QUIZ 1: MATH 13100

15 MINUTES

Show all work. You shouldn't need a calculator, but you can use one.

(1) Solve the following inequalities. Write the answer in interval notation.
 (a) $x^2 + 8x + 12 \geq 0$
 (b) $|2x + 3| \geq 5$

(2) Let $f(x) = \frac{x+3}{7x-1}$ and $g(x) = \frac{1}{x^2-9}$. What is
 (a) $(f \cdot g)(x)$ (product of f and g)?
 (b) $(f \circ g)(x)$ (composition of f and g)?
 (c) the natural domain of $f \cdot g$? Write your answer in interval notation.
 (d) the natural domain of $f \circ g$? Write your answer in interval notation.

1a $x^2 + 8x + 12 \geq 2$

$(x+6)(x+2) \geq 0$ Zeroes: $x = -2, -6$

-6 -2

Test points: $x = 0 \Longrightarrow (x+6)(x+2) = 12 \geq 0$

$x = -3 \Longrightarrow (x+6)(x+2) = (3)(-1) = -3 < 0$

$x = -7 \Longrightarrow (x+6)(x+2) = 5 > 0$

Answer: $(-\infty, -6] \cup [-2, \infty)$.

b) $|2x+3| \geq 5$

Equality occurs: $|2x+3| = 5$

$2x+3 = 5$ or $2x+3 = -5$

$x = 1$ $x = -4$

-4 1

Test points: $x = -5 \Longrightarrow |2x+3| = |-7| = 7 > 5$

$x = 0 \Longrightarrow |2x+3| = 3 < 5$

$x = 2 \Longrightarrow |2x+3| = 7 > 5$

Answer:

$(-\infty, -4] \cup [1, \infty)$

2a $(f \cdot g)(x) = f(x) \cdot g(x)$

$$= \frac{x+3}{7x-1} \cdot \frac{1}{x^2-9}$$

$$= \frac{x+3}{(7x-1)(x^2-9)}$$

b) $(f \circ g)(x) = f(g(x))$

$$= \frac{g(x)+3}{7g(x)-1}$$

$$= \frac{\frac{1}{x^2-9}+3}{7\frac{1}{x^2-9}-1}$$

$$= \frac{1+3(x^2-9)}{7-(x^2-9)} \cdot \frac{x^2-9}{x^2-9}$$

$$= \frac{3x^2-26}{16-x^2}$$

c) Natural Domain:

$7x-1 \neq 0 \Longrightarrow x \neq 1/7$
$x^2-9 \neq 0 \Longrightarrow x \neq \pm 3$

$$(-\infty,-3) \cup (-3,\tfrac{1}{7}) \cup (\tfrac{1}{7},3) \cup (3,\infty)$$

d) Natural Domain:

$16-x^2 \neq 0 \Longrightarrow x \neq \pm 4$
$x^2-9 \neq 0 \Longrightarrow x \neq \pm 3$

$$(-\infty,-4) \cup (-4,-3) \cup (-3,3) \cup (3,4) \cup (4,\infty)$$

Quiz 2: Math 13100

15 minutes

October 12, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. Determine whether the following limits exist. If they exist, compute them.
   (a) $$\lim_{x\to 3} x^2 - 3$$
   (b) $$\lim_{x\to 2} \frac{\sqrt{x}+3}{x-2}.$$
   (c) $$\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$$
2. Prove that
$$\lim_{x\to 2}(-3x) = -6,$$
using the ϵ-δ definition of limits.

1a $\lim_{x\to 3} x^2-3 = 3^2-3 = 6$

1b $\lim_{x\to 2} \frac{\sqrt{x}+3}{x-2}$ does not exist since $\sqrt{2}+3 \neq 0$ and $2-2=0$

1c $\lim_{x\to 1} \frac{x^2+4x-5}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+5)}{x-1}$

$= \lim_{x\to 1} x+5$

$= 1+5$

$= 6$

2 Let $\varepsilon > 0$ and pick $\delta = \frac{\varepsilon}{3}$. Then if $|x-2| < \varepsilon/3$, we have $|x-2| < \varepsilon/3 \implies |3x-6| < \varepsilon \implies |(-3x)-(-6)| < \varepsilon$.

Therefore, $\lim_{x\to 2} (-3x) = -6$.

