Test - Analiză Matematică, Gr. 216 - R1 (22.11.2022)

1. Să se determine $a \in (0, \infty)$ astfel încât seria $\sum_{n=1}^{\infty} C_{2n}^{n} \cdot a^n$ să fie convergentă.
2. Determinați derivata de ordin $n \in \mathbb{N}$ a funcției $f : \mathbb{R} \setminus \{-\frac{b}{a}\} \to \mathbb{R}$, $f(x) = \frac{1}{ax+b}$.
3. $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^2}{8^n \cdot (2n)!}}$.

* Punctaj: 3 x 3p + 1p oficiu = 10p. Succes!

Test - Analiză Matematică, Gr. 216 - R2 (22.11.2022)

1. Să se determine $a \in (0, \infty)$ astfel încât seria $\sum_{n=1}^{\infty} \frac{1}{C_{2n}^{n} \cdot a^n}$ să fie convergentă.
2. Determinați derivata de ordin $n \in \mathbb{N}$ a funcției $f : (-\frac{b}{a}, \infty) \to \mathbb{R}$, $f(x) = \sqrt{ax+b}$, $a, b > 0$.
3. $\lim_{n\to\infty} \frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\ldots+\sqrt{1+n^2}}{1+n^2}$.

* Punctaj: 3 x 3p + 1p oficiu = 10p. Succes!

Test - Analiză Matematică, Gr. 217 - R1 (22.11.2022)

1. Să se determine $a \in (0, \infty)$ astfel încât seria $\sum_{n=1}^{\infty} \frac{4^n}{n\sqrt{n} \cdot a^n}$ să fie convergentă.
2. Se consideră funcția $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \ln(1+x^2) - 2\operatorname{arctg} x$. Determinați punctele sale de extrem local. Se ating valorile extreme?
3. $\lim_{x\to 0} \frac{\ln(1-\cos x)}{\ln(\sin^2(x))}$.

* Punctaj: 3 x 3p + 1p oficiu = 10p. Succes!

Test - Analiză Matematică, Gr. 217 - R2 (22.11.2022)

1. Să se determine $a \in (0, \infty)$ astfel încât seria $\sum_{n=1}^{\infty} \frac{a^n}{n\sqrt{n} \cdot 3^n}$ să fie convergentă.
2. Determinați punctele de extrem local ale funcției $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x - \operatorname{arctg} x$. Se ating valorile extreme?
3. $\lim_{x\to 1} \frac{x \cdot (\ln x - 1) + 1}{(x-1) \cdot \ln x}$.

* Punctaj: 3 x 3p + 1p oficiu = 10p. Succes!

$\boxed{216, R_1}$

① $a \in (0,\infty)$ $\quad \sum_{n=1}^{\infty} C_{2n}^{n} \cdot a^{n}$ să fie C.

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{C_{2n}^{n} \cdot a^{n}}{C_{2n+2}^{n+1} \cdot a^{n+1}} = \frac{\dfrac{(2n)!}{n! \cdot n!}}{\dfrac{(2n+2)!}{(n+1)! \cdot (n+1)!} \cdot a} =$$

$\boxed{0,5}$

$$= \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \frac{1}{a} = \frac{n^2+2n+1}{4an^2+6an+2a} = \frac{1}{4a} \boxed{0,5}$$

$\boxed{0,5}$ (calcule)

$D > 1 \quad \frac{1}{4a} > 1 \iff 4a < 1 \Rightarrow a < \frac{1}{4}$ C $\boxed{1}$
Se conv.

$D < 1 \quad \frac{1}{4a} < 1 \iff 4a > 1 \Rightarrow a > \frac{1}{4}$ D
Se div.

