MEC302: Embedded Computer Systems

Theme I: Modeling Dynamic Behaviors

Lecture 2 – Continuous Dynamics

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Lecture outline

- Mathematical models and their types;
- Continuous dynamics modelling:
  - Newtonian (classical) mechanics;
  - Model order reduction;
  - Actor models;
- Model properties;
- Feedback control.

Mathematical models

What is a mathematical model and why do we need them?

- Mathematical models represent real-life problems in math terms and expressions (e.g., a system of equations, state machines);
- Mathematical models help us understand the real-world processes (i.e., physical systems, computer systems and both at the same time);
- Design of ECS require clear understanding of the interaction between the dynamics of ECS and its environment, which can be obtained from detailed modelling.

“All models are wrong, but some are useful.”

by George E. P. Box (British statistician)

Types of mathematical models

Mathematical models can be:

推论的

- Deductive, inductive or floating – derivation method (e.g., based on theory, from empirical observations or arbitrary (expected) structure).
- Static or dynamic – time dependency (time-invariant or time-dependent);
- Linear or nonlinear – dependency to varying parameters or variables;
- Explicit or implicit – output parameters can/cannot be directly derived from the input;
- Deterministic or probabilistic – if randomness cannot/can determine the model behavior;
- Discrete or continuous – operate with discrete or continuous quantities;

Continuous dynamics

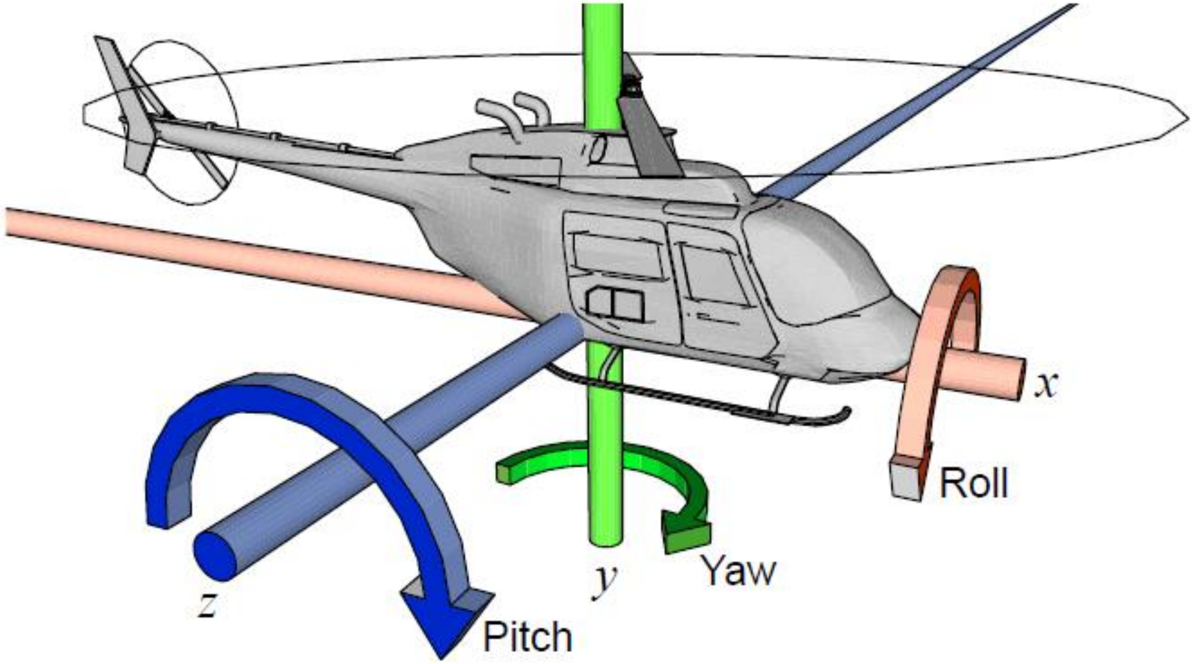
Continuous dynamics modelling:

- Studies dynamics of physical systems, which may include mechanical objects, electric circuits, chemical and biological processes, etc.;
- Represents evolution of an object or system in time according to a certain rule (i.e., physics laws);
- Modeled using differential or integral equations – accurate for continuous and “smooth” processes, where:
  - Differential equation: $F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}\right) = 0;$
  - Solution of a diff.eq.: $x = f(t).$

Newtonian (classical) mechanics

An object can represented with six degrees of freedom:

- Three to define a position:
  - Coordinated on axes: x, y, z;
- Three to define its orientation:
  - Rotation angles: $\theta_x, \theta_y, \theta_z$.



