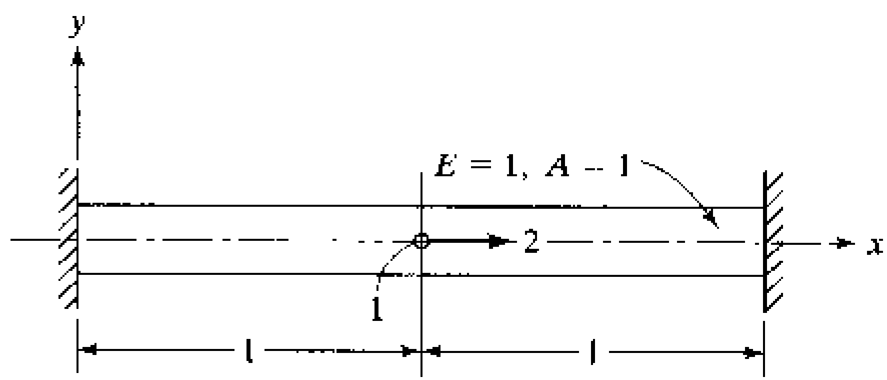
$$0 = a_1$$

$$0 = a_1 + 2a_2 + 4a_3$$

Hence,

$$a_2 = -2a_3$$

$$u = a_3(-2x + x^2) \qquad u_1 = -a_3$$



Then $du/dx = 2a_3(-1 + x)$ and

$$\Pi = \frac{1}{2}\int_0^2 4a_3^2(-1 + x)^2\, dx - 2(-a_3)$$

$$= 2a_3^2 \int_0^2 (1 - 2x + x^2)\, dx + 2a_3$$

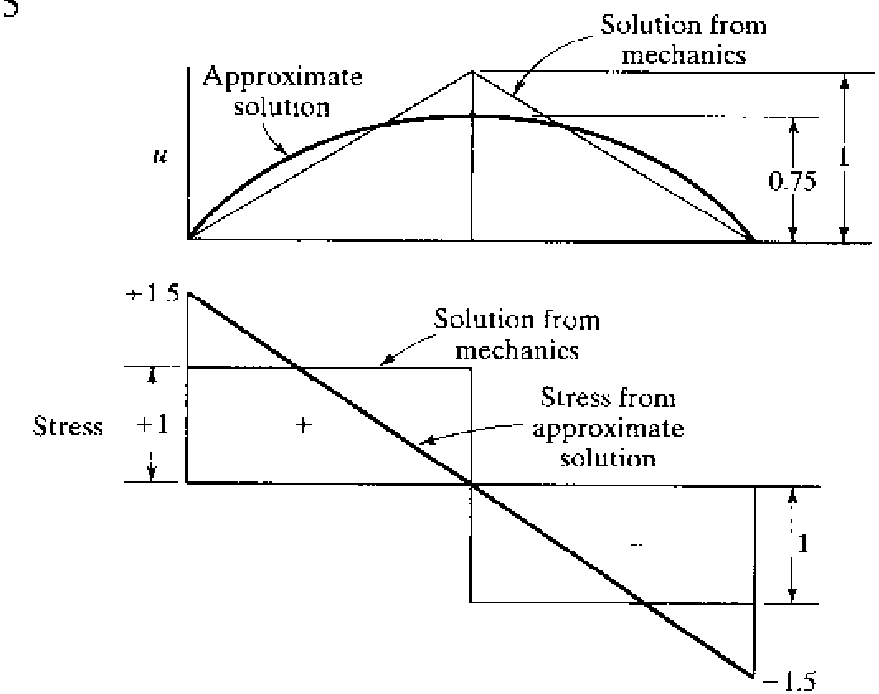
$$= 2a_3^2\left(\tfrac{2}{3}\right) + 2a_3$$

We set $\partial \Pi/\partial a_3 = 4a_3\left(\frac{2}{3}\right) + 2 = 0$, resulting in

$$a_3 = -0.75 \qquad u_1 = -a_3 = 0.75$$

The stress in the bar is given by

$$\sigma = E\frac{du}{dx} = 1.5(1 - x)$$



Weighted residual method

Residual of motion equation and boundary conditions

$$R_i = \sigma_{ij,j} + \bar{f}_i - \rho \ddot{u}_i \neq 0 \qquad \text{在}V\text{中}$$

$$\bar{R}_i = \sigma_{ij} n_j - \bar{T}_i \neq 0 \qquad \text{在}S_\sigma\text{上}$$

