Lecture 1 - Susy

Course	Supersymmetry
Date	@January 13, 2025
Status	Completed
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Intro to SUSY

$$Q_A \phi = \psi_A$$

Supercharges relate bosonic and fermionic DoF, and so in QFT must be spin-1/2 and carry a spinor index. Representations of supersymmetry are thus pairs of bosonic & fermionic fields separated by half spin.

The multiplet altogether, (ϕ, ψ_A), is known as a superfield. The SUSY algebra itself is an extension of the Poincare algebra (so Q_A in addition to generators $P_\mu, M_{\mu\nu}$).

Due to the spinor index, SUSY is a spacetime symmetry and not an internal symmetry. (Think about it – all internal symmetries act by mixing fields that are distinct in terms of spacetime, but spin is an angular momentum, and the components of a non-zero-spin field just encode behaviour under this angular momentum.)

Super-Poincare algebra

$$\begin{gathered}[P, P] = 0, \quad [P, M] \sim P, \quad [M, M] \sim M \\ [M, Q] \sim Q, \quad [P, Q] = 0\end{gathered}$$

Explicitly, a few examples:

$$\begin{gathered}[P_\mu, M_{\nu\gamma}] \sim \eta_{\mu\nu} P_\gamma - \eta_{\mu\gamma} P_\nu \\ [Q_A, M_{\mu\nu}] \sim (\gamma_{[\mu}\gamma_{\nu]})_A^{\ B} Q_B\end{gathered}$$

And most importantly, we have the anticommutation for the supercharges:

$$\{Q_A, Q_B\} \sim \gamma^{\mu}_{AB} P_{\mu}$$

Commutation relations would not work here, you couldn't get a consistent definition, essentially because of the spinor nature of the supercharges.

The explicit action of the supercharge will be:

$$\delta_Q \phi \sim \psi$$
$$\delta_Q \psi \sim \partial \phi$$

It should make sense that applying a supercharge twice can accomplish a derivative, due to the anticommutator above.

Note the algebra implies

$$[P^2, Q] = 0$$

I.e. the mass is invariant under supersymmetry, and the superpartners should have identical mass.

Susy in a field theory

As always, requiring additional symmetry means additional constraints on a theory, so susy theories are a subset of all theories, and we cannot turn every theory supersymmetric. We need to introduce superpartners of equal mass and related coupling.

If that is accomplished, though, susy theories can remove a lot of the UV divergences of the underlying theories, since the superpartner contribution has a different sign and, for equal mass and related couplings, does cancel the effect of many diagrams. We can even obtain finite QFTs with no UV divergences, i.e. $N = 4\ \mathrm{SYM}$, which has 3 complex scalars, a gauge boson, and 4 supercharges relating them to 4 fermionic superpartners.

Other benefits of susy:

1. No quadratic divergences of the kind $\int \frac{\mathrm{d}^4 p}{p^2 + m^2} \sim \Lambda^2$. This is related to the hierarchy problem, which is avoided in susy theories.
   In fermionic theories,
   $\delta m^2 \sim m^2 \log(M/m)$, whereas in bosonic theories $\delta m^2 \sim M^2$. Susy theories enforce the log divergent behaviour, which makes the hierarchy problem less significant – part of the hierarchy problem is why the presence

of different mass scales in the SM is stable under quantum corrections, and if the corrections are all logarithmic this stability is straightforward.
The MSSM thus avoids this problem, while also providing dark matter candidates. But it does prefer
$M_{\text{Higgs}} < 100\ \text{GeV}$, whereas it is actually 125, so it's not quite complete.

2. Susy improves the unification of strong-electro-weak – as of now, the couplings almost intersect at 10^{16}, but not quite.

Lecture 2 - Susy

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Spinor Rep of the Lorentz Group

Today the goal is to define notations.

Fermions are anticommuting spinors, and for today's discussion we focus on spinors with two complex components.

$\mathrm{SL}(2,\mathbb{C})$ is a double cover of $\mathrm{SO}(1,3)$, the Lorentz group, and the vectors of $\mathrm{SL}(2,\mathbb{C})$ become the spinor rep of Lorentz. $\mathrm{SL}(2,\mathbb{C})$ consists of 2×2 complex matrices with $\det = 1$ – so 6 real components, same as Lorentz.

For $S_\alpha^{\ \beta} \in \mathrm{SL}(2,\mathbb{C})$, we will use the Levi-Civita to raise and lower indices.

$$\begin{gathered}\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \\ \epsilon^{12} := 1 \\ \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} \equiv \delta_\alpha^{\ \gamma} \implies \epsilon_{12} = -1\end{gathered}$$

$\epsilon^{\alpha\beta}$ is an invariant tensor of $\mathrm{SL}(2,\mathbb{C})$, which is made clear by expressing the determinant condition in terms of it:

$$\begin{gathered}1 = \det S := \epsilon^{\beta\delta} S_\beta^{\ 1} S_\delta^{\ 2} = -\tfrac{1}{2}\epsilon_{\alpha\gamma}\epsilon^{\beta\delta} S_\beta^{\ \alpha} S_\delta^{\ \gamma} \\ \therefore \epsilon^{\beta\delta} S_\beta^{\ \alpha} S_\delta^{\ \gamma} = \epsilon^{\alpha\gamma}\end{gathered}$$

Then we can use the raising/lowering properties of Levi-Civita to write

$$(S^{-1})_\gamma^{\ \delta} = \epsilon^{\delta\beta} S_\beta^{\ \alpha} \epsilon_{\alpha\gamma}$$

The obvious fundamental rep is

$$\psi_\alpha \in \mathbb{C},$$
$$\psi'_\alpha = S_\alpha^{\ \beta}\psi_\beta$$

$\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$ is an equivalent rep (can be called the dual rep), which transforms with S^{-1}:

$$\psi'^\alpha = (S^{-1})_\beta^{\ \alpha}\psi^\beta$$

We can introduce another fundamental rep, $\bar{\psi}_{\dot{\alpha}}$, which transforms with S^*:

$$\bar{\psi}'_{\dot{\alpha}} = S_{\dot{\alpha}}^{*\dot{\beta}}\bar{\psi}_{\dot{\beta}}$$

Notice this relation is simply the complex conjugate of the relation $\psi'_\alpha = S_\alpha^{\ \beta}\psi_\beta$, so $\bar{\psi}_{\dot{\beta}} \mapsto \psi^*_\beta$. But it is useful to distinguish it as an independent representation, as we shall see later. We call this the conjugate rep.

The levi-civita works similarly, $\epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\beta}\dot{\alpha}}$. But note this is an *inequivalent representation* to the fundamental.

And its dual rep, $\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$:

$$\bar{\psi}'^{\dot{\alpha}} = (S^{*\,-1})_{\dot{\beta}}^{\ \dot{\alpha}}\bar{\psi}^{\dot{\beta}} = (S^{*\,-1\,T})^{\dot{\alpha}}_{\ \dot{\beta}}\psi^{\dot{\beta}}$$

Where we used:

$$(S^T)_\alpha^{\ \beta} =: S^\beta_{\ \alpha}$$

And similarly for other matrices.

The conjugate matrices also obey:

$$\epsilon^{\dot{\alpha}\dot{\gamma}}S_{\dot{\alpha}}^{*\dot{\beta}}S_{\dot{\delta}}^{*\dot{\gamma}} = \epsilon^{\dot{\beta}\dot{\delta}}$$

The two inequivalent reps will be the left- and right-handed spinors under Lorentz.

Now we introduce matrices to connect $\mathrm{SL}(2,\mathbb{C})$ fundamental reps to the $\mathrm{SO}(1,3)$ fundamental rep, which are the following **4** hermitian matrices:

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\sigma^0, \sigma^i)_{\alpha\dot{\alpha}}$$

Of which 3 are just the Pauli matrices. Note the indices.

$$\sigma^0 \equiv -\mathbb{I}_{2\times 2}$$

Also note our Minkowski metric is that of positive signature, $-+++$.

Using the epsilon tensors, the indices can be raised and we obtain:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\sigma^0, -\sigma^i)^{\dot{\alpha}\alpha}$$

Some properties:

$$\begin{aligned}\sigma^{\mu\dagger} &= \sigma^\mu \\ \bar{\sigma}^{\mu\dagger} &= \bar{\sigma}^\mu \\ \bar{\sigma}^\mu &= -\sigma_\mu\end{aligned}$$

The last equation is ofc obtained using the epsilon tensors and the flat metric.

Exercise: show that

1. $\sigma^{(\mu}\bar{\sigma}^{\nu)} = -\eta^{\mu\nu}\mathbb{I}$
2. $\bar{\sigma}^{(\mu}\sigma^{\nu)} = -\eta^{\mu\nu}\mathbb{I}$
3. $\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = -2\eta^{\mu\nu}$
4. $\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}$ (completeness relation of sorts – think $|i\rangle\langle j| = \delta_{ij}$)

We know that σ^μ form a basis on the vector space of 2×2 hermitian matrices – this is easy to show, 4 components there, 4 lin-ind herm matrices here.

But also, consider hermitian $X = X^\mu\sigma_\mu = X_\mu\sigma^\mu$, then

$$\begin{aligned}X^\dagger &= X \\ X' := SXS^\dagger &\quad S \in \mathrm{SL}(2,\mathbb{C}) \\ \implies X'^\dagger &= X'\end{aligned}$$

Thus $\mathrm{SL}(2,\mathbb{C})$ preserves the hermitivity. What happens to the determinant?

$$\begin{aligned}\det X' = \det X * \det S * \det S^\dagger &= \det X \\ \det X &= \det(X_\mu\sigma^\mu) \\ &= X_0^2 - X_1^2 - X_2^2 - X_3^2 \\ &= -X_\mu X_\nu \eta^{\mu\nu}\end{aligned}$$

Thus $\mathrm{SL}(2,\mathbb{C})$ also preserves the determinant, which can be understood as the Minkowski norm of the vector X_μ. And transformations which preserve the Minkowski norm of 4-vectors are, by definition, the Lorentz transformations!

Define

$$\begin{gathered} X'^\mu \equiv \Lambda^\mu_{\ \nu} X^\nu \\ \eta_{\mu\nu} X'^\mu X'^\nu = \eta_{mn} X^m X^n \\ \implies \Lambda^\mu_{\ m} \Lambda^\nu_{\ n} \eta_{\mu\nu} = \eta_{mn} \end{gathered}$$

And we have obtained the defining constraint for the Lorentz matrices!

Note, this only shows $\mathrm{O}(1,3)$, but we can then show that we can't obtain the inversions (time reversal, parity) from $\mathrm{SL}(2,\mathbb{C})$ and so it is only a double cover of $\mathrm{SO}(1,3)$.

And now we can connect these matrices with the original transformation S:

$$\begin{gathered} X'^\mu \sigma_\mu = (S\sigma_m S^\dagger) X^m \\ \implies \boxed{\Lambda^\mu_{\ \nu}\sigma_\mu = S\sigma_\nu S^\dagger} \end{gathered}$$

Look familiar yet? Recall a similar relation for the gamma matrices when discussing Dirac bispinors and their Lorentz transformation.

This relation is why we introduced the Pauli matrices with indices $\sigma^\mu_{\alpha\dot{\alpha}}$ – because S acts on α and $S^\dagger$ on $\dot{\alpha}$.

Hence we have a mapping from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SO}(1,3)$ – but note this maps $S, -S$ to the same Λ and is hence a double cover.

Now we can contract with $\bar{\sigma}^\nu$ and take the trace along the $\mathrm{SL}(2,\mathbb{C})$ indices, to get

$$\Lambda^\mu_{\ \nu} = -\tfrac{1}{2}\,\mathrm{Tr}\left(\bar{\sigma}^\mu S\sigma_\nu S^\dagger\right)$$

(Using $\mathrm{Tr}(\bar{\sigma}^\mu\sigma^\nu) = -2\eta^{\mu\nu}$.)

Lecture 3 - Susy

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Recall we showed that conjugation by $\mathrm{SL}(2, \mathbb{C})$ generates Lorentz transforms:

$$\Lambda^{\mu}{}_{\nu} = -\tfrac{1}{2}\operatorname{Tr}\left(\bar{\sigma}^{\mu} S \sigma_{\nu} S^{\dagger}\right)$$

From this it is straightforward to see that

$$\Lambda^{0}{}_{0} = \tfrac{1}{2}\operatorname{Tr}(SS^{\dagger}) > 0$$

Less straightforward, but still doable, is to show that

$$\det \Lambda = 1$$

Alternatively, show that

$$\operatorname{Tr}\Lambda = \Lambda^{m}{}_{m} > 0$$

The idea is that this excludes parity, but this isn't a complete proof. If one can show that $\Lambda^{i}{}_{i} \geq 0 \forall i$, then it is a complete proof.

Thus from the above relation, it can be shown that $\Lambda^{\mu}{}_{\nu} \in \mathrm{SO}^{+}(1,3)$ and not any superset of it.

General Reps of $\mathrm{SL}(2, \mathbb{C})$

$$T^{\alpha_1 \ldots \alpha_n \dot{\delta}_1 \ldots \dot{\delta}_p}_{\gamma_1 \ldots \gamma_l \dot{\rho}_1 \ldots \dot{\rho}_s}$$

Where lowered indices transform with S, S^* and raised with S^{-1}, S^{*-1}, and dots with conjugates.

This is a reducible rep. To find the irreps, we lower all the indices with $\epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}$. Then, since the antisymmetric part, for each pair of indices, is proportional to $\epsilon_{\alpha\beta}$ or $\epsilon_{\dot\alpha\dot\beta}$ times a lower-dim rep, no rep with antisymmetric components is irreducible, hence the only irreps are of the following kind:

$$T_{(\alpha_1\ldots\alpha_p)(\dot\beta_1\ldots\dot\beta_q)}$$

Where the dotted and undotted indices are symmetrised separately. Here, btw, the convention is $\cdot_{(mn)} = \frac{1}{2}(\cdot_{mn} + \cdot_{nm})$.

Then the irreps are characterised only by the two natural numbers (p, q). So the irreps:

- $(0,1) = \bar\psi_{\dot\alpha}$ → "right" spinor of Lorentz
- $(1,0) = \psi_\alpha$ → "left" spinor of Lorentz
- $(1,1)$ → Lorentz vector, $\Lambda^\mu{}_\nu \sigma_\mu = S\sigma_\nu S^\dagger$, thus $A^\mu = \bar\sigma^{\mu\dot\beta\alpha}T_{\alpha\dot\beta}$ is our Lorentz vector. It doesn't really matter if it's a covector or contravector, since they are equivalent reps of Lorentz.
  Note, this is a complex Lorentz vector.

Clearly, $(p+q)/2$ is the spin of the corresponding Lorentz rep.

- $(2,1)$ → spin-3/2, like a gravitino – specifically "left" gravitino, $T_{\mu\alpha}$.
- $(2,0)$ or $(0,2)$ → 3 complex components, what is this?

  Recall the Maxwell field strength tensor, 6 components, but that's not an irrep. The irreps are:

$$\begin{gathered} F^\pm_{mn} = F_{mn} \pm \tilde F_{mn} \\ \text{Where } \tilde F_{mn} = \frac{\iota}{2}\epsilon_{mnpq}F^{pq} \\ \tilde F^\pm = \pm F^\pm \\ \text{Note } \tilde{\tilde F} = F \because \epsilon_{mnpq}\epsilon^{m'n'pq} = -4\delta^{m'}_{[m}\delta^{n'}_{n]} \end{gathered}$$

  $\tilde F$ is the dual field strength tensor.

  To write $T_{\alpha\beta}$ in Lorentz rep form, we define

$$\begin{gathered} (\sigma_{mn})_\alpha{}^\beta = \frac{1}{2}\left(\sigma_{[m}\bar\sigma_{n]}\right)_\alpha{}^\beta \\ \implies \sigma_{mn} = \frac{\iota}{2}\epsilon_{mn}{}^{pq}\sigma_{pq} \end{gathered}$$

σ_{mn} is self-dual.

Thus

$$F^+_{mn} = (\sigma_{mn})^{\alpha\beta} T_{\alpha\beta}$$

Similarly,

$$\bar{\sigma}_{mn} = \frac{1}{2}\bar{\sigma}_{[m}\sigma_{n]}$$
$$\tilde{\bar{\sigma}}_{mn} = -\bar{\sigma}_{mn}$$

Note: In a Minkowski signature metric, $\eta_{00} = -1, \epsilon^{0123} = +1, \epsilon_{0123} = -1.$

- $(2, 2)$ → Spin-2, Graviton. 3*3=9 components, which is 10-1, which corresponds directly to the traceless part of the 4D metric (since the perturbations are in the traceless part).

Since $\sigma_{mn}, \bar{\sigma}_{mn}, \sigma_m, \bar{\sigma}_m$ absorb $2\alpha, 2\dot{\alpha}, 1\alpha\&1\dot{\alpha}, 1\alpha\&1\dot{\alpha}$ indices respectively, this means when $p+q$ is even we can obtain a Lorentz tensor, but when $p+q$ is odd, we can only obtain **spin-tensors** – non-integral spin objects, which have an unremovable $\mathrm{SL}(2, \mathbb{C})$ index.

4-Component spinors of Lorentz

Using the familiar Gamma matrices satisfying the Clifford algebra,

$$\{\gamma_n, \gamma_m\} = -2\eta_{nm} I$$

Then define

$$M^{\text{spinor}}_{mn} = \tfrac{\iota}{2}\gamma_{[m}\gamma_{n]}$$

And one can show these satisfy the Lorentz algebra.

Then the Lorentz transformation:

$$\hat{S} = \exp \frac{\iota}{2}\omega^{mn} M^{\text{spinor}}_{mn}$$

And one can show that

$$\hat{S}^{-1}\gamma^m \hat{S} = \Lambda^m_{\ n}\gamma^n$$

(Similarly, we can also write down $(M^{\text{vector}}_{mn})^A{}_B = \iota(\delta^A_m \eta_{nB} - \delta^A_n \eta_{mB})$ and show that it satisfies the Lorentz algebra.)

Lecture 4 - Susy

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Weyl Rep of Clifford Algebras

Weyl rep:

$$\gamma_m = \begin{pmatrix} 0 & \sigma_{m\alpha\dot{\beta}} \\ \bar{\sigma}_m^{\dot{\beta}\alpha} & 0 \end{pmatrix}$$

Let us look the details of this rep:

$$M_{mn} = \frac{\iota}{2}\gamma_{[m}\gamma_{n]} = \begin{pmatrix} \iota\sigma_{mn} & 0 \\ 0 & \iota\bar{\sigma}_{mn} \end{pmatrix}$$
$$\hat{S} = \begin{pmatrix} S_\alpha^{\ \beta} & 0 \\ 0 & \bar{S}^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix}$$

If we exponentiate M_{mn} and get the explicit form of $\hat{S}$, we obtain it in terms of $S = e^{-\frac{1}{2}\omega^{mn}\sigma_{mn}}$ and $\bar{S} = e^{-\frac{1}{2}\omega^{mn}\bar{\sigma}_{mn}}$. Since $\det S = \exp\frac{-1}{2}\omega^{mn}\,\mathrm{Tr}(\sigma_{mn}) = \exp 0 = 1$, S (and $\bar{S}$) are $\mathrm{SL}(2,\mathbb{C})$ matrices (and will generate Lorentz transforms like $\Lambda^\mu{}_\nu\sigma_\mu = S\sigma_\nu S^\dagger$).

Numerically,

$$\bar{\sigma}_{mn} = -\sigma^\dagger_{mn}$$
$$\bar{S}^\dagger = S^{-1}$$
$$\implies \bar{S} = S^{-1\dagger}$$

We are familiar with how this rep leads to the left- and right-components of the dirac bispinor to occupy the top and bottom halves of the bispinor.

The left- and right-components transform under different irreps.

$$\psi'_\alpha = S_\alpha^{\ \beta}\psi_\beta$$
$$\bar{\chi}'^{\dot{\beta}} = \bar{S}^{\dot{\beta}}_{\ \dot{\gamma}}\bar{\chi}^{\dot{\gamma}} = (S^{-1*T})^{\dot{\beta}}_{\ \dot{\gamma}}\bar{\chi}^{\dot{\gamma}}$$

Hence we write the bispinor as

$$\Psi = \begin{pmatrix}\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}$$

Note the raised indices for the right-component.

Gamma 5

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$$
$$P_{L/R} = \tfrac{1}{2}(\mathbb{I} \pm \iota\gamma^5)$$
$$\Psi_{L/R} := P_{L/R}\Psi$$

So in Weyl,

$$\iota\gamma_5 = \begin{pmatrix}\mathbb{I} & 0 \\ 0 & -\mathbb{I}\end{pmatrix}$$
$$\iota\gamma_5\Psi_{L/R} = \pm\Psi_{L/R}$$

Weyl Spinors

The left- and right-handed spinors we've already encountered were Weyl spinors. More generally, $\Psi_{L/R}$ are Weyl spinors expressed with 4 components in any representation of the Clifford algebra.

Weyl spinors have 4 real, or 2 complex, degrees of freedom.

Majorana Spinors

These also have 4 real components, but obey a different constraint,

$$\Psi_R = \Psi_L^*$$

In the Weyl representation, this becomes

$$\Psi = \begin{pmatrix}\psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}\end{pmatrix}, \quad \bar{\psi}_{\dot{\beta}} = \psi_\beta^*$$

Charge Conjugation

$$C^{-1}\gamma^\mu C \equiv -(\gamma^\mu)^T$$

Then

$$\Psi^c \equiv C\bar{\Psi}^T \quad (\bar{\Psi} = \Psi^\dagger\gamma^0)$$

Is the charge conjugated field.

For Ψ obeying the coupled Dirac equation,

$$\begin{aligned}(\not{p} + e\not{A})\Psi = 0 \\ \implies (\not{p} - e\not{A})\Psi^c = 0\end{aligned}$$

The charge conjugated field obeys the EoM with inverted charges.

We can also define the Majorana spinor as

$$\Psi_m = \Psi_m^c$$

In the Weyl rep, C is

$$C = \begin{pmatrix} \iota\sigma_2 & 0 \\ 0 & -\iota\sigma_2 \end{pmatrix} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

Grassmann Algebra

An algebra is a vector space with multiplication defined.

Given a basis/generators θ_i s.t.

$$\{\theta_i, \theta_j\} = 0$$

And for scalar e, $e\theta_i = \theta_i e$.

