Final Exam

Math 15100

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Name: ______________________________

Instructions:

1. Print your name clearly at the top of each page.
2. This midterm has 12 questions for a total of 300 points. The value of each part of each question is stated.
3. Please show your work in the space provided. Partial credit will be awarded for partial or incomplete solutions.
4. **Show all of your work.** If you need more space, you can use the final page, which is blank, or request additional paper.

Good Luck!

Question	Points	Score
1	25	
2	25	
3	40	
4	40	
5	20	
6	20	
7	25	
8	20	
9	20	
10	20	
11	20	
12	25	
Total:	300	

1. (25 points) Using mathematical induction, prove that

$$1 + 3 + \cdots + (2n-3) + (2n-1) = n^2$$

for all $n \geq 1$; that is, the sum of the first n odd numbers is n^2.

Base Case: $n=1$ implies $1 = 1^2$ ✓

Inductive Step:

Assume $1+3+\cdots+(2n-3)+(2n-1) = n^2$.

Then

$$\underbrace{1+3+\cdots+(2n-1)}_{} + (2(n+1)-1) -$$
$$= n^2 + 2(n+1) - 1$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$

Therefore the equality holds for all $n \geq 1$. $\boxtimes$

2. (25 points) Us the ϵ-δ definition of limit to prove that

$$\lim_{x \to -1} \left(x^2 - 3x + 5 \right) = 9.$$

Let $\varepsilon > 0$. Pick $\delta = \min\{1, \frac{\varepsilon}{6}\}$.

If $0 < |x+1| < \delta$, then

$$\begin{aligned}
|(x^2-3x+5)-9| &= |x^2-3x-4| \\
&= |x+1||x-4| \\
&< \delta|x-4| \\
&= \delta|(x+1)-1-4| \\
&= \delta|(x+1)+(-5)| \\
&\leq \delta(|x+1|+|-5|) \\
&< \delta(\delta+5) \\
&\leq \delta \cdot 6 \\
&\leq \varepsilon.
\end{aligned}$$

$\blacksquare$

3. Calculate the following limits or indicate if they do not exist.

(a) (10 points) $\lim_{x\to\frac{1}{2}} \left(x^3 - x^2 + \cos(\pi x)\right)$

$= \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + \cos\left(\frac{\pi}{2}\right)$

$= \frac{1}{8} - \frac{1}{4} + 0$

$= \boxed{-\frac{1}{8}}$

(b) (10 points) $\lim_{x\to -1} \frac{x^2 - x}{x^3 - x^2 + x - 1}$

Plug in $x=-1$, get

$\frac{(-1)^2-(-1)}{(-1)^3-(-1)^2+(-1)-1} = \frac{1+1}{-1-1-1-1} = -\frac{2}{4} = \boxed{-\frac{1}{2}}$

(c) (10 points) $\lim_{x\to -3} \frac{|x+3|}{x^2+3x}$

$\lim_{x\to -3^+} \frac{|x+3|}{x^2+3x} = \lim_{x\to -3^+} \frac{x+3}{x(x+3)} = \lim_{x\to -3^+} \frac{1}{x} = -\frac{1}{3}$

$\lim_{x\to -3^-} \frac{|x+3|}{x^2+3x} = \lim_{x\to -3^-} \frac{-(x+3)}{x(x+3)} = \lim_{x\to -3^+} -\frac{1}{x} = \frac{1}{3}$

Since, $-\frac{1}{3} \neq \frac{1}{3}$, the limit DNE.

(d) (10 points) $\lim_{x\to 0} \frac{1-\cos(2x)}{3x} = \lim_{x\to 0} \frac{1-\cos(2x)}{2x} \cdot \frac{2}{3}$