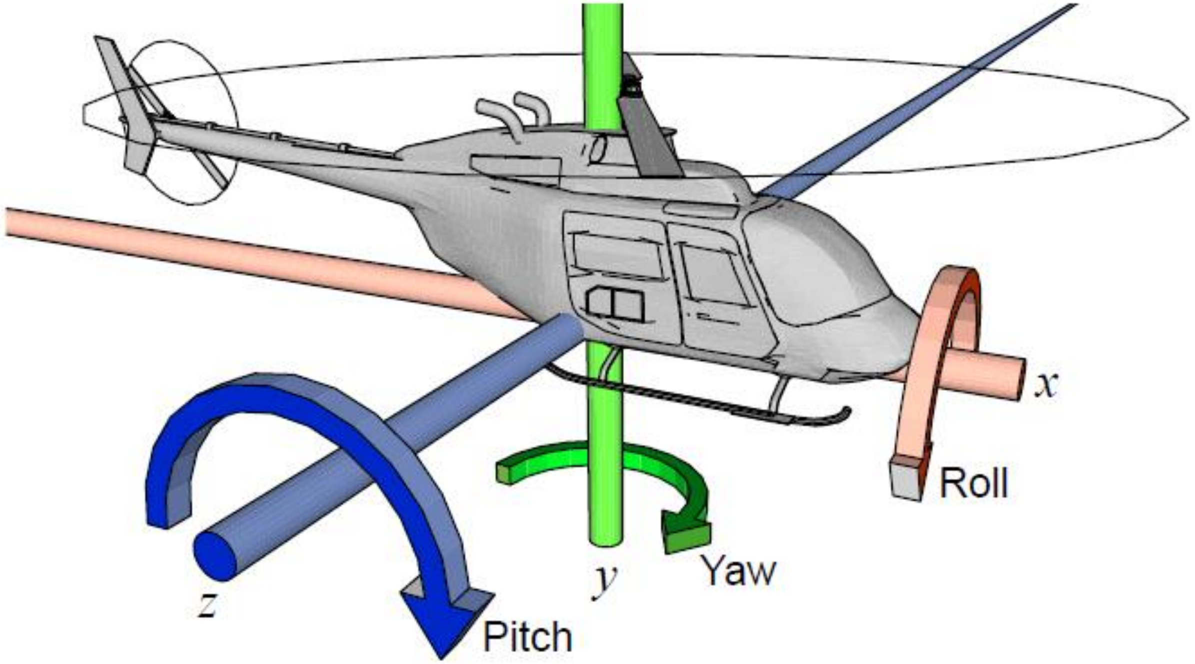
Effectively these can be represented with two vectors:

$$\boldsymbol{x}: R \to R^3 \text{ and } \boldsymbol{\theta}: R \to R^3$$

Newtonian (classical) mechanics

To track its position (along $\boldsymbol{x}: R \to R^3$) in time:

- Newton's second law: $\boldsymbol{F}(t) = M\ddot{\boldsymbol{x}}(t)$



Object's velocity:

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(0) + \int_0^t \ddot{\boldsymbol{x}}(\tau) d\tau$$

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(0) + \frac{1}{M}\int_0^t \boldsymbol{F}(\tau) d\tau \;\forall\, t > 0$$

Object's position:

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \dot{\boldsymbol{x}}(\tau) d\tau$$

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{x}(0) + t\,\dot{\boldsymbol{x}}\,(0) + \frac{1}{M}\int_0^t \int_0^\tau \boldsymbol{F}(\alpha)\, d\alpha\, d\tau \;\forall\, t > 0}$$

