EE16B — Midterm 2 Review

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Overview

State-Space Representations

Stability, Observability, and Controlability

Eigenvalue Placement

Linearization

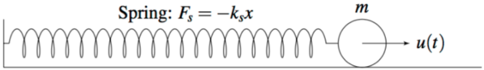
Singular Value Decomposition

Principle Component Analysis

Discretization

State Space Modeling: Example

Assume we have the following spring system:



We can model the system as a linear continuous time state space model:

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$

$\vec{y}(t) = C\vec{x}(t)$

in which:

$\vec{x}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

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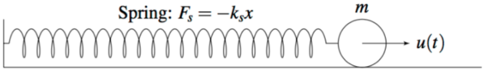

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State Space Modeling:

original space:

new space:

$$
\begin{tikzcd}
\vec{x}(t) \arrow[r, "A"] \arrow[d, bend right, "T^{-1}"'] & \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) \arrow[d, bend right, "T^{-1}"'] \\
\tilde{\vec{x}}(t) \arrow[r, "\tilde{A}"] \arrow[u, bend right, "T"'] & \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) \arrow[u, bend right, "T"']
\end{tikzcd}
$$

Always apply operations to right side first

$$\tilde{A} = T^{-1}AT$$

$$\tilde{\vec{x}}(t) = T^{-1}\vec{x}(t)$$

State Space Modeling Procedure:

1. Set up differential equation of the form: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$
2. Find λ_i of A; let $\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$
3. Find eigenvectors $\vec{v}_i$ of A; let $T = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$
4. Convert $\vec{x}(t)$ to $\tilde{\vec{x}}(t)$ using: $\tilde{\vec{x}}(t) = T^{-1}\vec{x}(t)$
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State Space Modeling:

Continuous time solution:

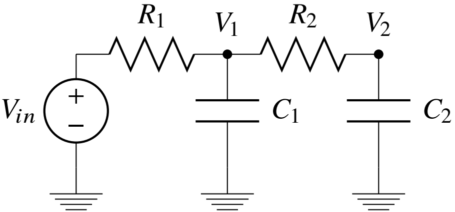
$$\tilde{x}(t) = e^{\lambda t}\tilde{x}(0) + \frac{e^{\lambda t} - 1}{\lambda}u(t) + w(t)$$

Discrete time solution:

$$x_d(i+1) = e^{\lambda \Delta}x_d(i) + \frac{e^{\lambda \Delta} - 1}{\lambda}u(i) + w(i)$$

State Space Modeling Example:

Given the following circuit:



in which $R_1 = 2\,\Omega$, $R_2 = \frac{8}{3}\Omega$, $C_1 = 1$ C, $C_2 = \frac{3}{2}$C
solve equations for V_1 and V_2

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Stability, Observability, Controllability:

given:

$\vec{x}(i+1) = A\vec{x}(i) + Bu(i)$

$\vec{y}(i) = C\vec{x}(i)$

in which:

$\vec{x}$ is our state,

$\vec{u}$ is our input,

$\vec{y}$ is what we can observe:

Stability (Discrete time):

Discrete time model:
if $|\lambda_i| < 1$ for all λ_i of A, system is stable
intuition: if any $|\lambda_i| >= 1$, state vector is increasing each time step will be infinitely magnified over time

Stability (Continuous time):

Continuous time model:
if the real parts of all eigenvalues of A are strictly negative, system is stable
intuition: if real part of eigenvalue is positive, state vector is increasing over time and will be infinitely magnified over time

Controllability:

if $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ spans R^n, then system is controllable

Feedback:

if system is controllable, we can set: $u(t) = K\vec{x}(t)$
plugging in, we get: $\vec{x}(t+1) = (A + BK)\vec{x}(t)$
we can find the eigenvalues of (A + BK) to check for stability

Observability:

if $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ spans R^n, system is observable

intuition: if observability matrix is full rank, it is invertible, and we can retrieve all the past states without loss of information

Stability, Controllability, Observability Example:

given the following system:

$$\vec{x}[t+1] = \begin{bmatrix} -5 & 0 \\ 7 & 6 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

$$\vec{y}[t] = \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x}[t]$$

Stability Check:

$\lambda = 6, -5$

System is unstable

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Controllability Check:

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Observability Check:

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \text{ which spans } R^n$$

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$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}$ which spans R^n

System is observable

Overview

Eigenvalue Placement

Why?

