Perturbative Chern-Simons

A Schwarz-type topological quantum field theory

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Abstract

Topological Quantum Field Theories are independent of the metric. The Chern-Simons theory is a topological quantum field theory constructed from a gauge field over a three-manifold. It is a useful tool for constructing knot invariants.

While it is exactly solvable, a perturbative treatment is also possible, using the Faddeev-Popov method and Feynman diagrams. This project studied the perturbative expansion of the Chern-Simons theory and the derivation of Vassiliev Invariants.

In addition, I studied methods to improve the efficiency and accuracy of the integral computations of Vassiliev Invariants and their parametrisation dependence.

If the vacuum expectation values of some selected operators and their products remain invariant under changes to the metric, the field theory is considered **topological** and these operators the *observables*.[3]

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \dots \mathcal{O}_{\alpha_n} \rangle = 0 \tag{1}$$

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- TQFTs are metric-independent, hence diffeomorphism invariant.
- Exactly solvable, but can also be studied perturbatively - useful toy models for studying QFTs
- Can create and study toplogical invariants for manifolds

Schwarz Type : The action and observables must be explicitly metric-independent, e.g. Chern-Simons theory

Witten Type : Require a symmetry which transforms some $G_{\mu\nu}$ to $T_{\mu\nu}$, then correlation functions maintain metric independence, eg Donaldson-Witten.

Let a manifold M and the tangent space $T_p(M)$ at any point p on it.

Definition (One-form)

A 1-form is a linear map $\omega : T_p(M) \to F$, where F is the field over which the vector space $T_p(M)$ is defined.

The wedge product $\wedge$ is an antisymmetric bilinear map on 1-forms and is a product that creates higher-dimensional forms.

The Hodge dual $*$ is a map from p-forms to $(n-p)$-forms, where n is the dimension of the manifold.

Definition (Knot)

A smooth non-intersecting closed curve embedded in a 3-manifold. It can be *oriented*.

Links are collections of non-intersecting knots.

Knot diagrams are 2D projections of knots.

Reidemeister moves are local moves on knot diagrams that formalise physical equivalence of knots.

Linking Numbers Count the number of times a knot winds around another.

- Gauss linking number : $\frac{1}{4\pi}\oint_{K_l} dx^\mu \oint_{K_m} dy^\nu \epsilon_{\mu\nu\sigma}\frac{(x-y)^\sigma}{|x-y|^3}$
- Self-linking number : $K_l = K_m$, but the integrand diverges - **framing** is required.

Polynomials The coefficients are knot-invariants, and often related by recursive relations (eg Skein relation).

- Jones
- Alexander
- HOMFLY

Vassiliev Invariants that can be extended to self-intersecting knots of various orders.

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- A simple, compact gauge group G (with corresponding gauge connection A)
- Integer parameter k

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Under a gauge transformation g, the gauge field A is a 1-form belonging to the Lie algebra $\mathfrak{g}$ (adjoint representation) which transforms as

$$A \mapsto A^g = g^{-1}(A + d)g \tag{2}$$

The Action

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \underbrace{\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)}_{CS[A]} \tag{3}$$

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Define 2-form $F = dA + A \wedge A$, then the equation of motion is

$$\frac{\delta S_{CS}}{\delta A} = 0 \tag{4}$$

$$\implies F = 0 \tag{5}$$

Under a gauge transformation,

$$A \mapsto A^g = g^{-1}(A + d)g \tag{6}$$

$$\implies CS[A^g] = CS[A] - CS[g^{-1}dg] - d\,\mathrm{Tr}(g^{-1}Adg) \tag{7}$$

$$S_{CS}[A^g] = S_{CS}[A] - 2\pi k S_{WZ}[g] \tag{8}$$

The Wess-Zumino functional S_{WZ} takes integer values [2], so the partition function is invariant under $A \mapsto A^g$ only if $k \in \mathbb{Z}$.

The trace of a holonomy along a closed path - measures the curvature effects via parallel transport on the path.

$$W_j[K] = \mathrm{Tr}_j \, P \exp\left(\iota n \oint_K dx^\mu A_\mu(x)\right) \tag{9}$$

$$W[L] = \prod_{i=1}^{s} W_j[K_i]^{n_i} \tag{10}$$

In non-abelian theories, $A_\mu := A_\mu^a T_a$, where T_a are the generators of the Lie algebra $\mathfrak{g}$.

