# Renormalisation Group Flows

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**Abstract**

Renormalisation Group Flows study how theories transform with changes of scale. In particular, quantum field theories generally have divergent behaviour due to a non-decoupling of high-energy and low-energy physics, and eliminating this behaviour using additional terms necessitates describing the theory using an arbitrary scale. The renormalisation group equation encodes how quantities in the theory transform with this scale. This idea has been developed in a distinct variety of formulations and interpretations, and has also found application in many areas within and beyond Physics. In particular, the renormalisation group is a powerful tool to understand criticality and universal behaviours in Statistical Physics. In this report I survey 3 formulations of the renormalisation group – the Wilsonian approach, the classical approach originating from eliminating divergences, and Stevenson's more general arguments based on dimensional analysis – and briefly discuss some consequences to High-Energy Physics and Statistical Physics.

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# 1 Introduction

## 1.1 History of the Renormalisation Group

From their inception, quantum field theories predicted divergent values for any non-classical corrections – Weisskopf gave the first correct calculation from QED in 1934 [1], which featured a logarithmic divergence caused by high-energy photons.

Many years later, in the late 1940s, Feynman [2], Schwinger [3], Tomonaga [4] and Dyson [5] pioneered a general method to remove divergences – what we now call Renormalisation.

Peterman–Stückleburg [6] and Gell-Mann–Low [7] independently developed the idea of the Renormalisation Group – the former were studying transformations between different ways of defining the coefficients of a perturbative expansion of the S-matrix, while the latter introduced the idea of the propagator varying with the energy scale at which the system is probed. Bogoliubov recognised the connection between these ideas and wrote about them in a textbook published with Shikov [8], which is also where the name "Renormalisation Group" was first introduced. [9] provides an in-depth discussion of these developments.

While the initial ideas of renormalisation were closely tied to regularisation (imposing a momentum cut-off) and quantum field theories, Wilson developed renormalisation in a more general sense [10], building on the block-spin renormalisation introduced by Kadanoff [11]. In the process, he developed operator-product expansions [12], showed how to qualitatively analyse QFTs using their RG flows' fixed points [13], and provided an understanding of confinement in strongly-coupled gauge theories (eg QCD) using RG flows [14]. In [15], he used these tools to solve the Kondo problem [16], setting the stage for the renormalisation group to be used in fields as wide as Cosmology [17], Neuroscience [18], Generative Diffusion Models [19], and of course, in Statistical Physics and the study of critical phenomena and phase transitions [20][21].

## 1.2 Divergences in Quantum Field Theories

The divergent behaviour of field theories can be better understood using the Feynman rules [22] for various processes. In a Feynman diagram, each particle carries a momentum, and interaction vertices enforce momentum conservation – then the possible configurations are integrated over. Often, it is possible to have a diagram where some momenta are completely unconstrained (see Fig 1) – the integral over the momentum can then diverge. Generally the propagator behaves like $\sim k^{-2}$ while the measure is $\sim k^d$ in $d$ dimensions. In the example of Fig 1 in $d = 4$, the integral diverges as $k^2$ for large $k$.

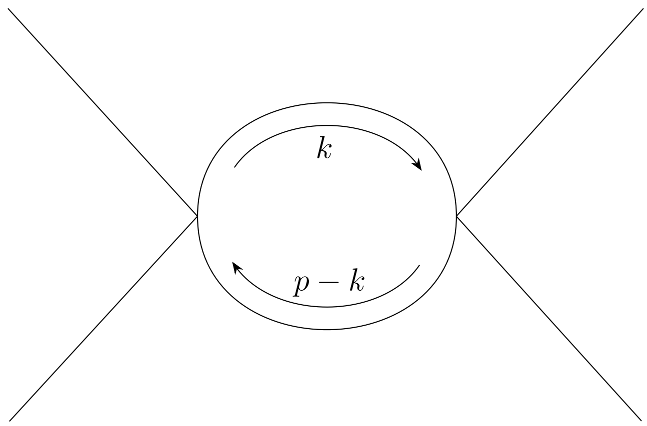



Figure 1: A 1-loop diagram in the $\phi^4$ QFT, which allows for only 4-particle interactions

Thus the divergent behaviour is arising due to configurations with large internal momenta, which is made possible by closed loops in the diagram. In the equivalent position-space representation, this divergence arises in the coincidence limit of the start and end points of a particle's trajectory, since the propagator behaves as $1/|x-y|^2$.

