Discrete Mechanical Vibrations SM32

Sino-European Institute of Aviation Engineering
October-November 2011

Professor: Alain BERLIOZ
alain.berlioz@isae.fr

Contents:

October 2011

Multi- degree of freedom Systems

MATRICES PROPERTIES:

Kinetic Energy

Potential Energy

Rayleigh Dissipative function

FREQUENCIES AND MODE SHAPES

Orthogonality relations for modes

Multiple roots (and rigid body mode)

RESPONSE TO EXCITATION

Modal Basis Approach:

Proportionnal viscous damping

Imposed Modal Damping

RAYLEIGH-RITZ METHOD

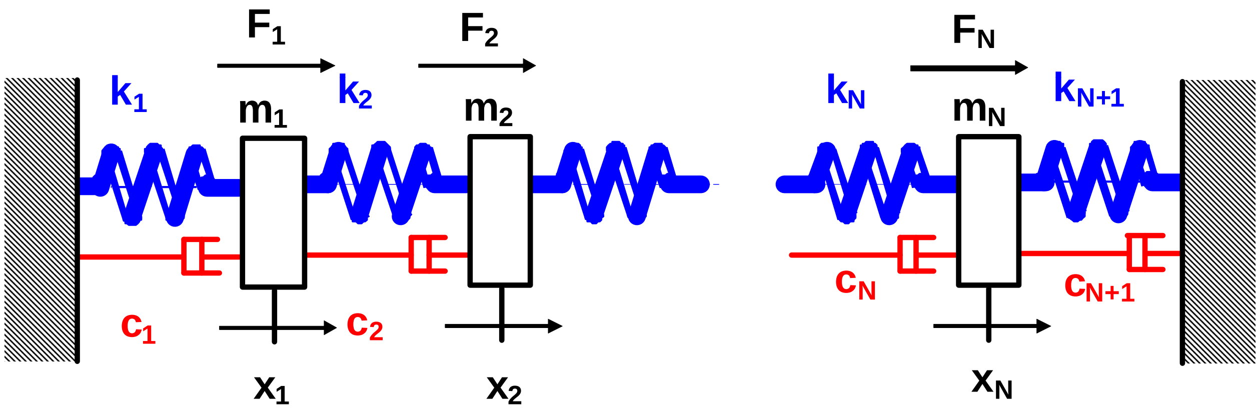
October 2011

Multi- degree of freedom Systems

The equations of motion for these systems can be obtained from either Newtonian or **Lagrangian** mechanics.

Theviscous damping will be considered; however, some of the system properties demonstrated are general and will be used in other applications.

This chapter serves as a better understanding for the prediction of the dynamic behavior of systems having a very large number of degrees of freedom, such as engineering structures which are modeled by the finite element method.



With the following compact matrix form:

$$\boxed{M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t)}$$

Matrices and Vectors

Mass Matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \vdots & m_{1N} \\ m_{21} & m_{22} & \vdots & m_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ m_{N1} & m_{N2} & \vdots & m_{NN} \end{bmatrix}$$

Stiffness Matrix

$$K = \begin{bmatrix} k_{11} & k_{12} & \vdots & k_{1N} \\ k_{21} & k_{22} & \vdots & k_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ k_{N1} & k_{N2} & \vdots & k_{NN} \end{bmatrix}$$

Damping Matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \vdots & c_{1N} \\ c_{21} & c_{22} & \vdots & c_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ c_{N1} & c_{N2} & \vdots & c_{NN} \end{bmatrix}$$

Displacements, Velocities and Accelerations Vectors

$$x^t = \begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \text{ and } \dot{x}^t = \begin{pmatrix} \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_N \end{pmatrix} \text{ and } \ddot{x}^t = \begin{pmatrix} \ddot{x}_1 & \ddot{x}_2 & \cdots & \ddot{x}_N \end{pmatrix}$$

Vector of External Forces

$$F^t = \begin{pmatrix} F_1(t) & F_2(t) & \cdots & F_N(t) \end{pmatrix}$$

If possible { , [, } ,] will be omitted in the following chapters

MATRICES PROPERTIES:

In the field of linear Vibrations, all the energies (kinetic, potential, or dissipation) can be expressed under a quadratic form. Considering a system with **n** parameters, it follows:

Kinetic Energy:

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{i,j}\dot{x}_i\dot{x}_j$$

$$= \frac{1}{2}\{\dot{x}\}^t[M]\{\dot{x}\}$$

$$= \frac{1}{2}\dot{x}^t M\dot{x}$$

or

$$T = \frac{1}{2}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix}^t \begin{bmatrix} m_{11} & m_{12} & \vdots & m_{1N} \\ m_{21} & m_{22} & \vdots & m_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ m_{N1} & m_{N2} & \vdots & m_{NN} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{Bmatrix}$$

with

$$\dot{x}^t = \begin{pmatrix} \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_N \end{pmatrix}$$

Bilinear Forms: (R^3)

For two vectors {a} and {b} :

$$\Phi = \{a\}^t [A] \{b\}$$

For the scalar Φ (i.e. an energy) and if {a} and {b} identical, then matrix A is symetric. (bi-linear symetric form or quadratique form).

$$\Phi = \{a\}^t [A] \{b\} = \{b\}^t [A]^t \{a\} = \Phi^t$$

$$\Phi = \{a\}^t [A] \{a\} = \{a\}^t [A]^t \{a\} = \Phi^t$$

The matrix A is symetric.

