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Substitution

Theorem (Substitution for Definite Integrals)

If f and u are functions, then

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Proof $\frac{d}{dx}\left[F(u(x))\right] = F'(u(x)) \cdot u'(x) = f(u(x))\, u'(x)$

$$\int_{u(a)}^{u(b)} f(u)\, du = F(u(b)) - F(u(a))$$
$$= \int_a^b f(u(x))\, u'(x)\, dx. \quad \blacksquare$$

Theorem (Indefinite Integral Substitution)

$$\int f(u(x))\, u'(x) dx = \int f(u)\, du$$

Example $\int x \cos(x^2)\, dx = \int \cos(u) \cdot \frac{1}{2} du$ $\quad u = x^2,\ du = 2x dx$

$$= \frac{1}{2}\sin(u) + C$$
$$= \frac{1}{2}\sin(x^2) + C \quad \blacksquare$$

Even and Odd Functions

A function $f(x)$ is **even** if $f(-x) = f(x)$.
is **odd** if $f(-x) = -f(x)$.

Theorem If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.

Proof $I = \int_{-a}^{a} f(x)\, dx$ $\quad u = -x,\ du = -dx$

$= \int_a^{-a} f(-u)\,(-du)$

$= -\int_a^{-a} -f(u)\, du$

$= -\int_{-a}^{a} f(u)\, du = -I$

Thus, $I = -I$
$2I = 0$
$I = 0. \quad \blacksquare$

Example 1) $\int_{-\pi}^{\pi} \sin(\sin(x))\, dx = 0$

2) $\int_0^{2\pi} \sin(\sin(x))\, dx$ $\quad u = x - \pi,\ du = dx$

$= \int_{-\pi}^{\pi} \sin(\sin(u+\pi))\, du$ $\quad \sin(u+\pi) = -\sin(u)$

$= \int_{-\pi}^{\pi} \sin(-\sin(u))\, du = -\int_{-\pi}^{\pi} \sin(\sin(u))\, du = 0$

3) $\int_0^{2\pi} \sin^3(x)\, dx = \int_{-\pi}^{\pi} \sin^3(u+\pi)\, du = \int_{-\pi}^{\pi} \sin^3(u)\, du = 0.$

Theorem If $f(x)$ is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$

Natural Logarithm

Some functions have antiderivatives that cannot be written in terms of functions already defined.

E.g. $\int \sin(\sin(x))\, dx$

$\int \tan(x^2)\, dx$

Definition The natural logarithm $\ln(x)$ is defined for all $x > 0$ by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

Properties 1) $\ln(1) = 0$

2) $\ln(ab) = \ln(a) + \ln(b).$

$$\int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt$$
$$= \ln(a) + \int_1^b \frac{1}{au}(a\, du) \quad u = \frac{t}{a},\ du = \frac{dt}{a}$$
$$= \ln(a) + \int_1^b \frac{1}{u} du$$
$$= \ln(a) + \ln(b).$$

3) $\ln(a^n) = n \cdot \ln(a)$

$$\ln(a^n) = \ln(\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}})$$
$$= \underbrace{\ln(a) + \ln(a) + \dots + \ln(a)}_{n \text{ times}}$$
$$= n \cdot \ln(a)$$

Indefinite Integral of $\frac{1}{x}$: $\int \frac{1}{x} dx = \ln|x| + C$

For $x > 0$, $\frac{d}{dx}[\ln x] = \frac{1}{x}.$

For $x < 0$, $\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$