Lecture 1 - 444

Course	PH 444 - EM Theory
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Next: Lecture 2 - 444

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Assessment:

- Quiz 1 - 10%
- Midsem - 20%
- Quiz 2 - 10%
- Project (group) - 20%
- Endsem - 40%
- Tut problems - ungraded

Reference: Classical Electrodynamics by J.D. Jackson

This will be a highly mathematical course, with not much intuition.

The first few classes will be a brief history and revision of first-year electromagnetism. I won’t make exhaustive notes for the history section.

- 1772, Cavendish performed the first electrostatics experiments
- 1785, Coulomb

- 1835, Faraday studies time-varying fields
- 1864, Maxwell
    - Very complicated, he tried to build analogies to mechanics because that was the rage.
- …
- 1990, Classical electrodynamics is shown to be derivable from the standard model, a limit of QED

Revising first-year electromagnetism

$$\begin{gathered}\nabla\cdot\vec{D}=\rho\\ \nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=\vec{J}\\ \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0\\ \nabla\cdot\vec{B}=0\end{gathered}$$

In vacuum,

$$\begin{gathered}\vec{D}=\epsilon_0\vec{E},\ \vec{B}=\mu_0\vec{H},\ c^{-2}=\epsilon_0\mu_0\\ \implies\ \nabla\cdot\vec{E}=\rho/\epsilon_0\\ \nabla\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}\\ \text{The other two are unchanged}\end{gathered}$$

These are the sourced Maxwell equations, and the other two are the source-free Maxwell equations. The sourced equations allow derivation fot he **continuity equation**.

$$\begin{gathered}\frac{\partial}{\partial t}(\nabla\cdot\vec{D})=\frac{\partial\rho}{\partial t}\\ \implies\ \nabla\cdot(\nabla\times\vec{H}-\vec{J})=\frac{\partial\rho}{\partial t}\\ \text{Div of a curl is 0,}\\ \frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0\end{gathered}$$

This encodes the conservation of charge. The integral version (obtained by integrating over a finite volume and using Gauss' Divergence Law) is intuitive, it says that the rate of change of charge in the volume is the negative of the outward flux of current from that volume.

This equation can also be written in 4-vector notation as $\partial_\mu j^\mu = 0$.

Spontaneous Emission

As covered in QM 2, when two-level systems are in an excited state, it tends to de-excite by emitting a photon.

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Spatial and Time averaging

If we want to treat a solid like a uniform object when probing it with light, we want the length and time scales of the light to be such that the microscopic fluctuations in the field due to spatial and temporal variation in the solid (the atomic lattice) are averaged over.

A wavelength of the order of 100Å works, but that gives a time scale of $10^{-8}\text{m}/c \sim 10^{-17}$, the same time scale as a bohr orbital period for an electron. So time averaging seems to be difficult to achieve. But (as we'll cover later) it ends up happening due to other reasons.

Familiar equations, modified

$$\begin{aligned} D_\alpha &= \epsilon_0 E_\alpha + \left[P_\alpha - \sum_\beta \frac{\partial Q_{\alpha\beta}}{\partial x_\beta} + \dots \right] \\ H_\alpha &= \frac{1}{\mu_0} B_\alpha + [-M_\alpha + \dots] \\ \implies D_\alpha(\vec{x}, t) &= \int d^3\vec{x}' \int dt' \epsilon_{\alpha\beta}(\vec{x}', t') E_\beta(\vec{x} - \vec{x}', t - t') \end{aligned}$$

That last integral is a convolution, for now integrated over $\mathbb{R}^4$ (we do have to insert causality later on, but not yet). What the last equation says is that the response of the material is dependent non-instantaneously and non-locally on the external electric field.

Since it is a convolution, it beautifully becomes multiplication in the fourier space (watch 3B1B's videos on convolutions). Thus we get

$$D_\alpha(\vec{k},\omega) = \epsilon_{\alpha\beta}(\vec{k},\omega)E_\beta(\vec{k},\omega)$$

Deriving Boundary Conditions

We use the Maxwell equations in their integral form and then apply them to special surfaces and volumes to obtain boundary conditions for some general cases.

$$\int_V \rho dV = \int_V \nabla \cdot \vec{D} dV = \int_S \vec{D} \cdot \hat{n} dA$$
$$0 = \int_V \nabla \cdot \vec{B} dV = \int_S \vec{B} \cdot \hat{n} dA$$

Take a small pillbox with faces parallel to some charged surface with volume V, face area Δa, face normal $\hat{n}$, surface densities $\sigma, \vec{K}$ (charge and current), and fields at the faces $\vec{E}_i, \vec{B}_i$ ($i = 1$ for inside, $i = 2$ for outside). Then

$$[\vec{D}_2 \cdot \hat{n} + \vec{D}_1 \cdot (-\hat{n})]\Delta a = \sigma \Delta a$$
$$\implies (\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma$$

Similarly we get that $(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$.

Then we use the Stoke's Theorem:

$$\int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot dA = \int_S (\nabla \times \vec{H}) \cdot dA = \oint_C \vec{H} \cdot d\vec{l}$$
$$-\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = \int_S (\nabla \times \vec{E}) \cdot d\vec{A} = \oint_C \vec{E} \cdot d\vec{l}$$

Take a small surface crossing the charged surface with normal $\hat{t}$, go around the boundary of the surface, use the integral form, and note that the fields are not diverging so their integral over an infinitesimal area will be zero, but $\vec{J} = \vec{K}\delta(z)$ so that integral does not go to zero. Hence

$$\Delta l(\hat{t} \times \hat{n}) \cdot (\vec{H}_2 - \vec{H}_1) = \vec{K} \Delta l \cdot \hat{t}$$
$$\implies \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$$

Where we could remove the $\hat{t}$ because it is arbitrary, we have not necessitated it to be in a particular direction.

Similarly, $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$.

Revision of Electrostatics

$$\begin{gathered}
\text{Coulomb} : \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2 (\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} \\
\vec{F}_{12} := q_1 \vec{E}_2 \\
\implies \vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 (\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3} \\
\xRightarrow{\text{Superposition}} \vec{E}(\vec{x}) = \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i (\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3} \\
\xrightarrow{\text{Generalises to}} \vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{\rho(x')(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}
\end{gathered}$$

Revision of the Delta Function

$$\begin{gathered}
g(x_i) = 0, g'(x_i) \neq 0 \\
\implies \int f(x)\delta(g(x))dx = \sum_i \frac{f(x_i)}{|g'(x_i)|} \\
\int f(x)\delta'(x-a)dx = -f'(a) \\
\int \delta(\vec{x} - \vec{x}')dV = 1 \implies \delta(\vec{x}) \text{ has dimensions } L^{-3}
\end{gathered}$$

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We will not be differentiating between fourier space and coordinate space functions except using the arguments and perhaps indices. E.g. Coordinate space permittivity will be written as $\epsilon_{\delta\rho}(\vec{x}, t)$.

Also may be worth looking deeper into the averaging ideas Sir mentioned.

Recall Gauss' Law, rederived in class for a point charge

Useful: $d\Omega = \frac{\cos\theta}{r^2} da$, where θ is the angle between $d\vec{a}$ and $\vec{r}$.

$$\int \vec{E} \cdot \hat{n} da = \int \frac{q}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} da = \frac{q}{\epsilon_0}$$

Scalar Potential

We can derive this for more general cases using the Helmholtz decomposition, but for electrostatics the derivation is briefly as follows:

$$\nabla \times \vec{E} = 0$$
$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3\vec{x}'$$
$$\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = -\nabla \left(\frac{1}{|\vec{x} - \vec{x}'|} \right), \text{ Hence}$$
$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \nabla \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$
$$\implies \vec{E} = -\nabla\Phi, \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

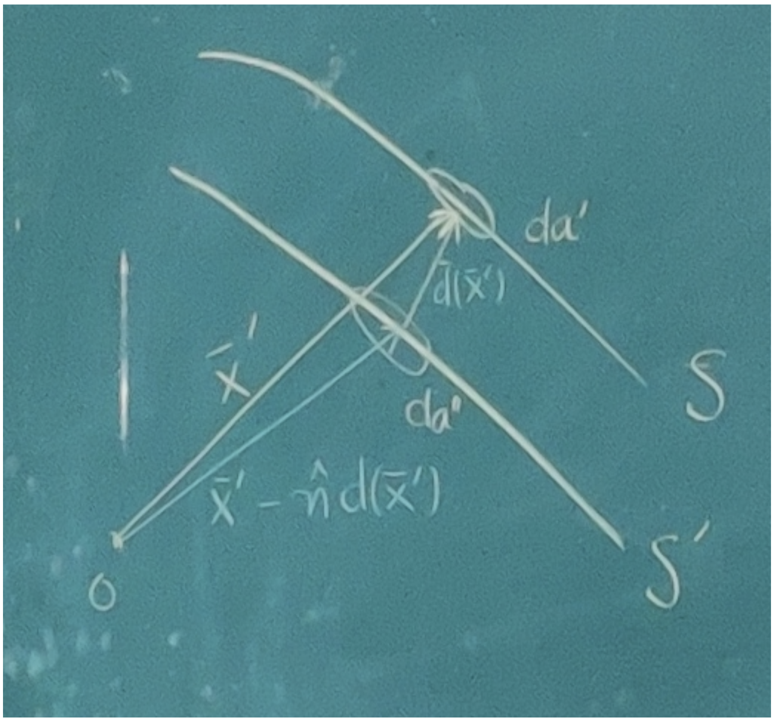
The work done moving a charge in an electrostatic field is hence

$$\vec{F} = q\vec{E}$$
$$W = -\int_A^B \vec{F} \cdot d\vec{l}$$
$$= \int_A^B q(\nabla\Phi) \cdot d\vec{l} = q(\Phi_B - \Phi_A)$$

Also note that for a surface charge density $\sigma(\vec{x})$, the potential is similarly $\frac{1}{4\pi\epsilon_0} \int da \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}$, can be rigorously derived using the delta function.

Dipole Layer Problem

Motivated by supercapacitors and wearable power generators.



Two layers S, S' with charge densities $\sigma(\vec{x}), -\sigma(\vec{x})$. We observe from point O (origin) a distance $\vec{x}$ from a point on S and $\vec{x} - \hat{n} \cdot d(\vec{x})$ from a point on S', $\hat{n}$ being the normal of these layers. $d(\vec{x})$ is the distance between the layers as a function of $\vec{x}$.

We want to use the limit $\lim_{d\vec{x}\to 0} \sigma(\vec{x})d(\vec{x}) = D(\vec{x})$. Write the potential as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|} da' - \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'+\hat{n}\cdot d(\vec{x}')|} da'$$

We can now use the multivariable taylor expansion:

$$f(x_i) = f(x_i^{(0)}) + \sum_i \frac{\partial f}{\partial x_i}\Delta x_i + \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \dots$$
$$\implies \frac{1}{|\vec{x}+\vec{a}|} = \frac{1}{|\vec{x}|} + \vec{a}\cdot \underbrace{\nabla_{\vec{a}}\left(\frac{1}{|\vec{x}+\vec{a}|}\right)_{\vec{a}=0}}_{\nabla\left(\frac{1}{|\vec{x}|}\right)} + \dots$$

Put this into the expression for potential, do some cancellations, and boom,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S D(\vec{x}')\hat{n}\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da'$$
$$\hat{n}\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \hat{n}\cdot\frac{-(\vec{x}'-\vec{x})}{|\vec{x}-\vec{x}'|^3}$$
$$= \frac{-\cos\theta da'}{r^2} = -d\Omega$$
$$\implies \Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int D(\vec{x}')d\Omega$$

We do this integral very close to the bilayer, so $D(\vec{x}') \to D$ and the integral just gives $2\pi D$.

Note that we did this integral such that $\vec{x}-\vec{x}'$ is sorta anti-parallel to $\hat{n}$. $\vec{x}-\vec{x}'$ points from the bilayer to the point where we're looking at for the potential. The side $\hat{n}$ points at is called the "outer". So the above calculation is for the "inner" side, and gives $\Phi_{\text{inner}} = \frac{-D}{2\epsilon_0}, \Phi_{\text{outer}} = \frac{D}{2\epsilon_0}$.

The discontinuity here (of D/ϵ_0) is the result of the small disk of the bilayer you have to cross to go from the outside to the inside. Without the disk, the potential would be continuous across the bilayer.

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▼ Understanding the dipole layer problem from another perspective.

Discussing the dipole layer problem from a different perspective,

Assume that $D(\vec{x})$ is continuous over length scale $\sim R'$ over the layer. We'll use the previously derived result,

$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int D(\vec{x}')d\Omega$$

Where $d\Omega$ is the solid angle subtended by the point $\vec{x}$ at the projection of the disk surface along $\vec{x}' - \vec{x}$. $d\Omega = \frac{\cos\theta}{|\Delta\vec{x}|^2}(rdrd\phi)$, then

$$\begin{aligned}\implies \Phi(\vec{x}) &= -\frac{1}{4\pi\epsilon_0}\left[2\pi\int_0^{R'} D(\vec{x}')r'\cos\theta dr' + \int_{R'}^{\infty} D(\vec{x}')d\Omega'\right] \\ &= -\frac{1}{2\epsilon_0}D\left(1-\frac{r^2}{r^2+R'^2}\right)^{\text{Wrong}} - \frac{1}{4\pi\epsilon_0}\int_{R'}^{\infty} D(\vec{x}')d\Omega'\end{aligned}$$

So as $r \to 0$, we obtain the $\pm\frac{D}{2\epsilon_0}$ result we obtained previously. And we neglect the second integral because the solid angle spanned by the second integral is negligible for small r.

Solving the Poisson Equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

We want to show that $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$ satisfies the Poisson equation, but without resorting to Dirac Deltas. So we instead look at $\Phi_a(\vec{x})$ and will later set $a \to 0$. Also, $r = |\vec{x} - \vec{x}'|$

$$\Phi_a(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{\sqrt{|\vec{x}-\vec{x}'|^2 + a^2}} d^3\vec{x}'$$
$$\nabla^2 f(r) = \frac{1}{r}\frac{d^2(rf(r))}{dr^2}, \frac{1}{r}\frac{d^2}{dr^2}(r(r^2+a^2)^{-1/2}) = \frac{-3a^2}{(r^2+a^2)^{5/2}}$$
$$\implies \nabla^2\Phi_a(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int \frac{3a^2\rho(\vec{x}')}{(|\vec{x}-\vec{x}'|^2+a^2)^{5/2}} d^3\vec{x}'$$

Choose an R such that $\rho(\vec{x})$ is almost constant in $r < R$. Then split the integral into two parts ($r < R, r > R$) and get the second term as 0 using $a \to 0$. Can't do that inside because the limit doesn't exist at $r = 0$. So

$$\nabla^2\Phi_a(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int_0^R r^2 dr \int d\Omega \frac{\rho(\vec{x}')3a^2}{(r^2+a^2)^{5/2}}$$

Then we Taylor expand $\rho(\vec{x}')$: ($\vec{r} = \vec{x}' - \vec{x}, r_1 = \Delta x$ and so on)

$$\rho(\vec{x}') = \rho(\vec{x}) + r_i\partial_i\rho + \frac{1}{2}r_ir_j\partial_i\partial_j\rho + \dots$$

We substitute into the integrand. Integrating over the sphere, all the r_i terms give 0 (odd functions of the angles). So we go to the third term, write in terms of θ, ϕ and integrate over the sphere. We obtain

$$-\frac{1}{4\pi\epsilon_0}\int_0^R r^2 dr \frac{3a^2}{(r^2+a^2)^{5/2}}\left(4\pi\rho(\vec{x}) + \frac{\pi r^2}{2}\frac{4}{3}\nabla^2\rho|_{\vec{x}}\right) + \Theta(a^2)$$

How $\Theta(a^2)$? These two terms go as $1/r^3$ and $1/r$ each, which means at small r they are significant even when $a \to 0$, but higher order r terms are finite and so go to 0 as a does, and so are negligible.

$$3a^2\int_0^R \frac{r^2}{(r^2+a^2)^{5/2}}dr = \frac{R^3}{(R^2+a^2)^{3/2}}$$
$$\text{Using } r = a\tan(\theta/2)$$

And hence we have the first term simplify to $-\frac{\rho(\vec{x})}{\epsilon_0}\left(1 + \frac{a^2}{R^2} + \dots\right)$.

