Asymptotic and BMS Symmetries Report

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Introduction

In a gauge theory - specifically those incorporating GR - the symmetry groups arising asymptotically at the boundary is what this topic is all about. In 4D GR, in addition to the Poincare group we get supertranslations, which combined are called the Bondi-Metzner-Sachs-van der Burg (BMS) group.

Defining the asymptotic group

The asymptotic group is intricately connected to the boundary conditions of the gauge theories.

Geometric Approach

Introduced by Penrose, the boundary conditions require certain data to be preserved at the boundary.
The weak definition of the asymptotic is a group isomorphic to the gauge transformations *induced on the boundary* which preserve this data. The strong definition is the quotient group of the data-preserving transformations by the trivial gauge transformations - the trivial

are defined as those with vanishing (Noether) charges. The connection between these being that the trivial transformations are not those induced at the boundary, I suppose. ^aa489e
Since we do not refer to coordinates in these boundary conditions, it is a manifestly gauge invariant approach, but it has the flaw that the boundary conditions are rigid, and often defined *a posteriori* - you can't start from BCs and obtain the asymptotes.

Gauge Fixing Approach

Related to Gauge Invariance and Theories.
Gauge fixing "quotients the field space" to eliminate redundancies in a gauge theory - mostly unphysical or "pure gauge" redundancies. ^fae226
However, the appropriate gauge fixing for a theory does not always remove all redundancy - see electromagnetism, where the Lorenz gauge $\partial_\mu A^\mu = 0$ still allows $A_\mu \to A_\mu + \partial_\mu \beta$, as long as $\partial_\mu \partial^\mu \beta = 0$.
After gauge fixing, the boundary conditions may be expressed in the chosen gauge.
Intuitively, gauge fixing is removing the pure gauge degrees of freedom - essentially the trivial ones with vanishing charges discussed above.
Similarly to above, the weak (strong) asymptotic group can be defined as the diffeomorphisms preserving the boundary conditions (with non-vanishing charges). But explicit coordinate expressions allow for flexibility in the boundary conditions, and gauge fixing is also local so global topological considerations are moot.
Consider the example of superrotations, an extension to BMS - they have poles at the celestial sphere, so the geometric approach would require a modified topology, a punctured celestial sphere - it just makes things difficult, as opposed to gauge fixing. But for any results of this approach, it is often needed to reproduce them with the geometric approach to show manifest gauge invariance.

Hamiltonian Approach

Perform calculations in a coordinate system - no gauge fixing needed - and study the symmetries at spatial infinity explicitly, with the trivial symmetries having charges vanishing identically over the phase space, and the asymptotic group being defined hence, the same way as in the other approaches (corresponding to the strong definitions).

Gauge Fixing

Gauge Symmetry

$$S[\Phi] = \int_M \mathbf{L}[\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \ldots] d^n x$$

A gauge transformation with parameter functions $F = (f^\alpha)$:

$$\begin{aligned} \delta_F \Phi &= R[F] \\ &= \sum_{k \geq 0} R_\alpha^{(\mu_1 \ldots \mu_k)} \partial_{\mu_1} \ldots \partial_{\mu_k} f^\alpha \end{aligned}$$

The Rs are local functions of the coordinates, the fields and their derivatives. This transformation is a symmetry if the lagrangian transforms as $\delta_F \mathbf{L} = d\mathbf{B}_F$, $B_F = B_F^\mu (d^{n-1}x)_\mu$ - a total derivative.

Examples

Vacuum electrodynamics

$$S[A] = \int_M \mathbf{F} \wedge *\mathbf{F}$$
$$\mathbf{F} = d\mathbf{A}$$
$$\delta_\alpha \mathbf{A} = d\alpha$$

For arbitrary α, this transformation is a symmetry.

General Relativity

Use the Einstein-Hilbert action (optionally include cosmological constant Λ):

$$S[g] = \frac{1}{16\pi G} \int_M (R - 2\Lambda)\sqrt{-g} d^n x$$
$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu}$$
$$= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\rho\nu}\partial_\mu \xi^\rho$$

For every vector field ξ^μ which generates diffeomorphisms, this transformation (where $\mathcal{L}_\xi$ is the Lie derivative) is a symmetry.

> *Sidenote*, the conditions for a vector field to generate diffeomorphisms are that it be complete, smooth, and proper (map compact to compact).

Worth noting is that these transformations have the common structure $\delta_F \Phi = R_\alpha f^\alpha + R_\alpha^\mu \partial_\mu f^\alpha$, involving at most first derivatives of the parameters.

Gauge Fixing

A constraint (generally algebraic or differential) on the fields to eliminate some part of the theory's redundancy, $G[\Phi] = 0$ (a set of n independent conditions).
This constraint must be satisfiable by a field configuration reachable by a gauge transformation $(n \leq \# f^\alpha)$, and to completely fix the gauge it must also use up all the available freedom $(n \geq \# f^\alpha)$ - hence it must have as many independent conditions n as the number of independent parameters f^α.

Note the common gauges used in

- **Electrodynamics**: Lorenz ($\partial^\mu A_\mu = 0$), Coulomb ($\partial^i A_i = 0$), temporal ($A_0 = 0$), axial ($A_3 = 0$).
  Used when $\Lambda \neq 0$,$x^\mu = (\rho, x^\alpha)$ where ρ is the expansion parameter, 0 at the spacetime boundary and positive in the bulk. The constraints:

$$\begin{aligned} g_{\rho\rho} &= -\frac{(n-1)(n-2)}{2\Lambda\rho^2} \\ g_{\rho a} &= 0 \end{aligned}$$

  Where ρ is a spacelike (timelike) for $\Lambda < 0(> 0)$.
- **Gravity**: ^0ab336
  - **de Donder (Harmonic) gauge** : $\Box x^\mu = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\partial^\nu x^\mu) = 0$. This is equivalent to ${\Gamma^\alpha}_{\beta\gamma} g^{\beta\gamma} = 0$, which is useful for working with gravitational waves, since it is also equivalent to $\partial_\gamma g^{\alpha\gamma} = \frac{1}{2}\partial^\alpha g$, which in the linear gravity approximation and perturbed metric $g \mapsto g + h$ becomes $\partial_\gamma h^{\alpha\gamma} = \frac{1}{2}\partial^\alpha h$.
  - **Fefferman-Graham gauge** : Used when $\Lambda \neq 0$,$x^\mu = (\rho, x^a)$ where ρ is the expansion parameter, 0 at the spacetime boundary and positive in the bulk. ρ is a spacelike (timelike) for$\Lambda < 0(> 0)$. The constraints:

$$\begin{aligned} g_{\rho\rho} &= -\frac{(n-1)(n-2)}{2\Lambda\rho^2} \\ g_{\rho a} &= 0 \end{aligned}$$

  - **Bondi gauge** : As the name suggests, this has special relevance to the symmetries we wish to study. Use coordinates (u, r, x^A), where x^A are the angular coordinates on the spatial unit $(n-2)$-sphere. Then the gauge:

$$\begin{aligned} g_{rr} &= 0 \\ g_{rA} &= 0 \\ \partial_r\left(\frac{\det g_{AB}}{r^{2(n-2)}}\right) &= 0 \end{aligned}$$

  Which implies that u labels the null hypersurfaces, x^A the null geodesics in any single hypersurface, and r the luminosity distance along a geodesic. The general metric becomes

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

  - **Newman-Unti gauge** : Similar to Bondi,

$$\begin{aligned} g_{rr} &= 0 \\ g_{rA} &= 0 \\ g_{ru} &= -1 \end{aligned}$$

Residual Gauge Transformations

Gauge transformations left over even after fixing the gauge - transformations with generators F s.t. $\delta_F G[\Phi] = 0$, F being local functions parametrised by $(n-1)$ coordinates.

