Asymptotic and BMS Symmetries

The IR limit of Gravity

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Outline

At null infinity, what symmetries does spacetime possess?

One would expect the Poincare group is recovered, but Bondi et al famously discovered that there are more.

How does this make sense? E.g. asymptotically flat:

$$\lim_{r\to\infty} g_{\mu\nu} = \lim_{r\to\infty} \eta_{\mu\nu} + \mathcal{O}(r^{-1}) \tag{1}$$

The asymptotic symmetries may not leave $\eta_{\mu\nu}$ completely invariant, but invariant up to subleading terms.

Geometric definition: Given boundary conditions, the asymptotic symmetry group

$$= \frac{\text{Gauge transformations preserving the boundary conditions}}{\text{Trivial gauge transformations}}$$

Gauge Fixing approach: After gauge fixing, the residual gauge transformations which preserve boundary conditions.

Can be defined for general gauge theories, we focus on gravity, where the transformations are diffeomorphisms.

Remark The *trivial* transformations can be defined in different ways, leading to the strong and weak definitions of the asymptotic symmetry group. Identical for our purposes.

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Coordinates (u, r, x^A), u labels null hypersurfaces (timelike), x^A the null geodesics in any single hypersurface (angular), and r the distance along a geodesic (radial).

n gauge conditions (from n coordinate choices):

$$g_{rr} = 0, \quad g_{rA} = 0, \quad \partial_r \left(\frac{\det g_{AB}}{r^{2(n-2)}} \right) = 0 \tag{2}$$

General form:

$$ds^2 = -Vdu^2 - 2e^{2\beta} dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{3}$$

For 4D flat spacetime:

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} \tag{4}$$

$$z = \frac{x^1 + ix^2}{x^3 + r} \tag{5}$$

- Foliate spacetime with null hypersurfaces $\Gamma_c : u = c$, label null generators with x^A and distance along them with r.
- Null normal on Γ_c is $n_\mu \equiv -\partial_\mu u$, so n^μ is null tangent.
- Define r with $n^\mu = e^{2\beta}\partial_r x^\mu$, so that $d\lambda = e^{-2\beta}dr$ is affine parameter along n^μ.
- Working in $x^\mu = (u, r, x^A)$, $n_\mu = (-1, 0, 0_A)$ and $n^\mu = (0, e^{-2\beta}, 0^A)$. Hence derive $g_{rr} = 0, \quad g_{rA} = 0$.
- $\theta = \nabla_\mu n^\mu = e^{-2\beta}\frac{D-2}{r}$ is the expansion of the transverse area to the null geodesics.
- Show that $\nabla_\mu n^\nu = \frac{1}{2}e^{-2\beta}\partial_r \log\det(g_{AB})$, thus derive $\partial_r\left(\frac{\det g_{AB}}{r^{2(n-2)}}\right)$.

Popularly studied: Asymptotically flat, asymptotically (A)dS

Expand the angular sub-metric in r-series:

$$g_{AB} = r^2 q_{AB} + rC_{AB} + D_{AB} + \mathcal{O}(r^{-1}) \tag{6}$$

And let $\mathring{q}_{AB}$ be the induced metric on a 2-sphere.

Asymptotically Flat:

$$\beta = \mathcal{O}(1) \tag{7}$$

$$V = \mathcal{O}(r^2) \tag{8}$$

$$U^A = \mathcal{O}(1) \tag{9}$$

$$q_{AB} = \mathring{q}_{AB} \tag{10}$$

Asymptotically (A)dS:

$$\beta = o(1) \tag{11}$$

$$V = \frac{2r^2\Lambda}{(n-1)(n-2)} + \mathcal{O}(r^2) \tag{12}$$

$$U^A = \mathcal{O}(1) \tag{13}$$

$$q_{AB} = \mathring{q}_{AB} \tag{14}$$

General Approach

1. Suitable gauge-fixing
2. Specify asymptotic boundary conditions
3. Write metric to leading order in r
4. Find diffeomorphisms which are asymptotically isometries - these are your symmetry-generating killing vector fields

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5. Evaluate symmetry algebras
6. Calculate conserved charges
7. Examine data at boundary from EoM (Einstein's equations)

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Consider (u, r, θ, ϕ) coordinates on $(\mathbb{R}^4, \eta)$, u is the retarded time from origin. Time and space translations:

$$u \mapsto u + \alpha Y_{00} \tag{15}$$

$$u \mapsto u + \sum_{m=-1}^{m=1} \alpha_m Y_{1m} \tag{16}$$

What if we generalize this to arbitrary l?

$$u \mapsto u + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta, \phi) = u + f(\theta, \phi) \tag{17}$$

