Homework 1

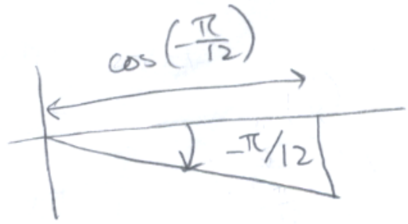
1. a) $\cos\left(-\frac{\pi}{12}\right)$. There are two ways to solve this.

1st Way Addition Identity

$$\cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \left(-\frac{\pi}{3}\right)\right)$$
$$= \cos\left(\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{2}+\sqrt{6}}{2}$$

2nd Way : Half-Angle Formula

$$\cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{-\pi/6}{2}\right)$$
$$= \pm\sqrt{\frac{1+\cos\left(-\frac{\pi}{6}\right)}{2}}$$
$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$
$$\boxed{\cos\left(-\frac{\pi}{12}\right) = \pm\frac{\sqrt{2+\sqrt{3}}}{2}}$$



Drawing a circle, we see that we take the + sign

$$\boxed{\cos\left(-\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2}}$$

b) $\tan\left(\frac{\pi}{8}\right)$. Here, you use the Half-Angle Formula.

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi/4}{2}\right)$$

$$= \pm\sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}}$$

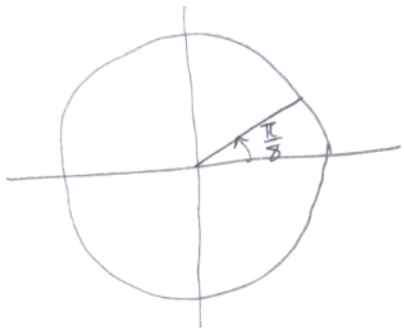
$$= \pm\sqrt{\frac{1+(\sqrt{2}/2)}{1-(\sqrt{2}/2)}}$$

$$= \pm\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$ (Multiply top and bottom by 2).

$$= \pm\frac{2+\sqrt{2}}{\sqrt{4-2}}$$ (By multiplying top and bottom by $\sqrt{2+\sqrt{2}}$)

$$= \pm(\sqrt{2}+1)$$

Drawing a circle!



We see we must take the + sign so

$$\boxed{\tan\left(\frac{\pi}{8}\right) = 1+\sqrt{2}}$$

2. a) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x}$

$= 1 \cdot 1$

$= 1$

b) $\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{4 \sin(4x)}{4x}$

$= 4 \cdot \lim_{x\to 0} \frac{\sin(4x)}{4x}$

$= 4 \cdot 1$

$= 4$

c) $\lim_{x\to 0} \left[(1-x)\cdot \csc(1-x^2)\right]$

$= \lim_{x\to 0} \frac{1-x}{\sin(1-x^2)} \cdot \frac{1+x}{1+x}$

$= \lim_{x\to 0} \frac{1-x^2}{\sin(1-x^2)} \cdot \frac{1}{1+x}$

$= \lim_{x\to 0} \frac{1-x^2}{\sin(1-x^2)} \cdot \lim_{x\to 0} \frac{1}{1+x}$

$= 1 \cdot 1$

$= 1$

Recall: If g is continuous at c and $\lim_{x\to g(c)} f(x)$ exists, then $\lim_{x\to c} (f\circ g)(x) = \lim_{x\to g(c)} f(x)$.

d) $\lim_{x\to 0} \frac{1-\cos x}{\tan x} = \lim_{x\to 0} (1-\cos x)\cdot(\cos x)\cdot\left(\frac{1}{\sin x}\right)$

$$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{x}{\sin x} \cdot \cos x$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \cos x$$

