Perturbative Chern Simons

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by

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Chapter 1

Theoretical Background

1.1 Topological Quantum Field Theories

Referring to

- Undergraduate Lecture Notes in Topological Quantum Field Theory by Vladimir G. Ivancevic, Tijana T. Ivancevic (Ivancevic and Ivancevic, 2008)
- Topological Quantum Field Theories - A Meeting Ground for Physicists and Mathematicians by Romesh Kaul (Kaul, 1999)
- Introduction To Chern-Simons Theories by Gregory W. Moore (Moore, 2019)
- Quantum Field Theory on the Plane by David Tong (Tong, 2006)

1.1.1 Introduction

What are TQFTs?

> Toplogical quantum field theories are independent of the metric of curved manifold on which these are defined; the expectation value of the energy-momentum tensor is zero, $\langle T_{\mu\nu} \rangle = 0$. *These possess no local propagating degrees of freedom; only degrees of freedom are topological.* Operators of interest in such a theory are also metric independent.

They have the interesting feature of being exactly solvable, no perturbation needed.

1.1.1.1 Motivation from Knot Theory (1.4)

Knot theory is concerned with the topological equivalence of knots and links. TQFTs arise from considering theories which provide such a description of the knots and links embedded in that space - so we want the observables over the knots to be metric independent.

1.1.2 Path Integral TQFT

Considering fields ϕ_i on a manifold M with metric $g_{\mu\nu}$, with an action $S[\phi_i]$ and some operators $\mathscr{O}_\alpha$ defined, we have vacuum expectation values defined as

$$\langle \mathscr{O}_\alpha \rangle = \int D[\phi_i] \mathscr{O}_\alpha(\phi_i) e^{\imath S[\phi_i]}$$

If the vacuum expectation values of some selected operators and their products remain invariant under changes to the metric, the field theory is considered **topological** and these operators the *observables*.

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathscr{O}_{\alpha_1} \ldots \mathscr{O}_{\alpha_n} \rangle = 0$$

Schwarz-type TQFTs guarantee this formally by requiring $S, \mathscr{O}_\alpha$ to be metric-independent.

An example, which I will cover in detail along with its relation to knot theory, is the Chern-Simons theory.

1.1.2.1 Chern-Simons Gauge Theory

Composed of

- A differentiable, compact 3-manifold M
- A simple, compact gauge group G (with corresponding gauge connection A)
- Integer parameter k (required to be integral for gauge invariance)

Then we have a Chern-Simons form, which integrates to give the action:

$$S_{CS}[A] = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

1.1.2.2 Witten-type TQFTs

The second way to ensure metric invariance of the action and observables, and also called cohomological of Witten-type. We require a symmetry with the infinitesimal transformation δ':

$$\delta' \mathscr{O}_\alpha(\phi_i) = 0 \tag{1.1}$$

$$T_{\mu\nu}(\phi_i) = \delta' G_{\mu\nu}(\phi_i) \tag{1.2}$$

$$\text{Where } T_{\mu\nu}(\phi_i) \equiv \frac{\delta}{\delta g_{\mu\nu}} S[\phi_i] \tag{1.3}$$

Note here $G_{\mu\nu}$ is some arbitrary tensor. Since δ' is a symmetry, $\delta' S = 0$ and $\delta' \mathscr{O}_\alpha(\phi_i) = 0$ under transformations $\delta' \phi_i$. Looking again at the vacuum expectation values,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathscr{O}_{\alpha_1} \dots \mathscr{O}_{\alpha_n} \rangle = i \int D[\phi_i] \mathscr{O}_{\alpha_1}(\phi_i) \dots \mathscr{O}_{\alpha_n}(\phi_i) e^{\imath S[\phi_i]} T_{\mu\nu}(\phi_i) \tag{1.4}$$

$$= \delta' \left(i \int D[\phi_i] \mathscr{O}_{\alpha_1}(\phi_i) \dots \mathscr{O}_{\alpha_n}(\phi_i) G_{\mu\nu}(\phi_i) e^{\imath S[\phi_i]} \right) \tag{1.5}$$

$$= 0 \tag{1.6}$$

Since an expectation value will not change under symmetry transformations, the final equality holds. We have also implicitly assumed the measure is invariant under the transformation. Note that the operators have been assumed metric-independent in the above proof, but it can be extended more generally to $\frac{\delta}{\delta g_{\mu\nu}}\mathscr{O}_\alpha(\phi_i) = \delta' O_\alpha^{\mu\nu}(\phi_i)$, where we have defined additional arbitrary tensor functionals.

δ' must also be a scalar symmetry, since it is a global symmetry and so has a constant parameter corresponding - if that were not a scalar, it would be a pretty harsh constraint to be satisfied on arbitrary manifolds.

Often, cohomological TQFTs satisfy $S = \delta'\Lambda$, which allows showing that any combination of observables is independent of the coupling constant, appearing in the theory as $\exp\left(\imath\frac{1}{g^2}S\right)$. A proof to first-order is given in the reference, simply take $1/g^2 \to 1/g^2 - \Delta$, assume the observables' form doesn't depend on the coupling, and a similar proof to the one for VEVs follows.

1.2 Differential Forms

Let a manifold M and the tangent space $T_p(M)$ at any point p on it.

1.2.1 One-forms

Definition 1.1. One-form
A 1-form is a linear map $\omega : T_p(M) \to F$, where F is the field over which the vector space $T_p(M)$ is defined.

Thus, a one form is an element of the dual vector space $(T_p(M))^*$, and can be written (for $M = \mathbb{R}^2$)as $w(\langle dx, dy\rangle) = adx + bdy$ (since it is linear). Note that this is exactly the differential element for the line integral over a vector field $(a,b)(x,y)$. The action of a 1-form is to scale the projection of its input on a particular line:

$$w(\langle x,y\rangle) = ax + by \tag{1.7}$$
$$= \langle a,b\rangle \cdot \langle x,y\rangle \tag{1.8}$$
$$= \|\langle a,b\rangle\| \cdot \text{Projection}_{\langle a,b\rangle}(\langle x,y\rangle) \tag{1.9}$$

1.2.2 The Wedge product and m-forms

We want to define a product $\wedge$ of one-forms which is a linear function and has some meaningful geometrical representation. This is not the most precise motivation for 2- or n-forms, but we'll build up to the more precise ones later - they have to do with antisymmetry and generalising the Stokes theorem. $w_1 \wedge w_2 : T_pM \times T_pM \to F$ is the map we're interested in - note that it takes two vectors as inputs. So we have the 4 components $w_i(v_j)$ to play with, and (going forward I refer to the one-form w_1 and the dual vector it projects vectors on, $\langle a,b\rangle$, interchangeably) these components are essentially the projections of v_1, v_2 onto w_1, w_2 - we can visualise these as two vectors, the projections of v_1, v_2 onto the w_1-w_2 plane. With two vectors in a plane, what operation could convert them to a scalar? Well, a dot product, but that does not encode the significance of this being a product - it still has length dimensions, in some sense, and has no antisymmetry. As we shall see later, we prefer something antisymmetric, something that does justice to its definition as a product - an area, $|v_1^{w_1\text{-}w_2} \times v_2^{w_1\text{-}w_2}|$:

$$w_1 \wedge w_2(v_1, v_2) := \det \begin{pmatrix} w_1(v_1) & w_2(v_1) \\ w_1(v_2) & w_2(v_2) \end{pmatrix}$$

Definition 1.2. 2-forms

The wedge product of two 1-forms - a bilinear, alternating map $w^{(2)} : T_pM \times T_pM \to F$.

1.2.2.1 Properties of the Wedge Product

- From the determinant structure, the two-form is **antisymmetric in inputs**: $w_1 \wedge w_2(v_1, v_2) = -w_1 \wedge w_2(v_2, v_1)$.
- Also from the determinant, the wedge product is **anti-commuting**:$w_1 \wedge w_2 = -w_2 \wedge w_1$.

Hence the wedge of a 1-form with itself is 0.

