January 20, 2019

Substitution

Theorem (Substitution for Definite Integrals)

If f and u are functions, then

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Proof $\frac{d}{dx}\left[F(u(x))\right] = F'(u(x))\cdot u'(x) = f(u(x))\,u'(x)$

$$\int_{u(a)}^{u(b)} f(u)\,du = F(u(b)) - F(u(a))$$
$$= \int_a^b f(u(x))\,u'(x)\,dx. \quad \blacksquare$$

Theorem (Indefinite Integral Substitution)

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

Example $\int x\cos(x^2)\,dx = \int \cos(u)\cdot\frac{1}{2}\,du$ $\quad u = x^2,\ du = 2x\,dx$

$$= \frac{1}{2}\sin(u) + C$$
$$= \frac{1}{2}\sin(x^2) + C \quad \blacksquare$$

Even and Odd Functions

A function $f(x)$ is *even* if $f(-x) = f(x)$.
is *odd* if $f(-x) = -f(x)$.

Theorem If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof $I = \int_{-a}^{a} f(x)\,dx$ $\quad u = -x,\ du = -dx$

$$= \int_a^{-a} f(-u)\,(-du)$$
$$= -\int_a^{-a} -f(u)\,du$$
$$= -\int_{-a}^{a} f(u)\,du = -I$$

Thus, $I = -I$
$2I = 0$
$I = 0. \quad \blacksquare$

Example 1) $\int_{-\pi}^{\pi} \sin(\sin(x))\,dx = 0$

2) $\int_0^{2\pi} \sin(\sin(x))\,dx$ $\quad u = x - \pi,\ du = dx$

$$= \int_{-\pi}^{\pi} \sin(\sin(u+\pi))\,du \qquad \sin(u+\pi) = -\sin(u)$$
$$= \int_{-\pi}^{\pi} \sin(-\sin(u))\,du = -\int_{-\pi}^{\pi} \sin(\sin(u))\,du = 0$$

3) $\int_0^{2\pi} \sin^3(x)\,dx = \int_{-\pi}^{\pi} \sin^3(u+\pi)\,du = \int_{-\pi}^{\pi} \sin^3(u)\,du = 0.$

Theorem If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

Natural Logarithm

Some functions have antiderivatives that cannot be written in terms of functions already defined.

E.g. $\int \sin(\sin(x))\,dx$

$\int \tan(x^2)\,dx$

Definition The *natural logarithm* $\ln(x)$ is defined for all $x>0$ by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

Properties 1) $\ln(1) = 0$

2) $\ln(ab) = \ln(a) + \ln(b).$

$$\int_1^{ab}\frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab}\frac{1}{t}\,dt$$
$$= \ln(a) + \int_1^b \frac{1}{au}(a\,du) \qquad u = \frac{t}{a},\ du = \frac{dt}{a}$$
$$= \ln(a) + \int_1^b \frac{1}{u}\,du$$
$$= \ln(a) + \ln(b).$$

3) $\ln(a^n) = n\cdot\ln(a)$

$$\ln(a^n) = \ln(\underbrace{a\cdot a\cdot a\cdot\cdots\cdot a}_{n\text{ times}})$$
$$= \underbrace{\ln(a) + \ln(a) + \cdots + \ln(a)}_{n\text{ times}}$$
$$= n\cdot\ln(a)$$

Indefinite Integral of $\frac{1}{x}$: $\int \frac{1}{x}\,dx = \ln|x| + C$

For $x>0$, $\frac{d}{dx}[\ln x] = \frac{1}{x}.$

For $x<0$,

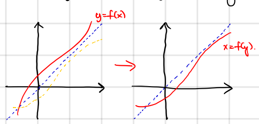
$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

§6.1 Integration with respect to y.

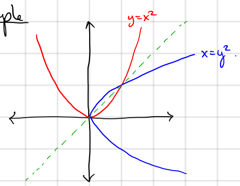
Functions in y

Sometimes it is convenient to express x as a function of y, e.g. $x = f(y)$.

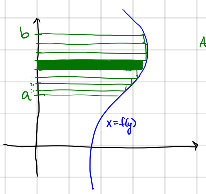
Interchanging x and y flips the graph along $x=y$.



