Renormalisation Group Flows

Why Theories Flow and Constants Run

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Outline

Divergences in QFT

From their inception, QFT predictions were divergent. [1]

Feynman diagrams illustrate this: Loops give rise to unconstrained momenta.

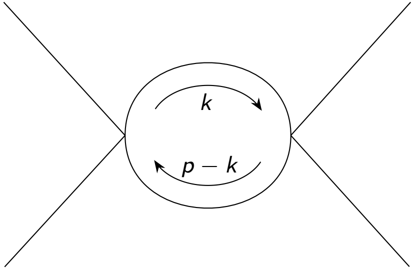


Figure: A 1-loop diagram in the ϕ^4 QFT, which allows for only 4-particle interactions

The divergence is due to large momenta – so they can be made finite by introducing a cut-off Λ in the integral over momentum.

Introduce Λ-dependent *counterterms* in the theory to cancel the divergent behaviour.

The new theory makes finite predictions independent of Λ.

Theories from which divergences can be eliminated in a cut-off-independent way are called **renormalisable**.

Decoupling of scales refers to the Physics at one scale not affecting a system at another scale.

Used implicitly all the time:

- Quantum phenomena can be neglected in mechanics
- Studying atomic physics without worrying about nuclear structure
- Inrafred radiation doesn't affect circuits

Decoupling suggests a theory can be correct at low-energy without knowledge of the exact high-energy Physics. Low-energy predictions should be independent of the cut-off. Thus decoupling suggests renormalisability – the Physics shouldn't change with the cut-off.

What of non-renormalisable theories? They are valid low-energy theories but don't have complete decoupling of scales.

Terms in the Lagrangian can depend on Λ, but physical quantities must not – demanding this gives a DE called the **Renormalisation Group Equation (RGE)**.

Changes in Λ (called dilatations) form the "Renormalisation Group" (technically a semi-group).

The trajectory of the Hamiltonian in the space of Hamiltonians is called the **Renormalisation Group Flow**.

Consider an observable F depending on cut-off Λ, dimensionless couplings $g_i(\Lambda)$ and energy scale Q. The 1**st RG equation** is

$$F(g_i(\Lambda); Q)_\Lambda = F(g_i(\Lambda'); Q)_{\Lambda'} \tag{1}$$

Rescale all masses in the theory by s: $F \to s^{d_F} F$, $Q \to sQ$, and set $\Lambda' = s\Lambda$ to get the 2**nd RG equation**:

$$F(g_i(\Lambda'), Q)_{\Lambda'} = s^{d_F} F(g_i(\Lambda'/s); sQ)_{\Lambda'/s} \tag{2}$$

A Bit of History

In the late 1940s, Feynman [2], Schwinger [3], Tomonaga [4] and Dyson [5] pioneered Renormalisation, a general method to remove divergences.

Peterman–Stückleburg [6] and Gell-Mann–Low [7] independently developed the idea of the Renormalisation Group. Bogoliubov recognised the connection between these ideas and wrote about them in a textbook published with Shikov [8], which is also where the name "Renormalisation Group" was first introduced.

While the initial ideas of renormalisation were closely tied to regularisation (imposing a momentum cut-off) and quantum field theories, Wilson developed renormalisation in a more general sense [9]. In [10], he used these tools to solve the Kondo problem [11], setting the stage for the renormalisation group to be used in fields as varied as Cosmology [12], Neuroscience [13], Generative Diffusion Models [14], and of course, in Statistical Physics and the study of critical phenomena and phase transitions [15][16].

Outline

Dimensionless Observable in Scale-Invariant Theory

Consider a scale-invariant theory, which must then involve no dimensionalful constants. Consider some dimensionless observable ρ being measured at an energy scale Q – this may be the centre-of-mass energy of a scattering experiment. A uniquely-defined dimensionless function must only depend on ratios of dimensionful objects – but this theory does not have any more dimensionful objects to provide.