$a = \frac{1}{4}$

$$R = \lim_{n\to\infty} \left(\frac{n^2+2n+1}{n^2+\frac{3}{2}n+\frac{1}{2}} - 1 \right) \cdot n = \lim_{n\to\infty} \frac{\cancel{n^2}+2n+1-\cancel{n^2}-\frac{3}{2}n-\frac{1}{2}}{n^2+\frac{3}{2}n+\frac{1}{2}} \cdot n$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2}n^2+\frac{1}{2}n}{n^2+\frac{3}{2}n+\frac{1}{2}} = \frac{1}{2} < 1 \Rightarrow D \ \boxed{0,5}$$

$$\text{Seria este} \begin{cases} C, & a \in (0, 1/4) \\ D, & a \in [\frac{1}{4}, \infty) \end{cases}$$

② $f: \mathbb{R} \setminus \{-\frac{b}{a}\} \to \mathbb{R}, \ f(x) = \frac{1}{ax+b}$.

$\boxed{1,5p}$
$$\begin{cases} f(x) = (ax+b)^{-1} \\ f'(x) = (-1) \cdot (ax+b)^{-2} \cdot a \\ f''(x) = (-1)\cdot(-2)\cdot a^2 \, (ax+b)^{-3} \end{cases}$$

$f'''(x) = (-1)\cdot(-2)\cdot(-3)\cdot a^3 \cdot (ax+b)^{-4}$

$f^{(n)}(x) = (-1)^n \cdot n! \cdot a^n \cdot (ax+b)^{-(n+1)}$

$\boxed{1p}$

$n \in \mathbb{N}$

$\boxed{0,5}$ Inducție: Pp. $P(n)$: $f^{(n)}(x) = (-1)^n \cdot n! \cdot (ax+b)^{-(n+1)} \cdot a^n, n \in \mathbb{N}$

$P(0)$: $f(x) = (-1)^0 \cdot 0! \cdot (ax+b)^{-1} \cdot a^0 = \frac{1}{ax+b}$ „A"

Pp. $P(k)$ adev, $k \in \mathbb{N}$ arb., și dem. $P(k+1)$ adev.

$P(k+1)$: $f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' =$

$= \left((-1)^k \cdot k! \cdot (ax+b)^{-(k+1)} \cdot a^k\right)' = (-1)^k \cdot k! \cdot a^k \cdot (ax+b)^{-(k+1)-1} \cdot a \cdot (-(k+1))$

$= (-1)^{k+1} \cdot (k+1)! \cdot a^{k+1} \cdot (ax+b)^{-(k+2)}$ „A"

$\overset{P.I.M}{\Rightarrow}$ $P(n)$ adev.

③ $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^2}{8^n \cdot (2n)!}}$. Fie $x_n = \frac{(n!)^2}{8^n \cdot (2n)!}$ $\boxed{0,5}$

$\boxed{0,5}$ $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{8^{n+1} \cdot (2n+2)!} \cdot \frac{8^n \cdot (2n)!}{(n!)^2} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{8 \cdot (2n+1)(2n+2)}$ $\boxed{1p}$ (calcule) $= \lim_{n\to\infty} \frac{n^2+2n+1}{32n^2+48n+16} = \frac{1}{32}$ $\boxed{0,5}$

Consecință $\Rightarrow$ Stolz-Cesaro $\lim_{n\to\infty} \sqrt[n]{x_n} = \frac{1}{32}$. $\boxed{0,5}$

$\boxed{216, R_2}$

(1) $a\in(0,\infty)$ a.î. $\sum_{n=1}^{\infty} \frac{1}{C_{2n}^{n}\cdot a^{n}}$ să fie C.

