# HOMEWORK SOLUTIONS MATH 21100

## 1. Homework Set 1

1.1. **Problem 1.** The series is given by $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. This series is convergence since the terms tend to 0 and alternate in sign. However, this series is not absolutely convergent since

$$\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{2k+1} \right| = \sum_{k=0}^{\infty} \frac{1}{2k+1} > \sum_{k=0}^{\infty} \frac{1}{2k+2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k},$$

and the series on the far right is the harmonic series which diverges to $\infty$.

1.2. **Problem 2.** Suppose $x \neq 1$. Consider the series $\sum_{k=0}^{\infty} x^k$ and the partial sums $S_N = \sum_{k=0}^{N} x^k$. For $N = 0$, we have $S_0 = 1 = \frac{1-x}{1-x} = 1$. This proves the base case for induction. Assume $S_N = \frac{1-x^{N+1}}{1-x}$. Then,

$$S_{N+1} = S_N + x^{N+1} = \frac{1 - x^{N+1}}{1-x} + \frac{1-x}{1-x} x^{N+1} = \frac{1 - x^{N+2}}{1-x}.$$

Hence, $S_N = \frac{1-x^{N+1}}{1-x}$ for all nonnegative integers $N$ by mathematical induction.

When $|x| > 1$, then $|x|^N$ tends to infinity. Thus, $\sum_{k=0}^{\infty} x^N$ cannot be divergent since the sequence of terms $x^N$ do not converge to 0.

When $x = 1$, we have that $S_N = \sum_{k=0}^{N} 1^k = N + 1$.

1.3. **Problem 3.** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ where $q > 1$. Here, we can use the integral test. We have the approximation

$$\sum_{n=1}^{N} \frac{1}{n^q} = 1 + \sum_{n=2}^{N} \frac{1}{n^q} \leq 1 + \int_1^N \frac{dx}{x^q} = 1 + \frac{1}{1-q} x^{1-q} \bigg|_1^N = \frac{1}{q-1} - \frac{N^{1-q}}{q-1}.$$

The sequence of partial sums is a monotone increasing sequence, since the terms are positive, and is bounded above by $\frac{1}{q-1}$. Hence, it converges.

1.4. **Problem 4.** The answer given by the program is never close to the exact value because of roundoff error. In computing the series, we must add 1 (the first term) to an extremely small number $\frac{(-25)^N}{N!}$ for some large $N$. Once this term gets below $2^{-52} \sim 2 \cdot 10^{-16}$, it is rounded to 0, so the sequence of partial sums (according to the program) becomes constant). This occurs for $N = 96$.

## 2. Homework Set 2

2.1. **Problem 1.** Let $\alpha$ and $\beta$ be real numbers. Suppose $f(h) = O(h^\alpha)$ as $h \to 0$. This means that $|f(h)| \leq C|h|^\alpha$ for some constant $C$. We then get that

$$|h^\beta f(h)| = |h|^\beta |f(h)| \leq C|h|^\beta |h|^\alpha = C|h|^{\alpha+\beta}$$

so that $h^\beta f(h) = O(h^{\alpha+\beta})$ as $h \to 0$.

*Date*: 9 June 2016.

2.2. **Problem 2.** For this problem, we wish to consider the error:

$$\left|\int_0^1 f(x)\,dx - h\sum_{k=0}^{N-1} f\left((k+1/2)h\right)\right| = \left|\sum_{k=0}^{N-1}\int_{kh}^{(k+1)h} f(x)\,dx - h\sum_{k=0}^{N-1} f\left((k+1/2)h\right)\right|$$
$$\leq \sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h} f(x)\,dx - hf\left((k+1/2)h\right)\right|.$$

We then expand $f$ into its Taylor series approximation around $(k+1/2)h$ under the integral:

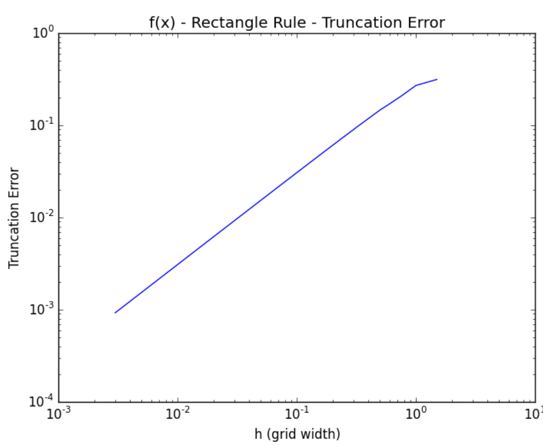
$$\begin{aligned}
&\sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h} f(x)\,dx - hf\left((k+1/2)h\right)\right| \\
&= \sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h} \Big[f((k+1/2)h) + (x-(k+1/2)h)f'((k+1/2)h) + O((x-(k+1/2)h)^2)\Big]\,dx - hf\left((k+1/2)h\right)\right| \\
&= \sum_{k=0}^{N-1}\left|\Big[hf((k+1/2)h) + O(h^3)\Big]\,dx - hf\left((k+1/2)h\right)\right| \\
&= \sum_{k=0}^{N-1} O(h^3) = \frac{1}{h}O(h^3) \\
&= O(h^2).
\end{aligned}$$

