Discrete Mechanical Vibrations SM32

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Contents:

Discretization of systems having an infinite number of degree of freedom.

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Flexural Motion of a beam

Hypotheses are based on theory of strength of materials

The motion is defined by:

V	lateral deflection	m
ψ	slope of neutral axis	rd
T	lateral shear force	N
M	flexural moment	mN

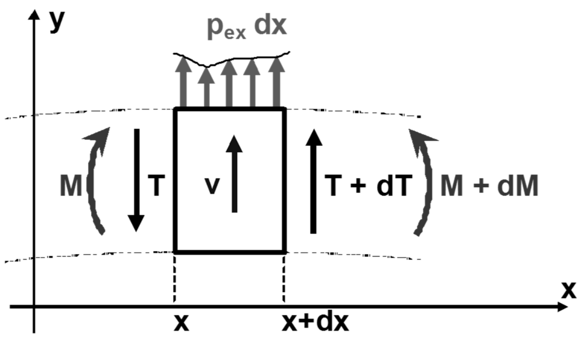
and mechanical or geometrical properties

E	Young's modulus	N/m^2
I	area moment of inertia of beam cross-section about the neutral axis	m^4
ρ	density	kg/m^3
S	area	m^2
L	length	m

and external force

p_{ex}	lateral external force per unit length	N/m

Derivation of equation of motion



The application of Newton's laws to the element depicted above in the lateral y-direction and about the z-direction gives

About y axis:

$$\rho S(x)\,dx\,\frac{\partial^2 v(x,t)}{\partial t^2} = -\cancel{T(x,t)} + \cancel{T(x,t)} + \frac{\partial T(x,t)}{\partial x}\,dx + p_{ex}\,dx$$

$$\rho S(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial T(x,t)}{\partial x} + p_{ex}$$

Around z axis:

$$0 = -\cancel{M(x,t)} + \cancel{M(x,t)} + \frac{\partial M(x,t)}{\partial x}\,dx + T(x,t)\,dx + p_{ex}\,\frac{dx^2}{2}$$

From the theory of strength of materials, the relations among **T**, **M** are

$$0 = \frac{\partial M(x,t)}{\partial x} + T(x,t) \qquad\qquad \frac{\partial M(x,t)}{\partial x} = -T(x,t)$$

and

$$M(x,t) = EI(x)\,\frac{\partial^2 v(x,t)}{\partial x^2}$$

which simplify to:

$$\boxed{\rho S(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\,\frac{\partial^2 v(x,t)}{\partial x^2}\right) = p_{ex}} \qquad \boxed{\rho S\,\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = 0}$$

This approach, in which some second order effects are ignored is called the Bernoulli-Euler approach. So, the classical obtained partial differential equation of a beam in motion of bending is called the **Bernoulli-Euler beam equation**.

Frequencies and Mode Shapes

Homogeneous previous equations are considered. The free-vibration solutions to these equations will be obtained by the method of separation of variables.

$$v(x,t) = \phi(x)\, f(t)$$

and substitute into EOM

$$EI\, \frac{d^4\phi(x)}{dx^4}\, f(t) + \rho S\, \phi(x)\, \frac{d^2 f(t)}{dt^2} = 0$$

After some manipulations, this gives

$$\frac{EI}{\rho S}\, \frac{1}{\phi(x)}\, \frac{d^4\phi(x)}{dx^4} = -\frac{1}{f(t)}\, \frac{d^2 f(t)}{dt^2} = C^{te}$$

which can be separated into two ordinary differential equations of motion, one in space and one in time:

$$\frac{d^2 f(t)}{dt^2} + C^{te}\, f(t) = 0$$ **time** equation

$$\frac{d^4\phi(x)}{dx^4} - C^{te}\, \frac{\rho S}{EI}\, \phi(x) = 0$$ **space** equation