Example



Integrating functions in y



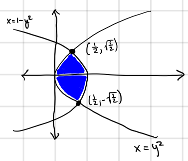
Area $\approx f(y_1)\Delta y_1 + f(y_2)\Delta y_2 + \dots + f(y_n)\Delta y_n$

where $\Delta y_i = y_i - y_{i-1}$.

As the partition $a = y_0 < y_1 < \cdots < y_n = b$ gets finer and finer, this sum converges to

$$\int_a^b f(y)\,dy.$$

Example



$1 - y^2 = y^2$

$1 = 2y^2$

$y = \pm\sqrt{\tfrac{1}{2}}$

As an integral in x, this area is given by

$$\int_0^{1/2} [\sqrt{x} - (-\sqrt{x})]\,dx + \int_{1/2}^{1} [\sqrt{1-x} - (-\sqrt{1-x})]\,dx$$

$$= 2\int_0^{1/2} \sqrt{x}\,dx + 2\int_{1/2}^{1} \sqrt{1-x}\,dx \qquad \begin{matrix} u = 1-x \\ du = -dx \end{matrix}$$

$$= 2\int_0^{1/2} \sqrt{x}\,dx + 2\int_{1/2}^{0} \sqrt{u}\,(-du)$$

$$= 4\int_0^{1/2} \sqrt{x}\,dx$$

$$= 4\left[\tfrac{2}{3} x^{3/2}\right]_0^{1/2}$$

$$= \tfrac{8}{3}\left(\tfrac{1}{2}\right)^{3/2}$$

$$= \tfrac{4}{3}\sqrt{\tfrac{1}{2}}$$

As an integral in y, this area is

$$\int_{-\sqrt{1/2}}^{\sqrt{1/2}} [(1-y^2) - y^2]\,dy = \int_{-\sqrt{1/2}}^{\sqrt{1/2}} [1 - 2y^2]\,dy$$

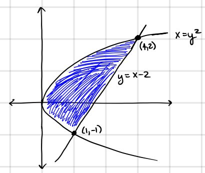
$$= \left(y - \tfrac{2}{3}y^3\right)\Big|_{-\sqrt{1/2}}^{\sqrt{1/2}}$$

$$= \left[\sqrt{\tfrac{1}{2}} - \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{3/2}\right] - \left[-\sqrt{\tfrac{1}{2}} + \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{3/2}\right]$$

$$= 2\sqrt{\tfrac{1}{2}} - \tfrac{2}{3}\sqrt{\tfrac{1}{2}}$$

$$= \tfrac{4}{3}\sqrt{\tfrac{1}{2}}$$

Example



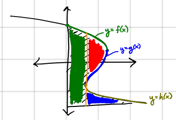
Calculating the area in 2 different ways, we see that

$$\int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 [\sqrt{x} - (x-2)]\,dx = \int_{-1}^{2} [(y+2) - y^2]\,dy$$

Calculating, we find they are both equal to $\frac{9}{2}$.

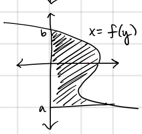
In the previous examples, much easier to express the area as an integral in y.

Example

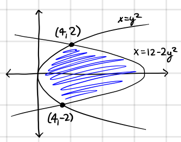


As an integral in x, need 3 separate definite integrals to evaluate area.

As an integral in y, need only 1.



Example



$12 - 2y^2 = y^2$

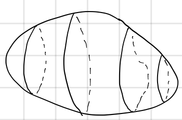
$12 = 3y^2$

$4 = y^2$

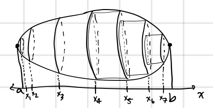
$y = \pm 2.$

§6.2 Volume by Cross-Sections

We have a solid S



To find the volume, we take slices



Each slice is approximated by a cylinder.

If $A(x)$ is the cross-sectional area of S, i.e. the area of the section obtained by intersecting S by a plane perpendicular to the x-axis through the point x, then the volume of the ith slice is approximately $A(x_i)\,\Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.