These divergences can thus be controlled by **regulating** the momentum – limiting the integral to be up to some large cut-off, which may be a sharp cut-off or a soft cut-off implemented via a *momentum regulator* in the propagator, such as $e^{-k^2/\Lambda^2}$. The modern viewpoint is that this implies the theory is valid up to some energy scale $\Lambda$ (equivalently, down to some length scale $1/\Lambda$), but is an approximation to some better theory valid at all energy scales (Chapter 1, [20]).

However, for some theories, it was possible to write them in a way that the cut-off dependence was completely removed from all physical observables, and the cut-off could then be safely taken to tend to $\infty$. This necessitated *counterterms* in the action which would cancel the cut-off dependence and hence the divergences. This is an expected result of the decoupling of Physics across large scales – the unknown high-energy Physics should not prevent a complete understanding of low-energy physics.

Quantum theories by their nature tend to disobey this, since all possible trajectories contribute to a quantum calculation, including those involving arbitrarily high energies. There is a small class of quantum field theories from which divergences can be eliminated in a cut-off-independent way. These theories are called **renormalisable**, and were once regarded as the only physical theories, but it is now understood that non-renormalisable theories are valid effective low-energy theories for systems without a complete decoupling of scales.

## 1.3 Renormalisation Group Flow

The various approaches to renormalisation all have one thing in common – they introduce an arbitrary scale, which is different from the cut-off scale. On demanding that physical quantities must not depend on this scale, a differential equation is obtained for how any quantity varies with this scale. This is known as the **Renormalisation Group Equation (RGE)**.

Changes in this scale, known as dilatations, thus leads to various changes in the Lagrangian or Hamiltonian. One may study this process (given by the RGE) as a dynamical system in the space of Hamiltonians. In this interpretation the trajectory of the Hamiltonian is called the **Renormalisation Group Flow**.

## 1.4 General Formulation

The various approaches to the renormalisation group provide different ways to eliminate divergences and provide a physical understanding of quantum field theories, but the central ideas can be put quite succinctly.

Consider a theory involving, or being measured at, some energy scale $Q$, where some arbitrary cut-off scale $\mu$ has already been introduced, perhaps in the process of regularisation. A physical quantity may be written as $F(g_i;Q)_\mu$, where $g_i$ are called "couplings" and parametrise the space of theories.

It is postulated that $F$ does not change with $\mu$, as long as the energy scale being probed at, $Q$, is smaller than $\mu$. This in general may require additional $\mu$ dependence appearing through $g_i \equiv g_i(\mu)$. Thus generally $g_i(\mu)$ will not be directly measurable. The 1**st RG equation** [23] is

$$F(g_i(\mu);Q)_\mu = F(g_i(\mu');Q)_{\mu'} \tag{1}$$

An important consequence comes from scaling all the masses in the theory by $s$. $F$ may have dimensions $d_F$ and change as $s^{d_F}$; $Q \to sQ$. We define $g_i$ to be massless – this can always be done by a redefinition $g' = \mu^{-d_g} g$. But this is equivalent to, say, redefining the kilogram. It should have no effect on the physics, and we can write

$$F(g_i(\mu);Q)_\mu = s^{d_F} F(g_i(\mu);sQ)_\mu \tag{2}$$

Now we use 1 with $\mu' = s\mu$ and obtain the 2**nd RG equation**:

$$F(g_i(\mu'),Q)_{\mu'} = s^{d_F} F(g_i(\mu'/s);sQ)_{\mu'/s} \tag{3}$$

While the RGE is with respect to some arbitrary scale, we are often interested in how quantities change with the actual scale of the experiment. 3 tells us how these two ideas are connected, and is the reason why, often, the behaviour with respect to $\mu$ is taken as a proxy for the behaviour with respect to $Q$ (a more complete explanation of this can be found in section 4.2).

Since the physical observables of the theory can be derived from the action, the action must also change in form with the RG flow, but at the same time remain invariant.

$$S[\sqrt{Z(\mu)}\phi;\mu,g_i(\mu)] = S[\sqrt{Z(\mu')}\phi;\mu',g_i(\mu')] \tag{4}$$

The *Wave-function Renormalisation* $Z(\mu)$ has been introduced – this is the rescaling of the field $\phi$ with the flow. In general, this is necessitated in interacting theories by the fact that $\phi$ acting on the vacuum does not create only single-particle states and is defined as $Z = |\langle\Omega|\phi|p\rangle|^2$ for the sum of single-particle states $|p\rangle$ [24].

# 2 Wilsonian Renormalisation Group

In 1965, Wilson pioneered a new approach to the renormalisation group [10]. He sliced momentum space into regions and analysed the ground states of the high-momenta modes to calculate their effect on the low-momenta modes. At each stage, the higher momenta were, in essence, integrated out of the problem. He ended up with a recursive relation between the coupling constants in each region. This approach was developed into what is now known as a Wilsonian Renormalisation Group Flow.