Frequencies and Mode Shapes

Time function

$$\frac{d^2 f(t)}{dt^2} + C^{te}\, f(t) = 0$$

The separation constant has been set equal to $+\omega^2$ so that the solutions will be bounded in time. It follows that

$$\frac{d^2 f(t)}{dt^2} + \omega^2\, f(t) = 0$$

Time solution is:

$$f(t) = A \sin \omega t + B \cos \omega t$$

Frequencies and Mode Shapes

Space function

$$\boxed{\frac{d^4\phi(x)}{dx^4} - \omega^2 \frac{rS}{EI}\phi(x) = 0}$$

Solutions are sought in the form $\mathbf{e^{rx}}$ resulting in the characteristic equations:

$$r^4 - \omega^2 \frac{\rho S}{EI} = 0$$

Roots are

$$r = \beta, \ -\beta, \ j\beta, \ -j\beta$$

with

$$\beta = \sqrt[4]{\frac{\rho S \omega^2}{EI}}$$

then, a typical solution is:

$$\phi(x) = \cdots e^{+\beta x} \ + \cdots e^{-\beta x} \ + \cdots e^{+j\beta x} \ + \cdots e^{-j\beta x}$$

Frequencies and Mode Shapes

Trigonometric form is more suitable, so for each value of β, one has a solution of the form:

$$\phi(x) = C \sin \beta x + D \cos \beta x + E \text{ sh } \beta x + F \text{ ch } \beta x$$

The frequencies $\boldsymbol{\omega}$ associated with each $\boldsymbol{\beta}$ are determined by application of the boundary conditions (*geometric* or *essential*). The most frequent boundary conditions for beams are:

Free (F):	M = 0,	T= 0
Clamped (C):	v = 0,	θ = 0
Simply-supported (S):	v = 0,	M = 0

Then for each value of ω one has a solution of the form

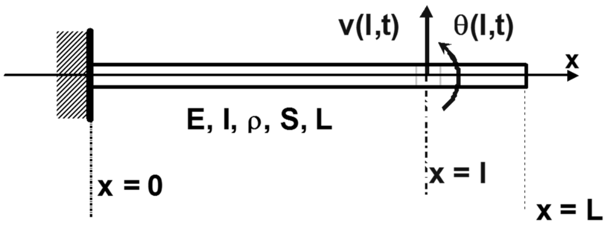
$$v(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\, f_n(t)$$
$$= \sum_{n=1}^{\infty} \left(A_n \sin \omega_n t + B_n \cos \omega_n t\right)$$
$$\left(C_n \sin \beta_n x + D_n \cos \beta_n x + E_n \text{sh } \beta_n x + F_n \text{ch } \beta_n x\right)$$

The constants $\mathbf{C_n}$, $\mathbf{D_n}$, $\mathbf{E_n}$ et $\mathbf{F_n}$ will be determined with boundary conditions. This classical procedure will lead to the frequencies of the beam. $\mathbf{A_n}$ and $\mathbf{B_n}$ are determined by initial conditions.

Application:

$$\phi(x) = C\sin\beta x + D\cos\beta x + E\,\mathrm{sh}\,\beta x + F\,\mathrm{ch}\,\beta x$$

Beam in bending



In that case, (Clamped-Free) boundary conditions are:

x = 0

Lateral displacement **V(0,t) = 0** $\Rightarrow$ $0 + D + 0 + F = 0$

rotation of section (dv/dx) **θ(0,t) = 0** $\Rightarrow$ $C + 0 + E + 0 = 0$

x = L

lateral shear force → (d^3v/dx^3) **T(L,t) = 0**

$$\Rightarrow \quad -C\cos\beta L + D\sin\beta L + E\,\mathrm{ch}\,\beta L + F\,\mathrm{sh}\,\beta L = 0$$

flexural moment → (d^2v/dx^2) **M(L,t) = 0**

$$\Rightarrow \quad -C\sin\beta L - D\cos\beta L + E\,\mathrm{sh}\,\beta L + F\,\mathrm{ch}\,\beta L = 0$$

$$\begin{array}{c} \\ \beta\,\times \\ \beta^2\times \\ \beta^3\times \end{array}
\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\beta L & -\cos\beta L & \mathrm{sh}\beta L & \mathrm{ch}\beta L \\ -\cos\beta L & \sin\beta L & \mathrm{ch}\beta L & \mathrm{sh}\beta L \end{bmatrix}
\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\begin{array}{l} \Leftarrow \text{Lateral displacement} \\ \Leftarrow \text{Slope of neutral axis} \\ \Leftarrow \text{Lateral shear force} \\ \Leftarrow \text{Flexural moment} \end{array}$$

If C, D, E and F are all zeroes, this is a non acceptable trivial solution. So, the determinant associated whith the matrix must be zero. After some manipulations,