Differenciation of a quadratic form.

{a} and **{b**} are identical and **A** is symetric, then it can be shown that:

$$\left\{\frac{\partial\Phi}{\partial a}\right\} = \left\{\begin{matrix}\dfrac{\partial\Phi}{\partial a_1} \\ \dfrac{\partial\Phi}{\partial a_2} \\ \vdots \\ \dfrac{\partial\Phi}{\partial a_N}\end{matrix}\right\} = 2\begin{bmatrix} A_{11} & A_{12} & \vdots & A_{1N} \\ A_{21} & A_{22} & \vdots & A_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ A_{N1} & A_{N2} & \vdots & A_{NN}\end{bmatrix}\left\{\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_N\end{matrix}\right\}$$

Hence for the kinetic energy:

$$\left\{\frac{\partial T}{\partial \dot{x}}\right\} = \left\{\begin{matrix} \dfrac{\partial T}{\partial \dot{x}_1} \\ \dfrac{\partial T}{\partial \dot{x}_2} \\ \vdots \\ \dfrac{\partial T}{\partial \dot{x}_N} \end{matrix}\right\} = \begin{bmatrix} m_{11} & m_{12} & \vdots & m_{1N} \\ m_{21} & m_{22} & \vdots & m_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ m_{N1} & m_{N2} & \vdots & m_{NN} \end{bmatrix} \left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{matrix}\right\}$$

Each term of the matrix can be expressed like:

$$\boxed{m_{i,j} = \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j}}$$

Remark: T= 0 if and only if (iff) $\dot{x}^t = (0)$

The mass matrix is called a **definite positive matrix.**

Potential Energy

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} k_{i,j} x_i x_j$$
$$= \frac{1}{2} x^t K x$$

Each term of the matrix can be expressed like:

$$\boxed{k_{i,j} = \frac{\partial^2 U}{\partial x_i \partial x_j}}$$

with

$$x^t = \begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix}$$

Potential energy (or stress energie) can be zero form some particular values of x_i.

The stiffness matrix is called a **semi-definite positive matrix**.

Rayleigh Dissipative function

In case of dissipative elements, the total Mechanical Energy is not conserved. Rayleigh proposed the following function (which is also a quadratic form in velocity).

$$R = \frac{1}{2}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{Bmatrix}^t \begin{bmatrix} c_{11} & c_{12} & \vdots & c_{1N} \\ c_{21} & c_{22} & \vdots & c_{2N} \\ \cdots & \cdots & \ddots & \cdots \\ c_{N1} & c_{N2} & \vdots & c_{NN} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{Bmatrix}$$

$$R = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{i,j}\dot{x}_i\dot{x}_j = \frac{1}{2}\{\dot{x}\}^t[C]\{\dot{x}\} = \frac{1}{2}\dot{x}^t C\dot{x}$$

This **R** function is called the **dissipative function of Rayleigh** and **C** the damping matrix. Note that :

$$\dot{x} = \left[\frac{m}{s}\right] \text{ and } C = \left[\frac{Ns}{m}\right] \Rightarrow R = \left[\frac{m}{s}\frac{Ns}{m}\frac{m}{s}\right] = \left[\frac{Nm}{s}\right]$$

Positive definite and positive semi-definite matrices

Finally:

The kinetic energy of the system is always positive unless all the velocities are zero.

The matrix **M** is said to be **positive definite.**

The dissipation function R and the strain energy U can be zero with one or more displacements not equal to zero; for example, for rigid-body motion of the system.

The matrices **C** and **K** are then said to be **positive semi-definite.**

FREQUENCIES AND MODE SHAPES

The conservative system associated is composed of **n** coupled equations. The compact matrix form is:

$$M\ddot{x}+Kx=0$$

with :

$$x^t = \begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix}$$

and

$$\ddot{x}^t = \begin{pmatrix} \ddot{x}_1 & \ddot{x}_2 & \cdots & \ddot{x}_N \end{pmatrix}$$

Solutions for $\mathbf{x_i}$ are sought in the form:

$$x_i = X_i e^{rt}$$

with:

$$X = (X_1 \quad X_2 \quad \cdots \quad X_N)^t$$

FREQUENCIES AND MODE SHAPES

Then

$$r^2 \, M \, X \, e^{rt} + K \; X \, e^{rt} = 0$$

It implies for any **t**

$$r^2 \, M \, X + K \; X = 0$$

or, alternatively,

$$\omega^2 \, M \, X = K \; X$$

in which, (the same notation was used for ONE or TWO systems)

$$r^2 \Rightarrow \omega$$

It can be shown that the solutions of the previous matrix equation are real since the matrix **M** is **symmetric, positive definite** and the matrix **K** is **symmetric, positive semi-definite.**

Solution of this generalized **eigenvalue** problem are eigenfrequencies. The associated eigenvectors are called the mode .