The integral for the second term is a little harder, we cover it in the next lecture.

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▼ A neat result from variable substitution

$$\nabla^2_{\vec{x}}\Phi_a = \frac{1}{4\pi\epsilon_0}\int d^3\vec{x}'\rho(\vec{x}')\nabla^2_{\vec{x}}\left(\frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+a^2}}\right)$$

$$(\vec{r}=\vec{x}-\vec{x}')$$

$$\sim \int d^3\vec{x}'\rho(\vec{x}')f(|\vec{r}|) = \int d^3(-\vec{r})\rho(\vec{x}-\vec{r})f(|\vec{r}|)$$

$$\stackrel{\vec{r}'=-\vec{r}}{=} \int d^3\vec{r}'\rho(\vec{x}+\vec{r}')f(|\vec{r}'|)$$

Continuing from the previous lecture,

$$\int_0^R \frac{r^4}{(r^2+a^2)^{5/2}}dr\left(\frac{-3a^2}{6\epsilon_0}\nabla^2\rho\right)$$

$$= \left(\underbrace{\ln\left|\frac{\sqrt{R^2+a^2}}{a}+\frac{R}{a}\right|}_{\Theta(a^2\ln a)+\Theta(a^2)} - \underbrace{\frac{R}{\sqrt{R^2+a^2}}}_{\Theta(a^2)} - \underbrace{\frac{1}{3}\frac{R^3}{(R^2+a^2)^{3/2}}}_{\Theta(a^2)}\right)\left(\frac{-3a^2}{6\epsilon_0}\nabla^2\rho\right)$$

Note that the Θ expressions are obtained including the a^2 term outside the brackets.

Hence in the limit $a\to 0$, this whole integral tends to 0. The first term considered in the previous lecture behaves as

$$\nabla^2\Phi = \lim_{a\to 0}\nabla^2\Phi_a = \lim_{a\to 0} -\frac{\rho(\vec{x})}{\epsilon_0}(1+\Theta(a^2)) = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Note that we have assumed ρ to be smooth, but that may not be true, eg for point charges. In such cases we take ρ to have gaussian behaviour around the point and have the width tend to 0.

Green's Theorem

Div theorem: $(S = \delta V$, surface)

$$\int_V \nabla \cdot \vec{A} dV = \oint_S \vec{A} \cdot \hat{n} da$$

Choose

$$\vec{A} = \phi \nabla \psi$$

Where ϕ, ψ are "nice" scalar fields.

Then

$$\begin{gathered}
\nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi \\
\implies \int_V (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi) dV = \oint_S \phi \, \partial_{\hat{n}} \psi \, da \\
\text{Exchange } \psi, \phi : \\
\int_V (\nabla \phi \cdot \nabla \psi + \psi \nabla^2 \phi) dV = \oint_S \psi \, \partial_{\hat{n}} \phi \, da \\
\text{And subtract from the first equation} \\
\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \oint_S (\phi \, \partial_{\hat{n}} \psi - \psi \, \partial_{\hat{n}} \phi) da
\end{gathered}$$

The equations in purple are Green's first and second identities respectively.

Put $\psi = \frac{1}{|\vec{x} - \vec{x}'|} = 1/R$ into the second identity, and integrate over the primed coordinates. Treat ϕ as an electric potential.

$$\int_V \left[-4\pi\phi(\vec{x}')\delta(\vec{x}-\vec{x}') - \frac{1}{R}\left(-\frac{\rho}{\epsilon_0}\right)\right] d^3\vec{x}' = \oint_S \left(\phi\frac{\partial}{\partial n'}\frac{1}{R} - \frac{1}{R}\frac{\partial\phi}{\partial n'}\right) da'$$

$$\implies \phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{R} d^3\vec{x}' + \frac{1}{4\pi}\oint_S \left(\frac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\frac{1}{R}\right) da'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R} d^3\vec{x}' + \frac{1}{4\pi\epsilon_0}\oint_S \left(\frac{\epsilon_0}{R}\frac{\partial\phi}{\partial n'} - \epsilon_0\phi\frac{\partial}{\partial n'}\frac{1}{R}\right) da'$$

$$\text{Where } \sigma(\vec{x}') = \epsilon_0\frac{\partial\phi}{\partial n'} \text{ is the surface charge density}$$

$$\text{And } \epsilon_0\phi \text{ is the dipole layer strength}$$

Hence this expression includes the potential contributions from both those features.

We take outward flux in Gauss' law, so $\hat{n}'$ is pointing to the outside of the volume.

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Next: Lecture 7 - 444

Uniqueness of Solutions

We'll prove uniqueness of the solution for the different boundary conditions:

1. Dirichlet → ϕ specified on the surface S
2. Neumann → $\phi_{,n}$ specified on S

If we have two solutions, $\phi_{1/2}$, define $U = \phi_1 - \phi_2$, then

- $\nabla^2 U = 0$ inside V
- U or $U_{,n}$ = 0 on S, for Dirichlet and Neumann respectively.

We want to prove either of these:

- $\int |U|^2 dV = 0$ → Much harder to prove
- $\int |\nabla U|^2 dV = 0$ → Put $\phi = \psi = U$ in the first Green's identity:

$$\int (\cancel{U\nabla^2 U} - |\nabla U|^2) dV = \cancel{\oint UU_{,n} da}$$

The last term is 0 because either of $U, U_{,n}$ is 0 on $S$for both boundary conditions. The first term is 0 because $\nabla^2 U$ is 0 inside V.

Hence $\int |\nabla U|^2 dV = 0$, which means $\nabla U = 0 \implies U = \text{constant}$, which we know to be 0 in the Dirichlet case. Hence in both cases the solutions are unique, up to an arbitrary additive constant in Neumann.

Overspecification of the potential

If both $\phi, \phi_{,n}$ are specified arbitrarily on the surface (Cauchy boundary conditions), in general this will not provide a solution, since that would require the unique Dirichlet and Neumann solutions in that case to coincide, which in general will not happen.

But what if the surface was, unlike as we have considered, open? The discussion is far longer and more interesting - so of course we skip it.

Green's Function approach

For the poisson equation, the Green's function is any function which satisfies

$$\begin{aligned}\nabla'^2 G(\vec{x},\vec{x}') &= -4\pi\delta(\vec{x}-\vec{x}') \\ \implies G(\vec{x},\vec{x}') &= \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}'), \nabla'^2 F = 0\end{aligned}$$

With this additional freedom of F, we can try and eliminate one of the surface integrals in our integral expression for the potential so as to use it to solve either of Dirichlet or Neumann problems.

$\phi = \Phi, \psi = G:$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')G(\vec{x},\vec{x}')d^3\vec{x}' + \frac{1}{4\pi}\oint_S \left(G\frac{\partial\phi}{\partial n'} - \phi\frac{\partial G}{\partial n'}\right)da'$$

Dirichlet

$$\begin{aligned}G_D(\vec{x},\vec{x}') &= 0 \forall \vec{x}' \in S \\ \implies \Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')G_D(\vec{x},\vec{x}')d^3\vec{x}' - \frac{1}{4\pi}\oint_S \phi\frac{\partial G_D}{\partial n'}da'\end{aligned}$$

Neumann

$\partial_{\hat{n}'}G_N = 0$ doesn't work, since that makes $\int \nabla G\cdot\hat{n}da = 0$, but that needs to be $\int \nabla G\cdot\hat{n}da = \int \nabla\cdot(\nabla G)dV = -\int 4\pi\delta(\vec{x}-\vec{x}')dV = -4\pi$.

$$\begin{aligned}\partial_{\hat{n}'}G_N &= -\frac{4\pi}{S} \Leftrightarrow \int \partial_{\hat{n}'}G_N da' = -4\pi \\ \implies \Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')G(\vec{x},\vec{x}')d^3\vec{x}' + \frac{1}{4\pi}\oint_S G\frac{\partial\phi}{\partial n'}da' - \underbrace{\int \frac{\phi}{S}da'}_{\langle\phi\rangle_S,\ \text{Generally constant}}\end{aligned}$$

This doesn't completely remove ϕ dependence, arguably.

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Next: Lecture 8 - 444

Electrostatic Potential Energy

$$W = q\Phi$$

$$\Phi_i(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}-\vec{x}_j|}$$

$$W_i = q_i\Phi_i(\vec{x}_i)$$

$$W = \frac{1}{4\pi\epsilon_0}\sum_i\sum_{j<i}\frac{q_iq_j}{|\vec{x}_i-\vec{x}_j|}$$

$$= \frac{1}{8\pi\epsilon_0}\sum_{ij}{}' \frac{q_iq_j}{|\vec{x}_i-\vec{x}_j|}$$

$$\sum{}' \text{ indicates that } i=j \text{ terms are not included}$$

$$\therefore W = \frac{1}{8\pi\epsilon_0}\int d^3\vec{x}\int d^3\vec{x}'\frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{2}\int d^3\vec{x}\rho(\vec{x})\Phi(\vec{x})$$

We want to derive the electrostatic energy density using this - so write ρ in terms of Φ:

$$\rho = -\nabla^2\Phi,$$

$$\nabla\cdot(fA) = f\nabla\cdot A + A\cdot\nabla f$$

$$\therefore \Phi\nabla^2\Phi = \nabla\cdot(\Phi\nabla\Phi) + |\nabla\Phi|^2$$

We like to write integrands as divergences, because Gauss' Theorem then allows us to turn it into a surface integral, which then goes to 0 if we have the surface going to infinity and the integrand going to 0 when it does.

We do that here, and so write

$$W = \frac{1}{2}\epsilon_0 \int d^3\vec{x}|\vec{E}|^2$$
$$u_E = \frac{1}{2}\epsilon_0|\vec{E}|^2$$

One notable issue - or feature - of this expression is that it is positive definite, unlike for the discrete case. It feels likely that this is caused by the other possible issue, that of counting the self-energy of charges. Let's explore this further.

A discrete example

q_i at $\vec{x}_i$, $i = 1, 2$, then

$$|\vec{E}|^2 = \frac{1}{(4\pi\epsilon_0)^2}\left(\underbrace{\sum_{i=1}^{2}\frac{q_i^2}{|\vec{x}-\vec{x}_i|^4}}_{\text{Self-energy terms}} + \underbrace{\frac{2q_iq_2(\vec{x}-\vec{x}_1)\cdot(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_1|^3|\vec{x}-\vec{x}_2|^3}}_{\text{Interaction term}}\right)$$

Let us find the interaction term's energy :

$$\frac{1}{32\pi^2\epsilon_0}2q_1q_2\int d^3\vec{x}\frac{(\vec{x}-\vec{x}_1)\cdot(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_1|^3|\vec{x}-\vec{x}_2|^3}$$
$$\vec{p} := \frac{\vec{x}-\vec{x}_1}{|\vec{x}_1-\vec{x}_2|}, \hat{n} = \frac{\vec{x}_1-\vec{x}_2}{|\vec{x}_1-\vec{x}_2|}, \frac{\vec{x}-\vec{x}_2}{|\vec{x}_1-\vec{x}_2|} = \vec{p}+\hat{n}$$
$$\text{Substitute in the integral, and cancel powers of } |\vec{x}_1-\vec{x}_2| :$$
$$\frac{q_1q_2}{16\pi^2\epsilon_0|\vec{x}_1-\vec{x}_2|}\int d^3\vec{p}\frac{\vec{p}\cdot(\vec{p}+\hat{n})}{|\vec{p}|^3|\vec{p}+\hat{n}|^3}$$
$$= \frac{q_1q_2}{16\pi^2\epsilon_0|\vec{x}_1-\vec{x}_2|}\int d^3\vec{p}\ \nabla_{\vec{p}}\left(\frac{1}{|\vec{p}|}\right)\cdot\nabla_{\vec{p}}\left(\frac{1}{|\vec{p}+\hat{n}|}\right)$$

We use $\nabla\cdot(f\vec{A}) = \vec{A}\cdot\nabla f + f\nabla\cdot\vec{A}$, set $f = \frac{1}{|\vec{p}+\hat{n}|}, \vec{A} = \nabla_{\vec{p}}\left(\frac{1}{|\vec{p}|}\right)$, then the integrand is the$\vec{A}\cdot\nabla f$ term, so we get

$$\int d^3\vec{p}\ \nabla_{\vec{p}}\left(\frac{1}{|\vec{p}|}\right)\cdot\nabla_{\vec{p}}\left(\frac{1}{|\vec{p}+\hat{n}|}\right)$$
$$= \underbrace{\cancel{\int d^3\vec{p}\ \nabla_{\vec{p}}\left(\frac{1}{|\vec{p}+\hat{n}|}\nabla_{\vec{p}}\frac{1}{|\vec{p}|}\right)}}_{\text{Surface term}} - \int d^3\vec{p}\,\frac{1}{|\vec{p}+\hat{n}|}\nabla^2_{\vec{p}}\frac{1}{|\vec{p}|}$$
$$= 4\pi\int d^3\vec{p}\frac{1}{|\vec{p}+\hat{n}|}\delta(|\vec{p}|) = 4\pi$$

Hence the interaction term's energy is

$$\frac{q_1 q_2}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|}$$

Which is the correct expression for the interaction potential energy of two charges. So the interaction term can be negative or positive. But compare with the self-energy terms, and you'll see that the sum cannot be negative. So the reason continuous energy density is positive definite must also be the same - that we're counting self-energy.

Capacitance

Surprise *****s

It's a tensor now

$$\begin{gathered} V_i = P_{ij} Q_j \\ \implies Q_i = C_{ij} V_j \\ C_{ii} \text{ are called capacitances} \\ C_{ij}, i \neq j \text{ are called inductance coefficients} \end{gathered}$$

Lecture 8 - 444

Course	PH 444 - EM Theory
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Next: Lecture 9 - 444

Method of Images

Point charge and grounded conducting plane

q at $(a, 0)$, plane is along $x = 0$, then we can replace the plane by $-q$ at $(-a, 0)$ and obtain the correct potential distribution.

When image charges are considered, they must satisfy a few rules:

1. The potential on the conductor must be 0 (unless non-grounded, which is a much harder case to solve)
2. The field lines must be normal to the conductor.
3. Take advantage of symmetry

Point charge and grounded conducting sphere

While it is straightforward from symmetry arguments and setting the potential at two locations to 0, one could also more formally write $\Phi(\vec{x})$ and then $\Phi(|\vec{x}| = a)$, and hence derive this.

$$q' = -q\frac{a}{y}$$
$$y' = \frac{a^2}{y}$$

▼ Formal proof

Let observation point $\vec{x} = x\hat{n}$, $\vec{y} = y\hat{n}'$, image charge q' at y' along the same radius as q.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$
$$\Phi(|\vec{x}| = a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|a||\hat{n} - \frac{y}{a}\hat{n}'|} + \frac{q'}{|y'||\frac{a}{y'}\hat{n} - \hat{n}'|}\right)$$

This must be 0 for any $\hat{n}, \hat{n}'$, so the $\hat{n}, \hat{n}'$ factors must be equal and the rest of the factors must be separately equal.

