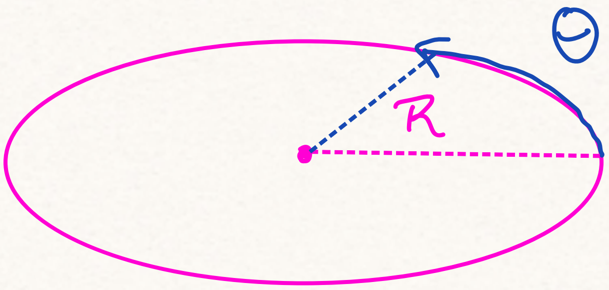
Rei Campo HOMEWORK DUE 9/15/25

Exercise 1: Calculate the moment of inertia tensor for an infinitely thin, homogenous disk.

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$



Because the disk is infinitely thin (using z as the height of the disk), height ~ ∂z

I am going to use cylindrical coordinates for this exercise.

Cylindrical

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$z = z$

$$\rho = \frac{\partial m}{\partial V} \Rightarrow \partial m = \rho\, \partial V$$

$$\partial m = \rho\, r\, \partial r\, \partial\theta\, \partial z$$

$$\partial m = \rho\, \partial z\, r\, \partial r\, \partial\theta$$

$$\partial m = \underbrace{\sigma}_{\text{Areal Density}}\, r\, \partial r\, \partial\theta$$

$$\sigma = \frac{m}{\pi R^2}$$

$$I_{xx} = \int (y^2 + z^2)\, dm$$

$$= \int_0^R \int_0^{2\pi} \left((r\sin(\theta))^2 + z^2 \right) \sigma\, r\, dr\, d\theta$$

$$= \sigma \int_0^R \int_0^{2\pi} r^3 \sin^2(\theta) + z^2 r \; dr\, d\theta$$

$$= \sigma \int_0^{2\pi} \frac{r^4}{4} \sin^2(\theta) + \frac{z^2 r^2}{2} \; d\theta \Big|_0^R$$

$$= \sigma \int_0^{2\pi} \frac{R^4}{4} \sin^2(\theta) + \frac{z^2 R^2}{2} \; d\theta$$

$$= \sigma \int_0^{2\pi} \frac{R^4}{4} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) \right) + \frac{z^2 R^2}{2} \; d\theta$$

$$= \sigma \left[\frac{R^4}{4} \left(\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) \right) + \frac{z^2 R^2}{2} \theta \right]_0^{2\pi}$$

$$= \frac{\sigma R^2}{2} \left[\frac{R^2}{2} \left(\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) \right) + z^2 \theta \right]_0^{2\pi}$$

$$= \frac{\theta R^2}{2}\left[\frac{R^2}{2}(\pi - 0) + 2\pi z^2\right] -$$

$$\cancel{\frac{\theta R^2}{2}\left[\frac{R^2}{2}(0-0) + 0\right]}$$

$$= \frac{\theta R^2}{2}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{\left(\frac{m}{\pi \cancel{R^2}}\right)\cancel{R^2}}{2}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{m}{2\pi}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{m\cancel{\pi}R^2}{4\cancel{\pi}} + \frac{\cancel{2}m\cancel{\pi}z^2}{\cancel{2}\cancel{\pi}}$$

$$\boxed{I_{xx} = \frac{mR^2}{4} + mz^2}$$

$$I_{xy} = -\int xy\, dm$$

$$= -\int_0^R \int_0^{2\pi} (r\cos(\theta))(r\sin(\theta))\, \sigma\, r\, dr\, d\theta$$

$$= -\sigma \int_0^R \int_0^{2\pi} r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta$$

$$= -\sigma \int_0^{2\pi} \frac{r^4}{4} \sin(\theta)\cos(\theta)\, d\theta \Big|_0^R$$