Thus, the volume of S is approximately

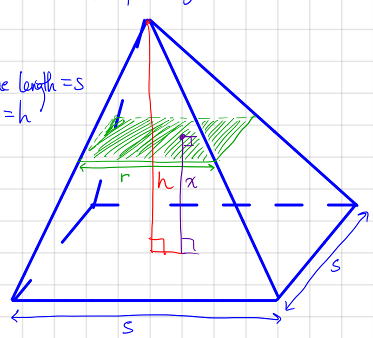
$$A(x_1)\Delta x_1 + A(x_2)\Delta x_2 + \dots + A(x_n)\Delta x_n.$$

Taking finer and finer partitions $\mathcal{P} = \{a = x_0 < x_1 < \dots < x_n = b\}$, we get that

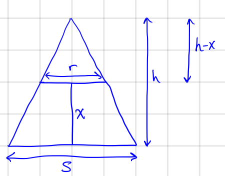
$$\boxed{\text{Volume }(S) = \int_a^b A(x)\,dx}$$

Example (Volume of Square Pyramid)

base side length $= s$
height $= h$



If we intersect the square pyramid by a plane parallel to the base, at a height x above the base, then the cross-section is a square of side length r and area $A(x)$.



By similar triangles,

$$\frac{s}{h} = \frac{r}{h-x}$$

$$r = \frac{s}{h}(h-x).$$

Thus, $A(x) = r^2 = \frac{s^2}{h^2}(h-x)^2$. Therefore,

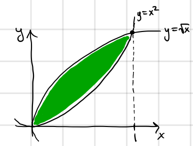
$$\text{Volume} = \int_0^h A(x)\,dx = \frac{s^2}{h^2}\int_0^h (h-x)^2\,dx$$

$$= \frac{s^2}{h^2}\left[-\frac{1}{3}(h-x)^3\right]_0^h$$

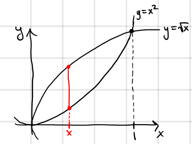
$$= \frac{s^2}{h^2}\left[\frac{1}{3}h^3\right]$$

$$\boxed{\text{Volume} = \frac{1}{3}s^2h}$$

Example Let S be the solid with base the region



And whose cross-sections by planes perpendicular to x-axis are squares.



The side-length of each square cross-section is $\sqrt{x} - x^2$ so the cross-sectional area is

$$A(x) = (\sqrt{x} - x^2)^2$$

The volume of this solid is

$$V = \int_0^1 (\sqrt{x} - x^2)^2\,dx$$

$$= \int_0^1 (x - 2x^{5/2} + x^4)\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{4}{7}x^{7/2} + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{1}{2} - \frac{4}{7} + \frac{1}{5}$$

$$= \frac{35}{70} - \frac{40}{70} + \frac{14}{70}$$

$$= \boxed{\frac{9}{70}}$$

Example Same example as above but cross-sections are semi-circles with diameter lying on xy-plane.

The diameter of each cross-section is $\sqrt{x} - x^2$

The radius is $\frac{1}{2}[\sqrt{x} - x^2]$.

The area is $A(x) = \frac{\pi}{4}[\sqrt{x} - x^2]^2$.

The volume is $V = \int_0^1 A(x)\,dx$

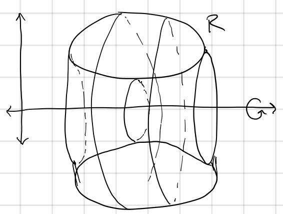
$$= \frac{\pi}{4}\int_0^1 [\sqrt{x} - x^2]^2\,dx$$

$$= \frac{\pi}{4}\cdot\frac{9}{70}$$

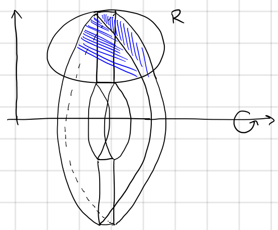
$$= \boxed{\frac{9\pi}{280}}$$

§6.2 Washer Method

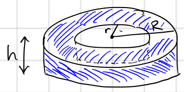
Take a region R and revolve it around the x-axis.



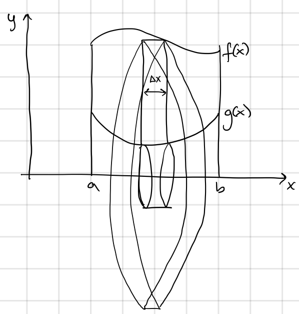
Vertical rectangles revolve into "washers".



Volume of Washer $= \pi(R^2 - r^2)h$



Suppose

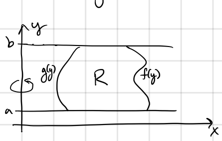


This washer has volume $\pi\left[f(x)^2 - g(x)^2\right]\Delta x$.

