Iul. 2019

$f:(-2,2)\to\mathbb{R}$, $f(x)=\ln\frac{x+2}{2-x}$. Seria Taylor, $x_0=0$

$$S=\sum_{n=0}^{\infty} f^{(n)}(x_0)\cdot\frac{(x-x_0)^n}{n!}=\sum_{n=0}^{\infty}\boxed{f^{(n)}(0)}\cdot\frac{x^n}{n!}$$

$$f(x)=\ln\frac{x+2}{2-x}=\ln(2+x)-\ln(2-x)$$

$$f'(x)=\frac{1}{2+x}+\frac{1}{2-x}=(2+x)^{-1}+(2-x)^{-1}$$

$$f''(x)=(-1)\cdot(2+x)^{-2}+(2-x)^{-2}$$

$$f'''(x)=(-1)(-2)(2+x)^{-3}+1\cdot 2\,(2-x)^{-3}$$

$$f^{(n)}(x)=(-1)^{n+1}\cdot(n-1)!\cdot(2+x)^{-n}+(n-1)!\,(2-x)^{-n},\ n\geq 1.$$

(Inducție)

$$f(0)=\ln 2-\ln 2=0$$

$$f^{(n)}(0)=(-1)^{n+1}\cdot(n-1)!\cdot 2^{-n}+(n-1)!\cdot 2^{-n}$$

$$=\frac{(n-1)!}{2^n}\left((-1)^{n+1}+1\right)=\begin{cases}0, & n \text{ par}\\ \frac{(n-1)!}{2^{n-1}}, & n \text{ impar}\end{cases}$$

$$S=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\cancel{!}}\cdot\frac{\cancel{(2n)!}}{2^{2n}}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^{2n}(2n+1)}$$

② $J(\alpha)=\int_0^1 \frac{x-1}{x^\alpha-1}\,dx$, $\boxed{\alpha>0.}$ $J(3)=?$

Anom pb. în 1.

$f:[0,1)\to\mathbb{R}_+$, $f(x)=\frac{x-1}{x^\alpha-1}$.

$$l=\lim_{\substack{x\to1\\x<1}}(1-x)^p\cdot\frac{x-1}{x^\alpha-1}\underset{p=0}{=}\lim_{\substack{x\to1\\x<1}}\frac{x-1}{x^\alpha-1}\left[\frac{0}{0}\right]\underset{\text{L'H}}{=}$$

$$=\lim_{\substack{x\to1\\x<1}}\frac{1}{\alpha\cdot x^{\alpha-1}}=\frac{1}{\alpha}<\infty, \text{ pt. că } \alpha>0$$

$$\left.\begin{array}{l}p=0<1\\ l=\frac{1}{\alpha}<\infty\end{array}\right\}\Rightarrow J(\alpha)\ \boxed{C},\ \forall\alpha>0.$$

$$J(3)=\int_0^1\frac{x-1}{x^3-1}\,dx=\int_0^1\frac{\cancel{x-1}}{\cancel{(x-1)}(x^2+x+1)}\,dx=$$

$$=\int_0^1\frac{1}{x^2+\underbrace{x}_{2\cdot x\cdot\frac{1}{2}}+1}\,dx=\int_0^1\frac{1}{x^2+x+\left(\frac{1}{2}\right)^2+\frac{3}{4}}\,dx=$$

$$=\int_0^1\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt3}{2}\right)^2}\,dx=\frac{2}{\sqrt3}\,\operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt3}{2}}\right)\Bigg|_0^1=$$

$$=\frac{2}{\sqrt3}\left(\operatorname{arctg}\sqrt3-\operatorname{arctg}\frac{1}{\sqrt3}\right)=\frac{2}{\sqrt3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{2}{\sqrt3}\cdot\frac{\pi}{6}=\frac{\pi}{3\sqrt3}.$$

③ $f: \overline{B}(0_2,1) \to \mathbb{R}$, $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$

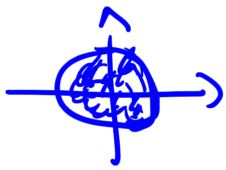
a) Ar. că f cont. în $(0,0)$.

f cont. în $(0,0) \iff \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$

$$\left|\frac{x^3+y^3}{x^2+y^2} - 0\right| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x^3|+|y^3|}{x^2+y^2} =$$

$$= \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} = \frac{|x|\cdot|x|^2}{x^2+y^2} + \frac{|y|\cdot|y|^2}{x^2+y^2} =$$

$$= |x|\cdot\underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} + |y|\cdot\underbrace{\frac{y^2}{x^2+y^2}}_{\le 1} \le |x|+|y| \xrightarrow[(x,y)\to(0,0)]{} 0$$



$\Rightarrow f$ cont. în 0_2.

b) Valorile extreme ale lui f. Se ating?

$\left.\begin{array}{l} A \text{ compactă (mărg.+închisă)} \\ f \text{ cont.}\end{array}\right\} \overset{\text{T. Weierstrass}}{\Longrightarrow}$ f își atinge extremele pe A.