A generic element may be written in a series

$$X = \sum a_{i_1 \dots i_n}\theta_{i_1}\dots\theta_{i_n}$$

Clearly, the sum is over all possible subsets of $[[1, N]]$, and $n \leq N$.

$$\theta_{i_1} \dots \theta_{i_n} = \epsilon_{i_1 \dots i_n} \theta_1 \dots \theta_n$$
$$\therefore a_{i_1 \dots i_n} = a_{[i_1 \dots i_n]}$$

Thus there are a total of 2^N field-valued coefficients describing the generic element.

Clearly, the generators can't be real numbers – they have to have some substructure, eg be matrices. We can construct an example out of the Euclidean gamma matrices.

$$\{\gamma_r, \gamma_s\} = 2\delta_{rs}\mathbb{I}, \quad r, s \in [[1, N]]$$
$$\theta_k \equiv \gamma_k + \iota\gamma_{k+1 \mod N}$$

Spinor Algebra

Grassmann algebra where the generators are spinors

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha$$

Similarly,

$$\bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} = -\bar{\psi}_{\dot{\beta}} \bar{\psi}_{\dot{\alpha}}$$

Introduce the contraction of spinors:

$$\psi\chi \equiv \psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \chi_\alpha$$
$$= (-1)^2 \epsilon^{\beta\alpha} \chi_\alpha \psi_\beta = \chi\psi$$

Thus the contraction becomes overall a symmetric product!

Note a different convention for contractions of the conjugate rep,

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}$$

Lecture 5 - Susy

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Spinor Algebra

$$\psi\chi = \chi\psi$$
$$\bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$$
$$\psi\psi = -2\psi_1\psi_2$$
$$\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\psi,$$
$$\psi^\alpha\psi^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi$$

Since ψ_α is now a Grassmann variable, so it has some substructure and can be thought of as a matrix, $\bar{\psi}$ is no longer a complex conjugate but rather a hermitian conjugate.

$$\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^\dagger$$
$$(\psi\chi)^\dagger = \chi^\dagger\psi^\dagger = \bar{\chi}\bar{\psi}$$

Some other contractions explicitly:

$$\chi\sigma^m\bar{\psi} = \chi^\alpha\sigma^m_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}$$
$$\therefore \chi\sigma^m\bar{\psi} = -\bar{\psi}\bar{\sigma}^m\chi$$

(Because $\sigma^m_{\alpha\dot{\alpha}} \equiv \bar{\sigma}^m_{\dot{\alpha}\alpha}$, since $\bar{\sigma}$ was defined to just be σ with raised and swapped SL(2,C) indices.)

$$(\chi\sigma^m\bar{\psi})^\dagger = \psi\sigma^m\bar{\chi}$$

Since σ^m are Hermitian.

Fierz Identities

Using the completeness of the sigma matrices:

$$\sigma_{m\alpha\dot{\delta}}\bar{\sigma}^{m\dot{\gamma}\beta} = \sigma^m_{\alpha\dot{\delta}}\bar{\sigma}^{\dot{\gamma}\beta}_m = -2\delta^\beta_\alpha\delta^{\dot{\gamma}}_{\dot{\delta}}$$

From these, we obtain the Fierz identities:

$$\begin{aligned}\phi^\alpha\psi_\beta\bar{\chi}^{\dot{\delta}} * \sigma^m_{\alpha\dot{\delta}}\bar{\sigma}^{\dot{\gamma}\beta}_m &= -2\delta^\beta_\alpha\delta^{\dot{\gamma}}_{\dot{\delta}} * \phi^\alpha\psi_\beta\bar{\chi}^{\dot{\delta}} \\ \implies (\phi\sigma^m\bar{\chi})(\bar{\sigma}_m\psi)^{\dot{\gamma}} &= -2(\phi\psi)\bar{\chi}^{\dot{\gamma}} \\ \implies (\phi\psi)\bar{\chi}_{\dot{\alpha}} &= -\tfrac{1}{2}(\phi\sigma^m\bar{\chi})(\psi\sigma_m)_{\dot{\alpha}}\end{aligned}$$

Contract another spinor for the second one:

$$(\phi\xi)(\bar{\chi}\bar{\eta}) = \tfrac{1}{2}(\phi\sigma^m\bar{\eta})(\bar{\chi}\bar{\sigma}_m\xi)$$

And some others:

$$\begin{aligned}\delta_{\alpha\dot{\beta}}\delta_{\gamma\dot{\delta}} &= \frac{1}{2}\left(\delta_{\alpha\dot{\delta}}\delta_{\gamma\dot{\beta}} + \sigma^i_{\alpha\dot{\delta}}\sigma^i_{\gamma\dot{\beta}}\right) \\ \implies (\phi\bar{\chi})(\psi\bar{\eta}) &= -\tfrac{1}{2}\left((\phi\bar{\eta})(\psi\bar{\chi}) + (\phi\sigma^i\bar{\eta})(\psi\sigma^i\bar{\chi})\right)\end{aligned}$$

$$\begin{aligned}\delta^\beta_\alpha\delta^\delta_\gamma &= \tfrac{1}{2}\left(\delta^\delta_\alpha\delta^\beta_\gamma - (\sigma^{mn})_\alpha^{\ \delta}(\sigma_{mn})_\gamma^{\ \beta}\right) \\ \implies (\theta\phi)(\chi\eta) &= -\tfrac{1}{2}\left((\theta\eta)(\chi\phi) - (\theta\sigma^{mn}\eta)(\chi\sigma_{mn}\phi)\right)\end{aligned}$$

Super-Poincare Algebra

A super-Lie algebra, also called a $\mathbb{Z}_2$-graded algebra.

Notation: We will use $[\cdot,\cdot\}$ to refer to a bracket that will be either the commutator or anticommutator, based on the context.

The super Lie algebra will be a vector space $V = V_B \oplus V_F$, and given generators X_B, X_F, a generic element is

$$X = x_B \cdot X_B + x_F \cdot X_F$$

x_B are numbers, x_F are Grassmannian.

$[\cdot,\cdot\}$ will be a commutator for X_B with X_B/X_F, and an anticomm only for X_F, X_F.

Requirements:

1. The bosonic subspace is closed, so $[X_B^i, X_B^j] = C^{ij}{}_k X_B^k$.
   a. $[X_B^i, X_F^\alpha] = \hat{C}^{i\alpha}_\beta X_F^\beta$
   b. $\{X_F^\alpha, X_F^\beta\} = \tilde{C}^{\alpha\beta}_i X_B^i$
2. They obey the super-Jacobi identity.

We can realise these algebras in representations using *supermatrices*, eg

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$a, b \text{ are numbers}$$
$$c, d \text{ are Grassmannian}$$

Super-Jacobi identity

$$[X_1, [X_2, X_3\}\} \pm [X_2, [X_3, X_1\}\} \pm [X_3, [X_1, X_2\}\} = 0$$

Where a sign is flipped only when that permutation flips the order of two fermionic generators. This means the only case with sign flips is two fermionic and one bosonic generators.

Super-Poincare

P_m, M_{mn}, and to get fermions from bosons, the supersymmetry transformation generators must have a spinor index, so we have Q_α and $\bar{Q}_{\dot{\alpha}} = Q_\alpha^\dagger$. Note, these are not Grassmannian, these are operators, but their coefficients will be Grassmannian, generally.

$$[P_m, P_n] = 0$$
$$[M_{mn}, P_k] = \iota(\eta_{mk}P_n - \eta_{nk}P_m)$$
$$[M_{mn}, M_{kl}] = \iota(\eta_{mk}M_{nl} - \eta_{ml}M_{nk} - \eta_{nk}M_{ml} + \eta_{nl}M_{mk})$$

And now for the new Lie brackets. As always, the Lie bracket relations must be Lorentz-invariant laws themselves, and so the structure constants must have the correct transformation laws.

The only objects we have that can convert between spinors and vectors are the sigma (or gamma) matrices, $\sigma^m_{\alpha\dot{\beta}}, \bar{\sigma}^{m\dot{\alpha}\beta}$.

Consider $[P^m, Q_\alpha]$. One may expect, given the spinor index, that this could be some $\bar{c}^m_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}} + c^m_{\alpha\beta}Q^\beta$ – and while we have $\bar{c}$-like objects in the pauli matrices, we don't have any c-like objects.

Thus

$$[P^m, Q_\alpha] = c\sigma^m_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}}$$

But as we will show next time, c is still forced to be 0. One may guess this, considering the spin-1 into spin-1/2 object on the lhs should give some sort of spin-3/2 object? Perhaps. That argument is not easy to make rigorous.

Lecture 6 - Susy

Course	Supersymmetry
Date	@January 23, 2025
Status	Completed
Next	Lecture 7 - Susy
Previous	Lecture 5 - Susy

Super-Poincare, continued

As before,

$$[P^m, Q_\alpha] = c\sigma^m_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}}$$

Take the hermitian conjugate,

$$[P^m, \bar{Q}^{\dot{\alpha}}] = -c^*\bar{\sigma}^{m\dot{\alpha}\beta}Q_\beta$$

The minus sign occurring due to $[A, B]^\dagger = -[A^\dagger, B^\dagger]$.

Then we use the Jacobi identity:

$$\begin{aligned}[P_n, [P_m, Q_\alpha]] + [P_m, [Q_\alpha, P_n]] + [Q_\alpha, \cancel{[P_m, P_n]}] = 0 \\ \implies -cc^*\left(\bar{\sigma}_n\sigma_m - \sigma_n\bar{\sigma}_m\right)Q_\alpha = 0\end{aligned}$$

Since $\left(\bar{\sigma}_n\sigma_m - \sigma_n\bar{\sigma}_m\right)Q_\alpha$ is not 0, c must be.

$$\bar{\sigma}_n\sigma_m - \sigma_n\bar{\sigma}_m = 2\bar{\sigma}_{nm} - [\sigma_n, \bar{\sigma}_m] = 2\bar{\sigma}_{nm} + \begin{cases} 2\iota\epsilon_{nmk}\sigma_k & n, m \neq 0 \\ 0 & n = 0 | m = 0\end{cases}$$

Thus

$$[P_m, Q_\alpha] = 0, \quad [P_m, \bar{Q}_{\dot{\alpha}}] = 0$$

And with the Lorentz generators

$$[M_{mn}, X^A] = C^A_{mnB} X^B$$

Eg

$$\begin{aligned} [M_{mn}, M_{pq}] &= \iota \left(\eta_{mp} M_{nq} + \text{terms}\right) \\ [M_{mn}, P_\mu] &= \iota \left(\eta_{m\mu} P_n - \eta_{\mu n} P_m\right) \end{aligned}$$

Thus, using the only correct objects we have, and using Jacobi to fix the coefficient,

$$\begin{aligned} [M_{mn}, Q_\alpha] &= -\iota \sigma_{mn\alpha}{}^{\beta} Q_\beta \\ [M_{mn}, \bar{Q}^{\dot{\alpha}}] &= -\iota \bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \\ \implies \ [M_{mn}, \hat{Q}] &= -\tfrac{\iota}{2} \gamma_{[m} \gamma_{n]} \hat{Q} \end{aligned}$$

In terms of Majorana bispinor $\hat{Q}$.

Again, using indices, we determine

$$\{Q_\alpha, Q_\beta\} = c \sigma^{mn}_{\alpha\beta} M_{mn}$$

Where Jacobi again forces $c = 0$.

$$\{Q_\alpha, Q_\beta\} = 0$$

Finally,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2 \sigma^m_{\alpha\dot{\beta}} P_m$$

The normalisation here is unconstrained by Jacobi, so we just set it to 2.

Note, We can add several pairs of Q_αs and $Q^\dagger_\alpha$.

The only impact this will have is another index labelling the theory and Kronocker delta of the indices in each commutator.

Consequences - Properties of this Algebra

Positivity of energy

Using the anticomm of $Q, \bar{Q}$, we can solve for P_0:

$$
\begin{gathered}
\mathrm{Tr}(\sigma_m \bar{\sigma}_n) = -2\eta_{mn}, \\
\therefore P^m = -\tfrac{1}{4}\bar{\sigma}^{m\dot{\alpha}\beta}\{Q_\beta, \bar{Q}_{\dot{\alpha}}\} \\
\therefore P^0 > 0
\end{gathered}
$$

Explicitly, we can evaluate

$$
\begin{gathered}
P^0 = \tfrac{1}{4}\delta^{\dot{\alpha}\beta}\{Q_\beta, \bar{Q}_{\dot{\alpha}}\} \\
= \tfrac{1}{4}\sum_{\alpha=1}^{2}\left(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha\right) \\
\therefore E = \langle\psi|\hat{E}|\psi\rangle = \frac{1}{4}\sum_{\alpha=1}^{2}\left(\left|Q_\alpha^\dagger|\psi\rangle\right|^2 + \left|Q_\alpha|\psi\rangle\right|^2\right) \geq 0
\end{gathered}
$$

Now consider a general supersymmetric transformation,

$$
\begin{gathered}
U = e^{\varepsilon Q + \bar{\varepsilon}\bar{Q}} \\
|\psi'\rangle = U|\psi\rangle \\
\therefore \delta|\psi\rangle = (\varepsilon Q + \bar{\varepsilon}\bar{Q})|\psi\rangle + \mathcal{O}(\varepsilon^2)
\end{gathered}
$$

If a state is invariant under all susy transformations,

$$
\begin{gathered}
Q|\psi\rangle = 0 \\
\bar{Q}|\psi\rangle = 0 \\
\Leftrightarrow \hat{E}|\psi\rangle = 0
\end{gathered}
$$

Thus every state invariant under susy transformations has the minimum possible energy, and is a vacuum.

What about spontaneous symmetry breaking? If the supersymmetry is broken, and there is no state invariant under susy, then the vacuum has some $E_{\text{vac}} > 0$. The converse is also true – non-zero energy vacuum implies susy breaking.

Equality of Boson and Fermion masses

Introduce $M^2 \equiv -P_m P^m$, then since P_m commutes with Q_α,

$$
[M^2, Q_\alpha] = 0
$$

Which means

$$
\begin{aligned}
Q_\alpha M^2|\psi_B\rangle &= M^2\underbrace{Q_\alpha|\psi_B\rangle}_{|\psi_F\rangle} \\
\implies m_B^2 Q_\alpha|\psi_B\rangle &= M^2|\psi_F\rangle \\
\implies m_B^2|\psi_F\rangle &= m_F^2|\psi_F\rangle \\
\implies m_B^2 &= m_F^2
\end{aligned}
$$

Lecture 7 - Susy

Course	Supersymmetry
Date	@January 27, 2025
Status	Completed
Next	Lecture 8 - Susy
Previous	Lecture 6 - Susy

Super-Poincare using 4-spinors

Regarding the super-poincare algebra,

$$\{Q_\alpha, Q_\beta\} = 0$$
$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m$$

While these are 2-component spinors, we can write these relations in terms of 4-component spinors:

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}_{\dot{\alpha}} \end{pmatrix}$$
$$\{Q, Q\} = 2\gamma^m P_m$$

Using γ^m in the Weyl representation (different reps would give different forms of Q in terms of $Q_\alpha, \bar{Q}_{\dot{\alpha}}$).

Generalisations of Susy

N-extended susy

$$Q^r_\alpha, \quad r \in [[1, N]]$$

Centrally-extended susy

$$\{Q^r_\alpha, Q^s_\beta\} = \epsilon_{\alpha\beta} Z^{rs}$$

The Zs are scalars that commute with all other generators. They are called the *extra bosonic* or *central charge* generators, and are antisymmetric:

$$Z^{rs} = -Z^{sr}$$

These appear when studying solitons in supersymmetric gauge theories.

These theories have $|Z|$ being a lower bound on the masses,

$$m \geq |Z|$$

The states with $m = |Z|$ are the supersymmetric BPS states.

Consequences of Super-Poincare, continued

Equality of number of bosonic and fermionic states per susy rep

$$n_B = n_F$$

We would like to show this formally.

Introduce the fermion number operator N_F, and the bosonic/fermionic indication operator ν:

$$\nu = (-1)^{N_F}$$
$$\nu|B\rangle = |B\rangle$$
$$\nu|F\rangle = -|F\rangle$$

So for example, the hydrogen atom, with two fermions, is a bosonic state.

Since $Q|B\rangle = |F\rangle$, it increases the number of fermionic states by 1, so

$$\{\nu, Q\} = 0$$

Define the trace over all states *in a particular representation*,

$$\mathrm{Tr}(...) = \sum_b^{n_B} \langle B_b| \ldots |B_b\rangle + \sum_f^{n_F} \langle F_f| \ldots |F_f\rangle$$

And compute

$$
\begin{aligned}
\mathrm{Tr}(\nu\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}) &= \mathrm{Tr}(\nu Q\bar{Q}) + \mathrm{Tr}(\nu\bar{Q}Q) \\
&= \mathrm{Tr}(\nu Q\bar{Q}) + \mathrm{Tr}(Q\nu\bar{Q}) \\
&= \mathrm{Tr}(\{\nu, Q\}\bar{Q}) = 0
\end{aligned}
$$

Since $\{\nu, Q\} = 0$. But also,

$$
\mathrm{Tr}(\nu\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}) = \mathrm{Tr}(\nu * 2P^m \sigma_{m\alpha\dot{\beta}})
$$

Consider

Massive susy rep

Then P^m evaluates to $(M, 0, 0, 0)$ for the bosonic and fermionic states – we choose the unique states in the trace's definition up to Lorentz transformations, so as to not overcount by a factor of a 3-sphere volume.

Then $\mathrm{Tr}(\nu * 2P^m \sigma_{m\alpha\dot{\beta}}) = 2M\sigma_0 \,\mathrm{Tr}(\nu)$.

Massless susy rep

Similar arguments, no centre-of-mass frame so instead $\mathrm{Tr}(\nu * 2P^m \sigma_{m\alpha\dot{\beta}}) = 2E(\sigma_0 - \sigma_3)\,\mathrm{Tr}(\nu)$.

Finally, note that

$$
\mathrm{Tr}(\nu) = n_B - n_F
$$

Thus when we have already shown that $\mathrm{Tr}(\nu\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}) = 0$, this gives us that $n_B = n_F$ *in each susy rep!*

Reps of Susy

Reps of $\mathfrak{so}(3)$

Recall reps of $\mathfrak{so}(3)$, given by any objects obeying:

$$
[J_i, J_j] = \iota\epsilon_{ijk} J_k
$$

In which case one can also define

$$J_\pm = J_1 \pm \iota J_2$$
$$J^2 = J_i J_i = J_+ J_- + J_3^2 - J_3$$
$$[J_\pm, J_3] = \pm J_3$$
$$[J_+, J_-] = 2J_3$$

J^2 is the quadratic Casimir – it commutes with everything. So we can label the reps by the J^2 eigenvalue.

J_3 is also sometimes called the linear Casimir, and it is used to label states within a rep.

$$J^2 |s, s_3\rangle = s(s+1)|s, s_3\rangle$$

s is half-integral.

Of course, $J_+^\dagger = J_-$, and they raise and lower s_3.

Reps of Poincare

Since $\mathrm{SO}(3)$ is the little group (i.e. the stabiliser subgroup of the mass states), the reps are labelled by the mass and the J^2 eigenvalue,

$$|m; s, s_3\rangle$$
$$M^2 = -P^2$$
$$M^2 |m; s, s_3\rangle = m^2 |m; s, s_3\rangle$$

The Casimir invariant is built out of the Pauli-Lubanski vector,

$$W^2 = W^\mu W_\mu$$
$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$

For $P^\mu = (m, 0, 0, 0)$, $W^0 = 0, W^i = mJ^i$ – thus for a particular massive rep in the CoM frame, W^i is simply J^i, so given a mass the reps are simply the SO(3) reps, and hence the labelling makes sense.

What about massless reps? $P^\mu = (E, 0, 0, E)$, $W^\mu = \lambda P^\mu$, where λ is the helicity:

$$\lambda = \frac{P_i J_i}{|\vec{P}|}$$

In this case $\lambda = J_3$, since the momentum is along z. The states are now labelled as

$$|E, \lambda\rangle$$

λ can take values $\pm s$.

Finally, super-poincare

Our generators:

$$Q \oplus (P, M)$$

So all we have to do is extend the generators like we did for Poincare from so(3). The representations of susy are merely multiplets of representations of poincare, fermionic paired with bosonic.

Massless reps

$$|E, \lambda\rangle \oplus |E, \lambda \pm \tfrac{1}{2}\rangle$$

More on this in the next lecture, tomorrow.

Lecture 8 - Susy

Course	Supersymmetry
Date	@January 28, 2025
Status	Completed
Next	Lecture 9 - Susy
Previous	Lecture 7 - Susy

Reps of Super-Poincare

Like for Poincare and SO(3), we can find a Casimir invariant object for the supersymmetry algebra (this is known and is called the super-Pauli-Lubanski vector) and use its eigenvalues to label the reps. But we'll simply construct them by using the spin-statistics theorem and exploiting our knowledge of supersymmetry being the boson-fermion symmetry (note this is not explicit in the definition of super-poincare!).

Massless Reps

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m = 2E(\mathbb{I} + \sigma_3) = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\{Q_\alpha, Q_\beta\} = 0$$
$$\therefore \text{For } a_\alpha := \frac{1}{\sqrt{4E}} Q_\alpha,$$
$$a^\dagger_\alpha = \frac{1}{\sqrt{4E}} Q^\dagger_\alpha = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{\alpha}}$$
$$\implies \{a_1, a^\dagger_1\} = 1, \{a_2, a^\dagger_2\} = 0$$
$$\{a_\alpha, a_\beta\} = 0$$

How do we build the states off of this? Let's review how we work with a Clifford algebra.

Clifford Algebra

$$\{a_i, a^\dagger_j\} = \delta_{ij}$$
$$\{a_i, a_j\} = 0$$

Then we construct the states by taking a vacuum – any state annihilated by all the a_is – and act the creation operators on it. Due to the anticomm relations, we only get 2^N states (for $i \in [[1, N]]$), up to the N-particle state $a_1^\dagger \dots a_N^\dagger |0\rangle$.

If $\langle 0|0\rangle = 1$, then all the other states are normalised too. This is simple to see using the anticomm relations.

One can also check that no a_i can be involved in a non-vacuum state.

Back to massless rep

We will require the states for the irrep to not be 0-normed. So we're left with a 1-dim clifford algebra. But we additionally require the vacuum to be annihilated by a_1 and a_2. Then

$$
\begin{gathered}
|0\rangle = |E, \lambda\rangle \\
|i\rangle \equiv a_i^\dagger |0\rangle \\
\langle 1|1\rangle = 1 \\
\langle 2|2\rangle = 0
\end{gathered}
$$

And clearly $a_1^\dagger a_2^\dagger |0\rangle$ is also a 0-normed state. So for each E, λ, we get two states in the susy rep, one annihilated by a_1, and one that is not.