$$= 0 \cdot 1 \cdot 1$$

$$= 0$$

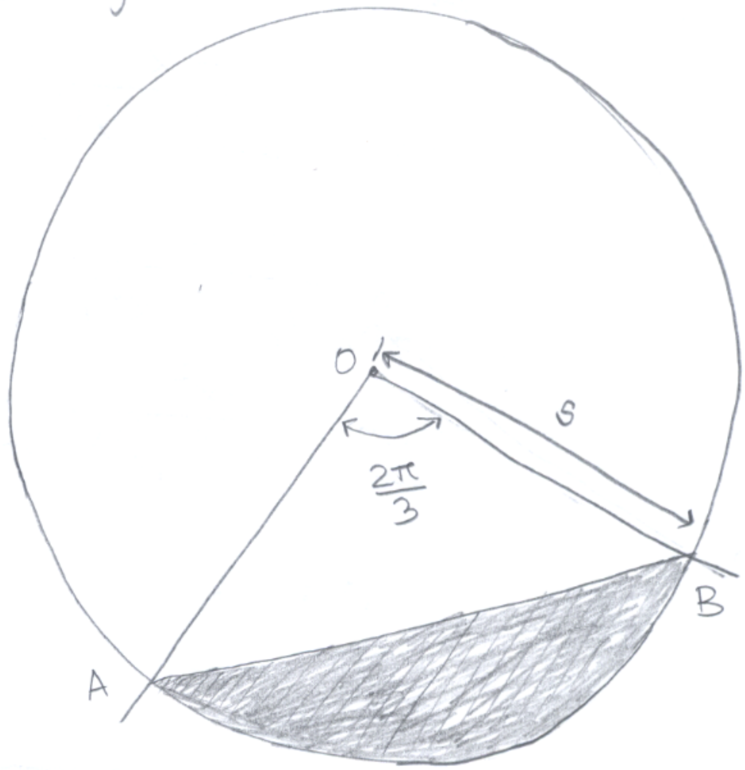
e) $\lim_{x\to 3} \frac{1-\cos^2(3x)}{x} = \frac{1-\cos^2(9)}{3}$

because $\cos(x)$ and hence $\frac{1-\cos^2(3x)}{x}$ is continuous at $x=3$.

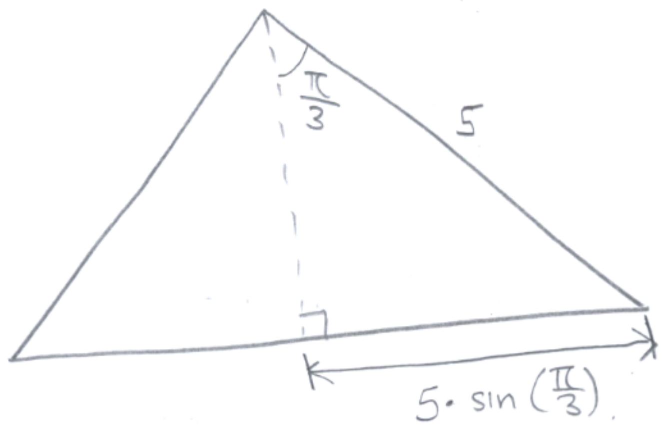
3. The area A of the shaded region is

$A =$ Area of Sector − Area of Triangle AOB.

The area of the sector is $r\theta$ where $r=5$ and $\theta = \frac{2\pi}{3}$, i.e. Area(sector) $= \frac{10\pi}{3}$.



The area of the white triangle can be determined as follows. Divide the triangle into two smaller triangles with the dotted line as shown



The base of one of these triangles has length $5 \sin\left(\frac{\pi}{3}\right)$ and the height is $5 \cos\left(\frac{\pi}{3}\right)$. Thus, the area of one triangle is

$$\frac{1}{2}\left(5 \sin\left(\frac{\pi}{3}\right)\right)\left(5 \cos\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{25}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$$

$$= \frac{25}{8}\sqrt{3}.$$

Homework 2

1. We have

$$\frac{d}{dx}\left[\cos x\right] = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos x}{h}$$

$$= \left(\lim_{h\to 0} \frac{\cos(x)\cos(h)-\cos(x)}{h}\right) - \left(\lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}\right)$$

$$= \left(\lim_{h\to 0} \cos(x)\cdot \frac{\cos(h)-1}{h}\right) - \left(\lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}\right)$$

$$= \cos(x)\left(\lim_{h\to 0} \frac{\cos(h)-1}{h}\right) - \sin(x)\cdot\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1$$

$$= -\sin(x).$$

2. a) $\frac{d}{dx}\left[\sin^4(x)-\cos^4(x)\right]$

$$= 4\sin^3(x)\cos(x) - 4\cos^3(x)(-\sin x) \qquad \text{via Chain Rule}$$

$$= 4\sin^3(x)\cos(x) + 4\cos^3(x)\sin(x)$$

$$= 4\sin(x)\cos(x)\left(\sin^2(x)+\cos^2(x)\right)$$

$$= 4\sin(x)\cos(x) \qquad \text{Pythagorean Identity}$$

You could have also chosen to write

$$\sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$= \sin^2 x - \cos^2 x$$

and differentiate this simpler function.

b) $\frac{d}{dx}\left[\frac{\sec^2(10x)}{1+\tan^2(10x)}\right]$.