Thus the volume of the solid S obtained by revolving R around x-axis is

$$\boxed{V = \int_a^b \pi\left[f(x)^2 - g(x)^2\right]dx}$$

If you have a region R



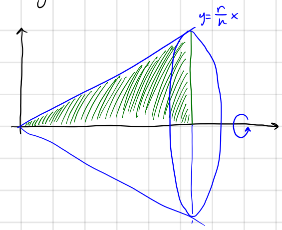
and the solid S is obtained by revolving around the y-axis, then the volume is

$$\boxed{V = \int_a^b \pi\left[f(y)^2 - g(y)^2\right]dy}$$

Example (Volume of a Cone)

Let r = radius of base

h = height.



The cone S is obtained by revolving $y = \frac{r}{h}x$, $0 \le x \le h$ about x-axis.

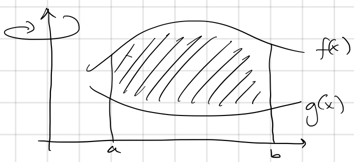
$$\text{Volume} = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi\frac{r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{r^2\pi}{h^2}\cdot\frac{1}{3}x^3\Big|_0^h$$

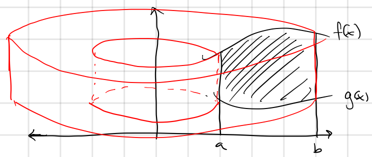
$$= \boxed{\frac{1}{3}\pi r^2 h.}$$

§6.3 Shell Method

Consider the following region R



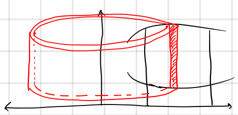
Revolve this region around y-axis



Obtain a solid S.

What is the volume of this solid?

The solid S can be approximated by shells



Let $R =$ outer radius

$r =$ inner radius

$h =$ height

Then $V_{shell} = \pi R^2 h - \pi r^2 h$

$= \pi (R+r)\underbrace{(R-r)}_{\text{thickness of wall}} h$

If we choose a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, then the dimensions of the shells are

$R = x_{i+1}$

$r = x_i$

$h = f(x_i) - g(x_i)$

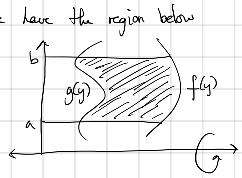
So the volume of the ith shell is approximately (assuming $\Delta x_i := x_{i+1} - x_i$ is small)

$V_i = 2\pi x_i \, [f(x_{i+1}) - g(x_i)] \, \Delta x_i$

Summing over these volumes, and taking the limit as the partitions tend to 0, gives

$$\boxed{\text{Volume}(S) = \int_a^b 2\pi x \, [f(x) - g(x)] \, dx}$$

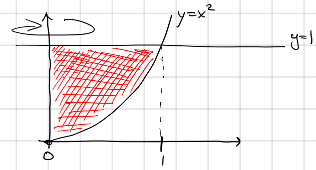
If we have the region below



The solid of revolution about the x-axis is

$$\boxed{V = \int_a^b 2\pi y \, [f(y) - g(y)] \, dy}$$

Example Find the volume of the solid obtained by revolving the region about the y-axis



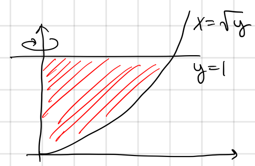
$V = 2\pi \int_0^1 x \, [1 - x^2] \, dx$

$= 2\pi \int_0^1 [x - x^3] \, dx$

$= 2\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^3 \right]_0^1$

$$\boxed{V = \frac{\pi}{2}}$$

Notice the we can also describe the region above as



Using the washer method, we find that

$\text{Volume} = \pi \int_0^1 [\sqrt{y}]^2 \, dy$

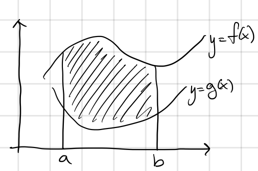
$= \pi \cdot \frac{1}{2} y^2 \Big|_0^1$

$$\boxed{V = \frac{\pi}{2}}$$

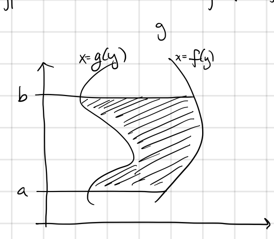
Just as above.

