Perturbative Chern Simons

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by

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Abstract

Topological Quantum Field Theories are independent of the metric. Among other interesting features, they are exactly solvable without the need for perturbative methods. Observables in these theories, being metric-independent, can be used to construct topological invariants of the manifold over which the theory is defined, or sub-manifolds of the same.

The Chern-Simons theory is a topological quantum field theory constructed from a gauge field and defined over a three-manifold. It has applications in Condensed Matter Physics when studying topological insulators and superconductors. Relevant to this project, it is closely linked to Knot Theory and is an incredible tool for constructing knot invariants.

While it is exactly solvable, a perturbative treatment is also possible, using the Faddeev-Popov method and Feynman diagrams. This project aims to study the perturbative expansion of the Chern-Simons theory and its applications to Knot Theory, specifically Vassiliev Invariants.

The computation of Vassiliev Invariants and other perturbation-based knot invariants is challenging and sensitive to the choice of knot parametrisation. I study methods to improve the efficiency and accuracy of these integral computations.

Contents

Part I

Theoretical Background

Chapter 1

Topological Quantum Field Theories

Referring to

- Undergraduate Lecture Notes in Topological Quantum Field Theory by Vladimir G. Ivancevic, Tijana T. Ivancevic [II08]
- Topological Quantum Field Theories - A Meeting Ground for Physicists and Mathematicians by Romesh Kaul [Kau99]
- Introduction To Chern-Simons Theories by Gregory W. Moore [Moo19]
- Quantum Field Theory on the Plane by David Tong [Ton06]

1.1 Introduction

What are TQFTs?

Toplogical quantum field theories are independent of the metric of curved manifold on which these are defined; the expectation value of the energy-momentum tensor is zero, $\langle T_{\mu\nu}\rangle = 0$. *These possess no local propagating degrees of freedom; only degrees of freedom are topological.* Operators of interest in such a theory are also metric independent.

They have the interesting feature of being exactly solvable, no perturbation needed.

1.1.1 Motivation from Knot Theory (4)

Knot theory is concerned with the topological equivalence of knots and links. TQFTs arise from considering theories which provide such a description of the knots and links embedded in that space - so we want the observables over the knots to be metric independent.

1.2 Path Integral TQFT

Considering fields ϕ_i on a manifold M with metric $g_{\mu\nu}$, with an action $S[\phi_i]$ and some operators $\mathcal{O}_\alpha$ defined, we have vacuum expectation values defined as

$$\langle \mathcal{O}_\alpha \rangle = \int D[\phi_i]\mathcal{O}_\alpha(\phi_i)e^{\iota S[\phi_i]} \tag{1.1}$$

If the vacuum expectation values of some selected operators and their products remain invariant under changes to the metric, the field theory is considered **topological** and these operators the *observables*.

$$\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{\alpha_1}\ldots\mathcal{O}_{\alpha_n}\rangle = 0 \tag{1.2}$$

Schwarz-type TQFTs guarantee this formally by requiring $S, \mathcal{O}_\alpha$ to be metric-independent. An example, which I will cover in detail along with its relation to knot theory, is the Chern-Simons theory.

1.2.1 Chern-Simons Gauge Theory

Composed of

- A differentiable, compact 3-manifold M
- A simple, compact gauge group G (with corresponding gauge connection A)
- Integer parameter k (required to be integral for gauge invariance)

Then we have a Chern-Simons form, which integrates to give the action:

$$S_{CS}[A] = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1.3}$$

1.2.2 Witten-type TQFTs

The second way to ensure metric invariance of the action and observables, and also called cohomological of Witten-type. We require a symmetry with the infinitesimal transformation δ':

$$\delta' \mathcal{O}_\alpha(\phi_i) = 0 \tag{1.4}$$

$$T_{\mu\nu}(\phi_i) = \delta' G_{\mu\nu}(\phi_i) \tag{1.5}$$

$$\text{Where } T_{\mu\nu}(\phi_i) \equiv \frac{\delta}{\delta g_{\mu\nu}} S[\phi_i] \tag{1.6}$$

Note here $G_{\mu\nu}$ is some arbitrary tensor. Since δ' is a symmetry, $\delta' S = 0$ and $\delta' \mathcal{O}_\alpha(\phi_i) = 0$ under transformations $\delta' \phi_i$. Looking again at the vacuum expectation values,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \dots \mathcal{O}_{\alpha_n} \rangle = i \int D[\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \dots \mathcal{O}_{\alpha_n}(\phi_i) e^{\iota S[\phi_i]} T_{\mu\nu}(\phi_i) \tag{1.7}$$

$$= \delta' \left(i \int D[\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \dots \mathcal{O}_{\alpha_n}(\phi_i) G_{\mu\nu}(\phi_i) e^{\iota S[\phi_i]} \right) \tag{1.8}$$

$$= 0 \tag{1.9}$$

Since an expectation value will not change under symmetry transformations, the final equality holds. We have also implicitly assumed the measure is invariant under the transformation. Note that the operators have been assumed metric-independent in the above proof, but it can be extended more generally to $\frac{\delta}{\delta g_{\mu\nu}}\mathcal{O}_\alpha(\phi_i) = \delta' O_\alpha^{\mu\nu}(\phi_i)$, where we have defined additional arbitrary tensor functionals.

δ' must also be a scalar symmetry, since it is a global symmetry and so has a constant parameter corresponding - if that were not a scalar, it would be a pretty harsh constraint to be satisfied on arbitrary manifolds.

Often, cohomological TQFTs satisfy $S = \delta'\Lambda$, which allows showing that any combination of observables is independent of the coupling constant, appearing in the theory as $\exp\left(\iota\frac{1}{g^2}S\right)$. A proof to first-order is given in the reference, simply take $1/g^2 \to 1/g^2 - \Delta$, assume the observables' form doesn't depend on the coupling, and a similar proof to the one for VEVs follows.

Chapter 2

Differential Forms

Let a manifold M and the tangent space $T_p(M)$ at any point p on it.

2.1 One-forms

Definition 2.1. One-form
A 1-form is a linear map $\omega : T_p(M) \to F$, where F is the field over which the vector space $T_p(M)$ is defined.

Thus, a one form is an element of the dual vector space $(T_p(M))^*$, and can be written (for $M = \mathbb{R}^2$) as $w(\langle dx, dy\rangle) = adx + bdy$ (since it is linear). Note that this is exactly the differential element for the line integral over a vector field $(a, b)(x, y)$. The action of a 1-form is to scale the projection of its input on a particular line:

$$
\begin{aligned}
w(\langle x, y\rangle) &= ax + by && (2.1)\\
&= \langle a, b\rangle \cdot \langle x, y\rangle && (2.2)\\
&= \|\langle a, b\rangle\| \cdot \text{Projection}_{\langle a,b\rangle}(\langle x, y\rangle) && (2.3)
\end{aligned}
$$

2.2 The Wedge product and m-forms

We want to define a product $\wedge$ of one-forms which is a linear function and has some meaningful geometrical representation. This is not the most precise motivation for 2- or n-forms, but we'll build up to the more precise ones later - they have to do with antisymmetry and generalising the Stokes theorem. $w_1 \wedge w_2 : T_pM \times T_pM \to F$ is the map we're interested in - note that it takes two vectors as inputs. So we have the 4 components $w_i(v_j)$ to play with, and (going forward I refer to the one-form w_1 and the dual vector it projects vectors on, $\langle a, b\rangle$, interchangeably) these components are essentially the projections of v_1, v_2 onto w_1, w_2 - we can visualise these as two vectors, the projections of v_1, v_2 onto the w_1 - w_2 plane. With two vectors in a plane, what operation could convert them to a scalar? Well, a dot product, but that does not encode the significance of this being a product - it still has length dimensions, in some sense, and has no antisymmetry. As we shall see later, we prefer something antisymmetric, something that does justice to its definition as a product - an area, $|v_1^{w_1\text{-}w_2} \times v_2^{w_1\text{-}w_2}|$:

$$
w_1 \wedge w_2(v_1, v_2) := \det \begin{pmatrix} w_1(v_1) & w_2(v_1) \\ w_1(v_2) & w_2(v_2) \end{pmatrix} \tag{2.4}
$$

Definition 2.2. 2-forms

The wedge product of two 1-forms - a bilinear, alternating map $w^{(2)} : T_pM \times T_pM \to F$.

2.2.1 Properties of the Wedge Product

- From the determinant structure, the two-form is **antisymmetric in inputs**: $w_1 \wedge w_2(v_1, v_2) = -w_1 \wedge w_2(v_2, v_1)$.
- Also from the determinant, the wedge product is **anti-commuting**:$w_1 \wedge w_2 = -w_2 \wedge w_1$.

Hence the wedge of a 1-form with itself is 0.

Lemma 2.1. The wedge product is distributive over addition

Proof:

$$w_1 \wedge (w_2 + w_3)[v_1, v_2] = w_1(v_1)(w_2 + w_3)(v_2) - w_1(v_2)(w_2 + w_3)(v_1) \tag{2.5}$$

$$= w_1 \wedge w_2[v_1, v_2] + w_1 \wedge w_3[v_1, v_2] \tag{2.6}$$

Because the determinant is distributive over addition in a single row/column.

Which leads to the following:

Corollary 2.1. On $\mathbb{R}^2$, $w_1 \wedge w_2 = Cdx \wedge d$ for $C \in F$

Proof:

$$w_1 \wedge w_2 = (Adx + Bdy) \wedge (Cdx + Ddy) \tag{2.7}$$

$$= AC\cancel{dx \wedge dx} + ADdx \wedge dy + BCdy \wedge dx + BD\cancel{dy \wedge dy} \tag{2.8}$$

$$= (AD - BC)dx \wedge dy \tag{2.9}$$

One can easily see that $dx \wedge dy$ gives the (signed) area between its two input vectors, and so any two-form simply scales this area by some constant.

Definition 2.3. m-forms

A multilinear and alternating $w : (T_pM)^m \to F$.

Multilinear - Linear in every argument

Alternating - Antisymmetric in any pair of arguments

One way to obtain an m-form is to construct it out of 1-forms and wedge products:

$$w(v_1, \ldots, v_2) = w_1 \wedge \cdots \wedge w_2(v_1, \ldots, v_2) \tag{2.10}$$
$$= \det(w_i(v_j)) \tag{2.11}$$

For coordinates x^i on M, any m-form on T_pM is a linear combination of $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$. Thus for an n-dim manifold, only $m \leq n$ forms are non-trivial.

Lemma 2.2. The wedge product is associative

Lemma 2.3. $\alpha \wedge \beta = (-1)^{m_\alpha m_\beta} \beta \wedge \alpha$

The proof is trivial - consider moving each component of β through all the components of α, each step is an anti-commutation.

Note how this means $\alpha \wedge \alpha = 0$ when m_α is odd, but not necessarily when it is even.

Lemma 2.4. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

We already saw that the wedge is distributive over addition of one-forms. Does this extend to m-forms?

First, consider the addition of m-forms. What does $\beta + \gamma$ mean?

$$(\beta + \gamma)(v_i) := \beta(v_i) + \gamma(v_i) \tag{2.12}$$

$$\alpha \wedge (\beta + \gamma) = \sum_i \sum_j a_i(b_j + c_j)(dx^{i_1} \wedge \cdots \wedge dx^{i_n}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_m}) \tag{2.13}$$

Clearly, assuming m-forms are spanned by $(dx^{j_1} \wedge \cdots \wedge dx^{j_m})$ makes the proof trivial. Can we prove this without the expansion? I'm not sure, but the proof would certainly hold even with the weaker assumption that the m-form space is spanned by the space of all wedge products of m 1-forms.

To work without the expansion, we need to define the wedge product on two m-forms more rigorously. Considering the definition that we want an $m_1 + m_2$ form, we can restrict the wedge product to be $\alpha \wedge \beta(\{v_i\}, \{w_j\}) = \det \begin{pmatrix} \alpha(\{v_i\}) & \beta(\{v_i\}) \\ \alpha(\{w_i\}) & \beta(\{w_i\}) \end{pmatrix}$. Then the proof is trivial - it is identical to that for 1-forms.

Remark

$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ for $I = \{i_j\}_{j=1}^m$. The space of all m-forms on a manifold, called $\Lambda^m M$, has a basis given by $\{dx_I\}$. -

$$dx_{\{i_k\}}(v^{(j)}) = \det(v_{i_k}^{(j)})_{1 \le j,k \le m} \tag{2.14}$$

- It's not hard to see that

$$\dim \Lambda^m M^n = \binom{n}{m} \tag{2.15}$$

Simply by considering that the basis is dx_I and I cannot have repeated indices.

2.3 Differential m-forms

These are m-forms but with differentiable functions as coefficients:

$$\omega := \sum_I f_I dx_I, \tag{2.16}$$

$$f_I : M \to F \forall I \tag{2.17}$$

$$\omega_p := \sum_I f_I(p) dx_I \tag{2.18}$$

So a differential m-form is really a map from M to $\Lambda_p^m M$ - a smooth tensor field over the manifold, mapping to the cotangent bundle. Alternatively, it could also map m vector fields to a scalar function on the manifold (tensor field innit).