We first write

$$1+\tan^2(10x) = \sec^2(10x) \quad \text{(Pythagerean Identity.)}$$

Hence, $\frac{\sec^2(10x)}{1+\tan^2(10x)} = \frac{\sec^2(10x)}{\sec^2(10x)} = 1$

where it is defined and so $\frac{d}{dx}(1) = 0$.

c). $\frac{d}{dx}\left[\cos(\cos(\sin(x)))\right]$.

$$= -\sin(\cos(\sin(x))) \cdot \frac{d}{dx}\left[\cos(\sin(x))\right]$$
$$= -\sin(\cos(\sin(x))) \cdot (-\sin(\sin(x))) \cdot \frac{d}{dx}(\sin(x))$$
$$= \sin(\cos(\sin(x))) \cdot \sin(\sin(x)) \cdot \cos(x).$$

e) $\frac{d}{dx}\left[\cot^3 x\right] = 3\cot^2 x \cdot \frac{d}{dx}\left[\cot x\right]$. by Chain Rule.

Thus, to calculate $\frac{d}{dx}\left[\cot x\right]$:

$$= \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{-\sin x \cdot \sin x - \cos x \cos x}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\csc^2 x.$$

Hence, $\frac{d}{dx}[\cot^3 x] = 3\cot^2 x(-\csc^2 x)$
$= -3\cot^2 x \csc^2 x.$

f) $\frac{d}{dx}\left[\tan(x)\csc(x)\left(1+\tan^2(x)\right)\right]$

First, note that

$$\tan(x)\csc(x)\left(1+\tan^2(x)\right)$$
$$= \tan(x)\csc(x)\cdot\sec^2(x)$$
$$= \frac{\sin(x)}{\cos(x)}\frac{1}{\sin(x)}\frac{1}{\cos^2(x)}$$
$$= \frac{1}{\cos^3 x}$$
$$= \sec^3 x$$

So, $\frac{d}{dx}[\sec^3 x] = 3\sec^2 x\frac{d}{dx}[\sec x]$
$= 3\sec^2 x(\sec x \tan x)$
$= 3\sec^3 x \tan x.$

g) $\frac{d}{dx}[\cot(\tan x)]$

$$= -\csc^2(\tan x)\cdot\frac{d}{dx}[\tan x]$$
$$= -\csc^2(\tan x)\cdot\sec^2 x.$$

h) $\frac{d}{dx}\left[(1+x^2)\sec(x)\right]$

$= \frac{d}{dx}(1+x^2) \cdot \sec(x) + (1+x^2)\frac{d}{dx}(\sec(x)).$

$= 2x\sec(x) + (1+x^2)\sec(x)\tan(x)$

i) $\frac{d}{dx}\left[\tan\left(\frac{1}{1+x^2}\right)\right]$

$= \sec^2\left(\frac{1}{1+x^2}\right) \cdot \frac{d}{dx}\left[\frac{1}{1+x^2}\right]$

$= \sec^2\left(\frac{1}{1+x^2}\right) \cdot \frac{-2x}{(1+x^2)^2}$

$= -\frac{2x\sec^2\left(\frac{1}{1+x^2}\right)}{(1+x^2)^2}$

j) $\frac{d}{dx}\left[\cos(x\sin x)\right]$

$= -\sin(x\sin(x)) \cdot \frac{d}{dx}(x\sin x)$

$= -\sin(x\sin x) \cdot (\sin x + x\cos x)$

3. a) $f(x) = \tan(x)$

$\tan(x+\Delta x) \approx \tan(x) + \sec^2(x) \cdot \Delta x$

Let $x = \frac{\pi}{3}$, $\Delta x = -\frac{\pi}{300}$.