Equivalent weak form

$$\int_V R_i v_i \mathrm{d}V + \int_{S_\sigma} \bar{R}_i \bar{v}_i \mathrm{d}S = 0$$

v_i $\bar{v}_i$ **are the test functions**

Approximate solution

$$u_i = \sum_{I=1}^{N} \phi_I a_{iI}$$

ϕ_I **is the trial functions**

$$v_i = \sum_{I=1}^{N} W_I b_{iI}$$

$$\int_V R_i W_I \mathrm{d}V = 0 \quad i = 1,2,3;\ I = 1,2,\cdots,N$$

$$\int_V R_i W_I \mathrm{d}V = 0 \quad i = 1,2,3;\ I = 1,2,\cdots,N$$

Collocation method

$$W_I = \delta(\boldsymbol{x} - \boldsymbol{x}_I) \quad I = 1,2,\cdots,N$$

$$R_i(\boldsymbol{x}_I) = 0 \quad i = 1,2,3;\ I = 1,2,\cdots,N$$

Subdomain method

$$W_I = \begin{cases} 1 & \boldsymbol{x} \in V_I \\ 0 & \boldsymbol{x} \notin V_I \end{cases} \qquad I = 1,2,\cdots,N$$

Least square method

$$\frac{\partial}{\partial a_{iI}} \int_V R_i^2 \mathrm{d}V = 2\int_V R_i \frac{\partial R_i}{\partial a_{iI}} \mathrm{d}V = 0$$

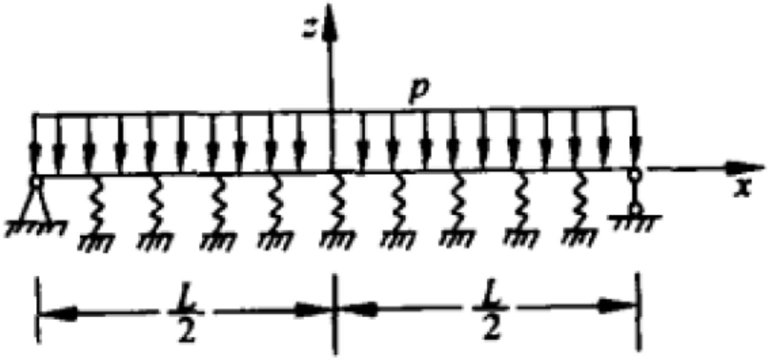
$$W_I = \frac{\partial R_i}{\partial a_{iI}} \quad I = 1,2,\cdots,N$$

Galerkin's method

$$\int_V R_i \phi_I \mathrm{d}V = 0 \quad i = 1,2,3;\ I = 1,2,\cdots,N$$

Example

$$\begin{cases} \dfrac{d^4 w}{dx^4} + \alpha w + 1 = 0 \quad -1 \le x \le 1 \\ w(-1) = 0 \\ w(1) = 0 \end{cases}$$



$$\phi_1 = -\frac{1}{24}(5 - x^2)(1 - x^2)$$

$$w_1(x) = \phi_1 a_1 = -\frac{a_1}{24}(5 - x^2)(1 - x^2)$$

$$R_1(x, a_1) = -a_1 - \alpha \frac{a_1}{24}(5 - x^2)(1 - x^2) + 1$$

Collocation method

$$R_1(0,a_1) = -a_1 - \frac{5\alpha}{24}a_1 + 1 = 0$$

$$a_1 = \left(1+\frac{5\alpha}{24}\right)^{-1}$$

Subdomain method

$$\int_{-1}^{1} R_1 \mathrm{d}x = -a_1 - \frac{2\alpha}{15}a_1 + 1 = 0 \qquad a_1 = \left(1+\frac{2\alpha}{15}\right)^{-1}$$