$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}} \implies |\hat{n} - c\hat{n}'| = \sqrt{1 + c^2 - 2c\cos\theta} = |c\hat{n} - \hat{n}'|$, so we need the same c for both terms. It is possible to write both terms as $|\hat{n} - c\hat{n}'|$, but that gives $y = y', -q = q'$, which is not the solution we seek. So we write one term as $|c\hat{n} - \hat{n}'|$, and obtain:

$$q' = -q\frac{a}{y}$$
$$y' = \frac{a^2}{y}$$

- Deriving surface charge density

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$
$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{x^2 + y^2 - 2xy\cos\gamma} - \frac{a/y}{x^2 + a^4/y^2 - 2xa\cos\gamma/y}\right)$$
$$\left.\frac{d\Phi}{dx}\right|_{x=a} = -\frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{y^3}(a - y\cos\gamma) - \frac{1}{ay}(1 - a\cos\gamma/y)}{\left[1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right]^{3/2}}\right)$$
$$\sigma = -\epsilon_0\left.\frac{d\Phi}{dx}\right|_{x=a} = -\frac{q}{4\pi a^2}\frac{a}{y}\frac{\left(1 - (a/y)^2\right)}{[1 + (a/y)^2 - 2(a/y)\cos\gamma]^{3/2}}$$

Lecture 9 - 444

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Next: Lecture 10 - 444

Calculating the Dirichlet Green's Function

▼ The problem we're trying to solve

We have a sphere S on which a potential is specified, so Dirichlet boundary conditions, and no charge outside the sphere. We want to know the potential outside the sphere. So

$$\Phi(\vec{x}) = \cancel{\frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')d^3\vec{x}'} - \frac{1}{4\pi}\oint_S \Phi\frac{\partial G_D}{\partial n'}da'$$

Note that n' is inwards here, because our V is the outside of the sphere.

We're trying to use the results from the method of images method to calculate the Dirichlet Green's function required above. We require $G_D(\vec{x},\vec{x}') = 0 \forall \vec{x}' \in S$, and we'll model the Green's function using a potential caused by charge $\rho(\vec{x}) = 4\pi\epsilon_0\delta(\vec{x}-\vec{y})$, where $\vec{y}$ is outside S.

Through the method of images we know the Φ and surface charge distribution due to a point charge outside a grounded sphere. The Green's function has mathematically identical constraints for $q = 4\pi\epsilon_0$, so using last lecture's results we know the Green's function:

$$G_D(\vec{x}, \vec{x}') = \left(\frac{1}{x^2 + x'^2 - 2xx' \cos\gamma} - \frac{a/x'}{x^2 + a^4/x'^2 - 2xa\cos\gamma/x'} \right)$$

$$\frac{dG_D}{dn'} = -\frac{dG_D}{dx'}\bigg|_{x'=a} = -\left(\frac{\frac{1}{a^3}(x - a\cos\gamma) - \frac{1}{xa}(1 - x\cos\gamma/a)}{\left[1 + \frac{x^2}{a^2} - 2\frac{x}{a}\cos\gamma\right]^{3/2}} \right)$$

$$= -\frac{x^2 - a^2}{x(x^2 + a^2 - 2xa\cos\gamma)^{3/2}}$$

Point charge and charged conducting isolated sphere

Use linear superposition for the potentials due to

1. External charge q and induced charge q' on a grounded sphere
2. $Q - q'$ on an isolated sphere with no external charge

Bada-boom bada-bing, you've got yourself a nice little potential. Now just don't try to quantise it or we'll have to deal with QFT fanboys. *shudders*

Point charge near a conducting sphere at potential V

We take the potential from the previous case and impose $\Phi(|\vec{x}| = a) = V$ to obtain Q, then we substitute that back into the potential expression. Hooray.

Lecture 10 - 444

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Next: Lecture 11 - 444

Grounded Conducting Sphere in a Uniform Electric Field

Let $E\hat{x}$ the field, sphere of radius a. We'll simulate the field by $-Q, Q$ at $R, -R$ on the x-axis, with R finally tending to ∞. (This is pretty neat, I like this.)

Then the field at the origin is gonna be

$$E = -\frac{(-Q)}{4\pi\epsilon R^2} + \frac{Q}{4\pi\epsilon R^2} = \frac{2Q}{4\pi\epsilon R^2}$$

Which we want to be constant as $R \to \infty$, so have $Q \to \infty$ as R^2 - we want Q/R^2 to remain constant.

The image charges due to these will be $\pm Qa/R$ located at $\pm a^2/R$

Aight, grounded sphere, take a point $\vec{r}$ making θ with $\hat{x}$, calculate the potential, take the limit. Lessgo.

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{\sqrt{r^2 + R^2 - 2rR\cos(\pi-\theta)}} - \frac{Q}{\sqrt{r^2+R^2-2rR\cos\theta}}\right.$$
$$\left. + \frac{Qa/R}{\sqrt{r^2 + (a^2/R)^2 - 2ra^2/R\cos\theta}} - \frac{Qa/R}{\sqrt{r^2 + (a^2/R)^2 - 2(ra^2/R)\cos(\pi-\theta)}}\right]$$

The θ terms are on the positive x axis, which are $-Q, Qa/R$, and the others on the negative x axis.

We know Q/R^2, so Taylor expand everything to obtain terms like Q/R^2:

$$\Phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0 R}\left[\left(\cancel{1} - \frac{r}{R}\cos\theta - \cancel{\frac{1}{2}\frac{r^2}{R^2}}\right) - \left(\cancel{1} + \frac{r}{R}\cos\theta - \cancel{\frac{1}{2}\frac{r^2}{R^2}}\right)\right.$$
$$\left. + \frac{a}{r}\left(\cancel{1} + \frac{a^2}{rR}\cos\theta - \cancel{\frac{1}{2}\frac{a^4}{r^2R^2}}\right) - \frac{a}{r}\left(\cancel{1} - \frac{a^2}{rR}\cos\theta - \cancel{\frac{1}{2}\frac{a^4}{r^2R^2}}\right)\right]$$
$$= \frac{Q}{4\pi\epsilon_0 R}\left(-\frac{2r}{R}\cos\theta + \frac{2a^3}{r^2R}\cos\theta\right)$$
$$= E_0\left(\frac{a^3}{r^2} - r\right)\cos\theta$$

Solving Problems using Expansions

Tl;dr : Don't encourage DLCs. But get this one.

Expanding the solution of a problem in terms of a class of functions is sometimes useful in obtaining the solution.

If we want a solution in $x \in (a, b)$ with some given boundary conditions, an expansion will consist of a class of functions $\{U_n(x)\}$ which satisfy the following nice properties:

1. Orthogonality : $\int_a^b U_n^*(x)U_m(x)dx = 0 \text{ at } n \neq m \text{ only}$
2. Completeness : $\sum_{n=1}^{\infty} U_n^*(x')U_n(x) = \delta(x - x')$

This is an interesting way of defining completeness. Jackson has a small semi-rigorous proof on how this translates to the intuitive idea of completeness.

The classic example - Fourier expansion

$x \in (-a, a)$, then our functions are $\sqrt{\frac{1}{a}}\sin\left(\frac{\pi m x}{a}\right)$, $\sqrt{\frac{1}{a}}\cos\left(\frac{\pi m x}{a}\right)$, with the scale factor to enforce orthonormality, which is not necessary but useful. Then

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m x}{a}\right) + B_m \sin\left(\frac{\pi m x}{a}\right)$$

Now this is the discrete 1D case. Extension to multiple dimensions is straightforward as $f(x, y) = \sum_{m,n} a_{mn}U_n(x)V_m(y)$.

Extension to the continuous case is slightly compex (get it?), but at this point we should all be familiar with it. Briefly, we let $\pi m/a \to k$, a continuous variable, and then the sums become integrals, maths happens, and out pops the famed **fourier transform**.

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Next: Lecture 12 - 444

Fourier Transform

Again.

$\frac{\pi m}{a} \rightarrow k, \Re(A(k)) = \sqrt{\frac{a}{\pi}} A_m, \sum_m \rightarrow \int dm = \frac{a}{\pi} \int dk$

And complexify $A(k)$ to include the $\sin$ term coefficients as well. Then

$$A(k) = \frac{1}{\sqrt{2\pi}} \int dx f(x) e^{-ikx}$$

I've greatly skipped over details, but then so did Sir. If you wish to revise the Fourier transform in detail, there are excellent resources one google search away.

Laplace Equation (Rectangular Coordinates)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, y, z) = 0$$

Variable separation - assume, find partial solutions, use the completeness of these solutions to construct general solutions.

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$
$$\implies \frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2$$
$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -\beta^2$$
$$\frac{1}{Z} \frac{d^2 Z}{zx^2} = \gamma^2 = \alpha^2 + \beta^2$$
$$\therefore \Phi \sim e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}$$

Typically α, β are not constants - another way to say that variable separation is not a valid solution in those cases. But then we simply decompose the boundary conditions into cases where it is, and sum over them - practically, summing over whereas α, β. The rigorousness of all this arises through the comlpeteness of a class of functions to expand in, specifically the fourier expansion/transform.

A brief solution of a standard example

Box of $a \times b \times c$, potential 0 on all sides except the $z = c$ face which has $V(x, y)$. Incorporating the rest of the boundary conditions, our solutions can be:

$$X = \sin \alpha x, \alpha_n = n\pi/a$$
$$Y = \sin \beta y, \beta_m = m\pi/b$$
$$Z = \frac{1}{2}(e^{\sqrt{\alpha^2+\beta^2}z} - e^{-\sqrt{\alpha^2+\beta^2}z})$$

With the last one being easy to understand by decomposing as $\sinh, \cosh$.

Obviously this doesn't satisfy the last boundary condition, but we take the Φ_{nm} this gives us and expand our solution in terms of Φ_{nm} terms such that the boundary condition be satisfied.

$$\Phi(x, y, z) = \sum_{nm} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh\left(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}z\right)$$
$$\gamma_{nm} := \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

Use orthogonality:

$$\int_0^a \sin(\alpha_n x) \sin(\alpha_m x) dx = \frac{a}{2}\delta_{nm}$$

Then at $z = c$, multiplying by appropriate functions and integrating,

$$\int_0^b \int_0^a V(x, y) \sin(\alpha_n x) \sin(\beta_m y) dx dy = A_{nm} \frac{a}{2}\frac{b}{2} \sinh(\gamma_{nm} c)$$

Which gives us every A_{nm}.

Laplace Equation (Spherical Coordinates)

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$
$$\Phi(r,\theta,\phi) = \frac{U(r)}{r}P(\theta)Q(\phi)$$

Which is a variable-separation ansatz selected due to hindsight. Some algebra (again, a standard and well-documented result) gives the following equations:

$$\frac{1}{U}\frac{d^2U}{dr^2} = \frac{l(l+1)}{r^2}$$
$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = m^2$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

Where we've given the constants $l(l+1), m^2$ particular forms due to hindsight.

The solutions of these maddening equations, then:

$$U = Ar^{l+1} + Br^{-l}$$

The DE for P can be rewritten with $\cos\theta = x$ to obtain a DE which has known solutions in the *Associated Legendre polynomials*.

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P = 0$$

The ordinary Legendre equation would have $m = 0$, and would have the Legendre polynomials as solutions:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Some important identities:

$$\text{Rodrigues formula}: P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$\text{Orthogonality}: \int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{lm}$$

Lecture 12 - 444

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Next: Lecture 13 - 444

Laplace's Equation in Spherical Coordinates

For the special case of azimuthal symmetry, we need merely the Legendre and not the associates Legendre polynomials, and the solution is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l R^{-(l+1)})P_l(\cos\theta)$$

Examples:

▼ Azimuthally symmetric spherical potential

$V(\theta)$ is the potential on a sphere, find potential inside, no ρ.

$$\begin{aligned} B_l &= 0 \text{ for nice behaviour at } r = 0 \\ V(\theta) &= \Phi(a,\theta) \\ \implies A_l &= \frac{1}{a^l}\frac{2l+1}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)V(\theta) \end{aligned}$$

Say

$$V(\theta) = \begin{cases} +V & 0 \le \theta < \frac{\pi}{2} \\ -V & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

Then we use two identities, the first being that Legendre polynomials are odd or even when l is odd or even. The second is

$$\int_0^1 P_l(x)dx = \frac{1}{2l+1}(P_{l+1}(x) - P_{l-1}(x))\Big|_0^1 = \frac{1}{2l+1}(P_{l-1}(0) - P_{l+1}(0))$$

Since $P_l(1) = 1$. This is derived from the tut problem from last week, the identity

$$\frac{d}{dx}(P_{l+1}(x) - P_{l-1}(x)) = (2l+1)P_l(x)$$

Then

$$A_l = \begin{cases} 0 & l \text{ is even} \\ \frac{V}{a^l}(P_{l-1}(0) - P_{l+1}(0)) & l \text{ is odd} \end{cases}$$

We also have another identity we can use,

$$P_l(0) = \begin{cases} \frac{(-1)^{l/2}}{2^l}\binom{l}{l/2} & l \text{ is even} \\ 0 & l \text{ is odd} \end{cases}$$

Thus we explicitly obtain every coefficient.

▼ Express $\frac{1}{|\vec{x}-\vec{x}'|}$ in an expansion of $P_l(\cos\gamma)$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

In general, find the potential on the symmetry axis. If it looks like $A_l r^l + B_l r^{-(l+1)}$, then anywhere away from the axis, merely multiplying each term by $P_l(\cos\gamma)$ will give the complete result, because of the uniqueness of the solution and the orthogonality of the Legendre polynomials.

For $\frac{1}{|\vec{x}-\vec{x}'|}$, when we're on the symmetry axis ($\gamma = 0$), it is simply $\frac{1}{|r-r'|}$ (for convenience we're defining z along $\vec{x}'$). Then

$$\frac{1}{|r-r'|} = \begin{cases} \frac{1}{r}|1 - r'/r|^{-1} = \frac{1}{r}\sum_{l=0}^{\infty}\frac{r'^l}{r^l} & r' < r \\ \frac{1}{r'}|1 - r/r'|^{-1} = \frac{1}{r'}\sum_{l=0}^{\infty}\frac{r^l}{r'^l} & r' > r \end{cases}$$

Then the result is straightforward.

Note: $r_> := \max(r, r'),\ r_< := \min(r, r')$

▼ Potential due to ring

Charge q, radius a, height b, z is the symmetry axis. Potential on the z-axis:
(Cylindrical coordinates, so $\hat{\rho}$ is radial in x-y)

$$r > c = \sqrt{a^2 + b^2} :$$

$$\Phi(z = r) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|r\hat{k} - b\hat{k} - a\hat{\rho}|} = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{(r-b)^2 + a^2}}$$

$$\tan\alpha = \frac{a}{b}, \text{ then write in } c,$$

$$\Phi(r\hat{k}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{c^2 + r^2 - 2cr\cos\alpha}} = \frac{q}{4\pi\epsilon_0} \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{c}{r}\right)^l P_l(\cos\alpha)$$

$$r < c : \Phi(r\hat{k}) = \frac{q}{4\pi\epsilon_0} \frac{1}{c} \sum_{l=0}^{\infty} \left(\frac{r}{c}\right)^l P_l(\cos\alpha)$$

$$\text{Hence } \Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$$

Lecture 13 - 444

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Next: Lecture 14 - 444

Next Extra: Quiz 1 Prep

Associated Legendre Polynomials & Spherical Harmonics

When azimuthal symmetry is broken, the complete equation for P is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)P=0$$
$$\implies \frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)P=0$$

The solutions to these equation are known as the Associated Legendre Polynomials - which, for $|x|<1$, require $l\in\mathbb{Z}$, $|m|\leq l$ and also an integer.

For positive m, we have (similar to the Rodrigues formula)

$$P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$
$$=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{m+l}}{dx^{m+l}}(1-x^2)^l$$

The last expression is actually valid for both positive and negative m.

For a negative m, we can relate to the positive m solutions neatly as

$$P_l^{-m}=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

And they have the orthongonality relation

$$\int_{-1}^{1} P_l^m(x) P_{l'}^m(x)\, dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

We would like to combine the θ, ϕ parts to construct fully orthonormal solutions on the unit sphere, which we will call the **Spherical Harmonics**.

$$\begin{gathered}
Q_m = e^{im\phi}, 0 \le \phi < 2\pi \\
Y_{lm}(\theta,\phi) = N_{lm} P_l^m(\cos\theta) Q_m(\phi) \\
\text{Want} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \\
\implies N^*_{l'm'} N_{lm} \int_0^{2\pi} d\phi\, e^{i(m-m')\phi} \int_0^{\pi} d\theta \sin\theta P_l^m(\cos\theta) P_{l'}^{m'}(\cos\theta) = \delta_{ll'}\delta_{mm'} \\
\implies |N_{lm}|^2 \cdot 2\pi \cdot \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} = 1 \\
\therefore N_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{l+m)!}}
\end{gathered}$$

Note that there are other definitions of spherical harmonics, sometimes having further negative signs, sometimes switching negative signs between P and N_{lm}, we will stick to this particular normalisation and organisation.