2.4 Integrating m-forms

2.4.1 Integrating 2-forms

Take a surface S embedded in M^n parametrised by $\phi : D \to M^n$, $D \subset \mathbb{R}^2$ and a differential 2-form ω. Consider the Reimann integral:

$$\iint_D f(x, y) dA = \lim_{\delta_x \to 0, \delta_y \to 0} \sum_{ij} f(x_i, y_j) \delta_x \delta_y \tag{2.19}$$

We've defined the integral by discretising space. Now we can take a point p in D and two points away from it, defining two vectors in $\mathbb{R}^2$, and map these to three points and two vectors in M^n. As the other points approach p in D, the vectors in M^n become tangent vectors at $\phi(p)$. We use this in the discrete space by taking $p = (u_i, v_j)$ and the other two points $(u_{i+1}, v_j), (u_i, v_{j+1})$.

Then

$$\int_S \omega := \lim_{\delta_u,\delta_v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)}(\phi(u_{i+1},v_j) - \phi(u_i,v_j), \phi(u_i,v_{j+1}) - \phi(u_i,v_j)) \tag{2.20}$$

$$= \lim_{\delta_u,\delta_v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)} \left(\frac{\phi(u_{i+1},v_j) - \phi(u_i,v_j)}{\delta_u}, \frac{\phi(u_i,v_{j+1}) - \phi(u_i,v_j)}{\delta_v} \right) \delta_u \delta_v \tag{2.21}$$

$$= \iint_D \omega_{\phi(u,v)} \left(\partial_u \phi, \partial_v \phi \right) dA \tag{2.22}$$

2.4.2 Generalising to m-forms

- $\omega = \sum_I f_I dx_I,\ I \in \{1,\ldots,n\}^m,\ dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ for $I = \{i_j\}_{j=1}^m$ - $S \subseteq M^n$ parametrised by $\phi : D \to M^n,\ D \subseteq \mathbb{R}^m$.

$$\int_S \omega := \int \cdots \int_D \omega_{\phi(u_1,\ldots,u_m)} \left(\partial_1 \phi, \ldots, \partial_m \phi \right) \underbrace{du_1 \ldots du_m}_{dV_m} \tag{2.23}$$

Note that $\partial_i \phi$ is an n-dim vector, and ω will act on it. The dx_I components of ω will extract the Ith components of each input vector, and take the determinant of all the components, m from each of the m vectors. The rest is merely multivariate integration.

Chapter 3

Chern Simons Theory

3.1 Introduction

Composed of

- A differentiable, compact 3-manifold (or else odd-manifold) M
- A simple, compact gauge group G (with corresponding gauge connection A)
- Integer parameter k

Under a gauge transformation g, the gauge field / connection A is a 1-form belonging to the Lie algebra $\mathfrak{g}$ which transforms as

$$A \mapsto A^g = g^{-1}(A + d)g \tag{3.1}$$

Then the 2-form $F := dA + A^2 \mapsto g^{-1}Fg$, and a conjugation-invariant quadratic on the lie algebra $p(F)$ becomes a 4-form, which is generally also a total derivative of a 3-form - we'll call this the generalised Chern-Simons form, $CS_p(A)$. Specifically, the usual Chern-Simons form is obtained from the simple example of

$$p(F) = \operatorname{Tr} F^2 \tag{3.2}$$

$$\operatorname{Tr} F^2 = dCS(A) \tag{3.3}$$

$$CS(A) = \operatorname{Tr}\left(AdA + \tfrac{2}{3}A^3\right) \tag{3.4}$$

Which integrates to give the action:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \operatorname{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{3.5}$$

$$Z = \int_{A/G} e^{\iota \int_M CS(A)} \tag{3.6}$$

The Partition function is obtained by integrating over A/G, the space of inequivalent gauge fields. Under a gauge transformation,

$$CS(A^g) = CS(A) - CS(g^{-1}dg) - d(\operatorname{Tr}(g^{-1}Adg)) \tag{3.7}$$

When p is "quantised" - for the trace, this refers to normalisation on a simple Lie algebra - and the manifold has no boundary, $CS(g^{-1}dg)$ is $2\pi k, k \in \mathbb{Z}$, so that the partition function is invariant. Often, $k \in \mathbb{Z}$ is instead written as a quantisation condition arising from the gauge invariance requirement.

Deriving the EoMs:

$$\delta CS(A) = 2\operatorname{Tr}(\delta A F) - d(\operatorname{Tr} A\delta A) \tag{3.8}$$

$$\implies F = 0 \tag{3.9}$$

(0 field strength, flat gauge fields.)

3.1.1 Abelian version

Another version of this action (in the abelian version, with some normalisation) is

$$S[A_\mu] = -\frac{k}{8\pi}\int_{S^3} d^3x\, \epsilon^{\mu\nu\sigma} A_\mu(x)\partial_\nu A_\sigma(x) \tag{3.10}$$

Recall vacuum expectation values from the path integral formulation:

$$\langle W\rangle = \frac{1}{Z}\int [dA] E e^{ikS} \tag{3.11}$$

$$Z = \int [dA] e^{ikS} \tag{3.12}$$

Are these metric-invariant now? They seem to be, but the gauge-fixing of A and the regularisation of the theory (mesh in spacetime - UV cutoff) are both metric dependent. Refer to[KR90] for how these dependences cancel out in some sense - our averages are still metric-invariant.

3.1.2 Motivation - Particle on a Ring

To skip motivation, go straight to Wilson Link Operators (Section 3.1.9). A simple example where the Chern-Simons action naturally arises is that of a charged particle on a ring through which a solenoid passes. It may seem slightly contrived, but it illustrates some important effects in QCD and QED, and also topological insulators. Consider a particle of mass m, charge q, on a ring of radius r, at angular position ϕ - then $K = \frac{1}{2}(mr^2)\dot{\phi}^2$. Also introduce a solenoid with magnetic field B. There is no magnetic field at the particle location, but there is a vector potential along the ring, since its curl must give the field inside the solenoid. Written as a differential form, $A = \frac{B}{2\pi}d\phi$ - essentially $A_\phi = \frac{B}{2\pi}$ and $A_z = A_r = 0$. The action becomes

$$S = \int K dt + \oint eA \tag{3.13}$$

$$= \int \left(\tfrac{1}{2}I\dot{\phi}^2 + \tfrac{eB}{2\pi}\dot{\phi}\right) dt \tag{3.14}$$

The second term is a *topological/θ term* - it is a total derivative, so classically has no impact, but quantum mechanically, it does.

3.1.3 Symmetries (Classical)

Rotations about the ring axis - an $O(2)$ symmetry, of which $SO(2)$ is encoded as $R(\alpha) : e^{i\phi} \mapsto e^{i\alpha}e^{i\phi}$, or the translation $\phi \mapsto \alpha + \phi$, or acting on a cartesian vector.

Parity $P : \phi \mapsto -\phi$. This does change the second term, but by a total derivative, so it has no effect. This is also part of $O(2)$.

The group elements combine as

$$R(\alpha)R(\beta) = R(\alpha + \beta) \tag{3.15}$$

$$P^2 = 1 \tag{3.16}$$

$$PR(\alpha)P = R(-\alpha) \tag{3.17}$$

And thus, with the action of $\langle P \rangle \cong \mathbb{Z}_2$ on $SO(2)$ defined, $O(2) = SO(2) \rtimes \mathbb{Z}_2$.

3.1.4 Field Theoretic approach

As an aside, note that the system can be treated field theoretically, with a field $e^{\iota\phi}$ mapping from the manifold $\mathbb{R}$ (encoding time) to the circle S^1. This is a 0+1 dimensional field theory. The parity operation becomes more like a charge conjugation, since it conjugates the $U(1)$-valued field. In addition the solutions have time translation and reversal symmetries (note though that the second term does not have time reversal symmetry), forming the group $\mathbb{R} \rtimes \mathbb{Z}_2$.

3.1.5 Symmetries (Quantum)

Conjugate (angular) momentum $L = I\dot{\phi} + \frac{eB}{2\pi}$. Legendre transform for the hamiltonian, and quantise $L \to -\iota\hbar\partial_\phi$:

$$H = L\dot{\phi} - \left(\tfrac{1}{2}I\dot{\phi}^2 + \tfrac{eB}{2\pi}\dot{\phi}\right) = \tfrac{1}{2}I\dot{\phi}^2 \tag{3.18}$$

$$\hat{H} = \frac{1}{2I}\left(\hat{L} - \frac{eB}{2\pi}\right)^2 \tag{3.19}$$

$$= \frac{\hbar^2}{2I}\left(-\iota\partial_\phi - \mathcal{B}\right)^2 \tag{3.20}$$

$$\text{Where } \mathcal{B} := \frac{eB}{2\pi\hbar} \tag{3.21}$$

This hamiltonian operator has eigenfunctions and eigenvalues $\psi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{\iota m\phi}$, $E_m = \frac{\hbar^2}{2I}(m-\mathcal{B})^2$. Check that out, an observable energy shift! And it's not a constant shift either, there's a $-2m\mathcal{B}$ term which is clearly measurable.

$\mathcal{B}$ also controls degeneracy - when integral, all but the ground state have two-fold degeneracy, and when half-integral, the ground state does too, and else none do. The *spectrum is periodic in $\mathcal{B}$*.

$$\text{Define unitary } U\psi_m := \psi_{m+1} \tag{3.22}$$

$$\text{Then } UH_{\mathcal{B}}U^{-1} = H_{\mathcal{B}+1} \tag{3.23}$$

3.1.6 Analogues in Higher Dimensional Field theories

1+1 dim Maxwell Theory

$$S = \frac{1}{e^2}\int F * F + \int \frac{\theta}{2\pi}F \tag{3.24}$$

The θ term is a coupling to the B field. Apply the Kaluza-lkein reduction, gauge choice $A_0 = 0$, then the only gauge invariant $e^{\iota\phi(t)} = e^{\iota\oint_{S^1} A}$ becomes $e^{\iota\oint_{S^1} A_1 dx^1}$. This can be done more generally in a 1+1 dimensional Yang-Mills theory too.

3+1 dim Maxwell $\theta = 2\pi\mathcal{B}$, and its value encodes the behaviour of, say, an effective electromagnetic theory in the presence of an insulator - the action is like

$$S = \int d^4x \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \int \frac{\theta}{(2\pi)^2} F \wedge F \tag{3.25}$$

If parity/time-reversal symmetry is present, then θ must be 0 or π, and $\theta = 0$ encodes normal insulators, whereas $\theta = \pi$ encodes topological insulators - $\mathcal{B}$ is half-integral, and the ground state is degenerate.

3.1.7 Back to Symmetries

When $\mathcal{B}$ is integral, the $O(2)$ symmetry holds, but if $\mathcal{B}$ is half-integral the group $\text{Pin}^+(2)$, a cover of $O(2)$, is a symmetry. The difference is the product rules of $O(2)$ elements no longer hold for the cover. And if $\mathcal{B}$ is neither, then only $SO(2)$ remains.

If we defined an angular momentum $\mathcal{L}^2/2I = E$, then for half-integral $\mathcal{B}$, this would take half-integral eigenvalues, as if the particle was spin-1/2. This is a more general phenomenon with Chern-Simons terms - the spins and statistics of particles can be shifted from classical values.

3.1.8 Symmetries of Chern-Simons

The presence of $\epsilon^{\mu\nu\rho}$ in the action, while ensuring Lorentz invariance, breaks both parity and time-reversal symmetry. Note, in odd dimensions - as Chern Simons is restricted to - parity is defined as a single coordinate's sign flip ($\vec{x} \mapsto -\vec{x}$ is merely a rotation in odd dim).

3.1.9 Wilson Link Operators

For link L made of knots $\{K_i\}_{i=1}^{s}$, the knot and link operators (n_i are integers denoting the charge on each loop):[II08]

$$W[K_i] = \exp\left(\iota n_i \oint_{K_i} dx^\mu A_\mu(x)\right) \tag{3.26}$$

$$W[L] = \prod_{i=1}^{s} W[K_i] \tag{3.27}$$

Since this is a non-interacting theory, only the 2-point correlators $\langle A_\mu A_\nu \rangle$ will feature in any expectation value, through the following 2-loop expectation values:

$$\left\langle \oint_{K_l} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(y) \right\rangle \tag{3.28}$$

3.1.9.1 Evaluating the 2-point correlators

Since we are working in a TQFT with mouldable knots, let us squish the link into a small region which we locally identify with $\mathbb{R}^3$, and hence solve for flat spacetime. Then $g_{\mu\nu} = \delta_{\mu\nu}$, and working in the Lorenz gauge $\delta^{\mu\nu}\partial_\mu A_\nu = 0$,

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{\iota}{k} \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3} \tag{3.29}$$

Which is simply the propagator in position space, easily read off the action. I derive it formally in Section 5.2.2. Then the 2-loop expectation values:

$$\left\langle \oint_{K_l} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(Y) \right\rangle = \frac{4\pi\iota}{k} \underbrace{\frac{1}{4\pi} \oint_{K_l} dx^\mu \oint_{K_m} dy^\nu \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}}_{\mathcal{L}(K_l, K_m)} \tag{3.30}$$

$\mathcal{L}$ is the *Gauss linking number* for two distinct knots - it measures how many times one knot passes through the other, and hence should be an integer. It measures orientation, so a right-handed link would have +1, and a left-handed -1 (see Hopf links). It can be given a physical interpretation too - it's the work done moving a magnetic monopole along one knot while a current runs in the other.