- Recall that we are always interested in determining if a given system is BIBO (bounded input bounded output) stable.
- More precisely, if we have a system described by $\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + \vec{\omega}(t)$ we would like the eigenvalues of $A \in \mathbb{R}^{n \times n}$, to satisfy the following property : $|\lambda_i| < 1$.
- So what if we have a λ that does not satisfy this property?
- This is where eigenvalue placement comes into play!
- Assuming the system is controllable, we will use closed loop controls to change the eigenvalues such that they satisfy this property.

How?

- Assume e.g. a DT system. Input: $u[t]$ If the system is controllable then we can use feedback, which means that we can let the input depend on the output, $\vec{x}[t]$.
- We would like to change the matrix multiplying $\vec{x}[t]$ such that $|\lambda_i| < 1$, so let's see what happens when we let $u[t] = K\vec{x}[t]$, where $K \in \mathbb{R}^{1\times n}$.
- Using this input we have:

$$\begin{aligned}\vec{x}[t+1] &= A\vec{x}[t] + Bu[t] + \vec{\omega}[t]\\ &= A\vec{x}[t] + BK\vec{x}[t] + \vec{\omega}[t]\\ &= (A+BK)\vec{x}[t] + \vec{\omega}[t]\end{aligned}$$

- Strategically choosing K allows us to have specific λ's for A + BK (Good!).
- This process is called coefficient matching.

Example

- Suppose we are given a controllable system defined by:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

- Is the system stable? No! $\lambda = 2, 1$
- What if we let

$$u[t] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$$

  Then we have:

$$\vec{x}(t+1) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$$

- Solve for the values of f_1 and f_2 such that $\lambda_1 = -0.25$ and $\lambda_2 = 0$
- The answer is $f_1 = -1.50$ and $f_2 = 0.25$
- Although the process is very messy hopefully you see why eigenvalue placement is very important to stabilize systems. What about bigger matrices?

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Controllable Canonical Form

- Controllable Canonical Form (CCF) for any controllable system is a special form that allows us to simplify the process of eigenvalue placement. It takes on the following form:

$$A^* = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \dots & \alpha_{n-1} \end{bmatrix} \qquad B^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The characteristic polynomial of A^* is $\lambda_n - \sum_{i=0}^{n-1} \alpha_i \lambda^i = 0$.
- So how does it help with eigenvalue placement? The last row of this matrix determines the eigenvalues of A^* so modifying the last row will allow us to (easily) modify the eigenvalues.

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How to convert to CCF

- Let A, B be the matrices in standard form and let A^*, B^* be the matrices in CCF.
- Recall the matrix we used to check controllability?

$$C = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

$$C^* = \begin{bmatrix} B^* & A^*B^* & \dots & A^{*n-1}B^* \end{bmatrix}$$

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Example

Consider the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[t]$$

(a) Is the system stable? Is it controllable?

(b) Using an appropriate transformation ($\vec{z}[t] = T\vec{x}[t]$), bring the system to controllable canonical form.

(c) Using the state feedback $u[t] =$

$$\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

$\vec{z}[t]$ bring the eigenvalues of the system to $0, 0.75, -0.25$.

