Midterm 2

Math 15200

Instructor: Reid Harris

November 22, 2019

Name: ______________________________

Instructions:

1. Print your name clearly at the top of each page.
2. This midterm has 7 questions for a total of 100 points. The value of each part of each question is stated.
3. Please show your work in the space provided. Partial credit will be awarded for partial or incomplete solutions.
4. The parts of the fourth question are true/false questions. In other questions please show your work. If you need more space, you can use the final page, which is blank, or request additional paper.

Good Luck!

Question	Points	Score
1	10	
2	5	
3	20	
4	20	
5	20	
6	10	
7	15	
Total:	100	

1. (10 points) **Use the limit definition of derivative** to evaluate the derivative of

$$f(x) = 2x + 3$$

at $x = 1$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(2(1+h)+3) - (2(1)+3)}{h}$$

$$= \lim_{h\to 0} \frac{2+2h+3-5}{h}$$

$$= \lim_{h\to 0} 2$$

$$= \boxed{2}$$

2. (5 points) Calculate the limit

$$\lim_{x\to 0} \frac{\sin(4x)}{5x}.$$

$$\lim_{x\to 0} \frac{\sin(4x)}{5x} = \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4}{5}$$

$$= (1) \cdot \frac{4}{5}$$

$$= \boxed{\frac{4}{5}}$$

3. Calculate the derivatives of the following functions.

(a) (5 points) $f(x) = 4x^5 - x^3 + \frac{1}{2}x^2 - \pi x + 12$

$$\boxed{f'(x) = 20x^4 - 3x^2 + x - \pi}$$

(b) (5 points) $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{(1)(x^2+1) - (x)(2x)}{(x^2+1)^2}$$

$$\boxed{f'(x) = \frac{1-x^2}{(1+x^2)^2}}$$

(c) (5 points) $f(x) = x^2 \sec(x)$

$$\boxed{f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x)}$$

(d) (5 points) $f(x) = \sin\left(\dfrac{1}{x^2+1}\right)$

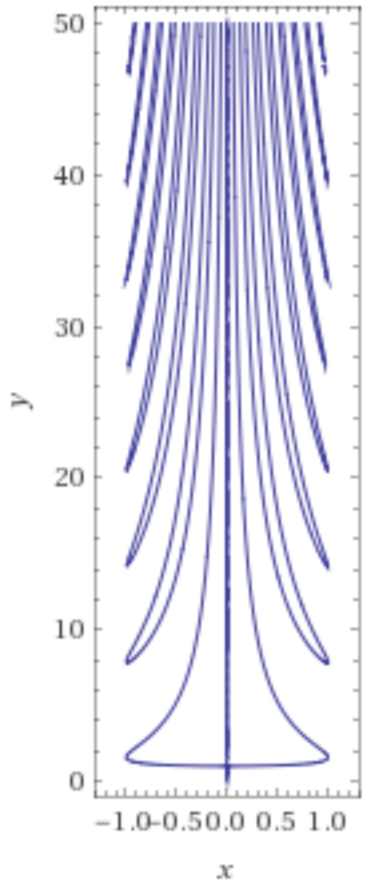
$$f'(x) = \cos\left(\frac{1}{x^2+1}\right) \cdot \left(\frac{-2x}{(x^2+1)^2}\right)$$

$$\boxed{f'(x) = -\frac{2x}{(x^2+1)^2} \cos\left(\frac{1}{x^2+1}\right)}$$

4. (20 points) Find the equation of the tangent line to the curve given by

$$\sin(xy) = x$$

at the point $\left(1, \frac{\pi}{6}\right)$.



$$\sin(xy) = x$$

$$(y + xy')\cos(xy) = 1$$

$$x\cos(xy)y' = 1 - y\cos(xy)$$

$$y' = \frac{1 - y\cos(xy)}{x\cos(xy)}$$

$$y'\left(1, \frac{\pi}{6}\right) = \frac{1 - \left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)}{(1)\cos\left(\frac{\pi}{6}\right)}$$

$$= \frac{1 - \frac{\pi}{6}\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{\sqrt{3}} - \frac{\pi}{6}$$

$$\boxed{y - \frac{\pi}{6} = \left(\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right)(x - 1)}$$

5. Evaluate the following.

50

(a) (10 points) $\frac{d^{70}}{dx^{70}}(\cos(x))$

Since 68 is divisible by 4, $\frac{d^{68}}{dx^{68}}(\cos x) = \cos x$,

$$= \frac{d^2}{dx^2}\left[\frac{d^{68}}{dx^{68}}(\cos x)\right] = \frac{d^2}{dx^2}(\cos x) = \boxed{-\cos x}$$

(b) (10 points) $\frac{d^{32}}{dx^{32}}(x^5 + 21x^2 + 7x - 4) = \boxed{0}$

6. (10 points) Let $f(x) = \cos(x) - x$. Which of the following intervals contains a solution to $f(x) = 2$?

