All Lectures - DG

Course: Differential Geometry

Status: Completed

Date: October 29, 2024

Properties

Next: Empty

Previous: Lecture 7 - DG

4 more properties

Contents

Introduction

This field rose in relevance to Physics with the advent of GR, which required the language of Differential Geometry to describe spacetime (and everything in it).

Spacetime was now described by coordinates x^μ, with (potentially tensorial) fields $\phi(x)$ living on it. The metric $g_{\mu\nu}(x)$, a rank 2 tensor, defined lengths, the distance between x^μ and $x^\mu + \mathrm{d}x^\mu$ is given by $\mathrm{d}s$, where $\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$.

This simplistic picture, however, breaks down for slightly more complex manifolds. Consider a 2-sphere. We need two patches to cover it, one coordinate patch cannot cover it completely (you can't turn a piece of paper into a closed ball, they're topologically different).

Also consider, singularities. For example, the following metric seems to have a singularity at $\rho = 0$:

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{\rho^4} + \frac{\mathrm{d}\theta^2}{\rho^2}$$

However, $\rho = 0$ is actually not even part of the manifold here. While it is alright to have coordinates which are limited to some subset of the real line, it often suggests (especially when single points are excluded) that are coordinate system is not the best choice. Notably, in this case, changing coordinates to $r = 1/\rho$ reveals that the metric is:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

Which is simply $\mathbb{R}^2$ in polar coordinates, a very simple and well-defined manifold.

However, not all singularities are such. Consider the Schwarzschild black hole – it has a singularity at $r = 0$, which is not removable by coordinate transformations.

References

Reference: Nakahara

Alternate:

- Isham, Modern differential geometry for physicists (World Scientific)
- Nash and Sen, Topology and Geometry for Physicists (Dover)
- Eduardo Nahmad-Achar, Differential Topology and Geometry with Applications to Physics (IoP). E-book available at https://iopscience.iop.org/book/mono/978-0-7503-2072-6

Manifolds

Heuristically, it is a space that locally looks like $\mathbb{R}^m$ (or $\mathbb{C}^m$) – this means calculus can (possibly) be extended to such a space via these local patches. Of course, not all spaces encompassed by the definitions we discuss will qualify as differentiable, and we will discuss differentiable manifolds too.

Manifolds can also be constructed to locally look like other fields, or multiplets of those other fields, but this often does not allow for calculus and related ideas to be extended to it. There are of course exceptions, but none very relevant to Physics.

Definition - Manifold

A topological space M that can be covered by patches U_i such that:

- $\exists \psi_i : U_i \to U_i' \in \mathbb{R}^m$, where $m =: \dim M$ is constant across the patches.
- ψ_i are invertible
- $\bigcup_i U_i = M$
- For any pair of patches i, j, on the intersection $U_i \cap U_j$, the descriptions in $\mathbb{R}^m$ are related by a smooth transformation. Mathematically,

$$\begin{gathered} \phi_{ij} := \psi_i \psi_j^{-1} \\ \phi_{ij} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j) \\ \phi_{ij} \text{ must be infinitely differentiable } (C^\infty) \end{gathered}$$

Definition - Atlas

(U_i, ψ_i) is known as a chart, and all the charts describing a manifold are together called an atlas.

Stereographic projection

Let us consider the example of S^2 embedded in $\mathbb{R}^3$, and the well-known stereographic projection from the North Pole: For $x^2 + y^2 + z^2 = 1$, we obtain the coordinates $X = \frac{x}{1-z}, Y = \frac{y}{1-z}$. But note that this covers everything except the North Pole! It's the sheet of paper, it almost wraps around but not quite.

We can define another patch by projecting lines from the South Pole (and noting where they intersect with the x–y plane, which is how we obtain the stereographic coordinates), $X' = \frac{x}{1+z}, Y' = \frac{y}{1+z}$. Two patches, ball is covered.

So we have our charts. Let's check the differentiability – on every point except NP and SP. With a little algebra:

$$\begin{gathered} X'/Y' = X/Y, \quad X^2 + Y^2 = 1/(X'^2 + Y'^2) \\ \therefore X'(X,Y) = \frac{X}{X^2+Y^2}, \quad Y'(X,Y) = \frac{Y}{X^2+Y^2} \end{gathered}$$

These transition functions are C^∞ everywhere except where $X^2 + Y^2 = 0$, which only happens at the south pole. Similarly, the inverse map is C^∞ except at the north pole. But we don't need either of those points, and so we are done!

Before we can discuss manifolds with any rigour, we must understand that they are topological spaces, and so we must understand topology first.

Refer to Lecture 1 - 406 and the ensuing notes for the same.

Topological Space

Professor Hull began with the usual discussion of the familiar idea of open sets on $\mathbb{R}$ and $\mathbb{R}^n$. In particular, extending from open intervals and unions of open intervals, we realise (and can prove) that all open sets in $\mathbb{R}^n$ can be formed by infinite unions and finite intersections of the open n-balls. To prove this we need to realise that balls can only be defined after defining a metric, and then we can use the metric-based definition of open sets to prove that the balls form a sort of basis for the open sets of $\mathbb{R}^n$ (take note, we will formalise this idea too, later).

We can also prove, using the metric-based definition of an open set, that finite/infinite unions and finite intersections of open sets always give open sets. But now we look for something more fundamental than the metric. We are trying to understand a space prior to the definition of a metric. So we define:

> **Definition - Topology**
> Consider a space (a set of points) X together with a collection of subsets of X called J, such that
> - $\phi, X \in J$ (null set and the whole space)
> - $\{U_\alpha\}_{\alpha \in I} \subseteq J \implies \bigcup_{\alpha \in I} U_\alpha \in J$ (up to union of infinite sets)
> - $\{U_\alpha\}_{\alpha=1}^{k} \subseteq J \implies \bigcup_{\alpha=1}^{k} U_\alpha \in J$ (only a finite number of intersections)
>
> Then J is called a **topology** on X.

> **Definition - Topological Space**
> A space X paired with a topology J on X is called a **topological space**.

> **Definition - Open Set**
> Any $U_\alpha \in J$ for a topological space is *defined* as an open subset of the space. There is no other definition of open set, and the topology *is* the collection of *all* the open sets in that space.

> **Definition - Closed Set**
> A subset of a top space which is a complement of an open set.

The open sets and metric-based definitions we are used to can all be derived in the:

Definition - Metric Topology
The metric topology is the collection of subsets $U \subseteq X$ such that $\forall x \in U, \exists r > 0 \text{ s.t. } B(x,r) \subseteq U$, where $B(x,r) := \{y \in X : d(x,y) < r\}$ is an open ball.

There is a unique topology for a given metric, because once a topology is defined, that is necessarily the set of all open subsets of X, which a metric uniquely defines.

But note that a topology does not always define a metric - there doesn't always exist a metric which can create a given topology.

One example of a topology is $J = \{\phi, X\}$, which is a topology for every set. Another topology is the power set. The former is the indiscrete topology, and the latter the discrete topology.

Continuity and Maps

Definition - Continuous Map
Between topological spaces X, Y, a map $f : X \to Y$ is **continuous** when the the pre-image of any open set in Y is an open set in X: $f^{-1}(V \in J_Y) = U \in J_X$.

Definition - Homeomorphism
An invertible map $f : X \to Y$ such that f and f^{-1} are continuous.

Definition - Homeomorphic spaces
If a homeomorphism exists between two top spaces, they are **homeomorphic** – *topologically equivalent*.

Topological Structure of a Manifold

Definition - Hausdorff
For $x, y \in X, x \neq y$, there must exist disjoint open sets containing x, y.

Thus every metric space is Hausdorff.

▶ Proof

This is useful because it helps us see if a topology can come from a metric by looking at any two points and seeing whether such disjoint open sets exist.

The Hausdorff property conveys an idea of distinguishability, or "topological smoothness". This is something we require for manifolds. Thus we rewrite (changes are italicised) the definition of manifolds more formally:

Definition - Manifold
A *Hausdorff* space M that can be covered by *open subsets* U_i such that:

- $\bigcup_i U_i = M$
- $\exists$ *homeomorphic* $\psi_i : U_i \to U_i' \in \mathbb{R}^m$

  (where $m =: \dim M$ is constant across the patches)
- For any pair of patches i, j, on the intersection $U_i \cap U_j$, the descriptions in $\mathbb{R}^m$ are related by a smooth transformation. Mathematically,

$$\phi_{ij} := \psi_i \psi_j^{-1}$$
$$\phi_{ij} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$$
$$\phi_{ij} \text{ must be infinitely differentiable } (C^\infty)$$

Thus U_i form an open cover of M.

Topological Invariants

Quantities which only depend on a manifold's topological structure – more formally, properties or quantities which are preserved under any topological isomorphism.

One of these is compactness – it is some idea of well-behavedness, not exactly finite, but more like finite detail. As we'll see, this matches the intuitive idea of a closed and bounded set in $\mathbb{R}^n$ with the usual topology, but a different definition turns out to be more useful for proving properties of arbitrary manifolds.

Definition - Boundedness
Given a metric space (X, d), a set $Y \subseteq X$ is bounded when $\exists R \in \mathbb{R}^+$ s.t. $d(y_1, y_2) < R \; \forall y_1, y_2 \in Y$.

Definition - Connected
A top space X is connected if it can't be written as the union of two disjoint open sets, i.e. $\nexists U, V \in \mathcal{T}_X$ with $U \cap V = \phi$ and $U \cup V = X$.

Compactness

Definition - Compact Topological Space
For every open cover (collection of open sets with union equal to the top space) there is a finite open sub-cover.

Theorem - Heine-Borel

For subsets of $\mathbb{R}^n$ (usual topology), compactness is equivalent to being closed and bounded (in extent).

Why compact? It's a powerful property, we can prove lots using it.

Constructing Topological Spaces

Definition - Subspace Topology

If (X, J) is a top space, $S \subseteq X$, then (S, J_s) is also a top space with the **subspace topology:**

$$J_S \equiv \{O_i \cap S : O_i \in J\}$$

Definition - Product Topology

Let X, Y be topological spaces, then the collection $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ forms a basis for what is known as the *product topology*.

The topology itself can be explicitly defined as the collection of all arbitrary unions and finite intersections of the elements of $\mathcal{B}$.

Relations

Definition - Relation

On set X, relation R is a subset $R \subseteq X \times X$ – a set of pairs from X.

$(a, b) \in R$ is written as $a \sim b$.

Definition - Equivalence Relation

A relation on X satisfying the following:

1. **Reflexive** → $a \in X \implies (a, a) \in R$
2. **Symmetric** → $(a, b) \in R \implies (b, a) \in R$
3. **Transitive** → $(a, b), (b, c) \in R \implies (a, c) \in R$

Definition - Equivalence Class

$$[a] \equiv \{x : x \in X, x \sim a\}$$

Lemma - Equivalence Classes
An equivalence relation on X gives a partition of X (collection of disjoint sets with union X) in terms of equivalence classes.

▶ Proof

a is a **representative** of class A iff $A = \{a\}$.

Quotient Topology

Definition - Quotient Space

The set of equivalent classes partitioning X defines a quotient space $X/\sim$, equipped with the projection map $\pi : X \to X/\sim, \pi(x) = [x]$.

This space inherits the following topology from X:

$$J_{X/\sim} := \{U \subseteq (X/\sim) : \pi^{-1}(U) \in J_X\}$$

Where π^{-1} is the pre-image map:

$$\pi^{-1}(U) = \{x \in X : \pi(x) \in U\}$$

Example → For the relation on $\mathbb{R}$, $x' \sim x$ when $x' = x + 2n\pi$, the quotient space is S^1.

Manifold : Complete Definition

A smooth m-dim real manifold is a set M such that:

1. M is a Hausdorff topological space with topology J.
2. M is equipped with an atlas $\{(U_i, \psi_i)\}_i, U_i \in J$, such that
    a. $\bigcup_i U_i = M$
    b. ψ_i are homeomorphisms to open sets $U_i' \in \mathbb{R}^m$
3. For any $U_i \cap U_j \neq \phi$,
   The transition functions $\phi_{ij} := \psi_i \psi_j^{-1} : U_j' \supseteq \psi_j(U_j \cap U_i) \to U_i'$ are smooth (C^∞).

Real Projective Space

$\mathbb{RP}^2$ is the space of all lines in $\mathbb{R}^3$, more formally defined as the quotient space achieved by partitioning by the equivalence classes of the relation $\vec{x} \sim \lambda\vec{x}, \lambda \in \mathbb{R} - \{0\}, \vec{x} \in \mathbb{R}^3 - \{0\}$. We need to exclude 0 from $\mathbb{R}^3$ else the relation is not an equivalence relation, or else $[0] = \{0\}$, depending on the domain of λ – we want neither, so we exclude 0.

This inherits the subspace, then the quotient topology. We also know that these inherited topologies are Hausdorff when the original one is, and the usual top on $\mathbb{R}^3$ is.

▶ Proof

Atlas on $\mathbb{RP}^2$

To construct the atlas, let's embed it back in $\mathbb{R}^3$.

Define $U_i = \{[\vec{x}] : \vec{x} \in \mathbb{R}^3 - \{0\}, x_i \neq 0\}$. Define the maps $\psi_i : U_i \to \mathbb{R}^2, \psi_i([\vec{x}]) = \left(\frac{x_{i+1}}{x_i}, \frac{x_{i+2}}{x_i}\right)$ (as usual, the indices are extended cyclically – $x_4 = x_1$ and so on – for notational convenience). Clearly, $\psi_i([\vec{x}]) = \psi_i([\lambda\vec{x}])$, as it should be.

Are these sets open? And are the transition maps C^∞?

In the regions where $x_i, x_j \neq 0$ $(U_i \cap U_j)$, the mappings are related by

$$\phi_{ij}\left(\frac{x_{j+1}}{x_j}, \frac{x_{j+2}}{x_j}\right) = \left(\frac{x_{i+1}}{x_i}, \frac{x_{i+2}}{x_i}\right)$$
$$\text{Let } j = 1, i = 2$$
$$\text{Then } (b_1, b_2) \xrightarrow{\phi_{ij}} (a_1, a_2) \implies a_1 = 1/b_1, a_2 = b_2/b_1$$

Which are C^∞ functions since in this region $b_1 \neq 0$ – and we can prove for any other i, j similarly.

Complex Smooth Manifolds

A smooth m-dim **complex** manifold is a set M such that:

1. M is a Hausdorff topological space with topology J.
2. M is equipped with an atlas $\{(U_i, \psi_i)\}_i, U_i \in J$, such that
   a. $\bigcup_i U_i = M$
   b. ψ_i are homeomorphisms to open sets $U_i' \in \mathbb{C}^m$
3. For any $U_i \cap U_j \neq \phi$,

   The transition functions $\phi_{ij} := \psi_i\psi_j^{-1} : U_j' \to U_i'$ are **analytic**

Analyticity is a stronger condition than smoothness. It requires continuous complex differentiability on its entire domain. From my CA notes,

> Analyticity is far more powerful and interesting than the humble definition suggests. This arises from the increased strength of the continuity and differentiability conditions in the complex plane. It is stronger than, for example, Fretchet differentiability, which (in finite dimensional normed spaces) is the existence of partial derivatives.
> Some results showcasing this power:
>
> - Differentiability implies analyticity
> - Analyticity implies infinite differentiability
> - Analyticity implies a power series expansion at all points (in the domain)

It can be shown that the Cauchy-Reimann conditions can be rewritten as:

$$\begin{gathered}\frac{\partial f}{\partial \bar{z}} = 0 \\ \Leftrightarrow \partial_x f + \iota \partial_y f = 0\end{gathered}$$

When extending this to $\mathbb{C}^m$, we simply require Cauchy-Reimann to be obeyed for each complex coordinate z^μ individually.

A complex manifold is a special case of a real manifold, since $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ and analyticity on $\mathbb{C}^m$ implies smoothness on $\mathbb{R}^{2m}$ – though the converse does not hold.

Examples

Recall S^2, with the stereographic projection coordinates, and the transition function between them:

$$x_N = \frac{x_S}{x_S^2 + y_S^2}, \quad y_N = \frac{y_S}{x_S^2 + y_S^2}$$

If we try $z_N = x_N + \iota y_N$ and similarly for S, clearly $z_N = 1/\bar{z}_S$, which is not analytic. But this can be easily remedied by defining $z_S = x_S - \iota y_S$, in which case $z_N = 1/z_S$ and is well-defined in the overlap region.

Thus the 2-sphere is in fact a smooth 1D complex manifold.

Complex Projective Space $\mathbb{CP}^n$

A very similar construction to the real case, this is the manifold of complex lines $(z = a + \lambda b, \lambda \in \mathbb{C})$ through the origin in $\mathbb{C}^n$.