Solutions to Example

(a) The characteristic polynomial is:
$\lambda^3 + 7\lambda^2 + 8\lambda = \lambda(\lambda^2 + 7\lambda + 8) = 0$, therefore the eigenvalues of A are $\{0, -5.56, -1.44\}$. As we can see there are $|\lambda_i| > 1$ therefore the system is not stable.
The controllability matrix $C =$

$$\begin{bmatrix} 1 & -2 & 8 \\ 0 & 2 & -12 \\ 0 & 0 & 2 \end{bmatrix}$$

C has full rank so the system is controllable

(b) As we previously mentioned the coefficients of the characteristic polynomial are closely related to the last row of the A^* matrix. Therefore, the CCF of the system is:

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -7 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[t]$$

Example Solutions Continued

(c) Our system then becomes:

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_1 & f_2 - 8 & f_3 - 7 \end{bmatrix} \vec{x}[t]$$

Which means its characteristic polynomial is :
$\lambda^3 - (f_3 - 7)\lambda^2 - (f_2 - 8)\lambda - f_1 = 0$.
Now, we know the characteristic polynomial should be $\lambda(\lambda - \frac{3}{4})(\lambda + \frac{1}{4})$, so we can equate the two and solve for the feedback vector $\vec{f}^{\,T} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{3}{16} \end{bmatrix}$.

Overview

Linearization

- Recall that if we have $\frac{dx}{dt} = \lambda x(t) + bu(t)$ we know a solution to this is:
$$x(t) = x(0)e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)}u(\tau)\, d\tau$$
- What if we had $\frac{dx}{dt} = f(x(t)) + bu(t)$, where f is nonlinear (e.g sin)?
- Big Picture: linearize f around an operating point and then treat it as a linear function in a small neighborhood of that point.
- Why linearization?
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Linearizing a Single-Variable Function

- Suppose we have $f(x)$ that is a non linear function. We can use a first order Taylor polynomial to linearize the function, this is equivalent to finding the slope of the tangent line of $f(x)$ at a particular point.
- From calculus: $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$.
- As long as we are within some (very small) δ neighborhood of x^* the linearization is valid.
- Example: Linearize $f(x) = 3e^{x^2+2}$ around x^*
- Solution:
  $f(x^*) = 3e^{x^{*2}+2}$
  $f'(x) = 3e^{x^2+2}(2x) = 6xe^{x^2+2}$
  $f'(x^*) = 6x^*e^{x^{*2}+2}$
  Therefore : $f(x) \approx 3e^{x^{*2}+2} + 6x^*e^{x^{*2}+2}(x - x^*)$

Linearizing Steps for $\frac{dx(t)}{dt} = f(x(t)) + bu(t)$

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(vii) Verify the linearization!
How do we know if the linearization is valid? Well, if we have $\frac{dx_l(t)}{dt} = \lambda x_l(t) + bu(t)$ we know the solution doesn't blow up if $\lambda < 0$ as we will have a term $e^{\lambda t}$.
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So what do we do if $m > 0$?
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Linearize $\frac{dx(t)}{dt} = \cos(x(t)) + bu(t)$ about $u^* = 0$.

Hint: $\cos(x^*) = 0$ has multiple solutions, which means that we can find numerous DC operating points, can you guess which one would result in a stable system ?

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Linearization of Vector Functions

What if we had $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, \vec{u})$? We will see that the process is very similar to the scalar case we just analyzed!

First, let's see how to linearize $\vec{f}(\vec{x})$ around a DC operating point $\vec{x}^*$. Where $\vec{f} \in \mathbb{R}^{n\times 1}$ is a vector of scalar functions.

The idea is to linearize individually each one of the f_i around the DC operating point.

For example: $f_1 \approx f_1(\vec{x}^*) + \frac{\partial f_1}{\partial x_1}(\vec{x}^*)(x_1 - x_1^*) + \ldots + \frac{\partial f_n}{\partial x_1}(\vec{x}^*)(x_n - x_n^*)$

Repeating this for all n functions in $\vec{f}$ we see we get a system of scalar, linearized, multivariate functions which makes you think, wouldn't it be nice to express it in a shorthand matrix notation?

