Vector Integrals

Lecture 4

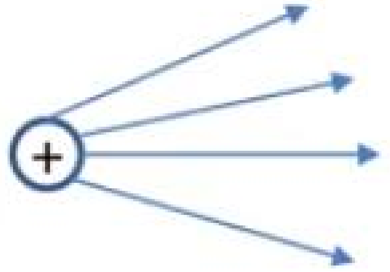
PH-122

Review of Vector Derivatives

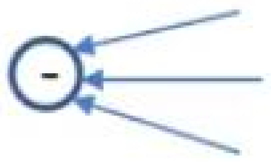
- Vectors can be differentiated to find other physical quantities; such as if we have a position vector '$\vec{r}$' we can differentiate it to find out velocity '$\vec{V}$' and differentiate velocity to find out acceleration '$\vec{a}$.'
- Similarly partial derivatives can also be found to give rate of change of any physical quantity along x-, y- or z-direction.
- Del operator is a operator based on partial derivatives and can be used to find gradient, divergence and curl.

Gradient, Divergence and Curl

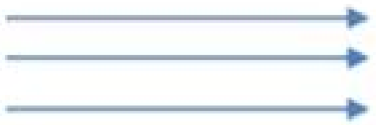
- Gradient is always founds for scalar quantities. It gives direction to scalar by defining a normal vector associated with scalar quantity, used to find directional derivative. It is defined by$\vec{\nabla}\varphi$.
- Divergence is always found for vector, defined by $\vec{\nabla}.\vec{A}$. If $\vec{\nabla}.\vec{A}$=0 field is solenoidal, $\vec{\nabla}.\vec{A} < 0$ field is convergent, $\vec{\nabla}.\vec{A} > 0$ field is divergent .



Divergent field



Convergent field



Solenoidal field

Gradient, Divergence and Curl

- Curl is also found for vectors. It is defined by $\vec{\nabla} X \vec{A}$. It is usually used to find if the field is conservative or non-conservative. As all conservative fields have straight lines of force that means that the curl of such fields will be zero. If $\vec{\nabla} X \vec{A} = 0$; field is conservative, $\vec{\nabla} X \vec{A} \neq 0$; field is non-conservative.

Ordinary Integrals of Vector

- Let $R(u) = R_1(u)\,i + R_2(u)\,j + R_3(u)k$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

- This is called an indefinite integral of R(u). where c is an arbitrary constant vector independent of u.

Continued…

- Integral can also be defined in a limit of a sum in a manner analogous to that of elementary integral calculus.

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

Problem

If $\mathbf{R}(u) = (u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (a) $\int \mathbf{R}(u)\,du$ and (b) $\int_1^2 \mathbf{R}(u)\,du$.

(a) $\int \mathbf{R}(u)\,du = \int \left[(u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}\right]du$

Continued…

(b) From (a), $\int_1^2 \mathbf{R}(u)\,du = \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}\Big|_1^2$

$$= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)\mathbf{i} + \frac{2^4}{2}\mathbf{j} - 3(2)\mathbf{k} + \mathbf{c}\right] - \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)\mathbf{i} + \frac{1^4}{2}\mathbf{j} - 3(1)\mathbf{k} + \mathbf{c}\right]$$

$$= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

Another Method.

$$\int_1^2 \mathbf{R}(u)\,du = \mathbf{i}\int_1^2 (u-u^2)\,du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du$$

$$= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Big|_1^2 + \mathbf{j}\left(\frac{u^4}{2}\right)\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2 = -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

Line Integral

- Let r(u) = x(u) i + y(u) j + z(u) k , where r(u) is the position vector of (x,y,z),define a curve C joining points P1 and P2 , where $u = u_1$ and $u = u_2$ respectively.
- We assume that C is composed of a finite number of curves for each of which r(u) has a continuous derivative. Let A(x,y,z) = $A_1 i + A_2 j + A_3 k$ be a vector function of position defined and continuous along C

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

- Then the integral of the tangential component of A along C from P1 to P2 , written as

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

Problem

Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

$$\text{Total work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) - 5(t^3)\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303$$

Practice Problems

The acceleration $\mathbf{a}$ of a particle at any time $t \geqq 0$ is given by $\mathbf{a} = e^{-t}\mathbf{i} - 6(t+1)\mathbf{j} + 3\sin t\,\mathbf{k}$. If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.
Ans. $\mathbf{v} = (1-e^{-t})\mathbf{i} - (3t^2+6t)\mathbf{j} + (3-3\cos t)\mathbf{k}$, $\mathbf{r} = (t-1+e^{-t})\mathbf{i} - (t^3+3t^2)\mathbf{j} + (3t-3\sin t)\mathbf{k}$

The acceleration $\mathbf{a}$ of an object at any time t is given by $\mathbf{a} = -g\,\mathbf{j}$, where g is a constant. At $t=0$ the velocity is given by $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + v_0\sin\theta_0\,\mathbf{j}$ and the displacement $\mathbf{r} = \mathbf{0}$. Find $\mathbf{v}$ and $\mathbf{r}$ at any time $t > 0$. This describes the motion of a projectile fired from a cannon inclined at angle θ_0 with the positive x-axis with initial velocity of magnitude v_0.
Ans. $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + (v_0\sin\theta_0 - gt)\mathbf{j}$, $\mathbf{r} = (v_0\cos\theta_0)t\,\mathbf{i} + [(v_0\sin\theta_0)t - \frac{1}{2}gt^2]\mathbf{j}$

If $\mathbf{F} = (5xy-6x^2)\mathbf{i} + (2y-4x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C in the xy plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$. *Ans.* 35

If $\mathbf{F} = (2x+y)\mathbf{i} + (3y-x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$. *Ans.* 11

Practice Problems

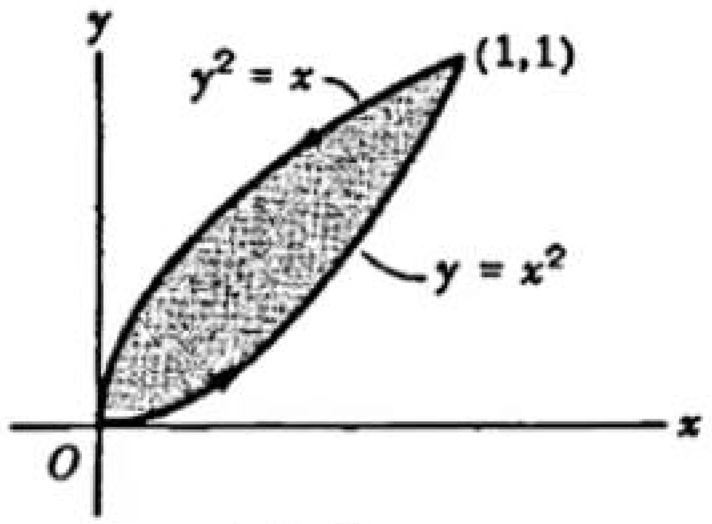


Fig. 2

Evaluate $\oint_C \mathbf{A} \cdot d\mathbf{r}$ around the closed curve C of Fig. 2 above if $\mathbf{A} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$. *Ans.* 2/3

Line Integral

- A very common use of line integrals is to find work done by variable force.
- Work done is given by $W = \vec{F}.\vec{d}$. This is applicable when force is constant for variable forces we use $\int \vec{F}.\overrightarrow{dr}$.