Recitation – Week 2

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Announcements

Office hour timings: Thursday, 1-2 PM, CDS Room 650

HW 2 due: 17th Sept 2019

Linear transformations

1. Project every vector $v \in \mathbb{R}^3$ onto the plane $z = 0$. How is this transformation defined? Is this a linear transformation? If yes, what's the matrix corresponding to this transformation? Also, what is the kernel and image of this transformation?
2. Which of the following functions are linear?
   a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v_1, v_2) = (v_2, 4v_1 + v_2, 0)$
   b) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $T(v_1, v_2) = v_1 - v_2$+5
   c) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $T(v_1, v_2) = \sqrt{\mathrm{v}_1^2 + v_2^2}$

3. Given a linear transformation $L: \mathbb{R}^m \rightarrow \mathbb{R}^n$, show that $\ker(L)$ is a subspace of $\mathbb{R}^m$ and $Im(L)$ is a subspace of $\mathbb{R}^n$

Linear transformations

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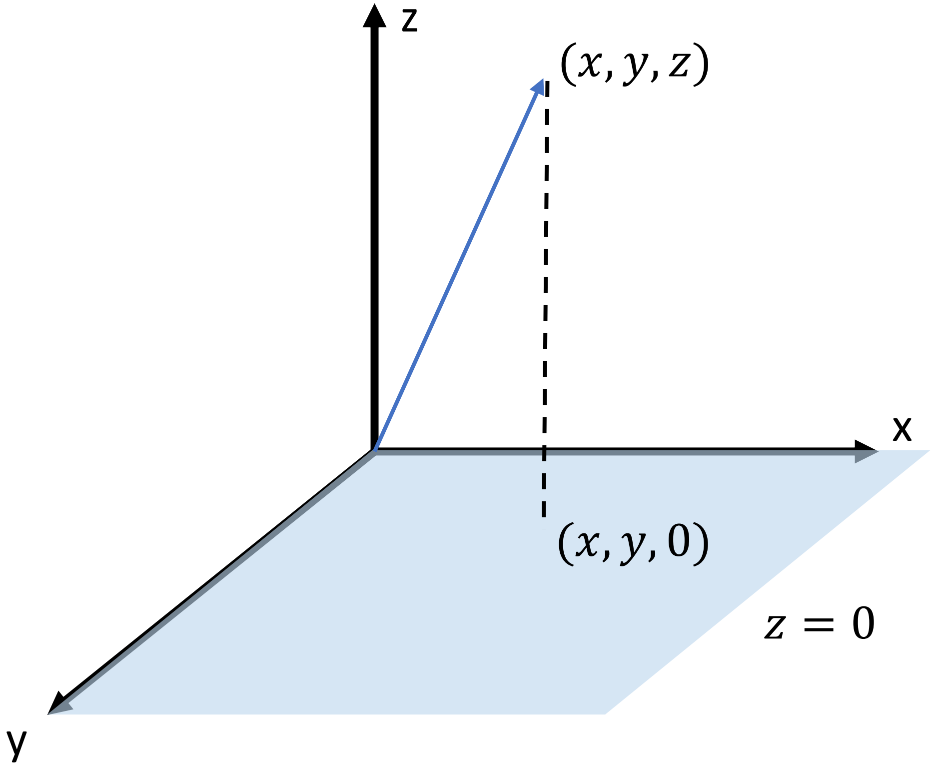

Matrix multiplication (Method 1)

- Let $A \in \mathbb{R}^{m \times n}$ with columns $a_1, \dots, a_n$ and let $x = [x_1, \dots., x_n]^T \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vdots \\ a_1 \\ \vdots \end{bmatrix} +$$

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$$Ax = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vdots \\ a_1 \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} \vdots \\ a_2 \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vdots \\ a_n \\ \vdots \end{bmatrix}$$

Matrix multiplication (Method 2)

- Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$and let $x = [x_1, \ldots., x_m] \; \in \mathbb{R}^m$

$$xA = [x_1 x_2 \ldots x_m] \begin{bmatrix} - \; a_1^T \; - \\ - \; a_2^T \; - \\ \vdots \\ - \; a_m^T \; - \end{bmatrix} = x_1 [\cdots a_1^T \cdots] +$$

Matrix multiplication (Method 2)

- Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x = [x_1, \ldots., x_m] \in \mathbb{R}^m$

$$xA = [x_1 x_2 \ldots x_m] \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} = x_1[\cdots a_1^T \cdots] + x_2[\cdots a_2^T \cdots] +$$

Matrix multiplication (Method 2)

- Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x = [x_1, \ldots., x_m] \in \mathbb{R}^m$

$$xA = [x_1 x_2 \ldots x_m] \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} = x_1[\cdots a_1^T \cdots] + x_2[\cdots a_2^T \cdots] + \cdots + x_m[\cdots a_m^T \cdots]$$

Matrix multiplication (2 new ways)

- Let $A \in \mathbb{R}^{m\times n}$ with columns $a_1, \ldots, a_n$ and let $x = [x_1, \ldots., x_n]^T \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} \vdots & \vdots & \vdots \\ a_1 & \cdots & a_n \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vdots \\ a_1 \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vdots \\ a_n \\ \vdots \end{bmatrix}$$

- Let $A \in \mathbb{R}^{m\times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x = [x_1, \ldots., x_m] \in \mathbb{R}^m$

$$xA = [x_1 \ldots x_m] \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} = x_1[\cdots a_1^T \cdots] + \cdots + x_m[\cdots a_m^T \cdots]$$

Both these methods can be easily extended for cases where x is a matrix

Matrix multiplication

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 4 \\ 5 & 2 & 1 \end{bmatrix}$
   a) How can you swap the the first and third column of A via matrix multiplication?
   b) How can you replace the second row with twice the first row added to the second row of A and then swap the obtained second row with the third row via matrix multiplication?
2. Fix $A \in \mathbb{R}^{4 \times 5}$. Describe the following set:

$$\left\{ Ax : x = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ c \end{bmatrix}, a, b, c \in \mathbb{R} \right\}$$

Linear transformations

1. If $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear transformation such that $\ker(T) = \{0\}$, and $v_1, \dots, v_k$ is a list of vectors in $\mathbb{R}^m$, then prove that $v_1, \dots, v_k$ is linearly independent iff $Tv_1, \dots Tv_k$ is linearly independent
2. If $L: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such:

$$L(1,2) = (1,3,0)$$
$$L(2,3) = (0,1,1)$$

   Write the matrix representation of L.

Revisiting Basis

1. Prove that any basis for $\mathbb{R}^n$ has length n

Revisiting Basis

Lemma 3.1: Let $v_1, .., v_m$ span $\mathbb{R}^n$ and suppose $w_1, \ldots, w_p \in \mathbb{R}^n$ with $p > m$. Then $w_1, \ldots, w_p$ are linearly dependent