The equivalence relation is that of scaling by a complex number λ, so we define charts by:

$$\begin{gathered}U_m : z_m \neq 0 \\ \psi_m : [\lambda(z_1, \ldots, z_n)] \to \left(\frac{z_1}{z_m}, \ldots, \frac{z_n}{z_m}\right) \equiv \left(Z_i^{(m)}\right)\end{gathered}$$

The transition functions then become (and these are well-defined in the overlap regions):

$$\phi_{21} : Z_1^{(2)} = 1/Z_1^{(1)}, Z_i^{(2)} = Z_i^{(1)}/Z_1^{(1)}$$

You may note that $\mathbb{CP}^1 \simeq S^2$, but this does not hold for higher n.

Half Space

$$H^m \subset \mathbb{R}^m : (x_1, \ldots, x_m), x_m \geq 0$$

This is the first manifold we have considered which has a boundary, which is written as ∂M for manifold M. In fact, this is how we define boundaries for more general manifolds.

Consider the half space inheriting the subspace topology from $\mathbb{R}^m$. We extend the definition of manifolds to include sets with charts mapping their open sets to open sets in H^m. The points in the manifold M which are, via the charts, mapped to the boundary of H^m, are *defined* to form ∂M.

Heine-Borel, more precisely

✱ **Theorem - Heine Borel**
Consider $S \subseteq \mathbb{R}^m$ with the subspace topology. Iff S is a closed and bounded subspace – bounded in extent, i.e. $\exists R$ s.t. $\forall x, y \in S, |x - y| \leq R$ – then S is compact.

Example : Orbifold

A topological space, but not a smooth manifold: Consider $\mathbb{R}^n/\mathbb{Z}_m$, or more particularly to visualise, $\mathbb{R}^2/\mathbb{Z}_m$. We can think of this as taking a section of the plane which, in polar coordinates, corresponds to $0 \leq \theta < \frac{2\pi}{m}$, and identifying $\theta \sim \theta + \frac{2\pi}{m}$ (this is the relation which quotients the space). We can see that this becomes a kind of cone shape, which is smooth everywhere except at 0 – so not a smooth manifold.

Differentiable Maps

Consider two smooth (C^∞) manifolds with atlases $M, \{(U_i, \psi_i)\}$ and $N, \{(V_a, \phi_a)\}$ with a map $f : M \to N$. The map is defined to be differentiable if $\phi_a \cdot f \cdot \psi_i^{-1}$ is a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$ for all i, a.

Nakahara refers to smooth maps as differentiable maps.

When both manifolds are complex, we can similarly define analytic maps.

Is this definition coordinate-dependent?

Short answer, no. Long answer?

Since composition of smooth maps is smooth, we can extend this from one coordinate system to any other *compatible* coordinate system.

Consider two compatible coordinate systems, then they can be combined into one atlas, so the transformation between coordinate systems is really just a transition function of the atlas, which – by definition of compatible coordinate systems – must be smooth. Then we can compose this with the original smooth map $\phi_a \cdot f \cdot \psi_i^{-1}$ to get $\phi_a \cdot f \cdot \psi_j^{-1}$, which will also be smooth – hence the idea of smooth map extends to all compatible coordinate systems.

Immersion and Embedding

If $\exists f : M \to N, \dim M \leq \dim N$, and the map f is injective and an *immersion*, then the map f is an embedding.

What is an *immersion*? The function f_*, corresponding to f's derivative (but not exactly that, we discuss more later), is also injective.

When f is an embedding, $f(M) \subseteq N$ is a submanifold of N.

Whitney Embedding Theorem

Any smooth real m-dim manifold (which includes Hausdorff and second countability in the requirements) can be smoothly embedded into $\mathbb{R}^{2m}$ $(m > 0)$.

Diffeomorphisms

If the map from one manifold to another is invertible, and both it and its inverse are smooth, then the map is a **diffeomorphism**. Diffeomorphisms only exist between equal-dimension manifolds. In fact, the existence of a diffeomorphism between two manifolds is understood as an equivalence between the manifolds.

The set of autodiffeomorphisms of M are known as $\text{Diff}(M)$.

Two manifolds with the same topology need not be diffeomorphic – this is where incompatible charts come in. It can be shown that S^7 allows for 28 unique differentiable structures, and S^{11} allows 992. Surprisingly, $\mathbb{R}^4$ allows an infinite number of unique differentiable structures – this result was proved by Donaldson, now a mathematician at Imperial, by studying the possible ways to set up Physics (a Yang-Mills theory) on $\mathbb{R}^4$.

Smooth Curves

Map C from $\mathbb{R}$ to M such that $\psi_i \cdot C$ is smooth $\forall i$.

A closed curve has the additional requirement that for the domain (a, b) of C, $C(a) = C(b)$. Then it makes more sense to think of it as $C' : S^1 \to M$.

Functions

The simplest smooth maps, $f : \mathcal{M} \to \mathbb{R}$. Smooth when $f \cdot \psi_i^{-1} : \mathbb{R}^m \to \mathbb{R}$ is smooth $\forall i$.

The set of functions on $\mathcal{M}$ is $\mathcal{F}(\mathcal{M})$, and inherits the ring structure of functions on $\mathbb{R}$.

Tangent Vectors

Considering a curve $C : \mathbb{R} \to \mathcal{M}$, we can define the *tangent* at any point: $\frac{\mathrm{d}}{\mathrm{d}\lambda}C(\lambda)$. Given a coordinate patch, so $C(\lambda) \equiv x^\mu(\lambda)$, the tangent is also a point in $\mathbb{R}^m$, $\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(\lambda)$.

We define an equivalence relation of curves passing through a point p for when they have identical tangents at the point.

An equivalence class of this relation is the formal definition of a **tangent vector** at p, V_p.

The set of all such tangent vectors is the tangent space at p, $T_p(\mathcal{M})$.

The local $\mathbb{R}^m$-ness of the manifold and hence the tangent space gives it a vector space structure. $\psi \cdot C : \mathbb{R} \to \mathbb{R}^m$, WLOG choose $\psi(p) = 0$, so

$$C_3 = \psi^{-1}[a(\psi \cdot C_1) + b(\psi \cdot C_2)]$$

is a valid curve through p. Then of course

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(C_3(\lambda))\bigg|_{\lambda=0} = a\,\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(C_1(\lambda))\bigg|_{\lambda=0} + b\,\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(C_2(\lambda))\bigg|_{\lambda=0}$$

Thus, the tangent space is isomorphic to $\mathbb{R}^m$. Being a vector space, we can give it a basis and write

$$V_p = v^\mu e_{p,(\mu)}$$

A basis transformation $e'_i = A_i{}^a e_a$ will lead to the components changing as $V'^i A_i{}^a = V^a$.

Dual Vectors

Consider linear functions $f : V \to \mathbb{R}$ on a vector space. They are of course completely defined by their action on a particular basis:

$$f(v) = v^i f(e_i)$$

And the space of linear functions is also a vector space over $\mathbb{R}$ in itself. (in general, $\mathbb{R}$ can be replaced everywhere by whatever field the vector space is over).

The space of all functions is called V^*, the dual vector space. It has the same dimensions, since $\{f(e_i)\}$ completely specifies any element of it. This is essentially an expansion in the basis e^i, defined such that

$$e^i(e_j) = \delta^i{}_j$$

This is called the basis dual to the basis $\{e_i\}$. Of course, another basis may be chosen.

$$\dim V^* = \dim V$$

Note that the dual basis will transform the same way the original vector space's vector components will, under a change of basis:

$$A_i{}^a e'^i = e^a$$

And between these vector spaces, an **inner product** – a bilinear map to the field – can be defined.

$$\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$$
$$\langle f, v\rangle := f(v) = f_i v^i$$

Since this can also be understood as defining the action of v on f, we realise that (for finite-dim vector spaces) $(V^*)^* = V$.

Cotangent Space

$T_p^*(\mathcal{M})$, dual to $T_p(\mathcal{M})$.

Cotangent Vectors

Elements of the dual vector space. AKA One-forms (especially on a manifold).

Tensors

Since covectors are linear maps on the vector space and vectors are linear maps on the covector space, we can generalise to higher-dimensional objects which are elements of some $(T_p(\mathcal{M}))^q \times (T_p^*(\mathcal{M}))^r$ and hence linear maps on $(T_p^*(\mathcal{M}))^q \times (T_p(\mathcal{M}))^r$.

These are tensors, and the above example would be called a (q, r) rank tensor. The space of all tensors of a particular rank is also a vector space – written as $\mathcal{J}^q_{r,p}(\mathcal{M})$.

The *tensor (outer) product* of any two vectors/covectors/tensors, denoted by $\otimes$, is a map taking a rank (q_1, r_1) and a rank (q_2, r_2) tensor and (by product of components, given a particular basis) giving a $(q_1 + q_2, r_1 + r_2)$ tensor. The basis independent definition is similar, the product function acts on $\sum_i q_i$ covectors and $\sum_i r_i$ vectors.

Contraction is the familiar idea of summing over one contravariant and one covariant index that have been identified. It's a trace along two of the dimensions (once a basis is defined, but the operation is independent of basis, because the resultant is also a tensor).

Tangents on functions

Two curves through a point are **tangent** to each other if their **tangent vectors** at that point – $\frac{\mathrm{d}x^\mu(C(\lambda))}{\mathrm{d}\lambda}$ – are equal.

Thus, the directional derivative can be defined as the derivative of a field along a curve:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}f\cdot C(\lambda)\bigg|_{p=C(\lambda_0)} = \frac{\partial}{\partial x^\mu}f(x)\bigg|_{x^\mu(p)}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(\lambda)\bigg|_{\lambda_0}$$

Where $x^\mu(\lambda) = \psi(C(\lambda))$, and $f(x)$ is shorthand for $f\cdot\psi^{-1}(x)$.

Note that this derivative is the same for any $C \in [C] = V_p$ ($\frac{\mathrm{d}x^\mu(C_1(\lambda))}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu(C_2(\lambda))}{\mathrm{d}\lambda}$), so we can uniquely define the directional derivative along a vector as well.

$$V_p[f] = \frac{\mathrm{d}}{\mathrm{d}\lambda}(f\cdot C(\lambda))\bigg|_p$$

We can easily define an operator space comprised of $\hat{X}_p : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ which accomplish the directional derivative along particular vectors, and show that this is an isomorphic vector space, $\hat{X}_p \cong X_p \in T_p\mathcal{M}$. Trivial.

Now recall that given a coordinate chart, we wrote $V_p = v^\mu e_\mu$, where

$$v^\mu = \frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu(\lambda)\bigg|_p$$

But then

$$\begin{aligned} V_p[f] &= v^\mu\frac{\partial}{\partial x^\mu}f(x)\bigg|_p \\ e_\mu[f] &= \frac{\partial}{\partial x^\mu}f(x)\bigg|_p \\ \implies \hat{e}_\mu &= \frac{\partial}{\partial x^\mu}\bigg|_p \end{aligned}$$

We do follow with an abuse of notation going forwards, and do not distinguish between e_μ and $\hat{e}_\mu$. One can formally show the equivalence by taking a curve through p given by

$$x^\mu_{(\nu)}(\lambda) = \psi^\mu(p) + \lambda\delta^\mu_\nu$$

The tangent vector of this curve is e_v, and the directional derivative along this curve, or any curve in the equivalence class of the tangent vector, is shown to be $\hat{e}_\nu f$.

Change of Coordinates

$\{x^\mu\} \to \{y^\alpha\}$, then

$$\begin{aligned}\hat{e}'_\mu = \frac{\partial}{\partial y^\mu} \text{ s.t. } V_p = v^\mu e_\mu = v'^\mu e'_\mu \\ \& \quad v'^\mu \left.\frac{\partial f(x(y))}{\partial y^\mu}\right|_{y(p)} = v'^\mu \left.\frac{\partial x^\nu(y)}{\partial y^\mu}\right|_{y(p)} \left.\frac{\partial f(x)}{\partial x^\nu}\right|_{x(p)} \\ \implies v'^\mu \left.\frac{\partial x^\nu(y)}{\partial y^\mu}\right|_{y(p)} = v^\nu\end{aligned}$$

Since $V_p[f]$ is defined coordinate-indendently for all functions f.
Going forwards, we can also refer to tangent vectors of a curve $C(\lambda)$ as $\frac{\mathrm{d}}{\mathrm{d}\lambda}$, which in a particular chart becomes

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} \left.\frac{\partial}{\partial x^\mu}\right|_{p=C(\lambda)}$$

Cotangent vectors as Differentials

For any $f : \mathcal{M} \to \mathbb{R}$, there is a natural definition of a cotangent vector ω at any point p, i.e. a linear map

$$\begin{aligned}\omega : T_p\mathcal{M} \to \mathbb{R} \\ V_p \xrightarrow{\omega} V_p[f]\end{aligned}$$

Given a coordinate basis,

$$V_p[f] = v^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{x(p)}$$

Using the dual basis defined by $e^i(e_j) = \delta^i{}_j$,

$$\begin{aligned}\omega = \omega_\mu e^\mu \implies \omega(V) = \omega_\mu V^\mu \\ \implies \omega_\mu = \left.\frac{\partial f}{\partial x^\mu}\right|_{x(p)}\end{aligned}$$

Now recall that the differential of a map $\mathbb{R}^m \to \mathbb{R}$ (which, recall, $f = f \cdot \psi^{-1}$ really is) is well-defined as

$$\mathrm{d}f|_p = \left.\frac{\partial f}{\partial x^\mu}\right|_{x(p)} \mathrm{d}x^\mu|_p$$

Which implies

$$\mathrm{d}f = \omega_\mu \mathrm{d}x^\mu$$

Recall the equivalence class of curves definition of tangent vectors. Similarly, one can define a cotangent vector as the equivalence class of functions with the same gradient $\left\{\frac{\partial f}{\partial x^\mu}\right\}$ at a point p. There is an isomorphism between linear maps from $T_p\mathcal{M}$ to $\mathbb{R}$ and such equivalence classes of functions. This extends to an isomorphism between cotangent vectors (the linear maps) and the differentials (clearly a vector space), mapping the covector $\omega = \omega_\mu e^\mu$ to the **one-form** $\omega_\mu \mathrm{d}x^\mu$ (such expressions are by definition one-forms). This is closely related to the isomorphism between vectors and directional derivatives, and we similarly identify the dual coordinate basis with the coordinate differentials at p,

$$e^\mu \equiv \mathrm{d}x^\mu|_p$$

Giving the sensible duality condition

$$\langle \mathrm{d}x^\mu, \partial_{x^\nu} \rangle = \delta^\mu_\nu$$

Through the equivalence classes,

$$\left\langle \mathrm{d}f|_p, \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_p \right\rangle = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_p$$

Which can then be used to derive the previous duality condition in a particular patch.

Change of Coordinates

$$\begin{aligned} \langle e'^\mu, e'_\nu \rangle = \langle e^\mu, e_\nu \rangle &= \delta^\mu_\nu \\ e'_\mu &= \left.\frac{\partial x^\nu}{\partial y^\mu}\right|_p e_\nu \\ \implies e'^\mu &= \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_p e^\nu \end{aligned}$$

Same as $\mathrm{d}y^\mu = \frac{\partial y^\mu}{\partial x^\nu}\mathrm{d}x^\nu$.

Tensors

$$T_p = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes \mathrm{d}x^{\beta_1} \otimes \dots \otimes \mathrm{d}x^{\beta_q}$$

This expansion connects our tensors to the more familiar definition of objects with certain transformation laws under diffeomorphisms:

$$T'^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} \frac{\partial y^{\mu_1}}{\partial x^{\alpha^1}} \cdots \frac{\partial y^{\mu_p}}{\partial x^{\alpha^p}} \frac{\partial x^{\beta_1}}{\partial y^{\nu^1}} \cdots \frac{\partial x^{\beta_q}}{\partial y^{\nu^q}}$$

Induced maps, tensor fields and flows

Push-forward

$f : \mathcal{M} \to \mathcal{N}$ induces a map on $T_p\mathcal{M}$,

$$\begin{gathered} f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N} \\ V \to f_*V \end{gathered}$$

This is called the push-forward. To define it, first define the pull-back of any function $g : \mathcal{N} \to \mathbb{R}$:

$$f^*(g) \equiv g \cdot f : \mathcal{M} \to \mathbb{R}$$

V can measure the pull-back's directional derivative. Equating that to directional derivatives of g gives us f_*V:

$$f_*V[g] \equiv V[g \cdot f] \quad \forall g$$

Importantly, this makes the push-forward a linear map on $T_p\mathcal{M}$, thus it maintains the same vector space structure. It is also interpretable as a linear approximation to f.