When $K_l = K_m$, it instead measures the *self-linking/framing number*, and is evaluated by creating a loop $K_f(\epsilon)$ displaced along the normal such that $K_f(0) = K$ (so K_f has coordinates $y^\nu = x^\nu(s) + \epsilon n^\nu(s)$, $n^\nu(s)$ is the principle normal (unit vector field) to the curve), integrate over K, K_f and limit $\epsilon \to 0$, so $\lim_{\epsilon \to 0} \mathcal{L}(K, K_f(\epsilon)) = \mathcal{SL}(K)$. K_f is known as K's frame.

Refer to the Calugareanu theorem (Section 4.3.1) for another calculation of $\mathcal{SL}$.

Framing is intrinsically connected to regularisation in field theories, avoiding the singularity in the two-point correlator (at $x \to y$) by point-splitting - this same divergence arises in the two-loop correlator seen above, which framing fixes - it provides a *toplogical regularisation*. It may seem like the self-linking depends on the frame, but the topological class of frames have a constant self-linking number.

$$\langle W[L] \rangle = \exp\left(-\frac{2\pi\iota}{k}\left[\sum_i^s n_i^2 \mathcal{SL}(K_i) + \sum_m^s \sum_{i \neq m}^s n_i n_m \mathcal{L}(K_i, K_m)\right]\right) \tag{3.31}$$

Thus topological invariants are connected with expectation values of certain field theory operators.

3.1.10 Wilson loops

An important invariant in this theory is the *Wilson loop*, the trace (in some rep R) of the holonomy of A along a path (1-cycle) γ:

$$\mathrm{Tr}_R(\mathrm{Hol}_\gamma A) = \mathrm{Tr}_R\left(Pe^{\int_\gamma A}\right) \tag{3.32}$$

These Wilson loops are labelled by representations R and the 1-cycles γ, embeddings of S_1 in M. It can be shown that these are knot and link invariants (Section 4.3).

Holonomies

A holonomy[II08] on a smooth manifold is a general geometrical consequence of the curvature of the manifold connection, measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported. Related to holonomy is a Wilson loop, which is a gauge-invariant observable obtained from the holonomy of the gauge connection around a given loop.

3.1.11 $U(1)$ Chern-Simons

Consider the 2+1d spacetime $S^1 \times S^2$ and the gauge group $U(1)$, so the gauge transformation is $A \mapsto A + d\omega$. If the gauge group is **compact** $U(1)$[Ton06], the magnetic flux is quantised to units of $\frac{1}{2\pi}\int_{S^2} F_{12} = 1$. Parametrise S^1 with $x^0 \in [0, 2\pi R)$, R being radius. We want to consider (large) gauge transformations which wind around S^1 and see what phase factors they pick up.

$$\omega = \frac{x^0}{R} \implies A_0 \mapsto A_0 + \frac{1}{R} \tag{3.33}$$

$$\text{And for matter fields of charge } q, \phi \mapsto e^{\imath q\tau/R}\phi \tag{3.34}$$

In the presence of unit magnetic flux and with $A_0 = a$, the action evaluates to (being careful about the topology of the spacetime)

$$S_{CS} = \frac{k}{4\pi}\int d^3x A_{(0}F_{12)} \tag{3.35}$$

$$= \frac{k}{2\pi}\int d^3x A_0 F_{12} = 2\pi k R a \tag{3.36}$$

Which under gauge transformations $S_{CS} \mapsto S_{CS} + 2\pi k$. Thus the action isn't gauge invariant, but for integral k the partition function is. Note the factor of two, arising from integrating by parts the A_1, A_2 terms before to obtain another $A_0 F_{12}$ term before setting $\partial_i A_0 = 0$, because our spacetime is topologically non-trivial.

3.2 Non-Abelian Chern-Simons Theory

Writing the Chern-Simons form a little more explicitly,

$$S = \frac{k}{4\pi} \int_{S^3} d^3x \, \epsilon^{\mu\nu\sigma} \, \mathrm{Tr} \left[A_\mu(x) \partial_\nu A_\sigma(x) + \frac{2}{3} A_\mu(x) A_\nu(x) A_\sigma(x) \right] \tag{3.37}$$

This becomes non-abelian by involving multiple vector fields A_μ^a, where a indexes the generators of a gauge group. The classic example of $SU(2)$: The generators are $\sigma^a/2\iota$, with $a \in \{1, 2, 3\}$, and the fields used in the action are $A_\mu = A_\mu^a \frac{\sigma^a}{2\iota}$. This theory now has, in addition to general coordinate invariance, an $SU(2)$ gauge invariance.

Every component A_μ takes, on a general knot, a spin-j (more generally j index) representation of its gauge group. Why? Well, it can't possibly take more than one - we have to integrate, sum, and different representations are in general different-dimensional matrices. And it must take some - and the results can differ based on which one, so it must be specified. A physical interpretation still eludes me, though. Before we write the Wilson loop operators, we need to define the

3.2.1 Path-ordered exponential

Used when exponentiating integrals of non-commutative algebras, similar to time-ordering but more general, so any product of operator/matrix/non-commutative object fields (created when expanding the exponential of an integral into a series of n-integrals) must be ordered such that objects at positions farther along the path (higher value of path parameter) occur first, or on the left.

$$P \exp \oint_K dx^\mu A_\mu = \prod_m \left(1 + dx_m^\mu A_\mu(x_m)\right) \tag{3.38}$$

3.3 Wilson loop operators (Non-abelian)

For an oriented knot on which the fields A_μ are in the j-spin representation,

$$W_j[K] = \mathrm{Tr}_j\, P \exp \oint_K dx^\mu A_\mu^a T_j^a \tag{3.39}$$

For a link L of knots K_i with spins j_i,

$$W[L] = \prod_{l=1}^{s} W_{j_l}[K_l] \tag{3.40}$$

3.3.1 Link Invariants from Wilson Loops

And the expectation value (aka functional averages, or link invariants):

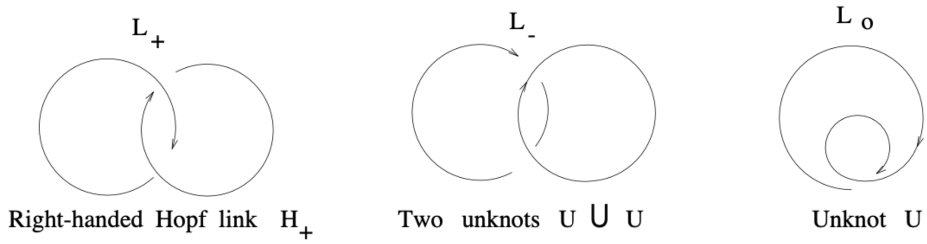
$$V[L] = \frac{1}{Z}\int [dA] W[L] e^{\iota k S},\ Z = \int [dA] e^{\iota k S} \tag{3.41}$$

The integrands are metric-independent, and [KR90] shows that $[dA]$ is too, so the expectation value depends only on the isotopy type of L and the representations $\{j_l\}$. These invariants can be obtained without perturbations! For example, Witten related links with all $j = 1/2$ as follows: For 3 links differing only in the presence & over/under type of one crossing, labelled as L_0, L_+, L_- respectively, the invariants relate as

$$qV_{1/2}[L_+] - q^{-1}V_{1/2}[L_-] = (q^{1/2} - q^{-1/2})V_{1/2}[L_0] \tag{3.42}$$

$q = \exp\frac{2\pi\iota}{k+h}$ is a root of unity related to the coupling constant k. The constant h is the *dual Coxter number* of any gauge group - it is 2 for SU(2). Here, those familiar with Knot theory would rejoice, because this is the generating Skein relation for the Jones polynomials, and indeed $V_{1/2}[L]$ is the one-variable Jones polynomial.

We can use this to evaluate $V_{1/2}[U]$, U being the unknot, by noting that the invariant for disjoint links is the product of their individual invariants, and that if $L_0 = U \cup U$, then $L_+ = L_- = U$, just with the opposite orientations. Then the skein relation gives $V_{1/2}[U] = q^{1/2}+q^{-1/2}$. Then we can build up to the Hopf link H_+ by $L_- = U \cup U$, $L_+ = H_+$,$L_0 = U$:



Then $V_{1/2}[H_+] = 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}$. Similarly, $L_+ = T_+$, a trefoil, can be solved for as $V_{1/2}[T_+] = \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q^3}} + \frac{1}{\sqrt{q^5}} - \frac{1}{\sqrt{q^9}}$.

Similarly, many new link invariants emerge from expectation values of the Wilson link operators with arbitrary spin representations. When these spins differ across knots, they are called *coloured polynomial invariants*. Of course, other semi-simple groups could be used instead of $SU(2)$, giving other new invariants. For these general cases, the recursion relations don't always exist, and aren't complete if they do.

3.4 Complete Solutions for $SU(2)$ Chern-Simons

Referring to [Kau92] A non-perturbative method which generalises Witten's formalism, it needs two ingredients:

1. Field Theoretic Ingredient - Bulk-Boundary Correspondence A Chern-Simons on a 3-manifold has a bulk-boundary correspondence with a 2D Wess-Zumino CFT, and the expectation values (in Chern-Simons) of Wilson operators along n lines ending at points on the boundary are related to the Wess-Zumino on the boundary with n punctures "carrying" the same spin representations. Essentially, the functional integral in CS relates to the n-point correlator in the WZ.

2. Braid Theory Ingredient An n-braid has two (horizontal) planes with n points on each, directly above/below corresponding points on the other plane.

The braid itself is the set of non-intersecting strands connecting points above with points below, never going upwards (backwards) for any section of the strand. This can be projected onto 2D with marked over- and under-crossings. The braids form a group. Identical strands give the ordinary braids, but we can also colour the strands separately (give them spins), which forms a groupoid instead of a group. Braids can become knots and links in a few ways:

1. **Closure** of a braid - connect the strands' ends for the n^{th} point in the above plane to the n^{th} point in the below plane.

   Alexander's theorem states any link/knot can be obtained from a braid's closure (not uniquely).

2. **Plaiting** of a braid - For a $2m$-braid with pairs of adjacent strands carrying the same colour (spin) but opposite orientations, connect these adjacent strands above and below. Birman's theorem states any coloured and oriented link can be represented as a braid's plat (not uniquely).

3.4.1 The solution

The boundary CFT creates matrix representations of braids from n-point correlators, and plats/closures correspond to specific elements of these matrices, which then give us the link invariant from the Wilson operator. Some results from the same:

$$V_j[U] = [2j+1]_q \tag{3.43}$$

Where square brackets denote q-numbers (see q-Analogues, Appendix A) using the square root of unity defined above.

$$V_j[T_+] = \sum_{m=0}^{\min\{2j,k-2j\}} [2m+1]_q (-1)^{2j+m} q^{-6C_j+\frac{3}{2}C_m} \tag{3.44}$$

Where $C_j = j(j+1)$, the (quadratic) Casimir invariant for a spin-j rep.

3.4.2 Framing

Recall framing from Section 3.1.9.1, ‘Evaluating the 2-point correlators’. The link invariants depend on the framing, and the above results are in the *standard framing*, in which every knot’s self-linking number is 0 - invariants are unchanged under all 3 Reidmeister moves, i.e. are ambient isotopic invariants.

Another important option is the *vertical framing*, where the frame is vertically displaced above the knot’s 2D projection - this gives regular isotopic invariants, unchanged under RII,III but not RI.

Chapter 4

Knot Theory

Definition 4.1. Knot

A smooth non-intersecting closed curve embedded in a 3-manifold. It can be *oriented*. A circle (and its equivalents) is an *unknot*.