We discard the 0-normed states because including them gives a reducible rep. An alternate explanation is that because of the energy positivity condition,

$$
\langle 0|\{a_2, a_2^\dagger\}|0\rangle = 0 \implies a_2^\dagger|0\rangle = 0
$$

In a Hilbert space, a 0-normed state must be 0, so this makes perfect sense.

In that sense, including them is a reducible rep, because it's including multiple copies of the trivial representation which is annihilated by all operators.

We can always construct the vacuum, because if $|\psi\rangle$ isn't the vacuum, $a_1|\psi\rangle$ is.

Why does the spin change?

Let's illustrate how we get the $\lambda \to \lambda + 1/2$ when going from $|0\rangle$ to $|1\rangle$.

$$
\begin{gathered}
[M_{mn}, \bar{Q}^{\dot{\alpha}}\} = -\iota \bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \\
\bar{\sigma}_{mn} = \frac{1}{4}(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m) \\
M_{12} = J_3, \bar{\sigma}_{12} = -\frac{\iota}{2}\sigma_3
\end{gathered}
$$

Thus

$$
\begin{gathered}
[J_3, \bar{Q}^{\dot{1}}] = -\tfrac{1}{2}\bar{Q}^{\dot{1}} \\
[J_3, \bar{Q}^{\dot{2}}] = +\tfrac{1}{2}\bar{Q}^{\dot{2}} \\
\text{Generally, } [J_3, \bar{Q}^{\dot{\alpha}}] = -\tfrac{1}{2}(\sigma_3 \bar{Q})^{\dot{\alpha}} \\
\implies [J_3, \bar{Q}_{\dot{1}}] = +\tfrac{1}{2}\bar{Q}_{\dot{1}}
\end{gathered}
$$

And one can similarly compute that the change in λ is as follows

	Q_1	Q_2	$\bar{Q}_{\dot{1}}$	$\bar{Q}_{\dot{2}}$
$\Delta\lambda$	$-1/2$	$1/2$	$1/2$	$-1/2$

Note, though the above calculation is valid, acting $\bar{Q}_{\dot{2}}$ on either of the states in the massless irrep will give 0.

Finally, the reps

So our supoincare irrep is built out of poincare reps in the following way:

$$|E, \lambda\rangle, |E, \lambda + \tfrac{1}{2}\rangle$$

For example, for $\lambda = 0$ we call this the scalar/chiral/on-shell multiplet.

$$|E, 0\rangle, |E, \tfrac{1}{2}\rangle$$

But notice, this is only one of the helicities – what about the other one? What about $|E, -\frac{1}{2}\rangle$?

That's because the transformation is not CPT invariant (helicity is flipped by parity). There is another rep with $|E, -\frac{1}{2}\rangle$, which can be accessed in the following *reducible* supoincare rep:

$$\left(|E, 0\rangle, |E, \tfrac{1}{2}\rangle\right) \oplus \left(|E, -\tfrac{1}{2}\rangle, |E, 0\rangle'\right)$$

Which is a CPT-invariant multiplet. Note that $|E, 0\rangle'$ may not be the same as $|E, 0\rangle$ – and generally won't be. The way to obtain it is to apply $Q_1^\dagger$ to $|E, -\frac{1}{2}\rangle$. This happens because we're working with a complex scalar – have to, because of the chirality of the corresponding fermion – and so there is another degree of freedom.

The **vector multiplet** is:

$$|E, \tfrac{1}{2}\rangle, |E, 1\rangle$$

And we can similarly construct a CPT-invariant multiplet:

$$(\tfrac{1}{2}, 1) \oplus (-1, -\tfrac{1}{2})$$

Where obvious shorthand was used.

Gravitino multiplet:

$$(1, \tfrac{3}{2})$$

Spin-2 multiplet

$$(\tfrac{3}{2}, 2)$$

Massive Reps

Now we build states out of $|m; s, s_3\rangle$. There is a notion of a super-Pauli-Lubanski vector that measures super-spin along with mass – but we will not go into detail about that.

$P^\mu = (m, 0, 0, 0)$, then

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu = 2m\delta_{\alpha\dot{\beta}}$$

Then this is just a Clifford algebra.

$$a_\alpha = \frac{1}{\sqrt{2m}} Q_\alpha$$
$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}$$

With $\alpha, \beta \in \{1, 2\} \therefore N = 2$.

As usual, define a normalised vacuum annihilated by a_αs, some $|m; s, s_3\rangle$ – in fact, several of them, labelled by s_3 – degenerate Clifford vacua. Pick any one, and we build the 4 states:

$$|0\rangle, \quad |1\rangle = a_1^\dagger|0\rangle, \quad |2\rangle = a_2^\dagger|0\rangle, \quad |3\rangle = a_1^\dagger a_2^\dagger|0\rangle$$

And we'll be using the same table as for $\Delta\lambda$, for Δs_3:

	Q_1	Q_2	$\bar{Q}_{\dot{1}}$	$\bar{Q}_{\dot{2}}$
Δs_3	$-1/2$	$1/2$	$1/2$	$-1/2$

Lecture 9 - Susy

Course	Supersymmetry
Date	@January 30, 2025
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Next	Lecture 10 - Susy
Previous	Lecture 8 - Susy

Reps of Super-Poincare, continued

Massive Reps

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu = 2m\delta_{\alpha\dot\beta}$$
$$a_\alpha = \frac{1}{\sqrt{2m}}Q_\alpha$$
$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}$$

Clifford algebra, so we define a Clifford vacuum, $|m; s, s_3\rangle$, annihilated by the a_αs.

This is actually a degenerate set of vacua, labelled by $|s_3| \leq s$.

Starting from any one vacua, we build the 4 states:

$$|0\rangle, \quad |1\rangle = a_1^\dagger|0\rangle, \quad |2\rangle = a_2^\dagger|0\rangle, \quad |3\rangle = a_1^\dagger a_2^\dagger|0\rangle$$

And these operators change Δs_3:

	Q_1	Q_2	$\bar{Q}_{\dot 1}$	$\bar{Q}_{\dot 2}$
Δs_3	$-1/2$	$1/2$	$1/2$	$-1/2$

Then

$$a_1^\dagger|m; s, s_3\rangle = \sum_{s'} C_{s'}|m; s', s_3 + \tfrac{1}{2}\rangle$$

We obtain a combination of states of different spins – $[J^2, \bar{Q}_1] \neq 0$. The general formula for this combination will involve the Clebsch-Gordan coefficients. But it will be limited to $s' \in \{s, s \pm 1/2\}$.

Scalar Multiplet

$$\begin{gathered} |0\rangle = |m; 0, 0\rangle \\ \implies |1\rangle = |m; \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ |2\rangle = |m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle \\ \implies |3\rangle = |m, 0, 0\rangle' \end{gathered}$$

Why do higher spin states not occur as part of $|3\rangle$? It can be argued that since there is no way to access $|s_3| \geq 1/2$ here, $s \geq 1/2$ will not occur either – since s_3 is just a projection along any axis, so all possible s_3 for a given s should appear in a spectrum – Lorentz invariance tells us this, since Lorentz transforms can change s_3 within $-s, \ldots, s$.

Also note that $|3\rangle \neq |0\rangle$ – this can be checked in any number of ways, but the simplest is that $a_2|3\rangle = |2\rangle$ and $a_2|0\rangle = 0$.

Spin-1/2 multiplet

$$\begin{gathered} |0\rangle = \{|m; \tfrac{1}{2}, \tfrac{1}{2}\rangle, |m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle\} \\ a_1^\dagger|m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\left(|m; 1, 0\rangle + |m; 0, 0\rangle\right) \\ a_2^\dagger|m; \tfrac{1}{2}, \tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\left(|m; 1, 0\rangle - |m; 0, 0\rangle\right) \\ a_1^\dagger|m; \tfrac{1}{2}, \tfrac{1}{2}\rangle = |m; 1, 1\rangle \\ a_2^\dagger|m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = |m; 1, -1\rangle \end{gathered}$$

The minus sign in the third equation is a result of an orthonormality requirement – though it can also be derived by an orthonormal definition of the supercharges.

$$\begin{gathered} a_1^\dagger a_2^\dagger|m; \tfrac{1}{2}, \tfrac{1}{2}\rangle = |m; \tfrac{1}{2}, \tfrac{1}{2}\rangle' \\ a_1^\dagger a_2^\dagger|m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = |m; \tfrac{1}{2}, -\tfrac{1}{2}\rangle' \end{gathered}$$

Just like before.

Clearly, the way this proceeds ensures that the total fermionic degrees of freedom equal the bosonic ones. Another check is that if we start with a spin-s vacuum we have $2s+1$ possible s_3, the single creation operator will give $s \pm$

$1/2$ spin states, which give a total of $2(s+1/2)+1+2(s-1/2)+1 = 4s+2$ states, and two creation operators give $2s+1$ states with the same quantum numbers as the vacua. QED.

Extended Supersymmetry Representations

(No central charges here)

$$\{Q^r_\alpha, \bar{Q}_{\dot{\beta}s}\} = 2\delta^r_s \sigma^m_{\alpha\dot{\beta}} P_m$$
$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$$

$2N$ supercharges.

Massless reps

$P^m = (E, 0, 0, E)$, then states $|E, \lambda\rangle$,

$$a^r_\alpha = \frac{1}{\sqrt{4E}} Q^r_\alpha$$
$$\{a^r_1, a^\dagger_{1s}\} = \delta^r_s, \text{The rest are } 0$$

A Clifford algebra with N generators and hence 2^N states, just like we discussed in the previous lecture. We discard the 0-normed states, so while we need ensure the vacuum is annihilated by $a_{\alpha r} \forall \alpha, r$, we only need to use $a^\dagger_{1r}$ for constructing the states.

$$a^\dagger_{1r}|E, \lambda\rangle = |E, \lambda + \tfrac{1}{2}\rangle_r$$
$$a^\dagger_{1r} a^\dagger_{1s}|E, \lambda\rangle = |E, \lambda + \tfrac{1}{2} * 2\rangle_{[rs]}$$
$$a^\dagger_{1s_1} \ldots a^\dagger_{1s_N}|E, \lambda\rangle = |E, \lambda + \tfrac{1}{2} * N\rangle_{[s_1 \ldots s_N]}$$

Because they anticommute, the states will be antisymm in the r, s indices.

Note that $\lambda_{\text{max}} = \lambda_{\text{min}} + N/2$, which means if we need to restrict ourselves to spin-1 and below particles, we can have at most $N = 4$ supersymmetry. Restricting to spin-2 gives $N = 8$.

Examples

$N = 2, \lambda_{\text{min}} = -1/2$

$$|E, -1/2\rangle \to \begin{cases} |E, 0\rangle \\ |E, 0\rangle' \end{cases} \to |E, 1/2\rangle$$

Notice how this is just like a combination of multiplets of $N = 1$, $(-\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ – a hypermultiplet. Thus the $N = 2, \lambda_{\text{min}} = -1/2$ irrep is parity invariant, and contains 1 massless Weyl fermion and two real scalars (or one complex scalar).

$N = 2, \lambda_{\text{min}} = 0$

$$|E, 0\rangle \to \begin{cases} |E, \frac{1}{2}\rangle \\ |E, \frac{1}{2}\rangle' \end{cases} \to |E, 1\rangle$$

Not CPT invariant, since we don't have the -1 state. But we can ofc pair this with the $\lambda_{\text{min}} = -1$ multiplet to get a total CPT-invariant multiplet.

Lecture 10 - Susy

Course	Supersymmetry
Date	@February 3, 2025
Status	Completed
Next	Lecture 11 - Susy
Previous	Lecture 9 - Susy

Extended Supersymmetry Representations, continued

$N = 4, \lambda_{\text{min}} = -1$ – Vector Multiplet

The CPT-invariant susy extension of Maxwell theory.

λ	-1	-1/2	0	1/2	1
#Fields	1	4	6	4	1

Thus we get a vector boson (2 on-shell DoF), 4 Weyl spinors (2 on-shell DoF each), and 6 scalars (1 DoF each).

When extended to an interacting Yang-Mills gauge theory, this is called the N=4 SYM.

This is the unique N=4 multiplet with $|\lambda| \leq 1$.

The R-symmetry group is $\text{Spin}(6) \simeq \text{SU}(4)$, though a-priori one may guess $\text{U}(4)$.

R-Symmetry

For N supercharges, one can multiply $Q_{\alpha r}$ by $U_s{}^r$ and $\bar{Q}_{\dot{\beta} s}$ by the hermitian conjugate, but this should leave the supoincare algebra unchanged, so U must be unitary. Thus naively the R-symmetry group is always $\text{U}(N)$.

> the R-symmetry is the largest subgroup of the automorphisms group of the supersymmetry algebra which commutes with the Lorentz group. In simple words, there exists a group which commutes with the Lorentz group AND leaves the susy algebra (the anticommutators) invariant. The largest such group is referred to as R-symmetry. (see http://arxiv.org/abs/hep-th/0212245 for instance)
>
> The important, but often neglected point, is that the R-symmetry needs NOT be a symmetry of the theory at all. Sometimes it is, sometimes it is NOT.

In theories like the vector multiplet, the $\text{U}(1)$ generator does not appear in the commutators, and the Lie algebra closes without it. One may impose this additional symmetry if they like.

Supergravity multiplets $(|\lambda| \leq 2)$

- N=1 → $(-2, -3/2) \oplus (3/2, 2)$ – This is called the N=1 sugra multiplet.
- N=2 → $(-2, -3/2, -1) \oplus (1, 3/2, 2)$
  - This includes the N=1 sugra multiplet along with other N=1 multiplets.
  - There are two DoF each at each $|\lambda| = 3/2$.
- Highest N allowed is $N = 8, \lambda_{\text{min}} = -2$ (any higher would give fields with spin>2).

λ	-2	-3/2	-1	-1/2	0	1/2	1
#Fields	1	8	28	56	70	56	28

Susy in Higher Dimensions

N=1 Maxwell multiplet in 10D, on dimensional reduction to 4D, gives us the N=4 Maxwell theory! Crucial to this is that the spinor partner to the vector is a Majorana-Weyl spinor, because that will have $2^{10/2-1}$ components, which in 4D become 4 Weyl spinors.

11D

If we go to very high dimensions, dimensional reduction gives higher spin fields. The claim is D=11 is the highest you can go for sugra without higher spin fields. The spinor will have $2^{[D/2]}$ components – so in 11D, that's 32 components, so 32×32 gamma matrices and 32 supercharges (maximal), since the supercharges form a spinor. $\{Q, Q\} = 2\gamma^\mu P_\mu$, so Q must have 32 components.

There is a unique rep with $|\lambda| \leq 2$ in 11D, which is $N = 1$, since on compactifying to 4D that gives $N = 8 \implies |\lambda| \leq 2$.

The fields are $(h_{MN}, \psi_M, A_{MNK})$ where the third field is a bosonic leftover which is totally antisymmetric.

- Graviton has $D(D-3)/2$ DoF, symmetric matrix minus gauge freedoms. Thus 44 off-shell components.
    - But ofc, there are only 2 on-shell components.
- Gravitino has $2^{[D/2]}(D-1)$ DoF, generally. In this case, $32 * 10$.
    - $2^{[D/2]}D$ is what the dimensions of the object look like, but we remove $2^{[D/2]}$ for gauge freedom.

Dimensional reduction of this to 4D, over some simple compact manifold like $M^{11} = M^4 \times T^7$, gives the $N = 8$ SYM in 4D.

The Lagrangian takes the form:

$$\mathcal{L} = R + \bar{\Psi}\gamma D\Psi + (\partial_{[m}A_{nkl]})^2 + \text{interactions}$$

The third term is a curl.

Massive Reps of Extended Susy

As usual we could consider a vacuum of the form $|m, s, s_3\rangle$, and creation operators $\bar{Q}_{\alpha r}$, $\alpha \in \{1, 2\}, r \in [[1, N]]$. We have our clifford algebra and creation operators will change s and s_3. But the coefficients will be even more complicated than Clebsch-Gordan.

Field Representations of Susy (on Minkowski)

Consider a (massless) scalar field with usual kinetic term for the Lagrangian, with familiar transformation under Poincare.

$$\begin{gathered} P_m \sim \partial_m \\ M_{mn} \sim x_m\partial_n - x_n\partial_m \\ Q \sim ? \end{gathered}$$

Using the susy algebra, we want to fix $\delta_Q\phi$.

Lecture 11 - Susy

Course	Supersymmetry
Date	@February 4, 2025
Status	Completed
Next	Lecture 12 - Susy
Previous	Lecture 10 - Susy

Field Transformations under Susy

Consider the $(-1/2, 0) \oplus (0, 1/2)$ representation, i.e. a complex scalar field and a Weyl spinor, obeying the Lagrangian

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* + \imath \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi$$

$\bar{\psi}$ is the usual charge conjugate, $\bar{\psi} = \psi^\dagger$ for Weyl spinors.

The left-chiral spinor obeys the corresponding Weyl equation:

$$\bar{\sigma}^m \partial_m \psi = 0$$

Which, combined with the right-chiral Weyl equation for the conjugate:

$$\sigma^m \partial_m \bar{\psi} = 0$$

Gives the massless Dirac equation:

$$\gamma^m \partial_m \Psi = 0$$

Where $\Psi = (\psi, \psi^*)^T$, and we're working in the Chiral representation of the Gamma matrices, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$.

Fields under Poincare

$$P_m = -\imath\partial_m \implies \delta\phi = -\imath a^m P_m\phi = -a^m\partial_m\phi$$
$$\implies \phi'(x) = e^{-a^m\partial_m}\phi(x) = \phi(x-a)$$

And similarly for $M_{mn} \sim x_{[m}\partial_{n]}$.

Extending to Superpoincare

Unlike with other transformations parametrised by bosonic numbers, the transformation will have to be parametrised by Grassmannian parameters, $\xi_\alpha, \bar{\xi}^{\dot{\alpha}}$, because classically ϕ are normal and ψ Grassmannian, and Q converts ϕ to ψ but $\delta\phi$ (which involves ξ) must remain bosonic – so another grassmannian parameter is necessary.

$$\{\xi_\alpha, \xi_\beta\} = \{\xi_\alpha, \bar{\xi}_{\dot{\beta}}\} = 0$$

They are further arranged into spinors simply because the supercharges are spinors.

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}}P_m$$
$$\delta_\xi \equiv \xi\cdot Q \equiv \xi Q + \bar{\xi}\bar{Q}$$
$$\xi\psi \equiv \xi^\alpha\psi_\alpha = -\xi_\alpha\psi^\alpha$$

δ must have mass dimension 0, and usually ϕ, ψ have mass dimensions 1, 3/2, so Q has 1/2 and ξ has -1/2. Since P_m has mass dimensions 1, $\bar{Q}, \bar{\xi}$ must have 1/2, -1/2 too. More generally, supersymmetry can be introduced as the operator Q being the “square root” of the hamiltonian, $\{Q, Q\} \sim H$, which demands mass dimension 1/2.

Bosons to Fermions

By convention, Q transforms in the left-handed representation of $\mathrm{SL}(2,\mathbb{C})$. Thus Q converts ϕ to the left-handed spinor ψ, and $\bar{Q}$ to the right-handed spinor $\bar{\psi}$. Then generally,

$$\delta_\xi\phi = \xi\cdot Q\phi = a\xi\psi + a'\bar{\xi}\bar{\psi}$$

1. Due to the dimensions, $Q\phi$ must be $\propto \psi$ and not any derivative of ψ.

   Also, ψ also satisfies KG, and applying this transformation to the KG equation for ϕ, we want to get a valid equation for the theory.

2. Since we are defining the representation here, we can choose that the second term does not occur. Since we can always define $\chi = a\phi + b\phi^*$ to obtain $\delta_\xi \chi \sim \psi + \bar{\psi}$, this is simply convention.

Complex conjugating,

$$\delta_\xi \phi^* = a^* \bar{\xi} \bar{\psi}$$

Fermions to Bosons

Both on dimensional grounds, and from realising that ϕ does not satisfy a first-order DE, the transformation must involve a derivative.

$$\delta_\xi \psi \sim \xi \partial \phi$$

Since ∂_m is a covector, we need our sigma matrix to convert to spinor indices.

$$\delta_\xi \psi_\alpha = c \sigma^m_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} \partial_m \phi$$

And note that $\bar{\xi}$ is necessary because we don't have an object which can convert from a vector and a spinor index (ξ^α) to another spinor index (ψ_α) – it needs to be in the dual rep and have a $\dot{\alpha}$-kind index.

Algebra Consistency Constraints

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \implies [\delta_\xi, \delta_\eta] = h^\mu(\xi, \eta) P_\mu$$

And now we will calculate h^μ.

$$\begin{gathered}
[\xi Q, \eta Q] = \xi^\alpha Q_\alpha \eta^\beta Q_\beta - \eta^\beta Q_\beta \xi^\alpha Q_\alpha \\
= \xi^\alpha \eta^\beta \{Q_\alpha, Q_\beta\} \\
\therefore [\delta_\xi, \delta_\eta] = [\xi Q, \bar{\eta}\bar{Q}] + [\bar{\xi}\bar{Q}, \eta Q] \\
= 2(\xi \sigma^m \bar{\eta} - \eta \sigma^m \bar{\xi}) P_m \\
= 2(\xi \sigma^m \bar{\eta} + \bar{\xi} \bar{\sigma}^m \eta) P_m
\end{gathered}$$

Lecture 12 - Susy

Course	Supersymmetry
Date	@February 6, 2025
Status	Completed
Next	Lecture 13 - Susy
Previous	Lecture 11 - Susy

Field Transformations under Susy, continued

$$[\delta_\xi, \delta_\eta]\phi = 2(\xi\sigma^m\bar{\eta} - \eta\sigma^m\bar{\xi})P_m\phi = -2\imath(\xi\sigma^m\bar{\eta} - \eta\sigma^m\bar{\xi})\partial_m\phi$$

But also,

$$\begin{gathered}\delta_\xi\phi = a\xi\psi \\ \delta_\xi\psi = c\sigma^m\bar{\xi}\partial_m\phi \\ \therefore [\delta_\xi, \delta_\eta]\phi = (ac)(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m\phi \\ \therefore ac = 2\imath\end{gathered}$$

What about $[\delta_\xi, \delta_\eta]\psi$?

$$\begin{gathered}\delta_\xi\delta_\eta\psi_\alpha = \delta_\xi(c(\sigma^m\bar{\eta})_\alpha\partial_m\phi) \\ = c(\sigma^m\bar{\eta})_\alpha\partial_m(a\xi\psi) = ac(\sigma^m\bar{\eta})_\alpha\xi^\beta\partial_m\psi_\beta\end{gathered}$$

Using the Fierz identity,

$$\begin{gathered}\sigma_{m\alpha\dot{\alpha}}\bar{\sigma}^{m\dot{\beta}\beta} = -2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}} \\ \therefore (\phi\psi)\bar{\chi}_{\dot{\alpha}} = -\tfrac{1}{2}(\phi\sigma^m\bar{\chi})(\psi\sigma_m)_{\dot{\alpha}}\end{gathered}$$

Thus

$$
\begin{aligned}
\bar{\eta}^{\dot{\beta}}(\xi\psi) &= -\frac{1}{2}(\bar{\sigma}_m\psi)^{\dot{\beta}}(\xi\sigma^m\bar{\eta}) \\
\therefore \delta_\xi\delta_\eta\psi_\alpha &= -\frac{ac}{2}\partial^m(\sigma_m\bar{\sigma}_n\psi)_\alpha(\xi\sigma^n\bar{\eta}) \\
&= -\frac{2\iota}{2}\partial^m((2\eta_{mn}\mathbb{I} - \sigma_n\bar{\sigma}_m)\psi)_\alpha(\xi\sigma^n\bar{\eta}) \\
&= -\iota(\xi\sigma^n\bar{\eta})(2\partial_n\psi - \sigma_n\bar{\sigma}_m\partial^m\psi)_\alpha
\end{aligned}
$$

But $\bar{\sigma}^m\partial_m\psi = 0$! That's the Weyl equation!