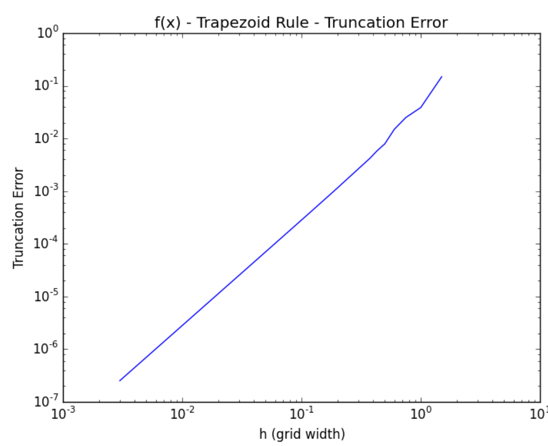
In the step from the second line to the third line, the term with $f'$ vanishes because its integral is 0. The second to last line follows since $N = \frac{1}{h}$.

2.3. **Problem 3.** Let $f(x) = \frac{x}{1+x^4}$. The relation between $h$ and $N$ now becomes $h = \frac{2-(-1)}{N} = \frac{3}{N}$. The choice of grid points becomes $x_j = -1 + hj$ for $j = 0, \ldots, N$. The value of $\int_{-1}^{2} f(x)\,dx$ can be found by running the rectangle method for large enough values of $N$ or by directly integrating using the substitution method. The value is

$$\frac{1}{2}(\tan^{-1}(4) - \tan^{-1}(1)) = 0.2702097501.$$

We have the following loglog plots for these methods.




Note that the rate of convergence can be detected from the slope of the line. This is because if the error $E$ is $O(h^k)$, that is $E \sim h^k$, then $\log E \sim k \log h$. The slopes of the plots (hence the rates of convergence) for the rectangle method is $O(h)$ and that of the trapezoidal method is $O(h^2)$.

2.4. **Problem 4.** Below are the two loglog plots of the error versus $h$ for the rectangle method and the trapezoidal method.

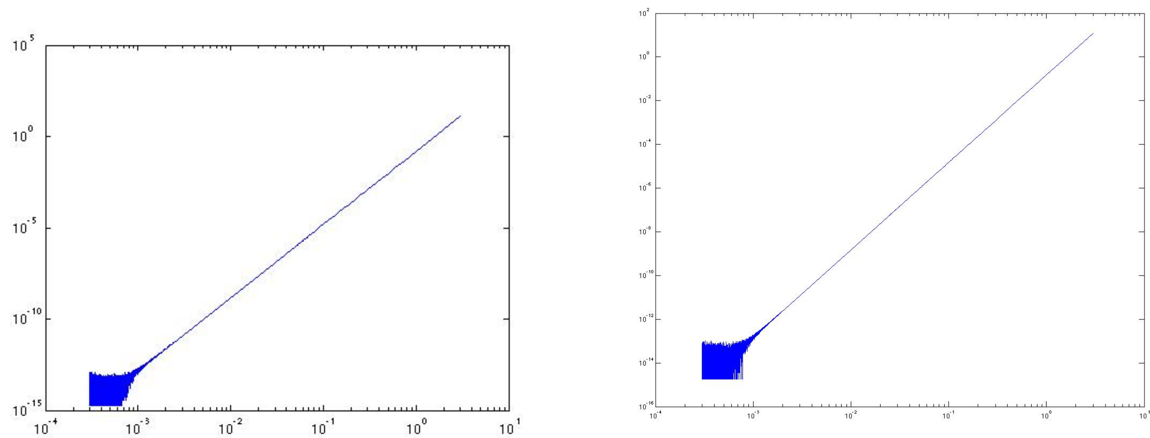



FIGURE 1. Rectangle (left) and Trapezoidal (right) Methods

Note that we can measure the slope of each of these plots to conclude that they each converge with rate $O(h^4)$. To explain why both the rectangle method and trapezoidal method give order 4 convergence, we first get, using the rectangle method,

$$\begin{aligned}
h\sum_{j=0}^{N-1}(hj)^3(hj-2)^2 &= h\sum_{j=0}^{N-1}(hj)^3((hj)^2-6hj+9) \\
&= h\sum_{j=0}^{N-1}(hj)^5-6(hj)^4+9(hj)^3 \\
&= h^6\sum_{j=0}^{N-1}j^5-6h^5\sum_{j=0}^{N-1}j^4+9h^4\sum_{j=0}^{N-1}j^3 \\
&\vdots \\
&= \frac{243(N^4-1)}{20N^4} = 12.15+O(h^4).
\end{aligned}$$

For the $\vdots$ I used mathematica to evaluate the $\sum_{j=0}^{N-1} j^k$ (which are polynomials in $N$) and used the relation $h=\frac{3}{N}$.

The trapezoidal rule has the same rate of convergence since these two methods only differ on the values of $g$ at the endpoints $-1$ and 2 and these values happen to be 0.