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Jacobian Matrix

We can use the Jacobian to express everything nicely and neatly.
The Jacobian is the name given to the matrix of partial derivatives of $\vec{f}$, and it is denoted by $J_{\vec{x}}$ or $\nabla_{\vec{x}}\vec{f}$.
Continuing from the previous slide we have:

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Linearization with Jacobians Example

Linearize $\vec{f}(\vec{x}(t)) = \begin{bmatrix} \sin(x_1(t) * x_2(t)) + 2x_1(t)x_3^2(t) \\ x_3(t)\cos(x_2(t)) + \frac{x_1(t)}{x_3(t)} \\ x_1(t) + 2x_3(t)x_2^3(t) \end{bmatrix}$ about $\vec{x}^* = \begin{bmatrix} 0 \\ 2\pi \\ \frac{2\pi}{3} \end{bmatrix}$

Solutions

Find the Jacobian:

$$\begin{bmatrix} x_2(t)\cos(x_1(t)*x_2(t)) + 2x_3^2(t) & x_1(t)\cos(x_1(t)*x_2(t)) & 4x_1(t)x_3(t) \\ \frac{1}{x_3(t)} & -x_3(t)\sin(x_2(t)) & \cos(x_2(t)) - \frac{x_1(t)}{x_3^2(t)} \\ 1 & 6x_3(t)x_2^2(t) & 2x_2^3(t) \end{bmatrix}$$

Evaluate the Jacobian about $\vec{x}^*$:

$$\begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix}$$

Linearize:

$$\vec{f}(\vec{x}(t)) \approx \begin{bmatrix} 0 \\ \frac{3\pi}{4} \\ 24\pi^4 \end{bmatrix} + \begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ x_2(t) - \frac{3\pi}{4} \\ x_3(t) - 24\pi^4 \end{bmatrix}$$

Steps to Linearize Vector ODE Systems

To linearize $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$ we use a similar procedure as we did for the scalar case.

(i) Solve $\vec{f}(\vec{x}^*, \vec{u}^*) = \vec{0}$ to determine the equilibrium point.

(ii) Let $\vec{x}_l(t) = \vec{x}(t) - \vec{x}^*$ and $\vec{u}_l(t) = \vec{u}(t) - \vec{u}^*$

(iii) Linearize $\vec{f}(\vec{x}, \vec{u})$ about $(\vec{x}^*, \vec{u}^*)$. That is:
$\vec{f}(\vec{x}(t), \vec{u}(t)) \approx \vec{f}(\vec{x}^*, \vec{u}^*) + J_{\vec{x}}\vec{x}_l(t) + J_{\vec{u}}\vec{u}_l(t)$

(iv) Plug (iv) back into the ODE: $\frac{d\vec{x}}{dt} \approx \cancel{\vec{f}(\vec{x}^*, \vec{u}^*)} + J_{\vec{x}}\vec{x}_l(t) + J_{\vec{u}}\vec{u}_l(t)$

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Linearizing Vector ODE Systems Example

Given a DC input $u^* = 1$, linearize:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} x_1^2(t) - x_2(t)u(t) \\ x_2^2(t)(1 + x_1(t)) + \sin(\pi x_1(t)u(t)) \end{bmatrix}$$

Solutions

Again, we will do this in steps:

(i) We are given $u^* = 1$

(ii) We need to find a DC operating point, this means solving the following system of equations:

$$x_1^{*2} - x_2^* u^* = 0 \tag{1}$$

$$x_2^{*2}(x_1^* + 1) + \sin(\pi x_1^* u^*) = 0 \tag{2}$$

The solution is $x_1^* = -1$ and $x_2^* = 1$.