Some other properties:

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}(\theta,\phi)$$

Completeness

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{ml}(\theta',\phi') Y_{ml}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$$

Examples:

$$\begin{gathered}
Y_{00} = \frac{1}{\sqrt{4\pi}}, Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}, Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)
\end{gathered}$$

And now we can finally write the complete solution to the laplace equation in spherical coordinates,

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) Y_{lm}(\theta,\phi)$$

Addition Theorem for Spherical Harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)$$

Where $\hat{n}' \cdot \hat{n} = \cos\gamma$.

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Continuing where we left off,

Addition Theorem for Spherical Harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Where $\hat{n}' \cdot \hat{n} = \cos\gamma$.

▼ Proof

For a particular l, m, $\Phi_{lm}(r,\theta,\phi) = R_l(r)Y_{lm}(\theta,\phi)$ will satisfy

$$\nabla^2\Phi_{lm} = 0$$
$$\implies \left[\frac{l(l+1)}{r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = 0$$
$$\implies r^2\nabla^2 Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

(Substitute the radial part of $\Phi_l(r,\theta,\phi)$ into the laplace equation.)

Consider that $P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_{l0}(\theta)$ also satisfied this equation.

Note that ∇^2 is the same in rotated coordinate frames - an intuition for this is that it has a dot product form, $\nabla \cdot \nabla f$, and dot products are invariant under rotations. This means that taking derivatives with respect to polar and azimuthal angles wrt some other axis would be the same operator overall.

Consider vectors $\vec{x}, \vec{x}'$, and we start in the rotated coordinate frame (call it S_1) with $\vec{x}'$ being the z-axis, $\vec{x}$ being at angles γ, β in this frame. In our original frame S_0 they have angles given by primed and unprimed θ, ϕ. Derivatives in frame S_i are written as ∇_i^2.

$$
\begin{aligned}
r^2\nabla_1^2 P_l(\cos\gamma) &= -l(l+1)P_l(\cos\gamma) \\
\implies r^2\nabla_{\mathbf{0}}^2 P_l(\cos\gamma) &= -l(l+1)P_l(\cos\gamma) \\
\implies P_l(\cos\gamma) &= \sum_{m=-l}^{l} b_{lm}(\theta', \phi')Y_{lm}(\theta, \phi)
\end{aligned}
$$

Note While b_{lm} is different for different l, we avoid the subscript l to avoid confusion.

Make the small note that rotating back into S_0 is the action which brings $\vec{x}'$ back to θ', ϕ'.

That last implication comes from two facts. The first is that equation 2 implies that $P_l(\cos\gamma)$ must be a linear combination of the spherical harmonics of l as functions of $\theta, \phi \to Y_{lm}(\theta, \phi)$, because they are known to satisfy that differential equation (which has ∇_0^2, so derivative with respect to $\vec{x}$'s angles in S_0, which are θ, ϕ).

Then we gave the coefficients primed angle dependence because γ is a function of the primed and unprimed angles and the harmonics will not include the primed angle dependence, so it has to go somewhere. Now try to evaluate these coefficients. Orthogonality gives:

$$
b_m(\theta', \phi') = \int P_l(\cos\gamma)Y_{lm}^*(\theta, \phi)d\Omega
$$

Now we can expand an arbitrary function in spherical harmonics, so why not spherical harmonics themselves?

If we expand $\sqrt{\frac{4\pi}{2l+1}}Y_{lm}^*(\theta, \phi)$ in the spherical harmonics $Y_{l'm'}(\gamma, \beta)$, then $b_m(\theta', \phi') = A_{l,0}$ (l,0th coefficient).

$$
f(\theta, \phi) = \sqrt{\frac{4\pi}{2l+1}}Y_{lm}^*(\theta, \phi) = \sum_{l',m'} A_{l'm'}Y_{l'm'}(\gamma, \beta)
$$

Now note that $A_{l,0}$ is independent of θ, ϕ (or γ, β, considering either pair is valid), so if we find $A_{l,0}$ at $\gamma = 0$ $(\theta, \phi = \theta', \phi')$ we'll have our proof done.

It's easy to derive that the expansion of a general function in $Y_{l'm'}(\gamma,\beta)$ reduces to the following when evaluated at $\gamma = 0$: ($P_l^m(1) = 0$ unless $m = 0$, then it's 1)

$$f(\theta,\phi)|_{\gamma=0} = \sum_{l'=0}^{\infty} \sqrt{\frac{2l'+1}{4\pi}} A_{l',0}$$

We substitute $f(\theta,\phi)|_{\gamma=0} = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta,\phi)|_{\gamma=0} = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta',\phi')$, and also note that $Y^*_{lm}(\theta,\phi)$ is just a rotated spherical harmonic as viewed in γ,β coordinates, and since the spherical harmonics for a particular l form an irreducible representation of the rotation group, they do not mix with other ls under rotation, only mix the ms. So we can be sure that the expansion of $Y^*_{lm}(\theta,\phi)$ will only have $l' = l$ terms. Thus

$$\begin{aligned}
\sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta',\phi') &= \sqrt{\frac{2l+1}{4\pi}} A_{l,0} \\
\implies b_m(\theta',\phi') &= \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') \\
\implies P_l(\cos\gamma) &= \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \\
\implies \frac{1}{|\vec{x}-\vec{x}'|} &= \sum_{l=0}^{\infty} \frac{r_>^l}{r_<^{l+1}} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)
\end{aligned}$$

Hence proved.

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- A bit about decomposing $\frac{1}{|\vec{x}-\vec{x}'|}$ by expanding it in the Legendre polynomials and then using the addition formula - useful when we can't have $\vec{x}$ define the z-axis, maybe the system has fixed coordinates or a different symmetry we also need to exploit, etc.

Laplace Equation in Cylindrical Coordinates

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\right)\Phi(\rho,\phi,z)=0$$
$$\text{Let } \Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$
$$\implies Z(z)=e^{\pm kz}, Q(\phi)=e^{\pm i\nu\phi}$$

ν must be an integer when the full azimuth is allowed, but if only solving over a strict subset of $[0,2\pi)$, then different coundary conditions must be used to constrain ν.

As for k, while we often work with a real k, it could be imaginary as well. Assuming real here, dealing with the imaginary later,

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\left(k^2-\frac{\nu^2}{\rho^2}\right)\right)R(\rho)=0$$

The solutions to this equation (for $\nu\in\mathbb{R}$) are the famous Bessel functions.

Bessel Functions

(Note : The details of the coefficients are not as relevent to us, focus on the general behaviour. Also, for the purposes of these coefficients, we have assumed $k = 1$. We can obtain the general form by putting $x \to kx$ in these formulae.)

Obtained by assuming a solution of the form $R(x) = x^\alpha \sum_{i=0}^{\infty} a_i x^i$. Substitute and obtain $\alpha = \pm\nu$, and $a_{2j} = -\frac{1}{4j(j+\alpha)} a_{2j-2}$. The coefficients of the odd powers vanish. Thus:

$$a_{2j} = \frac{(-1)^j \Gamma(\alpha+1)}{2^{2j} j! \Gamma(j+\alpha+1)} a_0$$
$$\text{Conventionally, } a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}$$

Then for the two values of α, we get two Bessel functions of the first kind of order ν.

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$
$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{i=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

These converge for all finite x, and are linearly independent if and only if ν is not an integer → $J_{-m}(x) = (-1)^m J_m(x)$ for integers.

Neumann Function

AKA the Bessel function of the second kind is defined as

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

Why introduce this? Because we want something to replace $J_{-\nu}$ that is linearly independent even as $\nu \to m \in \mathbb{Z}$. When $\nu = m$, then the Neumann function is not defined because there's a division by 0.

In summary, it's a useful replacement for $J_{-\nu}$.

Hankel Functions

Another variety of recombinations of the Bessel functions, still a solution to the Bessel equation, and known as the Bessel functions of the third kind,

$$H_\nu^{(1)}(x) = J_\nu(x) + \iota N_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - \iota N_\nu(x)$$

Limiting behaviour

Recall that we could use knowledge of which terms blow up in certain limits to set their coefficients to 0, eg with the radial parts of the laplace solution in spherical polar coordinates. There it was obvious - here we will have to investigate this to find out where we can use this.

Assuming $\nu \in \mathbb{R}, \geq 0$,

$$x \ll 1 : J_\nu(x) \to x^\nu \tag{1}$$
$$N_\nu(x) \to \begin{cases} \ln x & \nu = 0 \\ x^{-\nu} & \nu \neq 0 \end{cases} \tag{2}$$
$$x \gg 1 : J_\nu(x) \to \sqrt{\frac{1}{x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{3}$$
$$N_\nu(x) \to \sqrt{\frac{1}{x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{4}$$

Properties

Roots of a Bessel function

The asymptotic form makes clear that there are infinite roots for $J_\nu(x)$ - label them $x_{\nu n}, n \in \mathbb{N}$.

Orthogonality

$$\int_0^a \rho J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}(J_{\nu+1}(x_{\nu n}))^2 \delta_{nn'}, \nu \geq -1$$

Now this is a very unusual orthogonality, because it is not orthogonal over the bessel function index ν but rather the root index n. Also note that the RHS is not zero because $x_{\nu n}$ is not a root of $J_{\nu+1}$.

Utilisiing Bessel Functions

Thus one can expand a function satisfying Dirichlet boundary conditions as $f(0) = f(a) = 0$ in the ν^{th} Bessel function as

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu \left(x_{\nu n} \frac{\rho}{a} \right)$$
$$A_{\nu n} = \frac{2}{a^2 (J_{\nu+1}(x_{\nu n}))^2} \int_0^a \rho f(\rho) J_\nu \left(x_{\nu n} \frac{\rho}{a} \right) d\rho$$

Now if we're given Neumann boundary conditions, $\frac{\partial f}{\partial \rho} = 0$, at the endpoints $0, a$, then we can replace $x_{\nu n}$s by $y_{\nu n}$s, which are the roots of the derivatives of the Bessel functions. The derivatives also have certain interesting properties, look them up.

Modified Bessel Functions

For imaginary $k = i\kappa$, replace k^2 by $-\kappa^2$, and then the solutions are the Modified Bessel functions.

$$\left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \left(1 + \frac{\nu^2}{x^2} \right) \right) R(x) = 0$$
$$I_\nu = i^{-\nu} J_\nu(ix)$$
$$K_\nu = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

Limiting Behaviour

$$x \ll 1 : I_\nu(x) \to x^\nu \tag{5}$$
$$K_\nu(x) \to \begin{cases} -\ln x & \nu = 0 \\ x^{-\nu} & \nu \neq 0 \end{cases} \tag{6}$$
$$x \gg 1 : I_\nu(x) \to \frac{e^x}{x} \tag{7}$$
$$K_\nu(x) \to \frac{e^{-x}}{\sqrt{x}} \tag{8}$$

▼ Example

A right cylinder on the x-y plane, axis along z, given that the potential is 0 everywhere except the top face having $V(\rho, \phi)$. Radius a, height L.

Within the cylinder, what's the potential?

Using 0 boundary conditions, we get $Z_k(z) = \sinh kz, Q_m(\phi) = A \sin m\phi + B \cos m\phi, m \in \mathbb{Z}, R_{km} = CJ_m(k\rho) + \cancel{DN_m(k\rho)}$ (because the potential is finite at $\rho = 0$). Furthermore, to have zero potential on the curved surface, $J_m(ka) = 0 \implies k_{mn} = x_{mn}/a$.

Then we have

$$\Phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z)(A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

Continued in the next lecture.

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▼ Continuing the example from the last lecture,

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi)$$

At $z = L$, we have $\Phi = V(\rho,\phi)$. Then

$$A_{mn} = \frac{1}{\pi}\frac{1}{\sinh k_{mn}L}\frac{2}{a^2(J_{m+1}(ak_{mn}))^2}\int_0^a \rho J_m\left(k_{mn}\rho\right)d\rho \int V(\rho,\phi)\sin m\phi\, d\phi$$

$(k_{mn} = x_{mn}/a)$

And similarly we can evaluate B_{mn}.

Multipole Expansion

We start with some localised $\rho(\vec{x})$ and find a useful expansion for the potential at a distance R much larger than the region occupied by the charge.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$$

$$\text{Using } \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\text{(Obtained from the addition theorem after expanding 1/r in } P_l(\cos\theta))$$

$$\therefore\ \Phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\underbrace{\left[\int (r')^l Y^*_{lm}(\theta',\phi')\rho(\vec{x}')d^3\vec{x}'\right]}_{q_{lm}}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

We'll write some q_{lm} because kaam dhandha to hai nahi.

$$q_{00} = \frac{q}{\sqrt{4\pi}}$$
$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_x - \iota p_y)$$
$$q_{10} = \sqrt{\frac{3}{4\pi}}p_z$$
$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - Q_{22} - 2\iota Q_{12})$$
$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - \iota Q_{23})$$
$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{33}$$
$$\text{Where } Q_{ij} = \int (3x'_i x'_j - r'^2\delta_{ij})\rho(\vec{x}')d^3\vec{x}'$$

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Multipole Expansion

- $2l+1$ spherical multipoles per l
- Finer angular variations at higher orders

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{x}') d^3\vec{x}'$$
$$Y_{l,-m} = (-1)^m Y^*_{lm} \implies q_{l,-m} = (-1)^m q^*_{lm}$$

Cartesian Version

Using $\vec{r} = \vec{x} - \vec{x}'$,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (\vec{x}' \cdot \nabla_r)^l \frac{1}{r}\bigg|_{\vec{r}=\vec{x}}$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (\vec{x}' \cdot \nabla)^l \frac{1}{|\vec{x}|}$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \underbrace{(x'_{i_1} \dots x'_{i_l})}_{U} \cdot \underbrace{(\partial_{i_1} \dots \partial_{i_l})(\frac{1}{|\vec{x}|})}_{V}$$

Then U, V are two l-rank tensors being contracted together to give a scalar. They are both symmetric.

Recall that a second-rank tensor is reducible to the spherical tensors colloquially known as the scalar, the vector (antisymmetric tensor) and the symmetric traceless tensor. Similarly we can reduce an l-rank tensor into irreducible representations.

The trace of a tensor is generally not one scalar, but refers to all possible contractions of pairs of indices. So for a tensor V_{ijk}, the contractions $V^i{}_{ik}, V_k{}^i{}_i, V_{ik}{}^i$ for all k are the traces. Traceless means all these contractions should be 0. Because we're working in a region with large $r = |\vec{x} - \vec{x}'|$, V is traceless:

$$\mathrm{Tr}(V) = \partial_{i_3,r} \dots \partial_{i_l,r} \nabla_r^2 \frac{1}{r} = 0$$

This refers to all the traces because V is symmetric.

Also, for rank 2 matrices it is easy to show that $\sum_{ij} U_{ij} V_{ij} = \mathrm{Tr}(UV^T)$. The product we want to evaluate is $\sum_{i_1 \dots i_l} U_{i_1 \dots i_l} V_{i_1 \dots i_l}$, so we will assume that the formula extends to higher ranks (a formal proof is way beyond the scope of this course, but I will try to attach a link to one if I can find it).

Now note that we can simplify $\mathrm{Tr}(UV^T)$ using spherical tensor decomposition. The product of a scalar with a symmetric traceless or antisymmetric tensor is always traceless.

Furthermore, for an antisymmetric tensor multiplied by a symmetric tensor, when taking the trace (which is the contraction of two indices), if we exchange those indices in both tensors the antisymmetric tensor gives a total minus sign which means the trace of this product equals its own negative, and is hence 0.