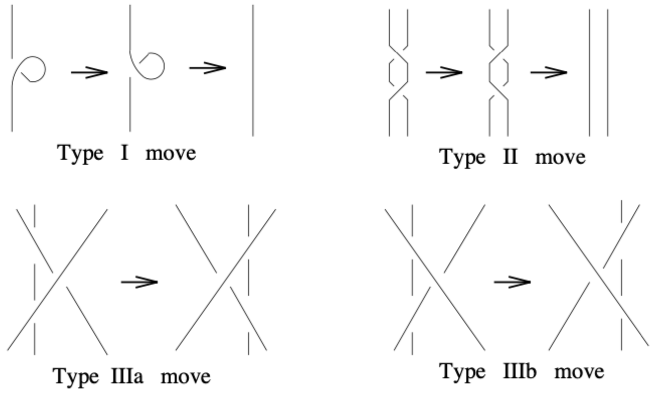
A *knot diagram* is a 2D projection of this with the minimum *double points* (self intersections) - count them to get the *crossing number*.

Definition 4.2. Link

Any collection of non-intersecting knots. It's oriented if the knots are. Define a *link diagram* similarly.

4.1 Reidmeister moves

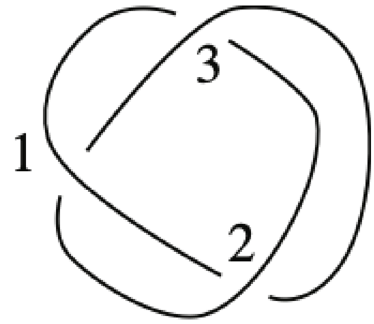
We're interested in the equivalence of knots and links as if they were ideal physical objects - so moving, stretching, shrinking is allowed, cutting isn't. How do we rigorously encode these? In the context of knot and link diagrams, the 3 Reidmeister moves suffice. These seem physically



sensible, and invariance under these has the technical name *ambient isotopy* - an isotopy is a continuous deformation of a sub-manifold. Some objects are invariant under types 2 and 3 but not 1 - they are called *regular* isotopic invariants. Equivalence of links or nots is not trivial to identify, and *link invariants* are a crucial aid for the same - see the Jones polynomial and generalisations. Note that mirror-images of knots are not considered equivalent per se, and while the Alexander polynomial doesn't distinguish between them, the Jones does. Distinct Jones imply distinct knots, but the converse need not be true.

4.2 Gauss Code

Label the crossings of a knot diagram with integers 1 to n, then the Gauss code is a cyclical double-occurrence list of the integers denoting the over and undercrossings encountered when moving along a knot. As always, we use the Trefoil for the non-trivial example. Label crossings $1, 2, 3$, then following the knot, one may start from the overcrossing at 1, undercross at 2, overcross at 3, then undercross at 1, and so on - the Gauss code is $O1U2O3U1O2U3$, O, U denoting over and undercrossings. An alternate notation is to denote overcrossings with the positive labels and undercrossings with negative labels, like so: $1, -2, 3, -1, 2, -3$. For the simplest forms of Knot diagrams, I expect Gauss codes cannot contain consecutive Os or Us - such a configuration could be simplified further with the Reidmeister moves.



4.3 Knot and Link Invariants

4.3.1 Linking Numbers

$$\underbrace{\frac{1}{4\pi}\oint_{K_l} dx^\mu \oint_{K_m} dy^\nu \epsilon_{\mu\nu\sigma}\frac{(x-y)^\sigma}{|x-y|^3}}_{\mathcal{L}(K_l,K_m)} \tag{4.1}$$

$\mathcal{L}$ is the *Gauss linking number* for two distinct knots - it measures how many times one knot passes through the other, and hence should be an integer. It measures orientation, so a right-handed link would have +1, and a left-handed -1 (see Hopf links). When $K_l = K_m$, it instead measures the *self-linking/framing number*, and is evaluated by creating a loop $K_f(\epsilon)$ displaced along the normal such that $K_f(0) = K$ (so K_f has coordinates $y^\nu = x^\nu(s) + \epsilon n^\nu(s)$, $n^\nu(s)$ is the principle normal (unit vector field) to the curve), integrate over K, K_f and limit $\epsilon \to 0$, so $\lim_{\epsilon\to 0}\mathcal{L}(K, K_f(\epsilon)) = \mathcal{SL}(K)$. K_f is known as K's frame.

Theorem 4.1. Calugareanu theorem

The self-linking number is the sum of the twist and writhe for any knot.

$$\mathcal{SL}(K) = T(K) + w(K) \tag{4.2}$$

$$T(K) = \frac{1}{2\pi} \int_K ds\, \epsilon_{\mu\nu\sigma} \frac{dx^\mu}{ds} n^\nu \frac{dn^\sigma}{ds} \tag{4.3}$$

$$w(K) = \frac{1}{4\pi} \int_K ds \int_K dt\, \epsilon_{\mu\nu\sigma}\, e^\mu \frac{de^\nu}{ds} \frac{de^\sigma}{dt} \tag{4.4}$$

$$e^\mu(s,t) = \frac{y^\mu(t) - x^\mu(s)}{|y(t) - x(s)|} \tag{4.5}$$

$e : K \otimes K \mapsto S^3$, $n^\mu(s)$ is the field of normal vectors to K at $x^\mu(s)$.

The twist and writhe/coil may not be integers, and they are not ambient isotopic invariants, but their sum is both.

One attempt at intuitively explaining this theorem was:

Twist This term represents the total "twist" of the curve. It corresponds to the sum of the signed angles between consecutive tangent vectors along the curve.

Writhe Writhe, in the context of the Calugareanu theorem, refers to the total "writhe" of the curve. It's the overall measure of how much the curve winds and coils in three-dimensional space.

Linking Number The linking number is a topological invariant that quantifies how many times one closed curve winds around another closed curve. It's a measure of their entanglement or linking.

4.3.2 Jones Polynomial

The Jones polynomial obeys the Skein relation:

$$(t^{1/2} - t^{-1/2})V[L_0] = tV[L_+] - t^{-1}V[L_-] \tag{4.6}$$

Where L_0, L_+, L_- are links differing in a single crossing change as follows:

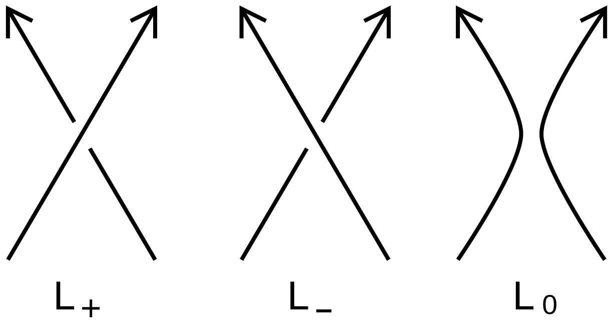


Figure 4.1: Image credits: Wikipedia

4.3.3 Alexander Polynomial

First discovered by J.W. Alexander in 1928, the Alexander polynomial was the only known polynomial invariant of knot types for over 50 years, until Jones polynomial was discovered by Vaughan Jones in 1984.

There are many equivalent ways to define it. One approach similar to the original definition uses the Wirtinger representation of the knot group associated with any knot to define the Seifert matrix. The ideal generated from the submatrices of this matrix is the Alexander ideal. This is a principle ideal, and any generator of this ideal is a valid Alexander polynomial (so these polynomials are unique up to factors of the form $\pm t^n,\ n \in \mathcal{Z}$).

The Alexander polynomial also obeys a slightly different Skein relation:

$$\Delta[L_+] - \Delta[L_-] + (t^{1/2} - t^{-1/2})\Delta[L_0] = 0 \tag{4.7}$$

Often, the more mathematical definition is forgone and the Skein relation is relied upon directly.

4.3.4 Vassiliev Invariants

The Vassiliev invariants were initially introduced by V. A. Vassiliev while he was studying the smooth maps from $S^1 \to S^3$ using knot theory, and are defined for the more general class of self-intersecting knots.

As I show in this project, they are also the coefficients of the perturbative expansion of the expectation value of the Wilson loop operators in the $SU(2)$ Chern-Simons theory.

4.4 Parametrisation and Examples

While knots are topological objects and most knot and link invariants can be computed without specifying a precise embedding, to compute the invariants from the Wilson operators, we need to specify a closed path in the manifold, with a parametrisation $\gamma : [0, 1] \to M$.

There exist many documented parametrisations for common knots, and some knots belong to the same family of parametrisations, the torus knots being the most famous example. While topologically equivalent, certain parametrisations fair better than others when trying to analytically or numerically compute the Wilson operators.

4.4.1 Torus Knots

The Torus knots are a family of knots defined by paths in 3D space parametrised by a pair of coprime integers (n, m). The parametrisation arises from the definition that an (n, m) torus knot is a path on the surface of an unknotted torus inside $\mathbb{R}^3$ which winds around the axis of the torus n times and around the circle centred inside the torus m times. Some simple examples are $(1, 0)$ and $(0, 1)$ torus knots being unknots, and $(2, 3)$ being the Trefoil.

For a torus with major radius R and minor radius r, working in cylindrical coordinates

(ρ, ϕ, z), the torus is defined as $(\rho - R)^2 + z^2 = r^2$. The parametrisation is then:

$$f_{n,m}(\theta) \equiv (x(\theta), y(\theta), z(\theta)) \tag{4.8}$$
$$x(\theta) = (R + r\cos(m\theta))\cos(n\theta) \tag{4.9}$$
$$y(\theta) = (R + r\cos(m\theta))\sin(n\theta) \tag{4.10}$$
$$z(\theta) = -r\sin(m\theta) \tag{4.11}$$
$$\theta \in [0, 2\pi]$$

In literature, often the torus with $R = 2, r = 1$ is used for simplicity. For numerical calculations, it can be useful to play with these values to improve the numerical stability of the integrals. Due to the ease of parametrisation and visualisation, I use these to test my calculations.

Some examples are:

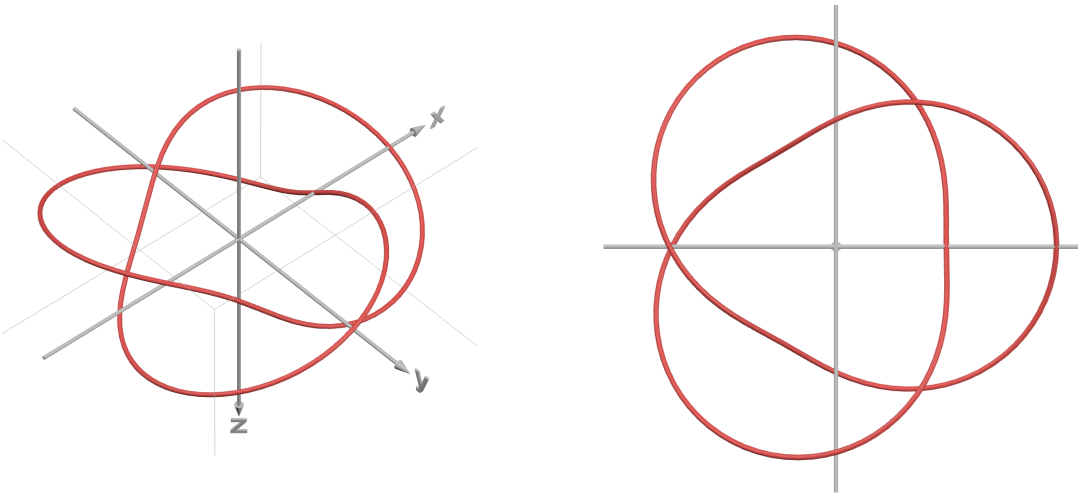
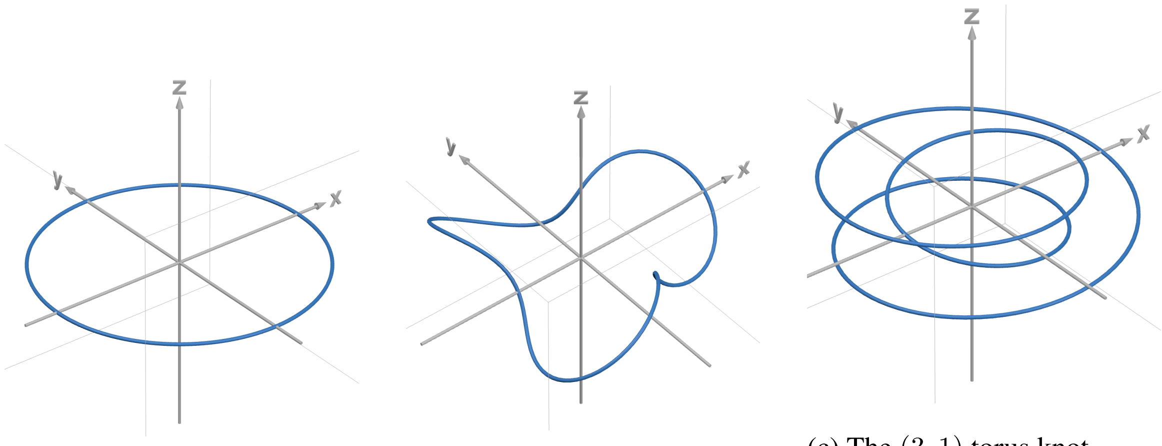


Figure 4.2: The trefoil knot, parametrised as $f_{2,3}(\theta)$

4.4.2 Torus Links

If the two integers n, m are not chosen to be coprime, we get a link instead, with as many loops as $\gcd(n, m)$. However, this link is not well-defined in the torus knot parametrisation, since the loops intersect.