We conclude $\rho(Q)$ cannot be uniquely-defined.

Stevenson proposes this can be done by defining ρ either recursively:

$$\rho(Q) = F\left(\frac{Q}{\mu}, \rho(\mu)\right) \tag{3}$$

$$\frac{\partial}{\partial\mu} F\left(\frac{Q}{\mu}, \rho(\mu)\right) = 0 \tag{4}$$

Or only via $\rho'(Q)$, which by dimensional analysis must be B/Q, where B is dimensionless and uniquely-defined:

$$\frac{\mathrm{d}}{\mathrm{d}Q}\rho(Q) = \frac{B(\rho)}{Q} \tag{5}$$

Both cases lead to a one-parameter ambiguity, and it is straightforward to show the cases are completely equivalent.

The recursive definition has the parameter explicit – in the second case the parameter occurs as a constant of integration:

$$\ln Q + C = \int \frac{\mathrm{d}\rho}{B(\rho)} \equiv K(\rho) \tag{6}$$

$$\therefore \ln \frac{Q}{\mu} = K(\rho(Q)) - K(\rho(\mu)) \tag{7}$$

$$\text{If } \exists \Lambda \text{ s.t. } K(\rho(\Lambda)) = 0, \tag{8}$$

$$\text{Then } \rho(Q) = K^{-1}\left(\ln \frac{Q}{\Lambda}\right) \tag{9}$$

A dimensionful constant has crept into our theory "because of the absence of a boundary condition" for $\rho(Q)$.

Consider any other quantity σ in the theory with mass dimension D. Usually, dimensional analysis would allow writing:

$$\sigma(x_1, \ldots, x_n) = x_1^D S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) \tag{10}$$

But with the theory's one-parameter ambiguity, if σ depends on ρ, it will also require to be written using μ, but in such a way that the μ-dependence cancels.

$$\sigma(x_1, \ldots, x_n) = x_1^D S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}; \frac{x_1}{\mu}, \rho(\mu)\right) \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu} S\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}; \frac{x_1}{\mu}, \rho(\mu)\right) = 0 \tag{12}$$

$$\implies \left(\frac{\partial}{\partial \mu} + \frac{\partial \rho(\mu)}{\partial \mu}\frac{\partial}{\partial \rho}\right) S = 0 \tag{13}$$

$$\implies \left(\mu\frac{\partial}{\partial \mu} + B(\rho)\frac{\partial}{\partial \rho}\right) S = 0 \tag{14}$$

$$\implies \boxed{\left(\frac{\partial}{\partial \ln \mu} + B(\rho)\frac{\partial}{\partial \rho}\right) \sigma = 0} \tag{15}$$

The equation can be extended to quantities which are unmeasurable and so are allowed to have explicit μ-dependence.

$$\Gamma(x_i;\mu) = \mu^{\gamma} G(x_i;\mu,\rho(\mu)) \tag{16}$$

Where G is independent of μ.

Considering $\gamma \in \mathbb{R}$ and Γ to have mass dimensions D,

$$G(x_i;\mu,\rho(\mu)) = x_1^{D-\gamma} F\left(\frac{x_2}{x_1},\ldots,\frac{x_n}{x_1};\frac{x_1}{\mu},\rho(\mu)\right) \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu}F = 0 \implies \left(\frac{\partial}{\partial \ln\mu} + B(\rho)\frac{\partial}{\partial\rho}\right)F = 0 \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\Gamma = \gamma\Gamma + 0 \implies \boxed{\left(\frac{\partial}{\partial \ln\mu} + B(\rho)\frac{\partial}{\partial\rho} - \gamma\right)\Gamma = 0} \tag{19}$$

Stevenson argues that even if a mass parameter or massive coupling constant is present in the theory, in the context of quantum field theories these bare parameters are necessarily divergent too – thus they cannot provide a mass scale in a way that could invaildate the dimensional-analysis-based arguments.