$$= -\sigma \int_0^{2\pi} \frac{R^4}{4} \sin(\theta)\cos(\theta)\, d\theta$$

$$u = \sin(\theta)$$

$$du = \cos(\theta)\, d\theta$$

$$= -\frac{\sigma R^4}{4} \int_0^{2\pi} u \, \partial u$$

$$= -\frac{\sigma R^4}{4} \left[\frac{u^2}{2} \right]_0^{2\pi}$$

$$= -\frac{\sigma R^4}{4} \left[\frac{\sin^2(\theta)}{2} \right]_0^{2\pi}$$

$$= -\frac{\sigma R^4}{4} \left[\frac{0}{2} - \frac{0}{2} \right]$$

$$\boxed{I_{xy} = 0}$$

Because $I_{xy} = I_{yx}$

$$\boxed{I_{yx} = 0}$$

$$I_{yy} = \int (x^2+z^2)\, dm$$

$$= \int_0^R \int_0^{2\pi} \left((r\cos(\theta))^2 + z^2 \right) \sigma r \, dr \, d\theta$$

$$= \sigma \int_0^R \int_0^{2\pi} r^3 \cos^2(\theta) + z^2 r \, dr \, d\theta$$

$$= \sigma \int_0^{2\pi} \frac{r^4}{4} \cos^2(\theta) + \frac{z^2 r^2}{2} \, d\theta \Big|_0^R$$

$$= \sigma \int_0^{2\pi} \frac{R^4}{4} \cos^2(\theta) + \frac{z^2 R^2}{2} \, d\theta$$

$$= \sigma \int_0^{2\pi} \frac{R^4}{4} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) \right) + \frac{z^2 R^2}{2} \, d\theta$$

$$= \frac{\sigma R^2}{2} \int_0^{2\pi} \frac{R^2}{2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) \right) + z^2 \, d\theta$$

$$= \frac{\sigma R^2}{2}\left[\frac{R^2}{2}\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + z^2\theta\right]_0^{2\pi}$$

$$= \frac{\sigma R^2}{2}\left[\frac{R^2}{2}\left(\frac{1}{2}(2\pi) + 0\right) + z^2(2\pi)\right] -$$

$$\cancel{\frac{\sigma R^2}{2}\left[\frac{R^2}{2}(0+0) + z^2(0)\right]}$$

$$= \frac{\sigma R^2}{2}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{\left(\frac{m}{\pi \cancel{R^2}}\right)\cancel{R^2}}{2}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{m}{2\pi}\left[\frac{\pi R^2}{2} + 2\pi z^2\right]$$

$$= \frac{m\cancel{\pi}R^2}{4\cancel{\pi}} + \frac{\cancel{2}\cancel{\pi}mz^2}{\cancel{2}\cancel{\pi}}$$

$$\boxed{I_{yy} = \frac{mR^2}{4} + mz^2}$$

$$I_{yz} = -\int yz \, dm$$

$$= -\int_0^R \int_0^{2\pi} (r\cos(\theta))(z)\, \sigma\, r \, dr \, d\theta$$

$$= -\sigma \int_0^R \int_0^{2\pi} z r^2 \cos(\theta) \, dr \, d\theta$$

$$= -\sigma \int_0^{2\pi} \frac{zr^3}{3} \cos(\theta) \, d\theta \Big|_0^R$$

$$= -\sigma \int_0^{2\pi} \frac{zR^3}{3} \cos(\theta) \, d\theta$$

$$= -\sigma \left[\frac{zR^3}{3} \sin(\theta) \right]_0^{2\pi}$$

$$= -\theta\left[\frac{zR^3}{3}(0) - \frac{zR^3}{3}(0)\right]$$

$$\boxed{I_{yz} = 0}$$

Because $I_{yz} = I_{zy}$,

$$\boxed{I_{zy} = 0}$$

$$I_{zz} = \int (x^2 + y^2)\, dm$$

$$= \int_0^R \int_0^{2\pi} (r\cos(\theta))^2 + (r\sin(\theta))^2 \, \sigma r\, dr\, d\theta$$