Instead, we can take inspiration from the torus knots and define *twin torus links* as links of



(a) The $(1, 0)$ torus knot

(b) The $(1, 3)$ torus knot

(c) The $(3, 1)$ torus knot, equivalent to unknot under Reidmeister I moves

Figure 4.3: Various parametrisations of the Unknot

two identical torus knots, with one rotated slightly about the torus axis. While one is parametrised as $f(n, m, t)$, the other is rotated by α and parametrised as:

$$g^{\alpha}_{n,m}(\theta) \equiv (x^{\alpha}(\theta), y^{\alpha}(\theta), z^{\alpha}(\theta)) \tag{4.12}$$

$$x^{\alpha}(\theta) = (R + r\cos(m\theta))\cos(n\theta + \alpha) \tag{4.13}$$

$$y^{\alpha}(\theta) = (R + r\cos(m\theta))\sin(n\theta + \alpha) \tag{4.14}$$

$$z^{\alpha}(\theta) = -r\sin(m\theta) \tag{4.15}$$

$$\theta \in [0, 2\pi]$$

We can also define torus-unknot links as the links between a torus knot and the circle centred inside the torus. I use these to test my calculations, since they are simple to visualise and calculate link invariants for.

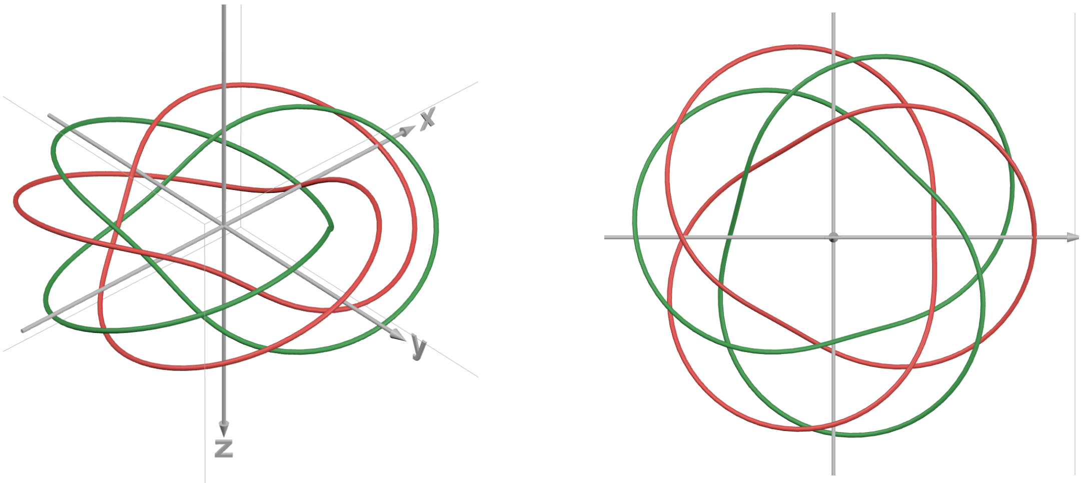


Figure 4.4: The twin trefoil link, parametrised as $g^0_{2,3}(\theta)$ and $g^{\pi/4}_{2,3}(\theta)$

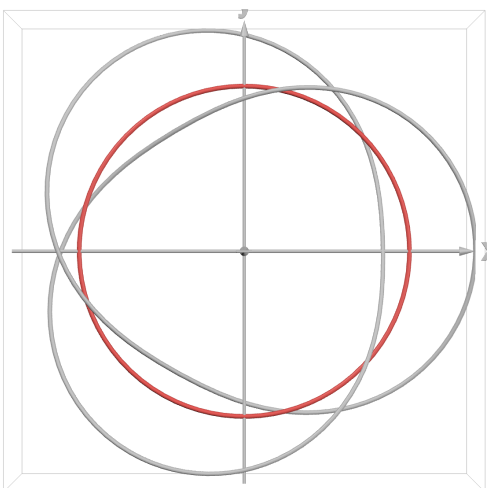


Figure 4.5: The trefoil-unknot link

Chapter 5

Perturbative Chern-Simons

Referring to

- Perturbative Aspects of the Chern Simons Topological Quantum Field Theory and other publications by Dror Bar-Natan, [Bar91] [Bar95b],[Bar95a]
- Field Theory, a Modern Primer by Pierre Ramond [Ram81]
- Link invariants from Chern-Simons theory, and other publications by Guadagnini, [GMM91], [GMM89], [Gua93],[GP94]

The Wilson operators can be expanded in a power series of the coupling constant, and the coefficients will be topological invariants of the knot. The CS theory is super-renormalizable, and the coupling constant k depends on the regularisations - for $SU(2)$ and certain regularisations, it changes as $k \mapsto k+2$, consistent with the effective coupling observed in non-perturbative studies. To first order, the theory reduces to its abelian cousin, so the Wilson invariant is simply the self-linking of the knot (up to factors based on the gauge group). Higher order contributions are the *Vassiliev invariants*, initially introduced when studying the smooth maps on $S^1 \to S^3$ -

based on the types of singularities, the maps can be classified, each class corresponding to a knot and hence characterised by a family of invariants. Perturbative CS in the Landau or light-cone gauges allowed covariant integral representations of these invariants, and in the temporal gauge, combinatorial formulae for them.

5.1 Faddeev-Popov Procedure

Consider a gauge theory with gauge connection A_μ transforming via representation R of a Lie group G. (Note that in this report we work in an adjoint representation, where $A_\mu = T^a A^a_\mu$, T^a being the generators of the corresponding Lie algebra $\mathfrak{g}$.) For every physically unique configuration A_μ there is a set of configurations related to it by a gauge transformation. In group theory language, this is known as the orbit of A_μ under G.

When we integrate over all possible field configurations in the path integral, we overcount the correct answer by a factor of the size of the orbit, which is infinite for Lie (continuous) groups. We could resolve this by fixing the gauge and integrating over the restricted configuration space, but this is difficult to implement in functional integrals, and does not lend itself well to the machinery already developed for these field theory integrals. In addition some gauge-fixing choices affect the manifest lorentz invariance of the integral.

One approach to resolve this is the Faddeev-Popov procedure. The steps are briefly as follows: 1. Define a functional $F[A]$ which has a single zero in every orbit under G - a zero once in every physically unique configuration. This is essentially rewriting any gauge-fixing condition as $F[A] = 0$. Thus, to fix the gauge, F must be an l-component functional, for the A-space having dimension l. 2. Insert $\delta(F)$ into the path integral, to restrict the integral to a single set of physical configurations. For the A-space having dimension l, this is an l-dim Dirac delta. 3. Divide by the volume of the G-orbit to correct for factors arising from the inserted $\delta(F)$. This is obtained as the factor $\det\left(\frac{\delta F^a}{\delta g}\right)$.

A more rigorous treatment follows in the non-abelian gauge theory section.

It is then convenient to rewrite the inserted factors as integrals of exponential integrands over some auxiliary fields (which are also acted upon by the representation R, so they can sum with A^μ terms). These integrands then combine with $e^{\iota S}$ to give additional terms to the Lagrangian, which involve interactions of the existing fields and the newly-introduced auxiliary fields. Thus the gauge-fixed path integral effectively becomes an integral over the entire configuration space of A^μ and over additional auxiliary fields, which are known as ghosts.

This modified path integral can be treated perturbatively to obtain Feynman rules for both the true fields of the theory and the ghost fields. Thus the gauge-fixed results can be obtained from Feynman diagrams if they also involve the pseudo-particles associated with ghost fields.

5.1.1 Abelian Gauge Theory

Consider a $U(1)$ gauge theory with connection A^μ that transforms as $A^\mu \to A^\mu + \partial^\mu \alpha(x)$, and a Lagrangian $\mathcal{L}[A, \phi] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu + \mathcal{L}[\phi]$. We can fix the gauge as $\partial_\mu A^\mu = 0$, which is equivalent to saying we can always find an α such that $\partial_\mu \partial^\mu \alpha(x) = \partial_\mu A^\mu$. Consider the following integral, which should remain invariant under the transformation $\pi(x) \to \pi(x)+\alpha(x)$:

$$f(\xi) = \int \mathcal{D}\pi \; e^{-\iota \int d^4x \frac{1}{2\xi}(\Box\pi)^2} \tag{5.1}$$

$$\overset{\pi \to \pi+\alpha}{=} \int \mathcal{D}\pi \; e^{-\iota \int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2} \tag{5.2}$$

Now we can multiply and divide by $f(\xi)$ in the path integral, after which we perform the transformation $A^\mu \to A^\mu = +\partial_\mu \pi$. We obtain the vacuum expectation values:

$$\langle \Omega | \mathcal{O}(x) | \Omega \rangle = \frac{1}{Z[0]} \left(\frac{1}{f(\xi)} \int \mathcal{D}\pi \right) \int \mathcal{D}A^\mu \mathcal{D}\phi \mathcal{D}\phi^* \; e^{\iota \int d^4x \; \mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2} \mathcal{O}(x) \tag{5.3}$$

The same factor of $\left(\frac{1}{f(\xi)} \int \mathcal{D}\pi \right)$ occurs in the modified $Z[0]$, so it cancels and we obtain the gauge-fixed integral. Alternatively, we don't perform the transformation to remove π and don't cancel the factors. Then we have an additional field to integrate over, which would also appear in the perturbative treatment and hence in Feynman diagrams. This is known as a Faddeev-Popov ghost field, of a pseudo-particle, which is merely a mathematical artifact arising when correcting the integral for gauge freedom.

5.1.2 Non-Abelian Gauge Theories

Define the integral over the orbit of A_μ:

$$\Delta_F^{-1}[A_\mu] = \int \mathcal{D}g\ \delta[F^b(A_\mu^g)] \tag{5.4}$$

Where g is a group element, F is a multi-component function that has a unique zero in every G-orbit, $A^g = g(A+d)g^{-1}$. Since the integral is over the entire group,

$$\Delta_F^{-1}[A_\mu^{g'}] = \Delta_F^{-1}[A_\mu] \quad \forall g' \in G \tag{5.5}$$

Thus we can insert the identity into the path integral to perform gauge-fixing via the Dirac delta, as previously discussed:

$$Z[0] = \int \mathcal{D}A_\mu\ \Delta_F[A_\mu] \int \mathcal{D}g\ \delta[F^b(A_\mu^g)] e^{\iota S[A_\mu]} \tag{5.6}$$

Now we need to evaluate $\Delta_F[A_\mu]$. Consider a change of integration variables from F^b to g, which introduces the factor of the Jacobian:

$$\Delta_F^{-1}[A_\mu] = \int DF \det\left(\frac{\delta g}{\delta F}\right) \delta[F] \tag{5.7}$$

$$\implies \Delta_F[A_\mu] = \det\left(\frac{\delta F}{\delta g}\right)\Bigg|_{F[A_\mu]=0} \tag{5.8}$$

5.1.2.1 Ghost Fields

We would like to rewrite the additional terms in the path integral as integrals over exponential functions of auxiliary fields. This is easily accomplished for the Dirac delta by a Fourier transform:

$$\delta^l[F] = \frac{1}{(2\pi)^l} \int \mathcal{D}\phi\ e^{\iota F^a[A_\mu]\phi_a} \tag{5.9}$$

The determinant term can be treated as such. Usually, the integral of a gaussian $e^{\bar{X}AX}$ over the vector X gives the determinant of the matrix A in the denominator. However, it is possible to obtain the determinant in the numerator instead, if the integrating variables are instead Grassmann variables - anti-commuting variables. It can be shown that for Grassmann variables $\{c_a\}_{a=1}^{l}, \{\bar{c}_a\}_{a=1}^{l}$,

$$\int \mathcal{D}c\mathcal{D}\bar{c}\, e^{\iota\bar{c}Jc} \propto \det J \tag{5.10}$$

Thus we can replace the determinant by a Gaussian integral over Grassmannian fields. Grassmannian fields play a special role in field theory - they are used to implement fermion fields. So this term gives rise to fermionic ghosts.