Then $x + \Delta x = \frac{99\pi}{300}$ and

$\tan\left(\frac{99\pi}{300}\right) \approx \tan\left(\frac{\pi}{3}\right) + \sec^2\left(\frac{\pi}{3}\right) \cdot \left(-\frac{99\pi}{300}\right)$

$= \frac{\sqrt{3}}{3} + (2)^2\left(-\frac{99\pi}{300}\right) = \frac{\sqrt{3}}{3} - \frac{99\pi}{75}$

b) $f(x) = \cos(x)$.

$$\cos(x+\Delta x) \approx \cos(x) - \sin(x) \cdot \Delta x$$

Let $x = \frac{\pi}{2}$, $\Delta x = \frac{\pi}{12}$.

Then $x + \Delta x = \frac{7\pi}{12}$ so

$$\cos\left(\frac{7\pi}{12}\right) \approx \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \cdot \left(\frac{\pi}{12}\right)$$

$$= -\frac{\pi}{12}.$$

c) $f(x) = \sin(x)$

$$\sin(x+\Delta x) \approx \sin(x) + \cos(x) \cdot \Delta x$$

Let $x = 0$, $\Delta x = \frac{\pi}{1000}$. Then $x + \Delta x = \frac{\pi}{1000}$ and

$$\sin\left(\frac{\pi}{1000}\right) \approx \sin(0) + \cos(0) \cdot \frac{\pi}{1000}$$

$$= \frac{\pi}{1000}$$

4. $f(x) = \cos x$

$\cos(x+\Delta x) \approx \cos(x) - \sin(x) \cdot \Delta x$ and $x + \Delta x = \frac{2\pi}{3} - \frac{\pi}{12} = \frac{7\pi}{12}$ we find

Using $x = \frac{2\pi}{3}$, $\Delta x = -\frac{\pi}{12}$

$$\cos\left(\frac{7\pi}{12}\right) \approx \cos\left(\frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) \cdot \left(-\frac{\pi}{12}\right)$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{12}$$

$$= \frac{\pi\sqrt{3}}{24} - \frac{1}{2}$$

The exact value of $\cos\left(\frac{7\pi}{12}\right)$ is found by

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{7\pi/6}{2}\right)$$

$$= \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

Multiply top and bottom by 2 and use $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

$$= \pm\frac{\sqrt{2+\sqrt{3}}}{2}$$

We see from the diagram that we use the $-$ sign so

$$\cos\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{2+\sqrt{3}}}{2} \approx -0.2588.$$

The approximate values we found were

$$-\frac{\pi}{12} \approx -0.2618 \quad \text{and} \quad \frac{\pi\sqrt{3}}{24} - \frac{1}{2} \approx -0.2733$$

The value found by approximating $\frac{7\pi}{12} \approx \frac{\pi}{2}$ is more accurate than the value found by approximating $\frac{7\pi}{12} \approx \frac{2\pi}{3}$ since $|f''(x)|$ is smaller at $x = \frac{\pi}{2}$ than at $x = \frac{2\pi}{3}$ and so the linear approximation

$$f(x+\Delta x) \approx f(x) + f'(x)\cdot\Delta x$$

is much better.

Note: It is possible to get a better-than-linear approximation using the formula

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2.$$

Homework 3

1. a) $f(x) = \sin^2 x - \sqrt{3}\cos x$ on $[0,\pi]$

$f'(x) = 2\sin x\cos x + \sqrt{3}\sin x$

Endpoints: $x = 0, \pi$

Singular Points: None

Stationary Points:

Solve $2\sin x \cos x + \sqrt{3}\sin x = 0$

$\sin x\,(2\cos x + \sqrt{3}) = 0$

$\sin x = 0$

$x = 0, \pi$

$2\cos x + \sqrt{3} = 0$

$\cos x = -\frac{\sqrt{3}}{2}$

$x = \frac{5\pi}{6}$.

Values: $f(0) = -\sqrt{3} \approx -1.73$

$f(\pi) = \sqrt{3} \approx 1.73$

$f\left(\frac{5\pi}{6}\right) = \sin^2\left(\frac{5\pi}{6}\right) - \sqrt{3}\cos\left(\frac{5\pi}{6}\right)$

$= \left[\frac{1}{2}\right]^2 + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{4} + \frac{3}{2}$

$= \frac{7}{4} = 1.75$

Max: $\frac{7}{4}$ at $x = \frac{5\pi}{6}$

Min: $-\sqrt{3}$ at $x = 0$.

b) $f(x) = \sin^4 x - \sin^2 x$ on $[0, \frac{2\pi}{3}]$.