$$= \sigma \int_0^R \int_0^{2\pi} r^3 \cos^2(\theta) + r^3 \sin^2(\theta) \, dr \, d\theta$$

$$= \sigma \int_0^R \int_0^{2\pi} r^3 \overbrace{\left(\cos^2(\theta) + \sin^2(\theta)\right)}^{1} \, dr \, d\theta$$

$$= \sigma \int_0^R \int_0^{2\pi} r^3 \, dr \, d\theta$$

$$= \sigma \int_0^{2\pi} \frac{r^4}{4} \, d\theta \Big|_0^R$$

$$= \sigma \int_0^{2\pi} \frac{R^4}{4} \, d\theta$$

$$= \sigma \left[\frac{R^4 \theta}{4} \right]_0^{2\pi}$$

$$= \frac{\cancel{2}\pi\sigma R^4}{\cancel{4}} = \frac{\pi \sigma R^4}{2}$$

$$= \frac{\cancel{\pi}\left(\frac{m}{\cancel{\pi}\cancel{R^2}}\right)\cancel{R^4}}{2}$$

$$\boxed{I_{zz} = \frac{mR^2}{2}}$$

$$I_{zx} = -\int zx\,dm$$

$$= -\int_0^R\int_0^{2\pi} z \cdot r\cos(\theta)\,\sigma r\,dr\,d\theta$$

$$= -\sigma\int_0^R\int_0^{2\pi} zr^2\cos(\theta)\,dr\,d\theta$$

$$= -\sigma\int_0^{2\pi} \frac{zr^3}{3}\cos(\theta)\,d\theta\Big|_0^{2\pi}$$

$$= -\sigma \int_0^{2\pi} \frac{(2\pi)^3 z}{3} \cos(\theta)\, d\theta$$

$$= -\frac{8\pi^3 \sigma z}{3} \int_0^{2\pi} \cos(\theta)\, d\theta$$

$$= -\frac{8\pi^3 \sigma z}{3} \Big[\sin(\theta)\Big]_0^{2\pi}$$

$$= -\frac{8\pi^3 \sigma z}{3} \Big[0 - 0\Big]$$

$$\boxed{I_{zx} = 0}$$

Because $I_{zx} = I_{xz}$,

$$\boxed{I_{xz} = 0}$$

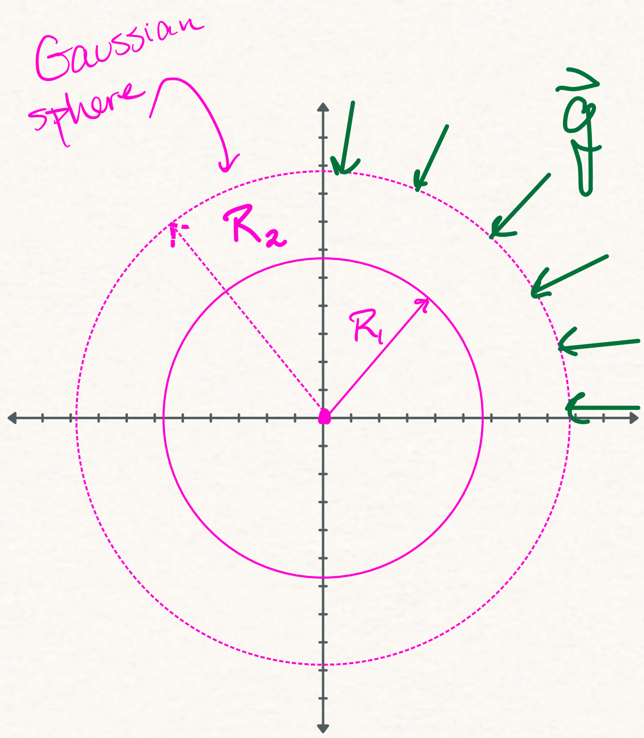
$$[I] = \begin{bmatrix} \frac{mR^2}{4} + mz^2 & 0 & 0 \\ 0 & \frac{mR^2}{4} + mz^2 & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$