Numrical methods are used for solving that problem i.e. evaluation of $\boldsymbol{\omega_i}$ and $\boldsymbol{\phi_i}$

$$\omega_i^2 \, M \, \phi_i = K \, \phi_i$$

Orthogonality relations for modes

Consider the modes or the mode shapes of the conservative, i.e .. undamped, system:

$$M\ddot{x}+Kx=0$$

Let ω_i , ϕ_i and ω_j , ϕ_j be two solutions,so that:

frequency and mode i: ω_i , ϕ_i $\Longleftrightarrow$ $\omega_i^2 M\phi_i = K\phi_i$

frequency and mode j: ω_j , ϕ_j $\Longleftrightarrow$ $\omega_j^2 M\phi_j = K\phi_j$

First equation is left-multiplied with ϕ_j^t and the second with ϕ_i^t , this yields

$$\omega_i^2\, \phi_j^t\, M\phi_i = \phi_j^t\, K\,\phi_i \qquad \text{and} \qquad \omega_j^2\, \phi_i^t\, M\phi_j = \phi_i^t\, K\,\phi_j$$

Alain Berlioz 2011/12

Orthogonality relations for modes

Transposition of the first equation yields :

$$\omega_i^2 \phi_j^t M \phi_i = \phi_j^t K \phi_i \quad \Longrightarrow \quad \omega_i^2 \phi_i^t M^t \phi_j = \phi_i^t K^t \phi_j$$

Since **M** and **K** are symmetric

$$\omega_i^2 \phi_i^t M^t \phi_j = \phi_i^t K^t \phi_j \quad \Longrightarrow \quad \omega_i^2 \phi_i^t M \phi_j = \phi_i^t K \phi_j$$

combining with second equation

$$\begin{array}{rll} & \omega_j^2 \phi_i^t M \phi_j & = \phi_i^t K \phi_j \\ - & & \\ & \omega_i^2 \phi_i^t M \phi_j & = \phi_i^t K \phi_j \\ \hline = & (\omega_j^2 - \omega_i^2) \phi_i^t M \phi_j = 0 & \end{array}$$

Alain Berlioz 2011/12

Assuming $\omega_j^2 \neq \omega_i^2$, it follows that

$$\boxed{\phi_i^t \, M \, \phi_j = 0} \quad \text{and} \quad \boxed{\phi_i^t \, K \, \phi_j = 0}$$

Previous equations are the orthogonality relations of the modes associated with the EOM.

It follows that if ϕ is the square modal matrix such that:

$$\boxed{\phi = \left[\phi_1, \phi_2, \cdots, \phi_n \right]}$$

then the modal matrices $\phi^t \, K \, \phi$ and $\phi^t \, M \, \phi$ are **diagonal**.

Multiple roots (and rigid body mode)

Consider now the important case of a system having several rigid-body modes for which, of course, the frequencies are zero. For the case of two rigid-body modes, i.e.

$$\omega_1 = 0 \qquad \text{and} \qquad \phi_1$$

$$\omega_2 = 0 \qquad \text{and} \qquad \phi_2$$

and then for k = 3,N

$$\omega_k \neq 0 \qquad \text{and} \qquad \phi_k$$

it follows that

$$\omega_1^2 M\phi_1 = K\phi_1 = 0 \qquad \text{because} \quad \omega_1 = 0$$

$$\omega_2^2 M\phi_2 = K\phi_2 = 0 \qquad \text{because} \quad \omega_2 = 0$$

and also for the last N-2 equations

$$\omega_k^2 M\phi_k = K\phi_k$$

Equations are left-multiplied with ϕ_k^t this yields

$$\phi_k^t K\phi_1 = \phi_k^t K\phi_2 = 0$$

then the transposed form of remaining equations are right-multiplied by ϕ_1 and ϕ_2 :

$$\omega_k^2 M\phi_k = K\,\phi_k \Rightarrow \omega_k^2\,\phi_k^t M = \phi_k^t K$$

the last N-2 equations

$$\omega_3^2\,\phi_3^t M\phi_1 = K\,\phi_1 = 0$$
$$\cdots$$
$$\omega_k^2\,\phi_k^t M\phi_1 = K\,\phi_1 = 0$$

and

$$\omega_3^2\,\phi_3^t M\phi_2 = K\,\phi_2 = 0$$
$$\cdots$$
$$\omega_k^2\,\phi_k^t M\phi_2 = K\,\phi_2 = 0$$

The classical orthogonality relations among modes are thereby verified between the rigid body modes and the others.