Newtonian (classical) mechanics

To track its orientation (along $\boldsymbol{\theta}: R \to R^3$) in time:

- Newton's second law (for rotation):

$$\boldsymbol{T}(t) = \frac{d}{dt}\left(\boldsymbol{I}(t)\,\dot{\boldsymbol{\theta}}(\boldsymbol{t})\right)$$

For a spherical object:

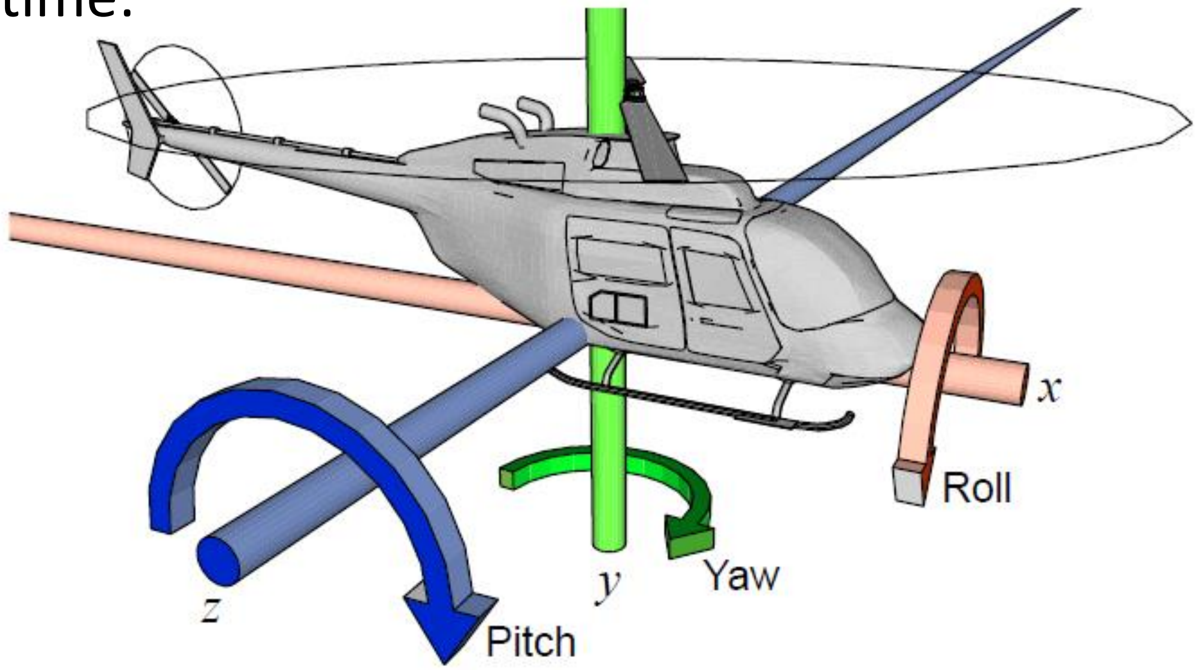
$$\boldsymbol{T}(t) = I\ddot{\boldsymbol{\theta}}(\boldsymbol{t})$$

Object's angular velocity:

$$\dot{\boldsymbol{\theta}}(t) = \dot{\boldsymbol{\theta}}(0) + \frac{1}{I}\int_0^t \boldsymbol{T}(\tau)d\tau\ \forall\ t > 0$$

Object's orientation:

