Vectors

1. Vectors
2. Vector differentiation
3. Gradient, Curl & Divergence

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NED LMS

Vectors:

- A vector is a quantity having both magnitude and direction such as displacement, velocity, force and acceleration.

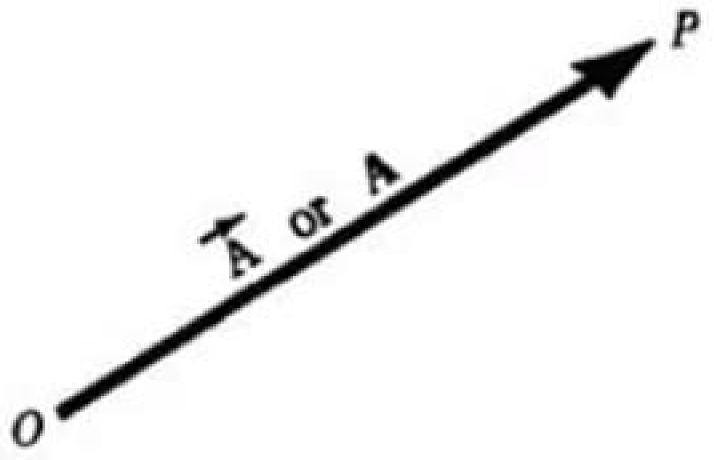

Fig.1

Continued...

- Graphically a vector is represented by an arrow OP (Fig.l) defining the direction, the magnitude of the vector being indicated by the length of the arrow.
- The tail end 0 of the arrow is called the origin or initial point.
- A vector is represented by a letter with an arrow over it, as A in Fig.1, and its magnitude is denoted by I AI or **A**

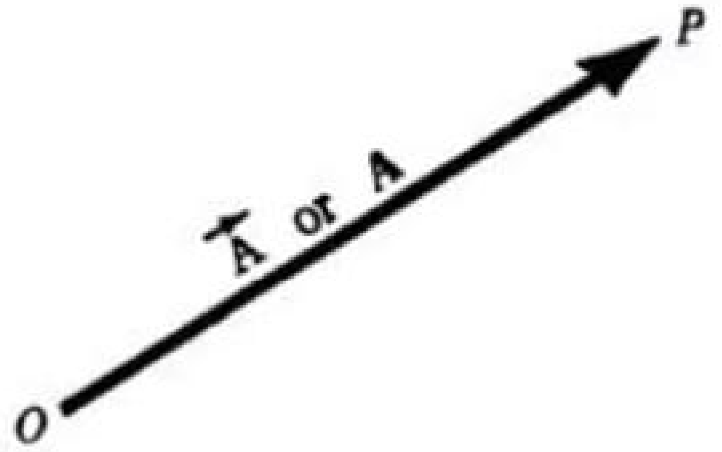

Fig.1

Scalars

- A scalar is a quantity having magnitude but no direction e.g. mass, distance, temperature and any real number.
- Scalars are indicated by letters in ordinary type as in elementary algebra.
- Operations with scalars follow the same rules as in elementary algebra.

Vectors Algebra:

- The operations of addition, subtraction and multiplication familiar in the algebra of numbers or scalars are, with suitable definition, capable of extension to an algebra of vectors.
- The following definitions are fundamental.

1. Two vectors A and B are equal if they have the same magnitude and direction regardless of the position of their initial points. Thus A= B in Fig.2.

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2. A vector having direction opposite to that of vector A but having the same magnitude is denoted by -A (Fig.3)