1. $[-2\pi, -\pi]$
2. $[-\pi, 0]$
3. $[0, \pi]$
4. $[\pi, 2\pi]$

$f(-2\pi) = 1 + 2\pi \approx 7.2$

$f(-\pi) = -1 + \pi \approx 2.1$

$f(0) = 1 = 1$

$f(\pi) = -1 - \pi \approx -4.1$

$f(2\pi) = 1 - 2\pi \approx -5.2$

Since 2 lies between $f(0)$ and $f(-\pi)$, the Intermediate Value Theorem says that $f(x) = 2$ has a solution in $[-\pi, 0]$

7. (15 points) Let

$$f(x) = \begin{cases} 1 + ax^2, & \text{for } x \le 3 \\ b + x, & \text{for } x > 3 \end{cases}$$

For what values of a and b is f differentiable at $x = 3$.

If f is differentiable at $x=3$, then

$$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h}.$$

We calculate

since $3+h>3$

$$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{(b+3+h)-(1+9a)}{h}$$

$$= \lim_{h\to 0^+} \frac{2+b-9a+h}{h}$$

This can only possibly exist if $\lim_{h\to 0^+}(2+b-9a+h) = 2+b-9a = 0.$

Thus, $b = 9a-2$ so

$$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} 1 = 1$$

Also, the left-handed limit is (because as $h\to 0^-$, $3+h<0$),

$$\lim_{h\to 0^-} \frac{1+a(3+h)^2-(1+9a)}{h} = \lim_{h\to 0^-} \frac{1+9a+6ah+ah^2-1-9a}{h}$$

$$= \lim_{h\to 0^-} (6a+ah)$$

$$= 6a.$$

This must be equal to the right-hand limit, so

$$\boxed{a = \frac{1}{6}}$$

$$b = 9a-2 = \frac{3}{2}-2$$

$$\boxed{b = -\frac{1}{2}}$$

In-Class Quiz 6

December 2, 2019

Name:

1. Prove that if
   - f and h are differentiable on $(-\infty, \infty)$,
   - h is an increasing function, and
   - f attains its global maximum/minimum at $x = c$, then

   $h \circ f$ attains its global maximum/minimum at $x = c$.

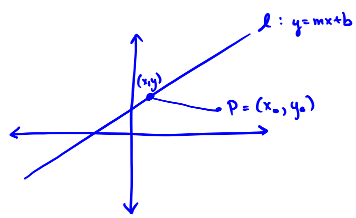
The maximum of $h(x)$ occurs when x is maximized, because h is increasing.

Thus, to maximize $h(f(x))$, we need to maximize the argument $f(x)$ which occurs at $x = c$. $\blacksquare$

2. Suppose we are given line ℓ and a point P not on ℓ. Show that if Q is the point on ℓ closest to P, then the line $\overline{PQ}$ is perpendicular to ℓ. (Just consider the case when ℓ is not vertical or horizontal.)

   Hint: If ℓ is given by the equation $y = mx + b$ and $f(x)$ is the distance from P to a point (x, y) on ℓ, then f attains its minimum exactly where f^2 attains its minimum (Why?). Also, two lines of slope m and m' are perpendicular if and only if $m \cdot m' = -1$.

Method 1 (straight-forward):



We wish to minimize the distance between $P = (x_0, y_0)$ and (x,y) on ℓ. This is the same as minimizing the square of the distance:

$$f(x)^2 = (x - x_0)^2 + (mx + b - y_0)^2$$

As $x \to \pm\infty$, $f(x)^2 \longrightarrow +\infty$ so the minimum occurs at a stationary or singular point. There are no singular points.

To find the stationary points, we have

$$\frac{d}{dx}\left[f(x)^2\right] = 2(x - x_0) + 2m(mx + b - y_0) = 0$$

$$(1 + m^2)x + (mb - my_0 - x_0) = 0$$

So, the coordinates of Q are

$$x_{min} = \frac{x_0 + my_0 - mb}{1 + m^2}$$

$$y_{min} = \frac{m}{1+m^2}(x_0 + my_0 - mb) + b$$

The slope of the line between (x_0, y_0) and (x_{min}, y_{min}) is

$$\frac{y_{min} - y_0}{x_{min} - x_0} = \frac{\frac{m}{1+m^2}(x_0 + my_0 - mb) + b - y_0}{\frac{1}{1+m^2}(x_0 + my_0 - mb) - x_0}$$

$$= \frac{m(x_0 + my_0 - mb) + (1+m^2)(b - y_0)}{x_0 + my_0 - mb - (1+m^2)x_0}$$

$$= \frac{mx_0 + \cancel{m^2y_0} - \cancel{m^2b} + b - y_0 + \cancel{m^2b} - \cancel{m^2y_0}}{\cancel{x_0} + my_0 - mb - \cancel{x_0} - m^2x_0}$$