5.1.3 For the Chern-Simons Theory

The Chern-Simons action is

$$cs[A] = \frac{k}{4\pi}\int_{M^3} \mathrm{Tr}(A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A) \tag{5.11}$$

A gauge transformation takes $A \mapsto g^{-1}(A+d)g$. Let us take the subset of gauge transformations path-connected to the identity transformation. Then, for an infinitesimal transformation,

$$g = 1 + \lambda_a T^a + \mathcal{O}(\lambda^2) \tag{5.12}$$

$$c := \lambda_a T^a, \quad g \approx 1 + c \tag{5.13}$$

$$\delta A \approx (1-c)(A+d)(1+c) - A \tag{5.14}$$

$$= \cancel{A} - cA + Ac - cAc + (1-c)\cancel{d(1)} + dc - cdc - \cancel{A} \tag{5.15}$$

$$= dc + [A, c] + \mathcal{O}(c^2) \tag{5.16}$$

$$\therefore \delta A = D^A c, \quad D^A c := dc + [A, c] \tag{5.17}$$

Note that while A is a 1-form, c is a 0-form, so their commutator is a well-defined 1-form. The Chern-Simons action is invariant under this transformation. To see this, first define the Lie bracket acting on one-forms as a two-form, $\frac{1}{2}[A, A] = A \wedge A$, with both sides of the equation being 0 if the gauge group is abelian. More precisely,

$$A \wedge A = (T^a A^a_\mu dx^\mu) \wedge (T^b A^b_\nu dx^\nu) \tag{5.18}$$

$$= T^a T^b A^a_\mu A^b_\nu \, (dx^\mu \wedge dx^\nu) \tag{5.19}$$

$$= \tfrac{1}{2}[T^a, T^b] A^a_\mu A^b_\nu \, (dx^\mu \wedge dx^\nu) \tag{5.20}$$

$$= \tfrac{1}{2}[A_\mu, A_\nu](dx^\mu \wedge dx^\nu) =: \tfrac{1}{2}[A, A] \tag{5.21}$$

Note an odd property of this bracket is that due to the inherent anticommuting behaviour of one-forms, it is symmetric, $[A, B] = [B, A]$. We still refer to it as the Lie bracket since on 0-forms it has antisymmetric behaviour. Also recall the differential form properties, $\alpha \wedge \beta = (-1)^{m_\alpha m_\beta} \beta \wedge \alpha$ and $d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^{m_\omega} \omega \wedge (d\mu)$. Using these, let us derive the changes in the two terms of the Chern-Simons form.

$$\delta(A \wedge dA) = \delta A \wedge dA + A \wedge d\delta A \tag{5.22}$$

$$d(A \wedge \delta A) = dA \wedge \delta A - A \wedge d\delta A, \tag{5.23}$$

$$dA \wedge \delta A = (-1)^{2*1} \delta A \wedge dA, \tag{5.24}$$

$$\implies \delta(A \wedge dA) = 2\delta A \wedge dA - d(A \wedge \delta A) \tag{5.25}$$

The total derivative will give neglectable surface terms once inside the integral.

And now for the second term,

$$\delta(A \wedge A \wedge A) = \delta A \wedge A \wedge A + A \wedge \delta A \wedge A + A \wedge A \wedge \delta A \tag{5.26}$$

$$\mathrm{Tr}(A \wedge B \wedge A) = \mathrm{Tr}(T^a T^b T^c) A^a_\mu B^b_\nu A^c_\rho \epsilon^{\mu\nu\rho} \tag{5.27}$$

$$= -\,\mathrm{Tr}(T^a T^b T^c) B^b_\nu A^a_\mu A^c_\rho \epsilon^{\nu\mu\rho} \tag{5.28}$$

$$= \mathrm{Tr}(T^a T^b T^c) B^b_\nu A^c_\rho A^a_\mu \epsilon^{\nu\rho\mu} \tag{5.29}$$

$$= \mathrm{Tr}(T^b T^c T^a) B^b_\nu A^c_\rho A^a_\mu \epsilon^{\nu\rho\mu} \tag{5.30}$$

$$= \mathrm{Tr}(B \wedge A \wedge A) \tag{5.31}$$

$$\text{Similarly } \mathrm{Tr}(A \wedge A \wedge B) = \mathrm{Tr}(B \wedge A \wedge A) = f^{abc} B^a_\mu A^b_\nu A^c_\rho \epsilon^{\mu\nu\rho} \tag{5.32}$$

$$\therefore \mathrm{Tr}(\delta(A \wedge A \wedge A)) = 3\,\mathrm{Tr}(\delta A \wedge A \wedge A) \tag{5.33}$$

Now we combine these to obtain:

$$\frac{4\pi}{k}\,\delta cs = 2\int \mathrm{Tr}(\delta A \wedge dA + \delta A \wedge A \wedge A) \tag{5.34}$$

The Euler-Lagrange equations for the Chern-Simons action give the on-shell constraint:

$$\left.\frac{\partial S_{CS}}{\partial A}\right|_{A=A_0} = 0 \tag{5.35}$$

$$\implies F^{A_0} \equiv dA_0 + A_0 \wedge A_0 = dA_0 + \tfrac{1}{2}[A_0, A_0] = 0 \tag{5.36}$$

The classical field configuration that satisfies this on-shell constraint is related to its orbit by the gauge transformations path-connected to the identity, so $\delta A = A_0^g - A_0$. This also means that the variation in the Chern-Simons form under these transformations goes to 0:

$$\frac{4\pi}{k}\,\delta cs = 2\int \mathrm{Tr}(\delta A \wedge F^A) = 2\int \mathrm{Tr}(\delta A \wedge 0) = 0 \tag{5.37}$$

Note that the invariance is only under a subset of the gauge group. Under a general gauge transformation, the Chern-Simons form is not invariant. The change is, however, an integral multiple of 2π (using the normalisation given here). This ensures that the path integral for the partition function or any expectation values is still gauge-invariant.

5.1.3.1 Lagrangian for non-stationary points

Consider a stationary point of the Chern-Simons form, B, so $F^B = 0$ (AKA an on-shell field configuration). We perturb around this point and write the generic connection as $A + B$. Under a gauge transformation, $A \mapsto g^{-1}(A + D^B)g$.

$$\mathcal{L}(A) \equiv cs(A+B) - cs(B) \tag{5.38}$$

$$= \frac{k}{4\pi}\int \mathrm{Tr}((A+B)\wedge d(A+B) + \tfrac{2}{3}(A+B)\wedge(A+B)\wedge(A+B)) - cs(B) \tag{5.39}$$

$$= \cancel{\frac{k}{4\pi}\int \mathrm{Tr}\left(B\wedge dB + \tfrac{2}{3}B\wedge B\wedge B\right) - cs(B)} \tag{5.40}$$

$$+ \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \tfrac{2}{3}(\underbrace{A\wedge A\wedge B + A\wedge B\wedge A + B\wedge A\wedge A}_{\frac{3}{2}A\wedge[B,A]}) + \tfrac{2}{3}A\wedge A\wedge A \right. \tag{5.41}$$

$$\left. + A\wedge dB + B\wedge dA + A\wedge[B,B]\right) \tag{5.42}$$

Now we use $d(A\wedge B) = B\wedge dA - A\wedge dB$ to remove the dA term from the Lagrangian. The total derivative leads to surface terms. Collect the terms linear in A:

$$A\wedge dB + B\wedge dA + A\wedge[B,B] = \cancel{d(A\wedge B)} + 2A\wedge dB + A\wedge[B,B] \tag{5.43}$$

$$= 2A\wedge(dB + B\wedge B) \tag{5.44}$$

$$= 2A\wedge F^B = 0 \tag{5.45}$$

Extend the covariant derivative to act on one forms using the extended definition of the Lie bracket:

$$D^B A = dA + [B,A] \tag{5.46}$$

Thus, our expression reduces to

$$\mathcal{L}(A) = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge D^B A + \tfrac{2}{3} A \wedge A \wedge A) \tag{5.47}$$

5.1.3.2 Gauge Fixing

For convenience, define the operator ad:

$$\mathrm{ad}\, A(c) = [A, c] \tag{5.48}$$

The covariant derivative acting on 0-forms gives a 1-form, and so can be written as a covariant vector:

$$D^B_\mu = \partial_\mu + \mathrm{ad}\, B_\mu \tag{5.49}$$

$$D^{B,\mu} = \sqrt{g} g^{\mu\nu} D^B_\nu \tag{5.50}$$

We use this to define the gauge condition:

$$\frac{k}{4\pi} D^B_\mu A^\mu = 0 \tag{5.51}$$

5.1.3.3 Faddeev-Popov

Now we have a gauge condition in the form of $F[A] = 0$. Define Lie-algebra valued fields $\phi, c, \bar{c}$.

From the gauge-fixing condition as a delta function in the integral, we get an additional term in the Lagrangian,

$$\frac{k}{4\pi} \int_{M^3} \mathrm{Tr}(\phi D^B_\mu A^\mu) \tag{5.52}$$

For the ghosts term, we need to look at the variation of the gauge condition with gauge transformation g. Recall that $\delta(A + B) = D^{A+B} \delta g$, and $\delta B = 0$ since it is a fixed configuration.

Then:

$$J = \frac{\delta F[A]}{\delta g} \tag{5.53}$$

$$= \frac{(\partial_\mu + \mathrm{ad}\, B_\mu)\delta A^\mu}{\delta g} \tag{5.54}$$

$$= D^B_\mu D^{A+B,\mu} \tag{5.55}$$

$$= D^B_\mu (D^{B,\mu} + \mathrm{ad}\, A^\mu) \tag{5.56}$$

So the additional term in the Lagrangian becomes

$$\frac{k}{4\pi} \int_{M^3} \mathrm{Tr}\, \bar{c} D^B_\mu (D^{B,\mu} + \mathrm{ad}\, A^\mu) c \tag{5.57}$$

And our total lagrangian is

$$\mathcal{L}(A, \phi, c, \bar{c}) = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge D^B A + \tfrac{2}{3} A \wedge A \wedge A + \phi D^B_\mu A^\mu + \bar{c} D^B_\mu (D^{B,\mu} + \mathrm{ad}\, A^\mu) c) \tag{5.58}$$

5.2 Feynman Rules

After applying the Faddeev-Popov formalism, we obtain the following Lagrangian, involving auxiliary 'ghost' fields to correct for the overcounting of contributions from physically identical field configurations connected by gauge transformations.

$$\mathcal{L}(A, \phi, c, \bar{c}) = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge D^B A + \tfrac{2}{3} A \wedge A \wedge A + \phi D^B_\mu A^\mu + \bar{c} D^B_\mu (D^{B,\mu} + \mathrm{ad}\, A^\mu) c) \tag{5.59}$$

We would like to evaluate the expectation values of Wilson Loop and Link operators, which take the form: (L is the loop comprised of knots $\{K_i\}$)

$$\langle W[L] \rangle = \int e^{\iota \mathcal{L}(A,\phi,c,\bar{c})} \prod_i e^{\iota\, \mathrm{Tr}_{j_i}\, P \oint_{K_i} dx^\mu A_\mu(x)} \tag{5.60}$$

To do so using perturbative Chern-Simons, we need to figure out the Feynman rules for this theory.

5.2.1 The Diagrams

Propagators

There are two quadratic forms in the lagrangian - one involving the fields A and ϕ, the other involving the fields $c, \bar{c}$. This means two propagators,

1. The bosonic field A represented by undirected dashed lines
2. The fermionic fields $c, \bar{c}$ represented by directed solid lines

Vertices

1. The $A \wedge A \wedge A$ term leads to a 3-point vertex of the gauge field.
2. The $\bar{c} D_{\mu}^{B} \operatorname{ad} A^{\mu} c$ term also gives a 3-point vertex between the boson A, the fermion c and its antiparticle $\bar{c}$.
3. There is also the factor of the Wilson loop operator. Expanding it in orders of A gives us vertices where the field A must end on the path being integrated over, i.e. the knot. We call these $X^2 A$ type vertices, where X represents components of the knot. This is similar to how a source term in the Lagrangian leads to external lines - the knot acts as a source for the bosonic field.

All our diagrams are drawn inside a larger circle or ellipse denoting the knot itself. We denote by the order m the diagrams with m internal loops, and hence $2m$ vertices (also counting the $X^2 A$ type vertices). To avoid clutter, since we only have trivalent vertices in our theory, propagators are allowed to cross each other since there is no 4-point vertex and hence unambiguously no interaction.

5.2.2 The Factors

Since the knot is provided in position space, we restrict ourselves to position-space Feynman rules to make integration easier. First, label the diagram.

1. Every field is Lie algebra valued, so every end of every propagator must have a group index. In addition, propagators of the gauge field also have a spatial index at every end.

2. Every vertex is labelled with a position in M^3. The positions of vertices on the knot will be integrated over the knot, whereas the positions of internal vertices will be integrated over all space.

Knot parametrisation : To integrate over the knot, we need to map the knot to an actual closed path in M^3. The path is known as $X : [0,1] \to M^3$, and points on the knot are labelled with s_i, where $X(s_i)$ is the actual point in space. When integrating over the points on the knot s_i, there cyclic order must be preserved.