When do we know when to use washer method versus the shell method?

Define Type I region as one of the following form:



Define a Type II re ion as one of the following form:
g

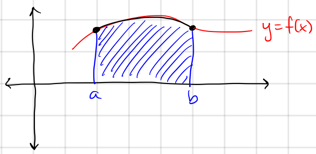


Which method should be used is detaild in the following chart.

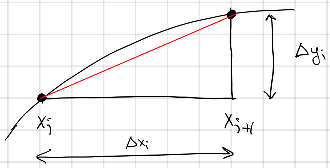
Type of Region / Axis of Rotation	Type I	Type II
x-axis	Washer	Shell
y-axis	Shell	Washer

Surface Area and Arc Length

Arclength Suppose I have a continuous function $f(x)$ on $[a,b]$.



What is the length of the arc of the graph of $f(x)$ from a to b?



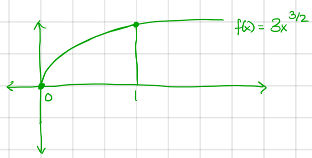
$$\text{Length} \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$$

Sum over i and take limit $\|P\| \to 0$, we get $\Delta x_i \to 0$ and so

$$\boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\, dx}$$

Example



What is the length of the blue arc?

$$f'(x) = \frac{9}{2} x^{1/2}$$

$$L = \int_0^1 \sqrt{1 + \frac{81}{4} x}\ dx$$

$u = 1 + \frac{81}{4}x$, $du = \frac{81}{4}dx$

$$\int_1^{\frac{85}{4}} \sqrt{u} \cdot \frac{4}{81}\, du$$

$$= \frac{4}{81}\left[\frac{2}{3} u^{3/2}\right]_1^{\frac{85}{4}}$$

$$= \frac{8}{243}\left[\left(\frac{85}{4}\right)^{3/2} - 1\right]$$

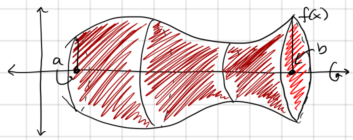
$$= \boxed{\frac{8}{243}\left[\frac{1}{8}(85)^{3/2} - 1\right]}$$

If $x = g(y)$ is rotated about y-axis, then arclength from $y=a$ to $y=b$ is

$$\boxed{L = \int_a^b \sqrt{1 + [g'(y)]^2}\, dy}$$

Area of Surface of Revolution

Let $f(x)$ be continuous on $[a,b]$.



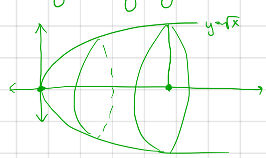
Rotate graph of $f(x)$ about x-axis.
Get a surface.

$$\boxed{\text{Surface Area} = A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx}$$

If $x = g(y)$ is rotated about y-axis, then surface area from $y=a$ to $y=b$ is

$$A = 2\pi \int_a^b g(y)\sqrt{1 + [g'(y)]^2}\, dy$$

Example Find the surface area of the surface from $x=0$ to $x=4$, obtained by revolving $y = \sqrt{x}$ about x-axis.



$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$$A = 2\pi \int_0^4 \sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}}\ dx$$

$$= 2\pi \int_0^4 \sqrt{x + \frac{1}{4}}\ dx$$

$u = x + \frac{1}{4}$, $du = dx$

$$2\pi \int_{1/4}^{17/4} \sqrt{u}\, du$$

$$= 2\pi \cdot \frac{2}{3} u^{3/2} \Big|_{1/4}^{17/4}$$

$$= \boxed{\frac{4}{3}\pi\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]}$$

§7.2 Natural Logarithm

Properties

1) $\ln(1) = 0$
2) $\ln(ab) = \ln(a) + \ln(b)$
3) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
4) $\ln\left(a^{p/q}\right) = \frac{p}{q}\ln(a)$

The range of $\ln(x)$ is $(-\infty, \infty)$. Therefore, by IVT, there is some e such that $\ln(e) = 1$.

The value of $e \approx 2.718281828459$

Example Let $f(x) = \ln|x|$ for $x \neq 0$. Then
$$f'(x) = \frac{1}{x} \text{ for } x \neq 0.$$

Rewritten in terms of integration $\int \frac{1}{x}\,dx = \ln|x| + C$.