Renormalisation in a Unified Theory

Any experiment can only ever measure a ratio of masses, lengths, etc. We cannot measure an absolute mass scale of the universe – and there is no reason to think such a thing should even exist.

A unified theory of Physics should then expected to be scale-invariant, and have at least one coupling which runs with energy. Stevenson says, "This theory would give a one-parameter Equivalence Class of universes with different (but unobservably so) absolute scales," and that it is this equivalence class, rather than any one of these theories, that would accurately describe our universe.

In 1965, Wilson pioneered a new approach to the renormalisation group [9]. He sliced momentum space into regions and analysed the ground states of the high-momenta modes to calculate their effect on the low-momenta modes. At each stage, the higher momenta were, in essence, integrated out of the problem. He ended up with a recursive relation between the coupling constants in each region. This approach was developed into what is now known as a Wilsonian Renormalisation Group Flow.

Given a Euclidean theory with a source term:

$$Z[J] = \int D\phi \; e^{-S(\phi)+\int \mathrm{d}^d x J(x)\phi(x)} \tag{20}$$

One may integrate over the high-energy modes:

$$\phi_{<\Lambda} \equiv \int_{|p|<\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\phi}(p) e^{\iota p x} \tag{21}$$

$$\phi_{>\Lambda} \equiv \int_{|p|>\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{\phi}(p) e^{\iota p x} \tag{22}$$

$$e^{-S_\Lambda(\phi)} \equiv \int D\phi_{>\Lambda} \; e^{-S(\phi)} \tag{23}$$

$$\implies Z[J] = \int D\phi_{<\Lambda} \; e^{-S_\Lambda(\phi_{<\Lambda})+\int \mathrm{d}^d x J(x)\phi_{<\Lambda}(x)} \tag{24}$$

S_Λ is known as the Wilsonian effective action and can contain all manner of operators:

$$S_\Lambda = \int \mathrm{d}^d x \left(\frac{1}{2} Z_{\phi,\Lambda} (\partial \phi)^2 + \sum_I \Lambda^{d-d_I} g_I O_I \right) \tag{25}$$

Where O_I has mass dimension $d_I \neq d$ and g_I are massless.

Since the divergences in our theory occur due to coincidence of points, O_I must be local [18].

$\sqrt{Z_{\phi,\Lambda}}$ is the wavefunction renormalisation.

Naively, due to the Λ^{d-d_I}, g_I will scale with Λ as Λ^{d_I-d}.

Instead, to dominant order, [19]

$$g_I(\Lambda) \sim \Lambda^{\Delta_I-d} \tag{26}$$

Δ_I is called the scaling or conformal dimension. This may differ from the mass dimension by an amount called the *anomalous dimension*:

$$\gamma_I \equiv \Delta_I - d_I \tag{27}$$

This difference is caused by factors of the wavefunction renormalisation $\sqrt{Z_{\phi,\Lambda}}$ arising from factors of ϕ in O_I.

Terms in the Action can be classified based on how they scale with the RG flow.

- **Relevant** if $\Delta_I < d$ – these terms dominate in the small Λ limit.
- **Irrelevant** if $\Delta_I > d$ – these terms are negligible in the small Λ limit.
- **Marginal** if $\Delta_I = d$ – these terms need to be analysed at higher loop order to determine their behaviour.

It is important to note that this classification is defined in the neighbourhood of a fixed point.

Define the β functions:

$$\beta_I(g_J) \equiv \frac{\mathrm{d}g_I}{\mathrm{d}\ln\Lambda} \tag{28}$$

Any physical observable $A(\Lambda, g_I, \phi)$ should be independent of Λ:

$$\boxed{0 = \frac{\mathrm{d}A}{\mathrm{d}\ln\Lambda} = \left(\frac{\partial}{\partial\ln\Lambda} + \sum_I \beta_I \frac{\partial}{\partial g_I} - 2\gamma_\phi \frac{\partial}{\partial \ln Z_{\phi,\Lambda}}\right) A} \tag{29}$$

This is the renormalisation group equation.

