Recitation – Week 3

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Announcements

- Office Hours: Thursday, 1-2 PM, CDS Room 650
- HW 3 due: September 24 2019

Kernel, rank and invertibility

1. Given a matrix $L \in \mathbb{R}^{m \times n}$, show that $Ker(L) = Ker(L^T L)$
2. Given a matrix $L \in \mathbb{R}^{m \times n}$, prove that $rank(L) = rank(L^T)$
3. Let $A \in \mathbb{R}^{n \times n}$ and $\mathrm{B} \in \mathbb{R}^{n \times n}$. Show that AB is invertible if and only if A and B are invertible
4. Let $A \in \mathbb{R}^{m \times n}$ where $m < n$. Under what conditions is there a matrix B such that $AB = I$ where I is the $m \times m$ identity matrix
5. Let $A \in \mathbb{R}^{m \times n}$ where $m > n$. Under what conditions is there a matrix B such that $BA = I$ where I is the $n \times n$ identity matrix

Kernel, rank and invertibility

1. Given a matrix $L \in \mathbb{R}^{m\times n}$, show that $Ker(L) = Ker(L^T L)$

Solution: $L: \mathbb{R}^n \to \mathbb{R}^m$

a. Let's assume $x \in Ker(L) \Rightarrow Lx = 0$

$$\Rightarrow L^T Lx = L^T 0 = 0$$

$$\Rightarrow L^T L(x) = 0$$

$$\Rightarrow x \in Ker(L^T L)$$

$$\Rightarrow Ker(L) \subset Ker(L^T L)$$

Kernel, rank and invertibility

b. Let's assume $x \in Ker(L^T L) \Rightarrow L^T Lx = 0$

$$\Rightarrow x^T L^T Lx = 0$$

$$\Rightarrow (Lx)^T Lx = 0$$

$$\Rightarrow Lx = 0 \Rightarrow x \in Ker(L)$$

$$\Rightarrow Ker(L^T L) \subset Ker(L)$$

From a and b, $Ker(L) = Ker(L^T L)$

Kernel, rank and invertibility

2. Given a matrix $L \in \mathbb{R}^{m\times n}$, prove that $rank(L) = rank(L^T)$

Solution:

From previous question, $Ker(L) = Ker(L^T L)$

Both $L\ and\ L^T L$ have $\mathbb{R}^n$ as their domain

$$\Rightarrow n - \dim(Ker(L)) = n - \dim(Ker(L^T L))$$

$$\Rightarrow Rank(L) = Rank(L^T L)$$

Similarly, $Rank(L^T) = Rank(LL^T)$

Also, using $Rank(A) \geq Rank(AB)$,

$$Rank(L) \geq Rank(LL^T)\ and\ Rank(L^T) \geq Rank(L^T L)$$

But we know that $Rank(L) = Rank(L^T L)\ and\ Rank(LL^T) = Rank(L^T)$

$$\Rightarrow Rank(L) \geq Rank(L^T)\ and\ Rank(L^T) \geq Rank(L) \Rightarrow Rank(L) = Rank(L^T)$$

Kernel, rank and invertibility

3. Let $A \in \mathbb{R}^{n \times n}$ and $\mathrm{B} \in \mathbb{R}^{n \times n}$. Show that AB is invertible if and only if A and B are invertible

Solution:

a. If A and B are invertible, then A^{-1} and B^{-1} exist

$$\Rightarrow B^{-1}\, A^{-1} AB = I$$

$\Rightarrow The\ inverse\ of\ AB\ is\ B^{-1}A^{-1}$

b. Given that AB is invertible, $Rank(AB) = n$ and $Ker(AB) = \{0\}$

$Rank(A) \geq Rank(AB) \Rightarrow Rank(A) \geq n$ but since $A \in \mathbb{R}^{n \times n}$, $rank(A) \leq n$

$\Rightarrow Rank(A) = n \Rightarrow A\ is\ invertible$

Also, if $x \in Ker(B) \Rightarrow Bx = 0 \Rightarrow ABx = A.0 = 0$

$\Rightarrow x \in Ker(AB) \Rightarrow Ker(B) \subset Ker(AB) = \{0\}$

$\Rightarrow Ker(B) = \{0\} \Rightarrow B$ is invertible

Kernel, rank and invertibility

4. Let $A \in \mathbb{R}^{m \times n}$ where $m < n$. Under what conditions is there a matrix B such that $AB = I$ where I is the $m \times m$ identity matrix

Solution:

$$A \begin{bmatrix} \vdots & & \vdots \\ b_1 & \cdots & b_m \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ e_1 & \cdots & e_m \\ \vdots & & \vdots \end{bmatrix}$$

$$\Rightarrow Ab_1 = e_1, \ldots, Ab_m = e_m$$

This means that $e_1, \ldots, e_m$ should be in the Im(A)

$$\Rightarrow Rank(A) \geq \mathrm{m}$$

But Rank(A)$\leq m$ because it only has m rows

$\Rightarrow Rank(A) = m$ for AB=I

Kernel, rank and invertibility

5. Let $A \in \mathbb{R}^{m \times n}$ where $m > n$. Under what conditions is there a matrix B such that $BA = I$ where I is the $n \times n$ identity matrix