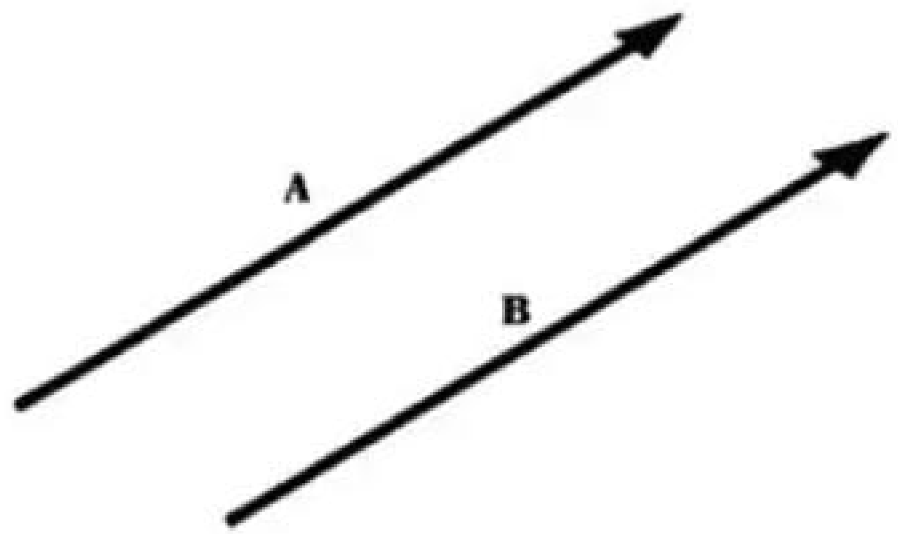


Fig. 2

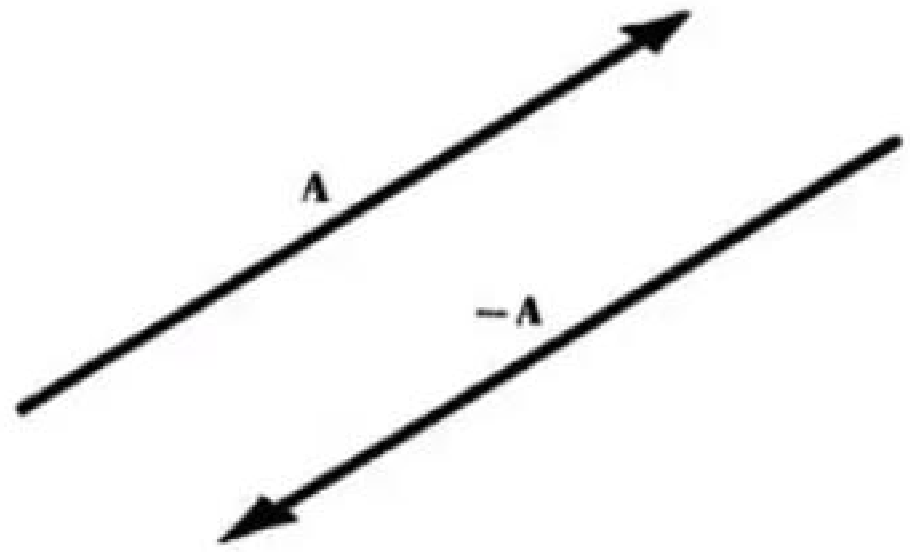


Fig. 3

Continued...

❖ Scalar Field

If to each point (x,y,z) of a region R in space there corresponds a number or scalar then is called a scalar function of position or scalar point function

Examples.

(1) The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

Continued...

1. State which of the following are scalars and which are vectors.

(a) weight (c) specific heat (e) density (g) volume (i) speed

(b) calorie (d) momentum (f) energy (h) distance (j) magnetic field intensity

Ans: (a) vector (c) scalar (e) scalar (g) scalar (i) scalar

(b) scalar (d) vector (f) scalar (h) scalar (j) vector

Formula:

- The sum of all three vectors is

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

- The magnitude of vector **A**

$$|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

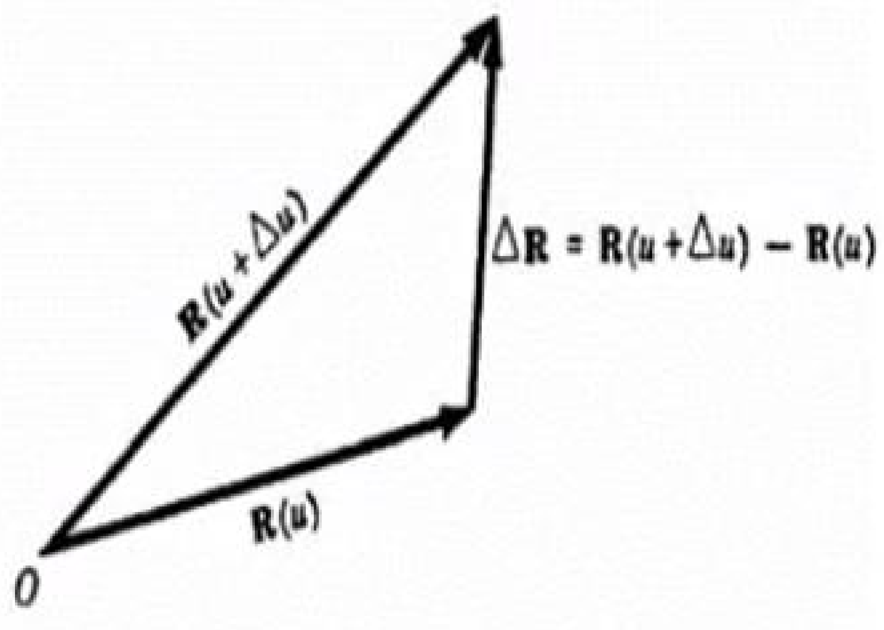
Vector Differentiation:

- There are two types of differentiation
- Ordinary derivative of a vector(vector depends on one variable)
- Partial derivatives of a vector(Vector depends on more than one variable)

Ordinary derivative of a vector

Let R(u) be a vector depending on a single scalar variable u.

$$\frac{\Delta \mathbf{R}}{\Delta u} = \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$



The ordinary derivative of the vector R(u) with respect to the scalar u is given by if the limit exists

$$\frac{d\mathbf{R}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathbf{R}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$

Continued...