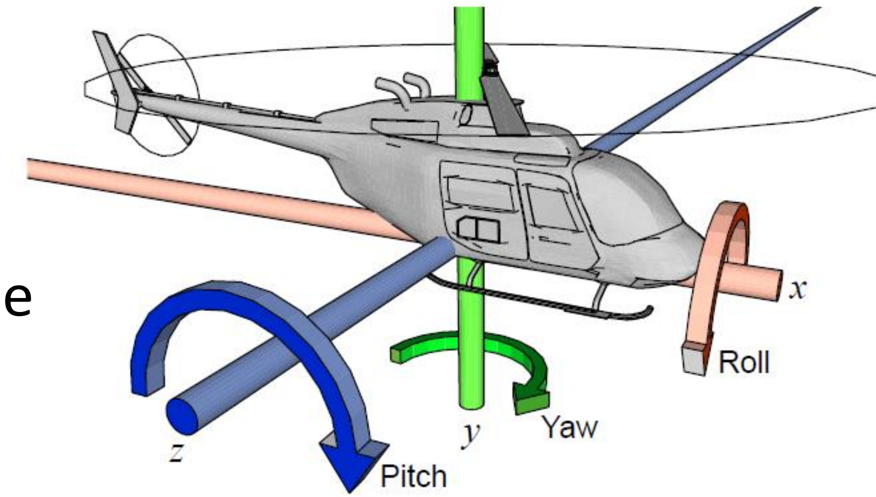
$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + t\,\dot{\boldsymbol{\theta}}(0) + \frac{1}{I}\int_0^t\int_0^\tau \boldsymbol{T}(\alpha)\ d\alpha\ d\tau\ \forall\ t > 0$$



Moment of inertia tensor for an arbitrary object:

$$\boldsymbol{I}(t) = \begin{bmatrix} I_{xx}(t) & I_{xy}(t) & I_{xz}(t) \\ I_{yx}(t) & I_{yy}(t) & I_{yz}(t) \\ I_{zx}(t) & I_{zy}(t) & I_{zz}(t) \end{bmatrix}$$

Model order reduction



Model order reduction is a formal procedure to reduce the number of model's degrees of freedom.

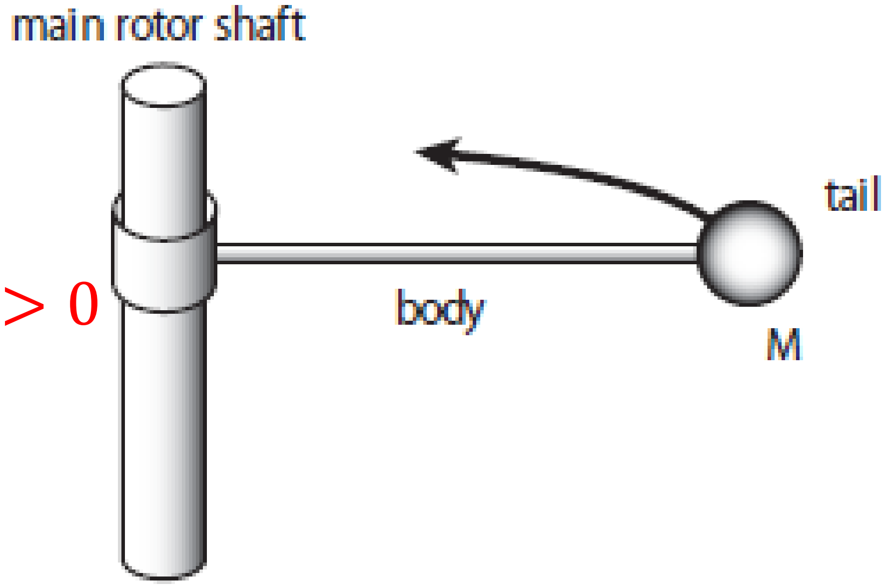
To diversify control channels (and of course for simplicity):

- For angular velocity around y-axis:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau \,\forall\, t > 0$$

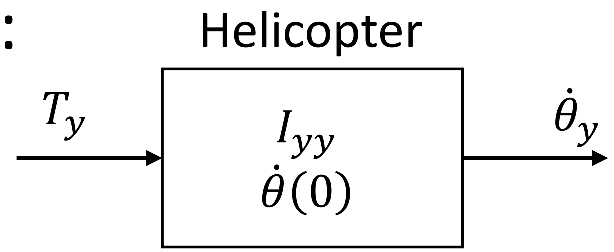
- For orientation around y-axis:

$$\theta_y(t) = \theta_y(0) + t\,\dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t \int_0^\tau T_y(\alpha)\, d\alpha\, d\tau \,\forall\, t > 0$$