Exercise 2a: Calculate the tidal shear tensor produced by a uniform sphere of radius R and mass M at any radius

$$\oint\!\!\oint \underbrace{\vec{g}} \cdot \underbrace{d\vec{A}}$$

This is the surface area of a sphere

This is the gravitational field



$$\oiint_{\partial V} \vec{g} \cdot d\vec{A} = -4\pi GM$$ ← Gravitational Flux at the surface of gaussian sphere

The dot product between these are $\vec{g}\, d\vec{A} \cos(\theta)$

$$\oiint_{\partial V} \vec{g}\, d\vec{A} \underbrace{\cos(\theta)}_{} = -4\pi GM$$

Because the angle between the $\vec{g}$ and $d\vec{A}$ is opposite ($180°$ or π), this turns negative

$$\oiint_{\partial V} -\vec{g}\, d\vec{A} = -4\pi GM$$

$$-g \oiint_{\partial V} d\vec{A} = -4\pi GM$$

→ Because we are choosing a sphere, it is a known geometry. This can be simplified right away

$$\cancel{-}g(\cancel{4}\cancel{\pi} R_2^2) = \cancel{-}\cancel{4}\cancel{\pi} GM$$

$$gR_2^2 = GM$$

$$g = \frac{GM}{R_2^2}\hat{r}$$

Now we want to find the gravitational field potential:

$$\vec{G} = -\vec{\nabla}\Phi$$

$$\frac{GM}{R^2} = -\vec{\nabla}\phi$$

We want to find Φ, so let's integrate:

$$\int_{\infty}^{R_2} \frac{GM}{R^2}\, dR = \int -\vec{\nabla}\phi$$

$$-\phi = \int_{\infty}^{R_2} \frac{GM}{R^2}\, dR$$

$$\cancel{-}\Phi = \cancel{-}\frac{GM}{R_2}$$

$$\boxed{\Phi = \frac{GM}{R_2}}$$

Now I have to find the tidal tensor using the Hessian matrix

$$J_{ab} = \frac{\boxed{\partial^2 U}}{\partial x^a \partial x^b}$$

This is the gravitational potential

Let $R_2 = r$

$$J_{rr} = \frac{\partial^2 \vec{G}}{\partial^2 r} = \frac{\partial^2}{\partial^2 r}\left(\frac{GM}{r}\right)$$

$$= \frac{\partial^2}{\partial^2 r}\left(GMr^{-1}\right)$$

$$= \frac{\partial}{\partial r}\left(-GMr^{-2}\right)$$

$$\boxed{J_{rr} = \frac{2GM}{r^3}}$$

$$J_{\theta\theta} = \frac{\partial^2}{\partial^2 \theta}\left(\frac{GM}{r}\right)$$

$$= \frac{\partial^2}{\partial^2 \theta}\left(GMr^{-1}\right)$$

$$\boxed{J_{\theta\theta} = 0}$$

$$J_{\phi\phi} = \frac{\partial^2}{\partial^2 \phi}\left(\frac{GM}{r}\right)$$

$$= \frac{\partial^2}{\partial^2 \phi}\left(GMr^{-1}\right)$$

$$\boxed{J_{\phi\phi} = 0}$$

Because of symmetry in the θ and ϕ direction, all off diagonal terms go to zero.