$= \sin^2 x(\sin^2 x - 1)$

$= -\sin^2 x \cos^2 x$

$f'(x) = 4\sin^3 x \cos x - 2\sin x \cos x$

Endpoints: $x = 0, \frac{2\pi}{3}$.

Singular: None

Stationary: Solve

$4\sin^3 x \cos x - 2\sin x \cos x = 0$

$2\sin x \cos x(2\sin^2 x - 1) = 0$

$\sin x = 0$

$x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}$

$2\sin^2 x - 1 = 0$

$\sin^2 x = \frac{1}{2}$

$\sin x = \pm\frac{\sqrt{2}}{2}$.

$x = \frac{\pi}{4}$

Evaluate: $f(0) = 0$

$f(\frac{\pi}{4}) = [\frac{\sqrt{2}}{2}]^4 - [\frac{\sqrt{2}}{2}]^2$

$= \frac{1}{4} - \frac{1}{2}$

$= -\frac{1}{2}$

$f(\frac{\pi}{2}) = \sin^4(\frac{\pi}{2}) - \sin^2(\frac{\pi}{2}) = 1 - 1 = 0$

$f(\frac{2\pi}{3}) = \sin^4(\frac{2\pi}{3}) - \sin^2(\frac{2\pi}{3}) = [\frac{\sqrt{3}}{2}]^4 - [\frac{\sqrt{3}}{2}]^2$

$= \frac{9}{16} - \frac{3}{4} = -\frac{3}{16}$

Max: 0 at $x = 0$, $x = \frac{\pi}{2}$

Min: $-\frac{1}{2}$ at $x = \frac{\pi}{4}$

c) $f(x) = \sin(x)$ on $[0, 5\pi]$

$f'(x) = \cos(x)$

Endpts: $0, 5\pi$

Singular: None

Stationary: Solve $\cos x = 0$ to get

$x = \frac{\pi}{2} + \pi n$ for any integer n.

The only values $\frac{\pi}{2} + \pi n$ which fall in $[0, 5\pi]$

are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$.

Evaluate:

$f(0) = 0$

$f(5\pi) = 0$

$f\left(\frac{\pi}{2}\right) = f\left(\frac{5\pi}{2}\right) = f\left(\frac{9\pi}{2}\right) = 1$.

$f\left(\frac{3\pi}{2}\right) = f\left(\frac{7\pi}{2}\right) = -1$.

Max: 1 at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

Min: -1 at $x = \frac{3\pi}{2}, \frac{7\pi}{2}$.

d) $f(x) = |\sin(x)|$ on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$f'(x) = \frac{\sin(x)}{|\sin(x)|} \cdot \cos(x)$$

Endpts: $x = \frac{\pi}{4}, \frac{3\pi}{4}$

Singular: Solve $|\sin(x)| = 0$

$\sin(x) = 0$

$x = \pi n$ for any integer n.

No such points fall in $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ so there are no singular points.

Stationary: Solve $f'(x) = \frac{\sin(x)}{|\sin(x)|} \cos(x) = 0$

$\sin(x) = 0$

None in $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

$\cos(x) = 0$

$x = \frac{\pi}{2}$

Evaluate: $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f\left(\frac{\pi}{2}\right) = 1.$

Max: 1 at $x = \frac{\pi}{2}$.

$\frac{\sqrt{2}}{2}$ at $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

e) $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x + 7$ on $[-5, 5]$

$f'(x) = x^2 - 4x - 5$
$= (x-5)(x+1)$.

Endpoints: $x = -5, 5$
~~Stationary~~ points: Solve $(x-5)(x+1) = 0$
$x = 5, -1$.
Singular Points: None.

Evaluate: $f(-5) = \frac{1}{3}(-5)^3 - 2(-5)^2 - 5(-5) + 7$
$= -\frac{179}{3}$

$f(5) = -\frac{79}{3}$

$f(-1) = \frac{29}{3}$.

Max: $\frac{29}{3}$ at $x = -1$
Min: $-\frac{179}{3}$ at $x = -5$

f) $f(x) = x^2 \sec(x)$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

$f'(x) = 2x\sec(x) + x^2\sec(x)\tan(x)$

Endpts: $x = -\frac{\pi}{4}, \frac{\pi}{4}$

Singular: same as singular points for $\tan(x)$ or $\sec(x)$

i.e. where $\cos(x) = 0$ which does not occur in $[-\pi/4, \pi/4]$.