Examples

- For the Lorenz gauge, we know $\Box\alpha = 0$ permits $\delta_\alpha A_\mu = \partial_\mu \alpha$ transformations.
- Fefferman-Graham : The transformations generated by ξ^μ must have $\mathcal{L}_\xi g_{\rho\mu} = 0, \mu = \rho, a$. The solutions of these are parametrised by n arbitrary functions (σ, ξ_0^a) of the $n-1$ coordinates x^a:

$$\begin{aligned}\xi^\rho &= \sigma(x^a)\rho \\ \xi^a &= \xi_0^a(x^b) + \frac{(n-1)(n-2)}{2\Lambda}\partial_b\sigma \int_0^\rho \frac{d\rho'}{\rho'} g^{ab}(\rho', x^c)\end{aligned}$$

- **Bondi gauge** : The residual transformations need to satisfy (ω an arbitrary function):

$$\begin{gathered}\mathcal{L}_\xi g_{rr} = 0 \\ \mathcal{L}_\xi g_{rA} = 0 \\ g^{AB}\mathcal{L}_\xi g_{AB} = 4\omega(u, x^A)\end{gathered}$$

  The solutions to these use the n functions of (u, x^A), ω, f, Y^A, with $\partial_r f = \partial_r Y^A = 0$. Also note $g = \det g_{AB}$ and $\mathcal{D}_A$ is the covariant derivative using g_{AB}.

$$\begin{gathered}\xi^\mu = f \\ \xi^A = Y^A + I^A \\ I^A = -\partial_B f \int_r^\infty dr'(e^{2\beta} g^{AB}) \\ \xi^r = -\frac{r}{n-2}(\mathcal{D}_A Y^A - 2\omega + \mathcal{D}_A I^A - \partial_B f U^B + \tfrac{1}{2} f g^{-1}\partial_u g)\end{gathered}$$

Boundary Conditions

In essence, requiring some constraints on the fields, in a neighbourhood of any given spacetime region. These are generally on the far-off *asymptotic region*, which again is often infinity (spacelike, timelike or null) but can be something else, like black hole horizons, as well.
Boundary conditions that are too strong - and hence bestow only a trivial asymptotic symmetry group - or too weak - giving divergent associated surface charges - are uninteresting. We want conditions endowing non-trivial symmetries with interesting charges (example properties are finite, integrable, conserved, generically non-vanishing.)

Asymptotic Flatness in Bondi

One boundary condition - with multiple definitions and variations - is ***asymptotic flatness** at null infinity* $(r \to \infty)$. All variations have as preliminary the following at $(r \to \infty)$:

$$\beta = o(1)$$
$$\frac{V}{r} = o(r^2)$$
$$U^A = o(1)$$
$$g_{AB} = r^2 q_{AB} + rC_{AB} + D_{AB} + \mathcal{O}(r^{-1})$$

With $(n-2)$-dim symmetric tensors q_{AB}, C_{AB}, D_{AB}, functions of (u, x^A). q_{AB} in particular becomes the transverse boundary metric.

Variation 1 (AF1)

Fix the determinant of q to some fixed (possibly time-dependent) volume element $\bar{q}$:

$$\sqrt{q} = \sqrt{\bar{q}}$$

AF2

Define $\mathring{q}_{AB}$ as the inherited $n-2$ unit sphere metric, and require q_{AB} to be conformally related:

$$q_{AB} = e^{2\phi} \mathring{q}_{AB}$$

For $n = 4$, this only needs a coordinate transformation to be satisfied (every metric on 2D is conformally flat).

AF3

The historical definition, and a sub-case of AF2,

$$q_{AB} = \mathring{q}_{AB}$$

Note the unique property of this definition to be asymptotically Minkowskian - leading orders of the metric tend to the Minkowski line element $ds^2 = -du^2 - 2dudr + r^2 \mathring{q}_{AB} dx^A dx^B$with $r \to \infty$.

Asymptotic (A)dS in FG & Bondi

Recap [(Anti) de-Sitter Spaces](). Briefly, they are maximally symmetric Lorentzian manifolds with a (negative) positive cosmological constant Λ measuring the curvature of spacetime everywhere.
Now we consider some boundary conditions which constrain the spacetime to tend to, instead of flat, a curved (A)dS behaviour in the asymptotic region.

Asymptotically Locally (A)dS

The preliminary condition, which can be augmented in various ways.

In the Fefferman-Graham Metric

$$g_{ab} = \mathcal{O}(\rho^{-2})$$
$$\Leftrightarrow g_{ab} = \frac{1}{\rho^2} g^{(0)}_{ab} + o(\rho^{-2})$$

This keeps the boundary metric $g^{(0)}_{ab}$ free, which differentiates asymptotically *locally* AdS from asymptotically AdS, the latter requiring the metric to tend to AdS on the whole boundary in the same coordinate system, which introduces the restriction that the induced boundary metric must be conformal to $\mathbb{R} \times S^{d-1}$.[1]
Note the difference in big-O and little-o notation - big-O is asymptotically tight, it requires equality in asymptotic growth rate, whereas little-o does not.

In the Bondi Gauge

The above conditions expressed in the Bondi gauge become

$$g_{AB} = \mathcal{O}(r^2) = r^2 q_{AB} + o(r^2)$$

Asymptotically AdS 1 (AAdS1)

A subcase of AlAdS.

Fefferman-Graham

$$g^{(0)}_{tt} = \frac{2\Lambda}{(n-1)(n-2)}$$
$$g^{(0)}_{ta} = 0$$
$$\det g^{(0)}_{ab} = \bar{q}\frac{2\Lambda}{(n-1)(n-2)}$$

Where $\bar{q}$ is any fixed (possibly time-dependent) volume form for the transverse $(n-2)$-dim space (not spacetime) in the boundary.

Bondi

$$\beta = o(1)$$
$$\frac{V}{r} = \frac{2r^2\Lambda}{(n-1)(n-2)} + o(r^2)$$
$$U^A = o(1)$$
$$\sqrt{q} = \sqrt{\bar{q}}$$

AAdS2

A subcase of AlAdS, dirichlet boundary conditions require $g^{(0)}_{ab}$ be fixed.
This is not very relevant for (A)dS spacetimes, however, since it restricts the bulk spacetime dynamics a little too strongly.

Fefferman-Graham

This is usually chosen to be the cylinder metric:

$$g^{(0)}_{ab}dx^a dx^b = \frac{2\Lambda}{(n-1)(n-2)}dt^2 + \mathring{q}_{AB}dx^A dx^B$$

$\mathring{q}_{AB}$ is the metric on the unit $(n-2)$-sphere, and as in Bondi, the capital indices are defined with $x^a = (t, x^A)$, taking values $\in [3, n]$.

Bondi

$$\beta = o(1)$$
$$\frac{V}{r} = \frac{2r^2\Lambda}{(n-1)(n-2)} + o(r^2)$$
$$U^A = o(1)$$
$$q_{AB} = \mathring{q}_{AB}$$

Note how AAdS1 and AAdS2 were similar to AF1 and AF3 respectively.
Also note how the Bondi gauge is well-suited in both flat and (A)dS asymptotes, whereas the Fefferman-Graham isn't defined for asymptotically flat.