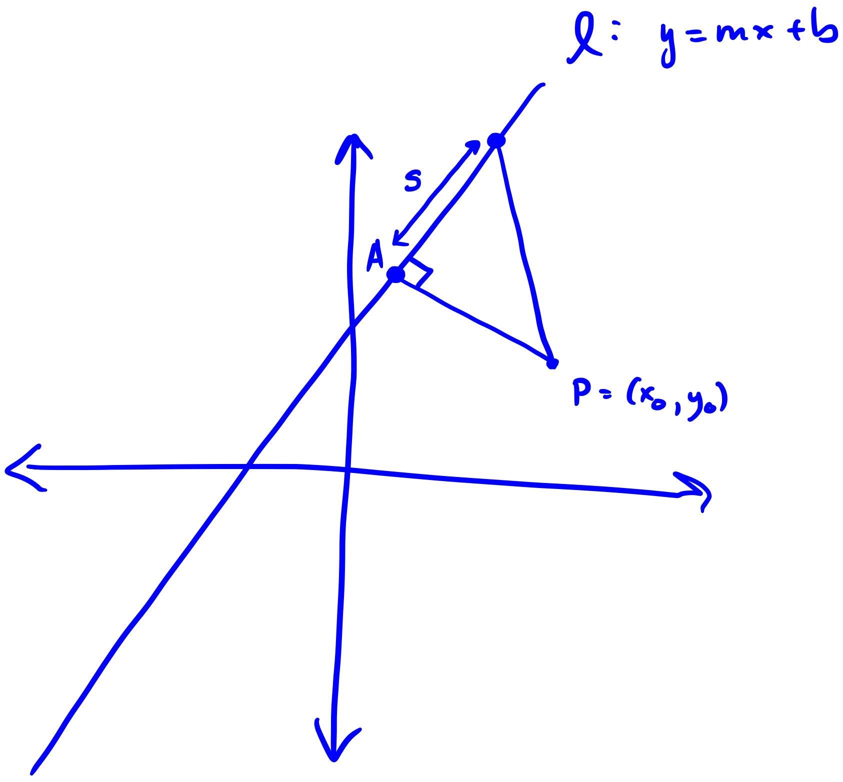
$$= \frac{mx_0 + b - y_0}{-m^2x_0 - mb + my_0}$$

$$= \frac{mx_0 + b - y_0}{-m(mx_0 + b - y_0)}$$

$$= -\frac{1}{m}$$

This implies $\overline{PQ}$ is perpendicular to ℓ. $\blacksquare$

Method 2.



Let A denote the point on ℓ such that $\overline{AP}$ is perpendicular to ℓ.

We wish to show that $A = Q$.

For a point B on ℓ, let $s \geq 0$ denote the distance from B to A.

Let d_0 denote the distance from A to P. This is a constant.

The distance from P to B is

$$f(s) = \sqrt{s^2 + d_0^2}$$

This is minimized exactly when

$$f(s)^2 = s^2 + d_0^2$$

is minimized. Since $f(s)^2$ is increasing, since $s \geq 0$, the minimum occurs at $s = 0$, i.e. A attains the minimal distance, so $A = Q$ ∎

In-Class Quiz 5

November 27, 2019

Name:

1. Find the local extrema of the function

$$f(x) = \sin(x) - \cancel{2}\tfrac{1}{2}x$$

on the interval $(0, \pi)$ and classify them as local minima or local maximua.

Since $f(x)$ is differentiable on $(0,\pi)$, there are no singular points. The stationary points occur at

$$f'(x) = 0$$

$$\cos(x) - \frac{1}{2} = 0$$

$$x = \frac{\pi}{3}$$

Then $f''(x) = -\sin(x)$

$$f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0$$

So $\boxed{x = \frac{\pi}{3} \text{ is a local maximum.}}$

Take-Home Quiz 5

Due Wednesday, November 20, 2019

1. Given that $|f'(x)| \leq 1$ for all real numbers x, show that
$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|$$
for all real numbers x_1 and x_2.

For $x_1 < x_2$, f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By the Mean Value Theorem, there is a point c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Longrightarrow |f'(c)| = \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq 1$$

$$\Longrightarrow |f(x_2) - f(x_1)| \leq |x_2 - x_1| \qquad \boxtimes$$

Take-Home Quiz 4

Due Monday, November 18, 2019

1. Show that the equation
$$x^n + ax + b = 0,$$
where n is an odd positive integer, has at most 3 distinct real roots.

Assume $f(x) = x^n + ax + b = 0$ has 4 distinct roots.

Call them $r_1 < r_2 < r_3 < r_4$. Then

$$f(r_1) = f(r_2) = f(r_3) = f(r_4) = 0,$$

so the MVT says there are points

$$c_1 \in (r_1, r_2)$$
$$c_2 \in (r_2, r_3)$$
$$c_3 \in (r_3, r_4)$$

such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$. Again, by the MVT, there are distinct points

$$d_1 \in (c_1, c_2)$$
$$d_2 \in (c_2, c_3)$$

such that $f''(d_1) = f''(d_2) = 0$.

However, $f''(x) = n(n-1)x^{n-2} = 0$

has only 1 solution $x = 0$.

This is a contradiction, so our original assumption that $f(x) = 0$ has 4 distinct solutions must have been false. Therefore, $f(x) = 0$ has at most 3 solutions. ∎