## 2.1 Wilsonian Effective Action

Aside : It is common in renormalisation approaches to perform a Wick rotation ($t \to \iota t$) and convert the Lorentzian manifold to a Euclidean one. This simplifies implementing the cut-off and makes the path integral convergent ($\iota S \to -S$), at least on a lattice [25]. Under certain assumptions, there is exact correspondence between QFTs related by a Wick rotation [26].

Given a Euclidean theory with a source term:

$$Z[J] = \int D\phi \; e^{-S(\phi)+\int \mathrm{d}^d x J(x)\phi(x)} \tag{5}$$

One may integrate over the high-energy modes:

$$\phi_{</>\Lambda} \equiv \int_{|p|</>\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\phi}(p) e^{\iota px} \tag{6}$$

$$e^{-S_\Lambda(\phi)} \equiv \int D\phi_{>\Lambda} e^{-S(\phi)} \tag{7}$$

Clearly $S_\Lambda(\phi) = S_\Lambda(\phi_{<\Lambda})$.

We define the source to have support in $|p| < \Lambda$ so the integration does not affect it and obtain:

$$Z[J] = \int D\phi_{<\Lambda} \; e^{-S_\Lambda(\phi_{<\Lambda})+\int \mathrm{d}^d x J(x)\phi{<}\Lambda(x)} \tag{8}$$

$S_\Lambda$ is known as the Wilsonian effective action and can contain all manner of non-renormalisable operators:

$$S_\Lambda = \int \mathrm{d}^d x \left( \frac{1}{2} Z_{\phi,\Lambda} (\partial\phi)^2 + \sum_I \Lambda^{d-d_I} g_I O_I \right) \tag{9}$$

Where $O_I$ has mass dimension $d_I \neq d$ and $g_I$ are massless.

Since the divergences in our theory occur due to coincidence of points, $O_I$ must be local [12].

We can also further integrate out the modes between $\Lambda$ and some $\mu < \Lambda$ to obain:

$$S_\mu = \int \mathrm{d}^d x \left( \frac{1}{2} Z_{\phi,\mu} (\partial\phi)^2 + \sum_I \mu^{d-d_I} g_I O_I \right) \tag{10}$$

This allows treating $\Lambda$ as a cut-off while $\mu$ is treated as an arbitrary scale which we adjust to our convenience.

## 2.2 Relevant and Irrelevant Terms

Terms in the Lagrangian – more generally, eigenvectors of the RG flow transformation – can be classified based on how they scale with the RG flow.

Naively, due to the $\mu^{d-d_I}$, $g_I$ will scale with $\mu$ as $\mu^{d_I-d}$. But generally, to dominant order,

$$g_I(\mu) \sim \mu^{\Delta_I - d} \tag{11}$$

[23] $\Delta_I$ is called the scaling or conformal dimension. This may differ from the mass dimension by an amount called the *anomalous dimension*:

$$\gamma_I \equiv \Delta_I - d_I \tag{12}$$

This difference is caused by factors of the wavefunction renormalisation $\sqrt{Z_{\phi,\mu}}$ arising from $n_I$ factors of $\phi$ in $O_I$.

$$\gamma_\phi \equiv -\frac{1}{2}\frac{\mathrm{d}\ln Z_{\phi,\mu}}{\mathrm{d}\ln\mu} \implies Z_{\phi,\mu} \sim \mu^{-2\gamma_\phi} \tag{13}$$

$$\therefore \gamma_I = -n_I \gamma_\phi \tag{14}$$

Thus terms can be classified by how the couplings scale:

**Relevant** if $\Delta_I < d$ – these terms dominate in the small $\mu$ limit.

**Irrelevant** if $\Delta_I > d$ – these terms are negligible in the small $\mu$ limit.

**Marginal** if $\Delta_I = d$ – these terms need to be analysed at higher loop order to determine their behaviour.

It is important to note that this classification is defined in the neighbourhood of a fixed point (see subsection 4.3).

When $\gamma_\phi = 0$ at first order, irrelevant terms are the ones with $g_I$ having negative mass dimensions, which are understood to be non-renormalisable. Thus the non-renormalisable terms can be neglected in this approach.