(iii) Let $\vec{x}_l(t) = \vec{x}(t) - \vec{x}^*$ and $\vec{u}_l(t) = \vec{u}(t) - \vec{u}^*$

(iv) Linearize,

$$\vec{f}(\vec{x}(t), u(t)) \approx \vec{f}(\vec{x}^*, 1) + \begin{bmatrix} -2 & -1 \\ 1-\pi & 0 \end{bmatrix} \vec{x}_l(t) + \begin{bmatrix} -1 \\ \pi \end{bmatrix} u_l(t)$$

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Solutions Continued

(v) Substitute linear approximation back into the system,

$$\frac{d\vec{x}(t)}{dt} \approx \cancel{\vec{f}(\vec{x}^*, 1)} + \begin{bmatrix} -2 & -1 \\ 1-\pi & 0 \end{bmatrix} \vec{x}_l(t) + \begin{bmatrix} -1 \\ \pi \end{bmatrix} u_l(t)$$

Break

GIVE US FEEDBACK!

`hkn.mu/feedback`

`https://github.com/hkntutoring/ee16b-review/issues`

Overview

SVD Theorem

Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into the product of three matrices

$$
\begin{aligned}
A &= U \Sigma V^T \\
U &: m \times m \\
\Sigma &: m \times n \\
V^T &: n \times n
\end{aligned}
$$

Such that U, V are unitary matrices and Σ only has nonnegative values along its main diagonal.

SVD: Compact Form

We can also express the SVD as

$$
\begin{aligned}
A &= \mathcal{U} S \mathcal{V}^T \\
\mathcal{U} &: m \times r \\
S &: r \times r \\
\mathcal{V}^T &: r \times n
\end{aligned}
$$

where r is the rank of A. The compact form matrices maintain properties of the original matrices, but have entries removed whenever they correspond to zero singular values.

SVD: Outer Product Form

Lastly, we can express

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

where $\vec{u}_i, \vec{v}_i$ are the columns of U, V, respectively, and σ_i are corresponding diagonal entry of the matrix Σ

Computing SVD with $A^T A$

$$\begin{aligned} A^T A &= U\Sigma V^T V \Sigma^T U^T \\ &= U \Sigma^2 U^T \end{aligned}$$

This is an eigen decomposition since Σ^2 is diagonal and $U^{-1} = U^T$. Thus solving for the eigenvalues and eigenvectors of $A^T A$ give $\lambda_i = \sigma_i^2$ with eigenvectors which correspond to the right singular vectors. We need to sort by decreasing σ_i.
Side note: $\Sigma^T \Sigma$ is not actually equal to Σ^2, but the former product yields a matrix with singular values squared on the diagonal entries, hence we call it Σ^2

Computing SVD with $A^T A$

Given a right singular vector $\vec{v}_i$ which we found from the previous part, we can apply it

$$\begin{aligned}
A\vec{v}_i &= \left(\sum_{k=1}^{r} \sigma_k \vec{u}_k \vec{v}_k^T\right)\vec{v}_i \\
&= \sum_{k=1}^{r} \sigma_k \vec{u}_k \vec{v}_k^T \vec{i} \\
&= \sigma_i \vec{u}_i \\
\vec{u}_i &= \frac{1}{\sigma_i} A\vec{v}_i
\end{aligned}$$

Computing SVD with AA^T

Similar calculations yield $\sigma_i = \sqrt{\lambda_i}$ of AA^T with eigenvectors as left singular vectors, and $\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$

Intepretation of SVD

- Unitary matrices act as rotation in a given space. A diagonal matrix stretches in a given coordinate space.
- SVD visualization (open in browser)

Intepretation of SVD

For a product $A\vec{x}$, we can decompose every vector $\vec{x}$ into a linear combination of right singular vectors

$$\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$$

Thus, we can see exactly which parts of $\vec{x}$ affect the output.

Compression of Low-Rank Matrices

- Suppose I had a matrix $A \in \mathbb{R}^{m \times n}$ with $m, n >> rank(A)$. How could I more efficiently store A and compute products like $A\vec{x}$?
- With the SVD, we only have to save r set of two vectors and a scalar, which saves us a lot of space if the rank is small with respect to the matrix. Also, less computation is carried out if we represent the matrix as the outer product form.