In contrast, between the two rigid-body modes, one has a relation like:

$$\phi_1^t K \phi_2 = 0$$

but, in general

$$\phi_1^t M \phi_2 \neq 0$$

Thus rigid-body modes can give rise to off-diagonal terms in the previous matrix product.

<u>6 rigid-body modes can exist on a structure</u>
<u>(see Finite-Elements)</u>

RESPONSE TO EXCITATION

Modal Basis Approach:

The number **n** of required frequencies is less than the number N of equations of the whole system, hence the ϕ matrix is not a squared matrix

$$
[\phi] = \begin{array}{ccccc|ccc}
\textbf{Vector 1} & & \textbf{Vector i} & & \textbf{Vector n} & & & \\
\downarrow & & \downarrow & & \downarrow & & & \\
\phi_{11} & \cdots & \phi_{i1} & \cdots & \phi_{n1} & \phi_{n+11} & \cdots\cdots & \phi_{N1} \\
\phi_{12} & \cdots & \phi_{i2} & \cdots & \phi_{n2} & \phi_{n+12} & \cdots\cdots & \phi_{N2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\
\phi_{1j} & \cdots & \phi_{ij} & \cdots & \phi_{nj} & \phi_{n+1j} & \cdots\cdots & \phi_{Nj} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\
\phi_{1N} & \cdots & \phi_{iN} & \cdots & \phi_{nN} & \phi_{n+1N} & \cdots\cdots & \phi_{NN} \\
\multicolumn{5}{c}{\underbrace{\hspace{6em}}_{\mathbf{n}}} & \multicolumn{3}{c}{\underbrace{\hspace{6em}}_{N-n}}
\end{array}
$$

$\boxed{n << N}$

N : size of the whole

n : nomber of node in the modal basis (computed)

Modal Basis Approach:

$$\phi^t M\phi\,\ddot{q} + \phi^t C\phi\,\dot{q} + \phi^t K\phi\,q = \phi^t F$$

Example:

N : size of the whole = 1000

n : number of node in the modal basis = 10

$$x = \phi\,q \Rightarrow [1000\times1] = [1000\times10]\times[10\times1]$$

$$\phi^t M\phi \Rightarrow [10\times1000]\times[1000\times1000]\times[1000\times10] \rightarrow [10\times10]$$

$$\phi^t C\phi \Rightarrow [10\times1000][1000\times1000]\times[1000\times10] \rightarrow [10\times10]$$

$$\phi^t K\phi \Rightarrow [10\times1000]\times[1000\times1000]\times[1000\times10] \rightarrow [10\times10]$$

$$\phi^t F \Rightarrow [10\times1000]\times[1000\times1] \rightarrow [10\times1]$$

$$\boxed{M\,\ddot{q} + C\,\dot{q} + K\,q = F}$$

Size of the new system is **n** : number of node in the modal basis (10)

Modal Basis Approach:

Ex: Matrices and Vectors in Modal Basis (N=1000, n=10)

Modal Mass Matrix

$$M = \begin{bmatrix} m_{11} & 0 & \vdots & 0 \\ 0 & m_{22} & \vdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \vdots & m_{nn} \end{bmatrix}$$

Modal Stiffness Matrix

$$K = \begin{bmatrix} k_{11} & 0 & \vdots & 0 \\ 0 & k_{22} & \vdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \vdots & k_{nn} \end{bmatrix}$$

Modal Damping Matrix

$$C = \begin{bmatrix} c_{11} & 0 & \vdots & 0 \\ 0 & c_{22} & \vdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \vdots & c_{nn} \end{bmatrix}$$

Modal Displacements, Velocities and Accelerations Vectors

$\mathrm{q}^{t} = (\mathrm{q}_1 \quad \mathrm{q}_2 \quad \cdots \quad \mathrm{q}_n)$ and $\dot{\mathrm{q}}^{t} = (\dot{\mathrm{q}}_1 \quad \dot{\mathrm{q}}_2 \quad \cdots \quad \dot{\mathrm{q}}_n)$ and $\ddot{\mathrm{q}}^{t} = (\ddot{\mathrm{q}}_1 \quad \ddot{\mathrm{q}}_2 \quad \cdots \quad \ddot{\mathrm{q}}_n)$

Vector of External Forces

$$F^{t} = (F_1 \quad F_2 \quad \cdots \quad F_n)$$

Proportionnal viscous damping

With,

$$\phi^t M\phi\ddot{q} + \phi^t(\alpha M + \beta K)\phi\dot{q} + \phi^t K\phi q = \phi^t(F_s \sin\Omega t + F_c \cos\Omega t)$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & m_n \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \dots \\ \ddot{q}_n \end{Bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & c_n \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dots \\ \dot{q}_n \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & k_n \end{bmatrix}\begin{Bmatrix} q_1 \\ \dots \\ q_n \end{Bmatrix} = \begin{Bmatrix} f_{s1} \\ \dots \\ f_{sn} \end{Bmatrix}\sin\Omega t + \begin{Bmatrix} f_{c1} \\ \dots \\ f_{cn} \end{Bmatrix}\cos\Omega t$$

All equations are un-coupled and solutions in q_n can be obtained with the results of the previous chapters.