Solution Space

Given a gauge fixing and boundary conditions, a field configuration $\tilde{\Phi}$ satisfying the boundary conditions and the Euler-Lagrange equations is a solution, and the set of all such is the solution space, parametrised by $\tilde{\Phi} \equiv \tilde{\Phi}(b)$, b being arbitrary functions of $(n-1)$ coordinates, since the solution space should lose a degree of freedom after the constraints.

$$\left.\frac{\delta\mathcal{L}}{\delta\Phi}\right|_{\tilde{\Phi}} = 0$$
$$G[\tilde{\Phi}] = 0$$

Asymptotically locally $(A)dS_4$ with Fefferman-Graham

The Einstein equations give the following limiting behaviour:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$
$$g_{ab} = \rho^{-2}g^{(0)}_{ab} + \rho^{-1}g^{(1)}_{ab} + g^{(2)}_{ab} + \rho g^{(3)}_{ab} + \mathcal{O}(\rho^2)$$

Note $g^{(i)}_{ab} \equiv g^{(i)}_{ab}(x^a)$, no ρ dependence. $g^{(0)}_{ab}$ is named the boundary metric, and along with $g^{(3)}_{ab}$ are the only free parameters. Define the stress-energy tensor:

$$T_{ab} \equiv \frac{\sqrt{3|\Lambda|}}{16\pi G}g^{(3)}_{ab}$$

And use the Einstein equations to obtain (Using $D^{(i)}_a$ as the covariant derivative defined using $g^{(i)}_{ab}$):

$$g^{(0)}_{ab}T^{ab} = 0$$
$$D^{(0)}_a T^{ab} = 0$$

Thus for $\Lambda \neq 0$, solution space is parametrised by:

- **AlAdS** boundary conditions: the 11 functions $g^{(0)}_{ab}, T_{ab}$.
- **AAdS1** boundary conditions: the 7 functions $g^{(0)}_{AB}, T_{ab}$.
- **AAdS2** boundary conditions: the 5 functions T_{ab}.
  Always implicit that T_{ab} satisfies the above constraints.

Asymptotically locally $(A)dS_4$ with Bondi

A diffeomorphism between the Bondi and Fefferman-Graham gauges has been worked out for asymptotically $(A)dS_4$ spacetimes in [2],[3]. This, combined with the validity of the Fefferman-Graham expansion, allows us to expand the functions in the metric in powers of r even in the Bondi gauge:

$$g_{AB} = r^2 q_{AB} + rC_{AB} + D_{AB} + \tfrac{1}{r}E_{AB} + \tfrac{1}{r^2}F_{AB} + \mathcal{O}(r^{-3})$$

. The defining condition of the Bondi gauge then implies the following:

$$\begin{aligned} g^{AB}\partial_r g_{AB} &= 4/r \\ \implies \det g_{AB} &= r^4 \det q_{AB} \\ \implies q^{AB}C_{AB} &= 0 \end{aligned}$$

And further conditions relating the other components of the expansion.
Using these results, we can solve the Einstein equations with this power-series ansatz. We obtain various constraints from the rr, rA components on the arbitrary functions which make up the solution. These involve logarithmic terms, however those vanish for $\Lambda \neq 0$, highlighted by the AB components of the Einstein equations.
The solution is parametrised by the 11 functions $\{\beta_0, U_0^A, q_{AB}, \mathcal{E}_{AB}, M, N^A\}_{\Lambda\neq 0}$, with the evolution of M, N^A with respect to u being constrained - hence the IVP is well-defined when given $\beta_0(u, x^C), U_0^A(u, x^C), \mathcal{E}_{AB}(u, x^C), q_{AB}(u, x^C), M(u_0, x^C), N^A(u_0, x^C)$. As with Fefferman-Graham, these reduce to 7 and 5 functions each in the AAdS1/2 boundary conditions : $\{q_{AB}, \mathcal{E}_{AB}, M, N^A\}$ with fixed $\det q_{AB}$, and $\{\mathcal{E}_{AB}, M, N^A\}$, respectively.

Asymptotically Flat with Bondi

Taking the limit $\Lambda \to 0$, the solutions are obtained:

$$G_{\mu\nu} = 0$$

The radial constraints are satisfied by $\beta_0 = 0, U_0^A = 0$ and the evolution constraint simplifies to $(\partial_u - l)q_{AB} = 0$. The evolution constraints on M, N^A remain unchanged. Various components like $C_{AB}, \mathcal{D}_{AB}, \mathcal{E}_{AB}$ become unconstrained or only time-evolution constrained, and hence are more free parameters. In particular, we need to impose $D^A\mathcal{D}_{AB} = 0$ to remove logarithmic terms from the expansion for $U^A(r)$.
Note that the assumption of g_{AB} being analytic in r and the power series expansion being valid, while unrestrictingly true for the (A)dS limit, is slightly restrictive for the flat limit.
As with (A)dS, different boundary conditions restrict the solution space in different ways.

For now, I have elected to avoid the details of this section on solution spaces and focus on the following sections. I may revisit it later.

Asymptotic Symmetry Algebras

Recall that asymptotic symmetries are the residual gauge transformations which preserve given boundary conditions. They are called *on-shell* when they do so, since they are closed transformations on the solution space. Considering the solution space as a subset of the larger parameter space traversed by all the possible gauge transformations, the on-shell transformations will be tangent to the solution space at every point (any normal component will exit the solution space).
The generators of these symmetries are called *asymptotic Killing vectors* in gravity contexts.

Asymptotic Symmetry Algebra

The gauge symmetry generators being F, the gauge transformations (on the solution space) being $R[F]$, their infinitesimal actions on the fields being δ_F, and $\approx$ denoting on-shell equality,

$$\begin{aligned}[R[F_1], R[F_2]] &= \delta_{F_1} R[F_2] - \delta_{F_2} R[F_1] \\ &\approx R[[F_1, F_2]_A]\end{aligned}$$

The bracket of gauge symmetry generators involves an arbitrary skew-symmetric bi-differential operator C. The two other terms account for the possible field-dependence of these generators. This bracket satisfies the Jacobi identity and so the symmetry generators do form a (solution-space-dependent) Lie algebra.

$$\begin{aligned}[F_1, F_2]_A &= C(F_1, F_2) - \delta_{F_1} F_2 + \delta_{F_2} F_1 \\ C(F_1, F_2) &= \sum_{k,l\geq 0} C^{(\mu_1 \ldots \mu_k)(\nu_1 \ldots \nu_l)}_{[\alpha,\beta]} \partial_{\mu_1} \ldots \partial_{\mu_k} F_1^\alpha \cdot \partial_{\nu_1} \ldots \partial_{\nu_l} F_2^\beta\end{aligned}$$

The algebra of the gauge transformations implies that $[\delta_{F_1}, \delta_{F_2}]\Phi = \delta_{[F_1,F_2]_A}\Phi$ - the infinitesimal actions of the symmetries on the fields creates a representation of the generator Lie algebra.

Consider the preliminary (A)dS boundary conditions in either gauge - neither of them add any further constraints to the residual gauge diffeomorphisms derived earlier.

AAdS1 boundary conditions, FH gauge

We discussed that the FH gauge symmetries are generated by a vector field ξ^μ. For these to preserve the AAdS1 boundary conditions, they must satisfy:

$$\begin{aligned}\mathcal{L}_\xi g^0_{t\mu} &= 0, \mu = t, a \\ g^{ab}_{(0)} \mathcal{L}_\xi g^{(0)}_{ab} &= 0\end{aligned}$$

^ed8925
These can be rewritten in the parameters of the metric:

$$\left(\partial_u - \frac{1}{2}l\right)\xi_0^t = \frac{1}{2}D_A^{(0)}\xi_0^A$$
$$\partial_u \xi_0^A = -\frac{\Lambda}{3}g_{(0)}^{AB}\partial_B\xi_0^t$$
$$\sigma = \frac{1}{2}(D_A^{(0)}\xi_0^A + \xi_0^t l)$$

where $l = \partial_u \ln\sqrt{q}$.
The lie bracket for ξ as generators is (known in this context as the *modified lie bracket*):

$$[\xi_1, \xi_2]_A = \mathcal{L}_{\xi_1}\xi_2 - \delta_{\xi_1}\xi_2 + \delta_{\xi_2}\xi_1$$

Using this and the the constraints on the components ξ^μ, we can derive the explicit asymptotic symmetry algebra.