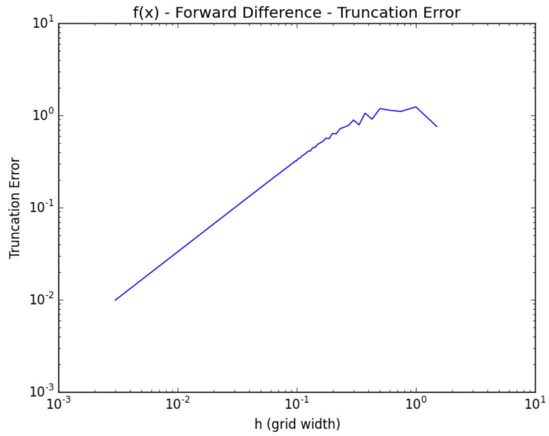
2.5. **Problem 5.** The exact value of $f'(1)$ is $-\frac{1}{2}$. You can get an approximation of this by running the code for small enough $h$. We have the following loglog plots for the forward difference and centered difference.

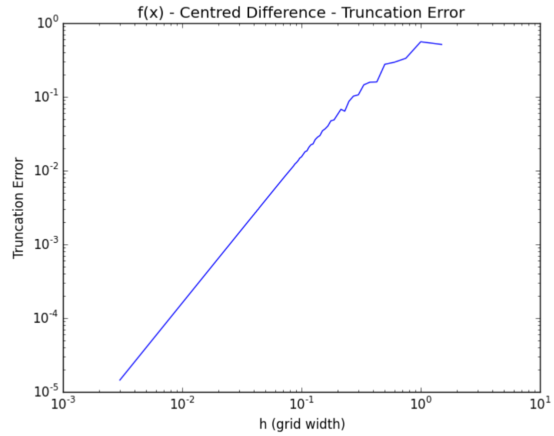
The slopes of these graphics show that the forward difference is $O(h)$ and centered difference is $O(h^2)$.

## 3. Homework 3

3.1. **Problem 1.** Here, we have that the Lagrange polynomials are

$$L_0(x)=\frac{(x-h)(x-2h)}{2h^2}, L_h(x)=\frac{x(x-2h)}{-h^2}, \text{ and } L_{2h}(x)=\frac{x(x-h)}{2h^2}$$




and so the interpolation quadratic is given by

$$\begin{aligned} p(x) &= f(0)L_0(x) + f(h)L_h(x) + f(2h)L_{2h}(x) \\ &= f(0)\frac{(x-h)(x-2h)}{2h^2} - f(h)\frac{x(x-2h)}{h^2} + f(2h)\frac{x(x-h)}{2h^2} \\ &= \left[\frac{f(0)-f(h)+f(2h)}{2h^2}\right]x^2 + \left[\frac{-3f(0)+4f(h)-f(2h)}{2h}\right]x + f(0). \end{aligned}$$

Taking the derivative of $p$ at $x = 0$, gives $p'(0) = \frac{1}{2h}[-3f(0) + 4f(h) - f(2h)]$. This approximates the derivative $f'(0)$.

Now, to show that the error is $O(h^2)$, we expand $f$ into its Taylor series around $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3).$$

Plugging in $x = h$ and $x = 2h$ gives

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(0) + O(h^3)$$

and

$$f(2h) = f(0) + 2hf'(0) + 2h^2 f''(0) + O(h^3).$$

We rewrite the first of these expression (for $x = h$) as

$$f'(0) = \frac{f(h)-f(0)}{h} - \frac{h}{2}f''(0) + O(h^2).$$

The error is then given by

$$\begin{aligned} |f'(0) - p'(0)| &= \left|\frac{f(h)-f(0)}{h} - \frac{-3f(0)+4f(h)-f(2h)}{2h} - \frac{h}{2}f''(0)\right| + O(h^2) \\ &= \left|\frac{f(0)-2f(h)+f(2h)}{2h} - \frac{h}{2}f''(0)\right| + O(h^2) \\ &= \left|\frac{f(0)}{2h} - \frac{1}{h}\left[f(0) + hf'(0) + \frac{h^2}{2}f''(0)\right] + \frac{1}{2h}\left[f(0) + 2hf'(0) + 2h^2 f''(0)\right] - \frac{h}{2}f''(0)\right| + O(h^2) \\ &= O(h^2). \end{aligned}$$

The $O(h^3)$ terms in the Taylor expansions in the third line are absorbed into the $O(h^2)$ term. This shows that this method is converges to $f'(0)$ at a rate of $O(h^2)$.

3.2. **Problem 2.** To make this problem much easier, it is worthwhile to note that both integration and Simpson's rule are linear. That is, fixing a set of grid points $x_j$, let $S(f)$ denote the result of applying Simpson's rule to the function $f$, then $S(af+bg) = aS(f) + bS(g)$ where $a$ and $b$ are any real numbers and $f$ and $g$ are any functions. Therefore, it suffices to consider the error on each of the terms in the quadratic polynomial $p(x) = ax^3 + bx^2 + cx + d$. For this, we already know that the quadratic, linear, and constant terms are approximated exactly by Simpson's rule. Therefore, we just need to check the function $x^3$. We have that

$$S(x^3) = \frac{h}{3}[(-h)^3 + (0)^3 + h^3] = 0 = \int_{-h}^{h} x^3\,dx.$$

Therefore, $S(p) = \int_{-h}^{h} p(x)\,dx$ which proves that Simpson's rule integrates cubic polynomials exactly.

