HOMEWORK SOLUTIONS MATH 20500

1. Homework Set 1

1.1. **Exercise 5.5.6.** Intuitively, we wish to partition the rectangle R into subrectangles by extending the sides of the subrectangles R_i.

Let $R \subset \mathbb{R}^n$ be a generalized rectangle with $v(R) > 0$ and let $R_1, \ldots, R_N$ be a finite collection of subrectangles each with nonzero volume. Define the partition $\mathcal{P}$ as follows. Each rectangle R_i is a product $[a_{i1}, b_{i1}] \times \cdots \times [a_{in}, b_{in}]$. For each fixed $1 \leq j \leq N$ (this index represents the jth dimension), order the set $A_j = \{a_{1j}, b_{1j}, a_{2j}, b_{2j}, \ldots, a_{Nj}, b_{Nj}\}$ and adjoin 0 and 1 to get the set $A'_j = \{x_{1j}, x_{2j}, \ldots, x_{2N,j}\}$ where $0 = x_{0j} \leq x_{1j} \leq x_{2j} \leq \cdots \leq x_{2N+1,j} = 1$. Now, define the partition $\mathcal{P}$ as the collection of rectangles

$$R_{i_1, i_2, \ldots, i_n} = [x_{i_1+1,1}, x_{i_1 1}] \times \cdots \times [x_{i_j+1,j}, x_{i_j j}] \times \cdots \times [x_{i_n+1,n}, x_{I_n n}]$$

where $(i_1, \ldots, i_n)$ ranges over all $0 \leq i_j \leq 2N$ for each j.

1.2. **Exercise 5.5.13.** Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions and let $\mathcal{P}'$ be their common refinement, i.e. by applying Exercise 5.5.6 to the union $\mathcal{P} \cup \mathcal{Q}$. From Exercise 5.5.11, we have that

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{Q}).$$

1.3. **Exercise 5.5.18.** Recall that

$$\underline{\int}_R f = \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } R\}$$

and

$$\overline{\int}_R f = \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } R\}.$$

Given a partition $\mathcal{P}$, the partition constructed from Exercise 5.5.6 from $\mathcal{P}$ is a regular partition and a refinement of $\mathcal{P}$. Hence, for any partition $\mathcal{P}$, there is a regular partition $\mathcal{P}'$ such that $L(f, \mathcal{P}') \geq L(f, \mathcal{P})$ and $U(f, \mathcal{P}') \leq U(f, \mathcal{P})$. By definition of the infimum and supremum, this gives the result.

2. Homework Set 2

2.1. **Problem 5.7.2.** This is just standard calculus:

$$\int_2^4 \int_1^3 x^2 y^3 \, dx \, dy = \int_2^4 \left[\frac{1}{3}x^3\right]_1^3 \cdot y^3 \, dy = \frac{26}{3} \int_2^4 y^4 \, dy = \frac{26}{12}(4^4 - 2^4) = 520.$$

2.2. **Problem 5.7.5.** Recall the function

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

It was possible to evaluate the upper and lower sums of this function by just integrating the constant functions $\underline{f}(x) = 0$ and $\overline{f}(x) = 1$ along whichever interval you care about. We want to do the same thing with this new function.

It is possible to evaluate the upper and lower integrals by first noting that on any rectangle R_0 of positive volume, the infimum and supremum of f on R_0 is exactly the infimum and supremum of the functions $\underline{f}$ and $\overline{f}$ given by

$$\underline{f}(x, y) = \begin{cases} 2y, & \text{if } y \leq 1/2 \\ 1, & \text{if } y \geq 1/2 \end{cases} \text{ and } \overline{f}(x, y) = \begin{cases} 1, & \text{if } y \leq 1/2 \\ 2y, & \text{if } y \geq 1/2 \end{cases},$$

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i.e.

$$\inf_{R_0} f = \inf_{R_0} \underline{f}$$

and

$$\sup_{R_0} f = \sup_{R_0} \overline{f}.$$

Hence, the lower and upper sums of f are equal to the lower and upper sums of $\underline{f}$ and $\overline{f}$, respectively, for any partition. Hence, the lower and upper integrals of f are equal to the lower and upper integrals of $\underline{f}$ and $\overline{f}$. Since $\underline{f}$ and $\overline{f}$ are continuous, they are integrable. Together, this shows that

$$\underline{\int}_R f = \int_R \underline{f} \text{ and } \overline{\int}_R f = \int_R \overline{f}.$$