This necessitates the path-ordering operator P.

The expectation value of the Wilson operators are knot and link invariants.

$$\langle W[L] \rangle = \frac{1}{Z} \int_{A/G} \mathcal{D}A \; W[L] e^{\iota S_{CS}[A]} \tag{11}$$

Witten derived that for $G = SU(2), j = \frac{1}{2}$, knots (and links) differing by over- and under-crossings have related expectation values.

$$qV_{1/2}[K_+] - q^{-1} V_{1/2}[K_-] = (q^{1/2} - q^{-1/2}) V_{1/2}[K_0] \tag{12}$$

$$\text{Where} \quad q := \exp\left(\frac{2\pi\iota}{k+2}\right) \tag{13}$$

This is the well-known Skein relation for the Jones polynomial!

While exactly solvable, the Chern-Simons theory can also be studied perturbatively. This is done by absorbing the level $\frac{k}{4\pi}$ into the gauge field A, which gives the interaction term $A \wedge A \wedge A$ an effective coupling constant of $\sqrt{\frac{4\pi}{k}}$.

In the perturbative expansion of the Wilson operators, each term is a Vassiliev knot invariant [2][5].

We can write Feynman rules and take the diagrammatic approach to calculating perturbative contributions.

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Faddeev-Popov Procedure

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5. Obtain Feynman rules in the gauge and auxiliary fields.

Dirac delta:

$$\delta^l[F] = \frac{1}{(2\pi)^l} \int \mathcal{D}\phi \; e^{\iota \,\mathrm{Tr} \int_M F[A_\mu]\phi} \tag{14}$$

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Volume term:

$$\Delta_F^{-1}[A_0] := \int \mathcal{D}g \; \delta[F[A_0^g]] \implies \Delta_F[A_0] = \det \left.\frac{\delta F[A_0^g]}{\delta g}\right|_{g=\mathbb{1}} \tag{15}$$

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And as an integral:

$$\det J \propto \int \mathcal{D}c\mathcal{D}\bar{c} \; e^{\iota \bar{c} J c} \tag{16}$$

Using Grassmannian numbers $c, \bar{c}$.

Generalise to Lie-algebra valued fields, so $\bar{c}Jc \mapsto \operatorname{Tr} \int_M \bar{c}Jc$.

Consider a stationary point of the Chern-Simons form, B, so $F^B = 0$. We perturb around this point as $A + B$. Then

$$D^B := d + \operatorname{ad} B = d + 2B\wedge \tag{17}$$

$$A \overset{g}{\mapsto} g(A + D^B)g^{-1} \tag{18}$$

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$$\mathcal{L}[A] \equiv CS[A + B] - CS[B] \tag{19}$$

$$\simeq \mathrm{Tr}\left(A \wedge D^B A + \frac{2}{3} A \wedge A \wedge A\right) \tag{20}$$

Divergence operator:

$$D_\mu^B = \partial_\mu + \mathrm{ad}\, B_\mu \tag{21}$$

$$D^{B,\mu} = \sqrt{g} g^{\mu\nu} D_\nu^B \tag{22}$$

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$$\frac{k}{4\pi}(*D^B *)A = \frac{k}{4\pi} D_\mu^B A^\mu = 0 \tag{23}$$

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Lagrange multiplier term:

$$\frac{k}{4\pi}\int_{M^3} \text{Tr}(\phi D_\mu^B A^\mu) \tag{24}$$

Volume term:

$$J = \frac{\delta F[A]}{\delta g} \tag{25}$$

$$= (\partial_\mu + \text{ad } B_\mu)\frac{\delta A^\mu}{\delta g} \tag{26}$$

$$= D_\mu^B D^{A+B,\mu} \tag{27}$$

$$= D_\mu^B (D^{B,\mu} + \text{ad } A^\mu) \tag{28}$$

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Ghost Lagrangian:

$$\frac{k}{4\pi}\int_{M^3} \operatorname{Tr} \bar{c} D^B_\mu (D^{B,\mu} + \text{ad } A^\mu) c \tag{29}$$

Propagators

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2. The $\bar{c} D_\mu^B \,\mathrm{ad}\, A^\mu c$ term also gives a 3-point vertex between the boson A, the fermion c and its antiparticle $\bar{c}$.
3. The Wilson operators leads to $X^2 A$ type vertices, where X represents components of the knot.