Thus

$$
[\delta_\xi, \delta_\eta]\psi = -2\iota(\xi\sigma^m\bar{\eta} - \eta\sigma^m\bar{\xi})\partial_m\psi = h^m(\xi,\eta)P_m\psi
$$

Thus the algebra is consistent on-shell, as expected.

Invariance of the Action

$$
S = \int \mathrm{d}^4x \left(-\partial_m\phi^*\partial^m\phi - \iota\psi\sigma^m\partial_m\bar{\psi}\right)
$$

Note the odd sign – this is because our metric is $-+^3$, unlike usual QFT.

$$
\begin{gathered}
\delta_\xi S = \int \mathrm{d}^4x(A_1\xi + A_2\bar{\xi}) \\
A_2 = -\partial^m\phi\partial_m(a^*\bar{\xi}\bar{\psi}) - \iota\bar{\psi}\bar{\sigma}^m\partial_m(c\sigma^n\bar{\xi}\partial_n\phi), A_1 = A_2^\dagger
\end{gathered}
$$

Both $\int A_i$ should be 0 independently, so

$$
\begin{gathered}
0 = -\int \mathrm{d}^4x\left(a^*\partial^m\phi\bar{\xi}\partial_m\bar{\psi} + \iota c\bar{\psi}\bar{\sigma}^m\sigma^n\bar{\xi}\partial_m\partial_n\phi\right) \\
= \cancel{\text{surface terms}} + \int \mathrm{d}^4x\left(a^*\bar{\xi}\bar{\psi}\Box\phi + \iota c\bar{\xi}\bar{\psi}\Box\phi\right) \quad (\bar{\sigma}^{(m}\sigma^{n)} = -\eta^{mn}) \\
\therefore a + \iota c = 0
\end{gathered}
$$

Along with $ac = 2\iota$, this forces:

$$
a = \sqrt{2},\ c = \iota\sqrt{2}
$$

Closing Susy algebra off-shell

We saw that the EoM was necessary to close the Susy algebra, and also that the bosonic and fermionic degrees of freedom match up on-shell, but off-shell neither of these happen.

Thus we need to add an auxiliary non-dynamical bosonic field F,

$$S = \int \mathrm{d}^4x \left(-\partial_m\phi^*\partial^m\phi - \iota\psi\sigma^m\partial_m\bar{\psi} + F^*F\right)$$

Its EoM is simply $F = 0$, and it has mass dimension 2.

We modify the susy transformations to include F,

$$\begin{gathered}\delta_\xi\phi = \sqrt{2}\xi\psi \\ \delta_\xi\psi_\alpha = \iota\sqrt{2}\sigma^m_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}}\partial_m\phi + \sqrt{2}\xi_\alpha F \\ \delta_\xi F = \iota\sqrt{2}\bar{\xi}\bar{\sigma}^n\partial_n\psi\end{gathered}$$

So that the algebra is consistent off-shell *and* on-shell – notice that $\delta_\xi F$ is proportional to the Weyl equation.

We can also check that the action is still invariant – the two new terms differ by a surface term.

Why do we care about the off-shell behaviour? Turns out it's really useful for constructing interactions if the algebra closes off-shell too.

Lecture 13 - Susy

Course	Supersymmetry
Date	@February 10, 2025
Status	Completed
Next	Lecture 14 - Susy
Previous	Lecture 12 - Susy

Adding mass terms

$$S_M = M \int \mathrm{d}^4x \left(\phi F + \phi^* F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\bar{\psi}\bar{\psi} \right)$$

And one can show that this term is also invariant under the supersymmetry transformations (off-shell too!) – though that requires the same mass for all fields.

We can clearly see that the fermionic terms contribute the correct term to the equations of motion,

$$-\iota\bar{\sigma}^m\partial_m\psi - M\bar{\psi} = 0$$

The EoM for F, ϕ are modified. Instead of $F = 0$, we get

$$F^* + M\phi = 0$$

So once we substitute the new values of the auxiliary field in the new EoM for ϕ, we again obtain the massive KG:

$$\partial^m\partial_m\phi - M^2\phi = 0$$

If we substitute values of F in the action, we will ofc get back the action with the usual mass terms, still invariant under susy – but the susy transformations laws will change to include mass terms, in accordance with the off-shell transformations.

Superfields and superspace

Susy transformations are like translations in a new direction, by grassmannian parameters. We encode this using superspace, parametrised by the usual Minkowski coordinates as well as grassmannian coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ (4 bosonic, 4 fermionic).

Susy translates θ:

$$\theta' = \theta + \xi$$
$$\bar{\theta}' = \bar{\theta} + \bar{\xi}$$

While a translation does not affect the θs, a susy transformation must change x^μ since $\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$.

$$x'^\mu = x^\mu + a^\mu(\xi, \bar{\xi}, \theta, \bar{\theta})$$

We'll use the result

$$[\delta_\xi, \delta_\eta] = -2\iota(\xi\sigma^m\bar{\eta} - \eta\sigma^m\bar{\xi})\partial_m$$

On θ,

$$[\delta_\xi, \delta_\eta]\theta = 0$$

Which is to be expected both since translations should commute, and since $\partial_m\theta = 0$.

But on x,

$$[\delta_\xi, \delta_\eta]x^m = -2\iota(\xi\sigma^m\bar{\eta} - \eta\sigma^m\bar{\xi})$$

Thus, a translation.

We postulate that

$$\delta_\xi x^\mu = \iota(\theta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\theta})$$

And we will show this next lecture.

General Superfield

Consider the scalar superfield $\Phi(x, \theta, \bar{\theta})$. Since any function of grassmann variables must be at most linear in each grassmann variable, we can expand this a la Taylor.

Furthermore, the $\theta^\alpha \theta^\beta$ terms will become $\sim \theta\theta\epsilon^{\alpha\beta}$, so the coefficient of those terms will effectively be a scalar. The $\theta^\alpha \bar{\theta}^{\dot{\alpha}}$ terms, on the other hand, can be simplified using $\sigma^\mu_{\alpha\dot{\alpha}}$ and the coefficients will transform as vectors. And so on, the effective general superfield is only composed of scalars, left and right chiral spinors, and a vector:

$$\begin{gathered}\Phi(x, \theta, \bar{\theta}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + (\theta\theta)M(x) + (\bar{\theta}\bar{\theta})N(x) \\ + (\theta\sigma^m\bar{\theta})V_m(x) + (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)\end{gathered}$$

The whole superfield is bosonic, so each term is bosonic, and coefficients of odd powers of Grassmann variables will be fermionic, like ψ and $\bar{\chi}$.

In general, the components of Φ will form a reducible representation of the superpoincare group. We can impose constraints to obtain sub-reps and irreps, such as a reality constraint, or some differential constraints.

We can also consider superfields which transform themselves as well – fermionic or vector superfields. Those will have different expansions in components.

For extended supersymmetry, we simply use more Grassmannian coordinates.

$$\begin{aligned}\delta_\xi \Phi &= \delta x^m \partial_m \Phi + \delta\theta^\alpha \partial_\alpha \Phi + \delta\bar{\theta}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \Phi \\ &= (\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\Phi\end{aligned}$$

Lecture 14 - Susy

Course	Supersymmetry
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Status	Completed
Next	Lecture 15 - Susy
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Superfields

Introduce (scalar) superfields $\Phi(x, \theta, \bar{\theta})$, which must transform as

$$\begin{aligned}\delta_\xi \Phi &= (\xi \hat{Q} + \bar{\xi}\hat{\bar{Q}})\Phi \\ &= \delta x^\mu \partial_\mu \Phi + \delta\theta^\alpha \partial_\alpha \Phi + \delta\bar{\theta}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} \Phi\end{aligned}$$

Note that these are scalar superfields since, in terms of supercoordinates $z = x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}$, under superpoincare,

$$\Phi'(z) = \Phi(z)$$

Where

$$\begin{aligned}\delta x^\mu &= \imath(\theta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\theta}) \\ \delta\theta^\alpha &= \xi^\alpha, \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\xi}^{\dot{\alpha}} \\ \partial_\alpha &\equiv \frac{\partial}{\partial\theta^\alpha}, \quad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\end{aligned}$$

These derivatives are not really limits of some difference. Since smooth functions of Grassmannian variables can be at most linear, these are just some algebraic operations fully determined by the rules:

$$\begin{aligned}\partial_\alpha \theta^\beta &= \delta_\alpha^\beta \\ \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} &= \delta_{\dot{\alpha}}^{\dot{\beta}}\end{aligned}$$

Grassmannian Calculus

The derivatives obey the following (anti)comm relations, since they are grassmannian:

$$\{\theta, \bar{\partial}\} = 0, \quad \{\bar{\theta}, \partial\} = 0$$
$$\{\partial_\alpha, \bar{\partial}_{\dot{\alpha}}\} = 0, \quad \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = 0$$
$$\{\theta^\alpha, \partial_\beta\} = \delta^\alpha_\beta, \quad \{\bar{\theta}^{\dot{\alpha}}, \bar{\partial}_{\dot{\beta}}\} = \delta^{\dot{\alpha}}_{\dot{\beta}}$$
$$[x^\mu, \partial_\nu] = \delta^m_n$$

We can raise and lower indices with the epsilon tensor:

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$$
$$\partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta$$

Also note the contraction:

$$\theta\theta = \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\alpha\theta_\beta$$

We can then verify, working with $\epsilon^{12} = 1$,

$$\theta^1 = \theta_2, \quad \theta^2 = -\theta_1$$

Which gives the relation for the partial derivatives.

Verify the anticomm by acting it on a test function,

$$\{\theta^\alpha, \partial_\beta\} f(\theta) = \theta^\alpha \partial_\beta f + \partial_\beta(\theta^\alpha f)$$
$$= \cancel{\theta^\alpha \partial_\beta f} + \partial_\beta(\theta^\alpha) f - \cancel{\theta^\alpha \partial_\beta f} = \delta^\alpha_\beta f$$

Note the minus sign in the product rule.

▼ **Exercise:** Verify $\partial_\alpha(\theta\theta) = 2\theta_\alpha$

$$\partial_\alpha(\theta\theta) = \epsilon_{\beta\gamma}(\partial_\alpha \theta^\beta \theta^\gamma - \theta^\beta \partial_\alpha \theta^\gamma)$$
$$= \epsilon_{\alpha\gamma}\theta^\gamma - \epsilon_{\beta\alpha}\theta^\beta = 2\theta_\alpha$$

Differential Operator Representation of Supercharges on Superspace

$$\hat{Q}_\alpha = \partial_\alpha - \iota\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$$
$$\hat{\bar{Q}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + \iota\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m$$

$$\delta_\xi x^\mu = (\xi \cdot \hat{Q} + \bar{\xi} \cdot \hat{\bar{Q}}) x^\mu = \iota(\theta \sigma^m \bar{\xi} - \xi \sigma^m \bar{\theta})$$

Then the anticomm

$$\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot{\alpha}}\} = 2\iota \sigma^m_{\alpha\dot{\alpha}} \partial_m = -2\sigma^m P_m$$

- My two cents on the sign difference with the original supoincare algebra

  Strictly speaking, these differential operators, while accomplishing translations in superspace, are not the representation of supoincare on superfields, but on the parameter space, where $-P_m$ is the representation of the momentum operator.

  However, in the same way that active and passive transformations differ, δ_ξ does not implement $\Phi(x, \theta) \to \Phi(x + h(\theta, \xi), \theta + \xi)$ but rather $\Phi(x, \theta) \to \Phi(x - h(\theta, \xi), \theta + \xi)$ – which means it is translating x by $h(\theta, \xi)$.

  If we're considering invariance under susy transformations, it works equally well. But strictly speaking, I think the covariant derivatives introduced next are the actual susy operators on the superfields.

  Wess-Bagger has a line on this but I don't quite understand what it means in terms of the abstract algebra and its representations.

  > Note, however, the change in sign, $P_m = -i\partial_m$. This stems from the fact that the product of successive group elements corresponds to a motion with the order of multiplication reversed. For example, $G(0,\xi_1,\bar{\xi}_1)G(0,\xi_2,\bar{\xi}_2)$ induces the motion $g(\xi_2,\bar{\xi}_2)g(\xi_1,\bar{\xi}_1)$.
  > We could have studied right multiplication instead of left multiplication

Introduce the covariant derivatives:

$$D_\alpha = \partial_\alpha + \iota \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m$$
$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - \iota \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m$$

Which have the commutation relations

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2\iota \sigma^m_{\alpha\dot{\alpha}} \partial_m = 2\sigma^m P_m$$
$$\{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{D_\alpha, Q_\beta\} = 0$$

And all other commutators of the covariant derivative with the susy generators are 0. The result? We can use these covariant derivatives to construct objects that are covariant under supersymmetry – eg acting on superscalars, we still get superfields.

Similarity to Einbeins

Recall how Einbeins work. We have a metric:

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n$$

Which we express as

$$g_{mn} = e_m^a(x)e_n^b(x)\eta_{ab}$$

So we treat the theory as flat space with the einbein field on it, of sorts?

Anyways, new covariant derivatives are introduced:

$$\hat{\partial}_a = e_a^m(x)\partial_m$$

These, however, don't commute with each other, but form some more complex algebra

$$[\hat{\partial}_a, \hat{\partial}_b] = C_{ab}^c\hat{\partial}_c$$

So why the covariant derivative?

$$\begin{gathered} D_\alpha = E_\alpha^M\partial_M \\ M = (m, \alpha, \dot{\alpha}) \end{gathered}$$

It's the same as the einbein, in some sense.

It is useful, because while $\partial_\mu\Phi$ is still a superfield, $\partial_\alpha\Phi$ is not – δ_ξ does not commute with ∂_α. But $[D_\alpha, \delta_\xi] = 0$! That's how it'll be useful.

Superfields from superfields

Addition and multiplication still gives superfields, and specifically superscalars.

$$\Phi_1 + \Phi_2, \quad \Phi_1 \cdot \Phi_2$$

Transformation of a General Scalar Superfield

$$\Phi = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x)$$
$$+ \dots + (\theta\theta)(\bar{\theta}\bar{\theta})\mathcal{D}(x)$$

Because $\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta$, we can always write $\mathcal{D}_{\alpha\beta\dot{\gamma}\dot{\delta}}\theta^\alpha\theta^\beta\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}}$ as $\theta^2\bar{\theta}^2\mathcal{D}$. As for cases like $\lambda_{\alpha\dot{\beta}}\theta^\alpha\bar{\theta}^{\dot{\beta}}$, they become $\theta\sigma^m\bar{\theta}V_m$, giving a vector index.

We *define* the variation of the various spacetime fields as

$$\delta\Phi = \delta\varphi(x) + \theta\delta\psi(x) + \bar{\theta}\delta\bar{\chi}(x)$$
$$+ \dots + (\theta\theta)(\bar{\theta}\bar{\theta})\delta\mathcal{D}(x)$$

Then we can compute the transformations, and we get something like

$$\delta\varphi = \xi^\alpha\psi_\alpha - \bar{\xi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}$$
$$\delta\psi_\alpha = \iota\bar{\xi}^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}\partial_m\varphi + \sigma^m_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}V_m + \dots$$

The first terms for $\delta\varphi, \delta\psi_\alpha$ will look familiar – those are the transformations laws for the chiral multiplet, up to some factors.

The general superfield and its transformation represent the most general multiplet for $\mathcal{N} = 1$ susy – its a reducible representation. We will have to impose some conditions to obtain irreps, specifically the chiral multiplet.

What conditions? We can impose differential constraints. But not on x alone, we wouldn't recover usual fields from that. We'll have to use θ derivatives. But those don't give susy-invariant equations!

And that's where D_α comes in, which we'll discuss next time.

Also note that the last component, $\mathcal{D}$, only changes by a total derivative, which means its integral over spacetime will be a susy invariant. This is the motivation for the first supersymmetric actions we'll construct in terms of superfields.

Lecture 15 - Susy

Course	Supersymmetry
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Chiral Superfields

The general scalar superfield was not an irrep of susy, so we want to impose some constraint which will isolate an irrep – specifically, we would like the CPT-invariant chiral irrep we have discussed in detail so far.

$$\Phi = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) \\ + \cdots + (\theta\theta)(\bar{\theta}\bar{\theta})\mathcal{D}(x)$$

The constraint must remove the vector $\theta\sigma^m\bar{\theta}V_m$, the spinor $\bar{\chi}$, etc – in short, remove all the $\bar{\theta}$ terms. $\bar{\partial}_{\dot{\alpha}}\Phi = 0$ accomplishes this!

But this is not a susy-invariant constraint, $\{\bar{\partial}_{\dot{\alpha}}, Q\} \neq 0$. However, a related constraint is:

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

One can show that this does not restrict x-dependence. Such a Φ is called a *chiral superfield*.

We could also impose both $\bar{D}\Phi = 0, D\Phi = 0$, or even $D^2\Phi = 0$, where $D^2 = D^\alpha D_\alpha$. But these would give a a linear differential equation on superspace, of the type $a^\mu\partial_\mu f = 0$, which would restrict the solution to living on a subspace of Minkowski — not something we want.

New variable

Back to $\bar{D}\Phi = 0$, introduce y:

$$y^\mu = x^\mu + \imath\theta\sigma^\mu\bar{\theta} \\ \bar{D}_{\dot{\alpha}}y^\mu = -\imath\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} + \imath\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} = 0$$

Thus a chiral superfield is a function of only y, θ:

$$\bar{D}\Phi = 0 \Leftrightarrow \Phi(x,\theta,\bar{\theta}) = \Phi(x + \iota\theta\sigma\bar{\theta}, \theta)$$

In terms of $(y, \theta, \bar{\theta})$,

$$\begin{aligned} Q_\alpha = \partial_\alpha, \quad \bar{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2\iota\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_{y^\mu} \\ \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}, \quad D_\alpha = \partial_\alpha + 2\iota\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{y^\mu} \end{aligned}$$

General form of Chiral superfield

Expanding in θ,

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$

Then we expand y back to $x, \theta, \bar{\theta}$. Fierz:

$$(\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta}) = -\tfrac{1}{2}\eta^{mn}\theta^2\bar{\theta}^2$$

Thus, discarding θ^3 or $\bar{\theta}^3$ terms,

$$\Phi = \varphi(x) + \iota\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box\varphi(x) + \sqrt{2}\theta\psi(x) - \tfrac{\iota}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2 F(x)$$

Anti-chiral superfield

$$\begin{aligned} D\Phi = 0, \quad \Phi = \Phi(\bar{y}, \bar{\theta}) \\ \bar{y}^\mu := (y^\mu)^\dagger = x^\mu - \iota\theta\sigma^\mu\bar{\theta} \end{aligned}$$

Chiral Superfield under Susy

$$(\xi \cdot \hat{Q} + \bar{\xi} \cdot \hat{\bar{Q}})\Phi = \delta\Phi = \delta\varphi(y) + \sqrt{2}\theta\delta\psi(y) + \theta^2\delta F(y)$$

Which reproduces the familiar transformations.

Lecture 16 - Susy

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Recap

We had the general superfields

$$\Phi(x,\theta,\bar{\theta})$$

To get chiral superfields we impose the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

We introduce a shifted spacetime variable

$$\begin{gathered} y^m = x^m + \iota\theta\sigma^m\bar{\theta} \\ \text{s.t. } \bar{D}_{\dot{\alpha}}y^m = 0 \end{gathered}$$

And we already had $\bar{D}_{\dot{\alpha}}\theta^{\beta} = 0$.

Thus, Φ can only depend on y, θ and not $\bar{\theta}$. Thus the chiral superfield takes the general form:

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$$

Action of Supercharges

We should recover the familiar transformations

$$\begin{gathered} \delta_\xi\varphi = \sqrt{2}\xi\psi \\ \delta_\xi\psi = \iota\sqrt{2}\sigma^m\bar{\xi}\partial_m\phi + \sqrt{2}\xi F \\ \delta_\xi F = \iota\sqrt{2}\partial_m(\bar{\xi}\bar{\sigma}^m\psi) \end{gathered}$$

$Q_\alpha, \bar{Q}_{\dot{\alpha}}$ are still defined in terms of derivatives of x,

$$\hat{Q}_\alpha = \partial_\alpha - \iota\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{x^m}$$
$$\hat{\bar{Q}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + \iota\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_{x^m}$$

If we rewrite in terms of derivatives of y,

$$\hat{Q}_\alpha = \partial_\alpha$$
$$\hat{\bar{Q}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2\iota\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_{y^m}$$

And similarly,

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}$$
$$D_\alpha = \partial_\alpha + 2\iota\sigma^m\bar{\theta}\partial_{y^m}$$

Now if we evaluate

$$\delta_\xi\Phi = (\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\left(\varphi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\right)$$

We can check that the familiar transformations are in fact recovered.

More chiral Superfields?

Since the constraint is now just a derivative

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

Any sum or product of chiral superfields satisfies this constraint and hence is also a chiral superfield.

More generally, the sums and products of functions of y, θ will still be functions of y, θ – no $x/\bar{\theta}$ dependence has been introduced. So any function of chiral superfields (with no other fields involved) can be a chiral superfield.