AAdS1 boundary conditions, Bondi gauge

The constraints are

$$(\partial_u - \tfrac{1}{2}l)f = \tfrac{1}{2}D_A Y^A$$
$$\partial_u Y^A = -\frac{\Lambda}{3}q^{AB}\partial_B f$$
$$\omega = 0$$

This gives rise to the asymptotic algebra

$$[\xi(f_1, Y_a^A), \xi(f_2, Y_2^A)]_A = \xi(f, Y^A):$$
$$f = \sum_{i=1}^{2} Y_i^A\partial_A f_{i+1} + \tfrac{1}{2}f_i D_A Y_{i+1}^A - \delta_{\xi_i} f_{i+1}$$
$$Y^A = \sum_{i=1}^{2} Y_i^B\partial_B Y_{i+1}^A - \frac{\Lambda}{3}f_i q^{AB}\partial_B f_{i+1} - \delta_{\xi_i}Y_{i+1}^A$$

This, known as the $\Lambda - \mathrm{BMS}_4$ algebra, is field-dependent, and contains area-preserving diffeomorphisms so is infinite itself. f generate the **supertranslations**, and Y^A generate the **superrotations**.

Differential Forms

> Differential Forms
>
> Let a manifold M and the tangent space $T_p(M)$ at any point p on it.
>
> ## One-forms
>
> One-form

A 1-form is a linear map $\omega : T_p(M) \to F$, where F is the field over which the vector space $T_p(M)$ is defined.

Thus, a one form is an element of the dual vector space $(T_p(M))^*$, and can be written (for $M = \mathbb{R}^2$)as $w(\langle dx, dy \rangle) = adx + bdy$ (since it is linear). Note that this is exactly the differential element for the line integral over a vector field $(a, b)(x, y)$.
The action of a 1-form is to scale the projection of its input on a particular line:

$$\begin{aligned} w(\langle x, y \rangle) &= ax + by \\ &= \langle a, b \rangle \cdot \langle x, y \rangle \\ &= \|\langle a, b \rangle\| \cdot \mathrm{Projection}_{\langle a,b \rangle}(\langle x, y \rangle) \end{aligned}$$

The Wedge product and m-forms

We want to define a product $\wedge$ of one-forms which is a linear function and has some meaningful geometrical representation.
This is not the most precise motivation for 2- or n-forms, but we'll build up to the more precise ones later - they have to do with antisymmetry and generalising the Stokes theorem.
$w_1 \wedge w_2 : T_pM \times T_pM \to F$ is the map we're interested in - note that it takes two vectors as inputs. So we have the 4 components $w_i(v_j)$ to play with, and (going forward I refer to the one-form w_1 and the dual vector it projects vectors on, $\langle a, b \rangle$, interchangeably) these components are essentially the projections of v_1, v_2 onto w_1, w_2 - we can visualise these as two vectors, the projections of v_1, v_2 onto the w_1 - w_2 plane. With two vectors in a plane, what operation could convert them to a scalar? Well, a dot product, but that does not encode the significance of this being a product - it still has length dimensions, in some sense, and has no antisymmetry. As we shall see later, we prefer something antisymmetric, something that does justice to its definition as a product - an area, $|v_1^{w_1 \text{-} w_2} \times v_2^{w_1 \text{-} w_2}|$:

$$w_1 \wedge w_2(v_1, v_2) := \det \begin{pmatrix} w_1(v_1) & w_2(v_1) \\ w_1(v_2) & w_2(v_2) \end{pmatrix}$$

2-forms >

The wedge product of two 1-forms.

Properties of the Wedge

- From the determinant structure, the two-form is **antisymmetric in inputs**: $w_1 \wedge w_2(v_1, v_2) = -w_1 \wedge w_2(v_2, v_1)$.
- Also from the determinant, the wedge product is **anti-commuting**: $w_1 \wedge w_2 = -w_2 \wedge w_1$.

- Hence the wedge of a 1-form with itself is 0.

The wedge product is distributive over addition

Proof:

$$\begin{aligned} w_1 \wedge (w_2 + w_3)[v_1, v_2] &= w_1(v_1)(w_2 + w_3)(v_2) - w_1(v_2)(w_2 + w_3)(v_1) \\ &= w_1 \wedge w_2[v_1, v_2] + w_1 \wedge w_3[v_1, v_2] \end{aligned}$$

Because the determinant is distributive over addition in a single row/column.

Which leads to the following:

on $\mathbb{R}^2$,$w_1 \wedge w_2 = Cdx \wedge dy$ for $C \in F$

Proof:

$$\begin{aligned} w_1 \wedge w_2 &= (Adx + Bdy) \wedge (Cdx + Ddy) \\ &= AC\,\cancel{dx \wedge dx} + ADdx \wedge dy + BCdy \wedge dx + BD\,\cancel{dy \wedge dy} \\ &= (AD - BC)dx \wedge dy \end{aligned}$$

One can easily see that $dx \wedge dy$ gives the (signed) area between its two input vectors, and so any two-form simply scales this area by some constant.

m-forms

A multilinear and alternating $w : (T_pM)^m \to F$.

- **Multilinear** - Linear in every argument
- **Alternating** - Antisymmetric in any pair of arguments

One way to obtain an m-form is to construct it out of 1-forms and wedge products:

$$\begin{aligned} w(v_1, \ldots, v_2) &= w_1 \wedge \cdots \wedge w_2(v_1, \ldots, v_2) \\ &= \det(w_i(v_j)) \end{aligned}$$

For coordinates x^i on M, any m-form on T_pM is a linear combination of $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$. Thus for an n-dim manifold, only $m \leq n$ forms are non-trivial.

The wedge product is associative

$\alpha \wedge \beta = (-1)^{m_\alpha m_\beta} \beta \wedge \alpha$

The proof is trivial - consider moving each component of β through all the components of α, each step is an anti-commutation.

Note how this means $\alpha \wedge \alpha = 0$ when m_α is odd, but not necessarily when it is even.

$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

We already saw that the wedge is distributive over addition of one-forms. Does this extend to m-forms?
First, consider the addition of m-forms. What does $\beta + \gamma$ mean?

$$(\beta + \gamma)(v_i) := \beta(v_i) + \gamma(v_i)$$
$$\alpha \wedge (\beta + \gamma) = \sum_i \sum_j a_i(b_j + c_j)(dx^{i_1} \wedge \cdots \wedge dx^{i_n}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_m})$$

Clearly, assuming m-forms are spanned by $(dx^{j_1} \wedge \cdots \wedge dx^{j_m})$ makes the proof trivial.
Can we prove this without the expansion? I'm not sure, but the proof would certainly hold even with the weaker assumption that the m-form space is spanned by the space of all wedge products of m 1-forms.
To work without the expansion, we need to define the wedge product on two m-forms more rigorously. Considering the definition that we want an $m_1 + m_2$ form, we can restrict the wedge product to be $\alpha \wedge \beta(\{v_i\}, \{w_j\}) = \det \begin{pmatrix} \alpha(\{v_i\}) & \beta(\{v_i\}) \\ \alpha(\{w_i\}) & \beta(\{w_i\}) \end{pmatrix}$.
Then the proof is trivial - it is identical to that for 1-forms.

$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ **for** $I = \{i_j\}_{j=1}^m$.

The space of all m-forms on a manifold, called $\Lambda^m M$, has a basis given by $\{dx_I\}$.

- $$dx_{\{i_k\}}(v^{(j)}) = \det(v_{i_k}^{(j)})_{1 \le j,k \le m}$$
- It's not hard to see that
$$\dim \Lambda^m M^n = \binom{n}{m}$$
Simply by considering that the basis is dx_I and I cannot have repeated indices.