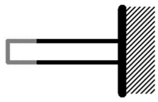
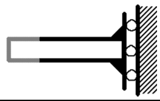
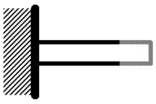
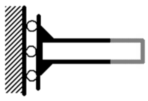
$$\boxed{1 + \cos\beta L\,\mathrm{ch}\,\beta L = 0}$$

Numerical solutions can be easily obtained:

βL	1.875	4.692	7.854	10.99	14.14
	3.516	22.03	61.69	120.9	199.8

Note: $\beta = \sqrt[4]{\dfrac{\rho S \omega^2}{EI}} \quad \Rightarrow \quad \omega = \dfrac{\beta^2}{L^2}\sqrt{\dfrac{EI}{\rho S}}$

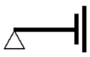
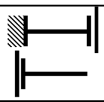
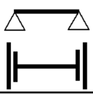
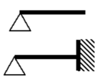
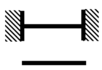
Using symmetries for boundary conditions, it remains 16 different cases. They are presented in the following table.

				
	$\sin\beta L = 0$	$\text{th}\,\beta L - \text{tg}\,\beta L = 0$	$\cos\beta L = 0$	$\text{th}\,\beta L - \text{tg}\,\beta L = 0$
		$1 - \cos\beta L\,\text{ch}\,\beta L = 0$	$\text{th}\,\beta L + \text{tg}\,\beta L = 0$	$1 + \cos\beta L\,\text{ch}\,\beta L = 0$
			$\sin\beta L = 0$	$\text{th}\,\beta L + \text{tg}\,\beta L = 0$
				$1 - \cos\beta L\,\text{ch}\,\beta L = 0$

Note that only 6 equations are obtained for all the cases.

The lowest values of X_n^2 are given in the following table

$$\boxed{\omega_n = \frac{X_n^2}{L^2}\sqrt{\frac{EI}{\rho S}}}$$

B.C.		X_1^2	X_2^2	X_3^2	X_4^2	X_5^2
	$\cos\beta L = 0$	2.467	22.21	61.68	120.9	199.9
	$1 + \cos\beta L\,\text{ch}\,\beta L = 0$	3.516	22.03	61.69	120.9	199.8
	$\text{th}\,\beta L + \text{tg}\,\beta L = 0$	5.593 0	30.22 5.593	74.63 30.22	138.8 74.63	222.7 138.8
	$\sin\beta L = 0$	9.869 0	39.47 9.869	88.82 39.47	157.9 88.82	246.7 157.9
	$\text{th}\,\beta L - \text{tg}\,\beta L = 0$	0 15.41	15.41 49.96	49.96 104.2	104.2 178.2	178.2 272.0
	$1 - \cos\beta L\,\text{ch}\,\beta L = 0$	22.37 0	61.67 22.37	120.9 61.67	199.8 120.9	298.5 199.8

Note: Rigid body modes occured for GF, GG and FF

Orthogonality relations

With the method of separation of variables, the EOM:

$$\rho S \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}(EI \frac{\partial^2 v}{\partial x^2}) = 0$$

becomes:

$$\frac{d^2}{dx^2}(EI \frac{d^2 \phi}{dx^2}) = \rho S \omega^2 \phi$$

This is true for each of the solution pairs**:** ω_i **,** ϕ_i and ω_j **,** ϕ_j (frequencies and mode shapes)

$$\frac{d^2}{dx^2}(EI \frac{d^2 \phi_i}{dx^2}) = \rho S \omega_i^2 \phi_i$$

$$\frac{d^2}{dx^2}(EI \frac{d^2 \phi_j}{dx^2}) = \rho S \omega_j^2 \phi_j$$

Multiplying the first equation by ϕ_j

$$\phi_j \frac{d^2}{dx^2}(EI \frac{d^2 \phi_i}{dx^2}) = \phi_j \rho S \omega_i^2 \phi_i$$

and the second by ϕ_I it becomes:

$$\phi_i \frac{d^2}{dx^2}(EI \frac{d^2 \phi_j}{dx^2}) = \phi_i \rho S \omega_j^2 \phi_j$$

That must be verified for the whole beam:

$$\int_0^L \phi_j \frac{d^2}{dx^2}\left(EI \frac{d^2 \phi_i}{dx^2}\right) dx = \int_0^L \phi_j \rho S \; \omega_i^2 \; \phi_i dx$$

$$\int_0^L \phi_i \frac{d^2}{dx^2}\left(EI \frac{d^2 \phi_j}{dx^2}\right) dx = \int_0^L \phi_i \rho S \; \omega_j^2 \; \phi_j dx$$

