Vector Spaces

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Visualizations in $\mathbb{R}^2$

Consider 2 vectors *v* and *w* in $\mathbb{R}^2$. Let $v = (1,1)$ and $\mathrm{w} = (-1,2)$. Interpret the following sets geometrically. Which of these are a subspaces of $\mathbb{R}^2$?

- Span(*v*)
- Span(*v*) ∪ Span(*w*)
- Span(*v*) ∩ Span(*w*)
- Span(*v, w*)
- Span(*v, w, x*) where $\mathrm{x} = (0,1)$
- $\{(1-t)v + tw: t \in (0,1)\}$
- $\{(1-t)v + tw: t \in \mathbb{R}\}$
- $\{av + bw: a, b \geq 0\}$
- $\{(a,b) \in \mathbb{R}^2: \mathrm{a}^2 + \mathrm{b}^2 \leq 25\}$
- $\{(a, a+5) \in \mathbb{R}^2: a \in \mathbb{R}\}$
- $\{(a, \mathrm{a}^2 - 6a) \in \mathbb{R}^2: a \in \mathbb{R}\}$

Linear Independence, Span, Basis and Dimension

1. Let $V \coloneqq \mathbb{R}^{n \times n}$ be the space of $n \times n$ matrices. Prove that V is a real vector space. Find the dimension of V
2. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ and $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for dim(Span(v_1, v_2, v_3, v_4))? No proof necessary
3. True or False: If B is a basis of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$, then a subset of B is the basis of W
4. Consider the non-empty set of functions $V \coloneqq \{p: \mathbb{R} \to \mathbb{R} \mid p(x) = \Sigma_{k=0}^{n} a_k x^k \; for \; a_k \in \mathbb{R}, x \in \mathbb{R}\}$. Define an addition operation $+: V \times V \to V$ and a scalar multiplication operation $\times: \mathbb{R} \times V \to V$ such that the triple $(V, +, \times)$ is a real vector space. Find a basis of this vector space and deduce its dimension
5. Suppose $(v_1, v_2, \ldots, v_m) \in \mathbb{R}^n$ be linearly dependent. Prove that for x $\in span(v_1, v_2, \ldots, v_m)$, there exist infinitely many $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ such that $x = \Sigma \alpha_i v_i$