Differential m-forms

These are m-forms but with differentiable functions as coefficients:

$$\omega := \sum_I f_I dx_I,$$
$$f_I : M \to F \forall I$$
$$\omega_p := \sum_I f_I(p) dx_I$$

So a differential m-form is really a map from M to $\Lambda_p^m M$ - a smooth tensor field over the manifold, mapping to the cotangent bundle. ^812167
Alternatively, it could also map m vector fields to a scalar function on the manifold (tensor field innit).

Integrating m-forms

Integrating 2-forms

Take a surface S embedded in M^n parametrised by $\phi : D \to M^n$, $D \subset \mathbb{R}^2$ and a differential 2-form ω.
Consider the Reimann integral:

$$\iint_D f(x,y) dA = \lim_{\delta_x \to 0, \delta_y \to 0} \sum_{ij} f(x_i, y_j) \delta_x \delta_y$$

We've defined the integral by discretising space. Now we can take a point p in D and two points away from it, defining two vectors in $\mathbb{R}^2$, and map these to three points and two vectors in M^n. As the other points approach p in D, the vectors in M^n become tangent vectors at $\phi(p)$. We use this in the discrete space by taking $p = (u_i, v_j)$ and the other two points $(u_{i+1}, v_j), (u_i, v_{j+1})$. Then

$$\int_S \omega := \lim_{\delta_u, \delta_v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)}(\phi(u_{i+1}, v_j) - \phi(u_i, v_j), \phi(u_i, v_{j+1}) - \phi(u_i, v_j))$$
$$= \lim_{\delta_u, \delta_v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)} \left(\frac{\phi(u_{i+1}, v_j) - \phi(u_i, v_j)}{\delta_u}, \frac{\phi(u_i, v_{j+1}) - \phi(u_i, v_j)}{\delta_v} \right) \delta_u \delta_v$$
$$= \iint_D \omega_{\phi(u,v)} (\partial_u \phi, \partial_v \phi) dA$$

And now m-forms

- $\omega = \sum_I f_I dx_I, \ I \in \{1, \ldots, n\}^m, \ dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m} \text{ for } I = \{i_j\}_{j=1}^m$
- $S \subseteq M^n$ parametrised by $\phi : D \to M^n, \ D \subseteq \mathbb{R}^m$.

$$\int_S \omega := \int \cdots \int_D \omega_{\phi(u_1, \ldots, u_m)} (\partial_1 \phi, \ldots, \partial_m \phi) \underbrace{du_1 \ldots du_m}_{dV_m}$$

Note that $\partial_i \phi$ is an n-dim vector, and ω will act on it. The dx_I components of ω will extract the Ith components of each input vector, and take the determinant of all the components, m from each of the m vectors. The rest is merely multivariate integration.

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Symmetries and Noether's Theorem

Symmetries and Noether's Theorem

Global Symmetry

Consider $\mathcal{L}[\Phi, \partial_\mu \Phi, \ldots]$ Lagrangian density with a transformation $\delta_Q \Phi = Q$, Q being a local function. This transformation is a symmetry if the Lagrangian changes only by a total derivative: $\delta_Q \mathcal{L} = dB_Q$, $B_q = B^\mu_Q (d^{n-1}x)_\mu$.

Gauge Symmetry

A symmetry depending on arbitrary spacetime functions $Q = R[F]$, $F = (f^\alpha)$. An (on-shell) equivalence class can thus be defined on the set of symmetries by collecting those differing only by gauge transformations:

$$Q \sim Q + R[F]$$

Classes $[Q]$ are called the global symmetries, with gauge symmetry being trivial ones.

Essentially, a gauge symmetry is one arising from various field configurations having the same physical observables.

Noether Current

Define a conserved current as an on-shell $n-1$ form, so $dj \approx 0$. For arbitrary $n-2$ forms K, we can again define equivalence classes

$$j \sim j + dK$$

The class $[j]$ is called a Noether current.

Gauge symmetries are the degeneracies of the Lagrangian, the redundancies in our description of the system. Their is a one-to-one correspondence between these symmetries and the Noether identities obeyed by the Lagrangian. Their corresponding Noether currents vanish on-shell, except for the boundary term K.

Noether's first theorem

There exists a one-to-one correspondence between global symmetries and Noether currents. The currents associated with gauge symmetries are trivial - the Noether identities.

We can then explicitly construct the Noether currents too,

$$\begin{aligned}\delta_Q\mathcal{L} &= \delta_Q\Phi\partial_\Phi\mathcal{L} + \delta_Q(\partial_\mu\Phi)\partial_{(\partial_\mu\Phi)}\mathcal{L} + \dots \\ &= Q\underbrace{\left(\partial_\Phi\mathcal{L} - \partial_\mu\partial_{(\partial_\mu\Phi)}\mathcal{L} + \dots\right)}_{\frac{\delta\mathcal{L}}{\delta\Phi}} + \partial_\mu\left(Q\partial_{(\partial_\mu\Phi)}\mathcal{L} + \dots\right)\end{aligned}$$

Since $\delta_Q\mathcal{L} = dB_Q$, we can combine the total derivative terms into $\partial_\mu j^\mu_Q = dj_Q$, which will then equal $Q\frac{\delta\mathcal{L}}{\delta\Phi}$ - which is 0 on-shell, hence the current is conserved on-shell.

Noether Representation Theorem

The Noether currents form a representation of the symmetries, when you define the bracket as

$$\begin{aligned}\{j_{Q_1}, j_{Q_2}\} &= \delta_{Q_1}j_{Q_2} \\ &\approx j_{[Q_1,Q_2]}\end{aligned}$$

where $[Q_1, Q_2] = \delta_{Q_1}Q_2 - \delta_{Q_2}Q_1$.

Proof

Operate δ_{Q_1} on the defining equation for the Noether current of Q_2. We'll use the fact that $[\delta_Q, \partial_\mu] = 0$ to write the RHS as $d\delta_{Q_1}j_{Q_2}$.
For the LHS, note that commutation relation between the variational derivative and the variation due to a transformation Q:

$$\left[\delta_Q, \frac{\delta}{\delta\Phi}\right]f = \sum_{k\geq 0}(-1)^{k+1}\partial_{\mu_1}\dots\partial_{\mu_k}\left(\frac{\partial Q}{\partial\Phi_{,\mu_1\dots\mu_k}}\frac{\delta f}{\delta\Phi}\right)$$

Also note that for a total derivative $f = dK$, the variational derivative is $\frac{\delta f}{\delta\Phi} = 0$.
Using these two, write the LHS as:

$$\delta_{Q_1}\left(Q_2\frac{\delta\mathcal{L}}{\delta\Phi}\right) = \delta_{Q_1}Q_2\frac{\delta\mathcal{L}}{\delta\Phi} + \cancel{Q_2\frac{\delta}{\delta\Phi}(\underbrace{\delta_{Q_1}\mathcal{L}}_{dB_{Q_1}})} - Q_2\sum\dots$$

The rest of the proof is too complex, so I will complete it later.

In the above proof, we required the:

Algebraic Poincare Lemma

Let $[\alpha^n]$ denote the equivalence class of n-forms relating $\alpha^n \sim \alpha^n + d\beta^{n-1}$, then the cohomology class for an operator d is

$$H^p(d) = \begin{cases} [\alpha^n] & p = n \\ 0 & 0 \leq p \leq n \\ \mathbb{R} & p = 0 \end{cases}$$

For further details, refer to Fibre Bundles and Jet Bundles.