We need the subleading terms:

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{18}$$

$$+\frac{2m_B}{r}du^2 + \left(rC_{zz}dz^2 + D^z C_{zz}dudz\right) + \text{c.c.} \tag{19}$$

$$+\frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_u m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + \text{c.c.} + \dots \tag{20}$$

- m_B is the Bondi mass aspect
- N_z is the angular momentum aspect
- C_{zz} encodes the gravitational waves, $N_{zz} = \partial_u C_{zz}$ is the Bondi news
- D_z is the covariant derivative with respect to $\gamma_{z\bar{z}}$

For diffeomorphism generated by ζ, demand that it preserves both the gauge conditions and the fall-off conditions - we should recover the metric's form up to subleading terms. Restrict considerations to

$$\zeta^u, \zeta^r \sim \mathcal{O}(1), \tag{21}$$

Then Lie derivatives of the metric components are:

$$\mathcal{L}_\zeta g_{ur} = -\partial_u \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \tag{22}$$

$$\mathcal{L}_\zeta g_{zr} = r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \tag{23}$$

$$\mathcal{L}_\zeta g_{z\bar{z}} = r\gamma_{z\bar{z}}(2\zeta^r + rD_z\zeta^r + rD_{\bar{z}}\zeta^{\bar{z}}) + \mathcal{O}(1) \tag{24}$$

$$\mathcal{L}_\zeta g_{uu} = -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}\left(\frac{1}{r}\right) \tag{25}$$

$$\vdots$$

Now demand gauge and fall-off conditions are preserved.

Thus we get the generators of Supertranslations:

$$\zeta = f\partial_u - \frac{1}{r}(D^z f\partial_z + D^{\bar{z}} f\partial_{\bar{z}}) + D^z D_z f\partial_r \tag{26}$$

The $f(z, \bar{z})\partial_u$ term is exactly the extension we saw earlier.

- Relate inequivalent vacua - on Minkowski, the data changes.
- BMS group is the semi-direct product of supertranslations with the Lorentz group.
- At next order, we obtain superrotations, related to N_z.

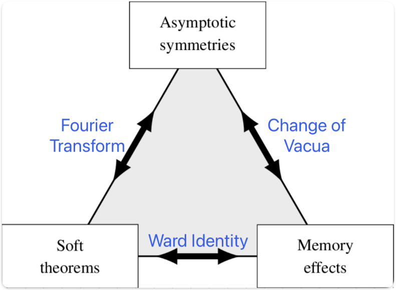
$$\mathcal{L}_f N_{zz} = f \partial_u N_{zz} \tag{27}$$

$$\mathcal{L}_f m_B = f \partial_u m_B + \frac{1}{4}(N^{zz} D_z^2 f + 2 D_z N^{zz} D_z f + \text{c.c.}) \tag{28}$$

$$\mathcal{L}_f C_{zz} = f \partial_u C_{zz} - 2 D_z^2 f \tag{29}$$

An interesting note here is the appearance of classically inequivalent vacua. From the above equations, if we begin with flat spacetime, $m_B = N_{zz} = C_{zz} = 0$, a supertranslation keeps 0 everything but C_{zz} - so while no mass or gravitational flux was created by the diffeomorphism, the vacuum changed.

The Infrared triangle of gauge theories refers to the connections between three seemingly unrelated topics - asymptotic symmetries, soft theorems, and memory effects.



Soft theorems are about scattering amplitudes involving massless particles with small momenta $q \to 0$ - then the scattering amplitude relates to that of the rest of the particles (to first order):

$$\mathcal{M}_{n+1}(q, p_i) = S^{(0)}\mathcal{M}_n(p_i) + \mathcal{O}(q^0), \; S^{(0)} \sim q^{-1} \tag{30}$$

In gauge theories, we refer to when a field is turned on as a result of a burst of energy passing through the region of interest, leading to an observable phenomenon.

In the context of gravity, gravitational waves can cause the *displacement memory effect*, wherein a permanent shift happens in the relative position of two (inertial) detectors.

1. Every symmetry transformation has a corresponding Ward-Takahashi identity equating scattering amplitudes related by that transformation. Supertranslations are symmetries of gravitational scattering processes, and the Ward identity for supertranslations is equivalent to the soft graviton theorem.
2. The displacement memory effect is equivalent to performing a supertranslation - it has the same effects on C_{AB} as a burst of gravitational waves. The supertranslation connects the two vacua related by the memory effect.
3. The soft theorem relates to the memory effect via a fourier transform. While the soft theorem concerns poles in momentum space ($S^{(0)} \sim q^{-1}$), the memory effect is a position-space permanent shift.

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