Anti-chiral Superfields

$$D_\alpha\Phi = 0$$
$$\implies \Phi \equiv \Phi(z, \bar{\theta})$$
$$z^m = x^m + \iota\bar{\theta}\bar{\sigma}^m\theta$$

$z = y^\dagger$, hence (and one can check this by expanding and conjugating), when Φ is a chiral superfield, $\Phi^\dagger$ is anti-chiral.

Product of Superfields

Chiral-Chiral

$$\Phi_i(y,\theta) = \varphi_i(y) + \sqrt{2}\theta\psi_i(y) + \theta\theta F_i(y)$$

Because the only grassmannian parameter is θ, no terms order 3 and above in θ can show up in the products.

$$\begin{gathered} \Phi_i\Phi_j = \varphi_{ij} + \sqrt{2}\theta\psi_{ij} + \theta\theta F_{ij} \\ \text{Where } \varphi_{ij} = \varphi_i\varphi_j \\ \psi_{ij} = 2\psi_{(i}\varphi_{j)} \\ F_{ij} = 2\varphi_{(i}F_{j)} - \psi_i\psi_j \end{gathered}$$

$$\begin{gathered} \Phi_i\Phi_j\Phi_k : \\ \varphi_{ijk} = \varphi_i\varphi_j\varphi_k \\ \psi_{ijk} = 3\psi_{(i}\varphi_j\varphi_{k)} = \psi_i\varphi_j\varphi_k + \psi_j\varphi_k\varphi_i + \psi_k\varphi_i\varphi_j \\ F_{ijk} = 3\varphi_{(i}\varphi_j F_{k)} - 3\psi_{(i}\psi_j\varphi_{k)} \end{gathered}$$

Anti Chiral-Chiral

$$V_{ij} \equiv \Phi_i^\dagger(y^\dagger, \bar\theta)\Phi_j(y,\theta)$$

We will have to substitute either or both of $y, y^\dagger$ in terms of θ to write the expression in terms of a single spacetime variable. This could be either $y, y^\dagger$ or x again. Then taylor expand, using the Grassmannian nature of θ to limit the expansion to up to 3 terms.

One can check that the product followed by the expansion gives the complete form of the general superfield discussed previously.

$$\begin{gathered} V_{ij} = \varphi_i^*(x)\varphi_j(x) + \sqrt{2}\theta\psi_j\varphi_i^* + \sqrt{2}\bar\theta\bar\psi_i\varphi_j + \theta^2\varphi_i^* F_j + \bar\theta^2 F_i^*\varphi_j \\ + \iota\theta\sigma^\mu\bar\theta(\varphi_i^*\partial_\mu\varphi_j - \varphi_j\partial_\mu\varphi_i^*) - 2\bar\theta\cdot\psi_i\theta\cdot\psi_j \\ + \theta^2\bar\theta\left[\tfrac{\iota}{\sqrt{2}}\sigma^\mu(\varphi_i^*\partial_\mu\psi_j - \partial_\mu\varphi_i^*\psi_j) - \sqrt{2}F_j\bar\psi_i\right] + \bar\theta^2\theta[\dots] + \theta^2\bar\theta^2\mathcal{D}_{ij}(x) \end{gathered}$$

$$\mathcal{D}_{ij}(x) = \frac{1}{4}\varphi_i^*\Box\varphi_j + \frac{1}{4}\varphi_i\Box\varphi_j^* - \frac{1}{2}\partial_m\varphi_i^*\partial^m\varphi_i \\ + \frac{\iota}{2}\partial_m\bar{\psi}_i\bar{\sigma}^m\psi_j - \frac{\iota}{2}\bar{\psi}_i\bar{\sigma}^m\partial_m\psi_i + F_i^*F_i$$

Super-Invariants

Recall that the last component of a generic superfield, $\mathcal{D}$, changes by a total derivative under susy. Same goes for F for the chiral superfield. This happens because under susy the only terms increasing in powers of θ are total derivatives, and those are the only ones which can contribute to $\delta\mathcal{D}$. Contributions from $\mathcal{D}$ itself must either increase or decrease in powers of θ (and increases will be discarded because θ is Grassmannian).

The same result can be achieved using dimensional arguments: $\mathcal{D}$ has mass dimension 4, so not only is it appropriate to be a Lagrangian density, but also, $\delta\mathcal{D}$ must involve derivatives since all other terms have lower mass dimensions.

This suggests that we can construct super-invariants:

$$\int \mathrm{d}^4x[V_{\text{general}}]_{\mathcal{D}} \\ \int \mathrm{d}^4x[\Phi_{\text{chiral}}]_F$$

How do we check or verify this? Well, we can write the free action for a chiral multiplet Φ_i as the $\mathcal{D}$ component of $\Phi_i^\dagger\Phi_i$:

$$S_0 = \int (\Phi_i^\dagger\Phi_i)_{\mathcal{D}}\,\mathrm{d}^4x = \int \mathrm{d}^4x\mathcal{L}_0 \\ \text{Since } (\Phi_i^\dagger\Phi_i)_{\mathcal{D}} = \mathcal{L}_0 = -\partial_m\varphi_i^*\partial^m\varphi_i - \iota\bar{\psi}_i\bar{\sigma}^m\partial_m\psi_i + F_i^*F_i$$

Up to total derivatives.

Superpotential

Consider an interaction term in the Lagrangian,

$$S_{\text{int}} = \int \mathrm{d}^4x\left([W(\Phi_i)]_F + [W^*(\Phi_i^\dagger)]_F\right)$$

Where W, the superpotential, can have terms of the form

$$W = \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k + \dots$$

We restrict to only till the cubic term for renormalisability.

In terms of the multiplet, we obtain the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = [W]_F = \lambda_i F_i + m_{ij}(\varphi_{(i} F_{j)} - \psi_i \psi_j) + g_{ijk}(\varphi_{(i} \varphi_j F_{k)} - \psi_{(i} \psi_j \varphi_{k)})$$

In general, W may be an arbitrary function of Φ_i, the only constraint being that it is also chiral.

Lecture 17 - Susy

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Susy Theories

Consider chiral superfields Φ_i. Let us try to write actions for theories in terms of this.

We can do this using terms of the kind

$$\int \mathrm{d}^4x\ [\text{general}]_{\mathcal{D}}$$
$$\int \mathrm{d}^4x\ [\text{chiral}]_F$$

We want renormalisable theories, so no couplings of negative mass dimension. We also want constant coefficients to maintain Poincare invariance.

This enforces that the interaction term is of the form:

$$\begin{aligned} S_{\text{int}} &= \int \mathrm{d}^4x \left([W(\Phi_i)]_F + [W^*(\Phi_i^\dagger)]_F\right) \\ W &= \lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k \\ [W]_F &= \lambda_i F_i + m_{ij}(2\varphi_{(i}F_{j)} - \psi_i\psi_j) + g_{ijk}(3\varphi_{(i}\varphi_j F_{k)} - 3\psi_{(i}\psi_j\varphi_{k)}) \end{aligned}$$

Since φ has mass dimensions 1, ψ has mass dimensions 3/2 and F has mass dimensions 2. One can also think of it by realising that θ has mass dimensions -1/2, so the F part of a chiral superfield will have mass dimension 1 more than that of the superfield. So for $[W]_F$ to have mass dimensions 4, W must have mass dimensions 3 — hence we are restricted to the Φ^3 term and no higher.

Kinetic Term

We take the kinetic term to be

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[\Phi_i^\dagger \Phi_i\right]_{\mathcal{D}}$$

Why? Well, consider a more general kinetic term, the $\mathcal{D}$ part of $K(\Phi, \Phi^\dagger)$,

$$K(\Phi, \Phi^\dagger) = h_i \Phi_i + h_i^* \Phi_i^\dagger + k_{(ij)} \Phi_i^\dagger \Phi_j + \dots$$

Where k_{ij} is symmetric because we want K to be real. K is known as the Kahler potential, and $\Phi_i^\dagger \Phi_i$ is the *canonical* Kahler potential.

For a polynomial Kahler potential, the $\mathcal{D}$ part of K can be obtained by its derivative with respect to Φ_i and $\Phi_i^\dagger$, then putting $\theta, \bar{\theta} = 0$ (so essentially derivatives with respect to φ_i, φ_i^* and selecting the term with no θs). This works because we expand $K(\Phi, \Phi^\dagger)$ in terms of θ using a Taylor-expansion-inspired approach, and are left with $\partial_\Phi K$ terms as coefficients.

The D-term is then

$$\begin{aligned} &G_{ij}(\varphi)\partial^\mu \varphi^i \partial_\mu \varphi^{*j} + \dots \\ &G_{ij} \sim \left(\frac{\partial^2 K}{\partial \Phi_i \partial \Phi_j^\dagger}\right)_{\theta, \bar{\theta}=0} \end{aligned}$$

The problem with these general Kahler potentials is that the theory becomes non-renormalisable – $\varphi(\partial\varphi)^2$ terms, etc. One could also consider derivatives $D\Phi$ in the Kahler potential – again, non-renormalisable.

In the non-linear sigma model, $G_{ij} = g_{ij}$ is independent of Φ and is called the Kahler metric. This behaves very similarly to working with the theory on a curved manifold. We will not go further here, but one can refer to 3.2.4 in David Tong's notes on supersymmetry: https://www.damtp.cam.ac.uk/user/tong/susy/susy.pdf.

Superpotential

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \lambda_i F_i + m_{ij}(\varphi_{(i}F_{j)} - \psi_i\psi_j) + g_{ijk}(\varphi_{(i}\varphi_j F_{k)} - \psi_{(i}\psi_j\varphi_{k)}) + \text{h.c.} \\ &= F_i \underbrace{(\lambda_i + m_{ij}\varphi_j + g_{ijk}\varphi_j\varphi_k)}_{\left(\frac{\partial W}{\partial \Phi_i}\right)_{\theta,\bar\theta=0}} + \text{fermions} + \text{h.c.}\end{aligned}$$

$$\therefore [W]_F = F_i \left(\frac{\partial W}{\partial \Phi_i}\right)_{\theta,\bar\theta=0} - \frac{1}{2}\psi_i\psi_j\left(\frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j}\right)_{\theta,\bar\theta=0}$$

Where the second term expands to $-\frac{1}{2}\psi_i\psi_j(m_{ij} + 2g_{ijk}\varphi_k)$, which contains the fermion mass term and the Yukawa coupling term.

We can write $\left(\frac{\partial W}{\partial \Phi_i}\right)_{\theta,\bar\theta=0} = \frac{\partial W(\varphi)}{\partial \varphi_i}$. Then the F terms in the action become

$$S = \int \mathrm{d}^4x \left(\cdots + F_i^* F_i + F_i \frac{\partial W(\varphi)}{\partial \varphi_i} + F_i^* \frac{\partial W^*(\varphi^*)}{\partial \varphi_i^*}\right)$$

$$\therefore \frac{\delta S}{\delta F_i^*} = 0 \implies F_i = -\frac{\partial W^*(\varphi^*)}{\partial \varphi_i^*}$$

Substituting this gives us the mass terms and everything else for both φ and ψ.

Notably, we can obtain the scalar potential (including the mass term) as

$$\begin{aligned} V[\varphi, \varphi^*] &= F_i F_i^* \\ = \left|\frac{\partial W(\varphi)}{\partial \varphi_i}\right|^2 &= \sum_i |\lambda_i + m_{ij}\varphi_j + g_{ijk}\varphi_j\varphi_k|^2\end{aligned}$$

Notice how the superpotential has brought in susy-consistent mass terms (Majorana mass for the fermion), a Yukawa interaction and a quartic interaction.

Also note $V \geq 0$: Energy must be positive in susy theories.

Integrals over Superspace

We can equivalently write the action as

$$S_{\text{kin}} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar\theta \, \Phi^\dagger \Phi$$

$$S_{\text{int}} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, W[\Phi] + \int \mathrm{d}^4x \, \mathrm{d}^2\bar\theta \, W^*[\Phi^\dagger]$$

Supersymmetry becomes manifest because susy transforms are equivalent to translations in the Grassmannian variables, (super)scalars are translation-invariant, and the measure is translation-invariant (though since x also changes when θ is shifted, there is a Jacobian one has to calculate – thankfully it ends up being 1).

Berezin Integrals

Integrating with respect to θ can be defined as well as the derivatives, but one must keep in mind these are formal algebraic transformations and not really measures of area. There are no limits of integration here.

We establish some axioms,

1. $\int \mathrm{d}\theta\ 1 = 0$
2. $\int \mathrm{d}\theta\ \theta = 1$
3. $\int \mathrm{d}\theta\ (a + b\theta) = b$

And from these, we can write the general fact

$$\int \mathrm{d}\theta f(a + b\theta) = bf'(a)$$

Lecture 18 - Susy

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Berezin Integrals, continued

$$\delta(\theta) \equiv \theta$$
$$\text{Let } f(\theta) = b + c\theta,$$
$$\int \mathrm{d}\theta \delta(\theta - a) f(\theta) = b - \int \mathrm{d}\theta a(b + c\theta)$$
$$= b + a \int \mathrm{d}\theta (b + c\theta) = b + ca = f(a)$$

Multiple Grassmannian Variables

$$\{\mathrm{d}\theta_\alpha, \theta_\beta\} = 0$$
$$\int \mathrm{d}\theta_\alpha \theta_\beta = \delta_{\alpha\beta}, \quad \int \mathrm{d}\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \delta_{\dot{\alpha}\dot{\beta}}$$
$$\therefore \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \theta_1 \theta_2 = -1$$
$$\theta^2 = \theta^\alpha \theta_\alpha = 2\theta_2\theta_1$$
$$\text{So } \mathrm{d}^2\theta \equiv \frac{1}{2} \mathrm{d}\theta_1 \mathrm{d}\theta_2$$
$$\text{Equivalently, } \mathrm{d}^2\theta \equiv -\frac{1}{4} \mathrm{d}\theta^\alpha \mathrm{d}\theta_\alpha, \quad \mathrm{d}^2\bar{\theta} \equiv -\frac{1}{4} \mathrm{d}\bar{\theta}_{\dot{\alpha}} \mathrm{d}\bar{\theta}^{\dot{\alpha}}$$
$$\text{s.t. } \int \mathrm{d}^2\theta \, \theta^2 = 1 \quad \left(\text{similarly, } \int \mathrm{d}^2\bar{\theta} \, \bar{\theta}^2 = 1\right)$$
$$\implies \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \theta^2 \bar{\theta}^2 = 1$$
$$\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \theta^2 = 0, \quad \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \bar{\theta}^2 \theta_\alpha = 0$$

Integrals over superspace selects invariants

All this to show that

$$\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, V(x,\theta,\bar{\theta}) = [V]_{\mathcal{D}}$$

Just like how integrating over translation-invariant Lagrangians gives a translation-invariant action, integrating over superfields selects a supersymmetry-invariant scalar (density).

While $\{x,\theta,\bar{\theta}\}$ is superspace, we can also define and integrate over the chiral superspace $\{x,\theta\}$ or the antichiral superspace $\{x,\bar{\theta}\}$. This is important for integrating over chiral/antichiral superfields, since integrating them over all superspace will give 0 (Well, not 0, but a total derivative — not what we want. We want the F term.)

$$\int \mathrm{d}^2\theta\, \Phi(y,\theta) = [\Phi]_F$$

Integrals as Derivatives

$$D^2 \equiv -\tfrac{1}{4} D^\alpha D_\alpha$$
$$\bar{D}^2 \equiv -\tfrac{1}{4} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}$$

We define this way to obtain

$$D^2\theta^2 = 1, \quad \bar{D}^2\bar{\theta}^2 = 1$$

Which we can check easily, since $\partial_m \theta = 0$.

$$D^2 = \frac{1}{2} D_1 D_2, \quad \theta^2 = 2\theta_2\theta_1$$
$$\partial_\alpha \theta^\beta = \delta_\alpha^\beta, \quad \partial_\alpha \theta_\beta = \partial_\alpha(\epsilon_{\beta\gamma}\theta^\gamma) = \epsilon_{\beta\alpha}$$

Explicitly,

$$D^2 = -\frac{1}{4}(\partial^\alpha\partial_\alpha - \bar{\theta}^2\partial^\mu\partial_\mu)$$
$$\bar{D}^2 = -\frac{1}{4}(\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + \theta^2\partial^\mu\partial_\mu)$$

Under $\int \mathrm{d}^4x$, we can drop the total derivative terms and

$$D^2 \to -\tfrac{1}{4}\partial^\alpha\partial_\alpha, \quad \bar{D}^2 \to -\tfrac{1}{4}\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}$$

Also,

$$[D^2, \bar{D}^2] = \Box + \tfrac{1}{2}\iota\sigma^n_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}D^\alpha\partial_n$$

Again, total derivative terms, so under the integral we can commute them.

Though we already knew that derivatives and integrals are equivalent for grassmannian variables, we can explicitly show it this way too. Specifically for superfields, this allows us to write,

Eg for a chiral superfield Φ, $\bar{D}_{\dot{\alpha}}\Phi = 0$,

$$[\Phi]_F = \int \mathrm{d}^2\theta\Phi = D^2\Phi$$

Up to total derivatives which are removed when integrating over spacetime.

And similarly for antichiral and general superfields.

This also makes it explicit why we can't have integrals over $\bar{\theta}$ for chiral superfields – its equivalent to a derivative, which is by definition 0.

Action for Chiral Superfields

$$\begin{aligned} S &= \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi^\dagger\Phi + \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,W[\Phi] + \int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,W^*[\Phi^\dagger] \\ &= \int \mathrm{d}^4x\left(D^2\bar{D}^2(\Phi^\dagger\Phi) + D^2W(\Phi) + \bar{D}^2W^*(\Phi^\dagger)\right) \end{aligned}$$

We can evaluate and verify this.

$$\begin{aligned} D^2W &= -\frac{1}{4}D^\alpha\left(\frac{\partial W}{\partial\Phi_i}D_\alpha\Phi_i\right) \\ &= -\frac{1}{4}\frac{\partial^2 W}{\partial\Phi_i\partial\Phi_j}D^\alpha\Phi_i D_\alpha\Phi_j + \frac{\partial W}{\partial\Phi_i}D^2\Phi_i \end{aligned}$$

Using the form of Φ,

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\psi(y)\theta + F(y)\theta^2$$

And defining

$$
\begin{aligned}
\frac{\partial W(\varphi)}{\partial \varphi_i} &\equiv \mathcal{V}_i(\varphi) \\
\frac{\partial^2 W(\varphi)}{\partial \varphi_i \partial \varphi_j} &\equiv \mu_{ij}(\varphi)
\end{aligned}
$$

Thus

$$
\begin{aligned}
D^2 W &= F_i(y)\mathcal{V}_i(\varphi) - \frac{1}{2}\mu_{ij}(\varphi)\psi_i(y)\psi_j(y) \\
\mathcal{L}_{\text{int}} &= F_i(y)\mathcal{V}_i(\varphi) - \frac{1}{2}\mu_{ij}(\varphi)\psi_i(y)\psi_j(y) + \text{h.c.} \\
S &= \int \mathrm{d}^4 x \left(-|\partial_m \varphi_i|^2 - \iota\bar{\psi}_i\bar{\sigma}^m\partial_m\psi_i + F_i^* F_i + \mathcal{L}_{\text{int}}(\varphi, F, \psi)\right)
\end{aligned}
$$

The EoM for F_i is

$$
F_i^* = -\mathcal{V}_i(\varphi)
$$

Which gives the scalar potential

$$
V = |F_i(\varphi)|^2 = |\mathcal{V}_i(\varphi)|^2
$$

And the rest of the interactions are mass and Yukawa terms.

In Bispinor notation

Using $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$,

$$
\begin{aligned}
\mathcal{L} = -|\partial\varphi_i|^2 - V(\varphi) - \frac{\iota}{2}\bar{\Psi}_i\gamma^m\partial_m\Psi_i - \frac{1}{2}\mu_{ij}(\varphi)\bar{\Psi}_i\Psi_j \\
-\frac{1}{2}g_{ijk}\varphi_k\bar{\Psi}_i(1-\gamma_5)\Psi_j - \frac{1}{2}g_{ijk}^*\varphi_k^*\bar{\Psi}_j(1+\gamma_5)\Psi_i
\end{aligned}
$$

Note, the γ_5 here is probably defined with the ι, whereas in a previous lecture we defined it without the ι. With the ι, it is hermitian.

Lecture 19 - Susy

Course	Supersymmetry
Date	@February 24, 2025
Status	Completed
Next	Lecture 20 - Susy
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Remark about Berezin integral identity

We had two identities about integrals and derivatives of Grassmann integrals,

$$\int \mathrm{d}\theta_\alpha \theta_\beta = \delta_{\alpha\beta}$$
$$\partial_\alpha \theta^\beta = \delta_\alpha^{\ \beta}$$

The latter is Lorentz covariant, but the former is not – the correct lorentz-covariant tensor with two lowered spinor indices is $\epsilon_{\alpha\beta}$. This is not a mistake – that integral is indeed not lorentz-covariant.

This does not matter for our use of Berezin integrals – we always do integrals over $\mathrm{d}^2\theta$, over scalar superfields – both Lorentz-invariant quantities. We never integrate over something with spinor indices for our action. And this makes sense – even before supersymmetry, we would not expect an action of the form $\int \mathrm{d}^4 x \mathcal{L}_\mu$ to be Lorentz invariant. The odd nature of the integral identity relates to the same idea for spinor indices.

Equations of Motion in terms of Superfields

Concise form of Action

Since $\int \mathrm{d}^2\theta = D^2$ as an operator, we write the action as a single integral by introducing $(D^2)^{-1}$ as simply the algebraic operator that cancels $\int \mathrm{d}^2\theta$:

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left(\Phi^\dagger \Phi + (\bar{D}^2)^{-1} W(\Phi) + (D^2)^{-1} W^*(\Phi^\dagger) \right)$$

Variation of the Action

Change of Variables to Unconstrained Superfield

It is complicated to take the variation with respect to a constrained quantity, and Φ_i is constrained by

$$\bar{D}_\alpha \Phi_i = 0$$

We often treat this using Lagrange multipliers, but here we can do something better.

The most general solution to the constrained equation is given by

$$\Phi_i = \bar{D}^2 X_i$$

Where X_i is a completely unconstrained superfield. This works because $D^3 = \bar{D}^3 = 0$ – but we did not discuss how to explicitly show that there is no other solution to the constraint.