Least square method

$$\frac{\partial R_1}{\partial a_1} = -1 - \frac{\alpha}{24}(5-x^2)(1-x^2) \qquad \int_{-1}^{1} R_1 \frac{\partial R_1}{\partial a_1}\mathrm{d}x = 0$$

$$a_1 = \left(1+\frac{2\alpha}{15}\right)\left(1+\frac{4\alpha}{15}+\frac{62\alpha^2}{2835}\right)^{-1}$$

Galerkin's method

$$\phi_1 = -\tfrac{1}{24}(5-x^2)(1-x^2) \qquad \int_{-1}^{1} R_1\phi_1 \mathrm{d}x = 0$$

$$a_1 = \left(1+\frac{31\alpha}{189}\right)^{-1}$$

α	精确解	配点法	子域法	伽辽金法	最小二乘法
1	0.1788	0.1724	0.1838	0.1790	0.1832
10	0.07836	0.06757	0.08929	0.07891	0.08304
100	0.01134	0.00954	0.01453	0.01197	0.05818
1000	0.001025	0.000995	0.001551	0.001262	0.006068

$$\int_V \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon}(\boldsymbol{\phi})\, dV - \int_V \boldsymbol{\phi}^{\mathrm{T}} \mathbf{f}\, dV - \int_S \boldsymbol{\phi}^{\mathrm{T}} \mathbf{T}\, dS - \sum_i \boldsymbol{\phi}^{i\mathrm{T}} \mathbf{P} = 0$$

$$\int_V \Bigg[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x\right)\phi_x + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y\right)\phi_y + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z\right)\phi_z\Bigg] dV = 0$$

where

$$\boldsymbol{\phi} = [\phi_x, \phi_y, \phi_z]^{\mathrm{T}}$$

is an arbitrary displacement consistent with the boundary conditions of $\mathbf{u}$. If $\mathbf{n} = [n_x, n_y, n_z]^{\mathrm{T}}$ is a unit normal at a point $\mathbf{x}$ on the surface, the integration by parts formula is

$$\int_V \frac{\partial \alpha}{\partial x}\theta\, dV = -\int_V \alpha \frac{\partial \theta}{\partial x}\, dV + \int_S n_x\, d\theta ds$$