Noether Charge

For an $n-1$ dim spacelike Σ with boundary $\delta\Sigma$,

$$H_Q[\Phi] := \int_\Sigma j$$
$$H'_Q[\Phi] = \int_\Sigma (j + dK) = H_Q[\Phi] + \cancelto{0}{\int_{\delta\Sigma} K}$$

Where the Stokes theorem was used.
Thus the Noether charge is some idea of a space-distributed quantity which is well-defined for a Noether current as an equivalence class.
Furthermore, it is conserved in time. Consider spacelike surfaces $\Sigma_i := (t_i = 0)$, then (denoting by $\Sigma_2 - \Sigma_1$ the volume enclosed by them)

$$H_Q[\Phi](t_2) - H_Q[\Phi](t_1) = \int_{\Sigma_2} j - \int_{\Sigma_1} j = \int_{\Sigma_2 - \Sigma_1} dj \approx 0$$

Note, this proof does require currents to vanish at infinity.

Like the currents, the charges too form a representation of the global symmetry algebra, and the proof too follows from the statement for the currents:

$$\{H_{Q_1}, H_{Q_2}\} := \delta_{Q_1} H_{Q_2} = \int_\Sigma \delta_{Q_1} j_{Q_2}$$
$$\implies \{H_{Q_1}, H_{Q_2}\} \approx H_{[Q_1, Q_2]}$$

Noether Identities

Consider $R[F]\frac{\delta\mathcal{L}}{\delta\Phi} = \partial_\mu j^\mu_F$. For a gauge symmetry, expand $R[F]$ in the derivatives of f^α:

$$R[F] = R_\alpha f^\alpha + R^\mu_\alpha \partial_\mu f^\alpha + R^{(\mu\nu)}_\alpha \partial_\mu \partial_\nu f^\alpha + \dots$$

And then write the LHS as $f^\alpha R^\dagger_\alpha$, $R^\dagger_\alpha = \sum_{n=0} (-1)^n \partial_{\mu_1} \dots \partial_{\mu_n} \left(R^{(\mu_1 \dots \mu_n)}_\alpha \frac{\delta\mathcal{L}}{\delta\Phi} \right)$ plus a total derivative $:= \partial_\mu S_F^\mu$.

$$f^\alpha R_\alpha^\dagger = \partial_\mu (j_F^\mu - S_F^\mu)$$

Now we can take the variational derivative wrt f^α, cause F is arbitrary, hence

$$R_\alpha^\dagger = 0$$

Thus each independent generator f^α creates such an identity. Also note that the identities satisfy off-shell as well.

Noether's Second Theorem

$$R[F]\frac{\delta \mathcal{L}}{\delta \Phi} = dS_F$$

Where $S_F = S_F^\mu (d^{n-1}x)_\mu$ is the (weakly) vanishing Noether current, $S_F \approx 0$.

Example - General Relativity, $\mathbf{L} = \frac{1}{16\pi G}(R - 2\Lambda)\sqrt{-g}\, d^n x$ (n-form lagrangian)

For a diffeomorphism generated by ξ^μ, the Noether identity is the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ and the Noether current is $\mathbf{S}_\xi = \frac{-\sqrt{-g}}{8\pi G}\xi_\nu (G^{\mu\nu} + g^{\mu\nu}\Lambda) d^{n-1}x$.

The Noether current associated with a gauge symmetry is trivial up to an exact $n-1$-form dK. We would like to define a conserved charge for gauge symmetries, but $H_F = \int_\Sigma j_F = \int_{\partial\Sigma} \mathbf{K}_F$, and K_F is arbitrary.

Definition - Reduciblity Parameter

Parameters $\bar{F}$ of gauge transformations satisfying $R[\bar{F}] \approx 1$ (the transformation is identity on-shell).
Two parameters are equivalent ($\bar{F} \sim \bar{F}'$) if they are the same on-shell ($\bar{F} \approx \bar{F}'$).
For a lot of gauge theories (EM, Yang-Mills, GR in ≥ 3dim), these equivalence classes can be mapped to the exact reducibility parameters $\bar{F}$ which give identity off-shell $R[\bar{F}] = 1$.

Generalised Noether's Theorem

$\exists$ a one-to-one mapping between the equivalence classes of reducibility parameters and the equivalence classes of conserved-on-shell $n-2$ forms:

$$[\bar{F}] \overset{1-1}{\longleftrightarrow} [\mathbf{K}]$$

Where the equivalence class $[\mathbf{K}]$ is defined by $\mathbf{K} \sim \mathbf{K}'$ iff $\mathbf{K} \approx \mathbf{K}' + d\mathbf{l}$ for an $n-3$ form $\mathbf{l}$.
So every identity transformation corresponds to a conserved $n-2$ form (both on-shell, and up to total derivatives). In other words, every symmetry gives a conserved $n-2$ form - this is not a conserved current, though, since currents are $n-1$ forms.
A non-trivial conserved current is $j \not\approx dK$, and reference 86 of Ruzziconi's lectures restates this theorem as mapping to the equivalence classes of non-trivial conserved currents.

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With the Barnich-Brandt procedure, we can construct the n-2 forms explicitly. Given exact reducibility parameters $\bar{F}$, we know (Noether's second theorem) that $d\mathbf{S}_{\bar{F}} = 0$ (the corresponding conserved current). While the details will have to wait until I cover appendix B of Ruzziconi's lectures, in essence there is a homotopy operator which allows the definition $\mathbf{k}_{\bar{F}}[\Phi, \delta\Phi] = -I^{n-1}_{\delta\Phi}\mathbf{S}_{\bar{F}}$, which then gives the conserved n-2 form $\mathbf{K}_{\bar{F}}[\Phi] = \int_\gamma \mathbf{k}_{\bar{F}}[\Phi, \delta\Phi]$.

Linearised General Relativity

$\bar{\xi}$, the diffeomorphism generators, are the exact reducibility parameters. They are the killing vectors of $g_{\mu\nu}$, so $\delta_{\bar{\xi}} g_{\mu\nu} = \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0$. There are no solutions to this for the general metric.

Consider linearised gravity, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, then

$$\delta_{\bar{\xi}} h_{\mu\nu} = \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0$$

The exact reducibility parameters are the diffeomorphism generators (Killing vectors) of the background metric. For a minkowski background, these are the Poincare generators, so we can explicitly construct the n-2 form via the Barnich-Brandt procedure and integrate it over an n-2 sphere at infinity. The constants obtained thus are the ADM charges of linearised gravity.

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Surface Charges

Associated with the generators of the asymptotic symmetries will be $n-2$ forms $\mathbf{k}_F[\Phi, \delta\Phi]$, and their construction mimics the Barnich-Brandt procedure. However, some of the assumptions, like the Noether current being conserved, the $n-2$ form being conserved on-shell, do not hold anymore. But the procedure can be recovered sufficiently for our purposes. Let us see how.

Barnich-Brandt $n-2$ form for asymptotic symmetries

$$\mathbf{k}_F[\Phi, \delta\Phi] = -I^{n-1}_{\delta\Phi}\mathbf{S}_F$$

For a first-order gauge theory ($\delta_{F=(f^\alpha)}\varPhi = R_\alpha f^\alpha + R^\mu_\alpha \partial_\mu f^\alpha$ and first-order EoMs for Φ), this becomes

$$\mathbf{k}_F = [\Phi, \delta\Phi] = -\frac{1}{2}\delta\Phi\frac{\partial}{\partial(\partial_\mu\Phi)}\left(\frac{\partial}{\partial dx^\mu}\mathbf{S}_F\right)$$
$$\mathbf{S}_F = R^\mu_\alpha f^\alpha \frac{\delta L}{\delta\Phi}(d^{n-1}x)_\mu$$