$$\boxed{J_{ab} = \begin{bmatrix} \frac{2GM}{R_2^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Now I will calculate the tidal shear tensor for the outside of the sphere

$$\Phi_{ab} = J_{ab} - \frac{1}{3} J^m_{\ m} \eta_{ab}$$

The trace of the matrix is given by:

$$J^m_{\ m} = \sum_{i=1}^{3} x_{ii}$$

$$= \frac{2GM}{R_2^3} + 0 + 0$$

$$J^{m}{}_{m} = \frac{2GM}{R_2^3}$$

$$\Phi_{ab} = \begin{bmatrix} \frac{2GM}{R_2^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{2GM}{3R_2^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\Phi_{\text{outside}} = \begin{bmatrix} \frac{4GM}{3R_2^3} & 0 & 0 \\ 0 & -\frac{2GM}{3R_2^3} & 0 \\ 0 & 0 & -\frac{2GM}{3R_2^3} \end{bmatrix}}$$

All of the derivatives above are for <u>outside</u> of the Gaussian sphere. I will now calculate the inside of the sphere:

$$\oint\!\!\oint_{\partial V} \vec{g}\, d\vec{A}$$

Mass of gaussian sphere

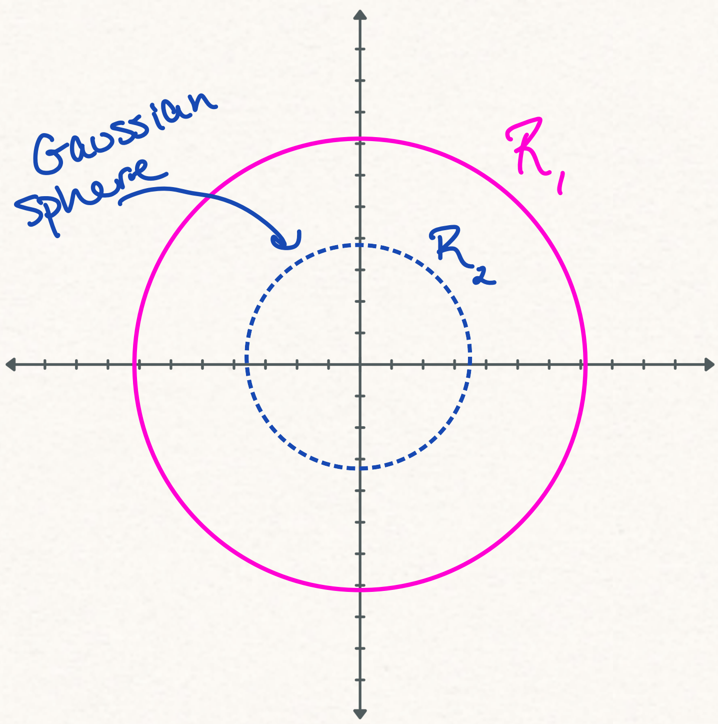
$$\oint\!\!\oint_{\partial V} \vec{g} \cdot d\vec{A} = -4\pi G M_2$$

$$\oint\!\!\oint_{\partial V} \vec{g}\, dA \cos(\theta) = -4\pi G M_2$$

$$\cancel{-}\, \vec{g}(\cancel{4}\cancel{\pi} R_2^2) = \cancel{-4}\cancel{\pi} G M_2$$

$$\vec{g} R_2^2 = G\left(\frac{M_1 R_2^3}{R_1^3}\right)$$

$$\vec{g} = G\left(\frac{M_1 R_2}{R_1^3}\right)$$



$$\rho_1 = \frac{M_1}{\frac{4}{3}\pi R_1^3} \qquad \rho_2 = \frac{M_2}{\frac{4}{3}\pi R_2^3}$$

$$\frac{3M_1}{4\pi R_1^3} = \frac{3M_2}{4\pi R_2^3}$$

We can set these equal b/c the density is uniform

$$\frac{M_1}{R_1^3} = \frac{M_2}{R_2^3} \Rightarrow \boxed{M_2 = \frac{M_1 R_2^3}{R_1^3}}$$

$$-\nabla\Phi = \vec{G}$$

$$\int -\nabla\Phi = \int_{R_1}^{R_2} G\left(\frac{M_1 R_2}{R_1^3}\right) \partial R_2$$