$= 0 \cdot \frac{2}{3}$

$= \boxed{0}$

4. Calculate the derivatives of the following functions.

(a) (10 points) $f(x) = x^7 - 3x^{5/2} + 5 - \dfrac{1}{x^{6/7}}$

$$f'(x) = 7x^6 - 3\cdot\frac{5}{2}x^{3/2} + 0 + \frac{6}{7}x^{-13/7}$$

$$\boxed{f'(x) = 7x^6 - \frac{15}{2}x^{3/2} + \frac{6}{7}x^{-13/7}}$$

(b) (10 points) $f(x) = \tan(x) - \cot(x)$

$$\boxed{f'(x) = \sec^2(x) + \csc^2(x)}$$

(c) (10 points) $f(x) = \dfrac{x^2}{x^4 - 1}$

$$f'(x) = \frac{2x(x^4-1) - x^2(4x^3)}{(x^4-1)^2}$$

$$f'(x) = \frac{2x^5 - 2x - 4x^5}{(x^4-1)^2}$$

$$\boxed{f'(x) = \frac{-2x^5 - 2x}{(x^4-1)^2}}$$

(d) (10 points) $f(x) = x\sin(x)$

$$\boxed{f'(x) = \sin(x) + x\cos(x)}$$

5. (a) (10 points) State the Intermediate Value Theorem, giving all hypotheses and conclusions.

Let $f(x)$ be continuous on $[a,b]$. For any value K between $f(a)$ and $f(b)$, there is some c in $[a,b]$ such that

$$f(c) = K.$$

(b) (10 points) Show that the equation

$$x^7 + 3x^5 + 2x - 1 = 0$$

has a solution in $[0, 1]$. How many solutions does this equation have in the entire real line? Explain.

Let $f(x) = x^7 + 3x^5 + 2x - 1$. Then

$$f(0) = -1$$

$$f(1) = 5.$$

Therefore, there is some c in $[0,1]$ such that $f(c) = 0$.

There are no other solutions since

$$f'(x) = 7x^6 + 15x^4 + 2 > 0$$

for all x.

6. (20 points) Calculate

$$\frac{d^2}{dx^2}\left[\frac{x}{x+1}\right].$$

$$\frac{d}{dx}\left[\frac{x}{x+1}\right] = \frac{(1)(x+1)-(x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$\frac{d^2}{dx^2}\left[\frac{x}{x+1}\right] = \boxed{\frac{-2}{(x+1)^3}}$$

7. (25 points) Find the equation of the tangent line to the curve given by

$$x^3 - y^2 - \tan(xy) = 1$$

at the point $(1, 0)$.

Slope

$$3x^2 - 2yy' - (y + xy')\sec^2(xy) = 0$$

At $x=1$, $y=0$,

$$3 - 0 - (0 + y')\sec^2(0) = 0$$

$$3 - y' = 0$$

$$y' = 3$$

Point-Slope Form

$$y - 0 = 3(x-1)$$

$$\boxed{y = 3x - 3}$$

8. (20 points) ~~Where~~ On what intervals is the function

$$f(x) = \frac{x^2}{x^4+16}$$

decreasing?

$$f'(x) = \frac{2x(x^4+16) - x^2(4x^3)}{(x^4+16)^2} = \frac{2x^5 + 32x - 4x^5}{(x^4+16)^2} = \frac{32x - 2x^5}{(x^4+16)^2} = 0$$

$$32x - 2x^5 = 0$$

$$2x(16 - x^4) = 0$$

$$2x(4 - x^2)(4 + x^2) = 0$$

$$2x(2-x)(2+x)(4+x^2) = 0$$

$x = 0$
$x = 2$
$x = -2$

-3 -2 -1 0 1 2 3

Sign of $f'(x)$: + − + −

$\boxed{f(x) \text{ is decreasing on } [-2, 0] \text{ and } [2, \infty)}$

9. (20 points) Find the global maximum and global minimum of the function

$$f(x) = \frac{x^{2/3}}{x^2+4}$$

on the interval $[-1, 10]$.

$$f'(x) = \frac{\frac{2}{3}x^{-1/3}(x^2+4) - x^{2/3}(2x)}{(x^2+4)^2}$$

$$= \frac{\frac{2}{3}x^{5/3} + \frac{8}{3}x^{-1/3} - 2x^{5/3}}{(x^2+4)^2}$$

Singular Points: $x=0$

Stationary Points: $\frac{2}{3}x^{5/3} + \frac{8}{3}x^{-1/3} - 2x^{5/3} = 0$

multiply through by $x^{1/3}$

$$\frac{2}{3}x^2 + \frac{8}{3} - 2x^2 = 0$$

$$-\frac{4}{3}x^2 = -\frac{8}{3}$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Endpoints: $x = -1, 10$

$f(-1) = \frac{(-1)^{2/3}}{1+4} = \frac{1}{5}$

$f(0) = 0$

$f(10) = \frac{10^{2/3}}{100+4} = \frac{100^{1/3}}{104}$

$f(\sqrt{2}) = \frac{2^{1/3}}{2+4} = \frac{2^{1/3}}{6}$

$\boxed{\text{Global Max: } x=\sqrt{2} \quad \text{Global Min: } x=0}$

10. (20 points) A spherical mothball of radius 3 cm is dissolving in water at a rate of 0.4 cm^3/s. At what rate is the mothball dissolving when the radius is 1 cm?