(The above proof is non-rigorous since we haven't specified how we're multiplying the two tensors - it has to be some contraction, but is it just one contraction or multiple? Sir did not specify and this extension of the formula seems to be non-standard. Will ask Sir about this. ***doubt***)

Thus

$$\begin{gathered} U = \underbrace{U^{(0)}}_{\text{scalar}} + \underbrace{U^{(1)}}_{\text{antisym}} + \underbrace{U^{(2)}}_{\text{sym,tr-less}} \\ \mathrm{Tr}(UV^T) = \mathrm{Tr}(U^{(0)}V^{(0)}) + \mathrm{Tr}(U^{(1)}V^{(1)T}) + \mathrm{Tr}(U^{(2)}V^{(2)}) \\ U^{(1)} = 0, V^{(0)} = 0, V^{(1)} = 0, \\ \therefore \mathrm{Tr}(UV^T) = \mathrm{Tr}(U^{(2)}V^{(2)}) \end{gathered}$$

So we need to evaluate the symmetric tracless part of U to obtain the product we want. We do that as:

$$\begin{gathered} U_{ij}^{(2)} = U_{ij} - \delta_{ij} \frac{\mathrm{Tr}(U_{ij})}{3} = x_i' x_j' - \frac{1}{3} r'^2 \delta_{ij} = \frac{r'^5}{3} V_{ij}(x') \\ U_{ijk}^{(2)} = x_i' x_j' x_k' - \frac{1}{3} (x_i' \delta_{jk} + x_j' \delta_{ki} + x_k' \delta_{ij}) r'^2 \end{gathered}$$

Looking at $l = 2$,

$$V_{ij}(x) = \partial_i \partial_j \left(\frac{1}{|\vec{x}|} \right) = \partial_i \left(\frac{-1}{|\vec{x}|^2} \cdot \frac{x_j}{|\vec{x}|} \right) = \frac{-1}{|\vec{x}|^3} \delta_{ij} + 3 \frac{x_j}{|\vec{x}|^4} \frac{x_i}{|\vec{x}|} = \frac{3x_i x_j - |\vec{x}|^2 \delta_{ij}}{|\vec{x}|^5}$$

In general, let the symmetric traceless part of U be written as $[x'_{i_1} \dots x'_{i_l}]_{\mathrm{ST}}$, then

$$V_{i_1 \dots i_l} = (-1)^l \frac{(2l-1)!!}{|\vec{x}|^{2l+1}} [x_{i_1} \dots x_{i_l}]_{\mathrm{ST}}$$

It is likely simple to show this using induction, though I have not attempted that yet.

Thus we can write the Cartesian multipole expansion as

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \frac{1}{|\vec{x}|^{2l+1}} [x_{i_1} \dots x_{i_l}]_{\mathrm{ST}} [x'_{i_1} \dots x'_{i_l}]_{\mathrm{ST}}$$

And the cartesian multipole moments are

$$Q_{i_1 \dots i_l} = (2l-1)!! \int d^3x' \rho(\vec{x}') [x'_{i_1} \dots x'_{i_l}]_{\mathrm{ST}}$$

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Equivalence of the cartesian and spherical multipole moments

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \frac{1}{|\vec{x}|^{2l+1}} [x_{i_1} \dots x_{i_l}]_{\text{ST}} [x'_{i_1} \dots x'_{i_l}]_{\text{ST}}$$

$$Q_{i_1 \dots i_l} = (2l-1)!! \int d^3x' \rho(\vec{x}') [x'_{i_1} \dots x'_{i_l}]_{\text{ST}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{[x_{i_1} \dots x_{i_l}]_{\text{ST}}}{|\vec{x}|^{2l+1}} Q_{i_1 \dots i_l}$$

Now one may question whether there are as many cartesian moments Q per l as there are spherical moments, which are $2l+1$. Let's see.

First looking at $[x'_{i_1} \dots x'_{i_l}]_{\text{ST}}$, dividing the power l among x, y, z gives $\binom{l+2}{2}$ possibilities.

Tracelessness gives $\binom{l}{2}$ constraints, so subtracting those, we get $2l+1$, as expected.

Expanding Spherical Harmonics

We can expand the spherical harmonics in the spherical cartesian basis as

$$r^l Y_{lm}(\theta, \phi) = \sum_{m_1, m_2, m_3} a(m_1, m_2, m_3)(x+iy)^{m_1}(x-iy)^{m_2} z^{m_3}$$

$$m_1 - m_2 = m, m_1 + m_2 + m_3 = l$$

Electric Field of a Dipole

Say we have a dipole,

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{Q^{(0)}}{r} + \underbrace{\frac{Q_i^{(1)}}{r^3}x_i}_{\frac{\vec{p}\cdot\vec{x}}{r^3}\text{, Dipole moment term}} + \frac{1}{2!}\frac{Q_{ij}^{(2)}}{r^5}x_i x_j + \dots\right]$$

$$\therefore \vec{E}_{\text{Dipole}}(\vec{r}) = -\nabla\frac{\vec{p}\cdot\vec{x}}{r^3} = -\vec{p}\cdot\nabla\frac{\vec{x}}{r^3} - \cancel{\vec{p}\times\left(\nabla\times\frac{\vec{x}}{r^3}\right)}$$
$$= p_i\partial_i\partial_j\frac{1}{r}$$

We can use either this, or the derivation given in Jackson, to derive

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{x}(\hat{x}\cdot\vec{p}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\delta(\vec{x}-\vec{x}_0)\right]$$

This will be a tutorial problem.

Multipole Expansion of Energy and Charge Density

(Assuming an external field)

$$W = \int \rho(\vec{x})\Phi_{\text{ext}}(\vec{x})d^3\vec{x}$$
$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2!}\sum_{ij} x_i x_j\frac{\partial E_j}{\partial x_i}(0) + \dots$$
$$\text{Subtract } \frac{r^2}{3}\nabla\cdot\vec{E}(0),$$
$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij}(3x_i x_j - r^2\delta_{ij})\frac{\partial E_j}{\partial x_i}(0) + \dots$$
$$\implies W = q\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \dots$$

Why is that subtraction valid? Well, if we assume the external field is not caused by charges present in the same region as the charge distribution we are calculating this for (which is where 0 is, the origin is the centre of charge), then the term we subtracted is 0.

Lecture 19 - 444

Course	PH 444 - EM Theory
Date	@February 16, 2023
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Next: Lecture 20 - 444

Next extra: Midsem prep - 444

Electrodynamics

Potential Formulation of Maxwell Equations

A brief derivation of the scalar and vector potential formulation of Maxwell's equations:

$$\nabla \cdot \vec{B} = 0 \implies \vec{B} = \nabla \times \vec{A}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$
$$\implies \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

Thus the source-free Maxwell equations have justified the formulation of a scalar and vector potential. Now we use the sourced equations to get some interesting results:

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \implies -\nabla^2 \Phi - \frac{\partial(\nabla \cdot \vec{A})}{\partial t} = \rho/\epsilon_0$$
$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$
$$\implies \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J}$$

These equations are a tad too coupled for us, especially since we're only just getting over Valentine's.

Gauge Freedom

We note that $\vec{A} + \nabla\Lambda$ would give the same $\vec{B}$ as $\vec{A}$, because a gradient's curl is 0. It seems like we have an extra degree of freedom here. But we need to ensure the electric field is unchanged as well, which we might be able to do if we change Φ in a certain way.

$$\vec{A}' = \vec{A} + \nabla\Lambda \implies \vec{B}' = \nabla \times \vec{A}' = \nabla \times \vec{A} = \vec{B}$$
$$\text{Want } \vec{E} = \vec{E}'$$
$$\implies -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi' - \frac{\partial \vec{A}'}{\partial t}$$
$$\implies \nabla\Phi' = \nabla\Phi - \frac{\partial \nabla\Lambda}{\partial t}$$
$$\implies \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

Hence we have the gauge freedom to change the potentials by the gradient and negative time derivative (respectively) of any scalar field.

Decoupling using the Gauge Freedom - Lorentz Gauge

We wanna set $\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$ (refer above), which means we shift to a gauge where it is so:

$$\nabla \cdot \vec{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0$$
$$\therefore \underbrace{\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\nabla \cdot \vec{A} - \frac{1}{c^2}\frac{\partial \Phi}{\partial t}}_{\text{Lorentz Gauge!}}$$

Using this transformation on the sourced potential equations above, we write

$$\nabla^2\Phi' - \frac{1}{c^2}\frac{\partial^2 \Phi'}{\partial t^2} = -\rho/\epsilon_0$$
$$\nabla^2 A' - \frac{1}{c^2}\frac{\partial^2 A'}{\partial t^2} = -\mu_0 \vec{J}$$

The decoupled wave equations arising from the lorentz gauge.

Coulomb Gauge

Another commonly used gauge, we set $\nabla \cdot \vec{A}' = 0 \implies \nabla \cdot \vec{A} = -\nabla^2\Lambda$.

$$\implies \nabla^2\Phi' = -\rho/\epsilon_0$$

So the electric potential becomes independent of time if the charge density is too, and more generally, the electric potential's temporal response is instantaneous, there is no memory of past charge densities.

$$\Phi'(\vec{x},t) = \int \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$$

Also look at the vector potential:

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}$$

As we'll see later, in exchange for the electric potential being instantaneous, we will have a retarded magnetic potential.

Solving the Lorentz Gauge Problem

Green's Function for Wave Equations

$$\begin{aligned}
&\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\vec{x},t) = -4\pi f(\vec{x},t) \\
&\xrightarrow[\text{Fourier}]{\mathcal{F}} (\nabla^2 + \underbrace{k^2}_{\omega^2/c^2})\tilde{\Psi}(\vec{x},\omega) = -4\pi\tilde{f}(\vec{x},\omega) \\
&\implies (\nabla^2 + k^2)G_k(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')
\end{aligned}$$

Lecture 20 - 444

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Status	Completed

Next: Lecture 21 - 444

Solving the Lorentz Gauge Problem, continued

Green's Function for Wave Equations

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$$

Solution for No Boundaries

Spherically symmetric,

$$\frac{1}{R}\frac{d^2}{dR^2}(RG) + k^2 G = -4\pi\delta(\vec{R})$$

For $R \neq 0$,

$$\frac{d^2}{dR^2}(RG_k) + k^2(RG_k) = 0$$
$$\implies G_k = \frac{1}{R}(Ae^{ikR} + Be^{-ikR})$$

Incoming and outgoing spherical waves, $G_k^{\pm}$.

For $R \to 0$, or $kR \ll 1$, the equation reduces to Poisson's equation, with the solution $\lim_{kR\to 0} G_k(R) = \frac{1}{R}$.

Now this is the Fourier-transformed solution. Let's go back to the time domain.

$$
\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')
$$
$$
\delta(t - t') = \int e^{-i\omega(t-t')}\frac{d\omega}{2\pi}
$$
$$
G(\vec{x}, t; \vec{x}', t') = \int G(\vec{x}, \omega; \vec{x}', t')e^{-i\omega t}\frac{d\omega}{2\pi}
$$
$$
\implies (\nabla^2 + k^2)G(\vec{x}, \omega; \vec{x}', t') = -4\pi\delta(x - x')e^{i\omega t'}
$$
$$
\text{With solutions } G_k^{\pm}(R)e^{i\omega t'}
$$

Thus we get the final solution,

$$
G^{\pm}(R, \tau) = \frac{1}{2\pi}\int \frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega
$$

k, ω not being independent.

Non-dispersive isotropic medium

$k = \omega/c$, then

$$
\begin{aligned}
G^{\pm}(R, \tau) &= \frac{1}{2\pi R}\int e^{-i\omega(\tau \mp \frac{R}{c})}d\omega \\
&= \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right) \\
\therefore G^{\pm}(\vec{x}, t; \vec{x}', t') &= \frac{1}{|\vec{x} - \vec{x}'|}\delta\left(t - t' \mp \frac{|\vec{x} - \vec{x}'|}{c}\right)
\end{aligned}
$$

These are the **retarded** and **advanced** Green's functions respectively. The governing equations don't impose an arrow of time, hence the advanced Green's function is a solution, though it may often be unphysical because it would require $t + |\vec{x} - \vec{x}'|/c = t'$, where the primed are supposed to be the source and the unprimed the observer, so it is not causal.

Lecture 21 - 444

Course	PH 444 - EM Theory
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Notes completed	☑
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Next: Lecture 22 - 444

Doubt: Is the Green's function still valid when $k \neq \omega/c$?

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$
$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \left(\frac{\partial \vec{J}}{\partial t} + \frac{\partial^2 \vec{P}}{\partial t^2}\right) + \nabla(\nabla \cdot \vec{E})$$

Use the polarisation:

$$\vec{P}(\vec{x}, t) = \epsilon_0 \int d^3\vec{x}' \int dt' \chi(\vec{x} - \vec{x}', t - t')\vec{E}(\vec{x}', t')$$

Imposing locality in space and homogeneity,

$$\vec{P} = \epsilon_0 \int dt' \chi(t - t')\vec{E}(\vec{x}, t')$$

Substituting into the wave equation and fourier transforming, we get

$$\nabla^2 \vec{E}(\omega) + \frac{\omega^2}{c^2}(1 + \chi(\omega))\vec{E}(\omega) = \mathcal{F}(\text{source terms})$$

So yes, still valid.

Green's Function for Wave Equations, continued

To solve (writing $\Box \equiv \left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)$)

$$\Box\psi(\vec{x}, t) = -4\pi f(\vec{x}, t)$$

Using

$$\Box G^{\pm}(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')$$

Multiply this by the source $f(\vec{x}', t')$ and integrate, then

$$\Box \int d^3\vec{x}' \int dt' G^{\pm} f(\vec{x}', t') = -4\pi f(\vec{x}, t)$$
$$\implies \psi(\vec{x}, t) = \int d^3\vec{x}' \int dt' G^{\pm} f(\vec{x}', t') + (\text{Homogenous term})$$

The Green's function DE is a hyperbolic equation - we need both boundary conditions (like Dirichlet or Neumann) and initial conditions (like Cauchy, which specifies $G, \partial_t G$ at all points at a particular time, or another example could be $G, \partial_t G < 0 \forall t < t_1$) to solve for the correct Green's function.

Reciprocity

$$\Box G(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')$$

Write the same equation in $\vec{x}, -t; \vec{x}'', -t''$. Multiply the first equation by $G(\vec{x}, -t; \vec{x}'', -t'')$ and the second by $G(\vec{x}, t; \vec{x}', t')$, then subtract and integrate over $(-\infty, T) \otimes \mathbb{R}^3$, $T > t', t''$.

The RHS becomes $4\pi(G(\vec{x}', -t'; \vec{x}'', -t'') - G(\vec{x}'', t''; \vec{x}', t'))$. Let's evaluate the LHS term-by-term. First term:

$$\int_{-\infty}^{T} dt \int d^3\vec{x} \left[G(\vec{x}, t; \vec{x}', t')\nabla^2 G(\vec{x}, -t; \vec{x}'', -t'') - G(\vec{x}, -t; \vec{x}'', -t'')\nabla^2 G(\vec{x}, t; \vec{x}', t')\right]$$
$$= \int_{-\infty}^{T} \int da\, \hat{n} \cdot (G(\vec{x}, t; \vec{x}', t')\nabla G(\vec{x}, -t; \vec{x}'', -t'') - G(\vec{x}, -t; \vec{x}'', -t'')\nabla G(\vec{x}, t; \vec{x}', t'))$$

(Used Green's identity from a few lectures ago.)

When both functions satisfy the same (Dirichlet or Neumann) boundary conditions, then this term becomes 0.

Second term:

$$\int d^3\vec{x} \int_{-\infty}^{T} dt\, \partial_t \left(G(\vec{x}, t; \vec{x}', t')\partial_t G(\vec{x}, -t; \vec{x}'', -t'') - G(\vec{x}, -t; \vec{x}'', -t'')\partial_t G(\vec{x}, t; \vec{x}', t')\right)$$
$$= \int d^3\vec{x} \left[G(\vec{x}, t; \vec{x}', t')\partial_t G(\vec{x}, -t; \vec{x}'', -t'') - G(\vec{x}, -t; \vec{x}'', -t'')\partial_t G(\vec{x}, t; \vec{x}', t')\right]_{-\infty}^{T}$$

The Green's function is causal, so using initial conditions $t' < T, t'' < T \implies -T < -t''$, $G, \partial_t G$ both go to 0 when the first time coordinate is lesser than the second time coordinate. Similarly for $t \to -\infty$, $t < t'$, hence the second term is also 0, and reciprocity is proved: $G(\vec{x}', -t'; \vec{x}'', -t'') = G(\vec{x}'', t''; \vec{x}', t')$.