Logarithmic Differentiation

Problem: Find $\frac{d}{dx}\left[g_1(x)\cdots g_n(x)\right]$.

Solution: Let $g(x) = g_1(x)\cdots g_n(x)$.

$$\ln g(x) = \ln g_1(x) + \dots + \ln g_n(x).$$

$$\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \dots + \frac{g_n'(x)}{g_n(x)}$$

$$\boxed{g'(x) = g(x)\left[\frac{g_1'(x)}{g_1(x)} + \dots + \frac{g_n'(x)}{g_n(x)}\right]}$$

Example 1) Let $g(x) = x(x-1)(x-2)(x-3)$.

Then
$$g'(x) = x(x-1)(x-2)(x-3)\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$

Integration of Trig Functions

Theorem 1) $\int \tan x\,dx = -\ln|\cos x| + C$

2) $\int \cot x\,dx = \ln|\sin x| + C$

3) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

4) $\int \csc x\,dx = \ln|\csc x + \cot x| + C$

Proof 1) Let $f(x) = -\ln|\cos x|$. Then

$$f'(x) = -\frac{1}{\cos x}\cdot(-\sin x)$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x.$$

Therefore, $\int \tan x\,dx = -\ln|\cos x| + C$.

2) $$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx \xrightarrow[du = \cos x\,dx]{u = \sin x} \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sin x| + C$$

3) $$\int \sec x\,dx = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$
$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$
$$\xrightarrow[du = (\sec x\tan x + \sec^2 x)\,dx]{u = \sec x + \tan x} \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|\sec x + \tan x| + C$$

Example 1) $$\int \frac{\sin x}{2 + \cos x}\,dx \xrightarrow[du = -\sin x\,dx]{u = 2 + \cos x} \int \frac{-du}{u}$$
$$= -\ln|u| + C$$
$$= \boxed{-\ln|2 + \cos x| + C}$$

2) $$\int \frac{1}{x\ln x}\,dx \xrightarrow[du = \frac{1}{x}dx]{u = \ln x} \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \boxed{\ln|\ln(x)| + C}$$

3) $$\int (1 + \sec x)^2\,dx = \int (1 + 2\sec x + \sec^2 x)\,dx$$
$$= \boxed{x + 2\ln|\sec x + \tan x| + \tan x + C}$$

§7.4 Exponential

Definition. The inverse of $\ln(x)$ is denoted $\exp(x)$.

Recall that e was a number such that $\ln(e) = 1$.

If $\frac{p}{q}$ is rational, then
$$\ln\left(e^{p/q}\right) = \frac{p}{q}\ln(e) = 1.$$

Therefore, $e^{p/q} = \exp\left(\frac{p}{q}\right)$ since they both solve the equation
$$\ln(x) = \frac{p}{q}.$$

If x is not rational, we don't know what e^x means, but we can define $e^x = \exp(x)$.

For this reason, $\exp(x) = e^x$ is called the exponential function

Properties of e^x

1) $e^0 = 1$
2) The domain of e^x is $(-\infty, \infty)$

   The range of e^x is $(0, \infty)$, so $e^x > 0$ for all x.
3) $e^{\ln x} = x$ for all $x > 0$.
4) $e^{a+b} = e^a \cdot e^b$ for all a, b
5) $\frac{d}{dx}\left[e^x\right] = e^x$
6) $\int e^x\,dx = e^x + C$

Proof 1) Since $\ln(1) = 0$, $e^0 = 1$.

2) The domain of e^x is the range of $\ln(x)$ and the range of e^x is the domain of $\ln(x)$.

3) This is a property of inverse functions.

4) Let c and d be positive real numbers.