Lemma 1.1. The wedge product is distributive over addition

Proof:

$$w_1 \wedge (w_2 + w_3)[v_1, v_2] = w_1(v_1)(w_2 + w_3)(v_2) - w_1(v_2)(w_2 + w_3)(v_1) \tag{1.10}$$

$$= w_1 \wedge w_2[v_1, v_2] + w_1 \wedge w_3[v_1, v_2] \tag{1.11}$$

Because the determinant is distributive over addition in a single row/column.

Which leads to the following:

Corollary 1.1. On $\mathbb{R}^2$, $w_1 \wedge w_2 = Cdx \wedge d$ for $C \in F$

Proof:

$$w_1 \wedge w_2 = (Adx + Bdy) \wedge (Cdx + Ddy) \tag{1.12}$$

$$= AC\cancel{dx \wedge dx} + ADdx \wedge dy + BCdy \wedge dx + BD\cancel{dy \wedge dy} \tag{1.13}$$

$$= (AD - BC)dx \wedge dy \tag{1.14}$$

One can easily see that $dx \wedge dy$ gives the (signed) area between its two input vectors, and so any two-form simply scales this area by some constant.

Definition 1.3. m-forms

A multilinear and alternating $w : (T_pM)^m \to F$.

Multilinear - Linear in every argument

Alternating - Antisymmetric in any pair of arguments

One way to obtain an m-form is to construct it out of 1-forms and wedge products:

$$w(v_1, \dots, v_2) = w_1 \wedge \dots \wedge w_2(v_1, \dots, v_2) \tag{1.15}$$

$$= \det(w_i(v_j)) \tag{1.16}$$

For coordinates x^i on M, any m-form on T_pM is a linear combination of $dx^{i_1} \wedge \dots \wedge dx^{i_m}$.

Thus for an n-dim manifold, only $m \leq n$ forms are non-trivial.

Lemma 1.2. The wedge product is associative

Lemma 1.3. $\alpha \wedge \beta = (-1)^{m_\alpha m_\beta} \beta \wedge \alpha$

The proof is trivial - consider moving each component of β through all the components of α, each step is an anti-commutation.

Note how this means $\alpha \wedge \alpha = 0$ when m_α is odd, but not necessarily when it is even.

Lemma 1.4. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

We already saw that the wedge is distributive over addition of one-forms. Does this extend to m-forms?

First, consider the addition of m-forms. What does $\beta + \gamma$ mean?

$$(\beta + \gamma)(v_i) := \beta(v_i) + \gamma(v_i) \tag{1.17}$$

$$\alpha \wedge (\beta + \gamma) = \sum_i \sum_j a_i (b_j + c_j)(dx^{i_1} \wedge \cdots \wedge dx^{i_n}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_m}) \tag{1.18}$$

Clearly, assuming m-forms are spanned by $(dx^{j_1} \wedge \cdots \wedge dx^{j_m})$ makes the proof trivial. Can we prove this without the expansion? I'm not sure, but the proof would certainly hold even with the weaker assumption that the m-form space is spanned by the space of all wedge products of m 1-forms.

To work without the expansion, we need to define the wedge product on two m-forms more rigorously. Considering the definition that we want an $m_1 + m_2$ form, we can restrict the wedge product to be $\alpha \wedge \beta(\{v_i\}, \{w_j\}) = \det \begin{pmatrix} \alpha(\{v_i\}) & \beta(\{v_i\}) \\ \alpha(\{w_i\}) & \beta(\{w_i\}) \end{pmatrix}$. Then the proof is trivial - it is identical to that for 1-forms.

Remark

$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ for $I = \{i_j\}_{j=1}^m$. The space of all m-forms on a manifold, called $\Lambda^m M$, has a basis given by $\{dx_I\}$. -

$$dx_{\{i_k\}}(v^{(j)}) = \det(v_{i_k}^{(j)})_{1 \leq j,k \leq m}$$

- It's not hard to see that

$$\dim \Lambda^m M^n = \binom{n}{m}$$

Simply by considering that the basis is dx_I and I cannot have repeated indices.

1.2.3 Differential m-forms

These are m-forms but with differentiable functions as coefficients:

$$\omega := \sum_I f_I dx_I, \tag{1.19}$$

$$f_I : M \to F \forall I \tag{1.20}$$

$$\omega_p := \sum_I f_I(p) dx_I \tag{1.21}$$

So a differential m-form is really a map from M to $\Lambda_p^m M$ - a smooth tensor field over the manifold, mapping to the cotangent bundle. Alternatively, it could also map m vector fields to a scalar function on the manifold (tensor field innit).

1.2.4 Integrating m-forms

1.2.4.1 Integrating 2-forms

Take a surface S embedded in M^n parametrised by $\phi : D \to M^n$, $D \subset \mathbb{R}^2$ and a differential 2-form ω. Consider the Reimann integral:

$$\iint_D f(x,y) dA = \lim_{\delta_x \to 0, \delta_y \to 0} \sum_{ij} f(x_i, y_j) \delta_x \delta_y$$

We've defined the integral by discretising space. Now we can take a point p in D and two points away from it, defining two vectors in $\mathbb{R}^2$, and map these to three points and two vectors in M^n. As the other points approach p in D, the vectors in M^n become tangent vectors at $\phi(p)$. We use this in the discrete space by taking $p = (u_i, v_j)$ and the other two points $(u_{i+1}, v_j), (u_i, v_{j+1})$.

Then

$$\int_S \omega := \lim_{\delta_u,\delta_v\to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)}(\phi(u_{i+1},v_j)-\phi(u_i,v_j),\phi(u_i,v_{j+1})-\phi(u_i,v_j)) \tag{1.22}$$

$$= \lim_{\delta_u,\delta_v\to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)} \left(\frac{\phi(u_{i+1},v_j)-\phi(u_i,v_j)}{\delta_u}, \frac{\phi(u_i,v_{j+1})-\phi(u_i,v_j)}{\delta_v}\right) \delta_u \delta_v \tag{1.23}$$

$$= \iint_D \omega_{\phi(u,v)} (\partial_u \phi, \partial_v \phi)\, dA \tag{1.24}$$

1.2.4.2 Generalising to m-forms

- $\omega = \sum_I f_I dx_I$, $I \in \{1,\dots,n\}^m$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ for $I = \{i_j\}_{j=1}^m$ - $S \subseteq M^n$ parametrised by $\phi : D \to M^n$, $D \subseteq \mathbb{R}^m$.

$$\int_S \omega := \int \cdots \int_D \omega_{\phi(u_1,\dots,u_m)} (\partial_1 \phi, \dots, \partial_m \phi) \underbrace{du_1 \dots du_m}_{dV_m}$$

Note that $\partial_i \phi$ is an n-dim vector, and ω will act on it. The dx_I components of ω will extract the Ith components of each input vector, and take the determinant of all the components, m from each of the m vectors. The rest is merely multivariate integration.

1.3 Chern Simons Theory

1.3.1 Introduction

Composed of

- A differentiable, compact 3-manifold (or else odd-manifold) M
- A simple, compact gauge group G (with corresponding gauge connection A)
- Integer parameter k

Under a gauge transformation g, the gauge field / connection A is a 1-form belonging to the Lie algebra $\mathfrak{g}$ which transforms as

$$A \mapsto A^g = g^{-1}(A+d)g$$

Then the 2-form $F := dA + A^2 \mapsto g^{-1}Fg$, and a conjugation-invariant quadratic on the lie algebra $p(F)$ becomes a 4-form, which is generally also a total derivative of a 3-form - we'll call this the generalised Chern-Simons form, $CS_p(A)$. Specifically, the usual Chern-Simons form is obtained from the simple example of

$$p(F) = \mathrm{Tr}\, F^2 \tag{1.25}$$

$$\mathrm{Tr}\, F^2 = dCS(A) \tag{1.26}$$

$$CS(A) = \mathrm{Tr}\left(AdA + \tfrac{2}{3}A^3\right) \tag{1.27}$$

Which integrates to give the action:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{1.28}$$

$$Z = \int_{A/G} e^{\imath \int_M CS(A)} \tag{1.29}$$

The Partition function is obtained by integrating over A/G, the space of inequivalent gauge fields. Under a gauge transformation,

$$CS(A^g) = CS(A) - CS(g^{-1}dg) - d(\mathrm{Tr}(g^{-1}Adg))$$

When p is "quantised" - for the trace, this refers to normalisation on a simple Lie algebra - and the manifold has no boundary, $CS(g^{-1}dg)$ is $2\pi k, k \in \mathbb{Z}$, so that the partition function is invariant. Often, $k \in \mathbb{Z}$ is instead written as a quantisation condition arising from the gauge invariance requirement.