The push-forward can be understood using charts:

$$
\begin{gathered}
V = v^\mu \partial_{x^\mu}|_p, \quad f_* V = w^\alpha \partial_{y^\alpha}|_{f(p)} \\
y = f(x) \\
f_* V[g(y)] = V[g \cdot f(x)] \quad \forall g \\
\implies w^\alpha \partial_{y^\alpha}[g(y)]|_p = \left(v^\mu \frac{\partial y^\alpha}{\partial x^\mu} \right) \partial_{y^\alpha}[g(y)]|_p \quad \forall g \\
\implies w^\alpha = v^\mu \frac{\partial y^\alpha}{\partial x^\mu} \bigg|_p
\end{gathered}
$$

And it generalises to tensor products,

$$
\begin{gathered}
f_* : T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M} \to T_{f(p)}\mathcal{N} \otimes \cdots \otimes T_{f(p)}\mathcal{N} \\
V \otimes \cdots \otimes W \to f_* V \otimes \cdots \otimes f_* W
\end{gathered}
$$

And hence to tensors:

$$
f_* : \mathcal{T}^q_{0,p}(\mathcal{M}) \to \mathcal{T}^q_{0,f(p)}(\mathcal{N})
$$

With components transforming correspondingly:

$$
w^{\alpha_1 \cdots \alpha_q} = v^{\mu_1 \cdots \mu_q} \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_q}}{\partial x^{\mu_q}} \bigg|_p
$$

And it distributes across compositions:

$$
\begin{gathered}
f : \mathcal{M} \to \mathcal{N}, \quad g : \mathcal{N} \to \mathcal{O} \\
\implies (g \cdot f)_* V = g_* \cdot f_* V
\end{gathered}
$$

Pull-Back

The pull-back of a function on $\mathcal{N}$ to a function on $\mathcal{M}$ was defined. This directly gives a definition of the pull-back on the cotangent space:

$$
\begin{gathered}
f^* : T^*_{f(p)}\mathcal{N} \to T^*_p\mathcal{M} \\
\omega \to f^* \omega
\end{gathered}
$$

But it remains to be shown that this is well-defined, i.e. $f^*[\mathrm{d}g] = [\mathrm{d}(f^* g)]$. Instead we use an equivalent definition and leave the proof of equivalence to the sorry sod who decides it worth their time.

Define the pull-back using the push-forward:

$$\langle f^*\omega, v\rangle = \langle \omega, f_* v\rangle$$

By definition, this is linear. Given charts,

$$\begin{aligned} f_* v &= \left(v^\mu \frac{\partial y^\alpha}{\partial x^\mu}\right) \partial_{y^\alpha} \\ \implies f^*\omega &= \left(\omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}\right) \mathrm{d}x^\mu \end{aligned}$$

This naturally extends to rank (0,r) tensors.

Tensor Fields

Denote the tensor bundle:

$$\mathcal{J}^q_r = \coprod_{p\in\mathcal{M}} \mathcal{J}^q_{r,p}$$

A tensor field is

$$\begin{aligned} A : \mathcal{M} &\to \mathcal{J}^q_r \\ p &\to A(p) \in \mathcal{J}^q_{r,p} \\ \text{s.t. } A(p) &\text{ is smooth} \end{aligned}$$

Note the requirement that the tensor associated with each point must belong to the tensor space *at that point*.

What does smooth mean, here? We need to define it.

Consider a vector field V, it acts on a function $f : \mathcal{M} \to \mathbb{R}$ and gives a directional derivative at each point, giving another $V[f] : \mathcal{M} \to \mathbb{R}$.

$$\begin{aligned} V[f] : \mathcal{M} &\to \mathbb{R} \\ p &\to V_p[f] \end{aligned}$$

Thus we define V as smooth if $V[f]$ is smooth when f is. Since $V[f](x) = v^\mu(x)\partial_{x^\mu} f(x)$, in a chart this implies all $v^\mu(x)$ must be smooth.

Similarly, a cotangent field takes a vector field and gives a real function, $\omega[V] : \mathcal{M} \to \mathbb{R}$, and is smooth when the real function is smooth for all smooth vector fields V. Since $\omega[V](x) = \omega_\mu(x)v^\mu(x)$, this simplifies to the smoothness of $\omega_\mu(x)$.

Similarly, using the definition of (q, r) tensors as linear functions on q, r cotangent and tangent vectors respectively, we define a smooth (q, r) tangent field as one giving smooth functions when acting on smooth covectors and vectors – and again this simplifies to having smooth components in any chart's coordinate basis.

Because we're on a manifold, for overlapping regions it is sufficient to show smoothness in any one chart.

Tensor fields inherit linearity over the field $\mathcal{F}(\mathcal{M})$:

$$\begin{gathered}A, B \in \mathcal{J}_r^q(\mathcal{M}), \quad a, b \in \mathcal{F}(\mathcal{M}) \\ C = aA + bB \in \mathcal{J}_r^q(\mathcal{M}) : \\ C(p) = a(p)A(p) + b(p)B(p) \forall p \in \mathcal{M} \\ C \text{ is smooth if } a, b, A, B \text{ are smooth.}\end{gathered}$$

Embeddings, Immersions

Consider $f : \mathcal{M} \to N, \dim \mathcal{M} \leq \dim \mathcal{N}$. f is an **immersion** if the corresponding push-forward is injective, or equivalently, $\operatorname{rank}(f_*) = m = \dim \mathcal{M}$. (This is equivalent because the push-forward is a linear operation – it would not be otherwise.)

f is an **embedding** if it is an injective immersion (i.e. both f, f_* are injective). Then $f(\mathcal{M})$ is defined to be a submanifold of $\mathcal{N}$ and is diffeomorphic to $\mathcal{M}$.

Induced maps on tensor fields

For a vector,

$$f_* V[g]|_{f(p)} = V[g \cdot f]|_p$$

To extend this to a vector field $V(p)$, it is necessary for f to be injective, otherwise $f_* V[g]$ may be multivalued for $f(p) = f(p')$. It isn't necessary here for f to be an embedding, but we would like $f(\mathcal{M})$ to be a manifold diffeomorphic to $\mathcal{M}$, probably to make some later step easier.

Even with an embedding, the push-forward can only define a vector field on its image $f(\mathcal{M})$. So we define the push-forward for a vector field V as:

$$f_* V[g](f(p)) = Y[g \cdot f](p) \quad \forall g$$

Since $f_* V[g]$ is a function on $f(\mathcal{M}) \subset \mathcal{N}$. Equivalently,

$$f_* Y[g] \cdot f = Y[g \cdot f] \quad \forall g$$

For a smooth vector field Y and smooth function f, $f_* Y[g]$ being smooth on $f(\mathcal{M}) \forall g$ implies that $f_* Y$ is a smooth vector field on $f(\mathcal{M})$. Here, we use the diffeomorphism between $\mathcal{M}, f(\mathcal{M})$ to be able to define an f^{-1} on $f(\mathcal{M})$.

The pull-back, on the other hand, is almost trivially extended to covector fields:

$$\langle f^* \omega, v \rangle|_p = \langle \omega, f_* v \rangle|_{f(p)}$$

Require this to be true for all $p \in \mathcal{M}$ and for all vector fields $v \in T_p\mathcal{M}$, and $f^*\omega$ is then a well-defined covector field, even if f is not injective. Smooth ω implies smooth $f^*\omega$.

If f is a diffeomorphism, then $\mathcal{M}, \mathcal{N}$ are diffeomorphic and the push-forward and pull-back can relate vector fields on $\mathcal{M}$ to those on $\mathcal{N}$ and vice versa.

Given charts on both manifolds with $f(x) = y(x)$, one can show

$$(f^*\omega)_\mu(x) = \omega_\alpha(y) \frac{\partial y^\alpha}{\partial x^\mu}$$

For a smooth f and smooth ω, these components are all smooth, hence so is $f^*\omega$.

Similarly, consider the push-forward on a vector in a coordinate representation:

$$w^\alpha = v^\mu \frac{\partial y^\alpha}{\partial x^\mu}\bigg|_p$$

When extending to vector fields, we'll get $w^\alpha(y)$ and $v^\mu(x(y))$ – but this requires $x(y) = f^{-1}(y)$ to be well-defined, which it only is if f is injective, and then too it is only well-defined on $f(\mathcal{M})$ and not the whole of $\mathcal{N}$.

Induced Maps and Diffeomorphisms

In general,

- We can't push covectors or pull vectors, since f isn't invertible
- We can push vectors/pull covectors at points, but not vector fields unless f is injective.

But for diffeomorphic f we can do much more. $(f^{-1})_*, (f^{-1})^*$ can pull vectors or push covectors, and hence any (q, r) tensor can be pushed or pulled. This is easily extended to tensor fields too, and smooth tensor fields are pushed to smooth tensor fields.

For an embedding, all of the above holds between $\mathcal{M}, f(\mathcal{M})$.

Flows

A vector field generates curves whose tangents at a point are defined by the vector field – these are called **integral curves**.

A curve has tangent $\frac{\mathrm{d}}{\mathrm{d}\lambda}|_p$, which must equal V, hence

$$\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} = V^\mu(x(\lambda))$$

Solving the ODE gives a family of curves (parametrised by the integration constant) which are the integral curves. Given a point p the curve goes through, there is a unique solution:

A smooth ODE has a unique solution for some finite interval around $\lambda = 0$. The vector field is **complete** if any integral curve through a point can be extended to be defined for all $\lambda \in \mathbb{R}$. On a compact manifold, every vector field is complete.

Consider the solution:

$$\begin{gathered} x^\mu(\lambda) = \sigma_V^\mu(\lambda, p) \\ [\sigma_V(0,p) = p] \\ \sigma_V : \mathbb{R} \times \mathcal{M} \to \mathcal{M} \end{gathered}$$

σ_V is called the **flow** defined by V, which is a smooth map when the vector field is smooth. (Sketch of proof: Mathematics Stack Exchange **Every smooth vector field generates a smooth flow**)

Properties:

- $\sigma_{cV}(\lambda) = \sigma_V(c\lambda)$
- $\sigma_V(\lambda + s, p) = \sigma_V(\lambda, \sigma_V(s,p))$
- At fixed λ, $\sigma_V(\lambda, \cdot)$ is an autodiffeomorphism with an abelian group structure over the set $\{\sigma_V(\lambda, \cdot) : \lambda \in \mathbb{R}\}$, given by
  - $\sigma_V(0) = \mathrm{Id}_{\mathcal{M}}$
  - $(\sigma_V(\lambda))^{-1} = \sigma_V(-\lambda)$
  - $\sigma_V(\lambda) \cdot \sigma_V(s) = \sigma_V(\lambda + s)$

This group is also the set of "active transformations" generated by V. This is made obvious by taylor expanding the flow for small ϵ using the ODE:

$$\begin{aligned} x'^\mu = \sigma_V^\mu(\epsilon, x^\mu) &= x^\mu + \epsilon V^\mu(x) + \mathcal{O}(\epsilon^2) \\ \implies f(x') &= f(x) + \epsilon V^\mu(x)\partial_{x^\mu} f(x) + \mathcal{O}(\epsilon^2) \\ &= f(x) + \epsilon V[f](x) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Here we introduced the generalisation of the action of a vector on a function, to the action of a vector field on a function – the **Lie derivative**.

$$
\begin{aligned}
V[f](p) &\equiv \lim_{\epsilon\to 0}\left(\frac{f(\sigma_V(\epsilon,p)) - f(p)}{\epsilon}\right) \\
&= V_p[f] = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_p = V^\mu(p)\frac{\partial f}{\partial x^\mu}(p)
\end{aligned}
$$

The Lie derivative is further generalised to act not only on scalars, but also vectors and more. First, new notation.

$$
\mathcal{L}_V[f]
$$

Acting on a vector field,

$$
\mathcal{L}_V[Y](p) \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma_V(-\epsilon,\cdot)_* Y(\sigma_V(\epsilon,p)) - Y(p)}{\epsilon}\right)
$$

Note that we need to push forward the vector field to evaluate it on the manifold obtained by acting with the flow. $\sigma_V(\epsilon,\cdot)$ maps from one copy of $\mathcal{M}$ to another, and to evaluate Y on the second one we really ought to evaluate its pull back to the first one. In fact, that is the only we actually can, because naively evaluating, $Y(p) \in T_p\mathcal{M}$ but $Y(\sigma_V(\epsilon,p)) \in T_{\sigma_V(\epsilon,p)}\mathcal{M}$! The pull back connects the tangent spaces. Note, we are pulling back using $(f^{-1})_*$.

Components of a Lie Derivative

(Henceforth, $\sigma_V(\lambda) \equiv \sigma_V(\lambda,\cdot)$.)

$$
\begin{gathered}
Y(p') = Y^\mu(x(p'))\ \left.\frac{\partial}{\partial x^\mu}\right|_p \\
\sigma_V(-\epsilon)_* Y(\sigma_V(\epsilon,p)) = Y^\mu(x(\sigma_V(\epsilon,p)))\frac{\partial x^\nu(p)}{\partial x^\mu(\sigma_V(\epsilon,p))}\ \left.\frac{\partial}{\partial x^\nu}\right|_p \\
Y^\mu(x(\sigma_V(\epsilon,p))) = Y^\mu(x(p)) + \epsilon V^\alpha(x(p))\partial_{x^\alpha}Y^\mu(x)|_{x(p)} + \mathcal{O}(\epsilon^2) \\
x^\mu(p) = x^\mu(\sigma_V(\epsilon,p)) - \epsilon V^\mu(x(\sigma_V(\epsilon,p))) + \mathcal{O}(\epsilon^2) \\
\implies \frac{\partial x^\nu(p)}{\partial x^\mu(\sigma_V(\epsilon,p))} = \delta^\nu{}_\mu - \epsilon\partial_{x^\mu}V^\nu(x)|_{x(\sigma_V(\epsilon,p))} + \mathcal{O}(\epsilon^2)
\end{gathered}
$$

But since for any function, $f(\sigma(\epsilon,x)) = f(x) + \mathcal{O}(\epsilon)$,

$$
\frac{\partial x^\nu(p)}{\partial x^\mu(\sigma_V(\epsilon,p))} = \delta^\nu{}_\mu - \epsilon\partial_{x^\mu}V^\nu(x)|_{x(p)} + \mathcal{O}(\epsilon^2)
$$

Substitute the horrifying monstrosities we have derived into the pull-back,

$$\sigma_V(-\epsilon)_* Y(\sigma_V(\epsilon, p)) = \left[Y^\nu + \epsilon \left(V^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu V^\nu\right) + \mathcal{O}(\epsilon^2)\right] \partial_\nu\big|_{x(p)}$$

Thus we can evaluate the Lie derivative:

$$\mathcal{L}_V[Y](p) = \left(V^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu V^\nu\right) \partial_\nu\big|_{x(p)}$$

Which has an interesting skew-symmetric structure that harkens back to the commutator. But it includes derivatives too. Let us define this as a new thing, the

Lie Bracket

$$\begin{gathered}[\cdot,\cdot] : \mathcal{J}_0^1 \times \mathcal{J}_0^1 \to \mathcal{J}_0^1 \\ X,\ Y \to [X, Y]\end{gathered}$$

The definition is that for all $g \in \mathcal{F}(\mathcal{M})$,

$$[X, Y][g] \equiv X[Y[g]] - Y[X[g]]$$

In a particular chart, this easily leads to the expression

$$[X, Y] = \left(X^\alpha \partial_\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu\right) \partial_\mu$$

Thus the lie derivative is

$$\mathcal{L}_V Y = [V, Y]$$

The Lie bracket is furthermore skew-symmetric and satisfies Jacobi:

$$\begin{gathered}[X, Y] = -[Y, X] \\ [[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0\end{gathered}$$

Under a diffeomorphism,

$$f_*([X, Y](p)) = [f_* X, f_* Y](f(p))$$

Proof.

$$
\begin{aligned}
& f_* X[g]\cdot f = X[g\cdot f] \\
\implies & X[Y[g\cdot f]] = X[f_*Y[g]\cdot f] = (f_*X)[(f_*Y)[g]]\cdot f \\
\implies & f_*([X,Y])[g]\cdot f = [X,Y][g\cdot f] = X[Y[g\cdot f]] - Y[X[g\cdot f]] \\
& = (f_*X)[(f_*Y)[g]]\cdot f - (f_*Y)[(f_*X)[g]]\cdot f \\
& = [f_*X, f_*Y][g]\cdot f
\end{aligned}
$$

This holds for all g, hence proved.

Commuting Flows

The commutator of vector fields measures the closure of their flows – does it matter if you perform one before or after the other?

Theorem. The diffeomorphisms $\sigma_X(\lambda)\cdot\sigma_Y(\epsilon), \sigma_Y(\epsilon)\cdot\sigma_X(\lambda)$ are equivalent iff $[X,Y]=0$.