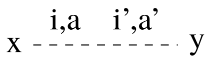


Figure 5.1: Gauge Propagator

Gauge Propagator For a propagator from x to y labelled i, a and i', a' at its ends, we want to derive the factor $V^{aa'}_{ii'}(x,y)$. This should be the inverse of the differential operator A is subject to in the Lagrangian - the inverse of the matrix M, when the Lagrangian is written as $A^T M A$. The differential operator is $t_{ab}\epsilon^{ijk}\partial_j$, since $\mathrm{Tr}(A \wedge D^B A) = t_{ab}\epsilon^{ijk} A^a_i \partial_j A^b_k$.

$$t^{ab}\epsilon^{ijk}\partial_j V^{cb}_{lk}(x,y) = 2\pi\iota\delta^c{}_a\delta^i{}_l\delta(x,y) \tag{5.61}$$

$$V^{ab}_{ij}(x,y) = \langle A^a_i(x) A^b_j(y)\rangle \tag{5.62}$$

We can perform a fourier transform on the equation to obtain the momentum-space propagator, and then use the inverse fourier transform to obtain the position-space propagator.

$$\partial_j \xrightarrow{\mathcal{F}} \iota p_j \tag{5.63}$$

$$\therefore\ \iota t_{ab}\epsilon^{ijk}p_j V_{lk}^{cb}(p) = 2\pi\iota\delta^c_{\ a}\delta^i_{\ l} \tag{5.64}$$

$$\delta^l_{\ i}\delta^i_{\ l} = 3, \quad t^{ad}t_{ab} = \delta^d_{\ b}, \quad t^{ad}\delta^c_{\ a} = t^{cd} \tag{5.65}$$

$$\implies \epsilon^{ijk}p_j V_{ik}^{cd}(p) = 6\pi t^{cd} \tag{5.66}$$

$$\text{By symmetry, each of the 6 contributing (i,j,k) terms equal } \pi t^{cd} \tag{5.67}$$

$$\implies V_{ik}^{ab}(p) = t^{ab}\epsilon_{ijk}\frac{\pi}{p_j} \tag{5.68}$$

$$\xrightarrow{\mathcal{F}^{-1}} V_{ik}^{ab}(r) = t^{ab}\epsilon_{ijk}\pi\int\frac{d^3p}{(2\pi)^3}\frac{e^{\iota\vec{p}\cdot\vec{r}}}{p_j} \tag{5.69}$$

$$= -\iota t^{ab}\epsilon_{ijk}\pi\partial_r^j\int\frac{d^3p}{(2\pi)^3}\frac{e^{\iota\vec{p}\cdot\vec{r}}}{p^2} \tag{5.70}$$

$$= \iota t^{ab}\epsilon_{ikj}\pi\partial_r^j\frac{1}{4\pi r} \tag{5.71}$$

$$= \iota t^{ab}\epsilon_{ikj}\frac{r^j}{4r^3} \tag{5.72}$$

Or, put concisely,

$$V_{ij}^{ab}(x,y) = 2\pi\iota\epsilon_{ijk}\partial_x^k\frac{t^{ab}}{4\pi|x-y|} \tag{5.73}$$

$$= \epsilon_{ijk}t^{ab}\frac{\iota}{2}\frac{(x-y)^k}{|x-y|^3} \tag{5.74}$$

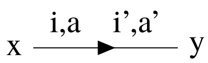


Figure 5.2: Ghost Propagator

Ghost Propagator The propagator is again the inverse of the quadratic term for the ghost fields, $G^{ab}(x,y)$. The differential operator is $t_{ab}D_iD^i$, so

$$t_{ab}D_iD^iG^{bc}(x,y) = -2\pi\delta^c_{\ a}\delta(x-y) \tag{5.75}$$

This is simply the laplacian, and gives the propagator

$$G^{ab}(x,y) = \frac{-\iota}{4\pi}\frac{t^{ab}}{|x-y|} \tag{5.76}$$

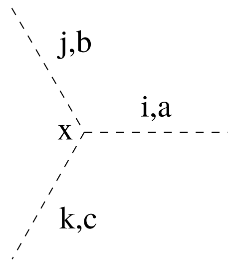


Figure 5.3: A^3 vertex

A^3 **vertex** These involve the normalised structure constants. The structure constants are $f_{ab}^{\ \ c}$, $[T_a, T_b] = f_{ab}^{\ \ c}\, T_c$, then use the normalisation of the bilinear form $t_{ab} = \mathrm{Tr}(T_a T_b)$ to 'lower' the raised index, like so:

$$t_{abc} := f_{ab}^{\ \ d}\, t_{dc} \tag{5.77}$$

An A^3 vertex consists of a position x, 3 spatial indices i, j, k and 3 group indices a, b, c. Then it contributes a factor of:

$$\frac{\iota}{2\pi}\int_{M^3} dx\ t_{abc}\epsilon^{ijk} \tag{5.78}$$

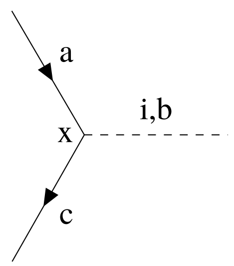


Figure 5.4: $cA\bar{c}$ vertex

$c A \bar{c}$ **vertex** Let the spatial index of the gauge propagator be i, and group indices a, b, c for $c, A, \bar{c}$, and vertex position x. Then using the covariant derivative with respect to x:

$$D_x^i = \sqrt{g} g^{ij} \frac{D}{Dx^j} \tag{5.79}$$

The factor for this vertex thus acts only on the spatial dependence of the gauge propagator, and is

$$\frac{1}{2\pi} \int_{M^3} t^{abc} D_x^i \tag{5.80}$$

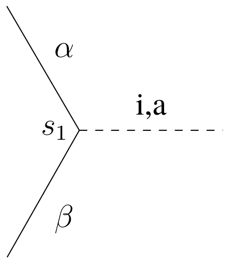


Figure 5.5: $X^2 A$ vertex

$X^2 A$ **vertex** We would have integrated over the position $d\vec{r}(s_i)$ of a point s_i on the loop, but that would also end in a substitution leading to a simpler integral over $\dot{X}(s_1) ds_1$. So we incorporate the $\dot{X}(s_i)$ as a factor in the Feynman diagrams themselves.

Also important is the group factor. Take the representation R to be spanned by $\{r_\alpha\}$, then $R(T^a) r^\alpha := R^\alpha_{a\beta} r^\beta$. The indices α, β could be called representation indices, going from 1 to $\dim R$. Essentially, for each generator T^a of the Lie algebra, we have a matrix $R^\alpha_{a\beta}$, which is the representation of T^a in the representation R.

Every component of the knot is labelled with a representation index at each end, so the $X^2 A$ vertex has two of those (α, β), and the gauge propagator has a spatial and a group index, i, a. Then the factor for this vertex:

$$-\int ds_1 R^\alpha_{a\beta} \dot{X}^i(s_1) \tag{5.81}$$

Additional Factors

1. For $\mathcal{E}$ closed loops of the ghost propagator, multiply a factor of $(-1)^{\mathcal{E}}$.

2. Divide each diagram by the number of symmetries it has. A symmetry is a self-map on the vertices and arcs of the knot which preserves the kinds of vertices, the kinds of propagators between them, and additionally the beginning and end points of arcs. Since the identity is always such a map, the minimum factor is 1.

5.3 1^{st} order Contributions

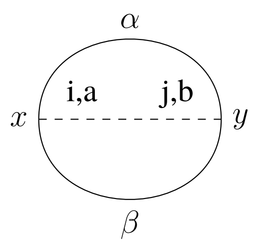


Figure 5.6: The only diagram contributing at 1^{st} order

Let us apply the Feynman rules to the only diagram at 1st order for a single knot. Since the ghosts can only couple to the gauge field, at 1st order with only two vertices available all we can have is a single gauge propagator connecting two points on the knot.

We need the following factors:

- Gauge propagator : $\epsilon_{ijk} t^{ab} \frac{\iota}{2} \frac{(x-y)^k}{|x-y|^3}$, where $x = X(s_1), y = X(s_2)$
- Vertex at x : $-\int ds_1 \; R^{\alpha}_{a\beta} \; \dot{X}^i(s_1)$
- Vertex at y : $-\int ds_2 \; R^{\beta}_{b\alpha} \; \dot{X}^j(s_2)$
- Combinatorial factor : 4 (identity, 180° rotation, two reflections)

Thus the complete integral contributing to the Wilson loop operator at 1^{st} order:

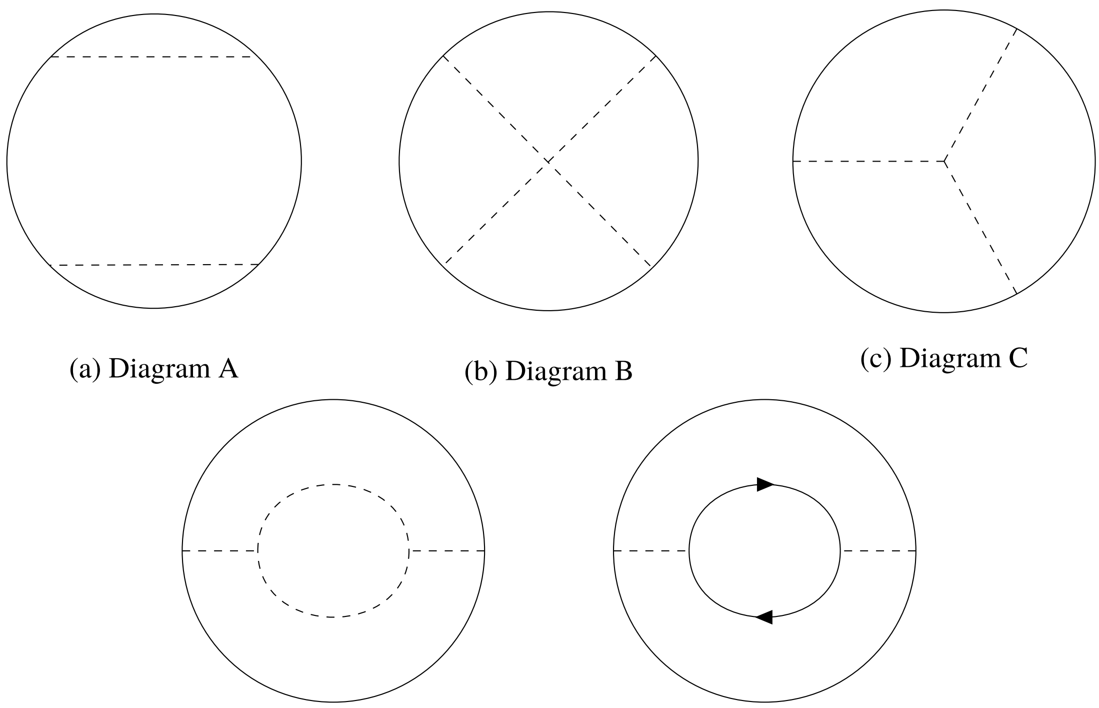
$$I[X] = \frac{\iota}{8} t^{ab} R^{\alpha}_{a\beta} R^{\beta}_{b\alpha} \iint_{s_1 < s_2}^{s_1, s_2 \in [0,1]} ds_1 ds_2 \; \epsilon_{ijk} \dot{X}^i(s_1) \dot{X}^j(s_2) \frac{(X(s_1) - X(s_2))^k}{|X(s_1) - X(s_2)|^3} \tag{5.82}$$

The integral requires the parametrisation X of the knot to be evaluated. Notably, the group factors and the geometrical factors separate. The integrand is undefined at $s_1 = s_2$, so to evaluate over the static limits $(s_1, s_2) \in [0, 1]^2$, we need to frame the knot.

Calugareanu [Cl61], Polykov [Pol88] and others have shown that the integral is not a knot invariant unless the torsion of X is added to it, which is well-defined only for framed knots.

Once framed, the integral simply becomes proportional to the Gauss self-linking number discussed in Section 3.1.9.1.

5.4 2^{nd} order Contributions



(a) Diagram A (b) Diagram B (c) Diagram C

(d) Diagram D (e) Diagram E

Figure 5.7: The diagrams contributing at 2^{nd} order

5 diagrams contribute at 2^{nd} order, including our first fermionic diagram. But we will not need to involve the ghost Feynman rules yet, since diagrams D and E's contributions exactly cancel [GMM89].

Furthermore, diagrams A and B would add to give a term proportional to the square of the self-linking number if it weren't for the group and combinatorial factors, so it is a simple exercise to write the contribution of diagram A as proportional to that of diagram B, plus a term proportional to the self-linking number squared.

Finally, we will only need to evaluate the contributions of diagrams B and C explicitly.

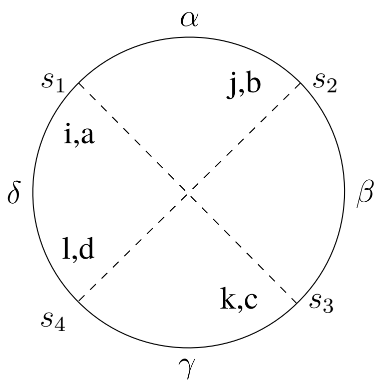
5.4.1 Evaluating the Feynman Diagrams

Diagram B

The diagram has 2 propagators and 4 vertices, but is really quite simple. We only need to be careful about the order of the points on the knot, both for the integral and for the group factors. For conciseness, $X_i := X(s_i)$ and $\dot{X}_i := \dot{X}(s_i)$, $d^4s = \prod_{i=1}^{4} ds_i$, and Δ_4 refers to the area to be integrated over, with the points on the knot in the order $s_1 < s_2 < s_3 < s_4$.