Why Logarithms?

1. Power law divergences are removed by analytic continuation and we are left with log divergences.
2. Resumming log divergences at all order gives a power law scaling. Thus logs are very natural to give nice scaling laws.
3. My favourite: Logs measure ratios, and we can only measure ratios.

Consequence: The RGE is wrt Λ, or μ, or Q, whichever you want – but really it's always in terms of s.

We can use the RGF to study the theory in the $\Lambda \to \infty$ and $\Lambda \to 0$ limits, known as the UV and IR limits.

As $\Lambda \to 0$, all masses become large wrt Λ. Theories with a mass gap have a trivial IR limit.

Theories without a mass gap comprise the **critical region**. These will limit to a non-trivial theory.

It is possible for there to be a fixed point in the space of theories,

$$g_I^* \text{ s.t. } \left.\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda} g_I\right|_{g_J^*} = \beta_I(g_J^*) = 0 \tag{30}$$

For the IR limit of a theory to be non-trivial, it will have to be a fixed point in theory space (excluding exotic situations with limit cycles [21]).

The fixed point theory has coupling constants which do not vary with scale – it is a scale-invariant theory and will be a Conformal Field Theory [22].

The irrelevant directions will span the critical surface while the relevant perturbations take the theory away from the fixed point.

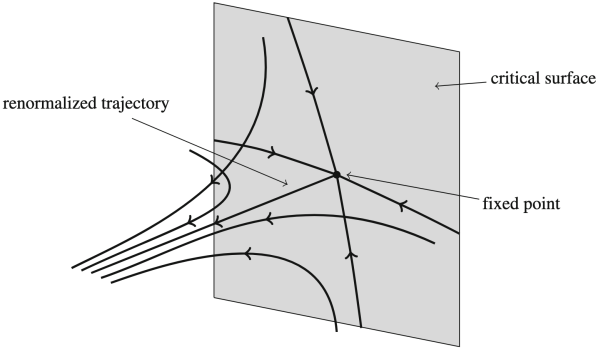


Figure: A theory space with 2 irrelevant and 1 relevant directions. The arrows point in direction of decreasing Λ.

In the UV limit, all masses become small relative to Λ – all theories asymptotically approach the critical region.

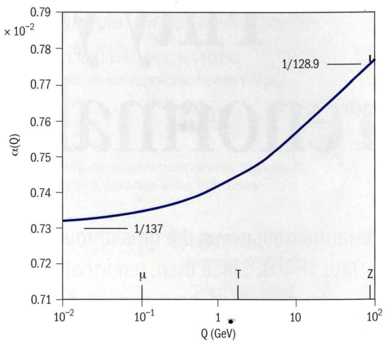
Recall that a renormalisable theory is one which has a well-defined UV limit as the cut-off is taken to infinity. From Fig. 2, we can see this implies that only theories lying on the flow generated by relevant perturbations to the fixed point are renormalisable.

Weinberg called this the **asymptotic safety** condition: that a QFT which is consistent at all scales must have a UV fixed point [24].

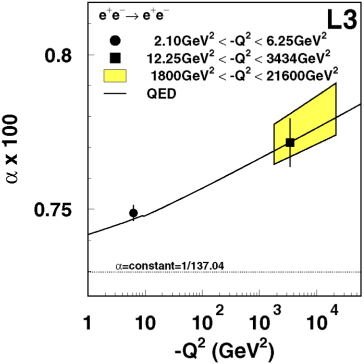
The coupling constants now depend on some arbitrary scale, but physical quantities like mass clearly should not.

For mass, this is solved by defining it as the pole of the 2-point function $G_2(p)$, given by the Källén–Lehmann spectral representation [25][26]. While G_2 depends on $m(\Lambda)$, it obeys the RGE and the total Λ dependence cancels, so the physical mass is independent of Λ as well.

