1 Divergences in QFT and the need for Renormalisation

From their inception, quantum field theories predicted divergent values for any non-classical corrections – Weisskopf gave the first correct calculation from QED in 1934 [1], which featured a logarithmic divergence caused by high-energy photons. The divergent behaviour of field theories can be better understood using the Feynman rules [2] for various processes. In a Feynman diagram, each particle carries a momentum, and interaction vertices enforce momentum conservation – then the possible configurations are integrated over. Often, it is possible to have a diagram where some momenta are completely unconstrained (see Fig. 1) – the integral over the momentum can then diverge. Generally the propagator behaves like $\sim k^{-2}$ while the measure is $\sim k^d$ in d dimensions. In the example of Fig. 1 in $d = 4$, the integral diverges as k^2 for large k.

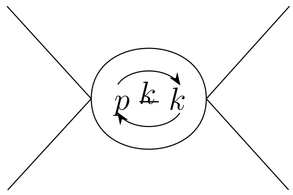

Figure 1: A 1-loop diagram in the ϕ^4 QFT, which allows for only 4-particle interactions

Thus the divergent behaviour is arising due to configurations with large internal momenta, which is made possible by closed loops in the diagram. In the equivalent position-space representation, this divergence arises in the coincidence limit of the start and end points of a particle's trajectory, since the propagator behaves as $1/|x - y|^2$.

These divergences can thus be controlled by **regulating** the momentum – limiting the integral to be up to some large cut-off, which may be a sharp cut-off or a soft cut-off implemented via a *momentum regulator* in the propagator, such as e^{-k^2/Λ^2}. The modern viewpoint is that this implies the theory is valid up to some energy scale Λ (equivalently, down to some length scale $1/\Lambda$), but is an approximation to some better theory valid at all energy scales (Chapter 1, [3]).

However, for some theories, it was possible to write them in a way that the cut-off dependence was completely removed from all physical observables, and the cut-off could then be safely taken to tend to ∞. This necessitated *counterterms* in the action which would cancel the cut-off dependence and hence the divergences. This is an expected result of the decoupling of Physics across large scales – the unknown high-energy Physics should not prevent a complete understanding of low-energy physics.

Quantum theories by their nature tend to disobey this, since all possible trajectories contribute to a quantum calculation, including those involving arbitrarily high energies. There is a small class of quantum field theories from which divergences can be eliminated in a cut-off-independent way. These theories are called **renormalisable**, and were once regarded as the only physical theories, but it is now understood that non-renormalisable theories are valid effective low-energy theories for systems without a complete decoupling of scales.

2 Dimensional Regularisation

A popular approach to the RGE is through explicit renormalisation using counterterms and analytic continuation in the spacetime dimension d.

Consider an action in dimensions $d = \lfloor d \rfloor - \varepsilon$,

$$S_0 = \int \mathrm{d}^d x \mathcal{L}(\phi_0, \partial\phi_0, m_0^2, g_0^n) \tag{1}$$

g_0^n is the coefficient of ϕ^n.

Since d is non-integral, the field and the couplings will have fractional dimensions. Introduce an arbitrary scale μ to absorb these fractional dimensions, then rescale the fields and couplings as such:

$$\phi_0 = \sqrt{Z}_\phi \phi_r, \quad m_0^2 = \frac{Z_m}{Z_\phi} m_r^2, \quad g_0^n = \frac{Z_{g^n}}{Z_\phi^{n/2}} \mu^{\varepsilon(n-2)/2} g_r \tag{2}$$

The bare action splits into

$$S_0 = S_r + S_{\mathrm{ct}} S_r \equiv \int \mathrm{d}^d x \mathcal{L}(\phi_r, \partial\phi_r, m_r^2, \mu^{\varepsilon(n-2)/2} g_r) \tag{3}$$

Now to any given order, one can compute the divergent contributions to observables and cancel them by choosing appropriate Z. For example, one may look at the 2-point vertex function:

$$\imath\Gamma_2 = \imath(-p^2 - m_r^2 + \imath\epsilon) - \imath\Sigma(k) \tag{4}$$

$\Sigma(k)$ is the self-energy, including contributions of $(Z_\phi - 1)p^2 + (Z_m - 1)m^2$ from S_{ct}, and loop diagram contributions from both S_r and S_{ct}.

Wick rotating and using some algebraic tricks developed by Schwinger and Feynman, the divergent integrals from the loops are written as integrals over parameters x, t. An example from ϕ^3 is:

$$\int_{t_0}^{\infty} \mathrm{d}t\, t^{1-d/2} e^{-t(p_E^2 x(1-x)+m^2)} \tag{5}$$

Where $t_0 \sim 1/\Lambda^2$ has been introduced as a regulator, since the divergence has been shifted to $t \to 0^+$ where it manifests as the Gamma function of negative integers (for high enough dimensions, including $d = 4$). p_E is a wick-rotated momentum.

Integration by parts further separates this into a finite number of power-law divergent terms $\sim t_0^- n \sim \Lambda^{2n}$ with coefficients of k^{2m}, and log divergences. The former can be removed by adding further counterterms to the action:

$$S_{\text{power-law ct}} = \int \mathrm{d}^d x \sum_m d_m \phi \Box^m \phi \tag{6}$$

The log divergence is the case when the integrand has $-1 + \mathcal{O}(\varepsilon)$ powers of t. This manifests as $\Gamma(\kappa\varepsilon) \sim \frac{1}{\kappa\varepsilon} - \gamma$ (γ is the Euler-Mascharoni constant). Finally, it is these

$1/\varepsilon$ divergences that are absorbed into Z_m, Z_ϕ, etc. At higher orders we need all the Z because the divergence must cancel for all momenta p.