1st integration by parts:

Putting:

with

$$u = \phi_j \quad \text{and} \quad v = \frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)$$

$$u' = \frac{d\phi_j}{dx} \quad \text{and} \quad v' = \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)$$

$$\int_0^L (u\,v\,)'\,dx = \int_0^L u'\,v\,dx + \int_0^L u\,v'\,dx$$

integrating by parts leads:

$$\int_0^L (u\,v\,)'\,dx \Leftrightarrow \left[\phi_j \frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)\right]_0^L$$

So, it becomes:

$$\left[\phi_j \frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)\right]_0^L = \int_0^L \frac{d\phi_j}{dx}\frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx + \int_0^L \phi_j \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx$$

So, using the following boundaries at **x** = 0 and **x = L**.

	$\phi(x)f(t)$	$\frac{d^2\phi(x)}{dx^2}f(t)$
Free	?	0
Supported	0	0
Clamped	0	?

hence

$$\left[\phi_j \frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)\right]_0^L = 0$$

Finally:

$$0 = \int_0^L \frac{d\phi_j}{dx}\frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx + \int_0^L \phi_j \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx$$

or:

$$\int_0^L \phi_j \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx = \underline{\underline{-\int_0^L \frac{d\phi_j}{dx}\frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx}}$$

2nd integration by parts:

Putting:

$$u = \frac{d\phi_j}{dx} \qquad \text{and} \qquad u' = \frac{d^2\phi_j}{dx^2}$$

$$v = EI\frac{d^2\phi_i}{dx^2} \qquad \text{and} \qquad v' = \frac{d}{dx}(EI\frac{d^2\phi_i}{dx^2})$$

hence

$$\int_0^L (u\,v\,)'dx \;\Leftrightarrow \left[\frac{d\phi_j}{dx}(EI\frac{d^2\phi_i}{dx^2})\right]_0^L$$

So, using the following boundaries at **x** = 0 and **x = L**.

	$\frac{d\phi(x)}{dx}f(t)$	$\frac{d^2\phi(x)}{dx^2}f(t)$
Free	?	0
Supported	?	0
Clamped	0	?

Substituting, it becomes:

$$-\left[\frac{d\phi_j}{dx}(EI\frac{d^2\phi_i}{dx^2})\right]_0^L = 0$$

and

$$0 = \int_0^L \frac{d^2\phi_j}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx + \int_0^L \frac{d\phi_j}{dx}\frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx$$

or:

$$\int_0^L \frac{d^2\phi_j}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx = \underline{\underline{-\int_0^L \frac{d\phi_j}{dx}\frac{d}{dx}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx}}$$

Finally, combining that result with first equation leads to:

$$\int_0^L \frac{d^2\phi_j}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)dx = \omega_i^2\int_0^L \rho S\phi_i\phi_j dx$$

$$\int_0^L \frac{d^2\phi_i}{dx^2}\left(EI\frac{d^2\phi_j}{dx^2}\right)dx = \omega_j^2\int_0^L \rho S\phi_i\phi_j dx$$

Subtracting:

$$0 = \left(\omega_i^2 - \omega_j^2\right) \int_0^L \rho S\, \phi_i\, \phi_j\, dx$$

As $\omega_i \neq \omega_j$

$$\boxed{\int_0^L \rho S\, \phi_i\, \phi_j\, dx = 0}$$

and

$$\boxed{\int_0^L EI\, \frac{d^2\phi_i}{dx^2}\, \frac{d^2\phi_j}{dx^2}\, dx = 0}$$

Previous equations are the **orthogonality conditions** for a continuous system deforming of a beam in flexion.

Modal Masses and modal stiffness matrices

As previously shown:

$$\omega_i^2 = \frac{\int_0^L EI\, \left(\frac{d^2\phi_i}{dx^2}\right)^2 dx}{\int_0^L \rho S\, \phi_i^2\, dx} = \frac{k_i}{m_i}$$

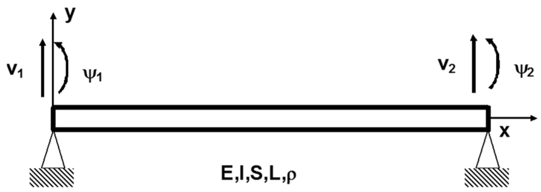
with:

$$\boxed{m_i = \int_0^L \rho S\, \phi_i^2\, dx} \quad \text{and} \quad \boxed{k_i = \int_0^L EI\, \left(\frac{d^2\phi_i}{dx^2}\right)^2 dx}$$

where $\mathbf{k_i}$ and $\mathbf{m_i}$ are the i^{th} **modal stiffness and modal mass** of this continuous system(beam in flexion).