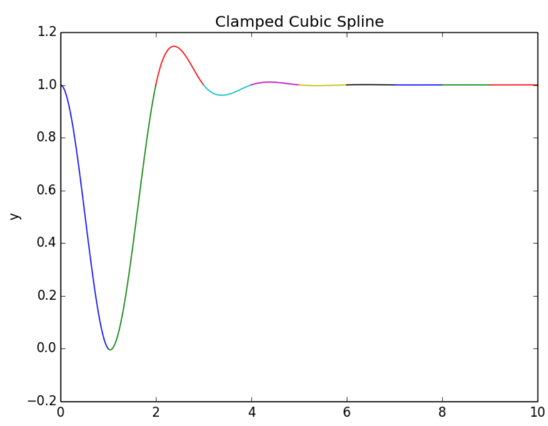
Now, we need to show that this method is $O(h^5)$. Let $f$ be any function $C^4$ on $[-h,h]$ and expand $f$ into it's Taylor series

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \frac{1}{24}f^{(4)}(0)x^4 + O(x^5).$$

Note that this has the form $f(x) = p(x) + Cx^4 + O(h^5)$, where $C$ is a constant and $p$ is a cubic polynomial so that $S(p) = \int_{-h}^{h} p(x)\,dx$ exactly. Then, the error of this method is

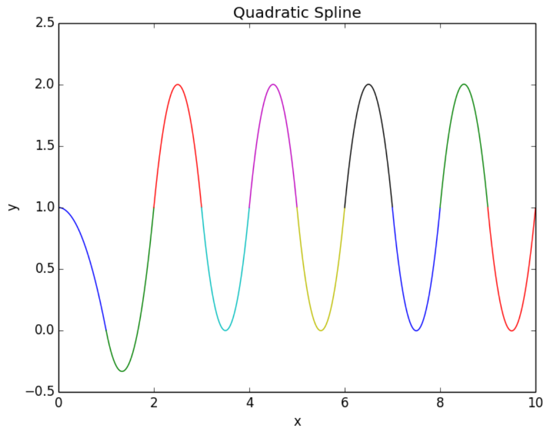
$$\begin{aligned}
\left|\int_{-h}^{h} f(x)\,dx - S(f)\right| &= \left|\int_{-h}^{h} p(x)\,dx + C\int_{-h}^{h} x^4\,dx + O(h^5) - S(p) - C\cdot S(x^4) + O(h^5)\right| \\
&= \left|C\int_{-h}^{h} x^4\,dx - C\cdot S(x^4) + O(h^5)\right| \\
&= \left|\frac{2C}{5}h^5 - C\cdot\frac{h}{3}[(-h)^4 + (0)^4 + (h)^4] + O(h^5)\right| \\
&= O(h^5).
\end{aligned}$$

3.3. **Problem 3.** The plot of the cubic spline is shown below.




3.4. **Problem 4.** The sequence of points $z_j, 0 \le j \le 10$ is given by $(0, -2, 4, -4, 4, -4, 4, -4, 4, -4)$. The plot of the quadratic spline is shown below.

As can be seen from the plot, there is quite a bit of oscillation even though we are approximating a constant function. This occurs for quadratics because on each interval, the quadratic spline is determined by just two points and we do not have enough degrees of freedom needed to control the derivative at each point of intersection as well.




## 4. Midterm

4.1. **Problem 1.** a) We expand $f$ as $f(x) = f(0) + f'(0)x + f''(0)x^2 + \dots$ and get that

$$\int_{-h}^{h} f(x)\,dx = 2hf(0) + 0 + f''(0)\int_{-h}^{h} x^2\,dx + O(h^5) = 2hf(0) + \frac{2}{3}hf''(0) + O(h^5).$$

Therefore, $\alpha = \frac{2}{3}h$.

b) We have that

$$f(-h) = f(0) - hf'(0) + \frac{1}{2}f''(0) + \dots$$

and

$$f(h) = f(0) + hf'(0) + \frac{1}{2}h^2f''(0) + \dots$$

Thus,

$$\begin{aligned} h\left[\frac{f(-h)}{2} + f(0) + \frac{f(h)}{2}\right] &= h\left[\frac{1}{2}(f(0) - hf'(0) + f''(0)h^2 + \dots) + f(0) + \frac{1}{2}(f(0) + hf'(0) + f''(0)h^2 + \dots)\right] \\ &= h\left[2hf(0) + f''(0)h^2 + O(h^4)\right] \\ &= 2h^2f(0) + h^3f''(0) + O(h^5). \end{aligned}$$

c) We can modify the trapezoidal rule by using the approximation

$$\int_{-h}^{h} f(x)\,dx = h\left[\frac{f(-h)}{2} + f(0) + \frac{f(h)}{2}\right] + (\alpha - \beta)f''(0) + O(h^5),$$

where $\alpha - \beta = -\frac{h^3}{3}$. This gives an error of $O(h^5)$ because this method has $h^3$ term $\beta f''(0) + (\alpha - \beta)f''(0) = \alpha f''(0)$ which is the same as the $h^3$ term for the exact solution.