Quiz 3: Math 13100

10 minutes

October 20, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. Determine the value of the limit (give either a real number, ∞ or $-\infty$). Show all steps just as I did in class.
   (a) $\displaystyle\lim_{x\to 2^-} \frac{x^2+x-6}{x^2-4x+4}$
   (b) $\displaystyle\lim_{x\to -\infty} \frac{2x^3-x+1}{-x^3+2x^2-x+2}$
2. Write the formal definition of the following using ϵ, δ and/or M. (Here, L and c are real numbers.)
   (a) $\displaystyle\lim_{x\to -\infty} f(x) = L$
   (b) $\displaystyle\lim_{x\to c^-} f(x) = L$

1a) $$\lim_{x \to 2^-} \frac{x^2+x-6}{x^2-4x+4} = \lim_{x \to 2^-} \frac{(x+3)(x-2)}{(x-2)^2}$$

$$= \lim_{x \to 2^-} \frac{x+3}{x-2} = \pm \infty$$

As x approaches 2 from the left, $x<2$, so $x+3>0$ and $x-2<0$. Therefore, $\frac{x+3}{x-2}<0$

so $$\lim_{x \to 2^-} \frac{x+3}{x-2} = -\infty.$$

b) $$\lim_{x \to -\infty} \frac{2x^3-x+1}{-x^3+2x^2-x+2} = \lim_{x \to -\infty} \frac{\frac{1}{x^3}(2x^3-x+1)}{\frac{1}{x^3}(-x^3+2x^2-x+2)}$$

$$= \lim_{x \to -\infty} \frac{2-\frac{1}{x^2}+\frac{1}{x^3}}{-1+\frac{2}{x}-\frac{1}{x^2}+\frac{2}{x^3}}$$

$$= \frac{2}{-1} = -2.$$

2 a) For every $\varepsilon>0$, there is an $M<0$ such that

$x<M \implies |f(x)-L|<\varepsilon$; in other words

if $x<M$, then $|f(x)-L|<\varepsilon$.

b) For every $\varepsilon>0$, there is a $\delta>0$ such that if $0<c-x<\delta$ then $|f(x)-L|<\varepsilon$.

Quiz 4: Math 13100

15 minutes

October 27, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. For each of the terms/phrases below, give the full definition.
   (a) "f is continuous at c"
   (b) "derivative of f"
2. If f is a continuous function such that $\lim_{x\to-\infty} f(x) = -1$ and $\lim_{x\to\infty} f(x) = 2$, must there be a real number c such that $f(c) = 1$? Why?
3. Find the derivative of $f(x) = \sqrt{2x+1}$, using the limit definition of derivative. (If you've had calculus before, do not use the Chain Rule.)

1a. A function f is continuous at a point c if the following three conditions hold:

1) $\lim_{x\to c} f(x)$ exists
2) $f(c)$ is defined
3) $\lim_{x\to c} f(x) = f(c)$.

b. The derivative of a function f is defined to be the function given by either of the following limits

$$f'(x) = \lim_{x_0\to x} \frac{f(x_0)-f(x)}{x_0-x} \quad \text{or} \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

2. Yes, because of the intermediate value theorem. A more complete proof would be as follows: Since $\lim_{x\to-\infty} f(x) = -1$, there is some point a such that $f(a) < \frac{1}{2}$. Since $\lim_{x\to\infty} f(x) = 2$, there is some point b such that $f(b) > \frac{3}{2}$. Therefore, by the intermediate value theorem, since f is continuous, there is a point c between a and b such that $f(c) = 1$, since $\frac{1}{2} < 1 < \frac{3}{2}$.

3. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

Multiply by $\dfrac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} = 1$:

$$= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \lim_{h\to 0} \frac{[2(x+h)+1] - [2x+1]}{h\left[\sqrt{2(x+h)+1} + \sqrt{2x+1}\right]}$$

$$= \lim_{h\to 0} \frac{2x+2h+1-2x-1}{h\left[\sqrt{2(x+h)+1} + \sqrt{2x+1}\right]}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \frac{2}{\sqrt{2x+1} + \sqrt{2x+1}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

Quiz 5: Math 13100

15 minutes

November 1, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. Calculate the derivatives of the following functions. You do not need to simplify the final expression you obtain.

   (a) $f(x) = (\sqrt{x} + 4x)(x^2 - 2)$

   (b) $f(x) = \dfrac{1}{(x+1)(x^2+3)}$

   (c) $f(x) = \dfrac{\sqrt{x}+2}{(\sqrt{x}+4x)(x^2-2)}$

   (Hint: You found the derivative of the denominator in part a.))

1a. Product Rule: Let $g(x) = \sqrt{x} + 4x$ $\quad g'(x) = \frac{1}{2\sqrt{x}} + 4$

$h(x) = x^2 - 2$ $\quad h'(x) = 2x$

Then $f(x) = g(x) \cdot h(x)$ so

$f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$

$$\boxed{f'(x) = \left[\frac{1}{2\sqrt{x}} + 4\right](x^2-2) + \left[\sqrt{x} + 4x\right] \cdot (2x)}$$

1b. Quotient Rule.

$f(x) = \dfrac{g(x)}{h(x)}$ where $g(x) = 1$, $h(x) = (x+1)(x^2+3)$

Then $g'(x) = 0$ and $h'(x) = (x^2+3) + 2x(x+1)$ via product rule.