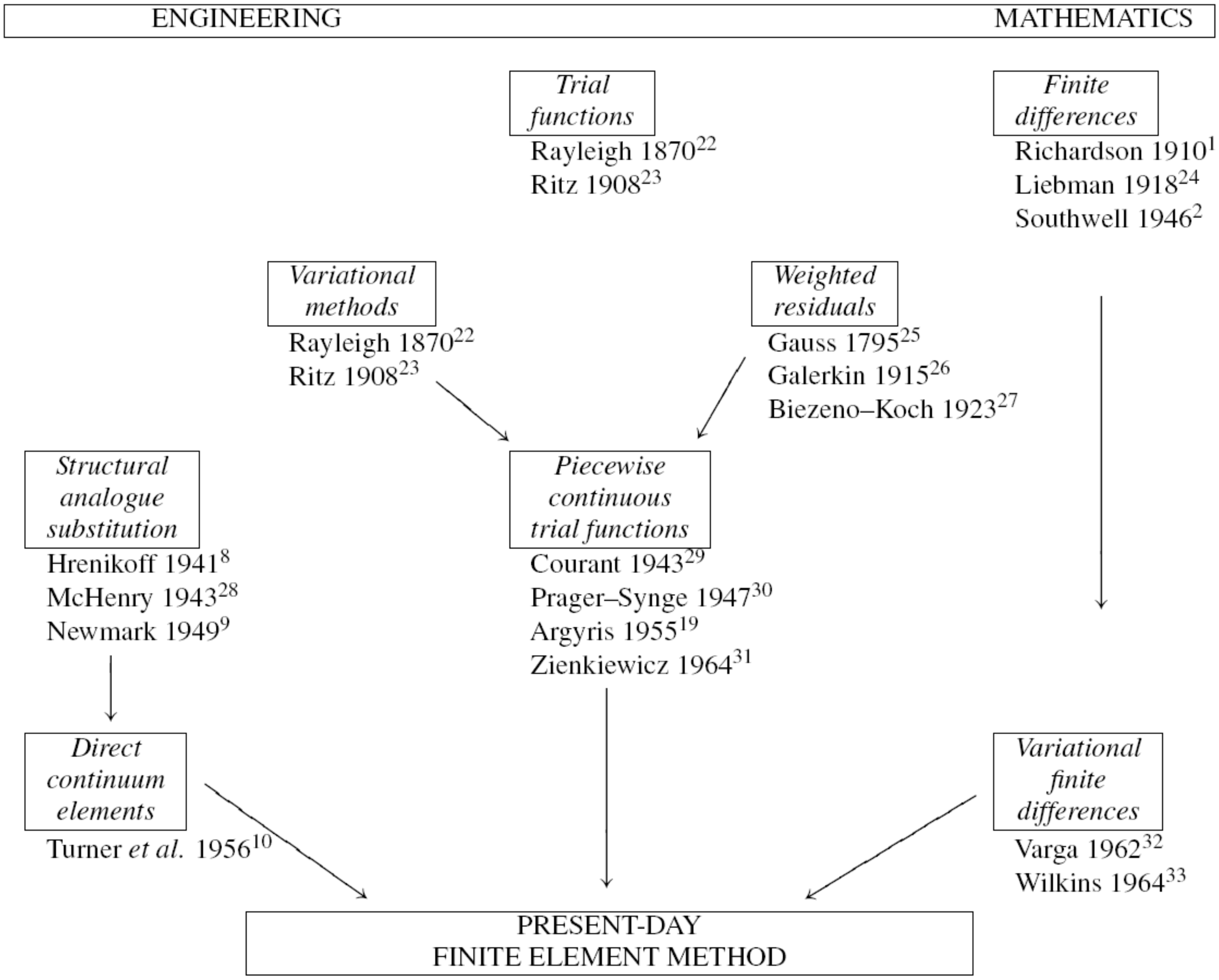
$$-\int_V \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon}(\boldsymbol{\phi})\, dV + \int_V \boldsymbol{\phi}^{\mathrm{T}} \mathbf{f}\, dV + \int_S [(n_x\sigma_x + n_y\tau_{xy} + n_z\tau_{xz})\phi_x + (n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{yz})\phi_y + (n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z)\phi_z]\, dS = 0$$

where

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \left[\frac{\partial \phi_x}{\partial x}, \frac{\partial \phi_y}{\partial y}, \frac{\partial \phi_z}{\partial z}, \frac{\partial \phi_y}{\partial z} + \frac{\partial \phi_z}{\partial y}, \frac{\partial \phi_x}{\partial z} + \frac{\partial \phi_z}{\partial x}, \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right]^{\mathrm{T}}$$

is the strain corresponding to the arbitrary displacement field $\boldsymbol{\phi}$.

History of approximate methods



❖ A typical structure built up from interconnected elements.

The forces acting on all the nodes

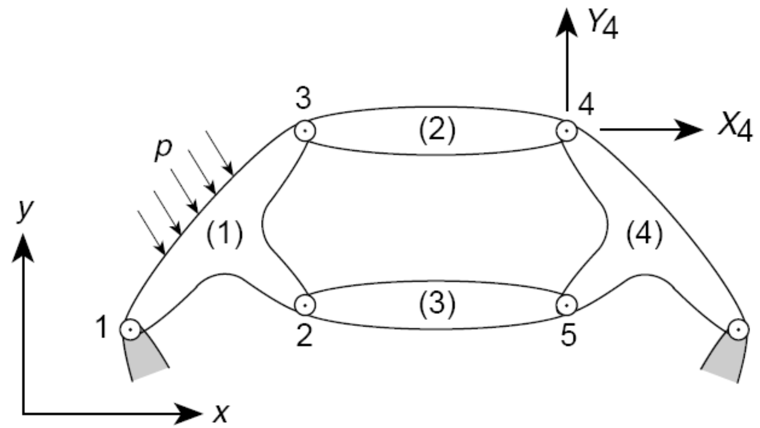
$$\mathbf{q}^1 = \begin{Bmatrix} \mathbf{q}_1^1 \\ \mathbf{q}_2^1 \\ \mathbf{q}_3^1 \end{Bmatrix} \qquad \mathbf{q}_1^1 = \begin{Bmatrix} U_1 \\ V_1 \end{Bmatrix}$$