Deriving the EoMs:

$$\delta CS(A) = 2\,\mathrm{Tr}(\delta A F) - d(\mathrm{Tr} A\delta A) \tag{1.30}$$

$$\implies F = 0 \tag{1.31}$$

(0 field strength, flat gauge fields.)

1.3.1.1 Abelian version

Another version of this action (in the abelian version, with some normalisation) is

$$S[A_\mu] = -\frac{k}{8\pi}\int_{S^3} d^3x\; \varepsilon^{\mu\nu\sigma} A_\mu(x)\partial_\nu A_\sigma(x)$$

Recall vacuum expectation values from the path integral formulation:

$$\langle W\rangle = \frac{1}{Z}\int [dA] E e^{ikS} \tag{1.32}$$

$$Z = \int [dA] e^{ikS} \tag{1.33}$$

Are these metric-invariant now? They seem to be, but the gauge-fixing of A and the regularisation of the theory (mesh in spacetime - UV cutoff) are both metric dependent. Refer to(Kaul and Rajaraman, 1990) for how these dependences cancel out in some sense - our averages are still metric-invariant.

1.3.1.2 Motivation - Particle on a Ring

To skip motivation, go straight to Wilson Link Operators (Section 1.3.1.9). A simple example where the Chern-Simons action naturally arises is that of a charged particle on a ring through which a solenoid passes. It may seem slightly contrived, but it illustrates some important effects in QCD and QED, and also topological insulators. Consider a particle of mass m, charge q, on a ring of radius r, at angular position ϕ - then $K = \frac{1}{2}(mr^2)\dot{\phi}^2$. Also introduce a solenoid with magnetic field B. There is no magnetic field at the particle location, but there is a vector potential along the ring, since its curl must give the field inside the solenoid. Written as a differential form, $A = \frac{B}{2\pi} d\phi$ - essentially $A_\phi = \frac{B}{2\pi}$ and $A_z = A_r = 0$. The action becomes

$$S = \int K dt + \oint eA \tag{1.34}$$

$$= \int \left(\tfrac{1}{2} I \dot{\phi}^2 + \tfrac{eB}{2\pi} \dot{\phi}\right) dt \tag{1.35}$$

The second term is a *topological/θ term* - it is a total derivative, so classically has no impact, but quantum mechanically, it does.

1.3.1.3 Symmetries (Classical)

Rotations about the ring axis - an $O(2)$ symmetry, of which $SO(2)$ is encoded as $R(\alpha) : e^{i\phi} \mapsto e^{i\alpha} e^{i\phi}$, or the translation $\phi \mapsto \alpha + \phi$, or acting on a cartesian vector.

Parity $P : \phi \mapsto -\phi$. This does change the second term, but by a total derivative, so it has no effect. This is also part of $O(2)$.

The group elements combine as

$$R(\alpha)R(\beta) = R(\alpha + \beta) \tag{1.36}$$

$$P^2 = 1 \tag{1.37}$$

$$PR(\alpha)P = R(-\alpha) \tag{1.38}$$

And thus, with the action of $\langle P \rangle \cong \mathbb{Z}_2$ on $SO(2)$ defined, $O(2) = SO(2) \rtimes \mathbb{Z}_2$.

1.3.1.4 Field Theoretic approach

As an aside, note that the system can be treated field theoretically, with a field $e^{\imath\phi}$ mapping from the manifold $\mathbb{R}$ (encoding time) to the circle S^1. This is a 0+1 dimensional field theory. The parity operation becomes more like a charge conjugation, since it conjugates the $U(1)$-valued field. In addition the solutions have time translation and reversal symmetries (note though that the second term does not have time reversal symmetry), forming the group $\mathbb{R} \rtimes \mathbb{Z}_2$.

1.3.1.5 Symmetries (Quantum)

Conjugate (angular) momentum $L = I\dot{\phi} + \frac{eB}{2\pi}$. Legendre transform for the hamiltonian, and quantise $L \to -\imath\hbar\partial_\phi$:

$$H = L\dot{\phi} - \left(\tfrac{1}{2}I\dot{\phi}^2 + \tfrac{eB}{2\pi}\dot{\phi}\right) = \tfrac{1}{2}I\dot{\phi}^2 \tag{1.39}$$

$$\hat{H} = \frac{1}{2I}\left(\hat{L} - \frac{eB}{2\pi}\right)^2 \tag{1.40}$$

$$= \frac{\hbar^2}{2I}\left(-\imath\partial_\phi - \mathscr{B}\right)^2 \tag{1.41}$$

$$\text{Where } \mathscr{B} := \frac{eB}{2\pi\hbar} \tag{1.42}$$

This hamiltonian operator has eigenfunctions and eigenvalues $\psi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{\imath m\phi}, E_m = \frac{\hbar^2}{2I}(m - \mathscr{B})^2$. Check that out, an observable energy shift! And it's not a constant shift either, there's a $-2m\mathscr{B}$ term which is clearly measurable.

$\mathscr{B}$ also controls degeneracy - when integral, all but the ground state have two-fold degeneracy, and when half-integral, the ground state does too, and else none do. The *spectrum is periodic in* $\mathscr{B}$.

$$\text{Define unitary } U\psi_m := \psi_{m+1} \tag{1.43}$$

$$\text{Then } UH_{\mathscr{B}}U^{-1} = H_{\mathscr{B}+1} \tag{1.44}$$

1.3.1.6 Analogues in Higher Dimensional Field theories

1+1 dim Maxwell Theory

$$S = \frac{1}{e^2}\int F * F + \int \frac{\theta}{2\pi}F$$

The θ term is a coupling to the B field. Apply the Kaluza-Klein reduction, gauge choice $A_0 = 0$, then the only gauge invariant $e^{\imath\phi(t)} = e^{\imath\oint_{S^1}A}$ becomes $e^{\imath\oint_{S^1}A_1 dx^1}$. This can be done more generally in a 1+1 dimensional Yang-Mills theory too.

3+1 dim Maxwell $\theta = 2\pi\mathscr{B}$, and its value encodes the behaviour of, say, an effective electromagnetic theory in the presence of an insulator - the action is like

$$S = \int d^4x \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \int \frac{\theta}{(2\pi)^2} F \wedge F$$

If parity/time-reversal symmetry is present, then θ must be 0 or π, and $\theta = 0$ encodes normal insulators, whereas $\theta = \pi$ encodes topological insulators - $\mathscr{B}$ is half-integral, and the ground state is degenerate.

1.3.1.7 Back to Symmetries

When $\mathscr{B}$ is integral, the $O(2)$ symmetry holds, but if $\mathscr{B}$ is half-integral the group $\mathrm{Pin}^+(2)$, a cover of $O(2)$, is a symmetry. The difference is the product rules of $O(2)$ elements no longer hold for the cover. And if $\mathscr{B}$ is neither, then only $SO(2)$ remains.

If we defined an angular momentum $\mathscr{L}^2/2I = E$, then for half-integral $\mathscr{B}$, this would take half-integral eigenvalues, as if the particle was spin-1/2. This is a more general phenomenon with Chern-Simons terms - the spins and statistics of particles can be shifted from classical values.

1.3.1.8 Symmetries of Chern-Simons

The presence of $\varepsilon^{\mu\nu\rho}$ in the action, while ensuring Lorentz invariance, breaks both parity and time-reversal symmetry. Note, in odd dimensions - as Chern Simons is restricted to - parity is defined as a single coordinate's sign flip ($\vec{x} \mapsto -\vec{x}$ is merely a rotation in odd dim).