## 2.3 Renormalisation Group Equations

Define

$$\beta_I(g_I) \equiv \frac{\mathrm{d}g_I}{\mathrm{d}\ln\mu} \tag{15}$$

$\mu$ is an arbitrary scale we introduced, and the partition function is unchanged, so any observable $A$ should be independent of $\mu$:

$$\boxed{0 = \frac{\mathrm{d}A}{\mathrm{d}\ln\mu} = \left(\frac{\partial}{\partial\ln\mu} + \sum_I \beta_I \frac{\partial}{\partial g_I} - 2\gamma_\phi \frac{\partial}{\partial \ln Z_{\phi,\mu}}\right) A} \tag{16}$$

This is the renormalisation group equation.

# 3 Dimensional Regularisation

There is another approach to the RGE, through explicit renormalisation using counterterms and analytic continuation in the spacetime dimension $d$.

Consider an action in dimensions $d = \lfloor d \rfloor - \varepsilon$,

$$S_0 = \int \mathrm{d}^d x \mathcal{L}(\phi_0, \partial\phi_0, m_0^2, g_0^n) \tag{17}$$

$g_0^n$ is the coefficient of $\phi^n$.

Since $d$ is non-integral, the field and the couplings will have fractional dimensions. Introduce an arbitrary scale $\mu$ to absorb these fractional dimensions, then rescale the fields and couplings as such:

$$\phi_0 = \sqrt{Z}_\phi \phi_r, \quad m_0^2 = \frac{Z_m}{Z_\phi} m_r^2, \quad g_0^n = \frac{Z_{g^n}}{Z_\phi^{n/2}} \mu^{\varepsilon(n-2)/2} g_r \tag{18}$$

The bare action splits into

$$S_0 = S_r + S_{\mathrm{ct}} S_r \equiv \int \mathrm{d}^d x \mathcal{L}(\phi_r, \partial\phi_r, m_r^2, \mu^{\varepsilon(n-2)/2} g_r) \tag{19}$$

Now to any given order, one can compute the divergent contributions to observables and cancel them by choosing appropriate $Z$. For example, one may look at the 2-point vertex function:

$$\iota\Gamma_2 = \iota(-p^2 - m_r^2 + \iota\epsilon) - \iota\Sigma(k) \tag{20}$$

$\Sigma(k)$ is the self-energy, including contributions of $(Z_\phi - 1)p^2 + (Z_m - 1)m^2$ from $S_{\mathrm{ct}}$, and loop diagram contributions from both $S_r$ and $S_{\mathrm{ct}}$.

Wick rotating and using some algebraic tricks developed by Schwinger and Feynman, the divergent integrals from the loops are written as integrals over parameters $x, t$. An example from $\phi^3$ is:

$$\int_{t_0}^{\infty} \mathrm{d}t \; t^{1-d/2} e^{-t(p_E^2 x(1-x) + m^2)} \tag{21}$$

Where $t_0 \sim 1/\Lambda^2$ has been introduced as a regulator, since the divergence has been shifted to $t \to 0^+$ where it manifests as the Gamma function of negative integers (for high enough dimensions, including $d = 4$). $p_E$ is a wick-rotated momentum.

Integration by parts further separates this into a finite number of power-law divergent terms $\sim t_0^- n \sim \Lambda^{2n}$ with coefficients of $k^{2m}$, and log divergences. The former can be removed by adding further counterterms to the action:

$$S_{\text{power-law ct}} = \int \mathrm{d}^d x \sum_m d_m \phi \Box^m \phi \tag{22}$$

The log divergence is the case when the integrand has $-1 + \mathcal{O}(\varepsilon)$ powers of $t$. This manifests as $\Gamma(\kappa\varepsilon) \sim \frac{1}{\kappa\varepsilon} - \gamma$ ($\gamma$ is the Euler-Mascharoni constant). Finally, it is these $1/\varepsilon$ divergences that are absorbed into $Z_m, Z_\phi$, etc. At higher orders we need all the $Z$ because the divergence must cancel for all momenta $p$.

The loop contributions must have factors from interactions, and all interactions carry factors of $\mu^{\varepsilon/2}$ in $S_r$.

$$\lim_{\varepsilon\to 0}\mu^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\ln\mu + \mathcal{O}(\varepsilon^2) \tag{23}$$

$$\lim_{\varepsilon\to 0}\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{24}$$

$$\therefore \lim_{\varepsilon\to 0}\Gamma\left(\frac{\varepsilon}{2}\right)\mu^{\varepsilon/2} = \frac{2}{\epsilon} + \ln\mu - \gamma + \mathcal{O}(\varepsilon) \tag{25}$$

Generally, one-loop contributions give factors of $\ln\mu$ and $n$-loop contributions give factors of $(\ln\mu)^n$.