- dR/du is itself a vector depending on u, we can consider its derivative with respect to u.
- If this derivative exists it is denoted by a d^2R/du^2. In like manner higher order derivatives are described.

Differential Formulas:

1. $\frac{d}{du}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$

2. $\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B}$

3. $\frac{d}{du}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \times \mathbf{B}$

4. $\frac{d}{du}(\phi \mathbf{A}) = \phi \frac{d\mathbf{A}}{du} + \frac{d\phi}{du} \mathbf{A}$

5. $\frac{d}{du}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} \times \frac{d\mathbf{C}}{du} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \times \mathbf{C} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B} \times \mathbf{C}$

6. $\frac{d}{du}\{\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\} = \mathbf{A} \times (\mathbf{B} \times \frac{d\mathbf{C}}{du}) + \mathbf{A} \times (\frac{d\mathbf{B}}{du} \times \mathbf{C}) + \frac{d\mathbf{A}}{du} \times (\mathbf{B} \times \mathbf{C})$

Differentiation Formulas

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(k) = 0, k \text{ is a constant}$$

$$\frac{d}{dx}(kx) = k, k \text{ is a constant}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}\left(e^{-x^2}\right) = -2x\, e^{-x^2}$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$$

Partial Differentiation Of A Vector

- If A is a vector depending on more than one scalar variable, say x, y, z for example, then we write A = A(x, y, z). The partial derivative of A with respect to x is defined as

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x+\Delta x, y, z) - \mathbf{A}(x,y,z)}{\Delta x}$$

if this limit exists. Similarly,

$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y+\Delta y, z) - \mathbf{A}(x,y,z)}{\Delta y}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x,y, z+\Delta z) - \mathbf{A}(x,y,z)}{\Delta z}$$

Rules for partial differentiation

1. $\dfrac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\dfrac{\partial \mathbf{B}}{\partial x} + \dfrac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}$

2. $\dfrac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\dfrac{\partial \mathbf{B}}{\partial x} + \dfrac{\partial \mathbf{A}}{\partial x}\times\mathbf{B}$

3. $\dfrac{\partial^2}{\partial y\,\partial x}(\mathbf{A}\cdot\mathbf{B}) = \dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})\right\} = \dfrac{\partial}{\partial y}\left\{\mathbf{A}\cdot\dfrac{\partial \mathbf{B}}{\partial x} + \dfrac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}\right\}$

$$= \mathbf{A}\cdot\frac{\partial^2 \mathbf{B}}{\partial y\,\partial x} + \frac{\partial \mathbf{A}}{\partial y}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\,\partial x}\cdot\mathbf{B}, \quad \text{etc.}$$

Problem

Q. If $\mathbf{A} = (2x^2y - x^4)\mathbf{i} + (e^{xy} - y\sin x)\mathbf{j} + (x^2\cos y)\mathbf{k}$, find: $\dfrac{\partial \mathbf{A}}{\partial x}, \dfrac{\partial \mathbf{A}}{\partial y}, \dfrac{\partial^2 \mathbf{A}}{\partial x^2}, \dfrac{\partial^2 \mathbf{A}}{\partial y^2}, \dfrac{\partial^2 \mathbf{A}}{\partial x\,\partial y}, \dfrac{\partial^2 \mathbf{A}}{\partial y\,\partial x}$.

Sol.

$$\frac{\partial \mathbf{A}}{\partial x} = \frac{\partial}{\partial x}(2x^2y - x^4)\mathbf{i} + \frac{\partial}{\partial x}(e^{xy} - y\sin x)\mathbf{j} + \frac{\partial}{\partial x}(x^2\cos y)\mathbf{k}$$

$$= (4xy - 4x^3)\mathbf{i} + (ye^{xy} - y\cos x)\mathbf{j} + 2x\cos y\,\mathbf{k}$$

$$= (4xy - 4x^3)\mathbf{i} + (ye^{xy} - y\cos x)\mathbf{j} + 2x\cos y\,\mathbf{k}$$

$$= 2x^2\mathbf{i} + (xe^{xy} - \sin x)\mathbf{j} - x^2\sin y\,\mathbf{k}$$

continued

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x}(4xy - 4x^3)\mathbf{i} + \frac{\partial}{\partial x}(ye^{xy} - y\cos x)\mathbf{j} + \frac{\partial}{\partial x}(2x\cos y)\mathbf{k}$$

$$= (4y - 12x^2)\mathbf{i} + (y^2e^{xy} + y\sin x)\mathbf{j} + 2\cos y\,\mathbf{k}$$

$$\frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y}(2x^2)\mathbf{i} + \frac{\partial}{\partial y}(xe^{xy} - \sin x)\mathbf{j} - \frac{\partial}{\partial y}(x^2\sin y)\mathbf{k}$$

$$= 0 + x^2e^{xy}\mathbf{j} - x^2\cos y\,\mathbf{k} = x^2e^{xy}\mathbf{j} - x^2\cos y\,\mathbf{k}$$

