Hw 3.

3-1.

$$p(p_1 \cdots p_K) \sim Dir(a_1, a_2 \cdots a_K).$$

$$\propto p_1^{a_1-1} p_2^{a_2-1} \cdots p_K^{a_K-1}$$

$$p(x_1 \cdots x_K \mid p_1 \cdots p_K) \sim Mult(n, p_1 \cdots p_K)$$

$$\propto p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$$

$$p(p_1 \cdots p_K \mid x_1 \cdots x_K) \propto p(p_1 \cdots p_K) \cdot p(x_1 \cdots x_K \mid p_1 \cdots p_K)$$

$$\propto p_1^{x_1+a_1-1} p_2^{x_2+a_2-1} \cdots p_K^{x_K+a_K-1}$$

Comparing the prior and $p(p_1 \cdots p_K \mid x_1 \cdots x_K)$,

we know the posterior $= Dir(x_1+a_1, x_2+a_2 \cdots x_K+a_K)$

3-2. $X_1 \cdots X_n \overset{iid}{\sim} Pois(\lambda)$. $p(x_i|\lambda) = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$

prior $\lambda \sim Gamma(a, \ rate = b)$.

$$f(\lambda) = \frac{b^a \lambda^{a-1}}{\Gamma(a)} e^{-b\lambda} \propto \lambda^{a-1} e^{-b\lambda}$$

posterior $p(\lambda | x_1 \cdots x_n)$

$$\propto p(x_1 \cdots x_n | \lambda) f(\lambda)$$

$$= f(\lambda) \prod_{i=1}^{n} p(x_i|\lambda)$$

$$= \frac{b^a \lambda^{a-1}}{\Gamma(a)} e^{-b\lambda} \cdot \prod_{i=1}^{n} \left[\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right].$$

$$\propto \lambda^{a-1} e^{-b\lambda} \prod_{i=1}^{n} \left[\lambda^{x_i} e^{-\lambda} \right].$$

$$= \lambda^{a + \left(\sum_{i=1}^{n} x_i\right) - 1} e^{-\lambda \cdot (b+n)}$$

$\lambda | x_1 \cdots x_n \sim Gamma(shape = a + \sum_{i=1}^{n} x_i, \ rate = b+n)$.

$E(\lambda | x_1 \cdots x_n) = shape/rate = \frac{a + \sum_{i=1}^{n} x_i}{b+n}$. (posterior mean).

$MLE(\lambda | x_1 \cdots x_n) = \text{sample mean} = \frac{1}{n} \sum_{i=1}^{n} x_i$

3-3 $x_i \overset{iid}{\sim} N(\mu, \frac{1}{\tau})$ var $= 1/\tau$

τ: precision.

~~$p(x_i) =$~~

$$p(x_i | \mu, \tau) = \frac{\tau^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} e^{-\frac{\tau}{2}(x_i - \mu)^2}$$

priors: $\mu \sim N(m, \sigma^2)$

$\tau \sim \text{Gamma}(a, \text{rate} = b)$

Posteriors $p(\mu | x_1 \cdots x_n)$, $p(\tau | x_1 \cdots x_n)$

are not in closed forms.

Full conditional dist'ns.

$$p(\mu | x_1 \cdots x_n, \tau) \propto \underbrace{p(\mu) \cdot p(x_1 \cdots x_n | \mu, \tau)}_{p(\mu|\tau) p(x_1 \cdots x_n | \mu, \tau)}.$$

$$\propto N(\mu | m, \sigma^2) \cdot \prod_{i=1}^{n} N(x_i | \mu, \tau).$$

$$\propto e^{-\frac{1}{2\sigma^2}(\mu - m)^2 - \frac{\tau}{2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

$$\propto e^{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2} + n\tau\right)\mu^2 - 2\mu\left(\frac{m}{\sigma^2} + \tau\sum_{i=1}^{n} x_i\right)\right]}$$

so $\mu | \{x_i\}, \tau \sim N\left(\left(\frac{1}{\sigma^2} + n\tau\right)^{-1}\left(\frac{m}{\sigma^2} + \tau\sum_{i=1}^{n} x_i\right), \text{var} = \left(\frac{1}{\sigma^2} + n\tau\right)^{-1}\right)$

$$p(\tau | x_1 \cdots x_n, \mu) \propto p(\tau | \mu) \, p(x_1 \cdots x_n | \tau, \mu).$$

$$= p(\tau) \, p(x_1 \cdots x_n | \tau, \mu)$$

$$= \text{Gamma}(\tau | a, b) \prod_{i=1}^{n} N(x_i | \mu, \tau).$$

$$\propto \tau^{a-1} e^{-b\tau} \prod_{i=1}^{n} \left[\tau^{\frac{1}{2}} e^{-\tau (x_i - \mu)^2} \right]$$

$$\propto \tau^{\frac{n}{2} + a - 1} e^{-\tau \left[b + \frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2 \right]}$$

Comparing with Gamma densities,

$$\tau | x_1 \cdots x_n, \mu \sim \text{Gamma}\left(\frac{n}{2} + a, \ \text{rate} = \frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2 + b \right)$$

$$E(\tau | x_1 \cdots x_n, \mu) = \frac{\frac{n}{2} + a}{\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2 + b}.$$

$$\tau = 1/\text{var}$$

$$\hat{\text{var}} = 1/\hat{\tau} = \frac{\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2 + b}{\frac{n}{2} + a}$$

If $a = b = 0$, $\hat{\text{var}} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2.$