So even when the divergent behaviour is cancelled, $\ln\mu$ terms remain. But $\mu$ was an arbitrary scale that should have no consequence in the $d = 4$ theory – so observables must obey:

$$\boxed{0 = \frac{\mathrm{d}A}{\mathrm{d}\ln\mu} = \left(\frac{\partial}{\partial\ln\mu} + \beta_{m^2}\frac{\partial}{\partial m^2} + \sum_n \beta_{g^n}\frac{\partial}{\partial g^n} - 2\gamma_\phi\frac{\partial}{\partial Z_\phi}\right)A} \tag{26}$$

$$\beta_{m^2} \equiv \frac{\mathrm{d}m^2}{\mathrm{d}\ln\mu}, \quad \beta_{g^n} \equiv \frac{\mathrm{d}g^n}{\mathrm{d}\ln\mu} \tag{27}$$

Once again, we have the RGE.

The dimensional regularisation approach highlights why the RGE is with respect to $\ln\mu$ – power-law divergences are easily removed, but log divergences are more subtle. In addition, when we resum the log behaviour at all orders, we expect polynomial behaviour, since:

$$k^\gamma = e^{\gamma\ln k} = 1 + \gamma\ln k + \frac{1}{2}\gamma^2(\ln k)^2 + \dots \tag{28}$$

So the RGE is well-suited to capture cases where the dominant behaviour of some $A(\mu)$ is $\mu^\gamma$:

$$\frac{\partial A}{\partial\ln\mu} = \gamma A + \dots \tag{29}$$

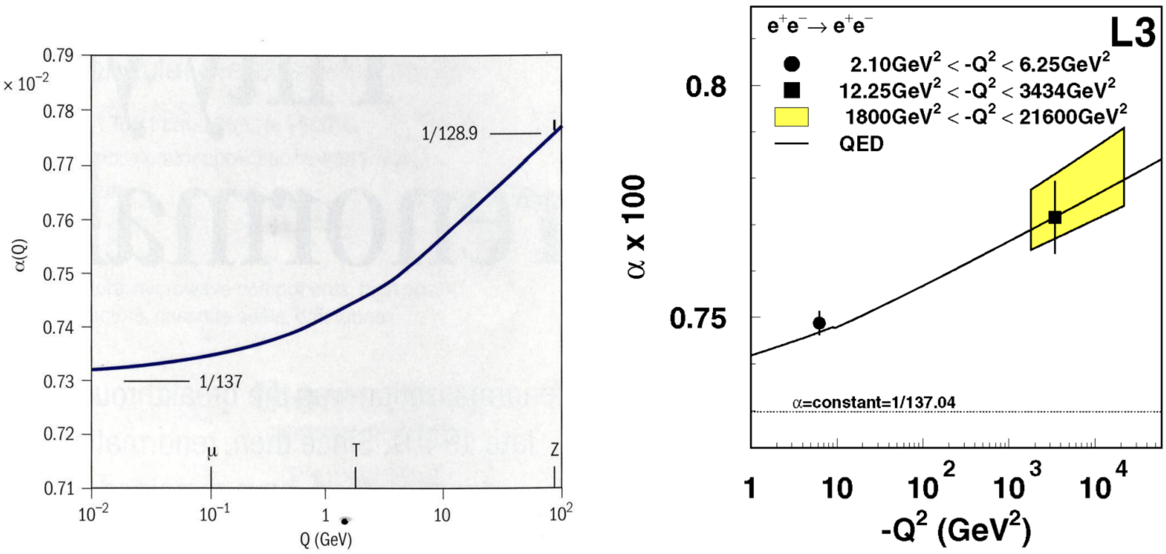
# 4 Consequences of Renormalisation Group

## 4.1 Running of Coupling Constants

The coupling constants now depend on some arbitrary scale, but physical quantities like mass clearly should not.

For mass, this is solved by defining it as the pole of the 2-point function $G_2(p)$, given by the Källén–Lehmann spectral representation [27][28]. While $G_2$ depends on $m(\mu)$, it obeys the RGE and the total $\mu$ dependence cancels, so the physical mass is independent of $\mu$ as well.

The other coupling constants do not have such a definition. $\alpha$, for example, is classically well-defined using the Coulomb interaction, but including quantum corrections, it must be defined using some scattering process. The scattering measurements will be $\mu$-independent, but not necessarily independent of the centre-of-mass. For example, if the electron charge is defined using the repulsion between two electrons, the quantum corrections to this process depend on momentum (see Fig 2).




(a) Predicted by theory [29] (b) Measured with the LEP at CERN [30]

Figure 2: Running of the Fine Structure constant $\alpha$ with energy $Q$

## 4.2 RG Improvement

An important use of the RG is that it allows improving perturbative results. Since $\mu$ is arbitrary, we can pick it to minimise loop corrections.