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{C_{2n+2}^{n+1}\cdot a^{n+1}}{C_{2n}^{n}\cdot a^{n}} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\cdot a}{\frac{(2n)!}{n!\cdot n!}}$ $\boxed{0,5}$

$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)\cdot a}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{4n^2\cdot a + 6n\cdot a + 2a}{n^2+2n+1} = 4a$ $\boxed{0,5}$

$\boxed{0,5}$ (calcule)

Se conv. dacă $D > 1 \Rightarrow 4a > 1 \Rightarrow a > \frac{1}{4}$ 1

Se div. dacă $D < 1 \Rightarrow 4a < 1 \Rightarrow a < \frac{1}{4}$

„$a = \frac{1}{4}$"

$R = \lim_{n\to\infty} n\cdot\left(\frac{n^2+\frac{3}{2}n+\frac{1}{2}}{n^2+2n+1} - 1\right) = \lim_{n\to\infty} \frac{\cancel{n^2}+\frac{3}{2}n+\frac{1}{2}-\cancel{n^2}-2n-1}{n^2+2n+1}\cdot n$

$= \lim_{n\to\infty} \frac{-\frac{1}{2}n^2 - \frac{1}{2}n}{n^2+2n+1} = -\frac{1}{2} < 1 \Rightarrow$ serie div. $\boxed{0,5}$

Seria este $\begin{cases} C, \text{ dacă } a\in(1/4,\infty) \\ D, \text{ dacă } a\in(0,1/4] \end{cases}$

(2) $f: \left(-\frac{b}{a},\infty\right)\to\mathbb{R}$, $f(x) = \sqrt{ax+b}$, $a,b>0$.

$\boxed{1,5p}$ $\begin{cases} f(x) = (ax+b)^{1/2} \\ f'(x) = \frac{1}{2}\cdot(ax+b)^{-1/2}\cdot a \\ f''(x) = \frac{1}{2}\cdot\frac{-1}{2}\cdot(ax+b)^{-3/2}\cdot a^2 \\ f'''(x) = \frac{(-1)\cdot(-3)}{2^3}\cdot(ax+b)^{-5/2}\cdot a^3 \end{cases}$

$f^{(IV)}(x) = \frac{(-1)\cdot(-3)\cdot(-5)}{2^4}\cdot(ax+b)^{-7/2}\cdot a^4$

$f^{(n)}(x) = \frac{(-1)^{n+1}\cdot(2n-3)!!}{2^n}\cdot a^n\cdot(ax+b)^{-\frac{(2n-1)}{2}}$ $\boxed{1p}$

$n\geq 2$

Inductie: $\boxed{0{,}5}$ Pp. $P(n): f^{(n)}(x) = \frac{(-1)^{n+1}\cdot(2n-3)!!}{2^n}\cdot a^n\cdot(ax+b)^{-\frac{(2n-1)}{2}}$, $n\geq 2$

$P(2)$: $f''(x) = \frac{(-1)^{2+1}\cdot 1!!}{2^2}\cdot a^2\cdot(ax+b)^{-\frac{3}{2}} = -\frac{a^2}{2^2}\cdot(ax+b)^{-3/2}$ „A"

Pp. $P(k)$ adev., $k\in\mathbb{N}$ arb., și dem. $P(k+1)$ adev.

$P(k+1)$: $f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' =$

$= \left(\frac{(-1)^{k+1}\cdot(2k-3)!!}{2^k}\cdot a^k\cdot(ax+b)^{-\frac{(2k-1)}{2}}\right)' =$

$= \frac{(-1)^{k+1}\cdot(2k-3)!!}{2^k}\cdot a^k\cdot\left(-\frac{(2k-1)}{2}\right)\cdot(ax+b)^{-\frac{(2k-1)}{2}-1}\cdot a$

$= \frac{(-1)^{k+2}\cdot(2k-1)!!}{2^{k+1}}\cdot a^{k+1}\cdot(ax+b)^{-\frac{(2k+1)}{2}}$ $\rightarrow = -\frac{(2(k+1)-1)}{2}$

„A"

$\overset{P.I.M.}{\Rightarrow}$ $P(n)$ „A", $\forall n\geq 2$.

③ $\lim\limits_{n\to\infty}\frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\ldots+\sqrt{1+n^2}}{1+n^2}$

Fie $a_n = \sqrt{1+2^2}+\ldots+\sqrt{1+n^2}$

$\boxed{0{,}5}$ $b_n = 1+n^2 \rightarrow$ crescător cu $\lim\limits_{n\to\infty} b_n = \infty$.

