Application of Gibbs sampling (MCMC)

— not required.

$$y_i = \alpha + \beta x_i + \varepsilon_i. \quad \varepsilon_i \overset{iid}{\sim} N(0, \text{precision} = \tau)$$

parameters: α. β. τ.

priors: $\alpha \sim N(0, \text{precision} = \tau_\alpha)$.

$\beta \sim N(0, \text{precision} = \tau_\beta)$.

$\tau_\alpha = \tau_\beta = 0.01$. i.e. var $= 100$.

$\tau \sim \text{Gamma}(\text{shape} = a, \text{rate} = b)$

$a = b = 0.001$.

Full conditional distributions:

$$p(\alpha \mid \{x_i, y_i\}_{i=1}^n, \beta, \tau) \propto p(\alpha) \cancel{p(y_i} \prod_{i=1}^n p(y_i \mid x_i, \alpha, \beta, \tau)$$

$$\propto N(\alpha \mid 0, \tau_\alpha) \cdot \prod_{i=1}^n N(y_i \mid \alpha + \beta x_i, \tau).$$

$\vdots$

$$\propto e^{-\frac{1}{2}(\tau_\alpha + n\tau)\left[\alpha^2 - 2 \cdot \frac{\tau \sum_{i=1}^n (y_i - \beta x_i)}{\tau_\alpha + n\tau} \alpha\right]}.$$

$$\alpha \mid \cdots \sim N\left(\frac{\tau \sum_{i=1}^n (y_i - \beta x_i)}{\tau_\alpha + n\tau}, \ \text{var} = (\tau_\alpha + n\tau)^{-1}\right).$$

Thm: If we have parameters $\theta_1, \theta_2 \cdots \theta_p$.

and data $x_1 \cdots x_n$, and we know

$$p(\theta_1 \mid \theta_2, \theta_3, \cdots \theta_p, x_1 \cdots x_n) \triangleq p_1$$

$$p(\theta_2 \mid \theta_1, \theta_3 \cdots \theta_p, x_1 \cdots x_n) \triangleq p_2$$

$$\vdots$$

$$p(\theta_p \mid \theta_1, \theta_2 \cdots \theta_{p-1}, x_1 \cdots x_n) \triangleq p_p$$

$\Rightarrow$ We draw $\theta_1^{(s)} \sim p_1$

$$\theta_2^{(s)} \sim p_2$$

$$\vdots$$

$$\theta_p^{(s)} \sim p_p.$$

and iterate $s = 1 \cdots S$. We get samples

$\{\theta_1^{(s)}, \theta_2^{(s)} \cdots \theta_p^{(s)}\}_{s=1}^{S}$, which is the

same as the posterior samples of $\{\theta_1 \cdots \theta_p\}$

if $S \to \infty$.

(Algorithm: Gibbs sampling
or Gibbs sampler).

$$p(\beta \mid \cdots) \propto p(\beta) \prod_{i=1}^{n} p(y_i \mid x_i, \alpha, \beta, \tau).$$

$$\vdots$$

$$\propto e^{-\frac{1}{2}\left(\tau_\beta + \tau \sum_{i=1}^{n} x_i^2\right)\left[\beta - \frac{\tau \sum_{i=1}^{n} x_i (y_i - \alpha)}{\tau_\beta + \tau \sum_{i=1}^{n} x_i^2}\right]^2}$$

$$\beta \mid \cdots \sim N\left(\frac{\tau \sum_{i=1}^{n} x_i (y_i - \alpha)}{\tau_\beta + \tau \sum_{i=1}^{n} x_i^2},\ var = \frac{1}{\tau_\beta + \tau \sum_{i=1}^{n} x_i^2}\right)$$

$$p(\tau \mid \cdots) \propto p(\tau) \prod_{i=1}^{n} p(y_i \mid x_i, \beta, \alpha, \tau).$$

$$\vdots$$

$$\propto \tau^{a + \frac{n}{2} - 1} e^{-\tau\left(b + \frac{1}{2}\sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2\right)}$$

$$\tau \mid \cdots \sim Gamma\left(a + \frac{n}{2},\ rate = b + \frac{1}{2}\sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2\right)$$