Practically, by solving the RGEs for the coupling constants at a certain order in perturbation theory, we obtain the coupling constants in terms of their values at some other energy:

$$m(\mu) = m(\mu, m_0, g_0, \mu_0), \quad g(\mu) = g(\mu, m_0, g_0, \mu_0) \tag{30}$$

Substituting these into any observable gives a better approximation for the observable instead of directly substituting $m_0, g_0$, and the $\mu$-dependence cancels.

Since we solve integral equations, this approach helps account for secular effects resulting from perturbative terms which have consistent behaviour over large scales – such terms, integrated, can contribute at lower orders.

## 4.3 Fixed Points and Critical Region

We can use the RGF to study the theory in the $\mu \to \infty$ and $\mu \to 0$ limits, known as the UV and IR limits. Let us first discuss the IR limit – though some ideas are common, since these are related by inverting the direction of the flow.

### 4.3.1 IR Limit

Using 3, we take the limit not by changing $\mu$ but by replacing it with $\mu' = \mu/s$ and letting $s \to \infty$. A consequence is that all mass scales in the theory, $Q \to sQ$, become large. If the theory has a mass gap [31] (i.e. no propagating massless states), then for low enough $\mu'$ the theory will have no propagating degrees of freedom.

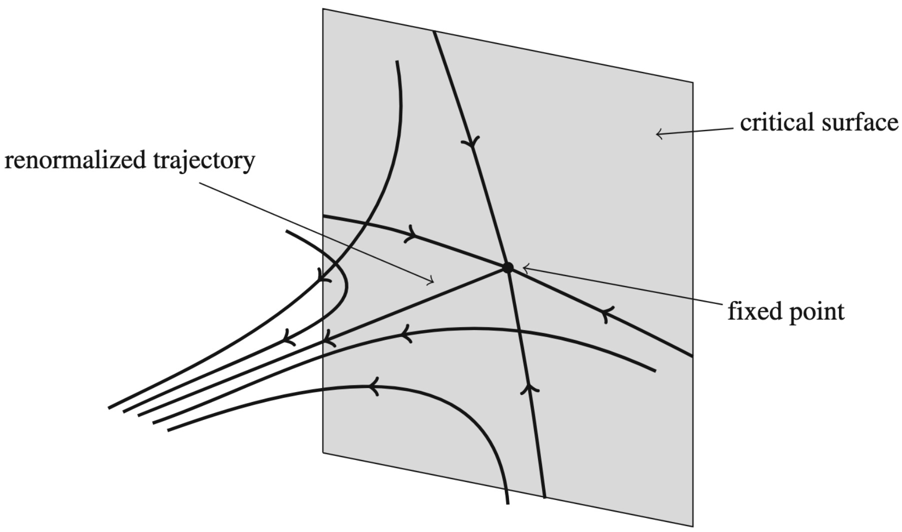



Figure 3: A theory space with 2 irrelevant and 1 relevant directions. The arrows point in direction of decreasing $\mu'$.

The IR limit is non-trivial, then, for the theories which do not have a mass gap. In the space of theories, the subspace with a vanishing mass gap provides a definition of the **critical region** – all theories in this region flow to a non-trivial theory in the IR limit.

### 4.3.2 Fixed Points

It is possible for there to be a fixed point in the space of theories,

$$g_i^* \text{ s.t. } \left.\frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_i\right|_{g_j^*} = \beta(g_j^*) = 0 \tag{31}$$

For the IR limit of a theory to be non-trivial, it will have to be a fixed point in theory space (excluding exotic situations with limit cycles [32]).

The fixed point theory has coupling constants which do not vary with scale – it is a scale-invariant theory. If we additionally assume Poincare invariance, unitarity, and a few other reasonable assumptions, this theory must also have conformal invariance [33] and will be a Conformal Field Theory [34] (NB: a non-perturbative proof for this still lacks, but no counterexamples have been found either).

It can be shown that the energy-momentum tensor of a Poincare- and scale-invariant theory is conformally invariant [23]. Conformal invariance is a powerful constraint on the correlators of the theory and often fixes them to obey power-law behaviours [33].

Recall the definitions of relevant and irrelevant operators in subsection 2.2. These are defined in terms of eigenvectors of the flow linearised about the fixed point. The irrelevant directions will span the critical surface while the relevant perturbations take the theory away from the fixed point (see Fig 3).

### 4.3.3 UV Limit

In the UV limit, the opposite happens – all masses become small relative to $\mu'$. Thus the flow must approach the critical surface – which is occupied by theories with propagating massless states – but it may only approach asymptotically and never cross the surface, see Fig 3.

