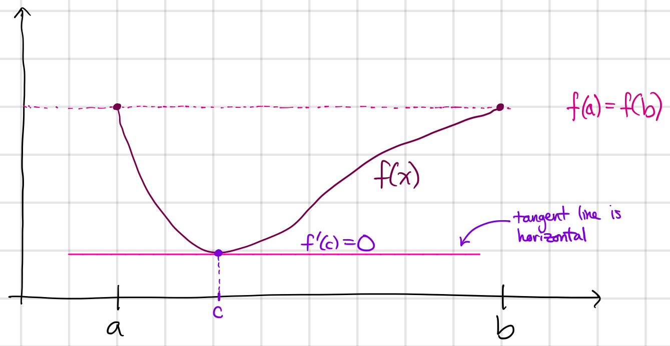
The Mean Value Theorem (§4.1)

Theorem (Rolle's Theorem) If f is

- continuous on $[a,b]$,
- differentiable on (a,b) and
- $f(a)=f(b)$.

Then there is some c in (a,b) such that
$$f'(c)=0.$$



"A car travelling along a road, starting and ending at the same point, must have stopped and turned around somewhere."

Example The function $f(x)=\sin(x)+\cos(x)$ is

- continuous on $[0,\frac{\pi}{2}]$
- differentiable on $(0,\frac{\pi}{2})$
- $f(0)=f(\frac{\pi}{2})=1$

Thus, there is some c in $(0,\frac{\pi}{2})$ such that
$$f'(c)=0$$

We may find this c:
$$f'(x)=\cos(x)-\sin(x)$$
$$f'(c)=\cos(c)-\sin(c)=0$$
$$\cos(c)=\sin(c)$$

This occurs exactly when $c=\frac{\pi}{4}$.

We generalize Rolle's Theorem to the Mean Value Theorem

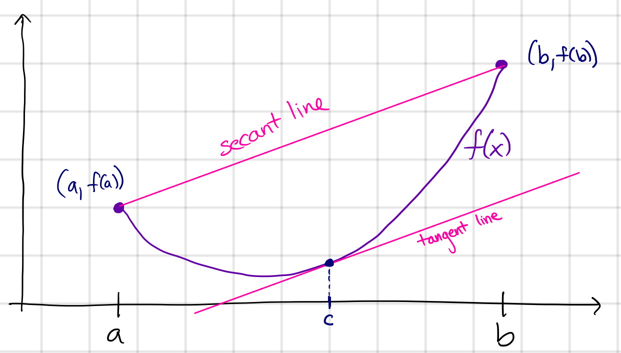
Theorem (Mean Value Theorem) Let f be

- continuous on $[a,b]$
- differentiable on (a,b).

Then there is some c in (a,b) such that
$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

slope of tangent line at $x=c$

slope of secant line through points $(a,f(a))$ and $(b,f(b))$



Proof (from Rolle's Theorem) Suppose f is

- continuous on $[a,b]$
- differentiable (a,b)

We define
$$g(x)=f(x)-\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$$

Then g is

- continuous on $[a,b]$
- differentiable (a,b)

But also, $g(a)=f(a)-\left(\frac{f(b)-f(a)}{b-a}\right)(a-a)$
$$=f(a)$$

and
$$g(b)=f(b)-\left(\frac{f(b)-f(a)}{b-a}\right)(b-a)$$
$$=f(b)-(f(b)-f(a))$$
$$=f(a)$$

So, $g(a)=g(b)$.

By Rolle's Theorem, there is some c in (a,b) such that
$$g'(c)=0$$

But then
$$g'(c)=f'(c)-\left(\frac{f(b)-f(a)}{b-a}\right)=0$$
$$f'(c)=\frac{f(b)-f(a)}{b-a}. \quad \blacksquare$$

Example Find c which satisfies the MVT for
$$f(x)=x^3-3x+5 \quad \text{on } [0,1]$$

First, f is continuous on $[0,1]$ and differentiable on $(0,1)$ since it is a polynomial.

By the MVT, there is some c in $(0,1)$ such that
$$f'(c)=\frac{f(1)-f(0)}{1-0}=\frac{3-5}{1-0}=-2$$

$$3c^2-3=-2$$
$$3c^2=1$$
$$c^2=\tfrac{1}{3}$$
$$c=\pm\sqrt{\tfrac{1}{3}}$$

Thus, we have two candidates: $\sqrt{\frac{1}{3}}$ and $-\sqrt{\frac{1}{3}}$.

But $-\sqrt{\frac{1}{3}}$ does not lie in the interval $(0,1)$. Hence, $c=\sqrt{\frac{1}{3}}$ satisfies the conclusion of the MVT.