Lecture 22 - 444

Course	PH 444 - EM Theory
Date	@March 2, 2023
Notes completed	☑
Status	Doubts

Next: Lecture 23 - 444

Solve wave equation with Green's function

$$
\begin{aligned}
\nabla'^2\psi(\vec{x}',t') - \frac{1}{c^2}\partial_{t'}^2\psi(\vec{x}',t') &= -4\pi f(\vec{x}',t') \\
\nabla^2 G(\vec{x},t;\vec{x}',t') - \frac{1}{c^2}\partial_t^2 G(\vec{x},t;\vec{x}',t') &= -4\pi\delta(\vec{x}-\vec{x}')\delta(t-t') \\
\xRightarrow{\text{Reciprocity}} \nabla^2 G(\vec{x}',-t';\vec{x},-t) - \frac{1}{c^2}\partial_t^2 G(\vec{x}',-t';\vec{x},-t) \\
&= -4\pi\delta(\vec{x}'-\vec{x})\delta((-t')-(-t)) \\
\xrightarrow[\text{Exchange primed and unprimed}]{-t\to t} \nabla'^2 G(\vec{x},t;\vec{x}',t') - \frac{1}{c^2}\partial_{t'}^2 G(\vec{x},t;\vec{x}',t') \\
&= -4\pi\delta(\vec{x}-\vec{x}')\delta(t-t')
\end{aligned}
$$

Multiply equations 1 and 4 by G, ψ respectively, subtract, and integrate over space and $t' \in [0, t^+], t^+ = t + \varepsilon$.

$$
\int_0^{t^+} dt' \int dV' \left[(G\nabla'^2\psi - \psi\nabla'^2 G) + \frac{1}{c^2}\left(\psi\partial_{t'}^2 G - G\partial_{t'}^2\psi\right)\right] = 4\pi\left(\psi(\vec{x},t) - \int_0^{t^+} dt' \int dV' f(\vec{x}',t')G\right)
$$

$$
\implies \psi(\vec{x},t) = \frac{1}{4\pi}\int_0^{t^+} dt' \oint da'\hat{n}'\cdot(G\nabla'\psi - \psi\nabla G) + \frac{1}{4\pi}\int\frac{dV'}{c^2}[\psi\partial_{t'}G - G\partial_{t'}\psi]_0^{t^+} + \int_0^{t^+} dt' \int dV' f(\vec{x}',t')G
$$

As in the previous lecture, causality causes $G, \partial_{t'}G$ to become 0 when putting in the limit $t' = t^+$ with $0 < t' < t < t^+$. Note the limits here are 0 to t^+ - we've chosen them because more often than not, the initial conditions we get match these limits.

The other limit of $t' = 0$ can be accounted for as a source term $I(\vec{x}',t')$ added to $f(\vec{x}',t')$. The integral of IG with time should give this other limit, which has a $\psi' G$ term and a $\psi G'$ term.

The former is easy to account for, just have a delta function to take $G(t')$ to $G(0)$ with some velocity coefficient.

The latter is little more involved, if you want the integral of G to become $G'(0)$, you need $\delta'(t')$ in the integrand - this is a standard property of the dirac delta function, but can be motivated intuitively as well.

- Intuition for using δ' - **incomplete**

  First, grasp the intuition of the dirac delta as an impulse source, and what kind of wave that creates - a travelling wave. The coefficient is some spatial distribution, which is associated with the vertical velocity (rate of change of displacement at each point, different from travelling velocity, which is just c) imparted at each point source. A localised impulse will have a spatial dirac delta as the coefficient, so that impulse is only at one point in space and time.

  doubt - I don't recall what the second half of the intuition was, I'll come back to it later.

$$
I(\vec{x}',t') = \frac{1}{c^2}\left(\psi(\vec{x}',0)\delta'(t') + v(\vec{x}',0)\delta(t')\right)
$$

$$
\psi(\vec{x},t) = \int_0^{t_+} dt' \int dV' G(\vec{x},t;\vec{x}',t')(f(\vec{x}',t' + I(\vec{x}',t')) + \int_0^{t_+} dt' \oint da \left(G\frac{\partial\psi}{\partial n'} - \psi\frac{\partial G}{\partial n'}\right)
$$

Where I only depends on the initial conditions.

Consider the unbounded case, where the surface goes to infinity, and we take our initial time to be $-\infty$, then

-

$$
\begin{aligned}
G^+ &= \frac{\delta(t' - (t - \frac{R}{c}))}{R}, R = |\vec{x} - \vec{x}'| \\
\psi(\vec{x}, t) &= \iint G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') d^3\vec{x}' dt' \\
&= \int \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|} d^3\vec{x}'
\end{aligned}
$$

Where $[f(t')]_{\text{ret}} = f(t - R/c)$.

This gives us the

Jeffimenko's Solutions

Using the Lorentz condition and solving the arising wave equations using the above Green's function approach (with an unbounded system and initial time $-\infty$.

$$
\begin{aligned}
\phi(\vec{x}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \frac{[\rho(\vec{x}', t')]_{\text{ret}}}{R} \\
\vec{A}(\vec{x}, t) &= \frac{\mu_0}{4\pi} \int d^3\vec{x}' \frac{[\vec{J}(\vec{x}', t')]_{\text{ret}}}{R}
\end{aligned}
$$

Lecture 23 - 444

Course	PH 444 - EM Theory
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Notes completed	☑
Status	Completed

Next: Lecture 24 - 444

Wave Equation, another way

Our first approach to solve the wave equation was through the potentials. We can try to do the same through the fields.

▼ Electric Field

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\implies \nabla \times (\nabla \times \vec{E}) = -\partial_t \left(\frac{1}{c^2} \partial_t \vec{E} + \mu_0 \vec{J} \right)$$
$$\implies \left(\nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \vec{E} = \frac{1}{4\pi\epsilon_0} \left(4\pi \nabla \rho + \frac{4\pi}{c^2} \partial_t \vec{J} \right)$$

▼ Magnetic Field

$$\nabla \times (\nabla \times \vec{B}) = -\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} + \mu_0 \nabla \times \vec{J}$$
$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = -4\pi \left(\frac{\mu_0}{4\pi} \nabla \times \vec{J} \right)$$

Using the two final wave equations, we apply the Green's function approach to them and obtain the results

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}' \cdot \frac{1}{r} \left[-\nabla' \rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret, so } t' = t - \frac{r}{c}}$$
$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3\vec{x}' \frac{[\nabla \times \vec{J}]_{\text{ret}}}{r}$$

These are a little annoying to solve, we would like to get rid of the gradients inside the retarded potentials, so write $[\nabla' f]_{\text{ret}}$ in terms of $\nabla'[f]_{\text{ret}}$. Simply apply the chain rule,

$$\nabla'[\rho]_{\text{ret}} = \nabla'\rho(\vec{x}, t - r/c) = [\nabla'\rho]_{\text{ret}} + \nabla'(t - r/c)\left[\frac{\partial \rho}{\partial t'}\right]_{\text{ret}}$$

The rule for curls is slightly more involved, note the minus sign:

$$\nabla' \times [\vec{J}]_{\text{ret}} = [\nabla' \times \vec{J}]_{\text{ret}} - \left[\frac{\partial \vec{J}}{\partial t}\right]_{\text{ret}} \times \nabla'(t - r/c)$$

▼ Proof

$$\begin{aligned}
(\nabla' \times [\vec{J}]_{\text{ret}})_i &= \epsilon_{ijk}\partial'_j([\vec{J}]_{\text{ret}})_k \\
&= \epsilon_{ijk}\left([\partial'_j J_k]_{\text{ret}} + \partial'_j(t - r/c)\left[\frac{\partial J_k}{\partial t'}\right]_{\text{ret}}\right) \\
&= \left([\nabla' \times \vec{J}]_{\text{ret}} - \left[\frac{\partial \vec{J}}{\partial t'}\right]_{\text{ret}} \times \nabla'(t - r/c)\right)_i
\end{aligned}$$

Note the useful $\nabla'(t - r/c) = \hat{r}/c$.

We substitute the expression for $[\nabla'\rho]_{\text{ret}}$ into the expression for $\vec{E}$, and once the gradient is outside the retarding brackets, we can perform integration by parts and assume the surface terms go to 0.

$$\int d^3\vec{x}'\,\frac{\nabla'[\rho]_{\text{ret}}}{r} = \cancel{\frac{1}{r}\oiint_{\delta V}[\rho]_{\text{ret}}d\vec{A}} - \int \frac{\hat{r}}{r^2}[\rho]_{\text{ret}}d^3\vec{x}'$$
$$=$$

Where we use the Gauss law, gradient form, for scalar fields:

$$\begin{aligned}
\int_V \nabla\phi dV &= \oint_{\partial V}\phi d\vec{S} \\
\text{From } \vec{k}\cdot\int_V \nabla\phi dV &= \int_V \nabla\cdot(\phi\vec{k}) \\
&= \oint_S \phi\vec{k}\cdot d\vec{S} \text{ (Gauss Law)} \\
&= \vec{k}\cdot\oint_S \phi d\vec{A}
\end{aligned}$$

Then, since this is valid for any $\vec{k}$, we can simply remove $\vec{k}$.

$$\implies \vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{x}'\left(\frac{\hat{r}}{r^2}[\rho(\vec{x}',t')]_{\text{ret}} + \frac{\hat{r}}{cr}\left[\frac{\partial\rho}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2 r}\left[\frac{\partial\vec{J}}{\partial t'}\right]_{\text{ret}}\right)$$

Similarly, we substitute $[\nabla' \times \vec{J}]_{\text{ret}}$ in the expression for $\vec{B}$, and, using the vector identity $\nabla \times (f\vec{A}) = f\nabla \times \vec{A} + \nabla f \times \vec{A}$, we can apply the generalised Stoke's Theorem:

$$\int_V \nabla \times \vec{A} dV = \oint \hat{n} \times \vec{A} dS$$

We have a $\int(\nabla \times [J])/R$ term in the $\vec{B}$ expression. We write that in terms of $\nabla \times ([J]/R)$ - which becomes a surface term from the Generalised Stoke's Theorem and goes to 0 - and a $\nabla(1/R)[J]$ term, which appears below.

$$\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3\vec{x}'\left([J]_{\text{ret}} \times \frac{\hat{r}}{r^2} + \left[\frac{\partial\vec{J}}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{r}}{cr}\right)$$

In the absence of a varying current, we simply obtain the Biot-Savart Law.

Combined, these are called the **Jefimenko's equations**. Yup, same name as the potential ones. It's the same thing in a different language.

Lecture 24 - 444

Course	PH 444 - EM Theory
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Next: Lecture 25 - 444

Evaluating Potentials for point charges

Imagine a situation like the following:

$$\phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{x}'\cdot\frac{q}{R}\cdot[\delta(\vec{x}'-\vec{r}_0(t')]_{\text{ret}}$$

When we substitute $t' = t - \frac{R}{c}$ for the retarded time, the delta function becomes quite non-trivial - how do we deal with this? How do we find the roots of $\vec{x}' - \vec{r}_0(t - R/c)$?

Revision : The Jacobian

How does one generally evaluate $\int d^3\vec{x}\ \delta(\vec{F}(\vec{x}))h(\vec{x})$?

$$\begin{gathered}\text{Let } \vec{u} = F(\vec{x}) \implies \vec{x} = F^{-1}(\vec{u}),\\ J[F]_{ij} = \frac{\partial u_i}{\partial x_j}, J[F^{-1}]_{ij} = \frac{\partial x_i}{\partial u_j},\\ dx_i = J[F^{-1}]_{ij}du_j\\ d^3\vec{x} = \det J[F^{-1}]d^3\vec{u}\end{gathered}$$

Where $J[F]$ is the Jacobian associated with the transformation. Also note that $J[F^{-1}] = J[F]^{-1}$, the proof of which is $(J[F^{-1}]J[F])_{ik} = \frac{\partial x_i}{\partial u_j}\frac{\partial u_j}{\partial x_k} = \delta_{ik}$.

We can use the Jacobian to write the integral:

$$\int \delta(F(\vec{x}))h(\vec{x})d^3\vec{x} = \int \delta(\vec{u})h(F^{-1}(\vec{u}))\frac{1}{\det J[F]}d^3\vec{u}$$

Back to the potentials

$$
\begin{aligned}
F(\vec{x}') &= \vec{x} - \vec{r}_0(t - |\vec{x} - \vec{x}'|/c) = \vec{u} \\
\phi(\vec{x}, t) &= \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\vec{u})}{|\vec{x} - F^{-1}(\vec{u})|} \frac{1}{\det J[F]} d^3\vec{u} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{x} - F^{-1}(0)|} \frac{1}{\det J[F]_{\vec{u}=0}}
\end{aligned}
$$

Existence and Uniqueness of solution

We want to show that $\vec{x}' - \vec{r}_0(t - |\vec{x} - \vec{x}'|/c) = 0$ has only one solution for $\vec{x}'$.

Let there be two solutions $\vec{x}^*_{1/2}$ at retarded times $t^*_i = t - |\vec{x} - \vec{x}^*_i|/c$, then write

$$
|\vec{x}^*_2 - \vec{x}^*_1| = v_{\text{avg}} |t^*_2 - t^*_1|
$$

Some people had a doubt as to whether an average velocity can be written - would it not depend on t^*_i in the general case? It could, but we nowhere do we have to worry about in the following steps:

$$
\begin{aligned}
|t^*_2 - t^*_1| = \frac{1}{c} \, ||\vec{x} - \vec{x}^*_1| - |\vec{x} - \vec{x}^*_2|| &\leq \frac{1}{c} |\vec{x}^*_2 - \vec{x}^*_1| \\
\implies |\vec{x}^*_2 - \vec{x}^*_1|(1 - v_{\text{avg}}/c) &\leq 0 \\
\implies v_{\text{avg}} &\geq c
\end{aligned}
$$

And hence we have our contradiction.

Calculating the Jacobian

$$
\begin{aligned}
u_i &= x'_i - r_{0,i}(t - |\vec{x} - \vec{x}'|/c) \\
\frac{\partial u_i}{\partial x_j} &= \delta_{ij} - \left| \frac{\partial r_{0,i}(t')}{\partial t'} \right|_{\text{ret}} \partial'_j (t - |\vec{x} - \vec{x}'|/c) \\
&= \delta_{ij} - v_i(t - |\vec{x} - \vec{x}'|/c) \frac{x_j - x'_j}{c|\vec{x} - \vec{x}'|} \\
\hat{n} &:= \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}, \vec{\beta} = \vec{v}/c, \\
J_{ij} &= \left. \frac{\partial u_i}{\partial x_j} \right|_{\vec{u}=0} = \delta_{ij} - \beta_i \hat{n}_j
\end{aligned}
$$

We need to find $\det J$.

Silvester's Theorem: $\det(\mathbb{I} - \vec{v} \otimes \vec{u}) = 1 - \vec{v} \cdot \vec{u}$

Can be used for evaluations like the one above.

Hence $\det J = 1 - \vec{\beta} \cdot \hat{n}$. Using this, we can write the potentials:

$$\phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}^*|}\frac{1}{|1-\vec{\beta}(t^*)\cdot\hat{n}(t^*)|}$$
$$\vec{A}(\vec{x},t) = \frac{q\mu_0}{4\pi}\frac{\vec{v}(t^*)}{|\vec{x}-\vec{x}^*||1-\vec{\beta}(t^*)\cdot\hat{n}(t^*)|}$$

Physical Intuition

If there is a source from x_1 to x_2, moving towards you (at x) with velocity v, then where you observe the source to be at a particular time will be $\tilde{x}_i$. Comparing elapsed times:

$$\frac{x-\tilde{x}_i}{c} = \frac{x_i-\tilde{x}_i}{v}$$
$$\implies \frac{\tilde{x}_2-\tilde{x}_1}{c} = \frac{(x_1-\tilde{x}_1)-(x_2-\tilde{x}_2)}{v}$$
$$\implies \Delta\tilde{x} = \frac{\Delta x}{1-\frac{v}{c}}$$

Thus the apparent length increases by $\frac{1}{1-\beta}$.