Actor model

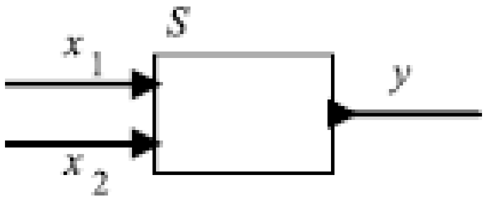
Helicopter's actor model:



$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

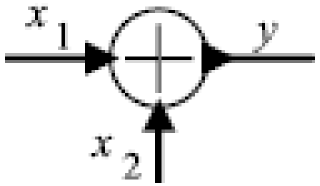
From control theory, actor models:

1. May have multiple inputs/outputs;



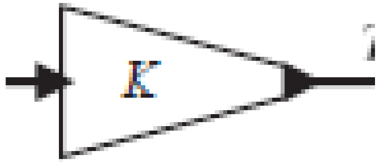
$$y = f(x_1, x_2)$$

2. Signal adder (particular example of 1.);



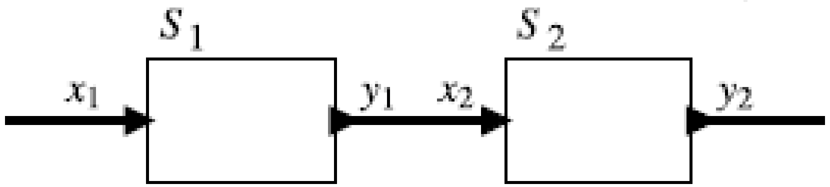
$$y = x_1 + x_2$$

3. Gain function (multiplication by *K*);



$$y = Kx$$

4. (De)composable



$$y_2 = f_2(f_1(x_1))$$

Model properties

Main model properties:

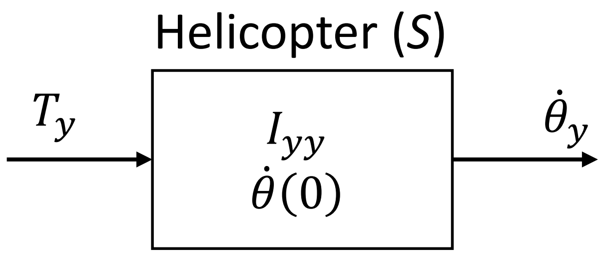
- **Causality** – output depends only on current and past inputs;
- **Memory** – output depends not only on the current inputs, but also on past inputs;
- **Linearity** – the outputs are proportional to the inputs;
- **Time Invariance** – model behavior does not depend on time;
- **Stability** – the output is bounded for all bounded input signals.

As a rule:

- To design an effective control, a model/object/system need to be Causal, Not Memoryless, Linear, Time Invariant and Stable

Model properties

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Causality – the output depends only on current and past inputs:

Causal model is defined as follows:

$$x_1 \Big|_{t \le \tau} = x_2 \Big|_{t \le \tau} \Rightarrow S(x_1) \Big|_{t \le \tau} = S(x_2) \Big|_{t \le \tau} \; \forall x_1, x_2 \in X$$

– *depends on past and present inputs!*

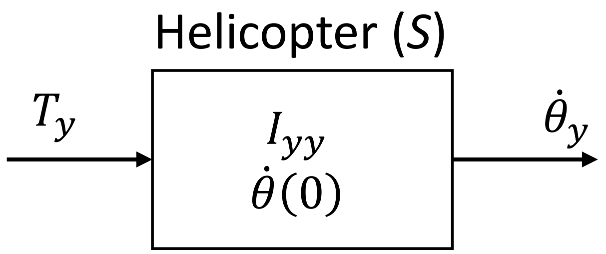
Strictly causal model is defined:

$$x_1 \Big|_{t < \tau} = x_2 \Big|_{t < \tau} \Rightarrow S(x_1) \Big|_{t \le \tau} = S(x_2) \Big|_{t \le \tau} \; \forall x_1, x_2 \in X$$

– *depends only on past inputs!*

Is the helicopter model causal, strictly causal or non-causal?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Model properties

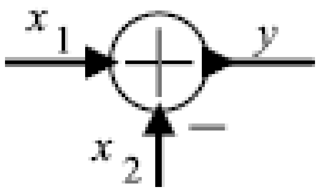
Memory – the output depends not only on the current input, but also on the past inputs.