Hint: The volume of a sphere is given by $V = \frac{4\pi}{3}r^3$.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\quad$ $\frac{dr}{dt}$ is constant

$0.4 = 4\pi(3)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{90\pi}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi(1)^2 \frac{1}{90\pi}$

$\frac{dV}{dt} = \frac{4\pi}{90\pi} = \boxed{\frac{2}{45} \text{ cm}^3/\text{s}}$

11. (20 points) A company is tasked with designing a cylindrical can to hold chicken noodle soup. They are only allowed to use 30 cm^2 of tin to make the can. What are the dimensions of the can that will hold the maximum amount of soup?

Hint: The volume and surface area of a cylinder are

$$V = \pi r^2 h \qquad \text{and} \qquad S = 2\pi r^2 + 2\pi r h.$$

$S = 2\pi r^2 + 2\pi r h = 30$

$h = \dfrac{30 - 2\pi r^2}{2\pi r}$

$V = \pi r^2 h$

$V = \pi r^2 \dfrac{30 - 2\pi r^2}{2\pi r}$

$V = \frac{1}{2} r (30 - 2\pi r^2)$

$\dfrac{dV}{dr} = \frac{1}{2}(30 - 2\pi r^2) + \frac{1}{2} r(-4\pi r)$

$= 15 - \pi r^2 - 2\pi r^2 = 0$

$15 = 3\pi r^2$

$r^2 = \dfrac{5}{\pi}$

$\boxed{r = \sqrt{\frac{5}{\pi}} \text{ cm}}$

$h = \dfrac{30 - 2\pi r^2}{2\pi r}$

$= \dfrac{30 - 2\pi\left(\frac{5}{\pi}\right)}{2\pi\sqrt{\frac{5}{\pi}}}$

$= \dfrac{30 - 10}{2\sqrt{5\pi}}$

$\boxed{h = \dfrac{10}{\sqrt{5\pi}} \text{ cm}}$

12. Determine whether each of the following statements is true or false. Circle true or false. If the statement is false, give a reason.

(a) (5 points) The function

$$f(x) = x^2 + 2$$

defined on $[0, 32)$ is continuous, hence by the Extreme Value Theorem, it attains both a global maximum and global minimum in $[0, 32)$.

True **False**

$[0,32)$ is not closed.

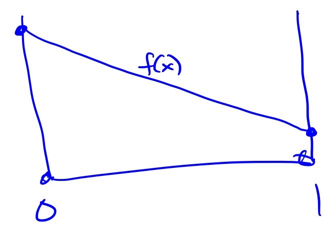
(b) (5 points) Let f and g be differentiable functions defined on $(-\infty, \infty)$. If $f(x) \le g(x)$ for all x, then $f'(x) \le g'(x)$.

True **False**

Consider $x^2 \geq 0$ for all x, but
$2x \geq 0$ is not true when x is negative.

(c) (5 points) Let f be continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $f'(x) < 0$ for all x in $(0, 1)$, then f attains its global maximum at $x = 1$.

True **False**



$x=1$ will be its global minimum since
$f'(x)<0 \Longrightarrow f$ is decreasing.

(d) (5 points) If f is a function such that $\lim_{x\to 0} f(x) = 3$, then $\lim_{x\to -2} f(x + 2) = 3$.

True **False**

(e) (5 points) Let

$$f(x) = x^{2/3} + 1.$$

Since $f(-1) = f(1) = 2$, Rolle's theorem implies that there is some point c in $[-1, 1]$ such that $f'(c) = 0$.

True **False**

Rolle's Theorem requires that f is differentiable in $(-1,1)$.
But $f(x) = x^{2/3}+1$ is not differentiable at $x=0$.