General Relativity, who could've guessed

Recall the weakly vanishing Noether current for GR $\mathbf{S}_\xi = \frac{-\sqrt{-g}}{8\pi G}\xi_\nu(G^{\mu\nu} + g^{\mu\nu}\Lambda)d^{n-1}x$, which gives

$$\mathbf{k}_\xi[g,h] = \frac{\sqrt{-g}}{8\pi G}(d^{n-2}x)_{\mu\nu}[\xi^\nu\nabla^\mu h + \xi^\mu\nabla_\sigma h^{\sigma\nu} + \xi_\sigma\nabla^\nu h^{\sigma\mu} + \frac{1}{2}(h\nabla^\nu\xi^\mu + h^{\mu\sigma}\nabla_\sigma\xi^\nu + h^{\nu\sigma}\nabla^\mu\xi$$

(Referring to $\delta g_{\mu\nu}$ as $h_{\mu\nu}$)

Theorem - Conservation Law

Define the *invariant presymplectic current*

$$\mathbf{W}[\Phi, \delta\Phi] = \frac{1}{2}I^n_{\delta\Phi}\left(\delta\Phi\frac{\delta\mathbf{L}}{\delta\Phi}\right)$$

Then we have the conservation law

$$d\mathbf{k}_F[\Phi, \delta\Phi] \approx \mathbf{W}[\Phi, R[F]]$$

Surface Charges

The infinitesimal surface charge is defined by integrating the conserved $n-2$ form of a symmetry over the $n-2$ boundary $\partial\Sigma$ of an $n-1$ surface Σ:

$$\not{\delta} H_F[\Phi] = \int_{\partial\Sigma}\mathbf{k}_F[\Phi, \delta\phi]$$

This infinitesimal charge is integrable if it is δ-exact, $\not{\delta} H_F[\Phi] = \delta H_F[\Phi]$, in which case we can integrate it over a path γ in the solution space from $\bar{\Phi}$ to Φ:

$$H_F[\Phi] = \int_\gamma \delta H_F[\Phi] + N[\bar{\Phi}]$$

Where $N[\bar{\Phi}]$ is the reference value at $\bar{\Phi}$, and the integral is independent of the chosen path, as long as the endpoints are the same.

Theorem - Charge representation

For integrable charges, the following algebra is satisfied:

$$\{H_{F_1}, H_{F_2}\} \approx H_{[F_1,F_2]_A} + K_{F_1,F_2}[\bar{\Phi}]$$
$$\text{Where } \{H_{F_1}, H_{F_2}\} := \delta_{F_2} H_{F_1} = \int_{\partial\Sigma} \mathbf{k}_{F_1}[\Phi, \delta_{F_2}\Phi]$$

$K_{F_1,F_2}[\bar{\Phi}]$ is called the *central extension*, which only depends on the reference solution field configuration $\bar{\Phi}$, is antisymmetric in the symmetries F_1, F_2, and satisfies the 2-cocycle condition, reminiscent of the Bianchi identities:

$$K_{[F_{(1},F_2]_A,F_{3)}} = 0$$
$$\implies K_{[F_1,F_2]_A,F_3} + K_{[F_3,F_1]_A,F_2} + K_{[F_2,F_3]_A,F_1} = 0$$

Thus, up to a central extension, the charges form a representation of the asymptotic symmetry algebra.

Properties of Surface Charges

Now we've obtained surface charges for a general set of boundary conditions, we'd like to impose some physical constraints on these charges and hence restrict ourselves to the interesting boundary conditions. But note that these conditions are violable, and we'll discuss these conditions in the context of BMS charges in 4D spacetime.
The charges should be

- **Finite** - Charges may diverge due to factors of the expansion parameter (which defines the asymptotic region), or from a divergence of the integral over the $n-2$ surface $\partial\Sigma$.
- **Integrable** - The infinitesimal charges, if integrable, allow us to define the *integrated* surface charges, which then form a representation of the asymptotic symmetry algebra. It is also the integrated charges that generate the symmetries on the solution space.
- **Generically non-vanishing** - Since they generate the symmetries, identically vanishing charges would give rise to trivial actions on the solution space, symmetries which would be considered trivial under the strong definition of the asymptotic symmetry group.
- **Conserved** - The integrated charge should be conserved in time. While we discussed a conservation law regarding the charges, this law can fail in certain conditions - the in-depth discussion of how the proof fails is outside the scope of this report, but I shall go through an example.

Violations of these properties

- For 4D spacetime on the manifold $\mathbb{R}^4$, since the surface to integrate over becomes infinite due to the divergence in radial coordinate r, the charges diverge. It's more sensible to consider a cut-off r and talk about the coefficients of the overleading orders of r.
- Non-integrable charges are dealt with by isolating an integrable part and integrating that. However, the representation theorem may not hold in this case. An alternate approach can be to work with the non-integrable expression and try to define a Lie bracket and representation for the same.
- In asymptotically flat spacetimes, at null infinity, the charges associated with time translation symmetry are not conserved with time. This is known as the *Bondi mass loss*, referring to the decrease of the mass at future null infinity due to radiation flux through the boundary. Similarly, non-conservation often encodes important information about the system's dynamics.

Example: The BMS Group in 4D Spacetime

Intuiting Supertranslations

Take a Minkowski spacetime, and identify points at $\mathcal{I}^+$ with the coordinates u, θ, ϕ, such that a beam of light shot from the origin in $\hat{n}(\theta, \phi)$ at time u reaches that point on $\mathcal{I}^+$ - essentially, u is the retarded time.
Then we have a few symmetries on the boundary. One is time translation, $u \to u + C$. The other is translation of the origin, $u \to u + \delta\vec{r} \cdot \hat{n}(\theta, \phi)$. Both of these could be written in terms of spherical harmonics as:

$$u \mapsto u + \alpha Y_{00}$$
$$u \mapsto u + \sum_{m=-1}^{m=1} \alpha_m Y_{1m}$$

With the condition that the sum over spherical harmonics is real.
One may wonder whether the natural extension of these to higher spherical harmonics is also a symmetry (while maintaining that the additional term must become real):

$$\begin{aligned} u \mapsto u + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta, \phi) \\ = u + f(\theta, \phi) \end{aligned}$$

Well, for the general Minkowski spacetime, clearly not, but at the boundary? It turns out, yes, they are - to be precise, they're symmetries of the asymptotic metric - and these are exactly the supertranslations of the BMS group.
One more way to intuit these is to realise that all the points at two different $\hat{n}(\theta, \phi)$ at the boundary are causally disconnected, so they cannot synchronise their times, and adding arbitrary time offsets at any θ, ϕ is a symmetry.

The Bondi Metric in 4D

The BMS group arises as the asymptotic symmetry in an asymptotically flat spacetime.
Let us work in the Bondi metric. Earlier, we described it as

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)$$
$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0$$

Where u labelled the null hypersurfaces - the lightcones. For 4D, we work in the bondi coordinates $(u, r, z, \bar{z})$ - z is complex and replaces our angular coordinates on the 2-sphere.

Minkowski space can be described in this metric by $\beta = 0, V = -r, U^A = 0$. Further, we abstract out the r dependence of metric on the 2-sphere by writing $g_{AB} = 2r^2\gamma_{z\bar{z}}$:

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$

The spacetime we are concerned with is asymptotically flat - so we want functions $\beta, V' = e^{2\beta}\frac{V}{r}, U^A$ such that in the asymptotic limit they give us the Minkowski metric. We discussed these requirements in Asymptotic Flatness in Bondi:

$$\beta = o(1)$$
$$\frac{V}{r} = o(r^2)$$
$$U^A = o(1)$$
$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + D_{AB} + \mathcal{O}(r^{-1})$$

The subleading order term in the metric, C_{AB}, describes gravitational waves - the term is order $1/r$ relative to leading order, and is transverse to $\mathcal{I}^+$.
Also introduce the Bondi mass aspect m_B occurring in the leading order expansion of the coefficients of du^2:

$$e^{2\beta}\frac{V}{r} + g_{AB}U^A U^B = -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2})$$

And introduce the angular momentum aspect, N_z, whose integrals over the sphere, contracted with a particular rotational vector field (equivalently, specifying an axis of rotation) give the total angular momentum.