Stationary: Solve $2x\sec(x) + x^2\sec(x)\tan(x) = 0$

$x\sec(x)(2 + x\tan(x)) = 0$

$x = 0$

$\sec(x) = 0$
No solutions

$2 + x\tan(x) = 0$.
No solutions in $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

In order to solve $2 + x\tan(x) = 0$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.
you need to look at the graph of the function $f(x) = 2 + x\tan(x)$. and notice it is never 0 in $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

2. The volume of a cylindrical can of height h and radius r is

$$V = \pi r^2 h.$$

It's surface area is

$$A = 2\pi r^2 + 2\pi r h$$
$$= 2\pi r (r+h).$$

The condition of the problem restricts $A=100$ so

$$2\pi r(r+h) = 100$$
$$r+h = \frac{100}{2\pi r}$$
$$h = \frac{100}{2\pi r} - r$$

relates the radius to the height.

Substituting this into the volume gives.

$$V = \pi r^2\left(\frac{100}{2\pi r} - r\right)$$
$$V = 50r - \pi r^3.$$

Taking derivatives gives.

$$\frac{dV}{dr} = 50 - 3\pi r^2.$$

Solving

$$\frac{dV}{dr} = 0$$
$$50 - 3\pi r^2 = 0$$
$$r = \sqrt{\frac{50}{3\pi}}$$

Gives stationary points.
There are no critical points.

For the endpoints, we need to figure out on what interval V is defined.

To find the interval on which V is defined, we note that in order to be a realistic value, we need

$$V \geq 0$$
$$r \geq 0$$
$$h \geq 0.$$

Already, we've restricted r to lie in $[0, \infty)$. Now,

$$h \geq 0$$

$$\Rightarrow \frac{100}{2\pi r} - r \geq 0$$ Multiply by r

$$\Rightarrow \frac{100}{2\pi} - r^2 \geq 0$$

$$\Rightarrow r \leq \sqrt{\frac{50}{\pi}}$$

Thus, the problem only makes sence if we are looking for a maximum in the interval $[0, \sqrt{\frac{50}{\pi}}]$. Thus, the endpoints are 0 and $\sqrt{\frac{50}{\pi}}$.

Evaluate: $V(0) = 0$

$$V\left(\sqrt{\frac{50}{\pi}}\right) = 0$$

$$V\left(\sqrt{\frac{50}{3\pi}}\right) = 50\sqrt{\frac{50}{3\pi}} - \pi\left(\frac{50}{3\pi}\right)^{3/2}$$

Therefore the max volume is

$$\left[50\sqrt{\frac{50}{3\pi}} - \pi\left(\frac{50}{3\pi}\right)^{3/2}\right] cm^3 = 76.776\ cm^3.$$

3. We have $h(t) = -16t^2 + 300\,000\,000\,t$.

$= -16t^2 + (3 \cdot 10^8)\,t$

$h'(t) = -32t + 3 \cdot 10^8$

$3 \cdot 10^8$ is just scientific notation for 3 followed by 8 zeroes.

- Endpoints: height above the Earth only makes sense if $h(t) \geq 0$ (unless Team Rocket is digging a hole). Thus, the interval we are interested in is the solutions to $h(t) \geq 0$, i.e.

$$-16t^2 + (3 \cdot 10^8)\,t \geq 0$$
$$t\left[3 \cdot 10^8 - 16t\right] \geq 0$$
$$t \geq 0 \text{ and } t \leq \frac{3 \cdot 10^8}{16}$$

so the interval we look at is $\left[0, \frac{3 \cdot 10^8}{16}\right]$ with endpoints 0 and $\frac{3 \cdot 10^8}{16}$.