1.3.1.9 Wilson Link Operators

For link L made of knots $\{K_i\}_{i=1}^{s}$, the knot and link operators (n_i are integers denoting the charge on each loop):(Ivancevic and Ivancevic, 2008)

$$W[K_i] = \exp\left(\imath n_i \oint_{K_i} dx^\mu A_\mu(x)\right) \tag{1.45}$$

$$W[L] = \prod_{i=1}^{s} W[K_i] \tag{1.46}$$

Since this is a non-interacting theory, only the 2-point correlators $\langle A_\mu A_\nu \rangle$ will feature in any expectation value, through the following 2-loop expectation values:

$$\left\langle \oint_{K_l} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(Y) \right\rangle \tag{1.47}$$

1.3.1.9.1 Evaluating the 2-point correlators Since we are working in a TQFT with mouldable knots, let us squish the link into a small region which we locally identify with $\mathbb{R}^3$, and hence solve for flat spacetime. Then $g_{\mu\nu} = \delta_{\mu\nu}$, and working in the Lorenz gauge $\delta^{\mu\nu}\partial_\mu A_\nu = 0$,

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{\imath}{k} \varepsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}$$

Which is simply the propagator in position space, easily read off the action. Then the 2-loop expectation values:

$$\left\langle \oint_{K_l} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(Y) \right\rangle = \frac{4\pi\imath}{k} \underbrace{\frac{1}{4\pi} \oint_{K_l} dx^\mu \oint_{K_m} dy^\nu \varepsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}}_{\mathscr{L}(K_l, K_m)} \tag{1.48}$$

$\mathscr{L}$ is the *Gauss linking number* for two distinct knots - it measures how many times one knot passes through the other, and hence should be an integer. It measures orientation, so a right-handed link would have +1, and a left-handed -1 (see Hopf links). It can be given a physical interpretation too - it's the work done moving a magnetic monopole along one knot while a current runs in the other.

When $K_l = K_m$, it instead measures the *self-linking/framing number*, and is evaluated by creating a loop $K_f(\varepsilon)$ displaced along the normal such that $K_f(0) = K$ (so K_f has coordinates

$y^\nu = x^\nu(s) + \varepsilon n^\nu(s)$, $n^\nu(s)$ is the principle normal (unit vector field) to the curve), integrate over K, K_f and limit $\varepsilon \to 0$, so $\lim_{\varepsilon\to 0} \mathscr{L}(K, K_f(\varepsilon)) = \mathscr{S}\mathscr{L}(K)$. K_f is known as K's frame.

Refer to the Calugareanu theorem (Section 1.4.3.1) for another calculation of $\mathscr{S}\mathscr{L}$.

Framing is intrinsically connected to regularisation in field theories, avoiding the singularity in the two-point correlator (at $x \to y$) by point-splitting - this same divergence arises in the two-loop correlator seen above, which framing fixes - it provides a *toplogical regularisation*. It may seem like the self-linking depends on the frame, but the topological class of frames have a constant self-linking number.

$$\langle W[L] \rangle = \exp\left(-\frac{2\pi\iota}{k} \left[\sum_i^s n_i^2 \mathscr{S}\mathscr{L}(K_i) + \sum_m^s \sum_{i\neq m}^s n_i n_m \mathscr{L}(K_i, K_m) \right] \right)$$

Thus topological invariants are connected with expectation values of certain field theory operators.

1.3.1.10 Wilson loops

An important invariant in this theory is the *Wilson loop*, the trace (in some rep R) of the holonomy of A along a path (1-cycle) γ:

$$\mathrm{Tr}_R(\mathrm{Hol}_\gamma A) = \mathrm{Tr}_R\left(Pe^{\int_\gamma A}\right)$$

These Wilson loops are labelled by representations R and the 1-cycles γ, embeddings of S_1 in M. It can be shown that these are knot and link invariants (Section 1.4.3).

Holonomies

A holonomy(Ivancevic and Ivancevic, 2008) on a smooth manifold is a general geometrical consequence of the curvature of the manifold connection, measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported. Related to holonomy is a Wilson loop, which is a gauge-invariant observable obtained from the holonomy of the gauge connection around a given loop.

1.3.1.11 $U(1)$ Chern-Simons

Consider the 2+1d spacetime $S^1 \times S^2$ and the gauge group $U(1)$, so the gauge transformation is $A \mapsto A + d\omega$. If the gauge group is **compact** $U(1)$(Tong, 2006), the magnetic flux is quantised to units of $\frac{1}{2\pi}\int_{S^2} F_{12} = 1$. Parametrise S^1 with $x^0 \in [0, 2\pi R)$, R being radius. We want to consider (large) gauge transformations which wind around S^1 and see what phase factors they pick up.

$$\omega = \frac{x^0}{R} \implies A_0 \mapsto A_0 + \frac{1}{R} \tag{1.49}$$

$$\text{And for matter fields of charge } q, \phi \mapsto e^{\imath q\tau/R}\phi \tag{1.50}$$

In the presence of unit magnetic flux and with $A_0 = a$, the action evaluates to (being careful about the topology of the spacetime)

$$S_{CS} = \frac{k}{4\pi}\int d^3x A_{(0}F_{12)} \tag{1.51}$$

$$= \frac{k}{2\pi}\int d^3x A_0 F_{12} = 2\pi kRa \tag{1.52}$$

Which under gauge transformations $S_{CS} \mapsto S_{CS} + 2\pi k$. Thus the action isn't gauge invariant, but for integral k the partition function is. Note the factor of two, arising from integrating by parts the A_1, A_2 terms before to obtain another $A_0 F_{12}$ term before setting $\partial_i A_0 = 0$, because our spacetime is topologically non-trivial.

1.3.1.12 Chern-Simons as a theory on a Boundary

1.3.2 Non-Abelian Chern-Simons Theory

Writing the Chern-Simons form a little more explicitly,

$$S = \frac{k}{4\pi}\int_{S^3} d^3x\, \varepsilon^{\mu\nu\sigma}\,\mathrm{Tr}\left[A_\mu(x)\partial_\nu A_\sigma(x) + \frac{2}{3}A_\mu(x)A_\nu(x)A_\sigma(x)\right]$$

This becomes non-abelian by involving multiple vector fields A^a_μ, where a indexes the generators of a gauge group. The classic example of $SU(2)$: The generators are $\sigma^a/2\imath$, with

$a \in \{1,2,3\}$, and the fields used in the action are $A_\mu = A_\mu^a \frac{\sigma^a}{2\imath}$. This theory now has, in addition to general coordinate invariance, an $SU(2)$ gauge invariance.

Every component A_μ takes, on a general knot, a spin-j (more generally j index) representation of its gauge group. Why? Well, it can't possibly take more than one - we have to integrate, sum, and different representations are in general different-dimensional matrices. And it must take some - and the results can differ based on which one, so it must be specified. A physical interpretation still eludes me, though. Before we write the Wilson loop operators, we need to define the

1.3.2.1 Path-ordered exponential

Used when exponentiating integrals of non-commutative algebras, similar to time-ordering but more general, so any product of operator/matrix/non-commutative object fields (created when expanding the exponential of an integral into a series of n-integrals) must be ordered such that objects at positions farther along the path (higher value of path parameter) occur first, or on the left.

$$P\exp \oint_K dx^\mu A_\mu = \prod_m \left(1 + dx_m^\mu A_\mu(x_m)\right)$$

1.3.3 Wilson loop operators (Non-abelian)

For an oriented knot on which the fields A_μ are in the j-spin representation,

$$W_j[K] = \mathrm{Tr}_j P\exp \oint_K dx^\mu A_\mu^a T_j^a$$

For a link L of knots K_i with spins j_i,

$$W[L] = \prod_{l=1}^{s} W_{j_l}[K_l]$$

1.3.3.1 Link Invariants from Wilson Loops

And the expectation value (aka functional averages, or link invariants):