4.2. **Problem 2.** Let $x_{-1} = -h, x_0 = 0$, and $x_1 = 2h$. We interpolate these values to find a degree 2 polynomial $p$ given by

$$p(x) = \frac{(x-x_0)(x-x_1)}{(x_{-1}-x_0)(x_{-1}-x_1)}f(-h) + \frac{(x-x_{-1})(x-x_1)}{(x_0-x_{-1})(x_0-x_1)}f(0) + \frac{(x-x_{-1})(x-x_0)}{(x_1-x_{-1})(x_1-x_0)}f(2h).$$

Multiplying out this monstrosity gives

$$p(x) = \left[\frac{1}{3h^3}f(-h) - \frac{1}{2h^2}f(0) + \frac{1}{6h^2}f(2h)\right]x^2 + \left[-\frac{2}{3h}f(-h) + \frac{1}{2h}f(0) + \frac{1}{6h}f(2h)\right]x + 1.$$

Taking

$$p''(0) = 2\left(\frac{1}{3h^3}f(-h) - \frac{1}{2h^2}f(0) + \frac{1}{6h^2}f(2h)\right) = \frac{2f(-h) - 3f(0) + f(2h)}{3h^2}.$$

The error associated with this method is found using Taylor series expansions

$$f(-h) = f(0) - hf'(0) + \frac{1}{2}h^2 f''(0) + O(h^3)$$

and

$$f(2h) = f(0) + 2hf'(0) + 2h^2 f''(0) + O(h^3).$$

Hence, we see that

$$\begin{aligned} p''(0) &= \frac{1}{3h^2}\left(2(f(0) - hf'(0) + h^2 f''(0)/2 + O(h^3)) - 3f(0) + f(0) + 2hf'(0) + 2h^2 f''(0) + O(h^3))\right) \\ &= f''(0) + O(h). \end{aligned}$$

Therefore, the order of accuracy of this method is $O(h)$.

## 5. Homework Set 4

5.1. **Problem 1.** We let $f(x) = x^3 - 3$ since the unique (real) zero of this function is $3^{1/3}$. Applying Newton's method amounts to iterating the function $\phi(x) = x - \frac{x^3-3}{3x^2} = \frac{2}{3}x + \frac{1}{x^2}$. Since Newton's method converges quadratically (the error $\epsilon_n$ at the $n$th step satisfies $\epsilon_n \leq C\epsilon_{n-1}^2$), we will need approximately 4 iterations to obtain 10 decimal places in accuracy (depending on your initial guess).

5.2. **Problem 2.** a) Suppose $|\phi'(x)| < 1$ for all $x$, and let $x, y$ be any elements such that $x < y$. By the Mean Value Theorem, there is a $\xi \in [x, y]$ such that $\phi'(\xi) = \frac{\phi(y)-\phi(x)}{y-x}$. Thus,

$$\left|\frac{\phi(y) - \phi(x)}{y - x}\right| = |\phi'(\xi)| < 1 \implies |\phi(y) - \phi(x)| < |y - x|.$$

Hence, $\phi$ is a contractive mapping.

b) The simplest of such functions is given by $\phi(x) = 2x$. This has the unique fixed point $x = 0$. Note that the fixed point iteration gives that $x_n = 2^n x_0$ where $x_0$ is the initial point. This sequence divergences to $\infty$ if and only if $x_0 \neq 0$ (where it is the constant sequence).

c) Let $f$ be a function with a single root $x^*$ and such that $f'(x) \neq 0$ in some neighborhood of $x^*$. Newton's method is exactly the iteration method applied to the function $\phi(x) = x - \frac{f(x)}{f'(x)}$. From part a), if $|\phi'(x)| < 1$ for all $x$ in the neighborhood of $x^*$, then we get convergence of the iteration method, hence Newton's method. This is equivalent to

$$\begin{aligned} |\phi'(x)| &= \left|1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}\right| \\ &= \left|\frac{f(x)f''(x)}{(f'(x))^2}\right| \\ &< 1. \end{aligned}$$

Hence, we see that if $|f(x)f''(x)| < (f'(x))^2$ for all $x$ in the neighborhood of $x^*$, then Newton's method converges.

Note: if $f$ is $C^2$, then if $|f(x^*)f''(x^*)| < (f'(x^*))^2$ we immediately get (by continuity) that $|f(x)f''(x)| < (f'(x))^2$ for all $x$ in some neighborhood of $x^*$ (although, we don't know how big this neighborhood is).

5.3. **Problem 3.** We take the Jacobian (or derivative) of the multivariable function

$$F(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 - 100 \\ x_1 x_2 x_3 - 1 \\ x_1 - x_2 - \sin x_3 \end{pmatrix}.$$

This gives us the matrix

$$DF(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \\ 1 & -1 & -\cos x_3 \end{pmatrix}.$$

The inverse of this matrix is

$$DF(x_1,x_2,x_3)^{-1} = \frac{1}{2(x_3(x_1-x_2)\cos x_3 - x_1x_2 + x_2^3)} \begin{pmatrix} \frac{x_1x_2 - x_1x_3\cos x_3}{x_1+x_2} & -\frac{2(x_2\cos x_3 - x_3)}{x_1+x_2} & -\frac{2(x_1x_2^2 - x_1x_3^2)}{x_1+x_2} \\ -\frac{x_1x_2 + x_2x_3\cos x_3}{x_1+x_2} & \frac{x_3 + x_1\cos x_3}{x_1+x_2} & \frac{2(x_1^2x_2 - x_2x_3^2)}{x_1+x_2} \\ x_3 & -2 & 2(x_2-x_1)x_3 \end{pmatrix}.$$