Lie Derivatives of Tensors

For scalars, it's the usual directional derivative:

$$
\mathcal{L}_V f \equiv V[f]
$$

For (q,r) tensors, we use the corresponding pull-back from $\mathcal{J}^q_{r,\sigma_V(\epsilon,p)}$ to $\mathcal{J}^q_{r,p}$:

$$
\begin{gathered}
\mathcal{L}_V[A](p) \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma_V(-\epsilon)_* A(\sigma_V(\epsilon,p)) - A(p)}{\epsilon}\right) \\
\mathcal{L}_V \mathcal{J}^q_r \to \mathcal{J}^q_r
\end{gathered}
$$

Exercise. Show that for a covector field $\omega_\mu(x)\mathrm{d}x^\mu$,

$$
\mathcal{L}_V\omega = (V^\alpha\partial_\alpha\omega_\mu + \omega_\alpha\partial_\mu V^\alpha)\,\mathrm{d}x^\mu
$$

And that the same result can be derived by demanding:

$$
\mathcal{L}_V\langle\omega, Y\rangle = \langle\mathcal{L}_V\omega, Y\rangle + \langle\omega, \mathcal{L}_V Y\rangle
$$

Check that the Lie derivative obeys the **Leibnitz rules**:

$$
\begin{gathered}
\mathcal{L}_V(fA) = V[f]A + f\mathcal{L}_V A \\
\mathcal{L}_V(A_1\otimes A_2) = (\mathcal{L}_V A_1)\otimes A_2 + A_1\otimes(\mathcal{L}_V A_2)
\end{gathered}
$$

These allow deriving the lie derivative for a tensor field much more easily.

Example: Active coordinate transformations in GR and symmetry

An active transformation takes point p to point p', whereas in a passive transformation the coordinates $x = \psi(p)$ for p are changed to $x' = \psi(p')$, so now $x' = \tilde{\psi}(p)$. The chart is changed.

For a tensor field, consider the new chart discussed above, and let the tensor field have components $T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$ in the old chart and $T'^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$ in the new chart, then

$$\delta T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} = T'^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} - T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$$
$$\therefore \delta T = -\epsilon \mathcal{L}_V T + \mathcal{O}(\epsilon^2)$$

If $\mathcal{L}_V T = 0$, then the diffeomorphism generated is a symmetry of the tensor field. For the metric, these are called *isometries*, and the vector fields satisfying $\mathcal{L}_V g = 0$ are called the *Killing vectors*.

Lie Groups

- Define groups
- Lie groups are groups on a manifold, with the multiplication and the inverse being differentiable maps.
  - Any continuous subgroup of a Lie group is Lie.
- $SU(2) \sim S^3, SO(3) \sim \mathbb{RP}^3$ (antipodal map, double cover, etc).

Group action

$$\Phi : G \times \mathcal{M} \to \mathcal{M}$$
$$(g, p) \to \Phi_g(p)$$

Where Φ is smooth and $\Phi_{g_1} \cdot \Phi_{g_2} = \Phi_{g_1 g_2}$.

Then Φ_g is a diffeomorphism (for any g).

- Every group acts on itself by multiplication.
- *Free action* → $\Phi_{g \neq e}$ leaves no points fixed.
- *Transitive action* → $\forall p, p' \in \mathcal{M}$, $\exists g \in G$ s.t. $p' = \Phi_g(p)$.
- *Orbit* → $\mathrm{Orb}_G(p) = \{\Phi_g(p) : g \in G\}$
- *Stabiliser* → $\mathrm{stab}_G(p) = H_p = \{g : p = \Phi_g(p)\}$. Always a subgroup, since it contains identity, inverses, and must be closed under group product.
  - For a freely acting group, $H_p = \{e\}$ is trivial.

Action on self

Left action

$$
\begin{gathered}
L_g : G \to G \\
h \to gh
\end{gathered}
$$

Right action

$$
\begin{gathered}
R_g : G \to G \\
h \to hg
\end{gathered}
$$

Conjugation

$$
\begin{gathered}
C_g : G \to G \\
h \to ghg^{-1}
\end{gathered}
$$

Conjugacy is also defined on subsets as

$$
C_g(H) = \{C_g(h) : h \in H\}
$$

- Two subgroups H_1, H_2 of G are conjugate if $\exists g \text{ s.t. } C_g H_1 = H_2$.
- Conjugate subgroups are isomorphic (converse may not be true).
- Subgroup H is normal if $C_g H \subseteq H \forall g \in G \iff gH = Hg \forall g$.
- The stabiliser groups of $p, p' = \Phi_g(p)$ are conjugate.
- Conjugacy is an equivalence relation on both the group and the space of subgroups.

Coset Submanifold

Given a subgroup H, define an equivalence relation $g \sim g'$ when $g' = gh, h \in H$. Thus the equivalence classes:

$$
[g] = gH
$$

And the coset submanifold:

$$
\begin{gathered}
G/H \equiv \{[g]\} \\
\dim G/H = \dim G - \dim H
\end{gathered}
$$

When H is normal, G/H is also a Lie group, $[g_1] \cdot [g_2] = [g_1 \cdot g_2]$.

Homogeneous Space

Given a compact manifold $\mathcal{M}$ on which G has a transitive action Φ, then

$$G/\operatorname{stab}_G(p) = \mathcal{M} \quad \forall p \in \mathcal{M}$$

Independent of p, hence "homogeneous".

Proof.

Pick any $p \in \mathcal{M}$.

$$\begin{gathered}\chi(g) \equiv \Phi_g(p) \\ \chi(gh) = \chi(g) \Longleftrightarrow h \in \operatorname{stab}_G(p) \\ \therefore \tilde{\chi}([g] \in G/\operatorname{stab}_G(p)) \equiv \chi(g)\end{gathered}$$

$\tilde{\chi} : G/H_p \to \mathcal{M}$ is injective because for $\tilde{\chi}([g']) = \tilde{\chi}([g''])$, then $\chi(g') = \chi(g'') \implies (g'')^{-1}g' \in \operatorname{stab}_G(p) \implies [g'] = [g'']$, hence proved.

Since Φ is transitive, $\tilde{\chi}$ is also surjective, hence bijective and invertible. It is also smooth, since Φ is, and hence it is a diffeomorphism. Thus G/H_p is diffeomorphic to $\mathcal{M}$.

4.8 Further Examples

Other commonly arising examples in physics are:

- $S^n = O(n+1)/O(n) = SO(n+1)/SO(n)$
- $S^{2n+1} = U(n+1)/U(n) = SU(n+1)/SU(n)$
- $\mathbb{CP}^n = U(n+1)/\left(U(n) \times U(1)\right) = SU(n+1)/U(n)$
- $\mathbb{RP}^n = SO(n+1)/O(n)$
- Various non-compact examples: de Sitter space dS_n, anti- de Sitter space AdS_n involve $SO(1,n)$ which is the Lorentz group in $n+1$ dimensions and $SO(2,n)$ which is the conformal group in in n dimensions consisting of matrices $A \in SL(n+2,\mathbb{R})$ preserving $A\eta' A^T = \eta'$ where $\eta' = \operatorname{Diag}\{-1,-1,1,1,\ldots,1\}$.
- $dS_n = SO(1,n)/SO(1,n-1)$
- $AdS_n = SO(2,n-1)/SO(1,n-1)$

Left-Invariant Vector Fields

Consider the left action of a group on itself, which is transitive. It is an autodiffeomorphism on the group, so we can push/pull vectors. In particular,

$$\begin{gathered} V \in T_e G \\ \implies L_{g*} V \in T_g G \\ X_V(g) \equiv L_{g*} V \end{gathered}$$

Thus we defined the **left-invariant vector field** using the unique extension of any vector at the identity to a vector field defined via push-forwards. Any vector field is left-invariant if

$$X(g) = L_{g*}(X(e))$$

In which case $V := X(e)$.

Verify group action compatibility:

$$\begin{gathered} X_V(gg') = (L_{gg'})_* V = (L_g \cdot L_{g'})_* V \\ = L_{g*} \cdot L_{g'*} V = L_{g*} X_V(g') \end{gathered}$$

Denote the set of all such vector fields by $\mathcal{L}(G) \subseteq \mathcal{J}_0^1$.

Lie Bracket on Left-invariant vector fields

It closes on $\mathcal{L}(G)$.

$$\begin{gathered} f_*([X, Y](p)) = [f_* X, f_* Y](f(p)) \\ \therefore L_{g*}([L_{h*} V, L_{h*} V'](h)) = [L_{gh*} V, L_{gh*} V'](gh) \\ h = e \implies L_{g*}([X_V, X_{V'}](e)) = [X_V, X_{V'}](g) \\ \implies [X_V, X_{V'}] \in \mathcal{L}(G) \end{gathered}$$

Hence proved.

Lie Algebra

Since linear combinations of left-invariant vector fields are also left-invariant vector fields, $\mathcal{L}(G)$ is a vector space, and we can further show that it is isomorphic to $T_e G$, since we have an invertible map between them, by definition.

$$\dim \mathcal{L}(G) = \dim T_e G = \dim G$$

The Lie bracket is a bilinear form on this vector space, hence promoting it to an algebra – called the Lie algebra.

Theorem. Every Lie algebra corresponds to a unique *simply connected* Lie group.

Structure Constants

Given a basis $\{e_a\}$ for T_eG, we get a basis for $\mathcal{L}(G)$,

$$L_a \equiv X_{e_a}$$

Since for any $v^a e_a = V \in T_eG$,

$$X_V = v^a X_{e_a} = v^a L_a$$

Expand the Lie bracket in the basis:

$$[L_a, L_b](g) = C_{ab}{}^{c} L_c(g)$$

$C_{ab}{}^{c}$ are the structure constants, independent of g. Properties:

- Antisymmetry in first two indices
- Jacobi identity: $C_{abc}C_{cde} + C_{dac}C_{cbe} + C_{bdc}C_{cae} = 0$

One-parameter Subgroups

A curve $g : \mathbb{R} \to G$ s.t.

$$g(t) \cdot g(s) = g(t+s)$$

Therefore

- $g(0) = e$
- $g(t)^{-1} = g(-t)$

This is an abelian subgroup of G.

For the flow generated on the group by any left-invariant vector field, the integral curve through the identity is a one-parameter subgroup.

$$\phi(t) = \sigma_{X_V}(t, e)$$

Proof.

Consider $\phi(t)$ defined as above. Also consider the curves:

$$\phi_1(t) = \sigma_{X_V}(t, \phi(s)) = \phi(t+s)$$
$$\phi_2(t) = \phi(s) \cdot \phi(t)$$

One is the integral curve through the identity, just parametrised in some displaced way. Due to the properties of flows created by smooth vector fields (see ¶ Flows), we have another expression for it. The second curve is generated by the group multiplication rule.

We'll show that they are the same curve.

Consider x^μ on the group, close to the identity.

$$X_V(g) = X^\mu(g)\, \frac{\partial}{\partial x^\mu}\bigg|_g$$

By the definition of a flow,

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\phi_1(t)) = X^\mu(\phi_1(t)) \\ \implies \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\phi_1(t))\, \frac{\partial}{\partial x^\mu}\bigg|_{\phi_1(t)} = X^\mu(\phi_1(t))\, \frac{\partial}{\partial x^\mu}\bigg|_{\phi_1(t)}\end{gathered}$$

Compare with ϕ_2,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\phi_2(t))\, \frac{\partial}{\partial x^\mu}\bigg|_{\phi_2(t)} &= \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\phi(s)\cdot\phi(t))\, \frac{\partial}{\partial x^\mu}\bigg|_{\phi(s)\cdot\phi(t)} \\ &= L_{\phi(s)*}\left(\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\phi(t))\, \frac{\partial}{\partial x^\mu}\bigg|_{\phi(t)}\right) \\ &= L_{\phi(s)*} X(\phi(t)) = X(\phi(s)\cdot\phi(t)) \\ &= X(\phi_2(t))\end{aligned}$$

▶ My problem with this proof (solved)

Thus $\phi_1(t), \phi_2(t)$ obey the same differential equation, and what's more, the initial conditions are $\phi(s)$ at $t = 0$ for both of them – by the uniqueness of ODE solutions, they must be the same function, hence

$$\phi(s+t) = \phi(s)\cdot\phi(t)$$

QED.

The converse, that any one parameter subgroup is associated with a left-invariant vector field, can also be proved.

Exponential Map

Define:

$$\begin{aligned}\exp : \mathcal{L}(G) &\to G \\ V &\to \exp(V) \equiv \phi_V(1)\end{aligned}$$

Using the one parameter subgroup $\phi_V(t)$ associated with the left-invariant vector field X_V generated by pushing forward V by the left action.

Recall:

$$\sigma_{tX_V}(\lambda) = \sigma_{X_V}(t\lambda)$$

Since $\phi_V(t) = \sigma_{X_V}(t)$, this means

$$\exp(tV) = \phi_{tV}(1) = \phi_V(t)$$

Thus the exponential map also gives us back the one-parameter subgroup associated with any Lie algebra vector.

Range of the Exponential

$\exp(\mathcal{L}(G))$ may be the whole group, but it may also be just a subgroup. Especially if two different groups have the same Lie algebra, the submanifolds generated by the exponential map must be diffeomorphic, but at least one of them will not equal the entire corresponding group.

Matrix Groups

The usuals. General linear, orthogonal, unitary, their special versions, complex and real entries, Lorentz group and other generalisations.

The Lie algebra can also be matrix-valued, since the coordinates can be expressed using the components of the matrices, $x^{ij}(g) = g_{ij}$.

A useful exercise is to show how a lie algebra element of, say, the general linear group, varies under the left action. We'll use the expression for a push-forward in a coordinate chart, ¶ Push-forward.

$$
\begin{aligned}
L_{g*}\left(\underbrace{v^{ij}\frac{\partial}{\partial x^{ij}}\bigg|_h}_{V\in T_hG}\right) &= v^{ij}\frac{\partial x^{ab}(gh)}{\partial x^{ij}(h)}\;\frac{\partial}{\partial x^{ab}(gh)}\bigg|_{gh} \\
&= v^{ij}\frac{\partial}{\partial x^{ij}(h)}x^{ak}(g)x^{kb}(h)\;\frac{\partial}{\partial x^{ab}(gh)}\bigg|_{gh} \\
&= \underbrace{x^{ai}(g)v^{ij}}_{v'^{aj}}\;\frac{\partial}{\partial x^{aj}(gh)}\bigg|_{gh} \in T_{gh}(G) \\
&\therefore v' = g\cdot v
\end{aligned}
$$

Note, since all matrix groups are submanifolds of the general linear groups, we have an embedding and corresponding push-forwards and pull-backs, so we can express the lie algebra elements and generators and all as matrices in all the other matrix lie groups too.

The Lie bracket is simply given by the commutator, and the structure constants become:

$$
C_{(ab)(cd)}{}^{(ij)} = \delta^{ai}\delta^{dj}\delta^{bc} - \delta^{bi}\delta^{cj}\delta^{ad}
$$

And the exponential map literally becomes the exponential, because:

$$
\begin{aligned}
g(s+t) &= g(s)\cdot g(t) \\
\implies g'(s+t) &= g(t)\cdot g'(s) \\
\implies g'(t) &= g(t)\cdot g'(0) \\
\implies g(t) &= \exp(tg'(0)) \\
\text{Where }\exp M &:= \sum_{n=0}^{\infty}\frac{M^n}{n!}
\end{aligned}
$$

Thus the one-parameter subgroup generated by $V = g'(0)$ is literally $\exp(tV)$, given by the matrix exponential. Hence the corresponding map from algebra to group is the same.

In fact, this proof holds more generally – the $\exp(tV) := \phi_V(t)$ definition implies $g'(t) = g(t)\cdot g'(0)$, so for whatever objects the solution to this differential equation is the series $\sum_n \frac{x^n}{n!}$, this definition of the exponential becomes equivalent to the familiar one. In fact, some might argue this definition is more fundamental.

Adjoint Map

Conjugation is an autodiffeomorphism on the group but does not move the identity, hence its push-forward is a linear map on the Lie algebra:

$$
\begin{gathered}
e = C_g e = geg^{-1} \\
\therefore C_{g*} : T_e G \to T_e G \\
\mathrm{Ad} : G \times \mathcal{L}(G) \to \mathcal{L}(g) \\
\mathrm{Ad}(g, X_V) = X_{C_{g*}V}
\end{gathered}
$$

In the matrix groups, this is simple:

$$
\begin{gathered}
C_g(e^V) = g \cdot e^V \cdot g^{-1} \\
= e^{g \cdot V \cdot g^{-1}} \\
\therefore \mathrm{Ad}(g, V) = g \cdot V \cdot g^{-1}
\end{gathered}
$$

Differential Forms

An order r diff form allows integration over an r dim surface in the manifold, and is essentially just a totally antisymmetric $(0, r)$ tensor field, so an element of $\mathcal{J}^0_r$. Total antisymmetry is formally written as, for any permutation P,

$$
T(V_1, \ldots, V_n) = \mathrm{sign}(P) T(V_{P_1}, \ldots, V_{P_n})
$$

For the conventions of this course, the wedge product is the antisym product:

$$
\mathrm{d}x_i \wedge \mathrm{d}x_j = \mathrm{d}x_i \otimes \mathrm{d}x_j - \mathrm{d}x_j \otimes \mathrm{d}x_i
$$

This is of course on one-forms, and generalises to antisymmetric parts of tensor products in general – but oftentimes we prefer talking about the wedge product as the defining construct for higher-dimensional forms, ensuring the multilinearity (required of (0,r) tensors as functions mapping (r,0) tensors to a field) and antisymmetry.