$$-\Phi = \frac{GM_1}{R_1^3}\int_{R_1}^{R_2} R_2 \, \partial R_2 = -\frac{GM_1}{R_1^3}\left(\frac{R_2^2}{2}\Bigg|_{R_1}^{R_2}\right)$$

$$= -\frac{GM_1}{R_1^3}\left[\frac{R_2^2}{2} - \frac{R_1^2}{2}\right]$$

$$= -GM_1\left[\frac{R_2^2}{2R_1^3} - \frac{\cancel{R_1^2}}{2R_1^{\cancel{3}}}\right]$$

$$= -GM_1\left[\frac{R_2^2}{2R_1^3} - \frac{1}{2R_1}\right]$$

$$\boxed{\Phi = -\frac{GM_1}{2}\left[\frac{R_2^2}{R_1^3} - \frac{1}{R_1}\right]}$$

Now I will calculate the Hessian since

I have the potential:

$$J_{ab} = \frac{\partial^2 U}{\partial x^a \partial x^b}$$

$$J_{rr} = \frac{\partial^2 U}{\partial x^2}$$

Taking derivative w.r.t R_2

$$= \frac{\partial^2}{\partial R_2^2}\left(-\frac{GM}{2}\left[\frac{R_2^2}{R_1^3} - \frac{1}{R_1}\right]\right)$$

R_1 is a constant

$$= \frac{\partial}{\partial R_2}\left(-GM\left[\frac{R_2}{R_1^3}\right]\right)$$

$$\boxed{J_{rr} = -\frac{GM}{R_1^3}}$$

$$J_{\theta\theta} = \frac{\partial^2}{\partial^2\theta}\left(-\frac{GM}{a}\left[\frac{R_2^2}{R_1^3} - \frac{1}{R_1}\right]\right)$$

$$\boxed{J_{\theta\theta} = 0}$$

$$J_{\phi\phi} = \frac{\partial^2}{\partial^2\theta}\left(\frac{GM}{a}\left[\frac{R_2^2}{R_1^3} - \frac{1}{R_1}\right]\right)$$

$$\boxed{J_{\phi\phi} = 0}$$

Because of the symmetry in the θ and ϕ direction, all non-diagonal terms go to zero.

$$\boxed{J_{ab} = \begin{bmatrix} -\frac{GM}{R_1^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Now I will calculate the tidal shear tensor for inside of the sphere

$$\Phi_{ab} = J_{ab} - \frac{1}{3} J^m_{\ m} \eta_{ab}$$

Calculating the trace of the matrix:

$$J^m_{\ m} = -\frac{GM}{R_1^3} + 0 + 0 = -\frac{GM}{R_1^3}$$

$$\Phi_{inside} = \begin{bmatrix} -\frac{GM}{R_1^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{GM}{3R_1^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\Phi_{inside} = \begin{bmatrix} -\frac{2GM}{3R_1^3} & 0 & 0 \\ 0 & \frac{GM}{3R_1^3} & 0 \\ 0 & 0 & \frac{GM}{3R_1^3} \end{bmatrix}}$$

Exercise 2b: Assuming these are the gravitational tidal tensors, can it spin the disk if it is at the center of the sphere? At the radius $r < R$? When $r > R$?

- If the disk was outside of the sphere, the disk can spin about the x, y, or z axis due to the gravitational tidal tensor. This is because the tidal tensor is non-zero across the diagonal (so the xx, yy, and zz indicies). However, because there is no moment of inertia in the off-diagonal (i.e.: zeroes are the only value found in the off-diagonal), the disk will not spin in those directions.

- If the disk is inside the sphere ($r<R$), it will spin in the x, y, and z directions because those are the only elements in the shear tidal tensor that are non-zero.
- If the disk is at the center, it will not be able to spin because when we set r (or in my case, R_1) equal to zero, there will be no tidal shear in any direction.