I'm not sure how this is related to the potentials and the EM theory above.

Lecture 25 - 444

Course	PH 444 - EM Theory
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Next: Lecture 26 - 444

Lienard-Wiechart Potentials

$$\phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}^*|}\frac{1}{|1-\vec{\beta}(t^*)\cdot\hat{n}(t^*)|}$$
$$\vec{A}(\vec{x},t) = \frac{q\mu_0}{4\pi}\frac{\vec{v}(t^*)}{|\vec{x}-\vec{x}^*||1-\vec{\beta}(t^*)\cdot\hat{n}(t^*)|}$$

To derive the electric and magnetic fields for a moving charge, we could take the derivatives of these potentials, or substitute the correct dirac delta function into the Jefimenko equations. Let's do both of them.

Differentiating the potentials

We can rewrite the potentials in a more convenient notation for this: $\vec{r} = \vec{x} - \vec{x}^*$, and $1 - \vec{\beta}(t^*)\cdot\hat{n}(t^*)$ is actually non-negative so drop the modulus. Also note that $\hat{n} = \hat{r}$. Hence,

$$\phi(\vec{x},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{rc-\vec{r}\cdot\vec{v}}$$
$$\vec{A}(\vec{x},t) = \frac{\vec{v}}{c^2}\phi(\vec{x},t)$$

Now let's derivative the hell outta these. Or maybe not so much, but a little bit, at least.

Use $t^* = t - \frac{r}{c} \implies r = c(t-t^*)$,

$$\nabla\phi = \frac{qc}{4\pi\epsilon_0}\frac{-1}{(rc-\vec{r}\cdot\vec{v})^2}(-c^2\nabla t^* - \nabla(\vec{r}\cdot\vec{v}))$$
$$\nabla(\vec{r}\cdot\vec{v}) = (\vec{r}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{r})$$

Note that $\vec{v} \equiv \vec{v}(t^*)$, so

$$
\begin{gathered}
(\vec{r} \cdot \nabla)\vec{v} = r_i \partial_i \vec{v} = \vec{a}(t^*)(\vec{r} \cdot \nabla t^*) \\
(\vec{v} \cdot \nabla)\vec{r} = (\vec{v} \cdot \nabla)\vec{x} - (\vec{v} \cdot \nabla)\vec{r}_0(t^*) = \vec{v} - \vec{v}(\vec{v} \cdot \nabla t^*) \\
\vec{r} \times (\nabla \times \vec{v}) = \hat{x}_i \epsilon_{ijk} r_j (\epsilon_{klm} \partial_l v_m(t^*)) \\
= \hat{x}_i \epsilon_{ijk} \epsilon_{klm} (r_j a_m(t^*) \partial_l t^*) = \vec{r} \times (\nabla t^* \times \vec{a}) \\
\vec{v} \times (\nabla \times \vec{r}) = \hat{x}_i \epsilon_{ijk} v_j (\epsilon_{klm} \partial_l (x_m - r_{0,m}(t^*)) \\
= \hat{x}_i \epsilon_{ijk} \epsilon_{klm} v_j (\cancel{\delta_{ml}} - v_m \partial_l t^*) = \vec{v} \times (\vec{v} \times \nabla t^*)
\end{gathered}
$$

Thus we have the derivative we were trying to evaluate,

$$
\nabla(\vec{r} \cdot \vec{v}) = \vec{a}(t^*)(\vec{r} \cdot \nabla t^*) + \vec{v} - \vec{v}(\vec{v} \cdot \nabla t^*) + \vec{r} \times (\nabla t^* \times \vec{a}) + \vec{v} \times (\vec{v} \times \nabla t^*)
$$

Using the identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$,

$$
\begin{gathered}
\nabla(\vec{r} \cdot \vec{v}) = \cancel{\vec{a}(t^*)(\vec{r} \cdot \nabla t^*)} + \vec{v} - \cancel{\vec{v}(\vec{v} \cdot \nabla t^*)} \\
+ (\vec{r} \cdot \vec{a})\nabla t^* - \cancel{(\vec{r} \cdot \nabla t^*)\vec{a}} + \cancel{(\vec{v} \cdot \nabla t^*)\vec{v}} - (\vec{v} \cdot \vec{v})\nabla t^* \\
= \vec{v} + \nabla t^*(\vec{r} \cdot \vec{a} - v^2)
\end{gathered}
$$

Substitute into the expression for $\nabla\phi$,

$$
\nabla\phi = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(rc - \vec{r} \cdot \vec{v})^2} (-\vec{v} - \nabla t^*(c^2 + \vec{r} \cdot \vec{a} - v^2))
$$

Now let's evaluate ∇t^*:

$$
\begin{gathered}
-c\nabla t^* = \nabla r = \nabla\sqrt{\vec{r} \cdot \vec{r}} \\
= \frac{1}{2\sqrt{r^2}} \nabla(\vec{r} \cdot \vec{r}) = \frac{1}{2r}(2(\vec{r} \cdot \nabla)\vec{r} + 2\vec{r} \times (\nabla \times \vec{r})) \\
\text{Again, let's solve term-by-term:} \\
(\vec{r} \cdot \nabla)\vec{r} = r_i \partial_i \vec{x} - r_i \partial_i \vec{r}_0(t^*) \\
= \vec{r} - \vec{v}(\vec{r} \cdot \nabla t^*) = \vec{r} + \frac{1}{c}\vec{v}(\vec{r} \cdot \nabla r) \\
\text{Term 2} : \vec{r} \times (\nabla \times \vec{r}) = \vec{r} \times (\vec{v} \times \frac{-1}{c}\nabla r) \\
= \frac{1}{c}(\vec{r} \cdot \vec{v})\nabla r - \frac{1}{c}(\vec{r} \cdot \nabla r)\vec{v} \\
\text{Substituting back,} \\
\nabla r = \frac{1}{cr}(c\vec{r} + \cancel{\vec{v}(\vec{r} \cdot \nabla r)} + (\vec{r} \cdot \vec{v})\nabla r - \cancel{(\vec{r} \cdot \nabla r)\vec{v}}) \\
\therefore \nabla r = \frac{c\vec{r}}{cr - \vec{r} \cdot \vec{v}}
\end{gathered}
$$

Now we substitute this back into $\nabla\phi$:

$$\nabla\phi = \frac{qc}{4\pi\epsilon_0}\frac{1}{(rc - \vec{r}\cdot\vec{v})^2}\left(\vec{v} - \frac{\vec{r}}{rc - \vec{r}\cdot\vec{v}}(c^2 + \vec{r}\cdot\vec{a} - v^2)\right)$$

And hence we have completed a single derivative in one lecture. Hurray.

Lecture 26 - 444

Course	PH 444 - EM Theory
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Notes completed	☑
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Next: Lecture 27 - 444

EM fields from the Lienard-Wiechart Potentials, continued

Time derivative of vector potential

$$\vec{A}(\vec{x},t) = \frac{\vec{v}(t_r)}{c^2}\phi(\vec{x},t)$$
$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{c^2}\left(\frac{\partial \vec{v}(t_r)}{\partial t}\phi + \vec{v}(t_r)\dot{\phi}\right)$$
$$\partial_t t_r = 1 - \frac{1}{c}\partial_t\sqrt{\vec{r}\cdot\vec{r}} = 1 - \frac{1}{c}\frac{1}{2r}2\vec{r}\cdot\frac{\partial(\vec{x}-\vec{r}_0(t_r))}{\partial t_r}\partial_t t_r$$
$$= 1 + \frac{1}{cr}\vec{r}\cdot\vec{v}\;\partial_t t_r$$
$$\implies \partial_t t_r = \frac{cr}{cr - \vec{r}\cdot\vec{v}}$$

We’ve seen this form often in the previous lecture as well. It becomes useful to define $\vec{u} = c\hat{r} - \vec{v}$, so that $cr - \vec{r}\cdot\vec{v} = \vec{r}\cdot\vec{u}$.

Aight, back to the potential.

$$
\begin{gathered}
\partial_t \vec{A} = \frac{1}{c^2} \partial_t t_r \left(\vec{a}\phi + \vec{v}\partial_{t_r} \phi\right) \\
= \frac{qr}{4\pi\epsilon_0 (\vec{r}\cdot\vec{u})} \left(\frac{\vec{a}}{\vec{r}\cdot\vec{u}} - \frac{\vec{v}}{(\vec{r}\cdot\vec{u})^2} \partial_{t_r} (\vec{r}\cdot\vec{u})\right) \\
\partial_{t_r} (\vec{r}\cdot\vec{u}) = c(-\hat{r}\cdot\vec{v}) - \vec{v}\cdot(-\vec{v}) - \vec{r}\cdot(\vec{a}) \\
= \frac{qr}{4\pi\epsilon_0 (\vec{r}\cdot\vec{u})^3} (\vec{a}(cr - \vec{r}\cdot\vec{v}) + \vec{v}(c\hat{r}\cdot\vec{v} - v^2 + \vec{r}\cdot\vec{a})) \\
\text{Adding and subtracting } rc\vec{v} \text{ inside the brackets,} \\
= \frac{qc}{4\pi\epsilon_0 (\vec{r}\cdot\vec{u})^3} \left((rc - \vec{r}\cdot\vec{v}) \left(\vec{a}\frac{r}{c} - \vec{v}\right) + \vec{v}\frac{r}{c}(c^2 - v^2 + \vec{r}\cdot\vec{a})\right)
\end{gathered}
$$

The point of all this manipulation is to write the field in a velocity and an accelaration-dependent term, so as to isolate their effects. Each term has different r dependence.

Finally, the Electric Field

$$
\begin{gathered}
\vec{E} = -\nabla\phi - \partial_t \vec{A} \\
= \frac{-qr}{4\pi\epsilon_0 (\vec{r}\cdot\vec{u})^3} \left(\vec{v}(rc - \vec{r}\cdot\vec{v}) - \vec{r}(c^2 + \vec{r}\cdot\vec{a} - v^2)\right. \\
\left. + (rc - \vec{r}\cdot\vec{v}) \left(\vec{a}\frac{r}{c} - \vec{v}\right) + \vec{v}\frac{r}{c}(c^2 - v^2 + \vec{r}\cdot\vec{a})\right) \\
= \frac{qr}{4\pi\epsilon_0 (\vec{r}\cdot\vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{r}\times(\vec{u}\times\vec{a})\right]
\end{gathered}
$$

Lecture 27 - 444

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Next: Lecture 28 - 444

It’s Magnetising Time

$$
\begin{aligned}
\vec{B} = \nabla \times \vec{A} = \frac{1}{c^2}\nabla \times (\vec{v}\phi) = \frac{1}{c^2}(\phi \nabla \times \vec{v} - \vec{v} \times \nabla \phi) \\
(\nabla \times \vec{v})_i = \epsilon_{ijk}(\partial_{t_r} v_k)\partial_j t_r = \nabla t_r \times \vec{a} = \left(\frac{-\vec{r}}{\vec{r}\cdot\vec{u}}\right) \times \vec{a} \\
\nabla \times \vec{A} = \frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\left[\vec{r}\cdot\vec{u}(\vec{a}\times\vec{r}) + (c^2 - v^2 + \vec{r}\cdot\vec{a})\vec{v}\times\vec{r}\right] \\
= \frac{-1}{c^2}\frac{qc}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\vec{r}\times\left[(\vec{r}\cdot\vec{u})\vec{a} + (\vec{r}\cdot\vec{a})\vec{v} + (c^2 - v^2)\vec{v}\right]
\end{aligned}
$$

Now we can add or subtract $\propto \hat{r}$ terms in the bracket because the cross product with $\vec{r}$ will be 0.

$\vec{u} = c\hat{r} - \vec{v}$, so $\vec{v} \to -\vec{u}$ in the brackets.

$$
\begin{aligned}
\vec{B}(\vec{x},t) = \frac{-1}{c^2}\frac{qc}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\vec{r}\times\left[(\vec{r}\cdot\vec{u})\vec{a} - (\vec{r}\cdot\vec{a})\vec{u} - (c^2 - v^2)\vec{u}\right] \\
= \frac{-1}{c^2}\frac{qc}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\vec{r}\times\left[\vec{r}\times(\vec{a}\times\vec{u}) - (c^2 - v^2)\vec{u}\right] \\
= \frac{\hat{r}}{c}\times\frac{qr}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\left[\vec{r}\times(\vec{u}\times\vec{a}) + (c^2 - v^2)\vec{u}\right] = \frac{\hat{r}}{c}\times\vec{E}(\vec{x},t)
\end{aligned}
$$

Beautiful.

These are known as the Lienard Wiechart potentials and fields, a special case of the Jeffimenko equations.

Discussing the form of the Electric field

$$
\frac{qr}{4\pi\epsilon_0}\frac{1}{(\vec{r}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{r}\times(\vec{u}\times\vec{a})\right]
$$

The first term depends only on v and not a, and goes as $1/r^2$. This is known as the *generalised Coulomb field*, or velocity field.

The second term falls as $1/r$, depends on a, and is known as the acceleration or radiation field. This dominates at large distances, and is responsible for the loss of energy of a charged particle, which brought up an issue with models of the atom prior to be Bohr model.

Fields of a moving charge from the Jefimenko equations

$$
\rho(\vec{x}', t') = q\delta(\vec{x}' - \vec{r}_0(t')), \vec{J}(\vec{x}', t') = \rho\vec{v}(t')
$$

$$
\vec{E}(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int d^3\vec{x}'\, \frac{\hat{R}}{R^2}[\delta(\vec{x}' - \vec{r}_0(t'))]_{\text{ret}} + \frac{\hat{R}}{cR}\left[\partial_{t'}\delta(\vec{x}' - \vec{r}_0(t'))\right]_{\text{ret}} - \frac{1}{cR^2}\left[\partial_{t'}(\vec{v}(t')\delta(\vec{x}' - \vec{r}_0(t'))\right]_{\text{ret}}
$$

Now the only subtlety here is $[\partial_{t'} f(t')]_{\text{ret}} = \partial_t [f(t')]_{\text{ret}}$.

$\vec{x}'$ has no time dependence inside the integral - it is only when the delta function collapses that it will. So R terms can be moved inside and outside the time derivative at will. So do that and then use the Jacobian method to solve the problems.

Lecture 28 - 444

Course	PH 444 - EM Theory
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Next: Lecture 29 - 444

Poynting Theorem and the Poynting Vector

→ *I never understood the uniqueness of this one, since all vectors do point.*

Poynting's Theorem

Rate of work done $= q\vec{v}\cdot\vec{E} \to \int \vec{J}\cdot\vec{E}dV$ (continuous distribution)

$$\int \vec{J}\cdot\vec{E}dV = \int dV\vec{E}\cdot(\nabla\times\vec{H} - \partial_t\vec{D})$$
$$\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}(\nabla\times\vec{H})$$
$$\implies \int \vec{J}\cdot\vec{E}dV = \int dV(\vec{H}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\partial_t\vec{D})$$

Assume

1. Linear medium
2. Non-dispersive and lossless

Which are quite restrictive constraints.

Then

$$-\int \vec{J}\cdot\vec{E}\, d^3\vec{x} = \int d^3\vec{x}\left[\partial_t\left(\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right) + \nabla\cdot(\vec{E}\times\vec{H})\right]$$
$$= \partial_t\int d^3\vec{x}\, u + \oint d\vec{A}\cdot\underbrace{(\vec{E}\times\vec{H})}_{\vec{S}}$$

Hence the Poynting vector $\vec{S}$ denotes the directional energy flux.

But you might note here that $\vec{S}$ is not fully determined here - one could add a curl-like term $\nabla \times \vec{V}$ to $\vec{S}$ and still get the same physical results. To fully determine $\vec{S}$, we will need relativity.

