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Mixed-effects model.

$$y_{ij} = \alpha + \beta x_{ij} + \gamma_i + \varepsilon_{ij}.$$

$$\gamma_i \sim N(0, s^2)$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

We estimate $\alpha, \beta, s^2, \sigma^2$ by MLE.

$$p(y_{ij} \mid \alpha, \beta, \gamma_i, \sigma^2) = N(\alpha + \beta x_{ij} + \gamma_i, \sigma^2).$$

$$p(\gamma_i \mid s^2) = N(0, s^2).$$

$$p(y_{ij} \mid \alpha, \beta, s^2, \sigma^2) = \int_{-\infty}^{+\infty} p(y_{ij} \mid \alpha, \underbrace{\beta, \gamma_i, \sigma^2) p(\gamma_i \mid s^2)} d\gamma_i$$

$$= \int_{-\infty}^{+\infty} p(y_{ij}, \gamma_i \mid \alpha, \beta, \sigma^2, s^2)\, d\gamma_i.$$

$$\rightarrow p(y_{ij} \mid \alpha, \beta, \gamma_i, \sigma^2)$$

$$= E_{\gamma_i} p(y_{ij} \mid \alpha, \beta, \gamma_i, \sigma^2)$$

$$= \frac{1}{N} \sum_{n=1}^{N} p(y_{ij} \mid \alpha, \beta, \gamma_i^{(n)}, \sigma^2).$$

A trick to make MLE valid:

when sampling $\gamma_i^{(n)} \sim N(0, s^2)$,

~~oth~~ we let $\gamma_i^{(n)} = s \cdot Z^{(n)}$, $Z^{(n)} \sim N(0,1)$.

$$p(y_{ij} \mid \alpha, \beta, \gamma_i, \sigma^2)$$

$$= \frac{1}{N} \sum_{n=1}^{N} p(y_{ij} \mid \alpha, \beta, s \cdot Z^{(n)}, \sigma^2). \quad (*)$$

we optimize $(*)$ and find $\hat{\alpha}, \hat{\beta}, \hat{s}^2, \hat{\sigma}^2$.

where $p(y_{ij} \mid \alpha, \beta, s \cdot Z^{(n)}, \sigma^2)$

$$= \phi(y_{ij} \mid \alpha + \beta x_{ij} + s \cdot Z^{(n)}, \sigma^2).$$