Theories which diverge to infinite couplings in the UV limit seem problematic, in the sense that they have no hope of being complete theories and must only be effective theories. Weinberg called this the **asymptotic safety** condition: that a QFT which is consistent at all scales must have a UV fixed point [35].

Recall that a renormalisable theory is one which has a well-defined UV limit as the cut-off is taken to infinity. From Fig 3, we can see this implies that only theories lying on the flow generated by relevant perturbations to the fixed point are renormalisable. This is also consistent with understanding irrelevant terms as non-renormalisable terms.

While asymptotically free field theories have asymptotic safety, scalars (like the Higgs) have a tendency to break asymptotic safety [23].

## 4.4 Universality and Critical Phenomena

Fixed points generally only have a finite number of relevant couplings [36], and this is usually a small number. A consequence is that the domain of attraction of these fixed points spans a large, infinite-dimensional subspace of the theory space. In the IR limit, most theories converge to a fixed point – consequently, at large length scales, the classes of theories attracted to the same fixed point exhibit very similar behaviours.

This is the origin of universality classes in theory space. In the theory of critical phenomena and phase transitions, near the transition temperature, the correlation length of the system diverges and large-scale collective behaviours are observed. Moreover, the macroscopic variables obey certain scaling laws with respect to temperature – the exponents in these power laws, at least to dominant order, are the same across groups of remarkably different microscopic theories [37]. This is called **universality**, and is explained using RGF to the IR limit fixed points.

Universality also occurs in the study of discrete dynamical systems with fixed points and limit cycles. Despite originating from very different ODEs, certain measures of their behaviour, such as the ratios of lengths between period-doublings, converge to universal values. This is also well-explained using RGF techniques [38].

# 5 Stevenson's Approach

In 1981, Stevenson presented another approach to the RGE [39]. Using only dimensional analysis and calculus on a general scale-invariant theory, he showed that observables must obey an RGE.

Consider a scale-invariant theory, which must then involve no dimensionalful constants. Consider some dimensionless observable $\rho$ being measured at an energy scale $Q$ – this may be the centre-of-mass energy of a scattering experiment. A uniquely-defined dimensionless function must only depend on ratios of dimensionful objects – but this theory does not have any more dimensionful objects to provide.

The formal resolution of this paradox is that in quantum theories, the Ward identity for scale-invariance has anomalies [40], but a simpler analysis is possible: We conclude $\rho(Q)$ cannot be uniquely-defined (unless it is a constant).

Stevenson proposes this can be done by defining $\rho$ either recursively:

$$\rho(Q) = F\left(\frac{Q}{\mu}, \rho(\mu)\right) \tag{32}$$

$$\frac{\partial}{\partial\mu} F\left(\frac{Q}{\mu}, \rho(\mu)\right) = 0 \tag{33}$$

Or only via $\rho'(Q)$, which by dimensional analysis must be $B/Q$, where $B$ is dimensionless and uniquely-defined:

$$\frac{\mathrm{d}}{\mathrm{d}Q}\rho(Q) = \frac{B(\rho)}{Q} \tag{34}$$

Both cases lead to a one-parameter ambiguity, and it is straightforward to show the cases are completely equivalent.

The recursive definition has the parameter explicit – in the second case the parameter occurs as a constant of integration:

$$\ln Q + C = \int \frac{\mathrm{d}\rho}{B(\rho)} \equiv K(\rho) \tag{35}$$

$$\therefore \ln\frac{Q}{\mu} = K(\rho(Q)) - K(\rho(\mu)) \tag{36}$$

$$\text{If } \exists\Lambda \text{ s.t. } K(\rho(\Lambda)) = 0, \tag{37}$$

$$\text{Then } \rho(Q) = K^{-1}\left(\ln\frac{Q}{\Lambda}\right) \tag{38}$$

It is important to distinguish $\mu$ and $\Lambda$. $\mu$ is comlpetely arbitrary, and is only necessary to be able to write down some of the quantities in the theory – its presence is thus similar to a gauge freedom. $\Lambda$, on the other hand, has physical significance. It is not predicted by the theory, since it is a massive constant – however, it can be measured by experiment. The ambiguity of the theory has been isolated to a measurable constant – but this holds no more information than 36. From 36, too, we can see that we need two experiments for one test of this equation. Both interpretations demand $N$ experiments for $N-1$ tests of the theory. $\rho$ is still independent of both $\mu$ and $\Lambda$.

A dimensionful constant has crept into our theory "because of the absence of a boundary condition" for $\rho(Q)$. (Note that such a boundary condition would also involve a massive constant, the $Q_0$ at which this boundary condition is defined.)