The integrals on the right hand side should be simple to evaluate by Fubini's theorem and breaking up $R = [0,1] \times [0,1]$ into $([0,1] \times [0,1/2]) \cup ([0,1] \times [1/2,1])$. For example,

$$\int_R \underline{f} = \int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^{1/2} 2y\,dy + \int_{1/2}^1 dy = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}.$$

2.3. **Problem 5.7.9.** We want to apply Fubini's theorem, but first we need to know that f is integrable on R. You can show this by using the fact that a function is integrable if and only if the set of discontinuities is countable: let S_g and S_h be the sets of discontinuities of g and h in R_1 and R_2. Then $S_g \times S_h$ contains the set of discontinuities of f and is countable, hence the set of discontinuities of f is countable and so f is integrable.

It is also possible to show that f is integrable by using partitions of R and directly using the definition. For any $\epsilon > 0$, we can build a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$ by using partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of g and h with the same properties and defining

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 = \{P_1 \times P_2 : P_1 \in \mathcal{P}_1 \text{ and } P_2 \in \mathcal{P}_2\}.$$

Check that this partition satisfies the definition for integrability.

Anyhow, f is integrable. Each of the functions $f_x(y) := f(x,y)$ is continuous since, fixing x, it is just multiplying h by a constant. We can then apply Fubini:

$$\int_R f = \int_{R_2} \int_{R_1} f(x,y)\,dx\,dy = \int_{R_2} \int_{R_2} g(x)h(y)\,dx\,dy = \left(\int_{R_2} h(y)\,dy\right)\left(\int_{R_1} g(x)\,dx\right).$$

3. Homework Set 3

3.1. **Exercise 5.8.7.** Suppose $o(f,x) = 0$. We want to show that f is continuous at x. Let $\epsilon > 0$. Pick $\delta > 0$ so that,

$$|M(f,x,\delta) - m(f,x,\delta) - o(f,x)| = |M(f,x,\delta) - m(f,x,\delta)| < \epsilon.$$

Then, for each $|x-y| < \delta$, we have that

$$|f(x) - f(y)| \leq |M(f,x,\delta) - m(f,x,\delta)| < \epsilon.$$

Hence f is continuous at x.

If f is continuous at x, let $\epsilon > 0$ and pick $\delta > 0$ so that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon/2$. Then, for any $\eta < \delta$,

$$\begin{aligned} |M(f,x,\eta) - m(f,x,\eta)| &= \sup_{x_1,x_2 \in B_\eta(x)} |f(x_1) - f(x_2)| \\ &\leq \sup_{x_1,x_2 \in B_\eta(x)} (|f(x) - f(x_1)| + |f(x) - f(x_2)| \\ &\leq \epsilon. \end{aligned}$$

By the $\epsilon - \delta$ definition of limits, this shows that $o(f,x) = 0$.

3.2. **Exercise 5.9.8.** i) The horizontal and vertical legs of the triangle $\partial\Omega$ are easily seen to have measure zero by covering them by increasingly thin rectangles. The hypotenuse also has measure zero. But we cannot see this by covering by a thin rectangle as with the other two sides because such a rectangle would have sides that are not parallel to the axes. Instead, we take the collection $\mathcal{C}_n$ of rectangles of the form $[i/n,(i+1)/n]\times[i/n,(i+1)/n]$ for $i=0,\ldots,n-1$. For each n, $\mathcal{C}_n$ covers the hypotenuse and the sum of the areas of the rectangles in $\mathcal{C}_n$ is $n\cdot(1/n)^2=1/n\to 0$. Hence, for each $\epsilon>0$, take n so that $n>1/\epsilon$. Then, the hypotenuse is covered by the union of the rectangles in $\mathcal{C}_n$ and they have total area $1/n<\epsilon$. This shows that each side has measure zero and finite unions of measure zero sets have measure zero, so we are done.

ii) You just set up the integral using Fubini as

$$\int_0^1\int_0^y \sin(y^2)\,dx\,dy=\int_0^1 y\sin(y^2)\,dy=\frac{1}{2}(1-\cos(1))$$

and solve using iterated integration.

iii) Trying to compute the other iterated integral given by Fubini,

$$\int_0^1\int_x^1 \sin(y^2)\,dy\,dx,$$

leads to problems since $\sin(y^2)$ has no integral in terms of elementary functions. By integrating with respect to x first, we introduce a factor of y which can then be used in conjunction with a standard change of variables.