The loop contributions must have factors from interactions, and all interactions carry factors of $\mu^{\varepsilon/2}$ in S_r.

$$\lim_{\varepsilon\to 0} \mu^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\ln\mu + \mathcal{O}(\varepsilon^2) \tag{7}$$

$$\lim_{\varepsilon\to 0} \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{8}$$

$$\therefore \lim_{\varepsilon\to 0} \Gamma\left(\frac{\varepsilon}{2}\right)\mu^{\varepsilon/2} = \frac{2}{\epsilon} + \ln\mu - \gamma + \mathcal{O}(\varepsilon) \tag{9}$$

Generally, one-loop contributions give factors of $\ln\mu$ and n-loop contributions give factors of $(\ln\mu)^n$.

So even when the divergent behaviour is cancelled, $\ln\mu$ terms remain. But μ was an arbitrary scale that should have no consequence in the $d = 4$ theory – so observables must obey:

$$\boxed{0 = \frac{\mathrm{d}A}{\mathrm{d}\ln\mu} = \left(\frac{\partial}{\partial\ln\mu} + \beta_{m^2}\frac{\partial}{\partial m^2} + \sum_n \beta_{g^n}\frac{\partial}{\partial g^n} - 2\gamma_\phi\frac{\partial}{\partial Z_\phi}\right)A} \tag{10}$$

$$\beta_{m^2} \equiv \frac{\mathrm{d}m^2}{\mathrm{d}\ln\mu}, \quad \beta_{g^n} \equiv \frac{\mathrm{d}g^n}{\mathrm{d}\ln\mu} \tag{11}$$

3 Fixed Points and Critical Region

We can use the RGF to study the theory in the $\mu\to\infty$ and $\mu\to 0$ limits, known as the UV and IR limits. Let us first discuss the IR limit – though some ideas are common, since these are related by inverting the direction of the flow.

3.1 IR Limit

We take the limit not by changing μ but by replacing it with $\mu' = \mu/s$ and letting $s\to\infty$. A consequence is that all mass scales in the theory, $Q \to sQ$, become large. If the theory has a mass gap [4] (i.e. no propagating massless states), then for low enough μ' the theory will have no propagating degrees of freedom.

The IR limit is non-trivial, then, for the theories which do not have a mass gap. In the space of theories, the subspace with a vanishing mass gap provides a definition of the **critical region** – all theories in this region flow to a non-trivial theory in the IR limit.

3.2 Fixed Points

It is possible for there to be a fixed point in the space of theories,

$$g_i^* \text{ s.t. } \left.\frac{\mathrm{d}}{\mathrm{d}\ln\mu}g_i\right|_{g_j^*} = \beta(g_j^*) = 0 \tag{12}$$

For the IR limit of a theory to be non-trivial, it will have to be a fixed point in theory space (excluding exotic situations with limit cycles [5]).

The fixed point theory has coupling constants which do not vary with scale – it is a scale-invariant theory. If we additionally assume Poincare invariance, unitarity, and a few other reasonable assumptions, this theory must also have conformal invariance [6] and will be a Conformal Field Theory [7] (NB: a non-perturbative proof for this still lacks, but no counterexamples have been found either).

4 Running the code

The code requires the FeynCalc package to work, but the command to install it is contained within the code. If FeynCalc is already installed, that command should be commented out and the next command (to import the package directly) should be uncommented.

The code also requires a set of model files, which have been included in the Phi4 folder with the code. This folder needs to be present in the same directory as the code.

The code is functioning perfectly on my device, but considering the multiple dependencies, there may be issues on other devices. Please contact me if any issues arise.

References

[1] V. Weisskopf. "Über die Selbstenergie des Elektrons". In: *Zeitschrift für Physik* 89.1 (1 Jan. 1934), pp. 27–39. DOI: `10.1007/BF01333228`.

[2] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Reading, USA: Addison-Wesley, 1995. DOI: `10.1201/9780429503559`.

[3] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. 4th ed. Oxford, New York: Clarendon Press, 2002. xx, 1054.

[4] Vakhtang Gogokhia and Gergely Gabor Barnaföldi. *The Mass Gap and Its Applications*. Hackensack, USA: World Scientific, 2013. DOI: `10.1142/9789814440714_bmatter`.

[5] K. M. Bulycheva and A. S. Gorsky. "Limit Cycles in Renormalization Group Dynamics - IOPscience". In: (Feb. 1, 2014). DOI: `10.3367/UFNe.0184.201402g.0182`.

[6] Alexander M. Polyakov. "Conformal Symmetry of Critical Fluctuations". In: *JETP Lett.* 12 (1970), pp. 381–383.

[7] Yu Nakayama. "Scale Invariance vs Conformal Invariance". In: *Physics Reports*. Scale Invariance vs Conformal Invariance 569 (Mar. 28, 2015), pp. 1–93. DOI: `10.1016/j.physrep.2014.12.003`.