The bondi mass aspect, the angular momentum aspect, and C_{AB} all are independent of r.
Also introduce the covariant derivative with respect to $\gamma_{z\bar{z}}$, D_z. With all these, we can write the metric to leading orders in r in the same convention followed by Bondi, van der Berg, Metzner, and Sachs: (note that c.c. implies replacing all z with $\bar{z}$, in the indices too, and not merely a complex conjugate)

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + \left(rC_{zz}dz^2 + D^zC_{zz}dudz\right) + \text{c.c.} \\ &+ \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_u m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + \text{c.c.} + \dots \end{aligned}$$

The eagle-eyed may notice that in Schwarzschild or Kerr spacetimes, the Bondi mass aspect $= GM$ and hence is proportional to the total mass of the system. More generally, the integral of $m_B \equiv m_B(u, z, \bar{z})$ over the 2-sphere gives the *total Bondi mass*. It contrasts with the ADM mass in that it excludes the energy carried by gravitational waves to infinity, and so it was key in proving that gravitational radiation decreases the mass of a system.

We also define the Bondi news tensor:

$$N_{zz} = \partial_u C_{zz}$$

This is the gravitational analogue of the field strength $F_{uz} = \partial_u A_z$, and its square is proportional to the energy flux across $\mathcal{I}^+$.

Deriving Supertranslations

Restricting to the diffeomorphisms with the asymptotic behaviours:

$$\xi^u, \xi^r \sim \mathcal{O}(1), \quad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}\left(\frac{1}{r}\right)$$

This eliminates 6 lorentz generators - the boosts and rotations which grow as $r \to \infty$.
Then, for a diffeomorphism generated by vector field ζ, the lie derivatives of the metric:

$$\begin{gathered}
\mathcal{L}_\zeta g_{ur} = -\partial_u \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_\zeta g_{zr} = r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_\zeta g_{z\bar{z}} = r\gamma_{z\bar{z}}(2\zeta^r + rD_z\zeta^r + rD_{\bar{z}}\zeta^{\bar{z}}) + \mathcal{O}(1) \\
\mathcal{L}_\zeta g_{uu} = -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}\left(\frac{1}{r}\right)
\end{gathered}$$

We want this diffeomorphism to be an isometry in the large r limit to find the killing vector field ζ corresponding to the asymptotic symmetries. This gives the general form:

$$\zeta = f\partial_u - \frac{1}{r}(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r + \dots$$

Leaving out the subleading order terms.
There is a caveat for the third term - it depends on the gauge choice for g_{ur}, and notably differs in the Newman-Unti gauge where $g_{ur} = -1$. But the first two terms are universal - these are the ones measured in gravitational memory. The first term, notably, is the symmetry $u \mapsto u + f(z, \bar{z})$ - the same supertranslations we intuited by extending time and space translations with spherical harmonics!

Applications

Holography

The *holographic principle* states that quantum gravity can be described in terms of lower-dimensional quantum field theories on the boundary.
An important extension of this duality is the AdS/CFT correspondence, which states that the gravitational theory in the $d+1$ dimensional asymptotically AdS spacetime (AAdS2) is dual to a CFT on the d dimensional boundary.
Among the relations between a bulk theory and its dual boundary theory, relevant to us is the correspondence between the gauge symmetries in the bulk theory and the global symmetries of the boundary theory. Now consider that the bulk theory has asymptotic symmetries like the ones we've discussed - this correspondence implies that there exist field theories on the boundary with the same symmetries as global.
The holographic principle is believed to hold in all types of asymptotics. In particular, in asymptotically flat spacetimes, the dual theory would have BMS as the global symmetry.

Infrared Triangle

The Infrared triangle of gauge theories refers to the connections between three seemingly unrelated topics - asymptotic symmetries, soft theorems, and memory effects.

Soft theorems are about scattering amplitudes involving massless particles with small momenta $q \to 0$ - then the scattering amplitude relates to that of the rest of the particles (to first order):

$$\mathcal{M}_{n+1}(q, p_i) = S^{(0)}\mathcal{M}_n(p_i) + \mathcal{O}(q^0),\ S^{(0)} \sim q^{-1}$$

$S^{(0)}$ is called the soft factor, particular to the soft particle involved and to some extent independent of the other particles - for example, it does not involve the spins of the rest of the particles. *Subleading soft theorems* have also been proposed, discussing the $S^{(1)} \sim q^0$ factor (also scaling $\mathcal{M}_n(p_i)$).
These theorems also imply that an infinite number of soft

Memory effects in general refer to permanent shifts in systems from cyclic processes. In gauge theories, we refer to when a field is turned on as a result of a burst of energy passing through the region of interest, leading to an observable phenomenon. In the context of gravity, gravitational waves can cause the *displacement memory effect*, wherein a permanent shift happens in the relative position of two (inertial) detectors. The spin and refraction memory effects also exist under gravity.
Consider the Bondi gauge in the AF3 boundary conditions - then the angular displacement of the two inertial observers, if they are in the asymptotic region, is proportional to the extent to which the field C_{AB} is turned on, $\Delta s^A \propto \Delta C_{AB}$. This could happen due to gravitational waves (Christodoulou effect) or a variation of the Bondi mass M (ordinary memory effect).

So how are these three related?

1. Every symmetry transformation has a corresponding Ward-Takahashi identity equating scattering amplitudes related by that transformation. The Ward identity for supertranslations is equivalent to the soft graviton theorem.

2. The displacement memory effect is equivalent to performing a supertranslation - it has the same effects on C_{AB} as a burst of gravitational waves. The supertranslation connects the two vacua related by the memory effect.
3. The soft theorem relates to the memory effect via a fourier transform. While the soft theorem concerns poles in momentum space ($S^{(0)} \sim q^{-1}$), the memory effect is a position-space permanent shift (sometimes called a DC shift). These are the same thing - the fourier transform of a pole is a step function.
   Furthermore, subleading triangles are also a topic of research - superrotations' ward identities are equivalent to the subleading soft graviton theorem, and the spin and refraction memory effects can be understood as superrotations too.

This infrared triangle is itself relevant to the black hole information paradox. In infrared - low energy - regimes, black holes produce a number of soft gravitons - through this correspondence, they relate to asymptotic symmetries and their surface charges, which then need to be accounted for as information storage - these charges are called *soft hairs*.

Black Hole Information Paradox

Following is an excerpt from an interview[4] with Strominger, where he explains in lay terms the relationship between asymptotic symmetries and the information paradox - the linchpin of the connection is that a black hole event horizon is, in some sense, expanding at the speed of light:

> The horizon of a black hole has the weird feature that it's a sphere and it's expanding outward at the speed of light... ... That's why nothing that is inside a black hole can get out—because the boundary of the black hole itself is already moving at the speed of light.
>
> There's this symmetry of a black hole that we all knew about in which you move uniformly forward and backward in time along all of the light rays. But there's another symmetry... ...in which the individual light rays are moved up and down. See, individual light rays can't talk to each other—if you're riding on a light ray, causality prevents you from talking to somebody riding on an adjacent light ray. So these light rays are not tethered together. You can slide them up and down relative to one another. That sliding is called a super-translation.
>
> ...It turns out that adding a soft graviton has an alternate description as a super-translation in which you move some of these light rays back and forth relative to one another.
>
> ...Super-translations were introduced in the 1960s, and they were talking not about the light rays that comprise the boundary of spacetime at the horizon of a black hole but the light rays that comprise the boundary of spacetime out at infinity.

Modern Research on Asymptotic Symmetries

1. Asymptotic symmetries at timelike and spacelike infinities.[5]

Bibliography

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