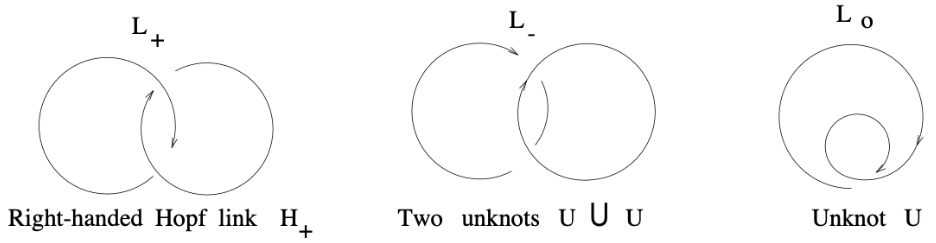
$$V[L] = \frac{1}{Z}\int [dA]W[L]e^{\imath kS},\; Z = \int [dA]e^{\imath kS}$$

The integrands are metric-independent, and (Kaul and Rajaraman, 1990) shows that $[dA]$ is too, so the expectation value depends only on the isotopy type of L and the representations $\{j_l\}$. These invariants can be obtained without perturbations! For example, Witten related links with all $j = 1/2$ as follows: For 3 links differing only in the presence & over/under type of one crossing, labelled as L_0, L_+, L_- respectively, the invariants relate as

$$qV_{1/2}[L_+] - q^{-1}V_{1/2}[L_-] = (q^{1/2} - q^{-1/2})V_{1/2}[L_0]$$

$q = \exp\frac{2\pi\imath}{k+h}$ is a root of unity related to the coupling constant k. The constant h is the *dual Coxter number* of any gauge group - it is 2 for SU(2). Here, those familiar with Knot theory would rejoice, because this is the generating Skein relation for the Jones polynomials, and indeed $V_{1/2}[L]$ is the one-variable Jones polynomial.

We can use this to evaluate $V_{1/2}[U]$, U being the unknot, by noting that the invariant for disjoint links is the product of their individual invariants, and that if $L_0 = U \cup U$, then $L_+ = L_- = U$, just with the opposite orientations. Then the skein relation gives $V_{1/2}[U] = q^{1/2} + q^{-1/2}$. Then we can build up to the Hopf link H_+ by $L_- = U \cup U$, $L_+ = H_+$,$L_0 = U$: Then $V_{1/2}[H_+] = 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}$. Similarly, $L_+ = T_+$, a trefoil, can be solved for as $V_{1/2}[T_+] =$



$\frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q^3}} + \frac{1}{\sqrt{q^5}} - \frac{1}{\sqrt{q^9}}$.

Similarly, many new link invariants emerge from expectation values of the Wilson link operators with arbitrary spin representations. When these spins differ across knots, they are called *coloured polynomial invariants*. Of course, other semi-simple groups could be used instead of $SU(2)$, giving other new invariants. For these general cases, the recursion relations don't always exist, and aren't complete if they do.

1.3.4 Complete Solutions for $SU(2)$ Chern-Simons

A non-perturbative method which generalises Witten's formalism, it needs two ingredients:

1. Field Theoretic Ingredient - Bulk-Boundary Correspondence A Chern-Simons on a 3-manifold has a bulk-boundary correspondence with a 2D Wess-Zumino CFT, and the expectation values (in Chern-Simons) of Wilson operators along n lines ending at points on the boundary are related to the Wess-Zumino on the boundary with n punctures "carrying" the same spin representations. Essentially, the functional integral in CS relates to the n-point correlator in the WZ.

2. Braid Theory Ingredient An n-braid has two (horizontal) planes with n points on each, directly above/below corresponding points on the other plane.

The braid itself is the set of non-intersecting strands connecting points above with points below, never going upwards (backwards) for any section of the strand. This can be projected onto 2D with marked over- and under-crossings. The braids form a group. Identical strands give the ordinary braids, but we can also colour the strands separately (give them spins), which forms a groupoid instead of a group. Braids can become knots and links in a few ways:

1. **Closure** of a braid - connect the strands' ends for the n^{th} point in the above plane to the n^{th} point in the below plane.

   Alexander's theorem states any link/knot can be obtained from a braid's closure (not uniquely).

2. **Plaiting** of a braid - For a $2m$-braid with pairs of adjacent strands carrying the same colour (spin) but opposite orientations, connect these adjacent strands above and below. Birman's theorem states any coloured and oriented link can be represented as a braid's plat (not uniquely).

1.3.4.1 The solution

The boundary CFT creates matrix representations of braids from n-point correlators, and plats/closures correspond to specific elements of these matrices, which then give us the link invariant from the Wilson operator. Some results from the same:

$$V_j[U] = [2j+1]_q$$

Where square brackets denote q-numbers (see q-Analogues, Appendix A) using the square root of unity defined above.

$$V_j[T_+] = \sum_{m=0}^{\min\{2j,k-2j\}} [2m+1]_q (-1)^{2j+m} q^{-6C_j+\frac{3}{2}C_m}$$

Where $C_j = j(j+1)$, the (quadratic) Casimir invariant for a spin-j rep.

1.3.4.2 Framing

Recall framing from 'Evaluating the 2-point correlators' (1.3.1.9.1). The link invariants depend on the framing, and the above results are in the *standard framing*, in which every knot's self-linking number is 0 - invariants are unchanged under all 3 Reidmeister moves, i.e. are ambient isotopic invariants.

Another important option is the *vertical framing*, where the frame is vertically displaced above the knot's 2D projection - this gives regular isotopic invariants, unchanged under RII,III but not RI.

1.4 Knot Theory

Definition 1.4. Knot

A smooth non-intersecting closed curve embedded in a 3-manifold. It can be *oriented*. A circle (and its equivalents) is an *unknot*.

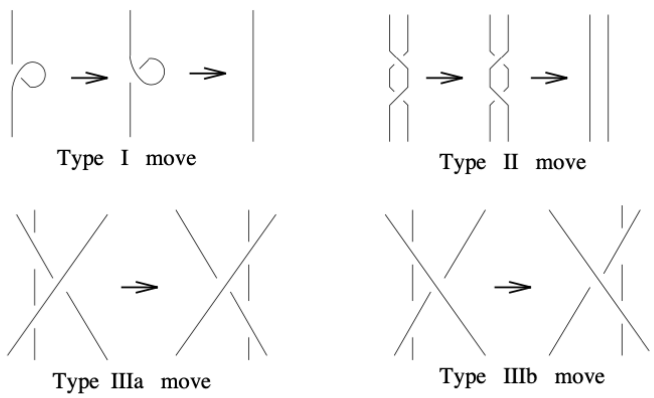
A *knot diagram* is a 2D projection of this with the minimum *double points* (self intersections) - count them to get the *crossing number*.

Definition 1.5. Link

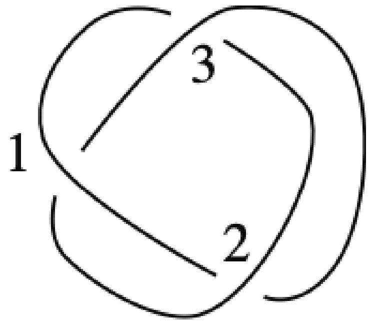
Any collection of non-intersecting knots. It's oriented if the knots are. Define a *link diagram* similarly.

1.4.1 Reidmeister moves

We're interested in the equivalence of knots and links as if they were ideal physical objects - so moving, stretching, shrinking is allowed, cutting isn't. How do we rigorously encode these? In the context of knot and link diagrams, the 3 Reidmeister moves suffice. These seem physically



sensible, and invariance under these has the technical name *ambient isotopy* - an isotopy is a continuous deformation of a sub-manifold. Some objects are invariant under types 2 and 3 but not 1 - they are called *regular* isotopic invariants. Equivalence of links or nots is not trivial



to identify, and *link invariants* are a crucial aid for the same - see the Jones polynomial and generalisations. Note that mirror-images of knots are not considered equivalent per se, and while the Alexander polynomial doesn't distinguish between them, the Jones does. Distinct Jones imply distinct knots, but the converse need not be true.

1.4.2 Gauss Diagrams

A more combinatorial alternative to knot diagrams.