Continued

$$\frac{\partial^2 \mathbf{A}}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial y}\right) = \frac{\partial}{\partial x}(2x^2)\mathbf{i} + \frac{\partial}{\partial x}(xe^{xy} - \sin x)\mathbf{j} - \frac{\partial}{\partial x}(x^2 \sin y)\mathbf{k}$$

$$= 4x\,\mathbf{i} + (xye^{xy} + e^{xy} - \cos x)\mathbf{j} - 2x \sin y\,\mathbf{k}$$

$$\frac{\partial^2 \mathbf{A}}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial x}\right) = \frac{\partial}{\partial y}(4xy - 4x^3)\mathbf{i} + \frac{\partial}{\partial y}(ye^{xy} - y\cos x)\mathbf{j} + \frac{\partial}{\partial y}(2x\cos y)\mathbf{k}$$

$$= 4x\,\mathbf{i} + (xye^{xy} + e^{xy} - \cos x)\mathbf{j} - 2x \sin y\,\mathbf{k}$$

Note that $\frac{\partial^2 \mathbf{A}}{\partial y\,\partial x} = \frac{\partial^2 \mathbf{A}}{\partial x\,\partial y}$, i.e. the order of differentiation is immaterial. This is true in general if $\mathbf{A}$ has continuous partial derivatives of the second order at least.

The Vector Differential Operator

- The vector differential operator or del is written as ∇

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

- This vector operator possesses properties analogous to those of ordinary vectors.

- It is useful in defining three quantities which arise in practical applications and are known as the gradient, the divergence and the curl

The Gradient

- Let ϕ(x, y, z) be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the gradient of ϕ) ,written ▼ϕ or grad, is defined by ϕ

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

- ▼ϕ defines a vector field.
- The component of ▼ϕ in the direction of a nit vector given by ▼ϕ .**a** and is called the directional derivative of ▼ϕ in the direction a.

Problem

Q. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

Sol.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k}$$

$$= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$$

The Divergence

- Let $V(x, y, z) = V_1 i + V_2 j + V_3 k$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. V defines a differentiable vector field).
- Then the divergence of V, written ▼. V or div V, is defined by

$$\nabla \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

- Note the analogy with $A .B = A_1B_1 + A_2B_2 + A_3B_3$. Also note that ▼.V≠ V. ▼

The Curl

- If V(x, y, z) is a differentiable vector field then the curl or rotation of V, written as ▼x V,
- Curl V or rot V, is defined by

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix}\mathbf{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Problem

Q. If $\mathbf{A} = xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k}$, find $\nabla \times \mathbf{A}$ (or curl $\mathbf{A}$) at the point $(1,-1,1)$.

Sol.

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\mathbf{k}$$

$$= (2z^4 + 2x^2y)\,\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

Practice questions

If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2,-2,-1)$. *Ans.* $10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $2\sqrt{93}$

If $\mathbf{A} = 2x^2\,\mathbf{i} - 3yz\,\mathbf{j} + xz^2\,\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.
Ans. 5, $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$

If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find (*a*) $\nabla(F+G)$ and (*b*) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (*a*) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (*b*) $-8\mathbf{j}$

Find $\nabla|\mathbf{r}|^3$. *Ans.* $3r\,\mathbf{r}$

Prove $\nabla f(r) = \dfrac{f'(r)\,\mathbf{r}}{r}$.

Evaluate $\nabla(3r^2 - 4\sqrt{r} + \dfrac{6}{\sqrt[3]{r}})$. *Ans.* $(6 - 2r^{-3/2} - 2r^{-7/3})\,\mathbf{r}$

Find $\mathbf{A}\times(\nabla\times\mathbf{B})$ and $(\mathbf{A}\times\nabla)\times\mathbf{B}$ at the point $(1,-1,2)$, if $\mathbf{A} = xz^2\,\mathbf{i} + 2y\,\mathbf{j} - 3xz\,\mathbf{k}$ and $\mathbf{B} = 3xz\,\mathbf{i} + 2yz\,\mathbf{j} - z^2\,\mathbf{k}$.
Ans. $\mathbf{A}\times(\nabla\times\mathbf{B}) = 18\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$, $(\mathbf{A}\times\nabla)\times\mathbf{B} = 4\mathbf{j} + 76\mathbf{k}$

Prove: (*a*) $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$
(*b*) $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi(\nabla\times\mathbf{A})$.

Thank you