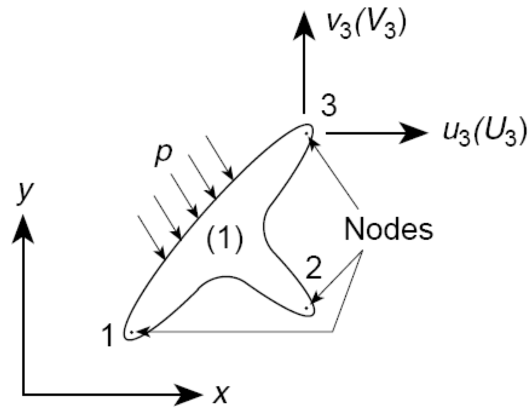
The corresponding nodal displacements

$$\mathbf{u}^1 = \begin{Bmatrix} \mathbf{u}_1^1 \\ \mathbf{u}_2^1 \\ \mathbf{u}_3^1 \end{Bmatrix} \qquad \mathbf{u}_1^1 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Assuming linear elastic behavior of the element, the characteristic relationship

$$\mathbf{q}^1 = \mathbf{K}^1\mathbf{u}^1 + \mathbf{f}^1$$





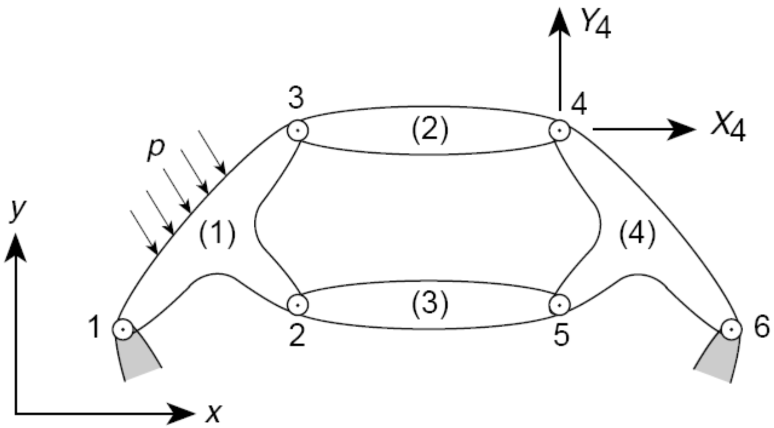
A typical element (1)

$\mathbf{f}^1$ **represents the nodal forces balanced any concentrated or distributed loads acting on the element.**

$\mathbf{K}^1\mathbf{u}^1$ **represents the forces induced by displacement of the nodes.**

$$\mathbf{q}^e = \begin{Bmatrix} \mathbf{q}_1^e \\ \mathbf{q}_2^e \\ \vdots \\ \mathbf{q}_m^e \end{Bmatrix} \quad \text{and} \quad \mathbf{u}^e = \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_m \end{Bmatrix}$$

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{11}^e & \mathbf{K}_{12}^e & \cdots & \mathbf{K}_{1m}^e \\ \mathbf{K}_{21}^e & \ddots & & \vdots \\ \vdots & \vdots & & \vdots \\ \mathbf{K}_{m1}^e & \cdots & \cdots & \mathbf{K}_{mm}^e \end{bmatrix}.$$



Assembly and analysis of a structure

Consider again the hypothetical structure of Fig. 1.1. To obtain a complete solution the two conditions of

(a) displacement compatibility and
(b) equilibrium

Any system of nodal displacements $\mathbf{u}$:

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{Bmatrix}$$

$$\sum_{e=1}^{m} \mathbf{q}_a^e = \mathbf{q}_a^1 + \mathbf{q}_a^2 + \cdots = \mathbf{0}$$

$$\left(\sum_{e=1}^{m} \mathbf{K}_{a1}^e\right)\mathbf{u}_1 + \left(\sum_{e=1}^{m} \mathbf{K}_{a2}^e\right)\mathbf{u}_2 + \cdots + \sum_{e=1}^{m} \mathbf{f}_i^e = \mathbf{0}$$

$$\mathbf{K}\mathbf{u} + \mathbf{f} = \mathbf{0}$$

$$\mathbf{K}_{ab} = \sum_{e=1}^{m} \mathbf{K}_{ab}^e \quad \text{and} \quad \mathbf{f}_a = \sum_{e=1}^{m} \mathbf{f}_a^e$$