Gauge propagator :

$$V_{ij}^{ab}(x,y) = 2\pi\iota\epsilon_{ijk}\partial_x^k \frac{t^{ab}}{4\pi|x-y|} \tag{30}$$

$$= \epsilon_{ijk} t^{ab} \frac{\iota}{2} \frac{(x-y)^k}{|x-y|^3} \tag{31}$$

i,a i',a'

x - - - - - - - - - - - - y

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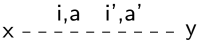
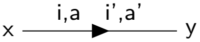


Ghost propagator :

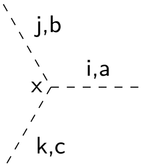
$$G^{ab}(x,y) = \frac{-\iota}{4\pi} \frac{t^{ab}}{|x-y|} \tag{32}$$



A^3 vertex:

$$\frac{\iota}{2\pi}\int_{M^3} dx \ t_{abc}\epsilon^{ijk} \tag{33}$$

$$t_{abc} := f_{ab}{}^{d} \ t_{dc}, \quad t_{ab} := \mathrm{Tr}(T_a T_b) \tag{34}$$



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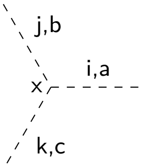
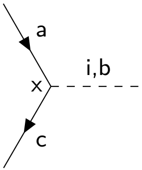


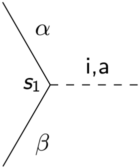
$A\bar{c}c$ vertex:

$$\frac{1}{2\pi}\int_{M^3} t_{abc} D^i_x \tag{35}$$



$X^2 A$ vertex:

$$-\int ds_1 R^{\alpha}_{a\beta}\ \dot{X}^i(s_1) \tag{36}$$



Where the matrix $R^{\alpha}_{a\beta}$ is T^a in the representation R, and $X : [0,1] \to M$ is the knot parametrisation.

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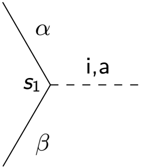

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Combinatorial factors:

1. For $\mathcal{E}$ closed loops of the ghost propagator, multiply a factor of $(-1)^{\mathcal{E}}$.
2. Divide each diagram by the number of symmetries it has.

1st Order Contributions

- Gauge propagator : $\epsilon_{ijk} t^{ab} \frac{\iota}{2} \frac{(x-y)^k}{|x-y|^3}$, where $x = X(s_1), y = X(s_2)$
- Vertex at x : $-\int ds_1 \; R^{\alpha}_{\;a\beta} \; \dot{X}^i(s_1)$
- Vertex at y : $-\int ds_2 \; R^{\beta}_{\;b\alpha} \; \dot{X}^j(s_2)$
- Combinatorial factor : 4 (identity, 180° rotation, two reflections)

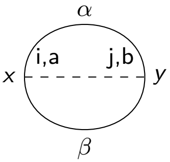

Figure: The only diagram contributing at 1st order

$$I[X] = \frac{\iota}{8} t^{ab} R^{\alpha}_{\;a\beta} R^{\beta}_{\;b\alpha} \iint_{s_1 < s_2}^{s_1, s_2 \in [0,1]} ds_1 ds_2 \; \epsilon_{ijk} \dot{X}^i(s_1) \dot{X}^j(s_2) \frac{(X(s_1) - X(s_2))^k}{|X(s_1) - X(s_2)|^3} \quad (37)$$

Once framed, the integral simply becomes proportional to the Gauss self-linking number.

Calugareanu [1], Polykov [6] and others have shown that the integral is not a knot invariant unless the torsion of X is added to it, which is well-defined only for framed knots.

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- Knot parametrisations for String theory
- Applications of Chern-Simons to Topological Quantum Computing
- Further exploration of Vassiliev invariants