3.3. **Exercise 5.9.11.** We are not allowed to use the change of variables formula. First, note that this problem is almost trivial if Ω is a rectangle: $v_n(\Omega)$ is the product of the side lengths of Ω which remain the same under the translation T.

For the general Ω with content zero boundary $\partial\Omega$, we let $\{\Omega_i\}_{i=1}^\infty$ be a collection of subrectangles of Ω such that each Ω_i is an open rectangle, $\Omega\subset\bigcup_{i=1}^\infty\overline{\Omega}_i$ and $\Omega_i\cap\Omega_j=\varnothing$. This implies that (Exercise 5.9.5)

$$v_n(\Omega)=\int_\Omega 1=\sum_{i=1}^\infty\int_{\Omega_i}1=\sum_{i=1}^\infty v_n(\Omega_i).$$

Then, $\{T(\Omega_i)\}_{i=1}^\infty$ is a similar collection of subrectangles of $T(\Omega)$ satisfying the same properties and so

$$v_n(\Omega)=\sum_{i=1}^\infty v_n(\Omega_i)=\sum_{i=1}^\infty v_n(T(\Omega_i))=v_n(T(\Omega)),$$

where the second equality comes from the special case in the previous paragraph.

4. Homework Set 9

4.1. **Exercise 6.5.27 iv.** The book states that the area form for a surface $S\subset\mathbb{R}^3$ is

$$\alpha_{\phi(s,t)}(v,w)=\frac{D_1\phi(s,t)(v)D_2\phi(s,t)(w)-D_2\phi(s,t)(v)D_1\phi(s,t)(w)}{\|D_1\phi(s,t)\|\|D_2\phi(s,t)\|}$$

and, if you're a reasonable person, you'd find no meaning in what this is. So, I'm going to take this time to go over what exactly this means. Actually, this definition uses incorrect notation.

If $\phi:U\to S$ is a parameterization of a surface, then the area form is defined by

$$\alpha(v,w)=\frac{1}{\|X\|\|Y\|}\left((X\cdot v)(Y\cdot w)-(X\cdot w)(Y\cdot v)\right),$$

where X and Y are the tangent vectors defined by $X=\frac{\partial\phi}{\partial s}$ and $Y=\frac{\partial\phi}{\partial t}$ (s and t are the coordinates on U). We can put this into the standard form $\alpha=f\,dx\wedge dy+g\,dx\wedge dz+h\,dy\wedge dz$ by noting that $f=\alpha(e_1,e_2)$, $g=\alpha(e_1,e_3)$ and $h=\alpha(e_2,e_3)$. This is what we do for 6.5.27 iv and 6.5.28 iv.

In this particular case, S is a rectangle and we can take $\phi(s,t) = (s,t,c-(c/b)t)$. Then, $X = e_1$ and $Y = e_2 - \frac{c}{b}e_3$. Then,

$$\begin{aligned}\alpha(v,w) &= \frac{1}{\sqrt{1+(c/b)^2}}\left((e_1\cdot v)\left(e_2\cdot w - \frac{c}{b}e_3\cdot w\right) - (e_1\cdot w)\left(e_2\cdot v - \frac{c}{b}e_3\cdot v\right)\right)\\ &= \frac{1}{\sqrt{1+(c/b)^2}}\left(v_1(w_2 - cw_3/b) - w_1(v_2 - cv_3/b)\right).\end{aligned}$$

where v_i and w_i are the coordinates of v and w respectively. We then get that $\alpha(e_1,e_2) = \frac{1}{\sqrt{1+(c/b)^2}}$ and $\alpha(e_1,e_3) = \frac{-c/b}{\sqrt{1+(c/b)^2}}$ and $\alpha(e_2,e_3) = 0$ so that

$$\alpha = \frac{1}{\sqrt{1+(c/b)^2}}dx\wedge dy - \frac{c/b}{\sqrt{1+(c/b)^2}}\,dx\wedge dz.$$