Variation wrt X_i

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left(D^2 X_i^\dagger \bar{D}^2 X_i + (\bar{D}^2)^{-1} W(\bar{D}^2 X_i) + (D^2)^{-1} W^*(D^2 X_i^\dagger) \right)$$

$$\therefore \frac{\delta S}{\delta X_i^\dagger} = 0 \implies D^2 \bar{D}^2 X_i + D^2 (D^2)^{-1} \frac{\partial W^*(D^2 X_i^\dagger)}{\partial (D^2 X_i^\dagger)} = 0$$

$$\implies D^2 \Phi_i + \frac{\partial W^*(\Phi_i^\dagger)}{\partial \Phi_i^\dagger} = 0$$

$$\text{Similarly, } \bar{D}^2 \Phi_i^\dagger + \frac{\partial W(\Phi_i)}{\partial \Phi_i} = 0$$

Spontaneous Susy Breaking

Going back to the component form of the Lagrangian,

$$\mathcal{L} = -\left|\partial \varphi_i\right|^2 - \iota \bar{\psi}_i \bar{\sigma}^m \partial_m \psi_i - \left|\frac{\partial W(\varphi)}{\partial \varphi_i}\right|^2 - \frac{1}{2} \left(\frac{\partial^2 W(\varphi)}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \text{h.c.} \right)$$

$$\mu_{ij} \equiv \frac{\partial^2 W(\varphi)}{\partial \varphi_i \partial \varphi_j}$$

This is obtained after putting $F_i^*(\varphi) = -\frac{\partial W}{\partial \varphi_i}$.

Now the scalar potential is

$$U(\varphi) = |F_i|^2 = \left|\frac{\partial W(\varphi)}{\partial \varphi_i}\right|^2 \geq 0$$

SSB happens when the action is invariant under a symmetry but the vacuum is not – so the VEVs of the fields must not be supersymmetric.

Let the vacuum fields be $\varphi_{0i}, \psi_{0i}, F_{0i}$.

Assuming the vacuum is still Poincare invariant, the scalars φ_{0i} must be constant and uniform. The same constraint also arises from minimising the kinetic energy, obvious in the hamiltonian formulation.

$\psi_{0i} = 0$, because it transforms under Lorentz, but we want the vacuum to be Lorentz-invariant, and there is no non-zero fixed point of the Lorentz group. Lorentz is global, so we can't escape this with some clever x dependence – plus we must minimise ψ's kinetic energy as well. So ψ must be constant, and 0.

Thus F_{0i} must also be constant and uniform – its x dependence completely comes from its φ dependence.

Recall the susy transformations:

$$\begin{gathered}\delta\varphi_i = \sqrt{2}\xi\psi_i \\ \delta\psi_i = \sqrt{2}\iota\sigma^m\xi\partial_m\varphi_i + \sqrt{2}\xi F_i \\ \delta F_i = \sqrt{2}\iota\bar{\xi}\bar{\sigma}^m\partial_m\psi_i\end{gathered}$$

Thus if $F_{i0} = 0$, a susy transformation leaves ψ_i invariant, $\delta\psi_i = 0$ – and the other two variations are already 0. So susy is indeed a symmetry of vacua with $F_{i0} = 0$.

The vacuum is given by the minimum energy, and

$$U(\varphi) = 0 \iff F_i(\varphi) = 0 \forall i$$

Thus, the condition for Spontaneous Susy breaking is that $F_i(\varphi) = 0$ has no solutions – if there any such solutions, then that is the vacuum, and a supersymmetric one at that. We need $F_i(\varphi) \neq 0$ for all $\varphi \in \mathbb{C}^n$ to get SSuB.

Consider then a φ_0 such that $F_{i0} \neq 0$ for some i. Since $\delta\psi_i \neq 0$ but the action must still be invariant, this implies that ψ_i cannot have mass. (The kinetic term shifts by a total derivative, but the mass term has a linear shift that we can't ignore.)

Since both F_{i0} and μ_{ij} arise from the same function W, there is clearly a connection. Let us prove it.

Super Goldstone Theorem

For each spontaneously broken supersymmetry, there is at least one massless fermion field, called a *goldstino*.

Proof of Super Goldstone Theorem

The equations of motion:

$$\Box\varphi_i = \frac{\partial W}{\partial \varphi_j}\frac{\partial^2 W^*}{\partial \varphi_j^* \partial \varphi_i^*}, \quad \Box\varphi_i^* = \frac{\partial W^*}{\partial \varphi_j^*}\frac{\partial^2 W}{\partial \varphi_j \partial \varphi_i}$$

Given φ is constant, we get the following set of linear equations:

$$\mu_{ij}(\varphi)F_j^*(\varphi) = 0$$

There are two ways this can go.

1. $\exists F_j(\varphi) = 0$ → Then susy is unbroken.
2. $\det \mu_{ij} = 0$ because $F_i(\varphi) \neq 0$ → Susy is broken, and there exists a massless fermion.

Case 2 proves the Super Goldstone Theorem.

Examples

Wess-Zumino Model

$$W = \lambda\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$$
$$-F^*(\varphi) = \frac{\partial W}{\partial \varphi} = \lambda + m\varphi + g\varphi^2$$

This is a quadratic and so will have a solution, real or complex – which means no susy breaking.

Generalised Wess-Zumino Model

$$W = \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k$$
$$-F_i^*(\varphi) = \frac{\partial W}{\partial \varphi_i} = \lambda_i + m_{ij} \varphi_j + g_{ijk} \varphi_j \varphi_k$$

When $\lambda_i = 0$, then $\varphi_i = 0$ is a solution to $F_i = 0$, so we must have some $\lambda_i \neq 0$. But that doesn't guarantee no solutions by itself. We discuss further next time.

Lecture 20 - Susy

Course	Supersymmetry
Date	@February 25, 2025
Status	Completed
Next	Lecture 21 - Susy
Previous	Lecture 19 - Susy

Spontaneous Susy Breaking, continued

Generalised Wess-Zumino

$$
\begin{aligned}
W &= \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \\
-F_i^*(\varphi) &= \frac{\partial W}{\partial \varphi_i} = \lambda_i + m_{ij}\varphi_j + g_{ijk}\varphi_j\varphi_k
\end{aligned}
$$

When $\lambda_i = 0$, then $\varphi_i = 0$ is a solution to $F_i = 0$, so we must have some $\lambda_i \neq 0$. But that doesn't guarantee no solutions by itself.

Example – O′ Raifeartaigh Model – GenWZ with $n = 3$

$$
\begin{aligned}
W &= \lambda\varphi_1 + m\varphi_2\varphi_3 + g\varphi_1\varphi_3^2 \\
\frac{\partial W}{\partial \varphi_i} &= 0 \to \begin{cases} \lambda + g\varphi_3^2 = 0 \\ m\varphi_3 = 0 \\ m\varphi_2 + 2g\varphi_1\varphi_3 = 0 \end{cases}
\end{aligned}
$$

Equations 2 and 3 imply $\varphi_3 = 0, \varphi_2 = 0$ (unless $m = 0$, but that would remove φ_2 from the theory). Thus if $\lambda \neq 0$, there are no solutions to eq 1, exactly as we want for SSB.

φ_1 is then arbitrary. The vacuum potential is

$$\mathcal{U}_0 = \left|\frac{\partial W}{\partial \varphi_i}\right|^2\Bigg|_{\varphi_2,\varphi_3=0} = |\lambda|^2$$

Let $\lambda = -gM^2 \neq 0$, with g, M both being real. Then $F_{01} = gM^2 \neq 0$, and ψ_1 must correspondingly be massless.

We can get the boson masses by expanding the Lagrangian around the vacuum. Let us expand around the $\varphi_0 = (0,0,0)$ vacuum just for convenience – the results will hold for other values of φ_{01} too.

$$\mathcal{U} = \mathcal{U}_0 + \lambda g\left(\varphi_3^2 + \varphi_3^{*2}\right) + m^2\varphi_3\varphi_3^* + m^2\varphi_2\varphi_2^* + \mathcal{O}(\varphi^3)$$

Thus φ_1 is massless, $\varphi_{2,3}$ are massive but need to be diagonalised.

Specifically, φ_2, φ_2^* have mass m, and for 3, consider massive real scalars a, b:

$$\begin{gathered}
\text{Let } \varphi_3 = \frac{1}{\sqrt{2}}\left(a + \iota b\right) \\
\varphi_3^2 + \varphi_3^{*2} = a^2 - b^2, \quad \varphi_3\varphi_3^* = \frac{1}{2}(a^2+b^2) \\
\therefore m_a^2 = m^2 - 2g^2M^2 \\
\&\ m_b^2 = m^2 + 2g^2M^2
\end{gathered}$$

Thus susy breaking causes a split in the boson masses, of order of the SSuB parameter g^2M^2.

$$\begin{gathered}
\mu_{ij} = \frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j} \implies \mu_{23} = \mu_{32} = m, \text{ rest } 0 \\
\text{Fermion mass terms}: -\frac{1}{2}\left(2m\psi_2\psi_3 + \text{h.c.}\right)
\end{gathered}$$

Thus ψ_1 is massless, and ψ_2, ψ_3 combine into a massive Dirac fermion of mass m.

Thus, the particle spectrum is not supersymmetric anymore. The only way to tell that the theory has susy is by examining the Yukawa coupling and the quartic coupling and finding a symmetry in those interactions – since if the Lagrangian itself is supersymmetric, that will show up in the interaction parameters being related, even in the component form.

There are other ways to break susy. One way is in supergravity – something called soft susy breaking, which occurs due to local susy breaking and causes some mass terms in the action. There is also the idea that breaking Poincare would break susy, such as in the context of branes in a higher-dim theory, but we did not discuss this in the lecture.

Supersymmetric QFT

Path Integral

$$Z = \int D\varphi D\psi \; e^{\iota S(\varphi,\psi)}$$

(Assuming we have already integrated out the auxiliary field F – it only has quadratic terms so we can do that.)

For the free theory, the bosonic and fermionic integrals decouple. The bosonic integrals have contributions of the form

$$\frac{1}{\sqrt{\det(\partial^2 - m^2)}}$$

Whereas fermionic terms, due to the Grassmannian (or more generally anticommuting) nature, the integral is instead

$$\sqrt{\det(\iota\bar{\sigma}^m\partial_m + m)}$$

With a similar factor for the Dirac adjoint fermion.

Since we have an equal number of bosons and fermions with the same masses, these factors will cancel, and we get

$$Z_{\text{free}} = 1$$

And the energy of the vacuum – the thermodynamic idea of free energy – is then 0!

Let's study this in more detail.

Vacuum Energy

$$-\iota\Gamma \equiv \log Z$$
$$\Gamma_{\text{free}} = 0$$

$$\Gamma_{B,\text{free}} = \operatorname{Tr}\log(\Box - m^2)$$
$$= \underbrace{\int \mathrm{d}^4 x}_{V_{\text{spacetime}}} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \log(p^2 + m^2)$$

This diverges – regularise it, $|p| \leq \mu$:

$$\Gamma_{B,\text{free}} \sim V_{\text{sp-t}} \mu^4$$

The μ^4 then acts as a sort of Cosmological constant – in gravity,

$$\int \mathrm{d}^4 x \sqrt{-g}(R + \Lambda)$$

The cosmological constant is the constant coefficient of the volume term. And this is one of the big issues of the infinite free energy QFT predicts.

However, in Susy, we also have the Γ_F terms, which are the exact same but with a minus sign – the free energy cancels!

$$\Gamma_B + \Gamma_F = 0$$

This holds more clearly for free theories, but it holds for interacting susy theories as well – though this is non-trivial to show. This is the perhaps susy theories' greatest attraction.

A way to understand it is to consider that in susy theories, the free energy is proportional not to the spacetime volume, but the superspace volume – which is:

$$\Gamma \propto \int \mathrm{d}^4 x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, 1 = 0$$

Since integral of constants over Grassmannian variables is 0.

Lecture 21 - Susy

Course	Supersymmetry
Date	@February 27, 2025
Status	Completed
Next	Lecture 22 - Susy
Previous	Lecture 20 - Susy

Quantum Properties of Chiral Multiplets

Assuming supersymmetry is unbroken,

$$Z = \int D\Phi e^{\iota S[\Phi]} = 1$$
$$E_{\text{vac}} = 0$$

This was easy to see for the free theory, but this can be proved for the interacting theory as well. The contributions cancel individually at each order, because the interactions are controlled by the same few parameters (eg the ϕ^4 and yukawa interactions both have the same coupling g).

Argument for $Z = 1$ in terms of Superfields

Consider an action with a single superfield

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \Phi^\dagger \Phi + \int \mathrm{d}^4x \mathrm{d}^2\theta W(\Phi) + \text{h.c.}$$

As usual we can set

$$\Phi = \bar{D}^2 X$$

So X is unconstrained, and the kinetic term becomes

$$\int \Phi^\dagger \Phi = \int X^\dagger D^2 \bar{D}^2 X + \text{surface terms}$$

The propagator is its inverse,

$$(D^2\bar{D}^2)^{-1}$$

Which will depend on two points in superspace, $z = (x, \theta, \bar{\theta})$ and a similar z'. Essentially this is the Green's function $G(z, z')$, such that

$$D^2\bar{D}^2 G(z, z') = \delta(z, z') = \delta^{(4)}(x, x')\delta^{(2)}(\theta, \theta')\delta^{(2)}(\bar{\theta}, \bar{\theta}')$$

Note for Grassmannian variables,

$$\delta(\theta) = \theta$$
$$\delta^{(2)}(\theta) = \theta^2$$

Since we know that $\int \mathrm{d}^2\theta \; \theta^2 = 1$.

Divergences and Renormalisation in Susy QFT

1. There will only be $\log$ UV divergences.
2. Only the $\mathcal{D}$ terms (integrals over all superspace) will be renormalised,

We introduce the effective quantum action, the generator of the 1PI diagrams, $\Gamma(\bar{\Phi})$. In general, this has divergent terms which will be integrals over the entire superspace, of this form:

$$\Gamma_{\text{divergent}} = \zeta(\Lambda) \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \Phi^\dagger \Phi$$

Where Λ is the the UV cutoff. The theory is renormalisable, so the divergences must be proportional to the original action.

ζ will only get log divergences, so

$$\zeta(\Lambda) = 1 + c_1(g) \log \frac{\Lambda}{m} + \mathcal{O}\left((\log \Lambda)^2\right)$$

There is no renormalisation in the superpotential terms – their contributions will be finite and non-local (in superspace).

The local part of Γ only comes from the integral over all superspace, and only local terms give divergences in QFT (since they occur when a propagator's beginning and end coincide – i.e. a loop exists).

Superpotential under Renormalisation

Since we do have wavefunction renormalisation from the kinetic term, that will affect the superpotential, but there is no necessity of renormalisation due to the superpotential.

$$
\begin{gathered}
W=\lambda\Phi+\frac{1}{2}m\Phi^2+\frac{1}{3}g\Phi^3 \\
=\lambda\zeta^{-1/2}\Phi_r+\frac{1}{2}m\zeta^{-1}\Phi_r^2+\frac{1}{3}g\zeta^{-3/2}\Phi_r^3 \\
m_r=m\zeta^{-1}, g_r=g\zeta^{-3/2}
\end{gathered}
$$

Divergent part of Effective Action

$$
\Gamma_{\text{divergent}}=\zeta(\Lambda)\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\Phi^\dagger\Phi+\text{finite terms}
$$

But the finite terms will depend on Φ non-trivially. We're interested in $\Gamma(0)$, i.e. no external legs, just vacuum diagrams, then as usual we get a volume factor – but the volume of superspace is 0!

$$
\Gamma(0)\sim\zeta\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}=0
$$

Supersymmetric Theories containing Vector Fields – Super-Maxwell Theory

So we want a (real) vector field $V_m(x)$ (we can deal with complex later, it is a simple generalisation). We know which (CPT-invariant) supermultiplet this is a part of: $(\frac{1}{2},1)\oplus(-1,-\frac{1}{2})$ (at least, on-shell). Off-shell we might need an auxiliary field again.

The EoMs:

$$
\partial_m F^{mn}=0
$$

With two helicity solutions, and

$$
\iota\bar{\sigma}^m\partial_m\lambda=0
$$

λ is our massless Weyl fermion.

On-shell, we have 2 and 2 DoF, all works out.

Off-shell, V_m has 3 DoF (though $m \in [[0,3]]$, due to gauge freedom there are only 3 DoF for the vector) and λ has 4, so we need another bosonic degree of freedom which vanishes off-shell.

So the vector multiplet is:

$$(V_m, \lambda, \mathcal{D})$$

Modulo the gauge freedom $V_m' = V_m - \partial_m \xi$.

$\mathcal{D}$ is a real scalar.

Action

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{mn}^2(V) - \iota \bar{\lambda} \bar{\sigma}^m \partial_m \lambda + \frac{1}{2} \mathcal{D}^2 \right)$$

Susy Transformations

$$\begin{aligned} \delta V_m &= \iota \left(\bar{\xi} \bar{\sigma}_m \lambda + \xi \sigma_m \bar{\lambda} \right) \underbrace{-\partial_m \alpha(x)}_{\text{gauge freedom}} \\ \delta \lambda &= \sigma^{mn} \xi F_{mn} + \iota \xi \mathcal{D} \\ \delta \mathcal{D} &= \bar{\xi} \bar{\sigma}^m \partial_m \lambda - \xi \sigma^m \partial_m \bar{\lambda} \end{aligned}$$

ξ is a constant parameter, a global symmetry, while $\alpha(x)$ is a local symmetry.

As before, this is a symmetry of the action, and the susy algebra holds off-shell.

Superspace description

We start with a scalar superfield – recall that the general scalar superfield does include a vector! – but we will require it to be real.

$$\mathrm{V}(x, \theta, \bar{\theta}) = \mathrm{V}^\dagger$$

Under that condition, the general superfield can be relabelled to be

$$
\begin{aligned}
\mathrm{V} = c(x) + \iota(\theta\chi(x) - \bar{\theta}\bar{\chi}(x)) - \theta\sigma^m\bar{\theta}V_m(x) + \frac{\iota}{2}\left(\theta\theta M(x) - \bar{\theta}\bar{\theta}M^*(x)\right) \\
+\iota\theta\theta\bar{\theta}\bar{\zeta}(x) - \iota\bar{\theta}\bar{\theta}\theta\zeta(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(\mathcal{D}(x) - \frac{1}{2}\Box c\right)
\end{aligned}
$$

These definitions allow us to explicitly express the vector supermultiplet. Note that c is a real scalar, M is a complex scalar and χ is a Weyl fermion.

With the definition

$$
\lambda \equiv \zeta - \frac{\iota}{2}\sigma^m\partial_m\bar{\chi}
$$

Then this superfield actually contains two supermultiplets, $(V_m, \lambda, \mathcal{D}) \oplus (c, \chi, M)$. They are closed under susy. But the second suplet is a little odd. Why does it occur?

The answer is the gauge symmetry. Until now we've not discussed gauge symmetry, but it must be incorporated somehow, to make the supermultiplet balanced off-shell. The gauge transformations will be parametrised by a real scalar, which will have to have its own supermultiplet. We continue next time.

Lecture 22 - Susy

Course	Supersymmetry
Date	@March 3, 2025
Status	Completed
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Super-Gauge-Invariance

Recall when trying to embed a real vector into a superfield, we ended up with the suplets $(V_m, \lambda, \mathcal{D}) \oplus (c, \chi, M)$.

Usually, the $\mathrm{U}(1)$ gauge symmetry acts as

$$V'_m = V_m - \partial_m \alpha(x)$$
$$\phi' = e^{\iota\alpha(x)}\phi$$
$$\nabla_m \phi = \partial_m \phi + \iota V_m \phi, \nabla_m \phi \text{ is invariant}$$

Gauging a superfield $\mathrm{U}(1)$

Let's start with our old chiral superfield (since it contains a scalar) and try to gauge the U(1) action on it.

The chiral superfield is some Φ s.t. $\bar{D}_{\dot{\alpha}}\Phi = 0$. We give it the action

$$S = \int \mathrm{d}^8 z \Phi^\dagger \Phi$$

U(1) is

$$\Phi' = e^{\iota\alpha}\Phi$$

This is a symmetry.

Gauged, we get

$$\Phi' = e^{\iota\alpha(x)}\Phi$$

This seems to be a symmetry, but the problem is, Φ' isn't even a chiral superfield anymore! It doesn't satisfy $\bar{D}_{\dot{\alpha}}\Phi' = 0$. We need to use a function of $y = x + \iota\theta\sigma\bar{\theta}$.

The product of chiral superfields is a chiral superfield, and a polynomial of chiral superfields is also one. So we can try:

$$\Phi' = e^{\iota\Lambda(y,\theta)}\Phi$$

Where $\Lambda(y, \theta)$ is a chiral superfield – the parameter of supergauge transformations, $\bar{D}\Lambda = 0$.

The action changes to

$$\int \mathrm{d}^8 z \Phi^\dagger e^{-\iota\Lambda^\dagger} e^{\iota\Lambda}\Phi$$

This isn't invariant either. Bummer. But this is a local transformation, we should expect needing another field, a vector in particular, to sort this out. Why? Because a vector is just a real superscalar, which is what $\iota(\Lambda^\dagger - \Lambda)$ is.

Introduce a vector-containing superfield into the kinetic term,

$$S = \int \mathrm{d}^8 z \Phi^\dagger e^{\mathrm{V}}\Phi$$

Then under a gauge transformation, we get

$$\mathrm{V}' = \mathrm{V} + \iota(\Lambda^\dagger - \Lambda)$$

Which preserves the reality of V.

The transformation parameter is expanded to

$$\Lambda = a(y) + \sqrt{2}\theta\mathcal{V}(y) + \theta\theta E(y)$$

Of which the case of all but $\Re(a)$ being 0 gives the usual gauge transformations on V_m. (Note a is usually complex.)

Explicitly, the transformation of V:

$$c' = c + \iota(a^* - a)$$
$$M' = M - E$$
$$\chi' = \chi - \sqrt{2}\mathcal{V}$$
$$V'_m = V_m - \partial_m(a + a^*) \to \text{ordinary gauge transformations}$$
$$\lambda' = \lambda$$
$$\mathcal{D}' = \mathcal{D}$$

This is why we defined λ and $\mathcal{D}$ the way we did in the previous lecture – these parts are invariant.

When we demand super-gauge-invariance, some of the DoF of the vector become unphysical. These are precisely c, M, χ and certain components of V_m – the same ones as with usual gauge invariance – since after we set c, M, χ to 0, we can still set $\Re(a)$ to be whatever we want.

Wess-Zumino Gauge

$$c = \chi = M = 0$$
$$\mathrm{V} = -\theta\sigma^m\bar{\theta}V_m + \iota\theta^2\bar{\theta}\bar{\lambda} - \iota\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2\mathcal{D}$$
$$\mathrm{V}^2 = -\tfrac{1}{2}\theta^2\bar{\theta}^2V^mV_m \quad \left((\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta}) = -\tfrac{1}{2}\eta^{mn}\theta^2\bar{\theta}^2\right)$$
$$\mathrm{V}^3 = 0$$
$$\therefore e^{\mathrm{V}} = 1 + \mathrm{V} + \tfrac{1}{2}\mathrm{V}^2$$

Thus the action is polynomial in this gauge.