Usually, the $1/r!$ factor is implicit in the wedge product, but in this course we take the different nomenclature:

$$
w = w_{\mu_1 \ldots \mu_r} \tfrac{1}{r!} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}
$$

The space of r-forms at a point is $\Omega^r_p(\mathcal{M})$. Generally we talk of fields of forms, elements of $\Omega^r(\mathcal{M}) = \coprod_{p \in \mathcal{M}} \Omega^r_p(\mathcal{M})$. $\Omega^r_p(\mathcal{M})$ is always empty for $r > m = \dim \mathcal{M}$, and more generally has dimensions:

$$
\dim \Omega^r_p(\mathcal{M}) = \frac{m!}{(m-r)!r!}
$$

Which is simply the number of components – the number of ways of choosing r coordinates of the total m to form the wedge product $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$. Order does not matter, because the antisymmetry always turns the order into a sign factor absorbed into the component itself.

We extend to 0-forms by making them just be the scalar functions, or more precisely, $\Omega_p^0(\mathcal{M}) = \mathbb{R}$ (or other field).

The wedge product is naturally extended:

$$\omega = \omega_{\mu_1 \dots \mu_r} \tfrac{1}{r!} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$$
$$\xi = \xi_{\nu_1 \dots \nu_s} \tfrac{1}{s!} \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_s}$$
$$\therefore \omega \wedge \xi = \omega_{\mu_1 \dots \mu_r} \omega_{\mu_1 \dots \mu_r} \tfrac{1}{r!s!} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_s}$$

Done! Simple.

- $\omega \wedge \xi = (-1)^{m_\xi m_\omega} \xi \wedge \omega$ (Graded commutativity)
- $\xi \wedge \xi = 0$ if m_ξ is odd
- The wedge product is associative
- r-forms naturally generalise to smooth r-form fields, eg $\Omega^0(\mathcal{M}) = \mathcal{F}(\mathcal{M})$.

Exterior Derivative

$$\mathrm{d}\omega = \partial_\mu \omega_{\alpha_1 \dots \alpha_r} \tfrac{1}{r!} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_r}$$

Graded Leibnitz rule:

$$\mathrm{d}(\omega \wedge \mu) = (\mathrm{d}\omega) \wedge \mu + (-1)^{m_\omega} \omega \wedge (\mathrm{d}\mu)$$

Nilpotent: $\mathrm{d}^2 = 0$ (because of mixed partials!)

Coordinate-free definition

$$(\mathrm{d}\omega)(X_1, \dots, X_{r+1}) \equiv \sum_{i=1}^{r+1} (-1)^{i+1} X_i[\omega(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{r+1})] + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, X_{r+1})$$

Eg

$$\mathrm{d}\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y])$$

Pullback

As with $(0, r)$ tensors. Additionally,

$$f^*(\omega \wedge \xi) = f^*(\omega) \wedge f^*(\xi)$$
$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$$

Interior Product

Operate the r-form on a vector X and call it $i_X\omega$. Also nilpotent.

$$(i_X\omega)(V_1, \ldots, V_{r-1}) \equiv \omega(X, V_1, \ldots, V_{r-1})$$

Lie Derivative

$$\mathcal{L}_X\omega = (\mathrm{d}i_X + i_X\mathrm{d})\omega$$

Closed, Exact

- Closed: $\mathrm{d}\omega = 0$
- Exact: $\omega = \mathrm{d}\sigma$
- Exact implies closed

So which forms are closed but not exact? This is where cohomology begins.

Electromagnetism

In the language of forms, the Bianchi identity (equivalent to the source-free Maxwell equations) states that:

$$\mathrm{d}F = 0$$

So it is closed. Does this imply that it is exact? It would be nice if it did – we would immediately have justification for writing solutions in terms of the 4-potential $F = \mathrm{d}A$. And in any contractible coordinate patch, any closed form is exact – we can do this. But not globally, not always. Depending on the topology of the manifold, we may not be able to. And this takes the form of the Ahranov-Bohm effect, among other things.

Orientation

Given a chart and a coordinate basis, the order of the coordinates defines an orientation. We can change this by changing the chart – swapping two coordinates should obviously flip the orientation, which we can encode as a sign change. Where else does a swap change signs, in the context of changing coordinate systems? The determinant of a matrix – specifically here, the Jacobian! The Jacobian for the coordinate transformation of a single swap is a permutation matrix and has determinant -1.

Thus we generalise the idea of orientation with the determinant of the Jacobian between coordinate systems. A transformation preserves the orientation if it has a positive determinant, and flips it if the determinant is negative (this also smoothly *integrates* with integrals).

Correspondingly, the manifold as a whole is **orientable** if there exists an atlas such that on all the overlaps between charts, the Jacobians of the transition functions have positive determinant.

Thus an orientable manifold can have a consistent orientation defined on the whole thing.

Top Forms, Volume Forms

A top form is simply the highest rank non-zero form possible on a manifold i.e. of rank the dimension of the manifold.

A volume form is a top form on an orientable manifold that is positive everywhere (given coordinate charts, ofc).

$$V = v(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

Let a coordinate change to $y(x)$ – this may be a different chart on the same patch or the transition function to another patch. Then

$$\begin{gathered} V = v(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \\ = v(x)\frac{\partial x^1}{\partial y^{\mu_1}}\mathrm{d}y^{\mu_1} \wedge \cdots \wedge \frac{\partial x^m}{\partial y^{\mu_m}}\mathrm{d}y^{\mu_m} \\ = v(x)\left(\epsilon_{\mu_1\dots\mu_m}\frac{\partial x^1}{\partial y^{\mu_1}}\cdots\frac{\partial x^m}{\partial y^{\mu_m}}\right)\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m \\ = v(x)\det J\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m \\ = v'(y)\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m \end{gathered}$$

Thus if a top form is positive definite in one patch, and it needs to be positive definite on all the other patches too, then it also needs to be positive definite in both coordinate charts in the overlap region – and that requires the determinant of the transition Jacobian to be positive in each overlap, i.e. an orientable manifold.

On a orientable manifold, infinitely many volume forms can be found, and a volume form defines an orientation on a manifold. For any open subset, the correct orientation is thus given by whichever ensures the given volume form is positive. Flipping the sign of the volume form gives the other orientation, which is then achieved by an atlas with odd permutations of the coordinates.

Integrating a top form on a chart

Consider an orientable manifold, an oriented atlas, a chart U_i with coordinates x, and a top form V, then define

$$\int_{U_i} V \equiv \int_{U_i} \mathrm{d}^m x\, v(x)$$

Is this consistent between charts? Yes, because the measure transforms as $|\det J^{-1}|$ and the top form as $\det J$.

$$\begin{aligned}\int_{U_i \cap U_j} \mathrm{d}^m y\, v'(y) &= \int_{U_i \cap U_j} \mathrm{d}^m y\, v(x) \det \frac{\partial x}{\partial y} \\ &= \int_{U_i \cap U_j} \mathrm{d}^m x\, v(x)\end{aligned}$$

Ofc the last step only holds if $\det J = |\det J|$. But we chose an oriented atlas so it's okay.

Now if this top form was a volume form, then we have obtained a positive definite integration measure.

And over the whole manifold?

Define smooth functions on the manifold that have a chart each as their support and sum to 1 at each point:

- $0 \leq \epsilon_i(p) \leq 1$
- $\epsilon_i(p) = 0$ if $p \notin U_i$
- $\sum_i \epsilon_i(p) = 1 \forall p \in \mathcal{M}$

This is called a partition of unity on the manifold, and decomposes functions as

$$f(p) = \sum_i \epsilon_i(p) f(p) \equiv \sum_i f_i(p)$$

Similarly, it can decompose the volume form, and any integral over the manifold, as

$$\int_{\mathcal{M}} V \equiv \sum_i \int_{U_i} \epsilon_i V$$
$$\int_{\mathcal{M}} fV \equiv \sum_i \int_{U_i} \epsilon_i fV$$

r-forms over sub-manifolds?

Suppose we have

- $\mathcal{M}$ embedded in $\mathcal{N}$ via f, so $f(\mathcal{M}) \subset \mathcal{N}$ is diffeomorphic to $\mathcal{M}$
- Orientable $\mathcal{M}$, then a volume form can be pushed forward to $f(\mathcal{M})$, inducing an orientation.

Then we may naturally integrate an m-form over $f(\mathcal{M})$ using the pullback:

$$\int_{f(\mathcal{M})} \omega \equiv \int_{\mathcal{M}} f^* \omega$$

Note that this extends to the boundary of $\mathcal{M}$, $\partial\mathcal{M}$, which is also naturally embedded as an $m-1$ submanifold in $\mathcal{N}$. We can integrate $m-1$ forms on that.

In simpler words, given an m-form and an m-dim submanifold, we can define integration of the form over the submanifold by treating the submanifold as a manifold itself. This can be done formally using the above approach, if necessary.

Stokes' Theorem and Cohomology

Cycles and Boundaries

We will talk about r-submanifolds – closed, oriented dim r submanifolds.

Given an r-submanifold V with a smooth embedding, its boundary ∂V is an $r-1$ dim oriented subset, and hence an $r-1$-submanifold or a disconnected sum of those.

An r-submanifold is:

- An r-**cycle** if it is closed and has no boundary, $\partial V = 0$
- An r-**boundary** if it is closed and the boundary of some other closed $r+1$-submanifold

Topology tells us that a boundary has no boundary – the ∂ operator is nilpotent, and this is the basis of homology theory.

Stoke's Theorem

For a closed oriented embedded r-submanifold V and an $r-1$ form ω with compact support:

$$\int_V \mathrm{d}\omega = \int_{\partial V} \omega$$

If the manifold is compact, then compact support of ω is guaranteed.

This can ofc be used alongside integration by parts:

$$\int_{\partial V} \omega \wedge \eta = \int_V \mathrm{d}\omega \wedge \eta + (-1)^{r-1} \int_V \omega \wedge \mathrm{d}\eta$$

Gauss's Law and Stoke's Law in 3D are just special cases of the general Stoke's theorem.

de Rham Cohomology

Define

- The space of closed r-forms, $Z^r(\mathcal{M})$
- The space of exact r-forms, $B^r(\mathcal{M})$

It can be easily shown that both are vector spaces.

Since all exact forms are closed, $B^r(\mathcal{M}) \subset Z^r(\mathcal{M})$. We want to understand the space of inexact closed forms, a quotient of $Z^r(\mathcal{M})$ by $B^r(\mathcal{M})$. We define an equivalence relation – being **cohomologous** – by

$$z_1 - z_2 \in B^r(\mathcal{M}) \quad (z_1, z_2 \in Z^r(\mathcal{M}))$$

Then this quotient, the set of equivalence classes, is the vector space $H^r(\mathcal{M})$, called the cohomology. To show the vector space structure,

$$\begin{gathered} \text{Let } z_i \sim w_i \\ \implies z_i = w_i + \mathrm{d}c_i \\ \implies \alpha z_1 + \beta z_2 = \alpha w_1 + \beta w_2 + \mathrm{d}(\alpha c_1 + \beta c_2) \sim \alpha w_1 + \beta w_2 \\ \implies [\alpha z_1 + \beta z_2] = \alpha[z_1] + \beta[z_2] \end{gathered}$$

Importantly, while $B^r(\mathcal{M}), Z^r(\mathcal{M})$ are infinite dimensional, $H^r(\mathcal{M})$ may be finite dimensional, and so we may give it a basis.

Examples

For any connected manifold, $H^0(\mathcal{M}) = \mathbb{R}$. This is because the only closed 0-forms are constant functions (hence $\mathbb{R}$) and there are no exact 0 forms.

Consider $H^1(\mathbb{R})$. All 1 forms must be closed, and any 1-form can be shown to have a differential primitive, so it must be exact, hence $H^1(\mathbb{R})$ is the singleton.

$$\omega = f(x)\mathrm{d}x = \mathrm{d}\left(\int_0^x f(t)\mathrm{d}t\right)$$

Now consider S^1 (embedded in $\mathbb{C}$). Every 1-form is still closed, but not every 1-form is exact. $F(\theta) = \int_0^\theta f(t)\mathrm{d}t$ can be defined locally for any 1-form $f(\theta)\mathrm{d}\theta$, and hence over a chart (say, $[0, 2\pi)$), but does that extend to a globally smooth function? Since $f(\theta)$ is a 1-form component, $F^{(n)}(\theta)$ are continuous and we only need enforce $F(2\pi) = F(0) = 0$ (the latter is clear from the integral definition). Thus

$$\int_{S^1} \omega = 0 \Leftrightarrow \omega \in B^1(S^1)$$

Now consider any two inexact closed forms ω, ω', then

$$a := \int \omega' / \int \omega,$$
$$\omega' - a\omega \in B^1$$

Thus the 1-form cohomology is the domain of a, which is $\mathbb{R}$.

Cohomology and Topology

The cohomology spaces of a manifold depend only on its topology. Hence, the rth Betti numbers are topological invariants:

$$b_r(\mathcal{M}) = \dim H^r(\mathcal{M})$$

And we define the Euler characteristic as

$$\chi = \sum_{i=0}^{\dim \mathcal{M}} (-1)^i b_i(\mathcal{M})$$

de Rham's Theorem: If $\mathcal{M}$ is compact without boundary then the cohomology spaces $H^r(\mathcal{M})$ are finite-dimensional, and in addition are dual to the corresponding Homology groups $H_r(\mathcal{M})$.

Cohomology with Stoke's Theorem

Consider integrating an exact form over a cycle:

$$\int_V \omega = \int_V \mathrm{d}\alpha = \int_{\partial V} \alpha = 0$$

Now consider integrating a closed form on a boundary:

$$\int_{\partial V} \beta = \int_V \mathrm{d}\beta = 0$$

These are important results. Combined with de Rham's theorem, this leads to:

Theorem. Consider a manifold compact without boundary and a closed r-form ω, then if $\int_V \omega = 0$ for all r-cycles in the manifold, then it is exact.

Given an r-cycle, we have the map

$$\Lambda_V : H^r \to \mathbb{R}$$
$$\Lambda_V([\omega]) \equiv \int_V \omega$$

Poincare's Lemma

An open set $U \subseteq \mathcal{M}$ is **contractible** if it can be smoothly contracted to a point, i.e.

$$\exists f : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$$
$$f(\lambda, p) = p'$$
$$\text{s.t. } f(0, U) = U,\ f(1, U) = p_0,\ f \text{ is } C^0$$

Eg discs are contractible, annuli are not.

Any open subset of a manifold is itself a manifold – it is diffeomorphic to an open subset of $\mathbb{R}^{\dim \mathcal{M}}$ – and we can define forms, and the vector spaces of closed/exact forms, over it as well.

Now we get to the lemma:

Poincare's Lemma: For a contractible open set $U \subset \mathcal{M}$, $Z^r(U) = B^r(U)$ (all closed forms are exact).

This relates to how the cohomology is a topological feature – locally, all closed forms are exact. It is only the global (topological) structure of the manifold that prevents closed forms from being exact – and this is exactly encoded in the de Rham cohomology.

Example: Electromagnetism

Recall we discussed that Maxwell's source-free equations are:

$$\mathrm{d}F = 0$$

So the two-form F is globally closed. Locally, then it can always be exact – within a particular open set. And since $H^r(\mathbb{R}^m) = 0$ (Poincare's lemma, and all of $\mathbb{R}^m$ is contractible), on any flat spacetime as well, F is exact – and hence can be written as $\mathrm{d}A$.

More generally, though, given a basis $\{[\omega^i]\}$ for $H^2(\mathcal{M})$,

$$F = \alpha_i \omega^i + \mathrm{d}A$$

Thus the vector potential formulation we use is a special case.

Suppose we are on a 2-cycle $\mathcal{M}$ which is not flat, say S^2, then

$$\int_{\mathcal{M}} F = 0 \text{ if } F \text{ is exact}$$

So if this flux does not vanish, F is not exact. Note this can only happen if the global space is not $\mathbb{R}^m$! We say the flux measures the magnetic charge contained within $\mathcal{M}$, and topologically magnetic charges are treated as obstructions to flatness. But that's a whole another discussion.

Poincare Duality

Define an inner product on the cohomologies of an m-dim compact manifold $\mathcal{M}$:

$$\begin{gathered}\langle \cdot, \cdot \rangle : H^r(\mathcal{M}) \times H^{m-r}(\mathcal{M}) \to \mathbb{R} \\ [\omega], [\alpha] \to \int_{\mathcal{M}} \omega \wedge \alpha\end{gathered}$$

Since $\mathcal{M}$ is compact, $\partial \mathcal{M} = 0$, hence the inner product is independent of the representative of the cohomology classes. This is easy to check.

Theorem (Poincare). This inner product is non-degenerate.

Note how with this inner product H^{m-r} is the dual vector space to H^r, which implied $b_r = b_{m-r}$, and hence a 0 Euler characteristic for odd-dimensional compact manifolds.

