HOMEWORK SOLUTIONS MATH 21100

1. Homework Set 1

1.1. **Problem 1.** The series is given by $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. This series is convergence since the terms tend to 0 and alternate in sign. However, this series is not absolutely convergent since

$$\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{2k+1} \right| = \sum_{k=0}^{\infty} \frac{1}{2k+1} > \sum_{k=0}^{\infty} \frac{1}{k},$$

and the series on the far right is the harmonic series which diverges to ∞.

1.2. **Problem 2.** Suppose $x \neq 1$. Consider the series $\sum_{k=0}^{\infty} x^k$ and the partial sums $S_N = \sum_{k=0}^{N} x^k$. For $N = 0$, we have $S_0 = 1 = \frac{1-x}{1-x} = 1$. This proves the base case for induction. Assume $S_N = \frac{1-x^{N+1}}{1-x}$. Then,

$$S_{N+1} = S_N + x^{N+1} = \frac{1 - x^{N+1}}{1-x} + \frac{1-x}{1-x} x^{N+1} = \frac{1 - x^{N+2}}{1-x}.$$

Hence, $S_N = \frac{1-x^{N+1}}{1-x}$ for all nonnegative integers N by mathematical induction.

When $|x| > 1$, then $|x|^N$ tends to infinity. Thus, $\sum_{k=0}^{\infty} x^N$ cannot be divergent since the sequence of terms x^N do not converge to 0.

When $x = 1$, we have that $S_N = \sum_{k=0}^{N} 1^k = N + 1$.

1.3. **Problem 3.** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ where $q > 1$. Here, we can use the integral test. We have the approximation

$$\sum_{n=1}^{N} \frac{1}{n^q} = 1 + \sum_{n=2}^{N} \frac{1}{n^q} \leq 1 + \int_1^N \frac{dx}{x^q} = 1 + \frac{1}{1-q} x^{1-q} \bigg|_1^N = \frac{1}{q-1} - \frac{N^{1-q}}{q-1}.$$

The sequence of partial sums is a monotone increasing sequence, since the terms are positive, and is bounded above by $\frac{1}{q-1}$. Hence, it converges.

1.4. **Problem 4.** The answer given by the program is never close to the exact value because of roundoff error. In computing the series, we must add 1 (the first term) to an extremely small number $\frac{(-25)^N}{N!}$ for some large N. Once this term gets below $2^{-52} \sim 2 \cdot 10^{-16}$, it is rounded to 0, so the sequence of partial sums (according to the program) becomes constant). This occurs for $N = 96$.

2. Homework Set 2

2.1. **Problem 1.** Let α and β be real numbers. Suppose $f(h) = O(h^\alpha)$ as $h \to 0$. This means that $|f(h)| \leq C|h|^\alpha$ for some constant C. We then get that

$$|h^\beta f(h)| = |h|^\beta |f(h)| \leq C|h|^\beta |h|^\alpha = C|h|^{\alpha+\beta}$$

so that $h^\beta f(h) = O(h^{\alpha+\beta})$ as $h \to 0$.

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2.2. **Problem 2.** For this problem, we wish to consider the error:

$$\begin{aligned}\left|\int_0^1 f(x)\,dx - h\sum_{k=0}^{N-1} f\left((k+1/2)h\right)\right| &= \left|\sum_{k=0}^{N-1}\int_{kh}^{(k+1)h} f(x)\,dx - h\sum_{k=0}^{N-1} f\left((k+1/2)h\right)\right| \\ &\leq \sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h} f(x)\,dx - hf\left((k+1/2)h\right)\right|.\end{aligned}$$

We then expand f into its Taylor series approximation around $(k+1/2)h$ under the integral:

$$\begin{aligned}&\sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h} f(x)\,dx - hf\left((k+1/2)h\right)\right| \\ &\quad= \sum_{k=0}^{N-1}\left|\int_{kh}^{(k+1)h}\left[f((k+1/2)h) + xf'((k+1/2)h) + O(x^2)\right]dx - hf\left((k+1/2)h\right)\right| \\ &\quad= \sum_{k=0}^{N-1}\left|\left[hf((k+1/2)h) + O(h^3)\right]dx - hf\left((k+1/2)h\right)\right| \\ &\quad= \sum_{k=0}^{N-1} O(h^3) = \frac{1}{h}O(h^3) \\ &\quad= O(h^2).\end{aligned}$$

In the step from the second line to the third line, the term with f' vanishes because its integral is 0. The second to last line follows since $N = \frac{1}{h}$.