- Singular: None
- Stationary: $-32t + 3 \cdot 10^8 = 0$

$$t = \frac{3 \cdot 10^8}{32}$$

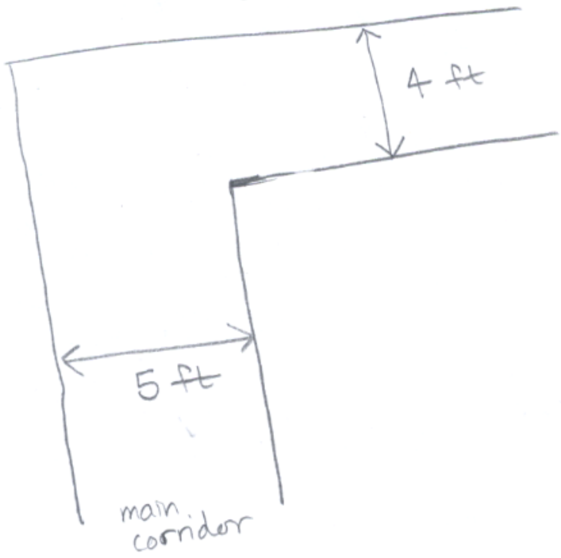
Evaluate: $h(0) = 0$

$$h\left(\frac{3 \cdot 10^8}{16}\right) = 0$$
$$h\left(\frac{3 \cdot 10^8}{32}\right) = 9{,}375{,}000$$

Thus, the maximum height is $9{,}375{,}000$ m.

Unfortunately, I forgot to say that the -16 in the problem has units of meters per second per second, which actually isn't true on Earth.

4. Consider the following diagram.



Strategy: We break this problem up into answering two questions.

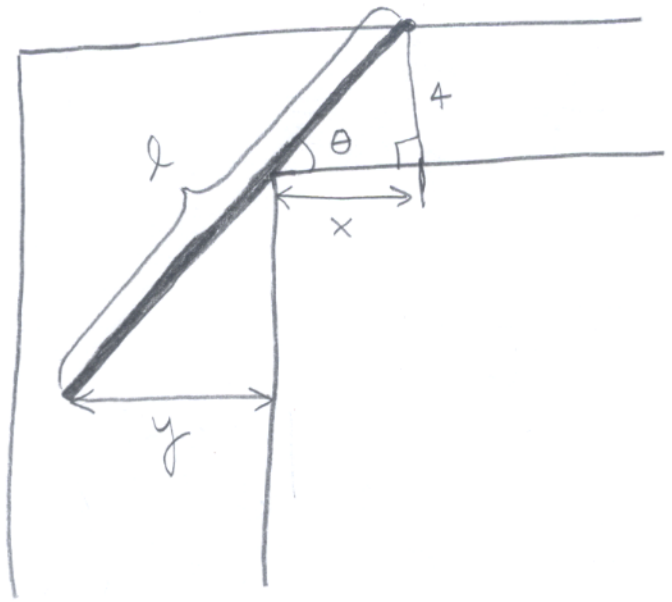
1) Fixing a length l for the pole, what is the furthest such a pole can "jutt out" from the right most wall of the main corridor (see diagram above). The amount of "jutting out" is measured by the length y in the diagram below.

2) Given the maximum distance y_{max} a pole of length l can "jutt out", we solve for l in the equation

$$y_{max} = 5$$

y_{max} is a function of l.

We have the digram



Let the variables θ, x, y be labelled as in the digram.

There are two ways to express y:

- y as a function of x and l (in class)
- y as a function of θ and l.

I will show both ways (I had intended for this to be a bonus on last quarter's final exam, so I solved it using the first method.)

- y as a function of x and l.

Using the pythagorean theorem to the upper triangle, in the diagram we can find the sides

Using the fact that the upper and lower triangles are similar, we have

$$\frac{x}{y} = \frac{\sqrt{16+x^2}}{l - \sqrt{16+x^2}}$$

so $$y = \frac{l - \sqrt{16+x^2}}{\sqrt{16+x^2}} \cdot x$$

- <u>y as a function of θ and l</u>

This may actually be simpler. The triangle in the top of the diagram is

We have

$$\frac{4}{\text{hypotenuse}} = \sin\theta$$

so hypotenuse $= 4\csc\theta$.

The triangle in the bottom is

Same angle because triangles are similar.

Thus, $$\frac{y}{l - 4\csc\theta} = \cos\theta$$

$$y = (l - 4\csc\theta)\cos\theta$$
$$= l\cos\theta - 4\cot\theta.$$

Much Simpler!

- In the first case (y as a function of x and l), we need to maximize y for x in $[0, l]$.
- In the second case, we maximize y when θ is in $[0, \frac{\pi}{2}]$.