Memoryless model is defined as follows:

$$\exists\, f: A \to B \mid f\big(x(t)\big) = \big(S(x)\big)(t)\ \forall\, t \in R$$

– model output depends only on the current input

For example, Adder ($y = x_1 + x_2$) is memoryless.



Is the helicopter model memoryless or not?

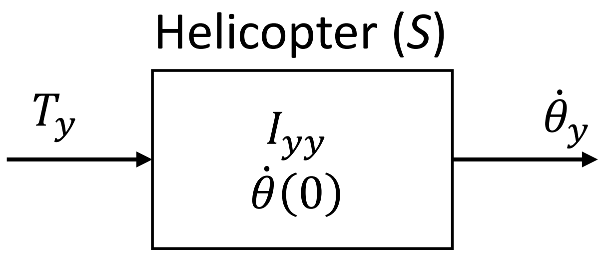
这个数学表达式是用于描述函数和变换的。让我们逐步分解这个表达式：

1. $\exists f: A \rightarrow B$：这表示存在一个函数f，它将集合A中的元素映射到集合B中的元素。
2. $f(x(t))$：这表示函数f作用于另一个函数x(t)，其中t是一个实数（属于实数集R）。
3. $(S(x))(t)$：这表示函数S作用于函数x之后，再作用于t。S可能是一个变换或运算符，例如微分、积分等。
4. $\forall t \in \mathrm{R}$：这表示对于所有属于实数集R的t值，表达式都成立。

整个表达式的意思是：存在一个函数f，它将集合A中的元素映射到集合B中的元素，对于所有属于实数集R的t值，当f作用于函数x(t)时，结果等于函数S作用于函数x后再作用于t。这个表达式通常用于描述一种函数或变换关系，可能涉及到微分方程、积分变换等领域。

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

Model properties



Linearity – the output(s) is proportional to the input(s):

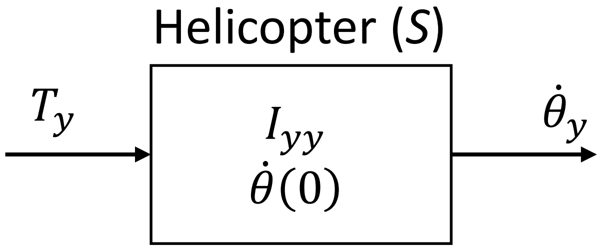
Linear model is defined as follows:

$$S(ax) = a\, S(x)\ \forall\ x \in X, a \in R$$

For example, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are linear.

Is the helicopter model linear or not?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Model properties

Time Invariance – model behavior does not depend on time.

Time invariant model is defined as follows:

$$S\big(D_\tau(x)\big) = D_\tau\big(S(x)\big)\ \forall\, x \in X, \tau \in R,$$

where $\big(D_\tau(x)\big)(t) = x(t - \tau)\ \forall\, x \in X, t \in R$.

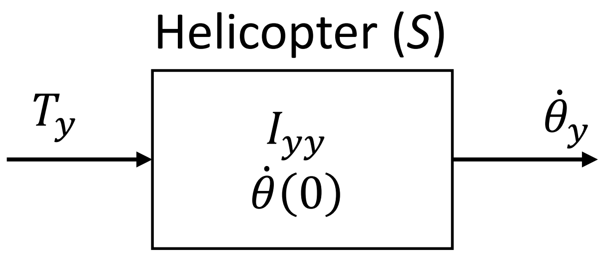
– the model is linear to delay function.

Obviously, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are time invariant.

Is the helicopter model time invariant or not?

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$

Model properties



Stability – the output is bounded for all bounded input signals.