2.3. **Problem 3.** Let $f(x) = \frac{x}{1+x^4}$. The relation between h and N now becomes $h = \frac{2-(-1)}{N} = \frac{3}{N}$. The choice of grid points becomes $x_j = -1 + hj$ for $j = 0, \ldots, N$. The value of $\int_{-1}^{2} f(x)\,dx$ can be found by running the rectangle method for large enough values of N or by directly integrating using the substitution method. The value is

$$\frac{1}{2}(\tan^{-1}(4) - \tan^{-1}(1)) = 0.2702097501.$$

We have the following loglog plots for these methods. [Insert Graphs] Note that the rate of convergence can be detected from the slope of the line. This is because if the error E is $O(h^k)$, that is $E \sim h^k$, then $\log E \sim k\log h$. The slopes of the plots (hence the rates of convergence) for the rectangle method is $O(h)$ and that of the trapezoidal method is $O(h^2)$.

2.4. **Problem 4.** Below are the two loglog plots of the error versus h for the rectangle method and the trapezoidal method.

Note that we can measure the slope of each of these plots to conclude that they each converge with rate $O(h^4)$. To explain why both the rectangle method and trapezoidal method give order 4 convergence, we first

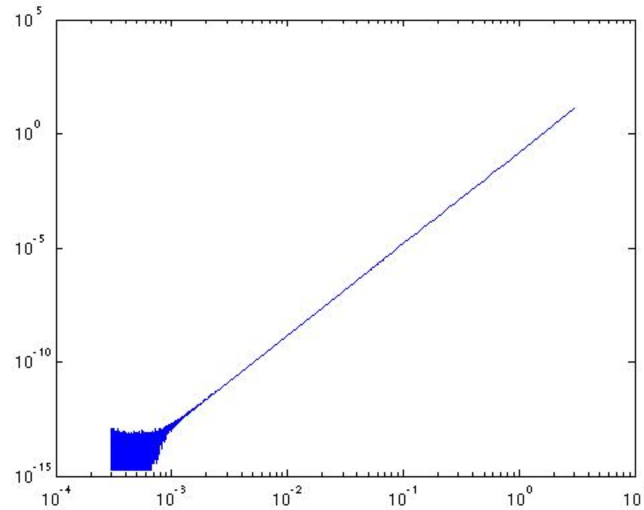

FIGURE 1. Rectangle Method

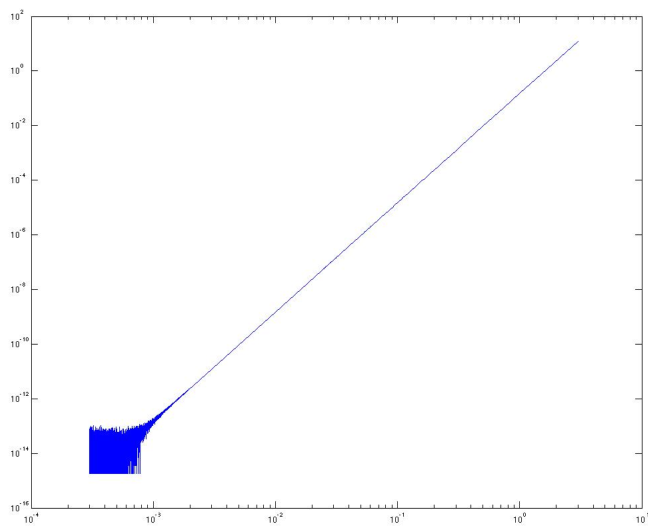

FIGURE 2. Trapezoidal Method

get, using the rectangle method,

$$\begin{aligned} h\sum_{j=0}^{N-1}(hj)^3(hj-2)^2 &= h\sum_{j=0}^{N-1}(hj)^3((hj)^2-6hj+9) \\ &= h\sum_{j=0}^{N-1}(hj)^5-6(hj)^4+9(hj)^3 \\ &= h^6\sum_{j=0}^{N-1}j^5-6h^5\sum_{j=0}^{N-1}j^4+9h^4\sum_{j=0}^{N-1}j^3 \\ &\vdots \\ &= \frac{243(N^4-1)}{20N^4} \\ &= 12.15+O(h^4). \end{aligned}$$

For the $\vdots$, I used mathematics to evaluate the $\sum_{j=0}^{N-1} j^k$ (which are polynomials in N) and used the relation $h = \frac{3}{N}$.

The trapezoidal rule has the same rate of convergence since these two methods only differ on the values of g at the endpoints -1 and 2 and these values happen to be 0.