Riemannian Geometry I

Metric

Given a manifold, the **Riemannian metric** is a (0,2) real tensor field which is symmetric and positive (in its action on vectors) at every point.

$$
\begin{gathered}
g \in \mathcal{T}_2^0 \\
g(U,V) = g(V,U) \\
g(U,U) > 0 \text{ except } g(0,0) = 0 \\
\forall p \in \mathcal{M}, U, V \in T_p\mathcal{M}
\end{gathered}
$$

$(\mathcal{M}, g)$ is then a *Riemannian manifold*.

A **Pseudo-Riemannian metric** is similar but relaxes the condition of positivity. It still requires, however, that

$$
g(U,V) = 0 \forall U \in T_p\mathcal{M} \implies V = 0
$$

Then $(\mathcal{M}, g)$ is a *Pseudo-Riemannian manifold*, and the metric divides $T_p\mathcal{M}$ into 3 classes:

$$
g(U,U) \begin{cases} < 0 & \text{Timelike} \\ = 0 & \text{Null} \\ > 0 & \text{Spacelike} \end{cases}
$$

Given a chart, the metric can be written as

$$
g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu
$$

The extra condition on pseudo-Riemannian metrics, or the positivity condition on Riemannian metrics, enforce the matrix $g_{\mu\nu}$ to be non-degenerate and hence invertible. Thus both cases allow defining the inverse metric, a (2,0) symmetric tensor field, whose components in any chart form the inverse matrix of $g_{\mu\nu}$.

$$
\begin{gathered}
g^{-1} \in \mathcal{T}_0^2 \\
g^{-1} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu \\
g^{\mu\alpha} g_{\alpha\nu} = \delta^\alpha_\beta
\end{gathered}
$$

The matrix $g_{\mu\nu}$ is real, symmetric and invertible, so it must have real non-zero eigenvalues. The *signature* is defined as a the number of negative and positive eigenvalues, $(\#\lambda < 0, \#\lambda > 0)$ and is independent of the point on the manifold (i.e. in $g_{\mu\nu}(x)$).

Riemannian metrics clearly have signature $(0, d)$ in d-dim. Pseudo-Riemmanian can have other signatures, but we are usually interested in those with a *Lorentzian* signature, $(1, d-1)$.

Inner Product

The metric defines a *non-degenerate* inner product on each $T_p\mathcal{M}$,

$$
\begin{gathered}
g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \\
U, V \to g(U, V)
\end{gathered}
$$

Since dual vectors also map vectors to $\mathbb{R}$, the metric provides a map from the vector space to the dual space

$$
\begin{gathered}
T_p\mathcal{M} \to T_p^* M \\
U \to g(\cdot, U)
\end{gathered}
$$

And similarly from the dual space to the vector space.

This is colloquially the raising and lowering of indices, and extends to general tensors straightforwardly.

Volume Element

On an orientable d-dim manifold, the metric also defines a canonical volume form:

$$
\Omega_g \equiv \sqrt{|\det g_{\mu\nu}|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d
$$

Though this is written with explicit coordinates, it is coordinate-invariant as a form should be (so long as the change of coordinates does not change orientation, but even then it is only a change of sign).

$x^\alpha \to y^\beta$ leads to a Jacobian factor

$$
\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d = \det \frac{\partial x^\alpha}{\partial y^\beta} \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^d
$$

Whereas the metric, being a tensor, transforms as

$$
\begin{gathered}
g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \underbrace{\left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu} \right)}_{\tilde{g}_{\alpha\beta}} \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta \\
\therefore |\det \tilde{g}_{\alpha\beta}| = \left| \det \frac{\partial x^\gamma}{\partial y^\alpha} \right|^2 |\det g_{\mu\nu}| \\
\therefore \sqrt{|\det g_{\mu\nu}|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d = \sqrt{|\det \tilde{g}_{\alpha\beta}|}\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^d
\end{gathered}
$$

Note that the equality only holds if the Jacobian is positive – changing the orientation will introduce a negative sign.

The volume element is thus a *pseudo-tensor* – it's tensorial up to its sign, which changes with a change of orientation. Other than that, it behaves like a tensor.

The other condition for being a volume form is to be positive everywhere. This is accomplished in part by the modulus on $g \equiv \det g_{\mu\nu}$, and in part by the metric being non-degenerate, so g is never 0.

Levi-Civita symbol

In m-dimensions, we can define

$$\epsilon_{\mu_1 \dots \mu_m} = \begin{cases} +1 & (\mu_1 \dots \mu_m) \text{ even permutation of } (1 \dots m) \\ -1 & (\mu_1 \dots \mu_m) \text{ odd permutation of } (1 \dots m) \\ 0 & \text{repeated indices} \end{cases}$$

The indices of this object may be raised using the metric, but it is not a tensor – its components do not transform at all with the coordinates. However, since it enforces antisymmetry, we can use it to construct the canonical volume form, which is a pseudo-tensor:

$$\Omega_g = \sqrt{|g|} \epsilon_{\mu_1 \dots \mu_m} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_m}$$

Thus the epsilon symbol is an example of a **tensor density** – an object which is a (pseudo)-tensor up to factors of g.

Some useful identities:

- $A_{\mu_1}^{\ \nu_1} \dots A_{\mu_m}^{\ \nu_m} \epsilon_{\nu_1 \dots \nu_m} = \det(A_\alpha^{\ \beta}) \, \epsilon_{\mu_1 \dots \mu_m}$
- $\epsilon_{\nu_1 \dots \nu_m} = \det g \, \epsilon^{\nu_1 \dots \nu_m}$
- $\epsilon^{\nu_1 \dots \nu_m} \epsilon_{\nu_1 \dots \nu_m} = m!$
- $\epsilon^{\nu_1 \dots \nu_r \beta_{r+1} \dots \beta_m} \epsilon_{\nu_1 \dots \nu_r \alpha_{r+1} \dots \alpha_m} = r!(m-r)! \delta_{\alpha_{r+1}}^{[\beta_{r+1}} \dots \delta_{\alpha_m}^{\beta_m]}$

Where square brackets on indices indicate normalised antisymmetrisation.

The final property can also be written in terms of the generalised Kronocker delta:

$$\begin{aligned} \epsilon^{\nu_1 \dots \nu_m} \epsilon_{\mu_1 \dots \mu_m} &\equiv \delta^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_m} \\ \epsilon^{\nu_1 \dots \nu_r \beta_{r+1} \dots \beta_m} \epsilon_{\nu_1 \dots \nu_r \alpha_{r+1} \dots \alpha_m} &= \delta^{\nu_1 \dots \nu_r \beta_{r+1} \dots \beta_m}_{\nu_1 \dots \nu_r \alpha_{r+1} \dots \alpha_m} = r! \delta^{\beta_{r+1} \dots \beta_m}_{\alpha_{r+1} \dots \alpha_m} \end{aligned}$$

The generalised Kronocker delta is NOT the correct object with which to change indices for forms – rather, $\delta_{\alpha_{r+1}}^{[\beta_{r+1}} \dots \delta_{\alpha_m}^{\beta_m]}$ is the correct object for that. Contracting with the generalised Kronocker delta gives an additional factor of $(m-r)!$.

Note that since it does not contain factors of g, it cannot raise or lower indices, but it can swap them.

Hodge Star

Since $\dim \Omega^r = \dim \Omega^{m-r}$, one may expect a correspondence to exist between them – and it does, once the metric is provided. It's called the Hodge star operator.

$$* : \Omega^r \to \Omega^{m-r}$$
$$*(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}) \equiv \frac{1}{(m-r)!}\sqrt{|g|}\epsilon^{\mu_1 \dots \mu_r}{}_{\mu_{r+1} \dots \mu_m} \mathrm{d}x^{\mu_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_m}$$

This easily generalises to $\omega = \omega_{\mu_1 \dots \mu_r} \frac{1}{r!} \mathrm{d}x^{\wedge \mu_1 \dots \mu_r}$. $*\omega$ then has components $\sqrt{|g|}\frac{1}{r!}\omega_{\mu_1 \dots \mu_r} \epsilon^{\mu_1 \dots \mu_r}{}_{\mu_{r+1} \dots \mu_m}$.

The canonical volume form can also be written as $*1 = \Omega_g$.

The Hodge star is involutory up to sign, which depends on r and the metric signature. In a metric with signature $(k, m-k)$,

$$**\omega = (-)^k (-)^{r(m-r)} \omega$$

▶ Proof

Inner Product on r-forms

Using the Hodge star, we obtain the natural m-form:

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \frac{1}{r!}(\alpha_{\mu_1 \dots \mu_r} \beta^{\mu_1 \dots \mu_r})\Omega_r$$

At a point, this would be sufficient for an inner product. To generalise to form fields, we do what m-forms are made for : integrate them.

$$(\cdot, \cdot) : \Omega^r(\mathcal{M}) \times \Omega^r(\mathcal{M}) \to \mathbb{R}$$
$$\alpha, \beta \to (\alpha, \beta) = \int_{\mathcal{M}} \alpha \wedge *\beta$$

Clearly, this inner product is symmetric. On Riemannian manifolds, it is also positive-definite.

Adjoint of the Exterior Derivative

$$\mathrm{d}^\dagger : \Omega^r(\mathcal{M}) \to \Omega^{r-1}(\mathcal{M})$$
$$\mathrm{d}^\dagger \omega \equiv (-)^{k+m(r+1)+1} * \mathrm{d} * \omega$$

Where k is the signature.

Nilpotency:

$$(\mathrm{d}^\dagger)^2 = *\mathrm{d}**\mathrm{d}* = \pm * \mathrm{d}^2 * = 0$$

The signs have been introduced to ensure the following holds (on any compact orientable manifold with metric without boundary):

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^\dagger \beta)$$

Which is also why this is named the adjoint derivative.

Why compact and without boundary? Well, when $\partial\mathcal{M} = 0$,

$$0 = \int_{\partial\mathcal{M}} \alpha \wedge *\beta$$

And then because it is compact, we can apply Stoke's theorem: (let β be an r-form)

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \mathrm{d}(\alpha \wedge *\beta) \\ \implies \int_{\mathcal{M}} \mathrm{d}\alpha \wedge *\beta + (-)^{r-1} \int_{\mathcal{M}} \alpha \wedge \mathrm{d} * \beta &= 0 \\ \implies \int_{\mathcal{M}} \mathrm{d}\alpha \wedge *\beta &= (-)^{r+k+(m-r+1)(m-(m-r+1))} \int_{\mathcal{M}} \alpha \wedge ** d * \beta \\ \implies (\mathrm{d}\alpha, \beta) &= (-)^{r+k+r(m-r)-(m-r)+r-1} (-)^{k+m(r+1)+1} (\alpha, \mathrm{d}^\dagger \beta) \\ &= (-)^{-r(r-1)} (\alpha, \mathrm{d}^\dagger \beta) = (\alpha, \mathrm{d}^\dagger \beta) \end{aligned}$$

QED

Exercise. Show that for a 1-form v_μ,

$$\mathrm{d}^\dagger v = -\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} v^\mu \right)$$

Generalising 3D Vector Calculus

In 3D space, with scalar fields as 0-forms and vector fields as 1-forms,

1. The gradient is the exterior derivative
2. The divergence is $-\mathrm{d}^\dagger$
3. The curl is $*\mathrm{d}$

Laplacian

This is defined more generally than in multivariable calculus as a derivative on r-forms.

$$
\begin{gathered}
\Delta : \Omega^r(\mathcal{M}) \to \Omega^r(\mathcal{M}) \\
\Delta \equiv \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} = (\mathrm{d} + \mathrm{d}^\dagger)^2
\end{gathered}
$$

For example, acting on a 0-form,

$$
\Delta f = -\frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|}\partial^\mu f(x)\right)
$$

Considering a Riemannian manifold without boundary, the inner product is positive definite, so

$$
(\omega, \Delta\omega) = (\omega, \mathrm{d}\mathrm{d}^\dagger\omega) + (\omega, \mathrm{d}^\dagger\mathrm{d}\omega) = (\mathrm{d}^\dagger\omega, \mathrm{d}^\dagger\omega) + (\mathrm{d}\omega, \mathrm{d}\omega) \geq 0
$$

Thus the Laplacian is a positive operator on such manifolds.

Example: Electromagnetism

While the source-free Maxwell equations were

$$
\mathrm{d}F = 0
$$

The sourced Maxwell equations can be written as

$$
\mathrm{d}^\dagger F = j
$$

Which also implies $\mathrm{d}^\dagger j = 0$ – current conservation.

This allows easy generalisation to higher-form electromagnetism, which is important in String Theory and higher dimensional gauge theories in general.

Hodge Theory

Recall we defined closed and exact forms in ¶ Closed, Exact. Similarly, we can define with the adjoint derivative,

- Co-closed : $\mathrm{d}^\dagger\omega = 0$
- Co-exact : $\omega = \mathrm{d}^\dagger\alpha$
- Harmonic : $\Delta\omega = 0$

The spaces of all such r-forms are $Z^{\dagger r}(\mathcal{M}), B^{\dagger r}(\mathcal{M})$ and $\mathrm{Harm}^r(\mathcal{M})$ respectively.

Harmonic is equivalent to being closed and co-closed.

▶ Proof

On compact Riemannian without boundary, $B^r, B^{\dagger r}$ and Harm^r are orthogonal – inner products between elements of these spaces are 0.

Harm^r is always a vector subspace of Ω^r, so there is an orthogonal subspace $\overline{\mathrm{Harm}^r}$, and a projector $P : \Omega^r \to \mathrm{Harm}^r$ can be defined.

Hodge Decomposition Theorem

On compact Riemannian without boundary, Ω^r admits a unique decomposition into the exact, co-exact and harmonic subspaces, and any r-form can be uniquely written as

$$\omega = \mathrm{d}\alpha + \mathrm{d}^\dagger\beta + \gamma$$
$$\Delta\gamma = 0$$

Incomplete proof:

Consider the Poisson equation

$$\Delta\phi = j$$

If j is orthogonal to Harm^r, then one can uniquely solve the equation with a ϕ also orthogonal to Harm^r. The rest of the solutions of the equation can be obtained by adding $\gamma \in \mathrm{Harm}^r$ to ϕ.

Why must j be orthogonal? Because $(\Delta\alpha, \gamma) = 0$ ($\Delta\alpha$ consists of an exact and co-exact component), so $(j, \gamma) = 0$ must be true too. On 0-forms this becomes the well-known constraint that the source integrates to 0 over the whole manifold (this can be violated, but then we must have a boundary).

Given ω, we can obtain $\omega' = \omega - P\omega$ orthogonal to the Harmonic subspace. Then we may uniquely write $\omega' = \Delta\phi = \mathrm{d}(\mathrm{d}^\dagger\phi) + \mathrm{d}^\dagger(\mathrm{d}\phi)$. The quantities in the brackets are α, β. Hence proved.

Harmonic Representatives

Consider a compact boundary-less Riemannian, and on it a closed form ω. Then

$$\omega = \mathrm{d}\alpha + \mathrm{d}^\dagger\beta + \gamma, \quad \Delta\gamma = 0 \implies \mathrm{d}\gamma = 0$$
$$\mathrm{d}\omega = 0 \implies \mathrm{d}\mathrm{d}^\dagger\beta = 0$$

Consider

$$0 = (\mathrm{d}\omega, \beta) = (\mathrm{d}^\dagger\beta, \mathrm{d}^\dagger\beta)$$
$$\therefore \omega = \mathrm{d}\alpha + \gamma, \quad \Delta\gamma = 0$$

Thus, every cohomology class has a unique Harmonic representative. For example, γ is the representative of the entire cohomology class $[\omega]$ (because the class is given by $\omega' = \omega + \mathrm{d}\eta$). Thus

$$\mathrm{Harm}^r \simeq H^r$$

And the dimension of Harm^r is given by the Betti number b_r, which is a topological invariant.

This also implies that the number of linearly independent solutions to $\Delta\omega = 0$ is a topological invariant!

Maxwell, once more

The equations:

$$\mathrm{d}F = 0, \quad \mathrm{d}^\dagger F = j$$

On a compact boundary-less Riemannian, we can then write

$$F = \omega + \mathrm{d}A$$

Where $[F] = [\omega]$, gauge transformations are $A' = A + \mathrm{d}\lambda \implies F' = F$, and j is *globally* a co-exact form.

Lorenz Gauge

$$\mathrm{d}^\dagger A = 0$$

If we start with A, then

$$\begin{gathered} A' = A + \mathrm{d}\lambda, \quad \mathrm{d}^\dagger A' = 0 \\ \implies \Delta\lambda = -\mathrm{d}^\dagger A \end{gathered}$$

Since $\mathrm{d}^\dagger A$ is co-exact, it is orthogonal to the harmonic forms, so we can invert the Laplacian and always find a λ to go to the Lorenz gauge.

Solving for the Potential

Since Harmonic forms are co-closed,

$$\mathrm{d}^\dagger(\omega + \mathrm{d}A) = j \implies \mathrm{d}^\dagger \mathrm{d}A = j$$

In Lorenz gauge, this becomes

$$\Delta A = j$$

Again, j is co-exact and hence orthogonal to the Harmonics, so this equation can always be solved. Boom.