Application :

Poutre de section constante sur deux appuis simple en flexion :



On connaît :

$$\phi_i(x) = C_i \sin\frac{i\pi x}{L} \qquad \text{et} \qquad \omega_i = \frac{(i\pi)^2}{L^2}\sqrt{\frac{EI}{\rho S}} \qquad \text{avec } i \neq 0 \quad \textbf{positif}$$

On vérifie bien que :

$$\left[\phi_j EI \frac{d}{dx}\frac{d^2\phi_i}{dx^2}\right]_0^L = -\left[EIC_jC_i\left(\frac{i\pi}{L}\right)^3 \sin\frac{j\pi x}{L}\cos\frac{i\pi x}{L}\right]_0^L$$
$$= 0$$

$$\left[EI\frac{d\phi_j}{dx}\frac{d^2\phi_i}{dx^2}\right]_0^L = -\left[EIC_jC_i\left(\frac{\pi}{L}\right)^3 i^2 j \cos\frac{j\pi x}{L}\sin\frac{i\pi x}{L}\right]_0^L$$
$$= 0$$

<u>*Raideurs modales ($i = j \neq 0$)*</u>

$$k_i = EI\int_0^L \left(\frac{d^2\varphi_i}{dx^2}\right)^2 dx = EIC_i^2\left(\frac{i\pi}{L}\right)^4 \int_0^L \sin^2\frac{i\pi}{L}x\,dx$$
$$= C_i^2 EI\left(\frac{i\pi}{L}\right)^4 \frac{L}{2}$$

soit

$$\boxed{k_i = C_i^2\frac{EI(i\pi)^4}{2L^3}}$$

<u>*Masses modales ($i = j \neq 0$)*</u>

$$m_i = \int_0^L \rho S\varphi_i^2 dx$$
$$= \rho SC_i^2\int_0^L \sin^2\frac{i\pi}{L}x\,dx$$

soit

$$\boxed{m_i = C_i^2\rho S\frac{L}{2}}$$

Pulsation propre :

$$\omega_i^2 = \frac{\int_0^L EI(\frac{d^2\phi_i}{dx^2})^2 dx}{\int_0^L \rho S \phi_i^2 dx} = \frac{k_i}{m_i}$$

$$= \frac{C_i^2 \frac{EI(i\pi)^4}{2L^3}}{C_i^2 \frac{\rho S L}{2}}$$

$$= \frac{i^4\pi^4}{L^4}\frac{EI}{\rho S}$$

Vérification

$$\omega_i = \sqrt{\frac{k_i}{m_i}} = \frac{i^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$$

La $i^{ème}$ équation (découplée) s'écrit :

$$m_i\ddot{q}_i + k_i q_i = 0$$

avec

$$m_i = C_i^2 \rho S \frac{L}{2} \qquad \text{et} \qquad k_i = C_i^2 \frac{EI(i\pi)^4}{2L^3}$$

Il est possible d'écrire (avec un choix judicieux des constantes) :

pour la $1^{ère}$ équation,

$$\rho S \frac{L}{2}\ddot{q}_1 + \frac{EI(\pi)^4}{2L^3} q_1 = 0$$

pour la $2^{ème}$ équation,

$$\rho S \frac{L}{2} f\ddot{q}_2 + \frac{EI(2\pi)^4}{2L^3} q_2 = 0$$

et sous forme matricielle pour **n** équations

$$\rho S \frac{L}{2}\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \end{Bmatrix} + \frac{EI(\pi)^4}{2L^3}\begin{bmatrix} 1 & & & 0 \\ & 16 & & \\ & & 81 & \\ 0 & & & \ddots \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \end{Bmatrix} = 0$$

<u>Rappel :</u>

$$\omega_i = \frac{(i\pi)^2}{L^2}\sqrt{\frac{EI}{\rho S}} \qquad \text{pour} \quad i = 1,2,\ldots$$