Newton's method for multivariable functions is given by the recurrence relation

$$\vec{x}_{n+1} = \vec{x}_n - (DF(\vec{x}_n))^{-1}F(\vec{x}_n).$$

5.4. **Problem 4.** a) We have that $F(x) = 1 + \int_0^x \arctan(y)\,dy$, $F'(x) = \arctan(x)$, and $F''(x) = \frac{1}{1+x^2}$. Therefore, $x^2 \geq 0$ for all $x$ and $1 + x^2 \geq 1 > 0$ for all $x$. Thus, $F''(x) = (1+x^2)^{-1} > 0$ for all $x$.

b) Newton's method for finding the minimum gives the recurrence relation

$$x_{k+1} = x_k - (x_k^2 + 1)\arctan(x_k).$$

The minimum of this function is at $x = 0$. If the largest interval around 0 for which Newton's method converges is $(-y, y)$, then using $x_0 = y$, we find that the sequence above is $x_n = (-1)^n y$ (this is because starting the sequence with starting value $y$ is at the edge of both convergence to 0 and divergence to $\infty$, and must oscillate between $y$ and $-y$; this also uses the oddness of the function $\arctan(x)$). Hence, $y$ solves the equation

$$-y = y - (y^2 + 1)\arctan(y).$$

The positive value for which this occurs is $y = 1.39175$. Hence, for any starting point in the interval $(-1.39175, 1.3975)$, Newton's method converges to 0, and for any starting point outside this interval, Newton's method diverges. These values can be checked numerically.

c) The foolproof method is exactly the bisection method applied to the function $F'$. Applying Newton's method with starting point either $a$ or $b$ will not work since it fails for the above function $F$ on the interval $[-2, 2]$. In fact, we are not even guaranteed that any such sequence $x_n$ obtained from Newton's method will lie in the interval $[a, b]$ where $F$ is defined.

## 6. Homework Set 5

6.1. **Problem 1.** a) First, we assume the solution has the form $y(t) = e^{rt}$ where $r$ is a constant. Substituting this into the equation $y''(t) + \omega^2 y(t) = 0$, we get that $r^2e^{rt} + \omega^2e^{rt} = (r^2 + \omega^2)e^{rt} = 0$. Hence, we require $r$ to be $\pm i\omega$, giving us the two solutions $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and $e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$. Hence a solution of this equation is a linear combination of these two:

$$y(t) = A(\cos(\omega t) + i\sin(\omega t)) + B(\cos(\omega t) + i\sin(\omega t)) = (A+B)\cos(\omega t) + i(A-B)\sin(\omega t) = c_1\cos(\omega t) + c_2\sin(\omega t).$$

The initial condition $y(0) = y_0$ gives $c_1 = y_0$ and $y'(0) = y_1$ gives $c_2 = y_1/\omega$. Hence, we obtain the solution

$$y(t) = y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t).$$

Letting $\omega = 1$, $y_1 = 1$ and $y_0 = 0$, we get the solution $y(t) = \sin t$.

b) By defining $z = y'$, we get that $z' = y'' = -y$ (from the differential equation). Hence,

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

Let $A$ denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

c) Let $\vec{y}$ denote the vector $\begin{pmatrix} y \\ z \end{pmatrix}$. Euler's forward method gives us the recurrence relation

$$\vec{y}_{n+1} = \vec{y}_n + hA\vec{y}_n = (I + hA)\vec{y}_n = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}\vec{y}_n.$$

We take a look at the eigenvalues $\lambda$ of the matrix $I + hA$. They satisfy the equation

$$\det\begin{pmatrix} 1-\lambda & h \\ -h & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + h^2 = \lambda^2 - 2\lambda + (h^2+1) = 0.$$

Using the quadratic formula, we find that

$$\lambda = 1 \pm ih.$$

The condition for stability is that each eigenvalue satisfy $|\lambda| < 1$ which gives the single equation $\sqrt{1+h^2} < 1$ which is never satisfied, hence stability for the forward Euler method is never acheived (for this particular ODE).

d) Same as above, only now we look at the eigenvalues of the matrix

$$(I - hA)^{-1} = \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix}^{-1} = \frac{1}{1+h^2} \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}.$$

The eigenvalues of this matrix satisfy

$$\det \begin{pmatrix} \frac{1}{1+h^2} - \lambda & \frac{h}{1+h^2} \\ -\frac{h}{1+h^2} & \frac{1}{1+h^2} - \lambda \end{pmatrix} = \left( \frac{1}{1+h^2} - \lambda \right)^2 + \frac{h^2}{(1+h^2)^2} = 0.$$

So the eigenvalues are

$$\lambda = \frac{1 \pm ih}{1+h^2}$$

which have absolute value $|\lambda| = \frac{1}{\sqrt{1+h^2}}$. For any choice of $h$, the eigenvalues satisfy $|\lambda| < 1$, so the backward Euler method is always stable.

e) The trapezoidal method gives rise to the recurrence relation

$$\vec{y}_{n+1} = \vec{y}_n + \frac{h}{2} A(\vec{y}_n + \vec{y}_{n+1}) = \frac{h}{2} A\vec{y}_n + \frac{h}{2} A\vec{y}_{n+1}.$$

Thus, we have that

$$\vec{y}_{n+1} = \left( I - \frac{h}{2} A \right)^{-1} \left( I + \frac{h}{2} A \right) \vec{y}_n.$$