1.4.2.1 Gauss Code

Label the crossings of a knot diagram with integers 1 to n, then the Gauss code is a cyclical double-occurrence list of the integers denoting the over and undercrossings encountered when moving along a knot. As always, we use the Trefoil for the non-trivial example. Label crossings $1,2,3$, then following the knot, one may start from the overcrossing at 1, undercross at 2, overcross at 3, then undercross at 1, and so on - the Gauss code is $O1U2O3U1O2U3$, O,U denoting over and undercrossings. An alternate notation is to denote overcrossings with the positive labels and undercrossings with negative labels, like so: $1,-2,3,-1,2,-3$. For the simplest forms of Knot diagrams, I expect Gauss codes cannot contain consecutive Os or Us - such a configuration could be simplified further with the Reidmeister moves, no?

1.4.3 Knot and Link Invariants

1.4.3.1 Linking Numbers

$$\underbrace{\frac{1}{4\pi}\oint_{K_l} dx^\mu \oint_{K_m} dy^\nu \varepsilon_{\mu\nu\sigma}\frac{(x-y)^\sigma}{|x-y|^3}}_{\mathscr{L}(K_l,K_m)}$$

$\mathscr{L}$ is the *Gauss linking number* for two distinct knots - it measures how many times one knot passes through the other, and hence should be an integer. It measures orientation, so a right-handed link would have +1, and a left-handed -1 (see Hopf links). When $K_l = K_m$, it instead measures the *self-linking/framing number*, and is evaluated by creating a loop $K_f(\varepsilon)$ displaced along the normal such that $K_f(0) = K$ (so K_f has coordinates $y^\nu = x^\nu(s) + \varepsilon n^\nu(s)$, $n^\nu(s)$ is the principle normal (unit vector field) to the curve), integrate over K, K_f and limit $\varepsilon \to 0$, so $\lim_{\varepsilon\to 0}\mathscr{L}(K, K_f(\varepsilon)) = \mathscr{S}\mathscr{L}(K)$. K_f is known as K's frame.

Theorem 1.1. Calugareanu theorem
The self-linking number is the sum of the twist and writhe for any knot.

$$\mathscr{S}\mathscr{L}(K) = T(K) + w(K) \tag{1.53}$$

$$T(K) = \frac{1}{2\pi}\int_K ds\, \varepsilon_{\mu\nu\sigma}\frac{dx^\mu}{ds} n^\nu \frac{dn^\sigma}{ds} \tag{1.54}$$

$$w(K) = \frac{1}{4\pi}\int_K ds \int_K dt\, \varepsilon_{\mu\nu\sigma}\, e^\mu \frac{de^\nu}{ds}\frac{de^\sigma}{dt} \tag{1.55}$$

$$e^\mu(s,t) = \frac{y^\mu(t) - x^\mu(s)}{|y(t) - x(s)|} \tag{1.56}$$

$e : K \otimes K \mapsto S^3$, $n^\mu(s)$ is the field of normal vectors to K at $x^\mu(s)$.
The twist and writhe/coil may not be integers, and they are not ambient isotopic invariants, but their sum is both.

One attempt at intuitively explaining this theorem was:

Twist This term represents the total "twist" of the curve. It corresponds to the sum of the signed angles between consecutive tangent vectors along the curve.

Writhe Writhe, in the context of the Calugareanu theorem, refers to the total "writhe" of the curve. It's the overall measure of how much the curve winds and coils in three-dimensional space.

Linking Number The linking number is a topological invariant that quantifies how many times one closed curve winds around another closed curve. It's a measure of their entanglement or linking.

1.4.3.2 Jones Polynomial

The Jones polynomial obeys the Skein relation:

$$(t^{1/2} - t^{-1/2})V[L_0] = tV[L_+] - t^{-1}V[L_-] \tag{1.57}$$

Where L_0, L_+, L_- are links differing in a single crossing change as follows:

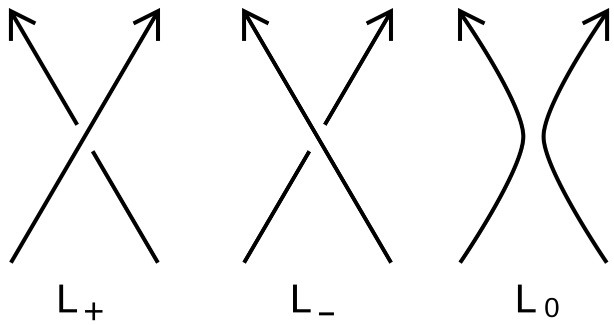


Figure 1.1: Image credits: Wikipedia

1.4.3.3 Vassiliev Invariants

The Vassiliev invariants were initially introduced by V. A. Vassiliev while he was studying the smooth maps from $S^1 \to S^3$ using knot theory, and are defined for the more general class of self-intersecting knots.

As I show in this project, they are also the coefficients of the perturbative expansion of the expectation value of the Wilon loop operators in the $SU(2)$ Chern-Simons theory.

1.4.4 Parametrisation and Examples

While knots are topological objects and most knot and link invariants can be computed without specifying a precise embedding, to compute the invariants from the Wilson operators, we need to specify a closed path in the manifold, with a parametrisation $\gamma : [0,1] \to M$.

There exist many documented parametrisations for common knots, and some knots belong to the same family of parametrisations, the torus knots being the most famous example. While topologically equivalent, certain parametrisations fair better than others when trying to analytically or numerically compute the Wilson operators.

1.4.4.1 Torus Knots

The Torus knots are a family of knots defined by paths in 3D space parametrised by a pair of coprime integers (n,m). The parametrisation arises from the definition that an (n,m) torus knot is a path on the surface of an unknotted torus inside $\mathbb{R}^3$ which winds around the axis of the torus n times and around the circle centred inside the torus m times. Some simple examples are $(1,0)$ and $(0,1)$ torus knots being unknots, and $(2,3)$ being the Trefoil.

For a torus with major radius R and minor radius r, working in cylindrical coordinates (ρ,ϕ,z), the torus is defined as $(\rho - R)^2 + z^2 = r^2$. The parametrisation is then:

$$f_{n,m}(\theta) \equiv (x(\theta), y(\theta), z(\theta)) \tag{1.58}$$

$$x(\theta) = (R + r\cos(m\theta))\cos(n\theta) \tag{1.59}$$

$$y(\theta) = (R + r\cos(m\theta))\sin(n\theta) \tag{1.60}$$

$$z(\theta) = -r\sin(m\theta) \tag{1.61}$$

$$\theta \in [0, 2\pi]$$

In literature, often the torus with $R = 2, r = 1$ is used for simplicity. For numerical calculations, it can be useful to play with these values to improve the numerical stability of the integrals. Due to the ease of parametrisation and visualisation, I use these to test my calculations.

Some examples are:

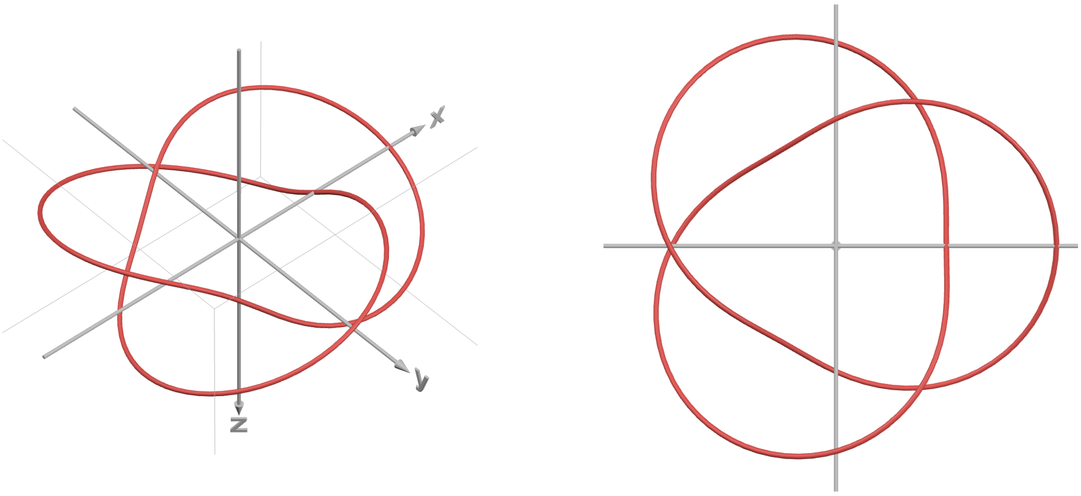
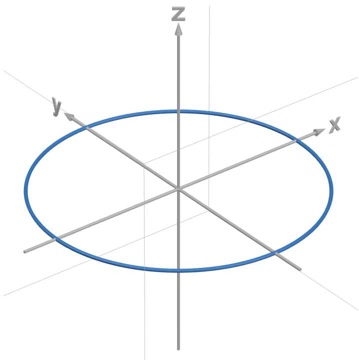
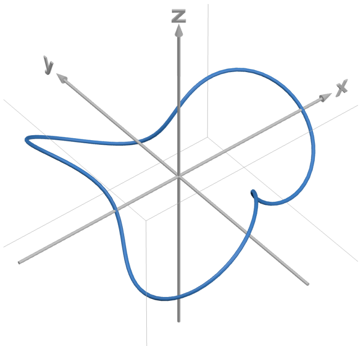