The boundary conditions

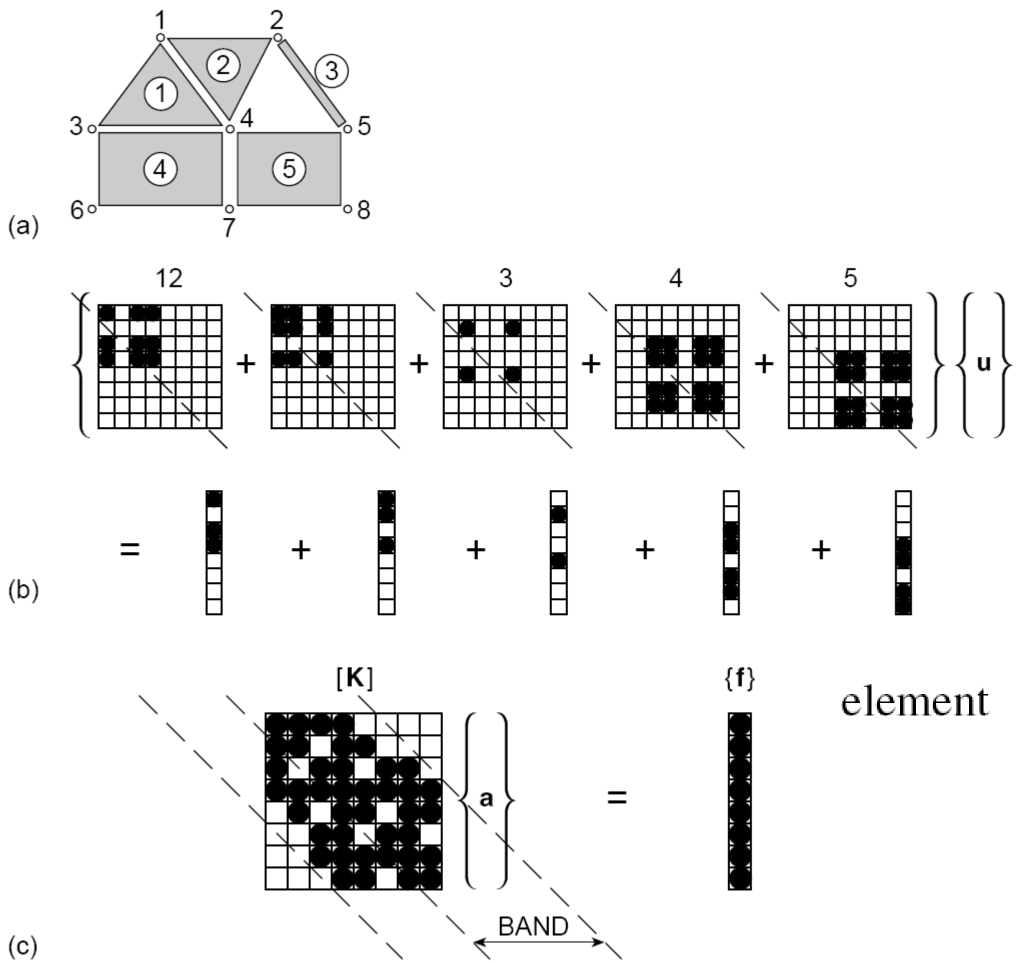
$$\mathbf{u}_1 = \mathbf{u}_6 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}_{11}\mathbf{u}_1 + \mathbf{K}_{12}\mathbf{u}_2 + \cdots + \mathbf{f}_1 = \mathbf{0}$$

$$\mathbf{K}_{21}\mathbf{u}_1 + \mathbf{K}_{22}\mathbf{u}_2 + \cdots + \mathbf{f}_2 = \mathbf{0}$$

etc.

The general pattern



element	1	connection	1	3	4	
	2		1	4	2	
	3		2	5		
	4		3	6	7	4
	5		4	7	8	5