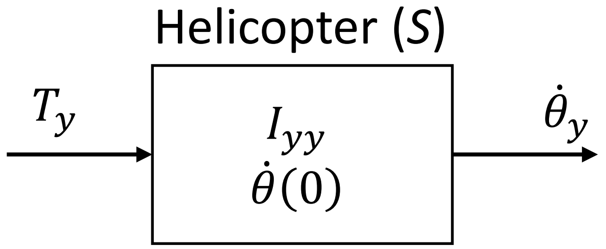
A model is bounded-input bounded-output (BIBO) stable if:

$$\exists |A|, |B| < \infty\colon \left|\big(S(x)\big)(t)\right| \le |A| \ \forall\ x(t) \le |B|, t \in R$$

Clearly, Adder($y = x_1 + x_2$) and Gain($y = Kx$) are BIBO stable.

Is the helicopter model BIBO stable?

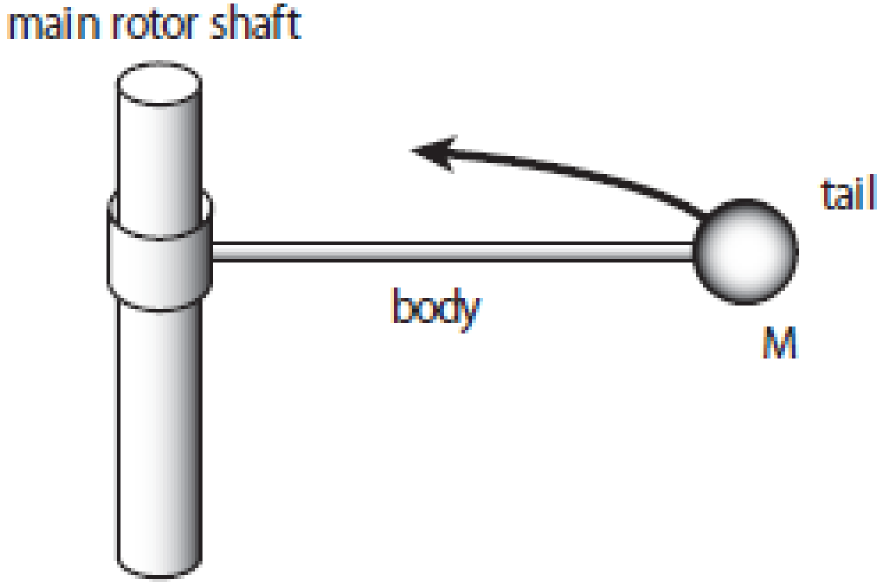
$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Helicopter model properties

Helicopter model is:

- Strictly causal;
- Not memoryless;
- Linear*
- Time invariant*
- BIBO unstable

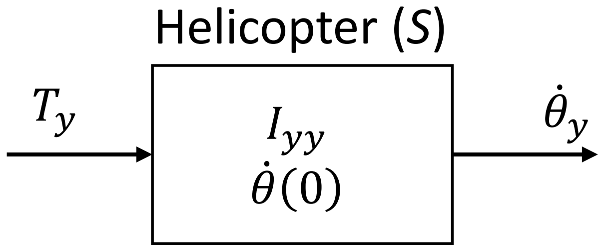


- Can we control such system?