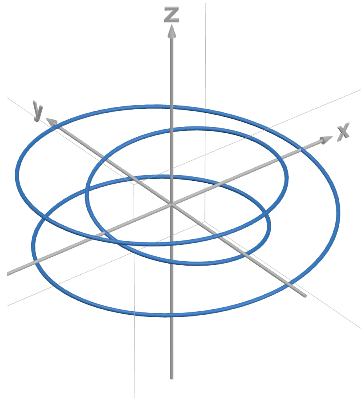
Figure 1.2: The trefoil knot, parametrised as $f_{2,3}(\theta)$



(a) The $(1,0)$ torus knot



(b) The $(1,3)$ torus knot



(c) The $(3,1)$ torus knot, equivalent to unknot under Reidmeister I moves

Figure 1.3: Various parametrisations of the Unknot

1.4.4.2 Torus Links

If the two integers n,m are not chosen to be coprime, we get a link instead, with as many loops as $\gcd(n,m)$. However, this link is not well-defined in the torus knot parametrisation, since the loops intersect.

Instead, we can take inspiration from the torus knots and define *twin torus links* as links of two identical torus knots, with one rotated slightly about the torus axis. While one is parametrised as $f(n,m,t)$, the other is rotated by α and parametrised as:

$$g^{\alpha}_{n,m}(\theta) \equiv (x^{\alpha}(\theta), y^{\alpha}(\theta), z^{\alpha}(\theta)) \tag{1.62}$$

$$x^{\alpha}(\theta) = (R + r\cos(m\theta))\cos(n\theta + \alpha) \tag{1.63}$$

$$y^{\alpha}(\theta) = (R + r\cos(m\theta))\sin(n\theta + \alpha) \tag{1.64}$$

$$z^{\alpha}(\theta) = -r\sin(m\theta) \tag{1.65}$$

$$\theta \in [0, 2\pi]$$

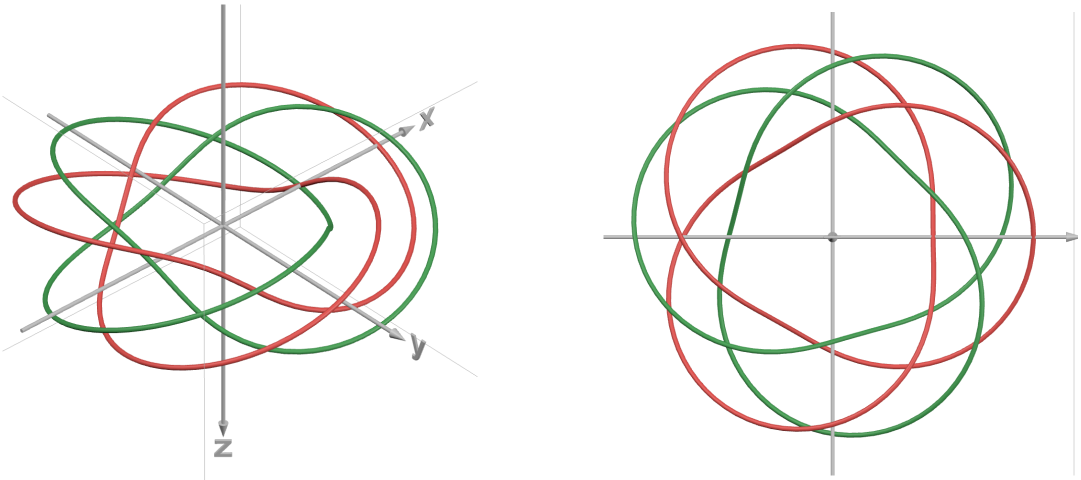


Figure 1.4: The twin trefoil link, parametrised as $g^{0}_{2,3}(\theta)$ and $g^{\pi/4}_{2,3}(\theta)$

We can also define torus-unknot links as the links between a torus knot and the circle centred inside the torus. I use these to test my calculations, since they are simple to visualise and calculate link invariants for.

1.5 Perturbative Chern-Simons

The Wilson operators can be expanded in a power series of the coupling constant, and the coefficients will be topological invariants of the knot. The CS theory is super-renormalizable, and the coupling constant k depends on the regularisations - for $SU(2)$ and certain regularisations, it changes as $k \mapsto k+2$, consistent with the effective coupling observed in non-perturbative studies. To first order, the theory reduces to its abelian cousin, so the Wilson invariant is simply

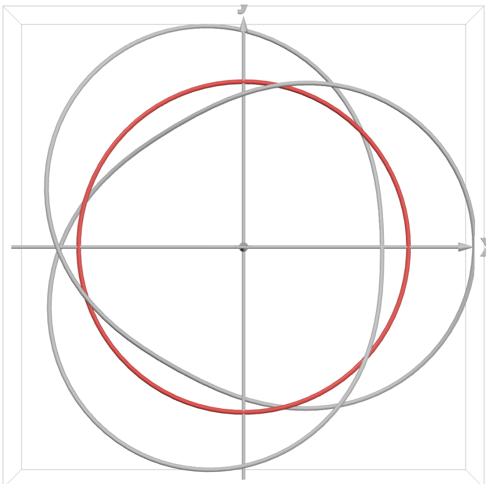
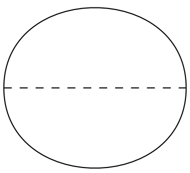


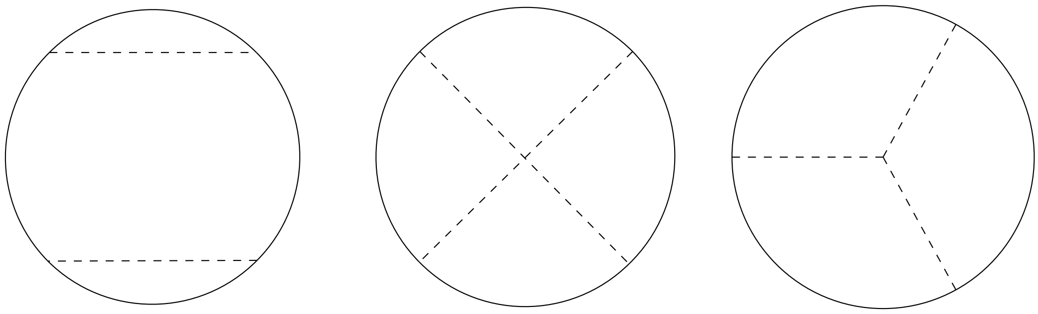
Figure 1.5: The trefoil-unknot link

the self-linking of the knot (up to factors based on the gauge group). Higher order contributions are the *Vassiliev invariants*, initially introduced when studying the smooth maps on $S^1 \to S^3$ - based on the types of singularities, the maps can be classified, each class corresponding to a knot and hence characterised by a family of invariants. Perturbative CS in the Landau or light-cone gauges allowed covariant integral representations of these invariants, and in the temporal gauge, combinatorial formulae for them.