<u>1st solution</u> Maximize $f(x) = y = \dfrac{(l - \sqrt{16+x^2})x}{\sqrt{16+x^2}}$ on $[0, l]$

$$= \frac{lx}{\sqrt{16+x^2}} - x$$

$$f'(x) = \frac{l}{\sqrt{16+x^2}} - \frac{lx^2}{(16+x^2)^{3/2}} - 1$$

$$= \frac{16l + lx^2}{(16+x^2)^{3/2}} - \frac{lx^2}{(16+x^2)^{3/2}} - 1$$

$$= \frac{16l}{(16+x^2)^{3/2}} - 1$$

Endpoints : $x = 0, l$

Singular : None

Stationary : Solve $f'(x) = 0$

$$\frac{16l}{(16+x^2)^{3/2}} - 1 = 0$$

$$16l = (16+x^2)^{3/2}$$

$$(16l)^{2/3} = 16 + x^2$$

$$x^2 = (16l)^{2/3} - 16$$

$$x = \sqrt{(16l)^{2/3} - 16}$$

Plugging this into the equation for y gives

$$y_{max} = \frac{\left(l - \sqrt{16 + [(16l)^{2/3} - 16]}\right)}{\sqrt{16 + [(16l)^{2/3} - 16]}} \left[(16l)^{2/3} - 16\right]^{1/2}$$

$$= \frac{l - (16l)^{1/3}}{(16l)^{1/3}} \left[(16l)^{2/3} - 16\right]^{1/2}$$

Set this equal to 5 and solve for l. It's best to use calculator. Then, we get. $l = 12.7017$

Note: After some algebraic manipulations, the equation

$$\frac{l - (16l)^{1/3}}{(16l)^{1/3}} \left((16l)^{2/3} - 16\right)^{1/2} = 5$$

can be written $(4l)^{2/3} - 4 = 10^{2/3}$
which can be solved explicity for

$$l = \frac{1}{4}\left(4 + 10^{2/3}\right)^{3/2}$$

2nd Solution Maximize $f(\theta) = y = l\cos\theta - 4\cot\theta$
on $[0, \frac{\pi}{2}]$.
$f'(\theta) = -l\sin\theta + 4\csc^2\theta$

End points: $\theta = 0, 2\pi$
Singular points: $\theta = 0$.
Stationary points: Solve $-l\sin\theta + 4\csc^2\theta = 0$

$$-l\sin\theta + \frac{4}{\sin^2\theta} = 0$$

$$-l\sin^3\theta + 4 = 0$$

$$\sin\theta = \sqrt[3]{\frac{4}{l}}$$

We find $\cos\theta = \sqrt{1-\sin^2\theta}$ by Pythagorean,

$$= \sqrt{1-(4/\ell)^{2/3}}$$

and

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-(4/\ell)^{2/3}}}{(4/\ell)^{1/3}}$$

Evaluate: At $\theta = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = 0$ and

$\lim_{\theta\to 0^+} f(\frac{\pi}{2}) = -\infty$ because $\lim_{\theta\to 0^+} \cot\theta = \infty$.

At the stationay value (there is only one),

$$f(\text{stationary value}) = \ell\cos\theta_s - 4\cot\theta_s$$

$$= \ell\sqrt{1-\left(\tfrac{4}{\ell}\right)^{2/3}} - 4\,\frac{\sqrt{1-\left(\frac{4}{\ell}\right)^{2/3}}}{\left(\frac{4}{\ell}\right)^{1/3}}$$

Thus, $\ell\sqrt{1-\left(\tfrac{4}{\ell}\right)^{2/3}} - 4\,\frac{\sqrt{1-\left(\frac{4}{\ell}\right)^{2/3}}}{\left(\frac{4}{\ell}\right)^{1/3}}$ is the maximum value, so we set this $=5$ and solve for ℓ:

$$= \ell\sqrt{1-\left(\tfrac{4}{\ell}\right)^{2/3}} - 4\,\frac{\sqrt{1-\left(\frac{4}{\ell}\right)^{2/3}}}{\left(\frac{4}{\ell}\right)^{1/3}} = 5$$

$$\ell - 4\left(\frac{\ell}{4}\right)^{1/3} = \frac{5}{\sqrt{1-\left(\frac{4}{\ell}\right)^{2/3}}}$$

Solve using calculator: $\ell = 12.701$ ft