Since eigenvalues multiply, we get that the eigenvalues of the matrix

$$\left( I - \frac{h}{2} A \right)^{-1} \left( I + \frac{h}{2} A \right) = \frac{1}{1+h^2/4} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 \end{pmatrix} = \frac{1}{1+h^2/4} \begin{pmatrix} 1-h^2/4 & h \\ -h & 1-h^2/4 \end{pmatrix}.$$

The characteristic equation here is then

$$\left( \frac{1-\frac{h^2}{4}}{1+\frac{h^2}{4}} - \lambda \right)^2 + \frac{h^2}{(1+\frac{h^2}{4})^2} = 0$$

giving that

$$\lambda = \frac{1-\frac{h^2}{4}}{1+\frac{h^2}{4}} \pm i \frac{h}{1+\frac{h^2}{4}}$$

with absolute value

$$|\lambda| = \frac{\sqrt{(1-h^2/4)^2 + h^2}}{1+h^2/4} = \frac{\sqrt{1+h^2/2+h^4/16}}{1+h^2/4} = 1.$$

Hence, this method is always unstable (or stable depending if you consider $\leq 1$ to be stable or $< 1$).

f) The matrix $B = \left(I - \frac{h}{2}A\right)^{-1} \left(I + \frac{h}{2}A\right)$ from the trapezoidal method has distinct eigenvalues (as long as $h \neq 0$) which have absolute values equal to 1. This implies that $B$ is a rotation of the 2-dimensional plane and hence must preserve the absolute values of vectors, i.e. $\|B\vec{v}\| = \|\vec{v}\|$ for any vector $v$. Since $\vec{y}_n = B^n \vec{y}_0$, we obtain that $\|\vec{y}_n\| = \|B^n \vec{y}_0\| = \cdots = \|y_0\| = 1$ and the sequence obtained from the trapezoidal rule fall on the unit circle.

6.2. **Problem 2.** a) We assume that $f(t, y) = \lambda y$ so that we obtain the recurrence relation $y_{n+1} = y_{n-1} + 2h\lambda y_n$. This recurrence relation can be restructured into a vector recurrence relation

$$\vec{y}_{n+1} = \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2h\lambda \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} = B\vec{y}_n.$$

The matrix $B$ has eigenvalues $\mu$ determined by the equation

$$\mu(\mu - 2h\lambda) - 1 = \mu^2 - 2h\lambda\mu - 1 = 0.$$

Hence, $\mu = h\lambda \pm \sqrt{(h\lambda)^2 + 1}$ When $\lambda$ is real, then we obtain the two conditions

$$|h\lambda \pm \sqrt{(h\lambda)^2 + 1}| = 1 \leq 1.$$

Multiplying these two conditions together gives $[(h\lambda)^2 + 1] - (h\lambda)^2 = 1 \leq 1$.

b) When $\lambda = i\nu$ for $\nu$ real, we have $|ih\nu \pm \sqrt{1 - (h\nu)^2}| < 1$. We have two cases: 1) $|h\nu| \leq 1$ and 2) $|h\nu| > 1$. For 1), this means that $\sqrt{1 - (h\nu)^2}$ is real so that

$$|ih\nu \pm \sqrt{1 - (h\nu)^2}| = 1$$

and so the method is stable for $|h\nu| \leq 1$. For 2), $\sqrt{1 - (h\nu)^2}$ is imaginary and we have that

$$|ih\nu \pm \sqrt{1 - (h\nu)^2}| = |h\nu \pm \sqrt{(h\nu)^2 - 1}| \leq 1.$$

However, taking the $\pm$ to have the same sign as that of $h\nu$, we get that $|h\nu \pm \sqrt{(h\nu)^2 - 1}| = |h\nu| + |\sqrt{(h\nu)^2 - 1}| \geq |h\nu| > 1$, a contradiction. Therefore, the stability region is $|h\lambda| \leq 1$ for $\lambda$ imaginary.

6.3. **Problem 3.** a) The forward Euler method gives the recurrence relation $y_{n+1} = y_n - hy_n^2$. Note that $hy_n^2 \geq 0$ for all $n$. Hence, $y_{n+1} \leq y_n \leq y_{n-1} \leq \cdots \leq y_0 = 100$.

b) We let $f(t, y) = -y^2$. Then $\lambda = \partial f/\partial y(0, 100) = -200$ so that the linearized differential equation becomes

$$y' = -200y.$$

The forward Euler method gives the condition $|1 + h\lambda| < 1$ for stability. Hence,

$$-1 < 1 - 200h < 1 \implies 0 < h < \frac{1}{100}.$$

d) The implicit recurrence relation is

$$y_{n+1} = y_n - hy_{n+1}^2 \implies hy_{n+1}^2 + y_{n+1} - y_n = 0 \implies y_{n+1} = \frac{-1 \pm \sqrt{1 + 4hy_n}}{2h}.$$

Thus, in order to use an implicit method, we need to be able to solve quadratic polynomials, which require other algorithms such as Newton's method.