We can use the parameterization ϕ to calculate $\int_S \alpha$ since $dx = ds$, $dy = dt$ and $dz = -(c/b)dt$. Thus,

$$\begin{aligned}\int_S \alpha &= \int_S dx\wedge dy\\ &= \int_S \frac{1}{\sqrt{1+(c/b)^2}}dx\wedge dy - \frac{c/b}{\sqrt{1+(c/b)^2}}\,dx\wedge dz\\ &= \frac{1}{\sqrt{1+(c/b)^2}}\left(\int_S dx\wedge dy - \frac{c}{b}\int_S dx\wedge dz\right)\\ &= \frac{1}{\sqrt{1+(c/b)^2}}\left(\int_0^b\int_0^a ds\wedge dt + \left(\frac{c}{b}\right)^2\int_0^b\int_0^a ds\wedge dt\right)\\ &= \sqrt{1+\left(\frac{c}{b}\right)^2}\int_0^b\int_0^a ds\wedge dt\\ &= ab\sqrt{1+\left(\frac{c}{b}\right)^2}\end{aligned}$$

Which we could expect since it is the area of the parallelogram S.

4.2. **Exercise 6.5.28 iv.** We use polar coordinates $x = \sin\phi\sin\theta$, $y = \sin\phi\cos\theta$, and $z = \cos\phi$ for $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. Hence, this is a parameterization $\Phi : [0,\pi/2]\times[0,2\pi] \to S$. The tangent spaces of the sphere are spanned by

$$X(\phi,\theta) := \frac{\partial\Phi}{\partial\phi} = \begin{pmatrix}\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{pmatrix} \text{ and } Y(\phi,\theta) := \frac{\partial\Phi}{\partial\theta} = \begin{pmatrix}-\sin\phi\sin\theta\\ \sin\phi\cos\theta\\ 0\end{pmatrix}.$$

The 2-form α is then defined by

$$\alpha(v,w) = \frac{(X\cdot v)(Y\cdot w) - (X\cdot w)(Y\cdot v)}{\|X\|\|Y\|}.$$

(This is the use of the formula above for α with different notation which I hope makes much clearer the operations involved.) We want to write α as a linear combination of $dx\wedge dy$, $dx\wedge dz$ and $dy\wedge dz$. Thus, we take v,w to be the unit vectors e_1,e_2,e_3 to get that

$$\alpha = \alpha(e_1,e_2)dx\wedge dy + \alpha(e_2,e_3)dy\wedge dz + \alpha(e_1,e_3)dx\wedge dz.$$

By plugging in, we get that

$$
\begin{aligned}
\alpha(e_1, e_2) &= \frac{(X \cdot e_1)(Y \cdot e_2) - (X \cdot e_2)(Y \cdot e_1)}{\|X\|\|Y\|} \\
&= \frac{\cos\phi \sin\theta \sin\phi \sin\theta + \sin\phi \cos\theta \cos\phi \cos\theta}{\sin\phi} \\
&= \cos\phi \cos^2\theta + \cos\phi \sin^2\theta \\
&= \cos\phi \\
&= z
\end{aligned}
$$

The other two components are similar. We get $\alpha = z\,dx \wedge dy - y\,dx \wedge dz + x\,dy \wedge dz$. This is the 2-form you are tasked with integrating over S. The procedure remains the same. By change of variables, with respect to the parameterization Φ, we obtain a new differential 2-form $\Phi^*\alpha$ on $[0, 2\pi] \times [0, \pi/2]$ which can be integrated. The chain rule gives

$$
\begin{aligned}
dx &= -\sin\phi \sin\theta\, d\theta + \cos\phi \cos\theta\, d\phi, \\
dy &= \sin\phi \cos\theta\, d\theta + \cos\phi \sin\theta\, d\phi,
\end{aligned}
$$

and

$$dz = -\sin\phi\, d\phi.$$

Thus,

$$
\begin{aligned}
\Phi^*\alpha &= z\,dx \wedge dy - y\,dx \wedge dz + x\,dy \wedge dz \\
&= (\cos^2\phi \sin\phi + \sin^2\phi \sin^2\theta \sin\phi + \sin^2\phi \cos^2\theta \sin\phi)\, d\phi \wedge d\theta \\
&= \sin\phi\, d\phi \wedge d\theta.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\int_S \alpha &= \int_{[0,2\pi]\times[0,\pi/2]} \Phi^*\alpha \\
&= \int_0^{2\pi} \int_0^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= 2\pi(-\cos\phi)_0^{\pi/2} \\
&= 2\pi.
\end{aligned}
$$

This agrees with the intuition that $\int_S \alpha$ should be the area of the surface S, in this case a half sphere.