The other coupling constants do not have such a definition. α, for example, is classically well-defined using the Coulomb interaction, but including quantum corrections, it must be defined using some scattering process.



(a) Predicted by theory [27]



(b) Measured with the LEP at CERN [28]

Figure: Running of the Fine Structure constant α with energy Q

Fixed points generally only have a finite number of relevant couplings [29], and this is usually a small number. A consequence is that the domain of attraction of these fixed points spans a large, infinite-dimensional subspace of the theory space.

In the theory of critical phenomena and phase transitions, near the transition temperature, the correlation length of the system diverges and large-scale collective behaviours are observed. Moreover, the macroscopic variables obey certain scaling laws with respect to temperature – the exponents in these power laws, at least to dominant order, are the same across groups of remarkably different microscopic theories [30]. This is called **universality**.

Universality also occurs in the study of discrete dynamical systems with fixed points and limit cycles. Despite originating from very different ODEs, certain measures of their behaviour, such as the ratios of lengths between period-doublings, converge to universal values.

[1] V. Weisskopf. "Über die Selbstenergie des Elektrons". In: *Zeitschrift für Physik* 89.1 (1 Jan. 1934), pp. 27–39. DOI: 10.1007/BF01333228.
[2] Richard P. Feynman. "Relativistic Cut-Off for Quantum Electrodynamics". In: *Physical Review* 74.10 (Nov. 15, 1948), pp. 1430–1438. DOI: 10.1103/PhysRev.74.1430.
[3] Julian Schwinger. "Quantum Electrodynamics. III. The Electromagnetic Properties of the Electron—Radiative Corrections to Scattering". In: *Physical Review* 76.6 (Sept. 15, 1949), pp. 790–817. DOI: 10.1103/PhysRev.76.790.
[4] Zirô Koba and Sin-itirô Tomonaga. "On Radiation Reactions in Collision Processes. I: Application of the Self-Consistent Subtraction Method to the Elastic Scattering of an Electron*". In: *Progress of Theoretical Physics* 3.3 (Sept. 1, 1948), pp. 290–303. DOI: 10.1143/ptp/3.3.290.
[5] F. J. Dyson. "The Radiation Theories of Tomonaga, Schwinger, and Feynman". In: *Physical Review* 75.3 (Feb. 1, 1949), pp. 486–502. DOI: 10.1103/PhysRev.75.486.
[6] E. C. G. Stueckelberg and A. Petermann. "La normalisation des constantes dans la théorie des quanta". In: (Sept. 15, 1953). DOI: 10.5169/SEALS-112426.
[7] M. Gell-Mann and F. E. Low. "Quantum Electrodynamics at Small Distances". In: *Physical Review* 95.5 (Sept. 1, 1954), pp. 1300–1312. DOI: 10.1103/PhysRev.95.1300.
[8] N. N. (Nikola Nikolaevich) Bogoliubov. *Introduction to the Theory of Quantized Fields*. New York : Interscience Publishers, 1959. 746 pp.

References II

[9] Kenneth G. Wilson. "Model Hamiltonians for Local Quantum Field Theory". In: *Physical Review* 140 (2B Oct. 25, 1965), B445–B457. DOI: 10.1103/PhysRev.140.B445.

[10] Kenneth G. Wilson. "The Renormalization Group: Critical Phenomena and the Kondo Problem". In: *Reviews of Modern Physics* 47.4 (Oct. 1, 1975), pp. 773–840. DOI: 10.1103/RevModPhys.47.773.

[11] Jun Kondo. "Resistance Minimum in Dilute Magnetic Alloys". In: *Progress of Theoretical Physics* 32.1 (July 1, 1964), pp. 37–49. DOI: 10.1143/PTP.32.37.

[12] Alessia Platania. "From Renormalization Group Flows to Cosmology". In: *Frontiers in Physics* 8 (May 27, 2020), p. 188. DOI: 10.3389/fphy.2020.00188. arXiv: 2003.13656 [gr-qc].