## 7. Homework Set 6

## 8. Homework Set 7

8.1. **Problem 1.** a) Let $g(x) = f(x/a)$. Then

$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx = \int_{-\infty}^{\infty} e^{-ikx} f(x/a)\, dx = \int_{-\infty}^{\infty} e^{-ikay} f(y)\, a \cdot dy = a\hat{f}(ak),$$

where we make the substitution $y = x/a$ in the second equality.

b) Let $g(x) = \overline{f(x)}$, then

$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{-ikx} \overline{f(x)}\, dx = \overline{\int_{-\infty}^{\infty} e^{ikx} f(x)\, dx} = \overline{f(-k)}.$$

c) Suppose $f$ is real and even, i.e. $f(x) = f(-x)$. Then,

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = -\int_{\infty}^{-\infty} e^{ikx} f(-x)\, dx = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx = \hat{f}(-k),$$

where the second equality comes from the substitution $x \mapsto -x$. Hence, $\hat{f}$ is even. To show that $\hat{f}$ is real, note that $e^{ikx} + e^{-ikx} = 2\cos(kx)$. Then,

$$2\hat{f}(k) = \hat{f}(k) + \hat{f}(-k) = \int_{-\infty}^{\infty} (e^{ikx} + e^{-ikx}) f(x)\,dx = 2\int_{-\infty}^{\infty} \cos(kx) f(x)\,dx.$$

Since $f(x)$ is real and $\cos(x)$ is real, we have that $\hat{f}(k)$ is real.

d) Suppose $f$ is real and odd, i.e. $f(-x) = -f(x)$. Then,

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = -\int_{\infty}^{-\infty} e^{ikx} f(-x)\,dx = -\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx = -\hat{f}(-k),$$

where the second equality comes from the substitution $x \mapsto -x$ and using the oddness of $f$. Hence, $\hat{f}$ is odd. To show that $\hat{f}$ is imaginary, note that $e^{ikx} - e^{-ikx} = 2i\sin(kx)$. Then,

$$2\hat{f}(k) = \hat{f}(k) - \hat{f}(-k) = \int_{-\infty}^{\infty} (e^{-ikx} - e^{ikx}) f(x)\,dx = -2i\int_{-\infty}^{\infty} \sin(kx) f(x)\,dx.$$

Since $f(x)$ is real and $\sin(x)$ is real, we have that $\hat{f}(k)$ is imaginary.

8.2. **Problem 2.** a) Let $f(x) = \frac{\sin x}{x}$. First, we break up the integral as

$$\int_{-\infty}^{\infty} \left|\frac{\sin x}{x}\right| dx = \sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx.$$

Then, on each interval $[k\pi, (k+1)\pi]$, where $k \in \mathbb{Z}$, we compare $|\sin x|$ with the function $\varphi_k$ given by

$$\varphi_k(x) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{for } x \in \left[\frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi\right] \\ 0, & \text{otherwise,} \end{cases}$$

noting that $|\sin(x)| \geq \varphi_k(x) \geq 0$ on $[k\pi, (k+1)\pi]$ (this is most easily seen by looking at the graphs of the two functions). Thus, for each $k$,

$$\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{\sqrt{2}} \int_{\frac{\pi}{4}+k\pi}^{\frac{3\pi}{4}+k\pi} \frac{dx}{|x|} \geq \frac{1}{\sqrt{2}\pi|(k+1)|}.$$

Hence, we obtain that

$$\int_{-\infty}^{\infty} \left|\frac{\sin x}{x}\right| dx = \sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{\sqrt{2}\pi} \sum_{k=-\infty}^{\infty} \frac{1}{|\frac{3}{4}+k|} \geq \frac{\sqrt{2}}{\pi} \sum_{k=0}^{\infty} \frac{1}{\frac{3}{4}+k} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Hence, $\sin(x)/x$ is not integrable.

b) Let $f(k) = \frac{\sin(k)}{k}$. Then the inverse Fourier transform of $f$ is given by

$$\varphi(x) = \begin{cases} 1/2, & \text{for } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

from the text (Example 17, Section 6.1). Thus,

$$\int_{-\infty}^{\infty} \frac{\sin(k)}{k}\,dk = 2\pi \cdot \varphi(0) = \pi.$$

8.3. **Problem 3.** Writing this integral equation as $u + K * u = f$, we take the Fourier transform and use the convolution theorem:

$$\hat{u} + \hat{K} \cdot \hat{u} = (1 + \hat{K})\hat{u} = \hat{f}.$$

Solving for $\hat{u}$, we obtain

$$\hat{u} = \frac{\hat{f}}{1 + \hat{K}},$$

so that in order to avoid dividing by zero, we must have that $\hat{K}(k)+1 \neq 0$ for any $k$. Taking the inverse Fourier transform, we obtain that

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}\hat{f}(k)}{1+\hat{K}(k)}\,dk.$$

8.4. **Problem 4.** We let $v_k(x) = ce^{-ikx}$ for $k \in \mathbb{Z}$. For $k, \ell \in \mathbb{Z}$, $k \neq \ell$, we have that

$$\langle v_k, v_\ell\rangle = \int_0^{2\pi} (ce^{-ikx})(ce^{i\ell x})\,dx = c^2\int_0^{2\pi} e^{i(\ell-k)x}\,dx = \frac{c^2}{i(\ell-k)}e^{i(\ell-k)x}\Big|_0^{2\pi} = 0,$$

since the function $e^{inx}$ is $2\pi$-periodic for $n$ an integer.

For $k$ any integer, we have that

$$\langle v_k, v_k\rangle = \int_0^{2\pi} c^2 e^{i(k-k)x}\,dx = c^2 \cdot 2\pi.$$

In order for this to equal 1, we need that $c = \frac{1}{\sqrt{2\pi}}$.