Then,

$$x = \phi q \qquad \text{for } i = 1, N$$

$$x_i(t) = \sum_{j=1}^{n} \phi_{ji}\, q_j(t)$$

with ϕ_{ij} j th term of the i th mode

Ex: Matrices and Vectors in Modal Basis (N=1000, n=10)

Imposed Modal Damping

$$\phi^t M \phi \ddot{q} + \{dampings\} + \phi^t K \phi q = \phi^t (F_s \sin \Omega t + F_c \cos \Omega t)$$

$$\{dampings\} \Rightarrow \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{Bmatrix}$$

With,

$$\boxed{\alpha_i = \frac{c_i}{2\sqrt{k_i m_i}}} \qquad \alpha_i = \frac{c_i}{c_{ci}} = \frac{c_i}{2\sqrt{k_i m_i}} = \frac{\text{current damping}_i}{\text{critical damping}_i}$$

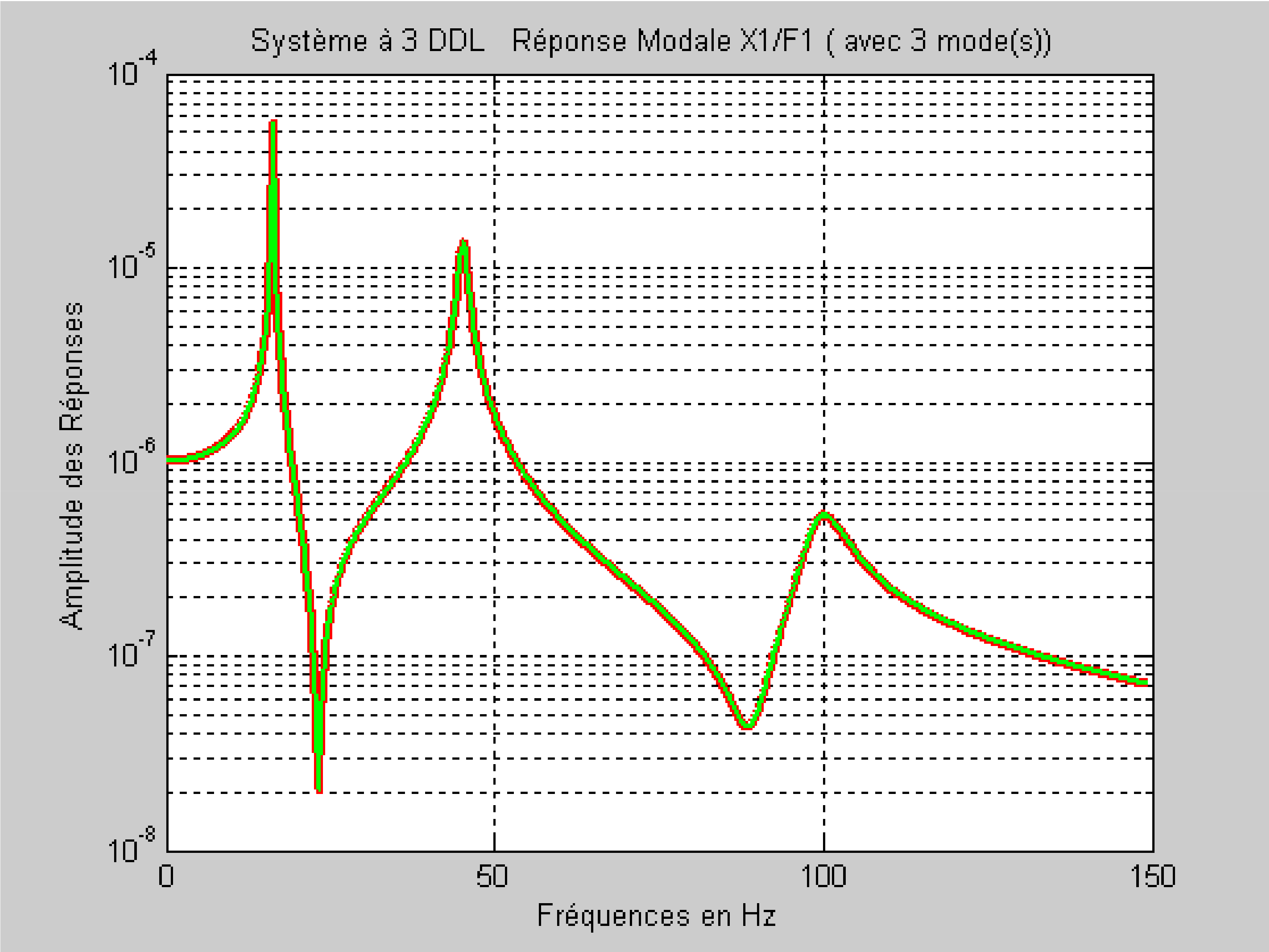
Then,

$$x = \phi q$$

$$x_i(t) = \sum_{j=1}^{n} \phi_{ji}\, q_j(t) \qquad \text{for } i = 1, N$$

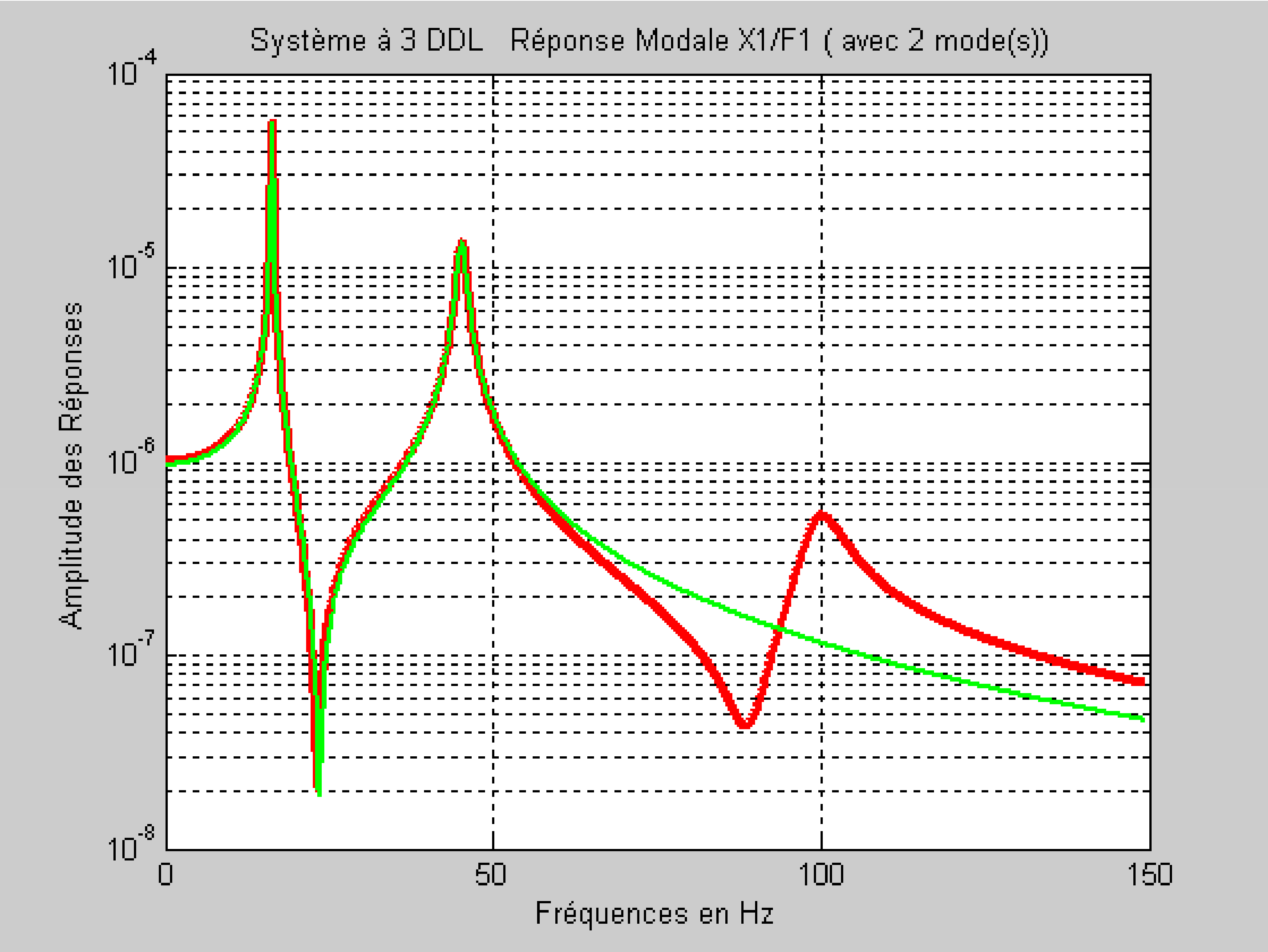
with ϕ_{ij} j^{th} term of the i^{th} mode