Since $\ln(cd) = \ln(c) + \ln(d)$, we can take exponentials of both sides to get
$$e^{\ln(cd)} = e^{\ln c + \ln d}$$
$$cd = e^{\ln(c) + \ln(d)}$$
$$e^{\ln c}\cdot e^{\ln d} = e^{\ln(c) + \ln(d)}$$

Since $a = \ln c$ and $b = \ln d$ can take on any real value, we have
$$e^a \cdot e^b = e^{a+b} \quad \text{for all real } a, b.$$

5) Recall that if f is one-to-one with inverse f^{-1}, then
$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

Using $f(x) = \ln x$ and $f^{-1}(x) = e^x$, we get
$$\frac{d}{dx}\left[e^x\right] = \frac{1}{1/e^x} = e^x.$$

6) Since $\frac{d}{dx}\left[e^x\right] = e^x$, the exponential is it's own antiderivative, and we have
$$\int e^x\,dx = e^x + C.$$

Examples 1) $$\int \frac{e^x}{e^x + 2}\,dx \xrightarrow[du = e^x dx]{u = e^x} \int \frac{du}{u} = \ln|u| + C$$
$$= \boxed{\ln|e^x + 2| + C}$$

2) $$\int \frac{e^x}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^x + e^{-x}}\cdot\frac{e^x}{e^x}\,dx$$
$$= \int \frac{e^{2x}}{e^{2x} + 1}\,dx$$
$$\xrightarrow[du = 2e^{2x}dx]{u = e^{2x} + 1} \int \frac{\frac{1}{2}du}{u}$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \boxed{\frac{1}{2}\ln|e^{2x} + 1| + C}$$

Exponential Growth and Decay

The exponential function is useful in describing the following scenario: Let $A(t)$ be some quantity depending on time t. Suppose that the rate of change $A'(t)$ is proportional to $A(t)$, i.e.

$$A'(t) = r\,A(t)$$

for some constant r.

Then $\boxed{A(t) = A(0)\cdot e^{rt}}$.

Examples 1) Let $P(t)$ = population at some time t. Then
$P'(t) = k\,P(t)$ where k is some constant.

2) (Radioactivity)

$$^{236}U \longrightarrow {}^{92}Kr + {}^{141}Br$$

Let $A(0)$ be starting amount of ^{236}U.

Let $A(t)$ = amount after time t.

Fact The rate of fission is proportional to the amount of fissable material present.

Thus, there is a constant $k>0$ such that

$$A'(t) = -k\,A(t)$$
$$\Updownarrow$$
$$A(t) = A(0)\,e^{-kt}.$$

There is a time, denoted λ, such that half the material remains:

$$A(\lambda) = \tfrac{1}{2}A(0)$$
$$\Rightarrow e^{-k\lambda} = \tfrac{1}{2}$$
$$\boxed{\lambda = \frac{\ln 2}{k}}$$

This λ is called the half life.

Theorem If $A(t)$ satisfies

$$A'(t) = kA(t)$$

then $A(t) = A(0)\,e^{kt}$.

Proof

$$A'(t) = k\,A(t)$$
$$\frac{A'(t)}{A(t)} = k$$
$$\int_0^t \frac{A'(z)}{A(z)}\,dz = \int_0^t k\,dz = kt$$
$$\ln|A(t)| = kt + C$$
$$A(t) = Ce^{kt}$$

Then $A(0) = Ce^{k(0)} = C$

so $A(t) = A(0)e^{kt}$.

Example You have 100 g of a radioactive substance. After 1 hr, there is 60 g of the substance left. How much is left after 1.5 hr?

$$A(t) = A(0)e^{-rt} \qquad A(0) = 100$$
$$A(1) = 100e^{-r} = 60$$
$$e^{-r} = \tfrac{3}{5}$$
$$-r = \ln\tfrac{3}{5}$$
$$r = \ln\tfrac{5}{3}.$$

Therefore, $A(t) = 100\left(\frac{5}{3}\right)^{-t}$

$$\boxed{A(1.5) = 100\left(5/3\right)^{-1.5}}$$

Example A population is 300 million. Ten years later, the population is 340 million. How long until it reaches 400 million?

$$P(t) = P(0)e^{kt} \qquad P(0) = 300$$
$$P(10) = 300\,e^{10k} = 340$$
$$e^{10k} = \tfrac{17}{15}$$
$$k = \tfrac{1}{10}\ln\left(\tfrac{17}{15}\right)$$

$$P(t) = 300\,e^{\frac{t}{10}\ln\left(\frac{17}{15}\right)} = 400$$
$$e^{t\ln(17/15)/10} = 4/3$$
$$\tfrac{1}{10}t\ln\left(\tfrac{17}{15}\right) = \ln\left(\tfrac{4}{3}\right)$$
$$\boxed{t = \frac{10\ln(4/3)}{\ln(17/15)}}$$