## 5.1 Renormalisation Group Equation

Consider any other quantity $\sigma$ in the theory with mass dimension $D$. Usually, dimensional analysis would allow writing:

$$\sigma(x_1, \ldots, x_n) = x_1^D S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) \tag{39}$$

But with the theory's one-parameter ambiguity, if $\sigma$ depends on $\rho$, it will also require to be written using $\mu$, but in such a way that the $\mu$-dependence cancels.

$$\sigma(x_1, \ldots, x_n) = x_1^D S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}; \frac{x_1}{\mu}, \rho(\mu)\right) \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu} S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}; \frac{x_1}{\mu}, \rho(\mu)\right) = 0 \tag{41}$$

$$\implies \left(\frac{\partial}{\partial \mu} + \frac{\partial \rho(\mu)}{\partial \mu}\frac{\partial}{\partial \rho}\right) S = 0 \tag{42}$$

$$\implies \left(\mu\frac{\partial}{\partial \mu} + B(\rho)\frac{\partial}{\partial \rho}\right) S = 0 \tag{43}$$

$$\implies \boxed{\left(\frac{\partial}{\partial \ln \mu} + B(\rho)\frac{\partial}{\partial \rho}\right) \sigma = 0} \tag{44}$$

Note the similarity with 16 and 27.

## 5.2 Unobservables and Anomalous Dimensions

The equation can be extended to quantities which are unmeasurable and so are allowed to have explicit $\mu$-dependence.

$$\Gamma(x_i; \mu) = \mu^\gamma G(x_i; \mu, \rho(\mu)) \tag{45}$$

Where $G$ is independent of $\mu$. Note that here, $\gamma \in \mathbb{R}$, but Stevenson also shows that the analysis generalises to more general functions of $\mu$, encoded by the *effective power*:

$$\Gamma(x_i; \mu) = \mu^{[\gamma]} G(x_i; \mu, \rho(\mu)) \tag{46}$$

$$\mu^{[\gamma]} \equiv \exp \int^\mu \mathrm{d}\mu' \frac{\gamma(\mu')}{\mu'} \tag{47}$$

Considering $\gamma \in \mathbb{R}$ and $\Gamma$ to have mass dimensions $D$,

$$G(x_i; \mu, \rho(\mu)) = x_1^{D-\gamma} F\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}; \frac{x_1}{\mu}, \rho(\mu)\right) \tag{48}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu} F = 0 \implies \left(\frac{\partial}{\partial \ln \mu} + B(\rho)\frac{\partial}{\partial \rho}\right) F = 0 \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln \mu}\Gamma = \gamma\Gamma + 0 \implies \boxed{\left(\frac{\partial}{\partial \ln \mu} + B(\rho)\frac{\partial}{\partial \rho} - \gamma\right) \Gamma = 0} \tag{50}$$

Thus we have an RGE that also accounts for the anomalous dimension introduced in 12.

## 5.3 Non-Scale-Invariant Theories

Stevenson argues that even if a mass parameter or massive coupling constant is present in the theory, in the context of quantum field theories these bare parameters are necessarily divergent too – thus they cannot provide a mass scale in a way that could invaildate the dimensional-analysis-based arguments.

Rather, such a theory attains a multi-parameter ambiguity, which is easily expressed in terms of the running mass and coupling constants $m(\mu), \alpha(\mu)$. Similar treatment to $\rho$ follows, and the $\beta$ function 15 appears instead of $B(\rho)$:

$$\mu m'(\mu) \equiv \beta_m \equiv -\gamma_m m(\mu) \tag{51}$$

This approach highlights that the quantities in terms of which the ambiguity of the theory is isolated – and the RGE written with – need not be the mass or coupling constants. It is equally valid to use any other quantity to isolate the ambiguity to, and it may even be preferable for non-perturbative applications.

## 5.4 Renormalisation in a Unified Theory

Any experiment can only ever measure a ratio of masses, lengths, etc. We cannot measure an absolute mass scale of the universe – and there is no reason to think such a thing should even exist.

A unified theory of Physics should then expected to be scale-invariant, and have at least one coupling which runs with energy. Stevenson says, "This theory would give a one-parameter Equivalence Class of universes with different (but unobservably so) absolute scales," and that it is this equivalence class, rather than any one of these theories, that would accurately describe our universe.

This interpretation also holds for all our previous discussions for quantum field theories, though they are not the unified theory of nature. The requirement of an arbitrary constant always indicates we are dealing with a theory that is an equivalence class of (more succintly-written) theories.

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