Super-Maxwell Action

Superfield Strength

We want a super-gauge-invariant kinetic term for V. Consider

$$W_\alpha \equiv \bar{D}^2D_\alpha\mathrm{V}$$

This is a **spinor superfield**, and is chiral. It is also super-gauge-invariant, shown using:

- $\bar{D}\Lambda = 0 = D\bar{\Lambda}$
- $\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2\iota\sigma^m_{\alpha\dot{\beta}}\partial_m$
- $[\bar{D}, \partial] = 0$

$$\therefore \bar{D}^2 D_\alpha \left(\iota\bar{\Lambda} - \iota\Lambda\right) = 0$$
$$W' = W$$

We would also like it to satisfy an equivalent of the Bianchi identities, which form the source-free Maxwell equations.

First, component form of W_α:

$$W_\alpha = -\iota\lambda_\alpha(y) + \theta_\beta \left(-\iota F^{mn}(y)(\sigma_{mn})_\alpha^{\ \beta} + \delta_\alpha^{\ \beta}\mathcal{D}(y)\right) + \theta^2\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\lambda}^{\dot{\alpha}}(y)$$

Bianchi-equivalent identity:

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$$

Let's show this – next lecture.

Lecture 23 - Susy

Course	Supersymmetry
Date	@March 4, 2025
Status	Completed
Next	Lecture 24 - Susy
Previous	Lecture 22 - Susy

Super-Maxwell and Super-Gauge-Invariance, continued

Recap

Vector superfield:

$$\begin{gathered}\mathrm{V}^\dagger = \mathrm{V}\\ \mathrm{V}' = \mathrm{V} + \iota(\Lambda^\dagger - \Lambda), \quad \bar{D}\Lambda = 0\end{gathered}$$

V contains $(V_m^\dagger, \lambda, \mathcal{D})$.

We can define

$$\begin{gathered}W_\alpha = \bar{D}^2 D_\alpha \mathrm{V}\\ \implies W'_\alpha = W_\alpha\\ \bar{W} \equiv W^\dagger\end{gathered}$$

Then W contains $F_{mn}, \lambda, \mathcal{D}$.

Bianchi Identity

$$D^\alpha W_\alpha - \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} = 0$$

Let's show this.

$$\begin{gathered}D^\alpha \bar{D}^2 D_\alpha \mathrm{V} = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} \mathrm{V}\\ \bar{D}^2 = -\tfrac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}, \quad D^2 = -\tfrac{1}{4} D^\alpha D_\alpha\\ \{D^2, \bar{D}_{\dot\alpha}\} = -4\iota\sigma^m_{\alpha\dot\alpha} D^\alpha \partial_m, \quad \{\bar{D}^2, D_\alpha\} = -4\iota\sigma^m_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_m\end{gathered}$$

$$\begin{gathered}\therefore D^\alpha \bar{D}^2 D_\alpha \mathrm{V} = 4D^2\bar{D}^2 \mathrm{V} - 4\iota\sigma^m_{\alpha\dot\alpha} D^\alpha \partial_m \bar{D}^{\dot\alpha}\mathrm{V}\\ = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}\mathrm{V} + 4\iota\sigma^m_{\alpha\dot\alpha} D^\alpha \partial_m \bar{D}^{\dot\alpha}\mathrm{V} - 4\iota\sigma^m_{\alpha\dot\alpha} D^\alpha\partial_m\bar{D}^{\dot\alpha}\mathrm{V}\\ = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}\mathrm{V}\end{gathered}$$

Hence proved, $DW = \bar{D}\bar{W}$.

$$W_\alpha(y,\theta) = -\iota\lambda_\alpha(y) + \left[-\iota F^{mn}(y)\sigma_{mn\alpha}{}^\beta + \delta_\alpha^\beta \mathcal{D}(y)\right]\theta_\beta + \theta\theta\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\lambda}^{\dot{\alpha}}(y)$$

Thus in component form this implies

1. $\varepsilon^{mnkl}\partial_n F_{kl} = 0$, the usual Bianchi equations.
2. $\mathcal{D}^* = \mathcal{D}$, but we already knew that, is trivial.

Action

$$S = \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^\alpha W_\alpha + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$$
$$\left[\tfrac{1}{2}W^\alpha W_\alpha\right]_F = -\tfrac{1}{4}F_{mn}F^{mn} - \iota\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \tfrac{1}{2}\mathcal{D}^2 - \tfrac{\iota}{8}\varepsilon^{mnkl}F_{mn}F_{kl}$$

Where

$$\varepsilon^{mnkl}F_{mn}F_{kl} = \partial_m(4\varepsilon^{mnkl}V_n\partial_k V_l)$$

Thus a total derivative which can be dropped.

Equations of Motion

$$\begin{aligned} S &= \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\; W^\alpha W_\alpha + \mathrm{h.c.} \\ &= \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\;(\bar{D}^2D^\alpha\mathrm{V})(\bar{D}^2D_\alpha\mathrm{V}) + \mathrm{h.c.} \\ &= \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\;\bar{D}^2\left[(D^\alpha\mathrm{V})(\bar{D}^2D_\alpha\mathrm{V})\right] + \mathrm{h.c.} \\ &= \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\;(D^\alpha\mathrm{V})(\bar{D}^2D_\alpha\mathrm{V}) + \mathrm{h.c.} \\ &\therefore \mathrm{EoM}: D^\alpha(\bar{D}^2D_\alpha\mathrm{V}) + \mathrm{h.c.} = 0 \\ &\implies \boxed{D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}}W^{\dot{\alpha}}} = 0 \end{aligned}$$

Combined with Bianchi, this gives $DW = 0, \bar{D}\bar{W} = 0$, which in component form become

1. $\mathcal{D} = 0$
2. $\varepsilon^{mnkl}\partial_n F_{kl} = 0$
3. $\partial_m F^{mn} = 0$
4. $\bar{\sigma}^m\partial_m\lambda = 0$

So Maxwell theory with an additional Weyl fermion.

Coupling to Matter

Consider a chiral scalar superfield Φ, then the super-gauge-invariant action coupling this to the gauge superfield is

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ \Phi^\dagger e^{\mathrm{V}}\Phi + \int \mathrm{d}^4x\mathrm{d}^2\theta\frac{1}{4}W^\alpha W_\alpha + \text{h.c.}$$
$$\mathrm{V}' = \mathrm{V} + \iota\left(\Lambda^\dagger - \Lambda\right)$$
$$\Phi' = e^{\iota\Lambda(y,\theta)}\Phi$$

We can write this in component form, but for that let's first assume Wess-Zumino gauge:

$$\mathrm{V} = -\theta\sigma^m\bar{\theta}\mathrm{V}_m(x) + \iota(\theta^2\bar{\theta}\bar{\lambda}(x) - \text{h.c.}) + \tfrac{1}{2}\theta^2\bar{\theta}^2\mathcal{D}(x)$$
$$\Phi = \varphi(x) + \iota\theta\sigma^m\bar{\theta}\partial_m\varphi - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box\varphi + \sqrt{2}\theta\psi(x) - \tfrac{\iota}{\sqrt{2}}\theta^2\partial_m\psi\sigma^m\bar{\theta} + \theta^2F(x)$$

Then the kinetic term simplifies to

$$\left[\Phi^\dagger e^{\mathrm{V}}\Phi\right]_{\mathcal{D}} = \left[\Phi^\dagger\left(1 + \mathrm{V} + \tfrac{1}{2}\mathrm{V}^2\right)\Phi\right]_{\mathcal{D}}$$
$$\mathrm{V}^2 = -\frac{1}{2}\theta^2\bar{\theta}^2V^mV_m$$

The V, V^2 term make up all the covariant derivative terms, as one would expect.

$$\nabla_m \equiv \partial_m + \frac{\iota}{2}V_m$$
$$\mathcal{L} = -\left|\nabla_m\varphi\right|^2 - \iota\bar{\psi}\bar{\sigma}^m\nabla_m\psi + F^*F - \tfrac{1}{4}F_{mn}F^{mn} - \iota\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \tfrac{1}{2}\mathcal{D}^2 + \text{interactions}$$

Interacting Matter

So far we only coupled Maxwell to free matter. What if the matter obeys a superpotential, eg

$$\mathcal{W}(\Phi) = \lambda\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$$

This is clearly not super-gauge-invariant. What is the solution?

MORE PHIs.

$$\mathcal{W}(\Phi) = \lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k$$

We generalise the gauge transformation to

$$\Phi_i' = e^{2\iota q_i\Lambda}\Phi_i$$

Previously, we had $q = 1/2$, which also showed up in the covariant derivative. Now,

$$\nabla_m \chi_i \equiv (\partial_m + \iota q_i V_m)\chi_i \quad \text{for } \chi = \varphi, \psi$$

q_is are essentially the charges.

This is accomplished by the gauge-invariant kinetic term:

$$\int \mathrm{d}^8 z \sum_i \Phi_i^\dagger e^{2q_i \mathrm{V}} \Phi_i$$

Constraints on Charges for super-gauge-invariance

We want $\mathcal{W}$ to be gauge invariant, so

1. If $\lambda_i \neq 0$, $q_i = 0$ (put another way, if any $q_i \neq 0$, then $\lambda_i = 0$)
2. If $m_{ij} \neq 0$, $q_i + q_j = 0$
3. If $g_{ijk} \neq 0$, $q_i + q_j + q_k = 0$

Lecture 24 - Susy

Course	Supersymmetry
Date	@March 6, 2025
Status	Completed
Next	Lecture 25 - Susy
Previous	Lecture 23 - Susy

Super-Maxwell + Matter Lagrangian

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\ \Phi^\dagger e^{\mathrm{V}} \Phi + \int \mathrm{d}^4x \mathrm{d}^2\theta \frac{1}{4} W^\alpha W_\alpha + \mathrm{h.c.}$$
$$\mathrm{V}' = \mathrm{V} + \iota \left(\Lambda^\dagger - \Lambda\right)$$
$$\Phi' = e^{\iota\Lambda(y,\theta)}\Phi$$

Then

$$\begin{gathered}\mathcal{L} = -|\nabla_m \varphi|^2 - \iota\bar{\psi}\bar{\sigma}^m \nabla_m \psi + FF^* \\ +\tfrac{1}{2}|\varphi|^2 \mathcal{D} - \tfrac{\iota}{\sqrt{2}}\left(\varphi^*\psi\lambda - \varphi\bar{\psi}\bar{\lambda}\right) \\ -\tfrac{1}{4}F_{mn}F^{mn} - \iota\bar{\lambda}\bar{\sigma}^m \partial_m \lambda + \tfrac{1}{2}\mathcal{D}^2\end{gathered}$$

Note the Yukawa interaction between the φ, ψ and λ. Furthermore, $\mathcal{D}$ is non-dynamical and will give rise to a scalar potential. It's EoM:

$$\begin{gathered}\mathcal{D} + \tfrac{1}{2}|\varphi|^2 = 0 \\ \therefore \mathcal{L}_\varphi = -|\nabla_m \varphi|^2 - U(\varphi), \\ U(\varphi) = \tfrac{1}{8}(\varphi^*\varphi)^2 \geq 0\end{gathered}$$

Thus, to couple to super-Maxwell, in addition to the modified derivative (minimal coupling with the gauge field), you must have the Yukawa interaction, and its natural partner, the quartic interaction.

Multiple Matter Fields

We generalise the gauge transformation to

$$\Phi_i' = e^{2\iota q_i \Lambda}\Phi_i$$

Previously, we had $q = 1/2$, which also showed up in the covariant derivative. Now,

$$\nabla_m \chi_i \equiv (\partial_m + \iota q_i V_m)\chi_i \quad \text{for } \chi = \varphi, \psi$$

q_is are essentially the charges.

This is accomplished by the gauge-invariant kinetic term:

$$\int \mathrm{d}^8 z \sum_i \Phi_i^\dagger e^{2q_i \mathrm{V}} \Phi_i$$

Super-gauge-invariant Superpotential

Each term of the superpotential must be invariant. For generalised Wess-Zumino,

$$\mathcal{W}(\Phi) = \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k$$

1. If $\lambda_i \neq 0$, $q_i = 0$ (put another way, if any $q_i \neq 0$, then $\lambda_i = 0$)
2. If $m_{ij} \neq 0$, $q_i + q_j = 0$
3. If $g_{ijk} \neq 0$, $q_i + q_j + q_k = 0$

Lagrangian

$$\begin{gathered}\mathcal{L} = -|\nabla_m \varphi|^2 - \iota\bar{\psi}\bar{\sigma}^m \nabla_m \psi + FF^* \\ + q_k |\varphi_k|^2 \mathcal{D} - \iota\sqrt{2} q_k \left(\varphi_k^* \psi_k \lambda - \varphi_k \bar{\psi}_k \bar{\lambda}\right) \\ - \tfrac{1}{4} F_{mn} F^{mn} - \iota\bar{\lambda}\bar{\sigma}^m \partial_m \lambda + \tfrac{1}{2}\mathcal{D}^2\end{gathered}$$

Potential

$$\mathcal{D}(\varphi) = -\sum_k q_k |\varphi_k|^2$$

$$\mathcal{U} = \frac{1}{2}\mathcal{D}^2(\varphi) = \frac{1}{2}\left(\sum_k q_k |\varphi_k|^2\right)^2 \geq 0$$

Thus the scalar potential is still positive.

SSuB

Before we look at the superpotential in this case, can we add a term to break the supersymmetry here? Perhaps some $[f(V)]_{\mathcal{D}}$ in the action? It would need to be super-gauge-invariant, i.e. invariant under the $V \to V + \iota(\Lambda^\dagger - \Lambda)$. We cannot use the covariant derivatives here, since Λ is chiral but $\Lambda^\dagger$ is anti-chiral. But the key realisation is that $\Lambda, \Lambda^\dagger$ have total derivatives for their $\mathcal{D}$ component, so we can have $f(V) = \kappa V$.

The additional term in the Lagrangian is then just $\frac{1}{2}\kappa\mathcal{D}$. The EoM is then

$$\mathcal{D}(\varphi) = -\sum_k q_k|\varphi_k|^2 - \frac{1}{2}\kappa$$
$$\mathcal{U} = \frac{1}{2}\left(\sum_k q_k|\varphi_k|^2 + \frac{1}{2}\kappa\right)^2$$

This causes SSuB even for just Super-Maxwell, and is known as a Fayet-Illiopoulos term (and FI mechanism for SSuB).

If $\kappa \leq 0$ and at least one positive charge, then $U(\varphi_k) = 0$ has a solution and there's no SSuB. But if $\kappa > 0$ and positive charges, the vacuum energy is $\frac{1}{8}\kappa^2$.

Since $\mathcal{D}$ is non-zero in the vacuum, and $\delta_\xi\lambda \sim \mathcal{D}$, λ must be massless to maintain the susy invariance of the action – this is our goldstino.

We can still also have the SSuB from the matter superpotential, where F takes a non-zero VEV.

For superpotential $\mathcal{W}$,

$$\mathcal{U} = \frac{1}{2}\left(\sum_k q_k|\varphi_k|^2 + \frac{1}{2}\kappa\right)^2 + \sum_k\left|\frac{\partial\mathcal{W}}{\partial\varphi_k}\right|^2$$

Thus the SSuB condition now is that the following set of equations should together have no solutions:

1. $\mathcal{D}(\varphi) = 0$
2. $F_k(\varphi) = 0$

Lecture 25 - Susy

Course	Supersymmetry
Date	@March 7, 2025
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Next	Lecture 26 - Susy
Previous	Lecture 24 - Susy

Super Yang-Mills Theory

I.e. Non-abelian super-gauge theory.

We'll have several vector multiplets $(V_m^a, \lambda^a, \mathcal{D}^a)$ which also form a representation of a symmetry group. This is called an internal symmetry (of the set of fields).

The Coleman-Mandula theorem says that spacetime and internal symmetries can only combine trivially – so incorporating susy, which is a spacetime symmetry, should be straightforward.

We expect, in the absence of matter, the straightforward generalisation will hold:

$$\mathcal{L} = -\frac{1}{4} F_{mn}^a F^{amn} - \iota \bar{\lambda}^a \bar{\sigma}^m \nabla_m \lambda^a + \frac{1}{2} \mathcal{D}^a \mathcal{D}^a$$
$$F_{mn}^a = \partial_m V_n^a - \partial_n V_m^a - g f^{abc} V_m^b V_n^c$$
$$\nabla_m \lambda^a = \partial_m \lambda^a - g f^{abc} V_m^b \lambda^c$$

F_{mn} and ∇_m are both covariant under the local gauge transformations.

We expect this theory will be invariant under the susy transformations:

$$\delta V_m^a = \iota \xi \sigma_m \bar{\lambda}^a + \text{h.c.}$$
$$\delta \lambda^a = \sigma^{mn} \xi F_{mn}^a + \iota \xi \mathcal{D}^a$$
$$\delta \mathcal{D}^a = \bar{\xi} \bar{\sigma}^m \nabla_m \lambda^a + \text{h.c.}$$

Matrix Notation

For gauge group $\mathrm{SU}(N)$,

$$\iota V_m \in \mathfrak{su}(N), \quad V_m^\dagger = V_m, \quad V_m = V_m^a t_a$$

$$\mathrm{SU}(N) \ni U = \exp(\iota \kappa^a t_a),\ U^\dagger U = \mathbb{I}$$
$$[t_a, t_b] = \iota f_{abc} t_c, \quad \mathrm{Tr}\, t_a = 0$$

More generally,

$$\begin{gathered}
\mathcal{U} : \mathbb{R}^4 \to G, \\
V_m' = \mathcal{U}^{-1} V_m \mathcal{U} + \mathcal{U}^{-1} \partial_m \mathcal{U} \\
F_{mn} \equiv \partial_m V_n - \partial_n V_m + \iota [V_m, V_n] \\
\implies F_{mn}' = \mathcal{U}^{-1} F_{mn} \mathcal{U} \\
\nabla_m \lambda \equiv \partial_m \lambda + \iota [V_m, \lambda], \quad \lambda \in \mathfrak{g} \\
\nabla_m (\iota t^a \lambda^a) = \iota t^a \partial_m \lambda^a - f^{abc} t^a V_m^b \lambda^c
\end{gathered}$$

The g appearing in the definitions in the previous section can be absorbed into V – then it disappears from F, ∇ but we get a $1/g$ factor in front of the Lagrangian.

Including Matter Fields – Superfield description

$$\mathrm{V} = t^a \mathrm{V}^a(x, \theta, \bar{\theta})$$

V^a are real superfields, and we are going to try and generalise the super-gauge-transformations to the non-abelian case. Earlier, we had $\mathrm{V}' = \mathrm{V} + \iota(\Lambda^\dagger - \Lambda)$.

Consider a set of superscalars Φ_i, transforming in the fundamental representation, so

$$\Phi' = e^{\iota\Lambda} \Phi \to \Phi'^i = U^i{}_j \Phi^j$$

We can write $U^i{}_j = e^{\iota\Lambda}$, but Λ are now matrices.

$$e^{\mathrm{V}'} = e^{\iota\Lambda^\dagger} e^{\mathrm{V}} e^{-\iota\Lambda}$$

This is non-trivial to simplify, since matrices don't commute, and we will generally have to use the Baker-Campbell-Hausdorff formula.

Instead, let's work via the gauge-covariant field strength. That has to be modified too, the new definition is:

$$W_\alpha = \bar{D}^2 \left(e^{-\mathrm{V}} D_\alpha e^{\mathrm{V}}\right)$$

And since Λ is still a matrix of chiral superfields, $\bar{D}\Lambda = 0, D\Lambda^\dagger = 0$, using which we can show the invariance:

$$\begin{gathered} W'_\alpha = \bar{D}^2 \left(e^{\iota\Lambda} e^{-\mathrm{V}} e^{-\iota\Lambda^\dagger} D_\alpha \left(e^{\iota\Lambda^\dagger} e^{\mathrm{V}} e^{-\iota\Lambda}\right)\right) \\ = e^{\iota\Lambda} \bar{D}^2 \left(e^{-\mathrm{V}} D_\alpha \left(e^{\mathrm{V}} e^{-\iota\Lambda}\right)\right) \\ = e^{\iota\Lambda} \bar{D}^2 \left(e^{-\mathrm{V}} D_\alpha e^{\mathrm{V}}\right) e^{-\iota\Lambda} + e^{\iota\Lambda} \bar{D}^2 \left(D_\alpha e^{-\iota\Lambda}\right) \end{gathered}$$

Since $\{\bar{D}, D\} \sim \partial_m,\ [\bar{D}, \partial_m] = 0$, and $\bar{D}\Lambda = 0$, the last term vanishes, and we're left with

$$W'_\alpha = \mathcal{U} W_\alpha \mathcal{U}^{-1}$$

We can also introduce the convenient notation:

$$\begin{gathered} \mathcal{A}_\alpha = e^{-\mathrm{V}} D_\alpha e^{\mathrm{V}} \\ \mathcal{A}'_\alpha = \mathcal{U} \mathcal{A}_\alpha \mathcal{U}^{-1} + \mathcal{U} D_\alpha \mathcal{U}^{-1} \end{gathered}$$

Thus $\mathcal{A}$ is a generalisation of the bosonic gauge potential.

Finally, the Action

$$S = \frac{1}{4g^2} \int \mathrm{d}^4x \left[\mathrm{Tr}(W^\alpha W_\alpha)\right]_F + \text{h.c.} + \int \mathrm{d}^4x \left[(\Phi^\dagger)_j (e^V)^j{}_i \Phi^i\right]_D$$

In terms of $\mathcal{A}$, the first term becomes:

$$\frac{1}{2g^2} \mathrm{Tr} \int \mathrm{d}^8 z \mathcal{A}^\alpha \bar{D}^2 \mathcal{A}_\alpha$$

And (without matter) the EoM will be:

$$\delta V : D^\alpha W_\alpha + [\mathcal{A}^\alpha, W_\alpha] = 0$$

The Bianchi identity still holds:

$$D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0$$

Lecture 26 - Susy

Course	Supersymmetry
Date	@March 10, 2025
Status	Completed
Next	Lecture 27, 28 - Susy
Previous	Lecture 25 - Susy

Super-gauge theories with Extended Supersymmetry

Constructing off-shell representations for extended susy, with auxiliary fields and all, becomes increasingly difficult with additional supercharges, so we will avoid that and discuss only on-shell representations, which only close with the use of the equations of motion.

$$\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\} = 2\sigma^m P_m {\delta^i}_j$$

We constructed massless reps by taking a minimum helicity vacuum $|E, \lambda\rangle$ and applying creation operators $\bar{Q}_{1i} \sim a_i^\dagger$ on it.