[13] J. Samuel Sooter et al. "Cortex Deviates from Criticality during Action and Deep Sleep: A Temporal Renormalization Group Approach". In: (June 1, 2024). DOI: 10.1101/2024.05.29.596499.

[14] Artan Sheshmani et al. *Renormalization Group Flow, Optimal Transport and Diffusion-based Generative Model*. Mar. 1, 2024. DOI: 10.48550/arXiv.2402.17090. arXiv: 2402.17090 [cond-mat]. URL: http://arxiv.org/abs/2402.17090 (visited on 03/10/2025). Pre-published.

[15] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. 4th ed. Oxford, New York: Clarendon Press, 2002. xx, 1054.

[16] Uwe C. Tauber. "Renormalization Group: Applications in Statistical Physics". In: *Nuclear Physics B - Proceedings Supplements* 228 (July 2012), pp. 7–34. DOI: 10.1016/j.nuclphysbps.2012.06.002. arXiv: 1112.1375 [cond-mat].

References III

[17] Konrad Osterwalder and Robert Schrader. "Axioms for Euclidean Green's Functions". In: ().

[18] Kenneth G. Wilson. "Non-Lagrangian Models of Current Algebra". In: *Physical Review* 179.5 (Mar. 25, 1969), pp. 1499–1512. DOI: 10.1103/PhysRev.179.1499.

[19] Timothy J Hollowood. *Renormalization Group and Fixed Points: In Quantum Field Theory*. SpringerBriefs in Physics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013. DOI: 10.1007/978-3-642-36312-2.

[20] Vakhtang Gogokhia and Gergely Gabor Barnaföldi. *The Mass Gap and Its Applications*. Hackensack, USA: World Scientific, 2013. DOI: 10.1142/9789814440714_bmatter.

[21] K. M. Bulycheva and A. S. Gorsky. "Limit Cycles in Renormalization Group Dynamics - IOPscience". In: (Feb. 1, 2014). DOI: 10.3367/UFNe.0184.201402g.0182.

[22] Yu Nakayama. "Scale Invariance vs Conformal Invariance". In: *Physics Reports*. Scale Invariance vs Conformal Invariance 569 (Mar. 28, 2015), pp. 1–93. DOI: 10.1016/j.physrep.2014.12.003.

[23] Alexander M. Polyakov. "Conformal Symmetry of Critical Fluctuations". In: *JETP Lett.* 12 (1970), pp. 381–383.

[24] Steven Weinberg. "Critical Phenomena for Field Theorists". In: DOI: 10.1007/978-1-4684-0931-4_1.

[25] Gunnar Källén. "On the Definition of the Renormalization Constants in Quantum Electrodynamics". In: *Helvetica Physica Acta* 25.IV (1952), p. 417. DOI: 10.5169/seals-112316.

[26] H. Lehmann. "Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder". In: *Il Nuovo Cimento (1943-1954)* 11.4 (4 Jan. 18, 2008), pp. 342–357. DOI: 10.1007/BF02783624.

[27] cern. "Fifty Years of the Renormalization Group". In: *CERN Courier* (Aug. 29, 2001).

[28] P. Achard et al. "Measurement of the Running of the Electromagnetic Coupling at Large Momentum-Transfer at LEP". In: *Physics Letters B* 623.1 (Sept. 8, 2005), pp. 26–36. DOI: 10.1016/j.physletb.2005.07.052.

[29] Damiano Anselmi. "Renormalization of a Class of Non-Renormalizable Theories". In: *Journal of High Energy Physics* 2005.07 (July 28, 2005), pp. 077–077. DOI: 10.1088/1126-6708/2005/07/077. arXiv: hep-th/0502237.

[30] P. Pfeuty, G. Toulouse, and Eytan Domany. "Introduction to the Renormalization Group and to Critical Phenomena". In: *Physics Today* 31.4 (Apr. 1, 1978), pp. 57–58. DOI: 10.1063/1.2994997.