Quantum Modular Forms in Knot Theory

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Motivation

Knot Theory formalises intuitive notions of knots and links. The central problem in Knot Theory is to characterise and differentiate knots using topological quantities which are invariant for a particular knot but can differ between knots. We explore the connection between knot invariants and curious mathematical objects called Quantum Modular Forms.

Modular Forms

- The **Modular group** $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm\mathbb{I}\}$ acts as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ on $z \in \mathbb{C} \cup \{\infty\}$.
- A **modular form** is a holomorphic function on H(upper half complex plane)$\cup\{\infty\}$ transforming as $f(\gamma z) = \epsilon(\gamma)(cz+d)^{2k} f(z)\ \forall \gamma \in \mathrm{PSL}_2(\mathbb{Z})$, $|\epsilon(\gamma)| = 1$

Quantum Modular Forms

A **quantum modular form** of weight k ($k \in \mathbb{Z}/2$) is a function $g : \mathbb{Q}/S \to \mathbb{C}$, for some discrete subset S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2, (\mathbb{Z})$ the functions :

$$h_{g,\gamma}(x) := g(x) - \epsilon(\gamma)^{-1}(cx+d)^{-2k} g(\gamma x) \tag{1}$$

satisfy a suitable property of continuity or analyticity in $\mathbb{R}$ (definitions vary).

Examples of QMFs

- Kontsevich (1997), Zagier (2001), Kashaev (1996): $F(\zeta_N) := \sum_{n=0}^{\infty} (\zeta_N)_n$, where $(x)_n = \prod_{k=0}^{n-1}(x-k)$ and $(\zeta_N)^N = 1$.
- Kashaev (1996): $J(\zeta_N) := \sum_{n=0}^{\infty} |(\zeta_N)_n|^2$.
- Lawrence - Zagier (1999): $W(\zeta_K) := \frac{-1-i}{\sqrt{120K}} \sum_{\substack{\beta \mod 60K, \\ \beta \neq 0 \mod K}} \frac{\sin(\frac{\pi\beta}{3K})\sin(\frac{\pi\beta}{5K})}{\cos(\frac{\pi\beta}{2K})} e^{-\pi i(\beta^2+1)/60K}$

Knot Theory

- A **knot** is a closed path embedded in a 3-manifold, generally $\mathbb{R}^3$.
- A **link** is a collection of knots, possibly intertwined.

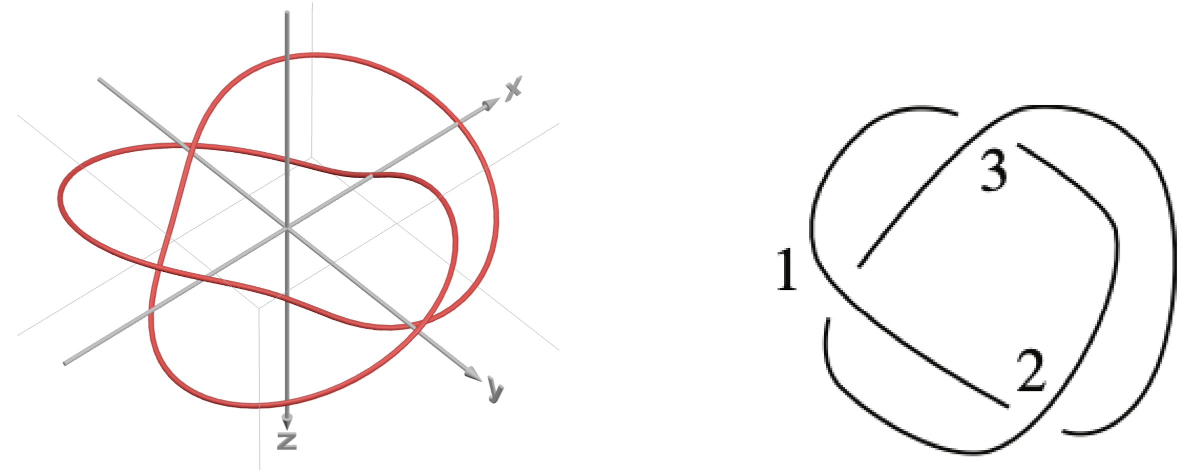

Figure 1. The trefoil knot and its 2D diagram with labelled crossings.

- A famous knot invariant is the **Jones Polynomial**, a Laurent polynomial generated via a recursive relation between knots related by the addition of a single crossing.
- Witten showed that the Jones polynomial for a knot can be obtained as the expectation value of the corresponding Wilson loop operator in a Chern-Simons field theory with gauge group $SU(2)$.
- The **coloured Jones polynomials** are generalisations obtained from the $SU(N)$ Chern-Simons theories (see below).
- **Torus knots** are knots embedded on the surface of a torus in $\mathbb{R}^3$. The (p, q) torus knot winds q times around the interior of the torus and p times around its axis - the trefoil is the $(2, 3)$ torus knot. (p, q) must be coprime. They can be parametrised as

$$(\cos p\theta(R + r\cos q\theta),\ \sin p\theta(R + r\cos q\theta),\ -r\sin q\theta) \tag{2}$$
$$\theta \in [0, 2\pi]$$

Chern-Simons Theory

Composed of

- A differentiable, compact 3-manifold M
- A simple, compact gauge group G (with corresponding gauge connection A, a 1-form)
- Integer parameter k (required to be integral for gauge invariance)

Then we have a Chern-Simons form, which integrates to give the action:

$$S_{CS}[A] = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{3}$$

The Wilson loop operator for a knot K is given by an integral over the knot:

$$W[K] = \exp\left(\iota n_i \oint_K dx^\mu A_\mu(x)\right) \tag{4}$$

The expectation value of a Wilson loop operator for gauge groups $SU(N)$ give the coloured Jones polynomials in the variable $q = \exp\frac{2\pi\iota}{k+h}$ for the knot K.

Torus Knots and Quantum Modular Forms

Example : Trefoil

The coloured Jones polynomial can be computed by associating tensorial factor to each crossing and contracting the indices of connected crossings. We write down the result for a trefoil in the fundamental representation of any $SU(N)$:

$$J_N(T_{2,3}, q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN}(q^{1-N})_n \tag{5}$$

This can be rewritten in the Kontsevich-Zagier series, a well-known Quantum Modular Form we discussed previously.

$$J_N(T_{2,3}, \zeta_N) = \zeta_N F(\zeta_N) \tag{6}$$

$T_{(2,2t+1)}$ Torus Knots

$$J_N(T_{(s,t)}, \zeta_N) = \zeta_N^{\frac{s^2t^2-s^2-t^2}{4st}} \tilde{\Phi}_{s,t}^{s-1,1}(\tfrac{1}{N}) \tag{7}$$

$$\tilde{\Phi}_{s,t}^{n,m}(\tau) := -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{s,t}^{n,m}(k) \exp\frac{2\pi\iota\tau k^2}{4st} \tag{8}$$

$$\chi_{s,t}^{n,m}(k) := \begin{cases} 1 & k = \pm(nt - ms) \mod 2st \\ -1 & k = \pm(nt + ms) \mod 2st \\ 0 & \text{otherwise} \end{cases} \tag{9}$$