$\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim\limits_{n\to\infty}\frac{\sqrt{1+(n+1)^2}}{1+(n+1)^2-n^2-1} = \lim\limits_{n\to\infty}\frac{\sqrt{n^2+2n+2}}{n^2+2n+1-n^2} =$

$\boxed{0{,}5}$ $\boxed{1p}$ (calcule)

$= \lim\limits_{n\to\infty}\frac{\sqrt{n^2+2n+2}}{2n+1} = \lim\limits_{n\to\infty}\frac{\cancel{n}\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{\cancel{n}\left(2+\frac{1}{n}\right)} = \frac{1}{2}$ $\boxed{0{,}5}$

$\overset{Stolz-Cesaro}{\Rightarrow}$ $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2}$. $\boxed{0{,}5}$

$\boxed{217, R_1}$

① $a \in (0,\infty)$ a.î. $\sum_{n=1}^{\infty} \frac{4^n}{n\sqrt{n}\cdot a^n}$ să fie C

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{4^n}{n\sqrt{n}\cdot a^n} \cdot \frac{a^{n+1}\cdot(n+1)\sqrt{n+1}}{4^{n+1}} =$ $\boxed{0,5}$

$\boxed{0,5}$ (calcule)

$= \lim_{n\to\infty} \frac{a}{4}\cdot\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = \frac{a}{4}$ $\boxed{0,5}$

$D>1$
Se conv. $\frac{a}{4} > 1 \Rightarrow a > 4$ C $\boxed{1}$

$D<1$
Se div. $\frac{a}{4} < 1 \Rightarrow a < 4$ D

$a = 4$.

$\sum_{n=1}^{\infty} \frac{\cancel{4^n}}{n\sqrt{n}\cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergentă $\boxed{0,5}$

(seria armonică generalizată cu $p = 3/2 > 1$)

S este $\begin{cases} C, & a \in [4,\infty) \\ D, & a \in (0,4) \end{cases}$

② $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \ln(1+x^2) - 2\,\mathrm{arctg}\, x$.

$f'(x) = \frac{2x}{1+x^2} - \frac{2}{1+x^2} = \frac{2(x-1)}{1+x^2}$ $\boxed{0,5}$

$f'(x) = 0 \Rightarrow x = 1$ $\boxed{0,5}$

1 - pct. de minim local $\boxed{1p}$

x	$-\infty$		1		∞
$f'(x)$		$- - -$	0	$+ + +$	
$f(x)$	∞	$\searrow$	$\ln 2 - \pi/2$	$\nearrow$	∞

$f(1) = \ln 2 - 2\,\mathrm{arctg}\,1 = \ln 2 - \frac{\pi}{2}$

$\lim_{x\to-\infty}\left[\ln(1+x^2) - 2\,\mathrm{arctg}(-x)\right] = \infty + 2\cdot\pi/2 = \infty$

$\inf = \ln 2 - \pi/2$ (se atinge) $\boxed{0,5}$

$\lim_{x\to\infty}\left[\ln(1+x^2) - 2\,\mathrm{arctg}\, x\right] = \infty - 2\cdot\pi/2 = \infty$

$\sup = \infty$ (nu se atinge) $\boxed{0,5}$

③ $\lim\limits_{x\to 0} \dfrac{\ln(1-\cos x)}{\ln(\sin^2 x)}$ $\left[\dfrac{\infty}{\infty}\right]$ $\underset{\boxed{0{,}5}}{\overset{l'H}{=}}$ $\lim\limits_{x\to 0} \dfrac{\frac{1}{1-\cos x}\cdot \sin x}{\frac{1}{\sin^2 x}\cdot 2\cancel{\sin x}\cos x}$ $\boxed{0{,}5}$

calcule $\boxed{0{,}5}$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{1-\cos x}\cdot\dfrac{\sin x}{2\cos x} = \lim\limits_{x\to 0}\dfrac{\sin^2 x}{2\cos x - 2\cos^2 x}$ $\left[\dfrac{0}{0}\right]$