1.5.1 One-Loop Contributions



1.5.2 Two-Loop Contributions



1.6 Resurgence

1.6.1 Introduction

Perturbative expansions form the basis of most practical applications of Quantum Field Theory. However, Pauli showed that these series could not be convergent. Ideas like Borel resummation suggested that non-perturbative terms - for example, exponential terms correcting a power series expansion - could remove the divergences in some cases. It was Jean Écalle who extended this idea to the complete approach of Resurgence, which methodically finds all the non-perturbative terms (and their associated series) from a perturbative series, to correct the same and reach the correct physical results. The corrected object is a sort of series of power series - called a *trans series*. There is further context about chaotic systems giving rise to problems with divergent but resurgence-correctable perturbative solutions, but I am currently not knowledgable enough to comment on the same.

References
An introduction to resurgence in quantum theory, by Marcos Marino(Marino, 2023)

1.6.2 Elementary Terminology

1.6.2.1 Asymptotic Series

Consider $\phi(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. This is a formal power series. We say it approximates the function f (written as $f(z) \sim \phi(z)$) if the Nth partial sum leaves errors of $\mathscr{O}(z^{N+1})$. Formally,

$$\lim_{z\to 0} \frac{1}{z^N}\left(f(z) - \sum_{n=0}^{N} a_n z^n\right) = 0\ \forall N > 0$$

That's a convergent series - contrast with asymptotic series, which don't converge, for example the Stirling series for the Gamma function:

$$\sqrt{\frac{z}{2\pi}}\left(\frac{z}{e}\right)^{-z}\Gamma(z) = 1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots$$

The partial sums here will approach $f(z)$, but not forever - after a point, they do diverge, allowing us to define an *optimal truncation*.

Example 1.1. Optimal Truncation - Factorial coefficients $a_n \sim A^{-n} n!$ is a frequently-encountered pattern in asymptotic series (think the number of nth order Feynman diagrams). Often, optimal truncation is practically accomplished by identifying the smallest term. Consider the Nth term:

$$|a_N z^N| = cN!\left|\frac{z}{A}\right|^N \tag{1.66}$$

$$\approx c\exp\left[N\left(\log N - 1 - \log\left|\frac{A}{z}\right|\right)\right] \tag{1.67}$$

(Using the Stirling approximation.) The minima is at $N_* = |A/z|$.

Assuming this is the most correct partial sum, we can estimate its error from the precise function value by evaluating the next term: $\varepsilon(z) = a_{N_*+1}|z|^{N_*+1} \sim e^{-|A/z|}$, which is also referred to as the resolution, or non-perturbative ambiguity.

Terms like $e^{-|A/z|}$, which are not analytic at $z = 0$, could not possibly arise from a perturbative expansion. Such behaviours are precisely what a power series expansion is missing. Another example worth working out, and one we will come back to since it is essentially a 0d

ϕ^4 theory, is the expansion of the quartic integral:

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz e^{-S(z)}, \; S(z) = \frac{z^2}{2} + g\frac{z^4}{4}$$

The power series in g turns out to have a 0 radius of convergence.

1.6.2.2 Borel Resummation

The *Borel Transform* acts on power series as:

$$\mathscr{B} : \mathbb{C}[[z]] \to \mathbb{C}[[z]] \tag{1.68}$$

$$z^n \mapsto \zeta^n / n! \tag{1.69}$$

Definition 1.6. A power series is **Gevrey-1** if $\exists M, \rho > 0$ s.t. $|a_n| < Mn!\rho^n \; \forall n$

Lemma 1.5. $\phi(z)$ is Gevrey-1 $\implies \mathscr{B}(\phi) = \widehat{\phi}$ is analytic in some neighbourhood of $\zeta = 0$.

Example 1.2. Consider the Gevrey-1 series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{\Gamma(b)} \left(\frac{z}{A}\right)^k \tag{1.70}$$

$$\implies \mathscr{B}(\phi)(\zeta) = (1 - \zeta/A)^{-b} \tag{1.71}$$

The transform has a singularity at $\zeta = A$, of type:

- $b = 1$: Pole
- $b = 0$: Logarithmic
- $0 < b < 1$: Branch point

The singularities of the Borel transform will guide us to the missing information of the series - the so-called additional sectors. We'll write a few definitions to focus on series where these ideas can (non-trivially) bear fruit:

Definition 1.7. Resurgent function

A Gevrey-1 series with a Borel transform such that any radial line through the origin encounters only a finite number of singularities, each of which can be avoided by deviating a little from the line.

(Formally, circumventing the singular points from above or below, following a path that differs from the line with only a finite number of such circumventions.)

Definition 1.8. Simple resurgent functions

All the singularities of the Borel transform are poles or logarithmic branch cuts.

We can locally expand the Borel transform around any singularity ζ_ω as follows:

Log singularity

$$\widehat{\phi}(\zeta_\omega+\xi) = -\frac{S}{2\pi}\log\xi \underbrace{\sum_{n\in\mathbb{N}\cup\{0\}} \widehat{c}_n\xi^n}_{\widehat{\phi}_\omega(\xi)=\mathscr{B}(\phi_\omega(z))} +\text{regular}$$

S is the scaling *Stokes constant*, and *regular* encodes the non-singular behaviour at that point with some analytical series like $\sum_{n\in\mathbb{N}\cup\{0\}} a_n(\zeta_\omega+\xi)^n$. $\widehat{\phi}_\omega(\xi)$ has a finite radius of convergence and is the Borel transform of a $\phi_\omega(z) = \sum_{n\in\mathbb{N}\cup\{0\}} (n!\widehat{c}_n)z^n$.

Poles Instead, the Borel transform simply gains a $-\frac{Sa}{2\pi}$ term, which corresponds to a harmonic term in the diverging series $\phi_\omega(z) = \frac{a}{z} + \sum_{n\geq 0} c_n z^n$.

Branch cuts For cuts of the form $(\zeta_\omega-\zeta)^{-b}, 0<\beta<1$,

$$\widehat{\phi}(\zeta_\omega+\xi) = \frac{1}{(-\xi)^b}\sum_{n\geq 0}\widehat{c}_n\xi^n + \text{regular}$$

Where $c_n = \Gamma(n+1-b)\widehat{c}_n$.

The key realisation here is that around every singularity, the local expansion of the Borel transform has an associated power series. For set of singularities Ω, associate $\phi(z) \to \{\phi_\omega(z)\}_{\omega\in\Omega}$.

Appendix A

q-Analogues

The q-analog of any mathematical object is a parameter-dependent generalisation which returns the original in some limit, generally $q \to 1$. This has significance to the interface between classical and quantum mechanics, where $\hbar \to 0$ should return known classical results. The following example for the non-negative numbers highlights how q-analogues can be applied to $\hbar \to 0$:

$$[n] = \frac{\sin n\hbar}{\sin \hbar} \tag{A.1}$$

$$= \frac{e^{\imath n\hbar} - e^{-\imath n\hbar}}{e^{\imath\hbar} - e^{-\imath\hbar}} = \frac{1 - e^{-2\imath n\hbar}}{1 - e^{-2\imath\hbar}} \tag{A.2}$$

$$q \equiv e^{-2\imath\hbar} \tag{A.3}$$

$$\implies [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \tag{A.4}$$

$$\lim_{q\to 1} [n]_q = \lim_{\hbar\to 0} [n] = n \tag{A.5}$$

Note that the hyperbolic sin function could be used instead to keep everything real, if preferred. It has no bearing on the further mathematics.

A.1 Q-Factorial and Combinatorics

The q-factorial often appears naturally - for example, when $n!$ counts the permutations, $[n]_q!$ counts the permutations while also keeping track of the number of inversions - it does this by being a polynomial in q.

$$[n]_q! := [1]_q[2]_q \dots [n]_q \tag{A.6}$$

$$= 1 \cdot (1+q) \cdot \dots \cdot (1+q+\dots+q^{n-1}) \tag{A.7}$$

For any permutation w, let $\mathrm{inv}(w)$ be the number of inversions, then

$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!$$

Where S_n is the set of all permutations (of length n). Clearly, derivatives wrt q give us the permutations for any inversion, and $q \to 1$ returns the usual factorial.

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

A.2 Exponentials and other functions

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

And similarly trignometric functions and fourier transforms.

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