Riemannian Geometry II

Induced Metric

Consider a manifold $\mathcal{M}$ with metric g and another manifold $\mathcal{N}$ embedded by a smooth map

$$f : \mathcal{N} \to \mathcal{M}$$

This then induces a metric by the pull-back, $g_{\mathcal{N}} = f^* g$.

Consider coordinates $\{x^\mu\}, \{y^\beta\}$ on $\mathcal{M}, \mathcal{N}$ with $x = f(y)$, then using the explicit metric we get the explicit induced metric:

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$
$$(f^* g) = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu} \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$$

The induced metric defines a volume form, which can be integrated to define a volume. But also, since $\mathcal{N}$ is diffeomorphic to $\mathcal{N}' = f(N)$, we also have an inherited metric $g_{\mathcal{N}'}$ – this will be the same as the push-forward $f_* g_{\mathcal{N}}$. The corresponding volume form $\Omega_{\mathcal{N}'}$ is the push-forward of $\Omega_{\mathcal{N}}$, so due to $\mathcal{N}, \mathcal{N}'$ being diffeomorphic spaces,

$$\int_{\mathcal{N}} \Omega_{\mathcal{N}} = \int_{\mathcal{N}'} \Omega_{\mathcal{N}'}$$

They have the same volume.

Lengths

Consider a curve $\mathcal{C}$ given by $x^\mu(\lambda)$ – a 1D embedded submanifold. The induced 1D metric, volume element and length of the curve are

$$
\begin{aligned}
g_{\mathcal{C}} &= g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \\
\Omega_{\mathcal{C}} &= \sqrt{|g_{\mathcal{C}}|}\mathrm{d}\lambda \\
s &= \int \Omega_{\mathcal{C}}
\end{aligned}
$$

We can also measure the length of parts of the curve $\mathcal{C}' \subseteq \mathcal{C}$,

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\lambda} &= \sqrt{|g_{\mathcal{C}}(\lambda)|} \\
L_{\mathcal{C}'} &= \int_{\mathcal{C}'} \Omega_{\mathcal{C}} = \int_{\mathcal{C}'} \frac{\mathrm{d}s}{\mathrm{d}\lambda}\mathrm{d}\lambda
\end{aligned}
$$

δs, for an infinitesimal shift $\delta\lambda$, is known as the line element and gives the distance measure along the curve. This is what the $\mathrm{d}s$ in the GR-style equations (like below) refers to:

$$
\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu
$$

Integration of n-forms

$\alpha \in \Omega^n(\mathcal{M})$, where $n = \dim \mathcal{N}$, then we know that the pull-back is defined such that the following works:

$$
\int_{f(\mathcal{N})} \alpha = \int_{\mathcal{N}} f^*\alpha
$$

The pull-back can be written in terms of the volume form:

$$
f^*\alpha = \left(\frac{1}{\sqrt{|\det g_{\mathcal{N}}|}}\alpha_{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1}}{\partial y^1}\cdots\frac{\partial x^{\mu_n}}{\partial y^n}\right)\Omega_{\mathcal{N}}
$$

Vector fields are mapped using

$$
t_\alpha = f_*\frac{\partial}{\partial y^\alpha} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial}{\partial x^\mu}
$$

t_α are then tangents to the submanifold $f(\mathcal{N})$.

Hypersurfaces

$m-1$ dimensional submanifolds in $\mathcal{M}$. Consider one called $\mathcal{N}' = f(\mathcal{N})$ with appropriate inherited volume form. There is then a 1-form:

$$n \in \Omega^1(\mathcal{M})$$
$$\Omega_{\mathcal{N}'} = *n$$

This will annihilate all vectors tangent to $\mathcal{N}'$:

$$V \in T_p\mathcal{N} \implies (n, f_*V) = 0$$

This is because the volume form is the wedge product of all basis 1-forms tangent to $\mathcal{N}'$, so n is the remaining basis vector and has a projection of 0 on $\mathcal{N}'$. For the same reason,

$$f^*n = 0$$

Furthermore, because it arises from the volume form, it is a smooth nowhere vanishing 1-form field on $\mathcal{N}'$.

We can raise its index to get $N^\mu = g^{\mu\nu}n_\nu, N \in T_{f(p)}\mathcal{M}$. Whether N is spacelike, timelike or null defines whether the hypersurface is spacelike, timelike or null.

Again, $g(N, f_*V) = 0$ (orthogonal to tangent vectors) and it is called a normal vector.

If it is null, then the induced volume form is also 0 – the induced metric has a vanishing determinant. Null hypersurfaces thus have 0 volume.

Additionally, the normal vector is normalised such that $g(N, N) = \pm 1$ (+ is spacelike, - timelike).

$$n \wedge *n = n^2\Omega_g, n^2 := g^{\mu\nu}n_\mu n_\nu$$
$$\Omega_{\mathcal{N}'} \wedge *\Omega_{\mathcal{N}'} = \Omega^2\Omega_g$$
$$\therefore \Omega^2 = (-)^k n^2$$

Recall this is for signature k.

For null, both are 0. Considering not null,

$$\Omega^2 = \frac{1}{(m-1)!}\Omega_{\mu_1 \dots \mu_{m-1}}\Omega^{\mu_1 \dots \mu_{m-1}}$$
$$\Omega_{\mu_1 \dots \mu_{m-1}} = \sqrt{|\det g_{\mathcal{N}}|}\epsilon_{\mu_1 \dots \mu_{m-1}}$$
$$\therefore \Omega^2 = \operatorname{sgn}(\det g_{\mathcal{N}}) = (-)^{k-1} \mp$$

+ for spacelike, - for timelike. Then $n^2 = \pm 1$.

$$g(N,N) = n^2 = \pm 1$$

This is inconsistent with the previous discussion.

1-forms integrated over the hypersurface

Given a general $w \in \Omega^1(\mathcal{M})$ with corresponding vector field W, we can use the ¶ Interior Product to write:

$$\begin{aligned} i_W \Omega_g &= *w \\ = i_W(n \wedge \Omega_{\mathcal{N}'}) &= n(W)\Omega_{\mathcal{N}'} - n \wedge i_W \Omega_{\mathcal{N}'} \end{aligned}$$

Thus, its pull-back:

$$f^*(*w) = n(W)\Omega_{\mathcal{N}}$$

Because $n \wedge i_W \Omega_{\mathcal{N}'}$ is an m-1 form which includes n, and the pull-back will only leave non-zero the m-1 form comprised of only tangential directions – because there should be only one m-1 form (up to factors) on $\mathcal{N}$.

$n(W) = n_\nu W^\nu$ is now a function on $\mathcal{N}$, and we get

$$\int_{f(\mathcal{N})} *w = \int_{\mathcal{N}} n(W)\Omega_{\mathcal{N}}$$

$n_\nu \Omega_{\mathcal{N}}$ is exactly the $\mathrm{d}\vec{S}$ one integrates over in Physics.

Electric/Magnetic Flux

$$F = F_{\mu\nu}\frac{1}{2}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

For a fixed time slice, $\mathrm{d}x^0 = 0$,

$$\begin{aligned} F_{ij} &= \epsilon_{ijk}B^k \\ (*F)_{ij} &= \epsilon_{ijk}E^k \end{aligned}$$

Thus $F, *F$ are the hodge duals of 1-forms B, E (lowered indices).

On a 2-surface $\mathcal{N}'$,

$$\int_{\mathcal{N}'} F = \int_{\mathcal{N}'} *B = \int_{\mathcal{N}} n(B)\Omega_{\mathcal{N}} = \int \vec{B} \cdot \mathrm{d}\vec{S}$$
$$\int_{\mathcal{N}'} *F = \int_{\mathcal{N}'} *E = \int \vec{E} \cdot \mathrm{d}\vec{S}$$

3D Vector Calculus again

Stoke’s theorem is obv just Stoke’s theorem for 1-forms:

$$\int_S \mathrm{d}v = \int_{\partial S} v = \int_{\partial S} \vec{v} \cdot \mathrm{d}\vec{l}$$
$$\int_S \mathrm{d}v = \int (*\mathrm{d}v)_i \mathrm{d}S^i = \int (\nabla \times v) \cdot \mathrm{d}\vec{S}$$

And Gauss’ theorem is Stoke’s theorem over 3-manifolds.

$$\int_V \mathrm{d}(*w) = \int_{\partial V} *w = \int_{\partial V} n(W)\Omega_{\partial V}$$
$$\int_V \mathrm{d}(*w) = \int_V - * d^\dagger w = \int_V (\nabla \cdot \vec{w})\mathrm{d}V$$
$$\therefore \int_V \nabla \cdot \vec{w} \mathrm{d}V = \int_{\partial V} \vec{w} \cdot \mathrm{d}\vec{S}$$

Which can be extended to other dimensions as well.

Connection

We want to define directional derivatives of non-scalar objects. For scalars, it is simply a vector’s action on the function, $V[f]$. We also have the All Lectures - Lie Derivative In Trash, but that depends on more than just the direction at a point. We specifically want a derivative that’s only directional.

Define the **affine connection**:

$$\nabla : \mathcal{J}_0^1(\mathcal{M}) \times \mathcal{J}_0^1(\mathcal{M}) \to \mathcal{J}_0^1(\mathcal{M})$$
$$X, Y \to \nabla_X Y$$
$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \quad \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$$
$$\nabla_{fX} Y = f\nabla_X Y, \quad \nabla_X(fY) = X[f]Y + f\nabla_X Y$$

$\nabla_{fX} Y = f\nabla_X Y$ is what makes it a directional derivative.

Since it is (bi)linear, we can define its action on basis vector fields $e_\mu = \partial_\mu$ using some components, called the connection components:

$$\nabla_\mu e_\nu = \Gamma_{\mu\nu}{}^{\alpha} e_\alpha$$

$\Gamma_{\mu\nu}{}^{\alpha}$ are smooth functions defined over every chart, and specify the connection once they are provided for all the charts.

Using these, we can write the derivative as

$$\begin{aligned}\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu e_\nu) \\ &= X^\mu (e_\mu [Y^\nu] e_\nu + Y^\nu \nabla_\mu e_\nu) \\ &= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\alpha}{}^{\nu} Y^\alpha \right) e_\nu\end{aligned}$$

Under a change of coordinates from $\{x^\mu\} \to \{y^\alpha\}$,

$$\tilde{\Gamma}_{\mu\nu}{}^{\rho} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \frac{\partial y^\rho}{\partial x^\sigma} \Gamma_{\alpha\beta}{}^{\sigma} + \frac{\partial^2 x^\alpha}{\partial y^\alpha \partial y^\nu} \frac{\partial y^\rho}{\partial x^\alpha}$$

Thus, this object is not a tensor – it does not have the requisite transformation laws. But the difference between two connections is a $(1, 2)$ tensor.

Torsion

$$\begin{gathered}\Gamma_{\mu\nu}{}^{\rho} = S_{\mu\nu}{}^{\rho} + \tfrac{1}{2} T_{\mu\nu}{}^{\rho} \\ S_{\mu\nu}{}^{\rho} \equiv \Gamma_{(\mu\nu)}{}^{\rho}, \quad T_{\mu\nu}{}^{\rho} \equiv 2\Gamma_{[\mu\nu]}{}^{\rho}\end{gathered}$$

Then the symmetric part still transforms like a connection, and in fact is a valid connection on its own, but the antisymmetric part transforms like a (1,2) tensor, called the **torsion tensor**.

The torsion tensor acts on two vectors to give another vector, and thus has the elegant basis-invariant definition:

$$\begin{gathered}\mathcal{T} : \mathcal{J}_0^1 \times \mathcal{J}_0^1 \to \mathcal{J}_0^1 \\ X, Y \to \mathcal{T}(X, Y) \\ \mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \\ T_{\mu\nu}{}^{\rho} = \langle e^\rho, \mathcal{T}(e_\mu, e_\nu) \rangle\end{gathered}$$

Note some important properties of the torsion map:

1. $\mathcal{T}(aX, bY) = ab\mathcal{T}(X, Y)$
2. $\mathcal{T}(e_\mu, e_\nu) = 2\Gamma_{[\mu\nu]}{}^{\rho} e_\rho$
3. $\mathcal{T}(X, Y) = -\mathcal{T}(Y, X)$

Parallel Transport, Geodesics

The Lie derivative was still only looking at vectors at a single point – the connection provides the additional structure needed to compare vectors in different tangent spaces, and hence provides additional structure to the complete tangent space of the manifold. This is why there isn't a unique choice of a connection on a manifold (though there is a unique choice of a torsion-free connection on a metric space). This is also why the 1-form gauge field is sometimes called a connection – just like how we have tangent spaces attached at each point on the manifold, in a gauge theory we have the gauge group attached at each point on the manifold, and the connection, the 1-form gauge field, dictates how to connect all those groups and provides a structure on the larger $\coprod_{p\in\mathcal{M}} G_p$.

Parallel Transport

Now that we have a connection, we can define **parallel transport** for vectors along a curve as

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} X\Big|_{p(\lambda)} = 0$$

I.e. the covariant derivative of vector field X along the curve given by tangent vectors $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ must be 0.

Since ∇ is a directional derivative, we only require X to be defined along the curve, and not elsewhere – with other derivatives, like the Lie derivative, you would run into problems if the vector field wasn't defined in, generically, some local m-dim subspace around a point. But here we only need $\frac{\mathrm{d}}{\mathrm{d}\lambda} X(x^\mu(\lambda))$:

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} X = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} X^\mu + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} X^\beta \Gamma_{\alpha\beta}{}^{\mu}\right) e_\mu$$

If we have X at one point, we can solve this IVP to get its parallely-transported versions at other points.

Geodesics

A curve is a **geodesic** if the tangent vector itself is parallely transported along the curve, i.e.

$$V = \frac{\mathrm{d}}{\mathrm{d}\lambda}, \quad \nabla_V V|_p = 0 \forall p \in \mathcal{C}$$

This is the analog of a straight line on curved manifolds.

In a coordinate patch, this becomes the **geodesic equation**:

$$\ddot{x}^\mu + \dot{x}^\alpha \dot{x}^\beta \Gamma_{\alpha\beta}{}^{\mu}(x) = 0, \quad \dot{x} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} x$$

Again, given a vector at a point, this is an IVP which we can solve for the geodesic passing through that point with that tangent vector.

For the trivial connection, this gives us the straight line, as expected.

Interpretation of Torsion

Note that the geodesic equation selects for the symmetric part of the connection, and hence is independent of the torsion. In fact, knowledge of all the geodesics on a manifold is equivalent to knowing the symmetric connection.

Then what does the torsion encode? It's effect does not disappear in all parallel transport equations. It can be interpreted as controlling how vectors orthogonal to the tangent transport.

Consider $X, Y \in T_p\mathcal{M}$. Flow by $\delta\lambda$ along the geodesic given by X to get $Y \to Y'$, then flow by $\delta\lambda$ along Y' to reach point p_{yx}. Vice versa to get p_{xy}. Torsion then measures the difference between these points:

$$x^\mu(p_{yx}) - x^\mu(p_{xy}) = \delta\lambda(T_{\alpha\beta}{}^\mu Y^\alpha X^\beta)|_p$$

(To linear order).

Note that we discussed how the commutator also measures something similar, but in the context of Lie derivatives and flows. This is incorporated here – recall that the torsion map includes the commutator.

Covariant Derivative

We extend the directional derivative to all tensor fields:

$$\begin{gathered}\nabla : \mathcal{J}_0^1 \times \mathcal{J}_r^q \to \mathcal{J}_r^q \\ X, T \to \nabla_X T\end{gathered}$$

We require it to still obey the following scaling rule and a Leibnitz rule over the tensor product:

1. $\nabla_{fX} T = f \nabla_X T$
2. $\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$

Furthermore, we require it to play nicely with tensor manipulations, specifically index contraction:

3. $(\nabla_X T)(\cdots, e^\mu, \cdots, e_\mu, \cdots) = \nabla_X (T(\cdots, e^\mu, \cdots, e_\mu, \cdots))$

On scalar fields, these properties are satisfied by the vector's action on the scalar:

$$\nabla_X f \equiv X[f]$$

This implies

$$\nabla_X \nabla_Y f - \nabla_Y \nabla_X f = ([X, Y])[f]$$

On vectors, these properties are satisfied by the affine connection, since its following property was the Leibnitz rule:

$$\nabla_X (fY) = X[f]Y + f\nabla_X Y$$

Because $fY = f \otimes Y$.

With these two defined, we can extend the covariant derivative to any tensor using the Leibnitz rule and the contraction rule. For example, take a covector:

$$\nabla_X (\omega \otimes V) = \nabla_X \omega \otimes V + \omega \otimes \nabla_X V$$

$\omega \otimes V$ contracts to $\langle \omega, V \rangle$, hence

$$\begin{gathered} \nabla_X \langle \omega, V \rangle = \langle \nabla_x \omega, V \rangle + \langle \omega, \nabla_X V \rangle \\ \therefore \langle \nabla_X \omega, V \rangle = X[\langle \omega, V \rangle] - \langle \omega, \nabla_X V \rangle \end{gathered}$$

Which is valid for all V and so defines $\nabla_X \omega$.