The group factors:

$$t^{ac}t^{bd}R^{\alpha}_{\ a\delta}R^{\beta}_{\ b\alpha}R^{\gamma}_{\ c\beta}R^{\delta}_{\ d\gamma} \tag{5.83}$$



The geometric factors:

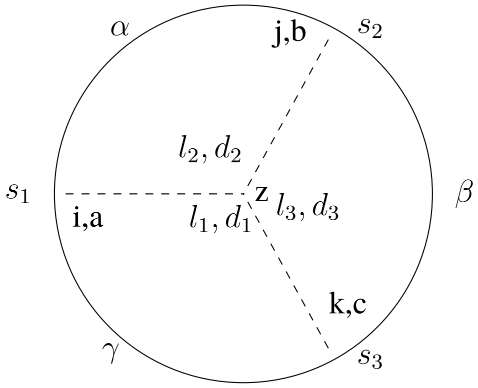
$$\frac{-1}{4}\int_{\Delta_4} ds\ \epsilon_{ikm}\epsilon_{jln}\dot{X}^i_1\dot{X}^j_2\dot{X}^k_3\dot{X}^l_4\frac{(X_1-X_3)^m}{|X_1-X_3|^3}\frac{(X_2-X_4)^n}{|X_2-X_4|^3} \tag{5.84}$$

The combinatorial factor is 4 - the identity, a 90° rotation, and two reflections.

Diagram C

This is our first diagram with an internal point. Keep in mind that the internal point is integrated over all space. While points on the knot have a 1D parametrisation and so give 1D integrals, internal points give 3D integrals. Combined with the unbound limits, this make internal points computationally expensive.

The group factors:



$$t_{d_1 d_2 d_3} t^{ad_1} t^{bd_2} t^{cd_3} R^{\alpha}_{a\gamma} R^{\beta}_{b\alpha} R^{\gamma}_{c\beta} \tag{5.85}$$

$$= f_{d_1 d_2}^{\ \ e}\, t_{ed_3} t^{ad_1} t^{bd_2} t^{cd_3} * R^{\alpha}_{a\gamma} R^{\beta}_{b\alpha} R^{\gamma}_{c\beta} \tag{5.86}$$

$$= f^{abc} R^{\alpha}_{a\gamma} R^{\beta}_{b\alpha} R^{\gamma}_{c\beta} \tag{5.87}$$

$$= f^{abc} * \mathrm{Rep}(T_a T_c T_b) = -f^{abc} * \mathrm{Rep}(T_a T_b T_c) \tag{5.88}$$

The geometric factors:

$$\frac{1}{16\pi} \int_{\Delta_3} ds \int_{M^3} d^3 z\, \epsilon_{il_1 m_1} \epsilon_{jl_2 m_2} \epsilon_{kl_3 m_3} \epsilon^{l_1 l_2 l_3} \dot{X}_1^i \dot{X}_2^j \dot{X}_3^k \frac{(X_1 - z)^{m_1}}{|X_1 - z|^3} \frac{(X_2 - z)^{m_2}}{|X_2 - z|^3} \frac{(X_3 - z)^{m_3}}{|X_3 - z|^3} \tag{5.89}$$

The combinatorial factor is 3 - the identity, a reflection and a 120° rotation.

5.4.2 Lie(algebra)s and Manipulations

Let us justify the claim that the contribution of diagram A is proportional to that of diagram B, plus a term proportional to the self-linking number squared.

The group factors of the two diagrams differ as:

$$R^{\alpha}_{a\delta} R^{\beta}_{b\alpha} R^{\gamma}_{c\beta} R^{\delta}_{d\gamma} (t^{ac} t^{bd} - t^{ad} t^{bc}) \tag{5.90}$$

$$\Longrightarrow\ t^{ac} t^{bd} R^{\alpha}_{a\delta} R^{\beta}_{b\alpha} (R^{\gamma}_{c\beta} R^{\delta}_{d\gamma} - R^{\gamma}_{d\beta} R^{\delta}_{c\gamma}) \tag{5.91}$$

The representation matrices $R^{\alpha}_{a\beta}$ must obey the Lie algebra commutation relations, so

$$t^{ac} t^{bd} (R^{\alpha}_{a\beta} R^{\beta}_{b\gamma} - R^{\alpha}_{a\beta} R^{\beta}_{b\gamma}) = f^{cde} R^{\alpha}_{e\gamma} \tag{5.92}$$

Thus we can obtain a relation between the group factors of diagrams A, B and C. Refer to diagram

A's group factors as $G[A]$ and so on, then

$$G[B] - G[A] = R^{\alpha}_{\ a\delta} R^{\beta}_{\ b\alpha} * t^{db} t^{ca} (R^{\delta}_{\ d\gamma} R^{\gamma}_{\ c\beta} - R^{\delta}_{\ c\gamma} R^{\gamma}_{\ d\beta}) \tag{5.93}$$

$$= R^{\alpha}_{\ a\delta} R^{\beta}_{\ b\alpha} * (f^{bae} R^{\delta}_{\ e\beta}) \qquad = f^{abc} R^{\beta}_{\ a\alpha} R^{\alpha}_{\ b\delta} R^{\delta}_{\ c\beta} = f^{abc} * \mathrm{Rep}(T_a T_b T_c) \quad = - \tag{5.94}$$

Let us refer to the geometrical factors for each diagram as $\mathcal{E}[A]$ and so on. Let us also refer to the complete contribution of the n^{th} order as $\mathcal{W}_n$. Recall that the geometrical factors of integrals A and B sum to give the geometrical factor of the 1st order contribution.

$$\mathcal{E}[A] + \mathcal{E}[B] = \mathcal{E}[1] = W_1^2/(4G[1])^2 \tag{5.95}$$

Where $G[1] = t^{ab} R^{\alpha}_{\ a\beta} R^{\beta}_{\ b\alpha}$ is the group factor of the 1st order contribution.

We would like to write W_2 in terms of W_1^2, $\mathcal{E}[B]$ and $\mathcal{E}[C]$ only. With the intention of removing all dependence on $\mathcal{E}[A]$, define:

$$\hat{W}_2 := W_2 - \frac{2G[A]}{4G[1]} W_1^2 \tag{5.96}$$

Then, to focus only on the geometrical factors, we note that since $G[B] - G[A] = -G[C]$, the group factors coefficiating $\mathcal{E}[B]$ are simply $G[C]$, and so we can write:

$$\tilde{W}_2 := \frac{1}{G[C]} \hat{W}_2 = -\frac{1}{4}\mathcal{E}[B] + \frac{1}{3}\mathcal{E}[C] \tag{5.97}$$

In the case of $G = SU(N)$, the exact factors give us the following relation between $\tilde{\mathcal{W}}_2$ and $\mathcal{W}_2$:

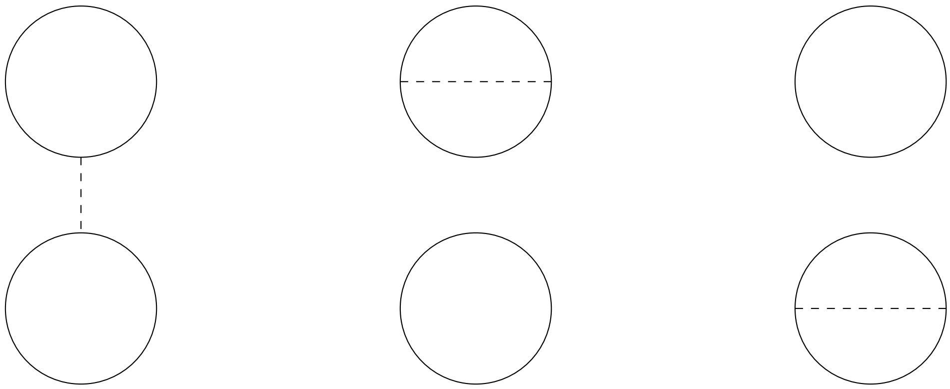
$$\tilde{\mathcal{W}}_2 = \frac{1}{N(N^2-1)} \left(\mathcal{W}_2 - \frac{1}{2N} \mathcal{W}_1^2 \right) \tag{5.98}$$

The combined integral becomes:

$$\tilde{\mathcal{W}}_2 = \frac{1}{16}\int_{\Delta_4} ds\ \epsilon_{ikm}\epsilon_{jln}\dot{X}_1^i\dot{X}_2^j\dot{X}_3^k\dot{X}_4^l\frac{(X_1-X_3)^m}{|X_1-X_3|^3}\frac{(X_2-X_4)^n}{|X_2-X_4|^3}$$
$$-\frac{1}{48\pi}\int_{\Delta_3} ds\int_{M^3} d^3z\ \epsilon_{il_1m_1}\epsilon_{jl_2m_2}\epsilon_{kl_3m_3}\epsilon^{l_1l_2l_3}\dot{X}_1^i\dot{X}_2^j\dot{X}_3^k\frac{(X_1-z)^{m_1}}{|X_1-z|^3}\frac{(X_2-z)^{m_2}}{|X_2-z|^3}\frac{(X_3-z)^{m_3}}{|X_3-z|^3} \tag{5.99}$$

5.5 2 Loop Diagrams

We can also apply these Feynman rules to a link of multiple knots. Let us examine the simplest 2-loop diagrams.



(a) Diagram 2.A (b) Diagram 2.B (c) Diagram 2.C

Figure 5.8: Contributing diagrams for 2 knots at 1st order

Since a loop with no vertices contributes a factor of 1, diagrams 2.B and 2.C only contribute the self-linking numbers of each knot, scaled by the group factors. Diagram 2.A is the only one we need to evaluate explicitly.

In many-loop diagrams, it becomes especially important to keep track of cyclic order when integrating, since the vertices on each knot will be cyclic among themselves. Each knot also has its own parametrisation X_i. Note the difference in notation here - $X(s_i) \neq X_i$ when multiple knots are involved.

In addition, the colors of the knots (the representations of the group on each knot) can be different, and must be kept track of too. Here, we assume both knots are colored identically.

The contribution from 2.A:

$$I[X] = \frac{\iota}{8} t^{ab} R^{\alpha}_{\ a\alpha} R^{\beta}_{\ b\beta} \iint_{s_1, s_2 \in [0,1]} ds_1 ds_2 \; \epsilon_{ijk} \dot{X}_1^i(s_1) \dot{X}_2^j(s_2) \frac{(X_1(s_1) - X_2(s_2))^k}{|X_1(s_1) - X_2(s_2)|^3} \qquad (5.100)$$

This integral has no divergences, and is in fact proportional to the Gauss linking number of the two knots discussed in Section 4.3.

Appendix A

q-Analogues

The q-analog of any mathematical object is a parameter-dependent generalisation which returns the original in some limit, generally $q \to 1$. This has significance to the interface between classical and quantum mechanics, where $\hbar \to 0$ should return known classical results. The following example for the non-negative numbers highlights how q-analogues can be applied to $\hbar \to 0$:

$$[n] = \frac{\sin n\hbar}{\sin \hbar} \tag{A.1}$$

$$= \frac{e^{\iota n\hbar} - e^{-\iota n\hbar}}{e^{\iota\hbar} - e^{-\iota\hbar}} = \frac{1 - e^{-2\iota n\hbar}}{1 - e^{-2\iota\hbar}} \tag{A.2}$$

$$q \equiv e^{-2\iota\hbar} \tag{A.3}$$

$$\implies [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \tag{A.4}$$

$$\lim_{q\to 1}[n]_q = \lim_{\hbar\to 0}[n] = n \tag{A.5}$$

Note that the hyperbolic sin function could be used instead to keep everything real, if preferred. It has no bearing on the further mathematics.

A.1 Q-Factorial and Combinatorics

The q-factorial often appears naturally - for example, when $n!$ counts the permutations, $[n]_q!$ counts the permutations while also keeping track of the number of inversions - it does this by being a polynomial in q.

$$[n]_q! := [1]_q[2]_q \dots [n]_q \tag{A.6}$$

$$= 1 \cdot (1+q) \cdot \dots \cdot (1+q+\dots+q^{n-1}) \tag{A.7}$$

For any permutation w, let inv(w) be the number of inversions, then

$$\sum_{w \in S_n} q^{\text{inv}(w)} = [n]_q!$$

Where S_n is the set of all permutations (of length n). Clearly, derivatives wrt q give us the permutations for any inversion, and $q \to 1$ returns the usual factorial.

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

A.2 Exponentials and other functions

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

And similarly trignometric functions and fourier transforms.

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