Solution:

$$BA = I \Rightarrow B^T A^T = I$$

$$A^T \in \mathbb{R}^{n \times m} \; with \; n < m$$

Now this is exactly same as last question

$$Rank(A^T) = Rank(A) = n \; for \; BA = I$$

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \dots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \; a_1^T \; - \\ - \; a_2^T \; - \\ \cdot \\ \cdot \\ \cdot \\ - \; a_m^T \; - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \dots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \; a_1^T \; - \\ - \; a_2^T \; - \\ \vdots \\ - \; a_m^T \; - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \\ \\ \end{bmatrix}$$

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m\times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \; a_1^T \; - \\ - \; a_2^T \; - \\ \cdot \\ \cdot \\ \cdot \\ - \; a_m^T \; - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \cdot \\ \cdot \\ \cdot \\ a_m^T x \end{bmatrix}$$

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \dots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \ a_1^T \ - \\ - \ a_2^T \ - \\ \cdot \\ \cdot \\ \cdot \\ - \ a_m^T \ - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \cdot \\ \cdot \\ \cdot \\ a_m^T x \end{bmatrix}$$

Matrix multiplication (3rd way)

$$AX = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ x & y \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1^T x & a_1^T y \\ a_2^T x & a_2^T y \\ \vdots & \vdots \\ a_m^T x & a_m^T y \end{bmatrix}$$

Matrix multiplication (3rd way)

$$AX = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ x & y \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1^T x & a_1^T y \\ a_2^T x & a_2^T y \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_m^T x & a_m^T y \end{bmatrix}$$

Matrix multiplication (4th way)

Let $A \in \mathbb{R}^{m\times n}$ and B $\in \mathbb{R}^{n\times k}$. Let $a_1, \dots, a_n$ be the columns of A and $b_1^T, \dots, b_n^T$ denote the rows of B

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} =$$

Matrix multiplication (4th way)

Let $A \in \mathbb{R}^{m\times n}$ and B $\in \mathbb{R}^{n\times k}$. Let $a_1, \dots, a_n$ be the columns of A and $b_1^T, \dots, b_n^T$ denote the rows of B

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} = a_1 b_1^T +$$

Matrix multiplication (4th way)

Let $A \in \mathbb{R}^{m\times n}$ and B $\in \mathbb{R}^{n\times k}$. Let $a_1, \dots, a_n$ be the columns of A and $b_1^T, \dots, b_n^T$ denote the rows of B

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} = a_1 b_1^T + a_2 b_2^T +$$

Matrix multiplication (4th way)

Let $A \in \mathbb{R}^{m\times n}$ and B $\in \mathbb{R}^{n\times k}$. Let $a_1, \dots, a_n$ be the columns of A and $b_1^T, \dots, b_n^T$ denote the rows of B

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} = a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

Matrix multiplication

1. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of xy^T? What is it's rank?
2. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of $y^T x$? What is it's rank?
3. True or False: Let $A \in \mathbb{R}^{3\times 2}$ and $B \in \mathbb{R}^{2\times 3}$, then the rank of AB can be 3
4. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$, then show that the matrix product AB can be expressed as: $AB = C_1 + \cdots + C_k$ such that $rank(C_i) \leq 1\ \forall i \in [1, k]$

Matrix multiplication

1. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of xy^T? What is it's rank?

Solution: Let $y^T = [y_1, \dots, y_n]$

$$\Rightarrow xy^T = \begin{bmatrix} \vdots & & \vdots \\ y_1 x & \cdots & y_n x \\ \vdots & & \vdots \end{bmatrix}$$

All the columns of xy^T are just scalar multiples of the column vector x

$$\Rightarrow Rank(xy^T) = 1 \; or \; 0(when \; x \; or \; y = 0)$$

Matrix multiplication

2. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of $y^T x$? What is it's rank?

Solution:

$$y^T x = \Sigma_i y_i x_i \in \mathbb{R}^{1\times 1}$$

$$\Rightarrow Rank(y^T x) = 1$$

Matrix multiplication

3. True or False: Let $A \in \mathbb{R}^{3\times 2}$ and $B \in \mathbb{R}^{2\times 3}$, then the rank of AB can be 3

Solution: False

$$Rank(AB) \leq Rank(A) \leq 2 \; since \; A \; has \; just \; 2 \; columns$$

Matrix multiplication

4. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$, then show that the matrix product AB can be expressed as: $AB = C_1 + \cdots + C_k$ such that $rank(C_i) \leq 1\ \forall i \in [1, k]$

Solution:

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_k \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_k^T & \cdots \end{bmatrix} = a_1 b_1^T + \cdots + a_k b_k^T = C_1 + \cdots + C_k$$

$$C_i = a_i b_i^T \ and\ from\ previous\ questions, we\ know\ that\ Rank(C_i) = Rank\left(a_i b_i^T\right) \leq 1$$