In a coordinate patch, let us determine how the covector basis e^μ transforms:

$$\begin{gathered} \langle \nabla_\mu e^\rho, e_\nu \rangle = e_\mu[\langle e^\rho, e_\nu \rangle] - \langle e^\rho, \nabla_\mu e_\nu \rangle \\ = \partial_\mu \delta^\rho_\nu - \langle e^\rho, \Gamma_{\mu\nu}{}^\sigma e_\sigma \rangle \\ = -\Gamma_{\mu\nu}{}^\rho \\ \therefore \boxed{\nabla_\mu e^\rho = -\Gamma_{\mu\nu}{}^\rho e^\nu} \end{gathered}$$

Now a general tensor may be decomposed in a coordinate patch as:

$$
\begin{gathered}
T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} e_{\mu_1} \otimes \dots \otimes e_{\mu_q} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_r} \\
\therefore \nabla_X T = X[T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r}] e_{\mu_1 \dots \mu_q}{}^{\nu_1 \dots \nu_r} + T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \sum_i e_{\mu_1} \otimes \dots \nabla_X e_{\mu_i} \otimes \\
\dots + T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \sum_j e_{\mu_1} \otimes \dots \otimes e_{\mu_q} \otimes \dots \nabla_X e^{\nu_j} \otimes \dots \\
\implies \nabla_X T = X^\alpha \partial_\alpha T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} e_{\mu_1 \dots \mu_q}{}^{\nu_1 \dots \nu_r} + \\
T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \sum_i X^\alpha \Gamma_{\alpha \mu_i}{}^\beta e_{\mu_1 \dots \mu_{i-1} \beta \dots \mu_q}{}^{\nu_1 \dots \nu_r} - \\
T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \sum_j X^\alpha \Gamma_{\alpha \beta}{}^{\nu_j} e_{\mu_1 \dots \mu_q}{}^{\nu_1 \dots \nu_{j-1} \beta \dots \nu_r}
\end{gathered}
$$

Metric Connection

Given a metric g, ∇ is the metric connection if

$$
\nabla_X g = 0 \;\; \forall X \in T_p\mathcal{M} \;\; \forall p \in \mathcal{M}
$$

This is equivalent to the metric being parallely-transported along any curve.

Given a chart, we can compute this condition to be:

$$
\partial_\mu g_{\alpha\beta} - \Gamma_{\mu\alpha}{}^\rho g_{\rho\beta} - \Gamma_{\mu\beta}{}^\rho g_{\alpha\rho} = 0 \; \forall \mu, \alpha, \beta
$$

We permute the indices μ, α, β to get other equations:

$$
\begin{gathered}
\partial_\beta g_{\mu\alpha} - \Gamma_{\beta\mu}{}^\rho g_{\rho\alpha} - \Gamma_{\beta\alpha}{}^\rho g_{\mu\rho} = 0 \\
\partial_\alpha g_{\beta\mu} - \Gamma_{\alpha\beta}{}^\rho g_{\rho\mu} - \Gamma_{\alpha\mu}{}^\rho g_{\beta\rho} = 0
\end{gathered}
$$

Add these two equations and subtract the first one, divide by 2. Collect the Γ terms on the RHS:

$$
\begin{gathered}
C_{\alpha\beta}{}^\rho g_{\rho\mu} = \Gamma_{(\alpha\beta)}{}^\rho g_{\mu\rho} - \Gamma_{[\mu\beta]}{}^\rho g_{\alpha\rho} - \Gamma_{[\mu\alpha]}{}^\rho g_{\beta\rho} \\
\boxed{C_{\alpha\beta}{}^\rho \equiv \frac{1}{2} g^{\rho\sigma} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})}
\end{gathered}
$$

We can rewrite C in terms of the symmetric connection and the torsion:

$$
C_{\alpha\beta}{}^\rho = S_{\alpha\beta}{}^\rho + g^{\rho\mu} g_{\nu(\alpha} T_{\beta)\mu}{}^\nu
$$

C is known as the **Christoffel connection**. Given a metric, the Christoffel is completely determined. Given a metric and the torsion, the metric connection is completely determined (since to go from S to Γ we still only need the torsion).

However, this tells us that unless torsion is vanishing, the Christoffel does not fully determine the geodesics, since we get a symmetric contribution from the torsion in the above equation.

Levi-Civita Connection

For vanishing torsion, there is a *unique metric connection*, called the *Levi-Civita connection* ${\Gamma_{\mu\nu}}^{\rho} = {C_{\mu\nu}}^{\rho}$. This is symmetric.

In the absence of Physics dynamically determining the torsion, this is what naturally arises on a metric space. For example, this is what arises from varying the Einstein-Hilbert action to obtain GR. It is then intuitive that this connection extremises the geodesic length, for all the metric connections given a metric.

The geodesic equation with the Levi-Civita connection can be derived by extremising the length of a curve, given a metric:

$$I[\mathcal{C}] \equiv \int_{\mathcal{C}} \mathrm{d}s = \int_{\mathcal{C}} \mathrm{d}\lambda \sqrt{\dot{x}^{\mu}(\lambda)\dot{x}^{\nu}(\lambda)g_{\mu\nu}(x(\lambda))}$$
$$\delta I = 0 \implies \ddot{x}^{\mu} + {C_{\alpha\beta}}^{\mu}\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

Curvature

$$R : \mathcal{J}_0^1 \times \mathcal{J}_0^1 \times \mathcal{J}_0^1 \to \mathcal{J}_0^1$$
$$X, Y, Z \to R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_y \nabla_X Z - \nabla_{[X,Y]} Z$$

This is antisymmetric in X, Y, trilinear $R(aX, bY, cZ) = abcR(X, Y, Z)\ \forall a, b, c \in \mathcal{F}(\mathcal{M})$ and defines a 1,3 tensor $R^{\mu}_{\nu\rho\sigma}$, called the Riemann curvature tensor:

$$R^{\mu}_{\sigma\nu\rho} = \langle e^{\mu}, R(e_{\nu}, e_{\rho}, e_{\sigma}) \rangle$$

In a coordinate basis, $[e_{\rho}, e_{\sigma}] = \partial_{\rho}(1)e_{\sigma} - \partial_{\sigma}(1)e_{\rho} = 0$, so

$$
\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \langle e^{\mu}, R(e_{\nu}, e_{\rho}, e_{\sigma}) \rangle \\
&= 2\langle e^{\mu}, \nabla_{[\nu}\nabla_{\rho]} e_{\sigma} \rangle \\
&= 2\langle e^{\mu}, \nabla_{[\nu}\nabla_{\rho]} e_{\sigma} \rangle \\
&= 2\langle e^{\mu}, \nabla_{[\nu}(\Gamma_{\rho]\sigma}{}^{\beta} e_{\beta}) \rangle \\
&= 2\left\langle e^{\mu}, \partial_{[\nu}\left(\Gamma_{\rho]\sigma}{}^{\beta}\right) e_{\beta} - \Gamma_{\beta[\nu}{}^{\alpha}\left(\Gamma_{\rho]\sigma}{}^{\beta}\right) e_{\alpha} \right\rangle \\
&= 2(\partial_{[\nu}\Gamma_{\rho]\sigma}{}^{\mu} - \Gamma_{\beta[\nu}{}^{\mu}\Gamma_{\rho]\sigma}{}^{\beta}) \\
&= \partial_{\nu}\Gamma_{\rho\sigma}{}^{\mu} - \partial_{\rho}\Gamma_{\nu\sigma}{}^{\mu} + \Gamma_{\nu\sigma}{}^{\beta}\Gamma_{\beta\rho}{}^{\mu} - \Gamma_{\rho\sigma}{}^{\beta}\Gamma_{\beta\nu}{}^{\mu}
\end{aligned}
$$

What is the geometric interpretation of this?

Consider a point x_p^{μ} in a chart, and two direction vectors $\epsilon^{\mu}, \delta^{\mu}$ (of small magnitude). We have curves through p:

$$
\begin{aligned}
\mathcal{C}_1 &: x_p^{\mu} + \lambda\epsilon^{\mu} \\
\mathcal{C}_2 &: x_p^{\mu} + \lambda\delta^{\mu}
\end{aligned}
$$

From the points at $\lambda = 1$, we can define two more curves:

$$
\begin{aligned}
\mathcal{C}_1' &: x_p^{\mu} + \epsilon^{\mu} + \lambda\delta^{\mu} \\
\mathcal{C}_2' &: x_p^{\mu} + \delta^{\mu} + \lambda\epsilon^{\mu}
\end{aligned}
$$

These meet at $p' : x_p^{\mu} + \epsilon^{\mu} + \delta^{\mu}$.

Consider a vector $V \in T_p$, transport it boh ways two $V_1, V_2 \in T_{p'}$, then

$$
V_1^{\mu} - V_2^{\mu} = R^{\mu}{}_{\rho\alpha\beta} V^{\rho}\delta^{\alpha}\epsilon^{\beta}
$$

The difference between these is the curvature.

Holonomy

Consider loops passing through a point p, and the parallel transport of a vector around the loop. The map from the initial to transported vector is a linear transformation on $T_p\mathcal{M}$ and is called the **holonomy** at p for given loop $\mathcal{C} : S^1 \to \mathcal{M}$.

Consider this effect at a point p in a contractible patch U, then we can write the linear transformation in terms of basis vectors:

$$
\begin{aligned}
(e_{\mu})_{\mathcal{C}} &= (M_{\mathcal{C}})_{\mu}^{\ \nu} e_{\nu} \\
X_{\mathcal{C}} &= X^{\mu}(M_{\mathcal{C}})_{\mu}^{\ \nu} e_{\nu}
\end{aligned}
$$

The reverse loop should clearly perform the inverse transformation, so the inverse must exist, and $M_{\mathcal{C}} \in \mathrm{GL}(m, \mathbb{R})$.

Since two curves combine to form a third curve, the set of all $M_{\mathcal{C}}$ through a point p will form a group, called the Holonomy group at p, H_p – this will be a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

In a similar fashion, for distinct points connected by a curve $\mathcal{D}$, one can define a linear transformation from $L_{\mathcal{D}} : T_p\mathcal{M} \to T_q\mathcal{M}$. We can combine the transport to q, a closed loop at q, and the inverse curve back to p – and this makes a closed curve starting and ending at p. Then

$$\begin{aligned} M_{\mathcal{C}_p} &= L_{\mathcal{D}}^{-1} M_{\mathcal{C}_q} L_{\mathcal{D}} \\ \implies H_p &\supseteq L_{\mathcal{D}}^{-1} H_q L_{\mathcal{D}} \end{aligned}$$

This way, every $M_{\mathcal{C}_q}$ can be mapped to a $M_{\mathcal{C}_p}$, but vice versa too. Thus

$$H_p = L_{\mathcal{D}}^{-1} H_q L_{\mathcal{D}}$$

And $H_p \simeq H_q$. This holds for any two points connected by a path.

If the manifold is path-wise connected, the holonomy group is independent of the point, $H_p = H \forall p \in \mathcal{M}$.

Berger's Classification

Given the following conditions:

1. Simply connected Riemannian manifold
2. The manifold is irreducible, i.e. not locally a product space
3. The manifold is non-symmetric, i.e. not locally a coset space G/H
4. The Levi-Civita connection is used

Then Berger showed that the holonomy groups can be completely classified into the following:

Holonomy	dim	Type of manifold	Comments
SO(*n*)	*n*	Orientable manifold	—
U(*n*)	2*n*	Kähler manifold	Kähler
SU(*n*)	2*n*	Calabi–Yau manifold	Ricci-flat, Kähler
Sp(*n*) · Sp(1)	4*n*	Quaternion-Kähler manifold	Einstein
Sp(*n*)	4*n*	Hyperkähler manifold	Ricci-flat, Kähler
G_2	7	G_2 manifold	Ricci-flat
Spin(7)	8	Spin(7) manifold	Ricci-flat

Kahler, hyperkahler and calabi-yau are all kinds of complex manifolds.

Non-coordinate Basis

Consider a basis $\hat{e}_a(p)$ for $T_p\mathcal{M}$ which is not arising from any coordinate patch – the only requirement beign that $\hat{e}_a(p)$ are smooth vector fields on each chart U.

This can ofc be written in terms of the coordinate basis:

$$\hat{e}_a = \hat{e}_a^{\ \mu} e_\mu, \quad \hat{e}_a^{\ \mu} \in \mathrm{GL}(m, \mathbb{R})$$

The matrices $\hat{e}_a^{\ \mu}$ are called zweibeins/dreibeins/vierbeins/vielbeins for manifolds of dimensions 2, 3, 4, >4.

A metric g can be given components $g_{ab} = g(\hat{e}_a, \hat{e}_b)$. At any point, this matrix form can be diagonalised to g'_{ab} with diagonal elements ± 1 – and so we can do it all points using a local basis transformation $\Lambda_a^{\ b}(x)$, which is a smooth function of x.

This gives us an orthonormal basis. In a pseudo-Riemannian manifold of signature (t, s),

$$g_{ab} = \eta_{ab} = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{t}, \underbrace{+1, \ldots, +1}_{s})$$

We also require that (generally) the non-coordinate basis have the same orientation as the coordinate basis, i.e. the vielbeins have positive determinant.

All this allows us to write the coordinate-basis metric in terms of the vielbeins which transform to the orthonormal basis:

$$
\begin{gathered}
g_{\mu\nu}\hat{e}_a^{\ \mu}\hat{e}_b^{\ \nu} = \eta_{ab} \\
\hat{e}^a_{\ \mu} \equiv (\hat{e}_a^{\ \mu})^{-1} \\
\implies g_{\mu\nu} = \hat{e}^a_{\ \mu}\hat{e}^b_{\ \nu}\eta_{ab}
\end{gathered}
$$

And other vectors' components may be written as $\hat{V}^a = \hat{e}^a_{\ \mu}V^\mu$ too.

Dual non-coordinate basis

$$
\begin{gathered}
\langle\hat{\theta}^a, \hat{e}_b\rangle = \delta^a_b \\
\implies \hat{\theta}^a = \hat{e}^a_{\ \mu}\mathrm{d}x^\mu \\
\omega = \omega_\mu \mathrm{d}x^\mu = \hat{\omega}_a\hat{\theta}^a \\
\hat{\omega}_a = \hat{e}_a^{\ \mu}\omega_\mu
\end{gathered}
$$

Form of the Metric

While the metric still takes the obvious form:

$$
g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b
$$

Something odd that happens is that, since this is not a coordinate basis,

$$
[e_\mu, e_\nu] = 0, \quad [\hat{e}_a, \hat{e}_b] = c_{ab}^{\ c}(p)\hat{e}_c \neq 0
$$

$c_{ab}^{\ c}(p)$ is called the *object of anholonomy*.

In the non-coordinate basis,

$$
\begin{gathered}
\Omega_g = \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^m \\
= \hat{e}^1_{\ \mu_1} \cdots \hat{e}^m_{\ \mu_m}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_m} \\
= \det(\hat{e}^a_{\ \mu})\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m
\end{gathered}
$$

Which is also we would obtain from evaluating $\sqrt{|\det g_{\mu\nu}|} = \det \hat{e}^a_{\ \mu}$, because of the orientation constraint.

Connections and curvature in NC basis

Define the connection:

$$
\nabla_{\hat{e}_a}\hat{e}_b \equiv \Gamma_{ab}^{\ \ c}\hat{e}_c
$$

Thus we can relate the connections:

$$\nabla_a \hat{e}_b = \hat{e}_a{}^{\mu} \nabla_\mu (\hat{e}_b^{\nu} e_\nu) = \hat{e}_a{}^{\mu} \left(\frac{\partial}{\partial x^\mu} \hat{e}_b{}^{\nu} + \hat{e}_b{}^{\alpha} \Gamma_{\mu\alpha}{}^{\nu} \right) e_\nu$$

$$\Gamma_{ab}{}^{c} = \hat{e}_a{}^{\mu} \hat{e}^c{}_{\nu} \left(\frac{\partial}{\partial x^\mu} \hat{e}_b{}^{\nu} + \hat{e}_b{}^{\alpha} \Gamma_{\mu\alpha}{}^{\nu} \right)$$

If we're using the metric connection,

$$\begin{aligned} 0 &= \nabla_a (\eta_{bc} \hat{\theta}^b \otimes \hat{\theta}^c) \\ &= (\eta_{dc} \Gamma_{ab}{}^{d} + \eta_{bd} \Gamma_{ac}{}^{d}) \hat{\theta}^b \otimes \hat{\theta}^c \\ &= 2\Gamma_{a(bc)} \hat{\theta}^b \otimes \hat{\theta}^c \\ \implies & \boxed{\Gamma_{abc} = -\Gamma_{acb}} \end{aligned}$$