Ex: Matrices and Vectors in Modal Basis (N=1000, n=10)



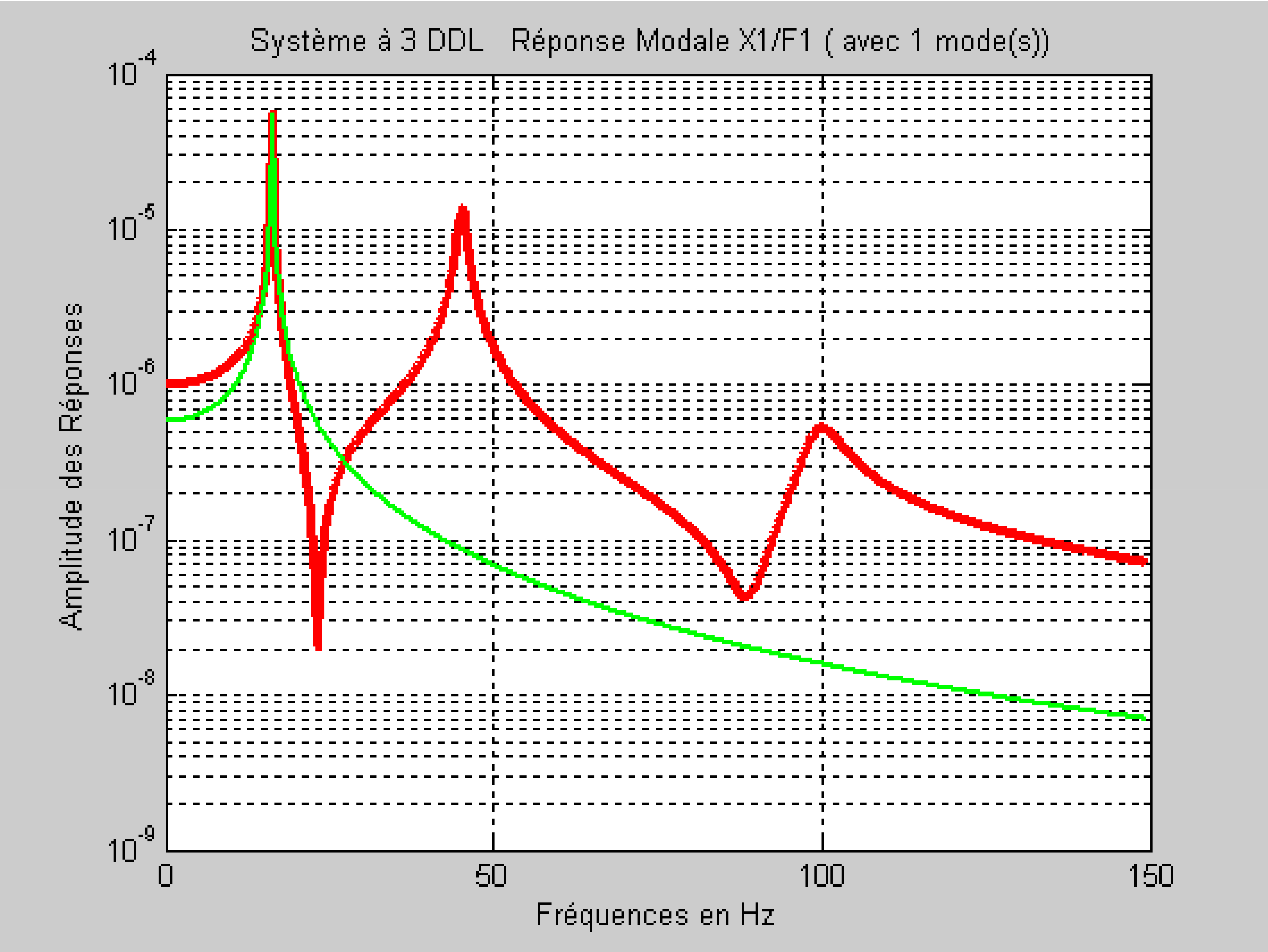
Three modes

Alain Berlioz 2011/12



Système à 3 DDL   Réponse Modale X1/F1 (avec 2 mode(s))
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
Amplitude des Réponses
0
50
100
150
Fréquences en Hz

Two modes


Système à 3 DDL   Réponse Modale X1/F1 (avec 1 mode(s))
Amplitude des Réponses
Fréquences en Hz
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0
50
100
150

TOne modes

The RAYLEIGH-RITZ METHOD

This is a generalization by Ritz of Rayleigh's method based on energy. It is used to reduce the number of degrees of freedom of the system and to estimate the lowest frequency.