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Overview

PCA

PCA is a linear dimensionality reduction tool. Given data $\vec{x}_i \in \mathbb{R}^d$, we can create a mapping $T : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}, d' < d$ such that the variance in the dataset is still captured

PCA — Computation

1. Store data row-major in $A \in \mathbb{R}^{n \times d}$
2. De-mean A
3. Take SVD: $A = U\Sigma V^T$
4. Create $V_{d'} \in \mathbb{R}^{n \times d'}$ from vectors of V corresponding to d' greatest signular values
5. To project data into the representative subspace: $T(x) := V_{d'}^T x$

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PCA: computation

The mapping T can then be expressed as

$$T(\vec{x}) = V_k^T \vec{x}$$

If we apply this transformation onto the entire dataset (which has row vectors), we can say

$$T(A) = B = AV_k$$

where $B \in \mathbb{R}^{n \times k}$

PCA: computation

If we were to show the projected vectors in the original space, we can multiply back with the projection vectors

$$A' = BV_k^T$$

Overview

Discretization: Q1

Note: this section follows hw8 q1 almost exactly. Suppose we have a scalar system

$$\frac{d}{dt}x(t) = \alpha x + \vec{\beta}^T \vec{u}(t)$$

and we apply a constant input $\vec{u}_n$ for times $t \in [nT, (n+1)T)$ for some $T > 0$. Given $x(nT)$ solve the differential equation

Discretization: Q1 Sol

From $t = nT$ to $t = (n+1)T$, $\vec{\beta}^T\vec{u}$ is a constant scalar. Thus, we can solve this like a normal differential equation. Let $x = x' - \frac{\vec{\beta}^T\vec{u}}{\alpha}$. Then

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \alpha(x' - \frac{\vec{\beta}^T\vec{u}}{\alpha}) + \vec{\beta}^T\vec{u}(t) \\
&= \alpha x' \\
x' &= Ae^{\alpha(x-nT)} \\
x + \frac{\vec{\beta}^T\vec{u}}{\alpha} &= Ae^{\alpha(x-nT)} \\
x &= Ae^{\alpha(x-nT)} - \frac{\vec{\beta}^T\vec{u}}{\alpha}
\end{aligned}
$$

Discretization: Q1 Sol

At which point we can use our initial condition to get

$$
\begin{aligned}
x(nT) &= A - \frac{\vec{\beta}^T \vec{u}}{\alpha} \\
A &= x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha} \\
x &= \left(x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha} \right) e^{\alpha(t-nT)} - \frac{\vec{\beta}^T \vec{u}}{\alpha}
\end{aligned}
$$

Discretization: Q2

Using the differential equation derived from question 1, create a discrete-time system to model the continuous time. In other words, if $x[n] = x(nT), \vec{u}[n] = \vec{u}(nT)$, find a relation such that

$$x[n+1] = A_d x[n] + B_d \vec{u}[n]$$

Discretization: Q2 Sol

We can solve the previous solution for $x((n+1)T)$

$$x((n+1)T) = \left(x(nT) + \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha}\right) e^{\alpha((n+1)T - nT)} - \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha}$$

$$x[n+1] = e^{\alpha T} x[n] + \frac{e^{\alpha T} - 1}{\alpha} \vec{\beta}^T \vec{u}[n]$$

We see that $A_d = e^{\alpha T}, B_d = ((e^{\alpha T} - 1)/\alpha)\vec{\beta}^T$

Discretization: Q3

Instead of a scalar, we instead have a diagonal matrix A such that

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$$

Discretize this system in the same was as Q2.

Discretiziation: Q3 Sol

Expanding the original system out line-by-line gives

$$\frac{d}{dt}x_i = a_i x_i + b_i\vec{u}_i$$

where x_i is the ith variable of $\vec{x}$, a_i is the diagonal entry of A, and b_i is the row of B.

Discretization: Generic Matrix

Math not shown, but we can perform a change of basis from our original space to our diagonal space, and then apply the results of the previous part.