$\mathcal{N} = 2$

The N=2 scalar multiplet is $(-\frac{1}{2}, 0, \frac{1}{2})$, which is CPT invariant. The N=2 vector multiplet starts from -1, it is $(-1, 2*(-\frac{1}{2}), 0)$, which isn't CPT invariant – we need to combine it with the multiplet starting from $\lambda = 0$ to obtain $N = 2$ Maxwell:

$$(-1, 2*(-\frac{1}{2}), 0) \oplus (0, 2*(\frac{1}{2}), 1)$$

In terms of $N = 1$, this is the CPT-invariant vector multiplet combined with the CPT-invariant chiral multiplet. And we have already discussed how to couple N=1 Maxwell to N=1 chiral! The difference now is that *the chiral multiplet is in the adjoint rep, not the fundamental.*

$$S = \frac{1}{4g^2} \mathrm{Tr} \int \mathrm{d}^4x \, (W^\alpha W_\alpha)_F + \mathrm{h.c.} + \mathrm{Tr} \int \mathrm{d}^4x \, \left(\Phi^\dagger e^{\mathrm{V}} \Phi e^{-\mathrm{V}}\right)_{\mathcal{D}}$$

Note the difference in the kinetic term – this is because of the adjoint representation.

$$\Phi' = e^{\iota\Lambda}\Phi e^{-\iota\Lambda}, \quad \Phi'^{\dagger} = e^{\iota\Lambda^{\dagger}}\Phi^{\dagger}e^{-\iota\Lambda^{\dagger}}$$
$$e^{\mathrm{V}'} = e^{\iota\Lambda^{\dagger}}e^{\mathrm{V}}e^{-\iota\Lambda}$$

Notice how Φ' has U^{-1} whereas V' has $U^{\dagger}$ — which are the same for $\mathrm{SU}(N)$, but not for all gauge groups.

More generally, N=2 gauge theories will be a coupling of N=2 Maxwell with N=2 hypermultiplets, eg the N=2 scalar multiplet — which is just two N=1 scalar multiplets, but they will both have to be in the same representation. So N=2 gauge theories are just a special case of N=1 gauge theories.

$\mathcal{N} = 4$

On-shell, the $\lambda_{\text{min}} = -1$ multiplet is $(-1, 4(-\frac{1}{2}), 6(0), 4(\frac{1}{2}), 1)$ – and that's the only one with $|\lambda| \leq 1$.

$\mathcal{N} = 3$

$$(-1, 3(-\frac{1}{2}), 3(0), \frac{1}{2})$$

Not CPT-invariant, so we need another of these with flipped helicities,

$$(-1, 3(-\frac{1}{2}), 3(0), \frac{1}{2}) \oplus (-\frac{1}{2}, 3(0), 3(\frac{1}{2}), 1)$$

In total, this is simply the unique $|\lambda| \leq 1$ $\mathcal{N} = 4$ multiplet.

Back to $\mathcal{N} = 4$

This can also be considered as $\mathcal{N} = 2$ SYM + an $\mathcal{N} = 2$ hypermultiplet in the adjoint rep. The former contains $(V_m, 2\psi, \phi)$ and the latter contains $(2\psi, 2\phi)$, where ψ is Weyl and ϕ complex. Ofc this can also be understood as $\mathcal{N} = 1$ SYM + 3 hypermultiplets.

Lagrangian

The scalars are ϕ_i^a, $i \in [[1, 3]]$ referring to the 3 hypermultiplets.

The fermion of the vector multiplet will be λ, and the ones in the chiral multiplet ψ_i.

$$\mathcal{L} = \Bigg[-\frac{1}{4g^2}(F^a_{mn})^2 - \nabla^m\phi_i^{a*}\nabla_m\phi_i^a - \iota\bar{\lambda}^2\bar{\sigma}^m\nabla_m\lambda^a - \bar{\psi}_i^a\bar{\sigma}^m\nabla_m\psi_i^a$$
$$- \frac{\iota}{\sqrt{2}}f^{abc}(\bar{\lambda}^a\bar{\psi}_i^c\phi_i^{b*} - \text{h.c.}) - \frac{\iota}{\sqrt{2}}f^{abc}(\epsilon_{ijk}\bar{\psi}_i^a\bar{\psi}_k^c\phi_j^b - \text{h.c.}) - U(\phi)\Bigg]$$
$$U(\phi) = \frac{1}{2g^2}(\mathcal{D}^a(\phi))^2 + |F^{a*}(\phi)|^2 = \frac{1}{2}(f^{abc}\phi_i^{b*}\phi_i^c)^2 + \frac{1}{2}f_{abc}f^a_{de}\epsilon_{ijk}\epsilon^{lmk}\phi_i^b\phi_j^c\phi_l^{d*}\phi_m^{e*}$$

These are the two quartic interactions. And at the end of the day, these 6 real scalar DoF should enter symmetrically, and we will be able to write it that way. The structure constants appear because all these terms arise from commutators of Lie algebra valued fields. Specifically, they arise from the following superpotential:

$$\mathcal{W} = \frac{\iota\sqrt{2}}{3!}\,\mathrm{Tr}\left(\Phi_i[\Phi_j,\Phi_k]\right)\epsilon^{ijk}$$
$$\Phi_i = \iota t^a\Phi_i^a, \mathrm{Tr}\, t^a t^b = 2\delta^{ab}$$
$$\implies \mathcal{W} \sim \frac{\sqrt{2}}{3!}f^{abc}\epsilon^{ijk}\Phi_i^a\Phi_j^b\Phi_k^c$$
$$F_i \sim \frac{\partial\mathcal{W}}{\partial\phi_i}, F_i^*F_i \text{ is the second term in } U(\phi)$$

$\mathcal{N}=4$ susy restricts not only the number and representation of multiplets, but also the superpotential, which is why this is the only superpotential we are considering. There is a unique $\mathcal{N}=4$ SYM (with renormalisability etc constraints taken into account).

We can also go from this to the $\mathcal{N}=2$ SYM by dropping $i=2,3$. This means the superpotential is also 0 — however, the scalar potential will still have the $\mathcal{D}^2$ term, albeit with only $i=1$.

Lecture 27, 28 - Susy

Course	Supersymmetry
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Next	Lecture 29 - Susy
Previous	Lecture 26 - Susy

$\mathcal{N} = 4$ SYM

Manifestly Symmetric Expression

R-Symmetry

The R-symmetry group is $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$. The fundamental rep acts on both the supercharges and the 4 fermions, while the 6D antisymm rep (equivalent to the fundamental so(6) rep) acts on the 6 scalars.

We will label the supercharges as Q_α^A, $A \in [[1, 4]]$, and the fermions as λ_α^A. The scalars will be labelled φ_{AB}, a complex antisymm matrix — that's 6 complex scalars, and it's also not an irrep. But since we're in 4D, we can impose an invariant constraint that limits us to an irrep with 6 *real* scalars:

$$\varphi_{AB} = \tfrac{1}{2}\varepsilon_{ABCD}\bar{\varphi}^{CD}, \quad \bar{\varphi} = \varphi^*$$

The vectors are unchanged, V_m.

Lagrangian

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\,\mathrm{Tr}\Big[&-\frac{1}{2}F_{mn}^2 - \nabla_m\varphi^{AB}\nabla^m\bar{\varphi}_{AB} - \iota\bar{\lambda}_A\bar{\sigma}^m\nabla_m\lambda^A \\ &- \lambda^A[\lambda^B, \bar{\varphi}_{AB}] + \bar{\lambda}_A[\bar{\lambda}_B, \varphi^{AB}] - 2[\varphi^{AB}, \varphi^{CD}][\bar{\varphi}_{AB}, \bar{\varphi}_{CD}]\Big]\end{aligned}$$

Thus the Yukawa terms are just commutators, and the scalar potential is just $\left|[\varphi^{AB}, \varphi^{CD}]\right|^2$.

The Susy transformations

1. $\delta V_m = \iota(\lambda^A\sigma_m\xi_A + \xi^A\sigma_m\bar{\lambda}_A)$
2. $\delta\lambda^A = -\frac{1}{2}F_{mn}\sigma^{mn}\xi^A + 4\iota\sigma^m\nabla_m\varphi^{AB}\bar{\xi}_B - 8[\bar{\varphi}_{BC}, \varphi^{CA}]\xi_B$

This is a little ugly, we have odd non-linear terms occurring. We'll soon see why that is.

3. $\delta\varphi^{AB} = \lambda^{[A}\xi^{B]} + \frac{1}{2}\varepsilon^{ABCD}\bar{\xi}_C\bar{\lambda}_D$

Another source of the same Lagrangian

In 10D, the $\mathcal{N}=1$ SYM is a unique theory with spins ≤ 1.

Q are Majorana-Weyl spinors in 10D,

$$\{Q, \bar{Q}\} \sim \Gamma^m P_m$$
$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}\mathbb{I}$$

In dimensions D, the Gamma matrices are generally matrices of size $2^{\left[\frac{D}{2}\right]}$, acting on spinors with $2^{\left[\frac{D}{2}\right]}$ complex dimensions – with the Majorana condition, $2^{\left[\frac{D}{2}\right]}$ real dimensions. The Weyl condition $(\Gamma_{10}\Psi = \Psi)$ additionally halves the number of components.

So in D=10, we have $2^4 = 16$ fermionic components, which is the same as the number of components in the 4 Weyl spinors in 4D $\mathcal{N}=4$.

The Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^a_{MN}F^{aMN} - \frac{\iota}{2}\bar{\lambda}^a\Gamma^M D_M\lambda^a$$

The claim is that the $\mathcal{N}=4, D=4$ SYM with $\mathrm{SU}(N)$ is the dimensional reduction of $\mathcal{N}=1, D=10$ SYM with $\mathrm{SU}(N)$.

Dimensional Reduction

Consider $X^N = (x^m, y^r), m \in [[0,3]], r \in [[1,6]]$. Then we can split every field over these, and we can also split vectors into 4D vectors and the rest:

$$V_M(X^M) = (V_m(x^m, y^r), \phi_r(x^m, y^r))$$

Then we assume that the r dimensions are compact and "small", or rather that all fields are slowly-varying with respect to the size of these dimensions, and so the y^r dependence of all fields can be dropped (we retain only the zero mode of the Fourier expansion).

Recovering the Lagrangian

We're left with 6 extra fields now. In particular, the field strength also splits:

$$
\begin{gathered}
F_{MN} = (F_{mn}, F_{mr}, F_{rs}) \\
F_{mn} = 2\partial_{[m}V_{n]} + [V_m, V_n] \\
F_{mr} = \partial_m V_r - \cancel{\partial_r V_m} + [V_m, V_r] = \nabla_m \phi_r \\
F_{rs} = [\phi_r, \phi_s]
\end{gathered}
$$

Thus

$$
F_{MN}F^{MN} = F_{mn}F^{mn} + 2\nabla_m \phi_r \nabla^m \phi_r + [\phi_r, \phi_s][\phi^r, \phi^s]
$$

Note that raising or lowering r doesn't matter since they don't have a time – that subspace is Euclidean.

Fermion Terms

The single spinor λ_I will split into $\lambda_\mu^A, A, \mu \in [[1, 4]]$.

$$
\begin{gathered}
D_m \lambda = (\nabla_m \lambda, D_r \lambda) \\
\nabla_m \lambda = \partial_m \lambda + [V_m, \lambda] \\
D_r \lambda = [\phi_r, \lambda]
\end{gathered}
$$

Therefore

$$
-\frac{\iota}{2}\bar{\lambda}^a \Gamma^M D_M \lambda^a = -\frac{\iota}{2}\bar{\lambda}_A \Gamma^m \nabla_m \lambda^A - \frac{\iota}{2}\bar{\lambda}_A \Gamma^r [\phi_r, \lambda^A]
$$

Thus we obtain the Yukawa term.

Some comments

1 – N=2 from 5/6D

Starting from N=1 in 5 or 6D, we can again obtain 4D N=2 using dimensional reduction. Which one you use depends on which fermions you use.

2 – Vacua?

For N=4 SYM,

$$U(\phi) \sim \mathrm{Tr}[\phi_r, \phi_s]^2$$

Assuming a compact group, the structure constants are positive and this potential is ≥ 0. What are the vacua of this theory?

If $[\phi_r, \phi_s] = 0$, the VEVs commute, then the vacuum has 0 energy and susy is unbroken. This is called the Coulomb branch of the vacua of the theory.

3 – Quantum Properties

$N = 4$ SYM has no UV divergences. The β function for $1/g^2$, which at 1-loop takes the general form:

$$\beta_{\text{1-loop}} = \frac{11}{3}N_V - \frac{2}{3}N_\psi - \frac{1}{6}N_S$$

In N=4, $N_V = N, N_\psi = 4N, N_S = 6N$, so this cancels – and in fact the β function cancels at all orders.

(Note that N is a Casimir invariant of the gauge group $\mathrm{SU}(N)$.

Recall that for N=1, the F terms in the action had no renormalisation, only the $\mathcal{D}$ terms did, because the F terms are non-local operators in superspace.

For N=1 SYM, the 1-loop β function seems to be non-zero – however, the Lagrangian is only the F term of $W^\alpha W_\alpha$, so what's going on?

The problem is that SYM is a gauge theory and requires ghosts, so the action does not have only non-local terms – the ghosts give rise to local terms – which is one of the reasons why the β function is non-zero and the coupling is renormalised. However, the non-local action still leads to the β function being exactly calculable.

For N=2, the β function is only non-zero at 1-loop order — this is the generalisation of the theorem for N=1, and can be seen using the explicit N=2 superfield construction, though we will not do so.

In the same vein, N=4 becomes UV-finite.

With N=4, the Lagrangian was

$$\mathrm{Tr}\,\frac{1}{g^2}\left[(W^\alpha W_\alpha)_F + \left(\Phi_i^\dagger e^V \Phi_i e^{-V}\right)_{\mathcal{D}} + c\left(\varepsilon^{ijk}\Phi_i[\Phi_j,\Phi_k]\right)_F\right]$$

Susy connects these three terms together. So while there is a D-term which could be renormalised, but the other two won't be renormalised. So assuming susy is conserved by the quantisation (which we don't actually know how to do), and neglecting the subtleties about ghosts and stuff, the D-term should also not be renormalised, and the theory is UV-finite.

Symmetries under dimensional reduction

Poincare in 10D is $\mathrm{ISO}(1,9)$, which after dimensional reduction reduces to $\mathrm{ISO}(1,3)\times\mathrm{SO}(6)$, which we see is 4D Poincare and the R-symmetry of N=4.

Minimal Supersymmetric Standard Model

Why?

The hierarchy problem. The SM is an effective low-energy theory, but whatever higher scale appears in the correct theory, eg the one which incorporates gravity, it will pull up the Higgs mass scale to itself during renormalisation. Why does that not happen?

$$\Delta M_{\text{scalar}}^2 \sim M_{\text{heavy}}^2$$

A solution is to incorporate susy. The N=1 susy, for example, has no quadratic divergences, only logarithmic, and the renormalisation would become more like:

$$\Delta M_{\text{scalar}}^2 \sim M_{\text{scalar}}^2 \log\Lambda$$

Which is much more manageable no matter what Λ is. It may be M_{heavy}.

How?

We introduce a susy theory with minimal extra fields over the standard model, then figure out a mechanism to break the susy such that the hierarchy problem remains solved but the superpartners are too heavy (or too weakly interacting) to have been observed.

The Particle Content

The matter fields are straightforward, the gauge bosons will use SYM.

Naively one Higgs chiral multiplet gives massless fermions – which we don't see. For a massive fermion, we need two chiral multiplets, so that's what we get.

Then we'll need to make the superpartners heavy, heavier than what we've been able to probe so far.

Lecture 29 - Susy

Course	Supersymmetry
Date	@March 13, 2025
Status	Completed
Previous	Lectures 27, 28 - Susy

MSSM

We have the Higgs scalar, but it is originally a doublet of scalars. In addition, we need its superpartner to be massive, which means a Dirac fermion, which means we need two chiral multiplets.

The fermions will each need their own chiral multiplets, and the gauge bosons will need their own vector multiplets.

Field Content

Superfields	Bosonic	Fermionic	Representation of SU(3)xSU(2)xU(1)_Y
L_i : Left leptons + sleptons	$\tilde{\nu}_{e_{L_i}}, \tilde{e}_{L_i}$	$\nu_{e_{L_i}}, e_{L_i}$	(1,2,-1)
Right leptons + sleptons	$\tilde{e}_{R_i}$	e_{R_i}	(1,1,-2)
Q_i : Left quarks + squarks	$\tilde{u}_{L_i}, \tilde{d}_{L_i}$	u_{L_i}, d_{L_i}	(3,2,1/3)
u_R : Right upper quarks + squarks	$\tilde{u}_{R_i}$	u_{R_i}	(3,1,4/3)
d_R : Right lower quarks + squarks	$\tilde{d}_{R_i}$	d_{R_i}	(3,1,-2/3)
Gluons + gluinos	A_m	$\tilde{A}$ (not a vector)	(8,1,0)
W-bosons + winos	$W_m^{\pm,3}$	$\tilde{W}^{\pm,3}$	(1,3,0)
B-bosons + bino	B_m	$\tilde{B}$	(1,1,0)
H_u : Higgs + Higgsino	H_u^+, H_u^0	$\tilde{H}_u^+, \tilde{H}_u^0$	(1,2,1)
H_d : Other Higgs + Higgsino	H_d^0, H_d^-	$\tilde{H}_d^0, \tilde{H}_d^-$	(1,2,1)

Superpotential

$$\mathcal{W} = y_{u,ij}\bar{u}_R^i Q_L^j H_u + y_{d,ij}\bar{d}_R^i Q_L^j H_d + y_{l,ij}\bar{e}_R^i L^j H_d + \mu H_u H_d$$

We also need to break susy in a *soft* way, i.e. without changing the UV behaviour. Spontaneous SuB is one way to do that — when the vacuum breaks the susy.

Supergravity

If we make supersymmetry local, we will have to introduce a gauge field for this. This will be a Majorana spin-3/2 particle, the gravitino. This can be intuited by the fact that the susy parameter is fermionic, a 4-component spinor,

$$\xi = \begin{pmatrix} \xi_\alpha(x) \\ \bar{\xi}_{\dot{\alpha}}(x) \end{pmatrix}$$

And the gauge field will transform as

$$\psi_m' = \psi_m + \partial_m \xi$$

So it is a product of a vector and a spinor rep, it must be the spin 3/2 rep.

It will have gauge freedom and have 12 off-shell DoF instead of 16 – equivalent to fixing the diagonal elements, like for the metric.

There are two on-shell multiplets in N=1 susy which contain this field, the $(1, 3/2)$ (and chiral partners), and the $(3/2, 2)$ (and chiral partners) multiplets. We are interested in the latter – we call that the on-shell linearised sugra multiplet.

Recall how the graviton occurred in the effective gravity theory. We linearised the Einstein-Hilbert action:

$$S = -\frac{1}{2\mathcal{H}^2}\int \mathrm{d}^4x\sqrt{-g}R(g)$$
$$g_{mn} = \eta_{mn} + h_{mn}$$
$$R_{mn} = 0 \implies$$
$$\Box h_{mn} - \partial_m\partial^k h_{kn} - \partial_n\partial^k h_{km} + \eta_{mn}\partial^k\partial^l h_{kl} + \partial_m\partial_n h - \eta_{mn}\Box h + \mathcal{O}(h^2) = 0$$
$$h = \eta^{mn}h_{mn}$$

The diffeomorphisms are then written as:

$$x'^m = x^m + a^m(x)$$
$$\therefore \delta h_{mn} = -\left(\partial_m a_n + \partial_n a_m\right)$$

One can show that the graviton only has 6 off-shell DoF, by fixing the gauge, eg choosing the de Donder gauge:

$$\partial_m(h^{mn} - \tfrac{1}{2}\eta^{mn}h) = 0$$
$$\iff \Box h = 0, \Box h_{mn} = 0$$

We still have residual gauge freedom, called the on-shell gauge freedom, in terms of parameters satisfying:

$$\Box a_m(x) = 0$$

Using this, we can also set $h = 0$ and set 3 more components of h_{mn} to be 0. So from 6 off-shell components, we go down to 2 on-shell components.

The gravitino is more challenging. We introduce the Rarita-Schwinger action:

$$\gamma^{mnk} \equiv \gamma^{[m}\gamma^{n}\gamma^{k]}$$
$$\mathcal{L} = \iota\bar{\psi}_m\gamma^{mnk}\partial_n\psi_k$$

This is the Dirac analog for spin-3/2 fields. The antisymmetry ensures invariance under

$$\psi'_m = \psi_m + \partial_m\xi$$

▼ Identities for γ^{mnk}

1. $\gamma^{mnk} =$
2. $\gamma^m\gamma^n\gamma^k = \gamma^{mnk} + \eta^{mn}\gamma^k - \eta^{mk}\gamma^n + \eta^{nk}\gamma^m$ (this is easy to show using $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$)

Then

1. Get EoM

$$\gamma^{mnk}\partial_n\psi_k$$

We can contract this with a γ to obtain

$$\partial^m \psi_m - \gamma^m \partial_m (\gamma^p \psi_p) = 0$$

2. Choose either of the following as the gauge choice or the on-shell consequence:

$$\gamma^m \psi_m = 0 \Longleftrightarrow \partial^m \psi_m = 0$$

This will still not get us down to 2 DoF, but instead 8. We will get additional conditions from on-shell gauge invariance, and only then will we get 4 components. Then it's the same as, say, Majorana fermions – 4 components in a 1st order equation lead to 2 helicities.

The on-shell gauge freedom is

$$\gamma^m \partial_m \xi = 0 \implies \Box \xi = 0$$

Full non-linear sugra

This is easiest to do in the connection and veilbein form:

$$\int \mathrm{d}^4 x \epsilon^{mnkl} \epsilon_{abcd} e_m^a e_n^b R_{kl}^{cd}(\omega)$$

Where ω is the connection and R the Reimann tensor, $R \sim \partial\omega + \omega^2$.

Then the gravitino part is simply the Rarita-Schwinger action with the covariant derivative

$$D_m = \partial_n + \frac{1}{4} \omega_n^{ab} \gamma_{ab}$$

Then we can vary and eliminate ω, and we'll get

$$\omega = \omega(e) + \bar{\psi}\gamma\psi$$