$\overline{B}(0_2,1) = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$

• A închisă, $A = \text{int} A \cup \text{fr} A$

$\text{int} A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 < 1\}$, $\text{fr} A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = 1\}$

$$\boxed{f(x,y) = \frac{x^3+y^3}{x^2+y^2}}$$

intA: I. $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} \frac{3x^2(x^2+y^2) - 2x(x^3+y^3)}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2} \\ \frac{y^4+3x^2y^2-2x^3y}{(x^2+y^2)^2} \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x-0} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = 1$$

$$\begin{cases} x^4 + 3\cancel{x^2}y^2 - 2xy^3 = 0 \\ y^4 + 3\cancel{x^2}y^2 - 2x^3y = 0 \end{cases} \ominus$$

$$\underbrace{x^4-y^4} - 2xy^3 + 2x^3y = 0 \Rightarrow (x-y)(x+y)(x^2+y^2) - 2xy(\underbrace{y^2-x^2}_{=0})$$

$$= (x-y)(x+y)(x^2+y^2) + 2xy(x-y)(x+y) = 0$$

$$(x-y)(x+y)^3 = 0$$

$$\begin{cases} x = y \Rightarrow x^4 + 3x^4 - 2x^4 = 0 \Rightarrow 2x^4 = 0 \Rightarrow x = y = 0 \\ x = -y \Rightarrow x^4 + 3x^4 + 2x^4 = 0 \Rightarrow x = y = 0 \end{cases}$$

dar (0,0) nu e pct. critic.

Nu avem pct. critice în interior.

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frA: $\text{fr}A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = 1\}$

$F(x,y) = x^2+y^2-1.$

$\text{fr}A = \{(x,y)\in\mathbb{R}^2 \mid F(x,y) = 0\}$

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot F(x,y) = \frac{x^3+y^3}{x^2+y^2} + \lambda(x^2+y^2-1)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = \dfrac{x^4+3x^2y^2-2xy^3}{\underbrace{(x^2+y^2)^2}_{=1}} + 2x\lambda = x^4+3x^2y^2-2xy^3+2x\lambda=0 \\ \dfrac{\partial L}{\partial y} = y^4+3x^2y^2-2x^3y+2y\lambda=0 \\ \dfrac{\partial L}{\partial \lambda} = x^2+y^2-1=0 \end{cases}$$

$$\begin{cases} x^4+3x^2y^2-2xy^3+2x\lambda=0 \\ y^4+3x^2y^2-2x^3y+2y\lambda=0 \end{cases} \quad \ominus$$

$$x^4-y^4-2xy(y^2-x^2)+2\lambda(x-y)=0$$

$$(x-y)(x+y)(x^2+y^2)+2xy(x-y)(x+y)+2\lambda(x-y)=0$$

$$(x-y)(x+y)\cdot(\underbrace{x^2+y^2+2xy}_{(x+y)^2})+2\lambda(x-y)=0$$

$$(x-y)(x+y)^3+2\lambda(x-y)=0$$

$$(x-y)\left[(x+y)^3+2\lambda\right]=0$$

$\boxed{x=y} \xRightarrow{Ec3} 2x^2-1=0$

$x^2=\frac{1}{2}$

$x=\pm\frac{1}{\sqrt{2}}$

$y=\pm\frac{1}{\sqrt{2}}$

$2\lambda=-(x+y)^3$

$P_1\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $P_2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$* \; 2\lambda=-(x+y)^3$

$\xRightarrow{Ec1} x^4+3x^2y^2-2xy^3-x(x+y)^3=0$

$$x^4+3x^2y^2-2xy^3-x\left(x^3+3x^2y+3xy^2+y^3\right)=0$$

$$\cancel{x^4}+\cancel{3x^2y^2}-2xy^3-\cancel{x^4}-3x^3y-\cancel{3x^2y^2}-xy^3=0$$

$$-3xy^3-3x^3y=-3xy\underbrace{\left(y^2+x^2\right)}_{=1}=0$$

$$-3xy=0$$

Ec.3

$$\Rightarrow\begin{cases} x=0 \Rightarrow y=\pm 1 \\ y=0 \Rightarrow x=\pm 1\end{cases}$$

$P_3(0,1,\lambda)$, $P_4(0,-1,\lambda)$

$P_5(1,0,\lambda)$, $P_6(-1,0,\lambda)$

$$f\left(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)=\frac{\left(-\frac{1}{\sqrt2}\right)^3+\left(-\frac{1}{\sqrt2}\right)^3}{\frac12+\frac12}=\frac{-\frac{2}{(\sqrt2)^3}}{1}=-\frac{1}{\sqrt2}$$

$$f\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)=\frac{1}{\sqrt2}$$

$$\left.\begin{aligned} f(0,1)&=1\\ f(1,0)&=1\end{aligned}\right\}\ \text{max}$$

$$\left.\begin{aligned} f(0,-1)&=-1\\ f(-1,0)&=-1\end{aligned}\right\}\ \text{min.}$$

4) a) șir fundamental de nr. reale $\Rightarrow$ vezi curs

b) Ex. șir fundamental și nemonoton.

$$x_n=(-1)^n\cdot\frac{1}{n}\ \to \text{nemonoton}$$

Fie $\varepsilon>0$.

$x_n\to 0$ (convergent) $\Downarrow$ fundamental.

$$|x_n-0|=\left|\frac{(-1)^n}{n}\right|=\frac1n<\varepsilon \quad \text{pt. } n\ge n_0$$

$$\frac1n\le\frac{1}{n_0}<\varepsilon\Rightarrow n_0>\frac1\varepsilon$$

$$n_0=\left[\frac1\varepsilon\right]+1$$