$\underset{\boxed{0{,}5}}{\overset{l'H}{=}} \lim\limits_{x\to 0}\dfrac{2\sin x\cos x}{-2\sin x + 4\cos x\sin x} = \lim\limits_{x\to 0}\dfrac{\cancel{2\sin x}\cdot\cos x}{\cancel{2\sin x}(-1+2\cos x)} =$

calcule $\boxed{0{,}5}$

$= \lim\limits_{x\to 0}\dfrac{\cos x}{\cancel{4}(-1+2\cos x)} = \dfrac{1}{\cancel{4}(-1+2)} = \cancel{\tfrac{1}{4}} = 1$ $\boxed{0{,}5}$

$\boxed{217, R2}$

1) $a \in (0,\infty)$ a.î. $\sum_{n=1}^{\infty} \frac{a^n}{n\sqrt{n} \cdot 3^n}$ să fie C.

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{a^n}{n\sqrt{n}\cdot 3^n} \cdot \frac{3^{n+1}}{a^{n+1}\cdot(n+1)\sqrt{n+1}} =$

$\boxed{0,5}$ $\boxed{0,5}$ (calcule)

$= \lim_{n\to\infty} \frac{3}{a} \cdot \frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = \frac{3}{a}$ $\boxed{0,5}$

S e conv. $D > 1 \Rightarrow \frac{3}{a} > 1 \Rightarrow$ $a < 3$, C $\boxed{1}$

S e div. $D < 1 \Rightarrow \frac{3}{a} < 1 \Rightarrow a > 3$, D

$a = 3$

$\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n\sqrt{n}\cdot \cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ C, deoarece e seria armonică $\boxed{0,5}$

gen. cu $p = 3/2 > 1$.

S este $\begin{cases} C, \text{ dacă } a \in (0,3] \\ D, \text{ dacă } a \in (3,\infty) \end{cases}$

2) $f : (0,\infty) \to \mathbb{R}$, $f(x) = \ln x - \operatorname{arctg} x$.

$f'(x) = \frac{1}{x} - \frac{1}{x^2+1} = \frac{x^2 - x + 1}{x(x^2+1)}$ $\boxed{0,5}$

$f'(x) = 0 \Rightarrow x^2 - x + 1 = 0 \Rightarrow \nexists$ răd. reale. $\boxed{0,5}$

$\Delta = 1 - 4 < 0$

x	0 $\quad\quad\quad\quad$ ∞
$f'(x)$	$+ + + + +$
$f(x)$	$-\infty \nearrow \infty$

$\nexists$ pct. de extrem local $\boxed{1p}$

$\inf = \lim_{\substack{x\to 0 \\ x>0}} (\ln x - \operatorname{arctg} x) = -\infty$ $\boxed{0,5}$

$\sup = \lim_{x\to\infty} (\ln x - \operatorname{arctg} x) = \infty - \pi/2 = \infty$ $\boxed{0,5}$

nu se ating val. extreme

③ $\lim\limits_{x\to 1} \dfrac{x\cdot(\ln x - 1) + 1}{(x-1)\ln x} \underset{\mathcal{L}'H}{\overset{\left[\frac{0}{0}\right]}{=}} \lim\limits_{x\to 1} \dfrac{\ln x - 1 + x\cdot\frac{1}{x}}{\ln x + (x-1)\cdot\frac{1}{x}} =$

$\boxed{0,5}$ $\boxed{0,5}$

$= \lim\limits_{x\to 1} \dfrac{\ln x}{\ln x + 1 - \frac{1}{x}} \underset{\mathcal{L}'H}{\overset{\left[\frac{0}{0}\right]}{=}} \lim\limits_{x\to 1} \dfrac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \dfrac{1}{1+1} = \dfrac{1}{2}$ $\boxed{0,5}$

calcule $\boxed{0,5}$ $\boxed{0,5}$ calcule $\boxed{0,5}$