As before, a reasonable hypothesis about the displacement of the system is made. Ritz suggested that this hypothesis be taken in the form of an expansion; for example, in the form:

$$x = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = \gamma p \qquad\qquad x = \begin{bmatrix} \gamma_1 & \vdots & \vdots \\ \gamma_2 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \gamma_n & \vdots & \vdots \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \gamma p$$

$$\langle N,1 \rangle = \langle N,1 \rangle \langle 1,1 \rangle \qquad\qquad \langle N,1 \rangle = \langle N,n \rangle \langle n,1 \rangle$$

Rayleigh's method — Rayleigh-Ritz's method

where γ_i are **N**-dimensional vectors with **n** << **N**. The assumption must satisfy the geometry boundary conditions of the system, (i.e. boundary conditions on displacements).

The RAYLEIGH-RITZ METHOD

Substituting into energy equations leads:

$$T = \frac{1}{2}\dot{x}^t M \dot{x} = \frac{1}{2}\dot{p}^t \gamma^t M \gamma \dot{p}$$

$$U = \frac{1}{2}x^t K x = \frac{1}{2}p^t \gamma^t K \gamma p$$

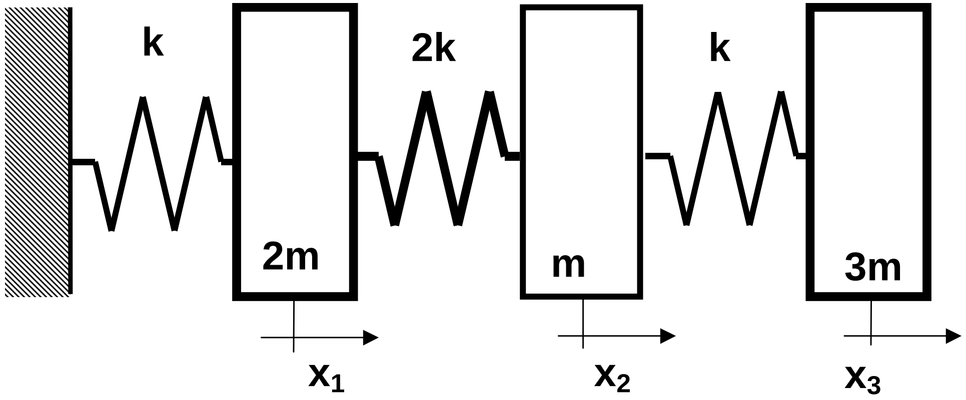
and then using Lagrange's equations:

$$\gamma^t M \gamma \ddot{p} + \gamma^t K \gamma p = 0$$

The order of the new system is **n**, which is much lower than that of the original system of size **N**.

That method is based on an approximation and the solution will not be exact but generally correct.

The RAYLEIGH-RITZ METHOD



Exact Solutions

$$\omega_1 = 0.3243\sqrt{\frac{k}{m}} \qquad \omega_2 = 0.8992\sqrt{\frac{k}{m}} \qquad \omega_3 = 1.980\sqrt{\frac{k}{m}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 1.395 \\ 2.038 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ 0.6914 \\ -0.4849 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 1 \\ -2.420 \\ 0.2249 \end{Bmatrix}$$

Alain Berlioz 2011/12

The RAYLEIGH-RITZ METHOD

$$x = \gamma p \qquad \boxed{\gamma \quad \Rightarrow \quad \begin{Bmatrix} 1 \\ 1.5 \\ 2.5 \end{Bmatrix}}$$

$$T = \frac{1}{2}\dot{p}^t \gamma^t M \gamma \dot{p} \;= \frac{1}{2}\dot{p}^t \begin{bmatrix} 1 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{bmatrix} \begin{Bmatrix} 1 \\ 1.5 \\ 2.5 \end{Bmatrix} \dot{p}$$

$$= 11.5 m \dot{p}^2$$

$$U = \frac{1}{2} p^t \gamma^t K \gamma p \;= \frac{1}{2} p^t \begin{bmatrix} 1 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 1.5 \\ 2.5 \end{Bmatrix} p$$

$$= 1.25 k p^2$$

The RAYLEIGH-RITZ METHOD

Lagrange's Equation ➔

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \left(\frac{\partial T}{\partial p}\right) + \left(\frac{\partial U}{\partial p}\right) = 0$$

$$\frac{\partial T}{\partial \dot{p}} = 2*11.5\,m\,\dot{p}(t)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) = 23\,m\,\ddot{p}(t)$$

$$\left(\frac{\partial T}{\partial p}\right) = 0$$

$$\left(\frac{\partial U}{\partial p}\right) = 2*1.25\,k\,p(t)$$

$$\boxed{23m\,\ddot{p} + 2.5k\,p = 0}$$

$$\omega_1 = 0.3297\sqrt{\frac{k}{m}}$$

$$\omega_1 = 0.3243\sqrt{\frac{k}{m}}$$

$\sim 2\%$

Alain Berlioz 2011/12