Mechanical work done

If no particles leave or enter the volume V, then the $\int \vec{J} \cdot \vec{E}\, d^3\vec{x}$ is the rate of mechanical work done by the fields. Consider the energy held by the fields themselves to be $u = \frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D})$, then

$$\partial_t E_{\text{mech}} + \partial_t E_{\text{fields}} = -\oint \hat{n} \cdot \vec{S} da$$

Is the energy conservation equation.

Conservation of Momentum

We're looking for a an equation of the form $\frac{d}{dt}\vec{P}_{mech} + \frac{d}{dt}\vec{P}_{field} =$ a flux, so we want a divergence term on the RHS since we're doing volume integrals.

$$\frac{d}{dt}\vec{P}_{\text{mech}} = \int (\rho\vec{E} + \vec{J} \times \vec{B}) d^3\vec{x}$$
$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0(\nabla \cdot \vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0 \partial_t \vec{E}\right) \times \vec{B}$$
$$= \epsilon_0 \left[(\nabla \cdot \vec{E})\vec{E} + c^2\epsilon_0(\nabla \cdot \vec{B})\vec{B} + c^2(\nabla \times \vec{B}) \times \vec{B} - \vec{E} \times (\nabla \times \vec{E}) - \partial_t \vec{S}\right]$$

(Using $\partial_t \vec{E} \times \vec{B} + \vec{E} \times \partial_t \vec{B} = \partial_t(\vec{E} \times \vec{B})$)

So now we associate. Let's simplify the rest of the terms.

$$[(\nabla \cdot \vec{E})\vec{E} - \vec{E} \times (\nabla \times \vec{E})]_i$$
$$= E_i \partial_j E_j - \epsilon_{ijk} E_j (\epsilon_{klm} \partial_l E_m)$$
$$= \partial_j (E_i E_j) - E_j \partial_j E_i - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l E_m$$
$$= \partial_j (E_i E_j) - E_j \partial_j E_i - E_j \partial_i E_j + E_j \partial_j E_i$$
$$= \partial_j (E_i E_j - \frac{1}{2}\delta_{ij} E^2)$$

Do exactly the same with the $\vec{B}$ terms, and we get

$$\frac{d}{dt}\vec{P}_{\text{mech}} + \frac{d}{dt}\vec{P}_{\text{field}} = \int_V \partial_\beta T^{\alpha\beta} d^3\vec{x} = \oint T^{\alpha\beta} n_\beta da$$
$$\text{Where } T^{\alpha\beta} = \epsilon_0[E^\alpha E^\beta + c^2 B^\alpha B^\beta - \frac{1}{2}\delta^{\alpha\beta}(E^2 + c^2 B^2)]$$

Where we have defined $\vec{P}_{\text{field}}$ as $\int_V \frac{1}{c^2}\vec{S}\ d^3\vec{x}$.

This T is the Maxwell's stress tensor, and $T^{\alpha\beta}n_\beta$ is the αth component of momentum flow per unit area across the surface S into the volume V.

Lecture 29 - 444

Course	PH 444 - EM Theory
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Next: Lecture 30 - 444

(Dipole) Radiation

I.e. the pointy makes some wavy.

Consider wobbly (localised harmonic) sources $\rho(\vec{x},t) \equiv \rho(\vec{x})e^{-i\omega t}, \vec{J}(\vec{x},t) \equiv \vec{J}(\vec{x})e^{-i\omega t}$.

Working in the Lorenz gauge (i.e. applying the Green's function approach to the Lorenz gauge wave equation),

$$\begin{aligned}\vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3\vec{x}'\int dt'\frac{\vec{J}(\vec{x}')e^{-i\omega t'}}{|\vec{x}-\vec{x}'|}\delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)\\ &= \frac{\mu_0}{4\pi}\int d^3\vec{x}'\vec{J}(\vec{x}')e^{-i\omega t+ik|\vec{x}-\vec{x}'|}\end{aligned}$$

Where $k = \omega/c$, and the $e^{ik|\vec{x}-\vec{x}'|}$ can be intuited as a spherical wave somehow, and the fields will be:

$$\begin{aligned}\vec{H}(\omega) &= \frac{1}{\mu_0}\nabla\times\vec{A}(\omega)\\ \vec{E}(\omega) &= \frac{i\sqrt{\mu_0}}{k\sqrt{\epsilon_0}}\nabla\times\vec{H}(\omega)\\ z_0 &= \sqrt{\frac{\mu_0}{\epsilon_0}}\end{aligned}$$

Assume localised source

Source dimension $\sim d$, wavelength λ, then we have 3 spatial reasons to consider.

$$\text{Near Zone}: d \ll r \ll \lambda$$
$$\text{Intermediate Zone}: d \ll r \sim \lambda$$
$$\text{Far (radiation) Zone}: d \ll \lambda \ll r$$

Near Zone

$k \propto 1/\lambda \implies kr \ll 1$, so put $e^{-ikr} \approx 1$.

Thus we have a constant $\vec{A}(\vec{x})$. We can expand $1/|\vec{x} - \vec{x}'|$ in the spherical harmonics and evaluate this better.

The fields will then only have oscillatory behaviour from $e^{-i\omega t}$, since $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$ and so on.

Far zone

$$|\vec{x} - \vec{x}'| = r\left(1 - \frac{2\hat{r}\cdot\vec{x}'}{r} + \frac{(r')^2}{r^2}\right)^{1/2}$$
$$\approx r - \hat{r}\cdot\vec{x}'$$

Thus we approximate

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\vec{x}'\,\vec{J}(\vec{x}')e^{-ik\hat{r}\cdot\vec{x}'}$$

$\hat{r}\cdot\vec{x}'$ in the exponential, so like an almost spherical wave with angular dependence.

Using $d \ll \lambda$, expand

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int d^3\vec{x}'\,\vec{J}(\vec{x}')(\hat{r}\cdot\vec{x}')^n$$

Take only the first term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\vec{x}'\,\vec{J}(\vec{x}')$$
$$\nabla(\vec{x}\cdot\vec{J}) = \vec{J} + \vec{x}(\nabla\cdot\vec{J}), \text{ and } \nabla\cdot\vec{J} = -\partial_t\rho$$
$$\text{Here, } \partial_t\rho \equiv -i\omega\rho(\vec{x})$$
$$\therefore \int d^3\vec{x}'\,\vec{J}(\vec{x}') = -i\omega\int \vec{x}'\rho(\vec{x}')d^3\vec{x}' \equiv -i\omega\vec{p}$$

Where $\vec{p}$ is the dipole moment. (We put surface terms to 0 because we're far from the source.) Thus

$$\vec{A}(\vec{x}) = \frac{-i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\vec{p}$$

Deriving the magnetic field

$$B_i = \epsilon_{ijk}\partial_j A_k$$
$$\partial_j\left(p_k\frac{e^{ikr}}{r}\right) = \left(\frac{ik}{r} - \frac{1}{r^2}\right)e^{ikr}\underbrace{\frac{x_j}{r}}_{n_j}$$
$$\therefore \vec{B} = \frac{\mu_0 ck^2}{4\pi}(\hat{n}\times\vec{p})\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}$$

And now the electric field

$$\vec{E} = i\frac{z_0}{k}\nabla\times\vec{H}$$

Writing the curl and cross product using Levi-Civita, this becomes trivial enough that we skip the exact calculation in class and end up at the result,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r} + \left(\frac{1}{r^3} - \frac{ik}{r^2}\right)(3(\hat{n}\cdot\vec{p})\hat{n} - \vec{p})\right]$$

The first term takes energy away to infinity, whereas the second keeps some energy localised. We haven't derived it yet, but the same expression is valid for near zone.

We can further take the far zone approximation for both $\vec{H}, \vec{E}$ by neglecting terms falling faster than $1/r$ to get:

$$\vec{H} = \frac{ck^2}{4\pi}\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}$$
$$\vec{E} = \frac{k^2}{4\pi\epsilon_0}(\hat{n}\times\vec{p})\times\hat{n}\frac{e^{ikr}}{r} = z_0\vec{H}\times\hat{n}$$

Lecture 30 - 444

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Power Loss via Radiation

Working in the far (radiation) zone and neglecting $\mathcal{O}(r^{-2})$ terms,

$$\vec{H} = \frac{ck^2}{4\pi}\hat{n} \times \vec{p}\,\frac{e^{ikr}}{r}$$
$$\vec{E} = z_0\vec{H} \times \hat{n}$$

Power per unit solid angle:

$$\frac{dP}{d\Omega} = r^2\hat{n} \cdot \underbrace{\frac{1}{2}\Re(\vec{E} \times \vec{H}^*)}_{\vec{S}}$$
$$= \frac{z_0}{2}r^2|\vec{H}|^2 = \frac{z_0}{2}\frac{c^2k^4}{16\pi^2}|\hat{n} \times \vec{p}|^2$$
$$\left(= \frac{z_0}{2}\frac{c^2k^4}{16\pi^2}|(\hat{n} \times \vec{p}) \times \hat{n}|^2\right)$$
$$= \frac{z_0c^2k^4}{32\pi^2}|\vec{p}|^2\sin^2\theta$$

Thus dipoles mainly emit power in an equatorial lobe. $\hat{n}$ is the observer direction from the source.

We can derive the total power emitted by integrating over $\sin\theta d\theta d\phi$:

$$P = \frac{c^2z_0k^4}{12\pi}|p|^2$$

Near Zone

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3\vec{x}' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$
$$= \frac{\mu_0}{4\pi} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \int \vec{J}(\vec{x}') r'^l Y^*_{lm}(\theta', \phi') d^3\vec{x}'$$
$$\text{Using just the } l = 0 \text{ term:}$$
$$\vec{A}(\vec{x}) \approx \frac{\mu_0}{4\pi} \frac{1}{r} \int J(\vec{x}') d^3\vec{x}'$$

Because near-zone approximation.

Intermediate Zone

In this region, we can't make any approximations,

$\frac{e^{ikr}}{r}$ is expanded in terms of the Bessel functions

some bullshit\

Some more bullshit

$$\vec{E} = (3\vec{p}['pjf'43jfgr3ot$$

Power radiated by moving point charges

$$\vec{E}(\vec{x}, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r} \cdot \vec{u})^3} [(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a})]$$
$$\vec{B}(\vec{x}, t) = \frac{1}{c} \hat{r} \times \vec{E}(\vec{x}, t)$$
$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0 c} (|\vec{E}|^2 \hat{r} - (\vec{E} \cdot \hat{r})\vec{E})$$

Taking the limit of large r, the velocity term in $\vec{E}$ is negligible compare to the acceleration term, which is also annihilated in the $\hat{r} \cdot \vec{E}$, so we get

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c} E^2_{\text{rad}} \hat{r}$$
$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r} \cdot \vec{u})^3} \vec{r} \times (\vec{u} \times \vec{a})$$

In the non-relativistic limit, $v \ll c \implies \vec{u} \approx c\hat{r}$,

$$
\begin{aligned}
\vec{E}_{\text{rad}} &= \frac{q}{4\pi\epsilon_0}\frac{r}{(cr)^3}c[(\vec{r}\cdot\vec{a})\hat{r} - r\vec{a}] \\
&= \frac{\mu_0 q}{4\pi}\frac{1}{r}[(\hat{r}\cdot\vec{a})\hat{r} - \vec{a}] \\
\vec{S}_{\text{rad}} &= \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi r}\right)^2 (a^2 - (\hat{r}\cdot\vec{a})^2)\hat{r} \\
&= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}
\end{aligned}
$$

The total radiated power is then given by the Larmor formula,

$$
\begin{aligned}
P = \oint \vec{S}_{\text{rad}} \cdot d\vec{a} &= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \cdot 2\pi \cdot \frac{4}{3} \\
&= \frac{\mu_0 q^2 a^2}{6\pi c}
\end{aligned}
$$

Lecture 31 - 444

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We discussed the doubt of why the dipole moment and charge density have been taken to be complex in previous lectures. The answer is we're working with the complex envelope of the charge density and hence derived quantities - it's an implicit complexification trick, nothing more.

Picking up where we left off, power radiated due to moving point charge - the Larmor Formula

We derived in the non-relativistic limit,

$$P_{tot} = \frac{\mu_0 q^2 a^2}{6\pi c}$$
$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\sin^2\theta$$

Where θ is the angle with respect to $\vec{a}$.

When we go towards the relativistic limit, the lobes of emitted power will align more and more with $\vec{a}$. But how do we do this?

Relativistic limit - Boost to a moving frame

The rate at which energy passes through a sphere around the charge is not the same as the rate at which energy left the charge - the difference is in the time it took to travel to the sphere, hence the former is $\frac{dW}{dt}$, the non-relativistic limit quantity we have already derived, and the the latter is $\frac{dW}{dt_r} = \frac{dW}{dt}\frac{\vec{r}\cdot\vec{u}}{rc}$.

Now we can substitute this into $\frac{dP}{d\Omega}$ as we calculated last lecture:

$$\frac{dP}{d\Omega} = \frac{\vec{r}\cdot\vec{u}}{rc}\frac{1}{\mu_0 c}E_{rad}^2 r^2$$
$$= \frac{\vec{r}\cdot\vec{u}}{rc}\frac{1}{\mu_0 c}\frac{q^2}{(4\pi\epsilon_0)^2}\frac{r^2}{(\vec{r}\cdot\vec{u})^6}|\vec{r}\times(\vec{u}\times\vec{a})|^2 r^2$$
$$= \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5}$$
$$\xrightarrow{\text{Integrate}} P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

Where $\gamma = 1/\sqrt{1-\beta^2}$.

To perform the integration in general is a very difficult task, but we will derive the results for certain simple cases below.

Bremsstrahlung Effect

When a charged particle suddenly comes to rest - or suddenly starts moving, though that is physically difficult - the sudden acceleration leads to radiation.

For electrons hitting a metal, this is the Bremsstrahlung Effect, used to produce X-rays.

Here, $\vec{v}, \vec{a}$ are collinear, so

$$\vec{u}\times\vec{a} = (c\hat{r}-\vec{v})\times\vec{a} = c\hat{r}\times\vec{a}$$
$$r^2|\vec{r}\times(\vec{u}\times\vec{a})|^2 = c^2|\vec{r}\times(\vec{r}\times\vec{a})|^2$$
$$= c^2|(\vec{r}\cdot\vec{a})\vec{r} - a^2r^2|^2$$
$$= c^2r^4(a^2 - (\hat{r}\cdot\vec{a})^2) = c^2r^4a^2\sin^2\theta$$
$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$
$$P = \frac{\mu_0 q^2\gamma^6 a^2}{6\pi c}$$

Thus $\frac{dP}{d\Omega}$ is maximised at $\sin\theta_{\max} = \sqrt{\frac{1-\beta}{2}}$.

Synchrotron Radiation

Electrons move at very high speeds through the Synchrotron tube, which is octagonal with circular arcs at the vertices - magnets ensure the electrons follow the tube path, and at the points of acceleration X-rays are emitted.

This is a good example of a case where $\vec{v}, \vec{a}$ are perpendicular.

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5}$$
$$\hat{r}\cdot\vec{u} = c(1-\beta\cos\theta)$$
$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = |(\hat{r}\cdot\vec{a})\vec{u}-(\hat{r}\cdot\vec{u})\vec{a}|^2$$
$$= (\hat{r}\cdot\vec{a})^2u^2+(\hat{r}\cdot\vec{u})^2a^2-2(\hat{r}\cdot\vec{a})(\hat{r}\cdot\vec{u})(\vec{u}\cdot\vec{a})$$
$$\text{Do the substitution on your own, he said.}$$
$$= a^2c^2((1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\phi)$$

Put this in the expression and we're done - or not? We can integrate to get the total power, just substitute $x = 1-\beta\cos\theta$.

Cerenkov Radiation

When charged particles move through a polarised material, like water, at speeds faster than the speed of light in the medium, we see radiation - blue for water - in what is geometrically like a sonic boom.

Cone angle is $\sin\theta = \frac{c/n}{v} = \frac{1}{\beta n}$, derived like in sonic booms. The emission angle is $90^\circ - \theta$.