Then $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2} = \dfrac{0 - (1)\left[(x^2+3) + (2x)(x+1)\right]}{(x+1)^2(x^2+3)^2}$

$$\boxed{f'(x) = \frac{-3x^2 - 2x - 3}{(x+1)^2(x^2+3)^2}}$$

1c Product Rule: Let $g(x) = \sqrt{x} + 2$

$h(x) = (\sqrt{x} + 4x)(x^2 - 2)$.

Then $g'(x) = \frac{1}{2\sqrt{x}}$ and from problem 1a, we have

$$h'(x) = \left[\frac{1}{2\sqrt{x}} + 4\right](x^2-2) + \left[\sqrt{x} + 4x\right] \cdot (2x).$$

Thus, $f(x) = \frac{g(x)}{h(x)}$ and

$$f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{[h(x)]^2}$$

$$\boxed{f'(x) = \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+4x)(x^2-2) - (\sqrt{x}+2)\left[\left(\frac{1}{2\sqrt{x}}+4\right)(x^2-2) + (\sqrt{x}+4x)(2x)\right]}{(\sqrt{x}+4x)^2(x^2-2)^2}}$$

Quiz 6: Math 13100

10 minutes

November 10, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. Find the third derivative of the function $f(x) = \sqrt{3x^2 - 1}$.

$$f(x) = \sqrt{3x^2 - 1}$$

Chain Rule : $$f'(x) = 6x \cdot \frac{1}{2\sqrt{3x^2-1}} = 3x(3x^2-1)^{-1/2}$$

Chain + Product : $$f''(x) = 3(3x^2-1)^{-1/2} + 3x \cdot \left[-\frac{1}{2}(3x^2-1)^{-3/2} \cdot (6x)\right]$$

$$= 3(3x^2-1)^{-1/2} - 9x^2(3x^2-1)^{-3/2}$$

Simplify $$= \left[3(3x^2-1) - 9x^2\right](3x^2-1)^{-3/2}$$

$$= -3(3x^2-1)^{-3/2}$$

Chain Rule : $$f'''(x) = -3 \cdot \left[-\frac{3}{2}(3x^2-1)^{-5/2} \cdot 6x\right]$$

$$= 27x(3x^2-1)^{-5/2}$$

Quiz 7: Math 13100

10 minutes

November 17, 2016

Show all work. You shouldn't need a calculator, but you can use one.

1. (10 points) Use implicit differentiation to find $\frac{dy}{dx}$ as a function of x and y.
   (a) $y^5 + 5xy^2 - x^4y = 2x^2$.
   (b) $x = \sqrt{y^2 - 1}$.

2. (10 points) Jack is walking along the shore of Lake Michigan and Jill is driving a boat 1 mile from shore in a direction parallel to the shore. They are both moving in opposite directions. Jack is walking at a pace of 3 mph and Jill is driving at a pace of 30 mph. How fast are they moving apart when they are 4 miles away from each other (i.e. when the distance between Jack and Jill is 4 miles)?

1a. $y^5 + 5xy^2 - x^4y = 2x^2$

$(5y^4y') + (5y^2 + 10xyy') - (4x^3y + x^4y') = 4x$

$[5y^4 + 10xy - x^4]\,y' = 4x - 5y^2 + 4x^3y$

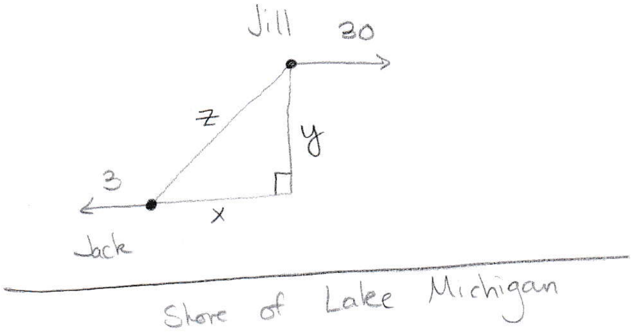
$$\boxed{y' = \frac{4x - 5y^2 + 4x^3y}{5y^4 + 10xy - x^4}}$$

1b $x = \sqrt{y^2-1}$

$1 = 2y \cdot \frac{1}{2\sqrt{y^2-1}}\, y'$

$1 = \frac{y}{\sqrt{y^2-1}} \cdot y' \Longrightarrow \boxed{y' = \frac{\sqrt{y^2-1}}{y}}$

2.



$x^2 + y^2 = z^2$ $\qquad y = 1$ (y is a constant).

$x^2 + 1 = z^2$

when $z = 4$, $x = \sqrt{15}$. Also, $\frac{dx}{dt} = 33$.

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$2(\sqrt{15})(33) = 2(4)\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{33\sqrt{15}}{4} \text{ mph.}$$