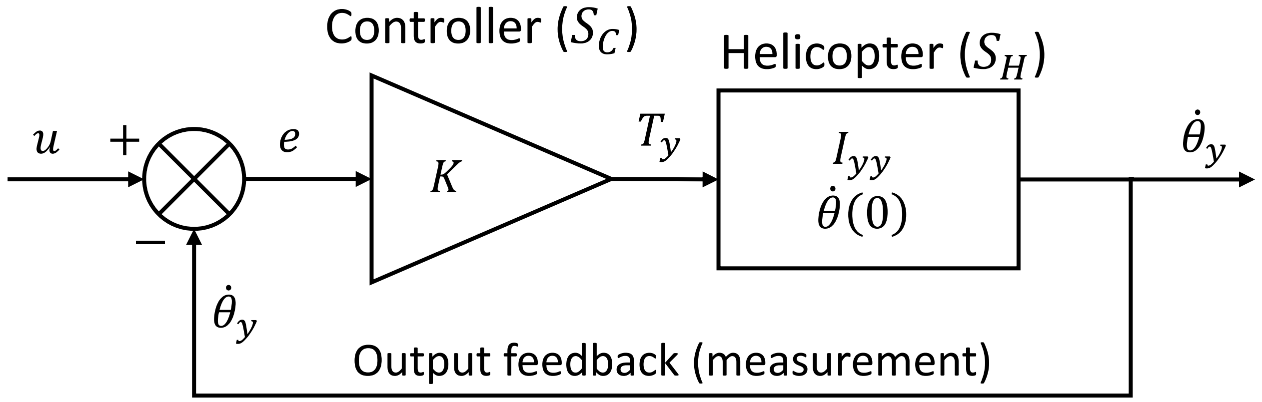
* – if $\dot{\theta}_y(0) = 0$

Feedback Control

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Let us introduce a feedback control loop:



Effective Transfer Function (TF) becomes:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t \left(u(\tau) - \dot{\theta}_y(\tau) \right) d\tau$$

Discrete difference equation (for $\Delta t \to 0$):

$$\dot{\theta}_y(s+1) = \dot{\theta}_y(s) + \frac{K}{I_{yy}} \sum_{s=1}^{S} \left(u(s) - \dot{\theta}_y(s) \right) \Delta t$$

To check if it is stable, we can:

- Model it using LabView or Simulink;
- Brute force – reformulate TF into a difference equation and solve it iteratively;
- Solve the differential equation and check if $\dot{\theta}_y(t)$ is bounded.

Feedback Control

Feedback control model TF:

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{K}{I_{yy}} \int_0^t \left(u(\tau) - \dot{\theta}_y(\tau)\right) d\tau$$

To find if the system is stable, let's solve its differential equation ($\dot{\theta}_y(t) = f(t)$):

1. Consider $u(t) = U$, then helicopter ODE is as follows: ODE，常微分方程ordinary differential equation

$$\frac{d\dot{\theta}_y(t)}{dt} = \frac{K}{I_{yy}}\left(U - \dot{\theta}_y(t)\right)$$

2. Let's rearrange/separate ODE variables:

$$\frac{1}{U - \dot{\theta}_y(t)} d\dot{\theta}_y(t) = \frac{K}{I_{yy}} dt$$

3. Now we can integrate both parts separately:

$$-\ln\left(U - \dot{\theta}_y(t)\right) - \ln C = \frac{K}{I_{yy}} t$$

4. Manipulate with it (get rid of logarithm):

$$\left(U - \dot{\theta}_y(t)\right) C = e^{\frac{-K}{I_{yy}} t}$$

5. Express $\dot{\theta}_y(t)$:

$$\dot{\theta}_y(t) = U - \frac{1}{C} e^{\frac{-K}{I_{yy}} t}$$

6. For initial conditions ($\dot{\theta}_y(0) = 0$), $C = \frac{1}{U}$:

$$\dot{\theta}_y(t) = U\left(1 - e^{\frac{-K}{I_{yy}} t}\right)$$

7. Thus, $\lim_{t \to \infty} \dot{\theta}_y(t) = U \ \forall \ K > 0$

To sum up

- Mathematical models help us understand the real-world processes (e.g., physical systems, computer systems and both at the same time);
- Continuous dynamics modelling uses differential or integral equations to model continuous and "smooth" processes (e.g., Newtonian mechanics can be used to model mechanical systems);
- Actor model provides a graphical representation of the mathematical model (e.g., physical system), which can be effective for the analysis and system design;
- Model properties (i.e., Causality, Memorylessness, Linearity, Time Invariance, and Stability) can show if a system is suitable for control:
  - I.e., Causal, Not Memoryless, Linear, Time Invariant and Stable system is deemed to be suitable for control.
- Feedback control is effective to improve system properties (e.g., Stability).

The End

See you next time (March 6)