Midterm 2 Review

13 November 2018

1. Differentiate the following functions at the given points using the limit definition of derivative (difference quotient).

   (a) $f(x) = x^3 - 4x$ at $x = 1$

   $$f'(1) = \lim_{h\to0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0} \frac{[(1+h)^3 - 4(1+h)] - [-3]}{h}$$
   $$= \lim_{h\to0} \frac{[1^3 + 3h + 3h^2 + h^3 - 4 - 4h] + 3}{h}$$
   $$= \lim_{h\to0} \frac{h^3 + 3h^2 - h}{h}$$
   $$= \lim_{h\to0} (h^2 + 3h - 1)$$
   $$= \boxed{-1}$$

   (b) $f(x) = \dfrac{1}{x+4}$ at $x = 4$

   $$f'(4) = \lim_{h\to0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to0} \frac{\frac{1}{(4+h)+4} - \frac{1}{8}}{h}$$
   $$= \lim_{h\to0} \frac{\frac{1}{h+8} - \frac{1}{8}}{h}$$
   $$= \lim_{h\to0} \frac{\left[\frac{8-(h+8)}{8(h+8)}\right]}{h}$$
   $$= \lim_{h\to0} \frac{-h}{8h(h+8)}$$
   $$= \lim_{h\to0} \frac{-1}{8(h+8)}$$
   $$= \boxed{-\frac{1}{64}}$$

   (c) $f(x) = \sqrt{x+4}$ at $x = 3$

   $$f'(3) = \lim_{h\to0} \frac{\sqrt{(3+h)+4} - \sqrt{7}}{h}$$
   $$= \lim_{h\to0} \frac{\sqrt{h+7} - \sqrt{7}}{h} \cdot \frac{\sqrt{h+7} + \sqrt{7}}{\sqrt{h+7} + \sqrt{7}}$$
   $$= \lim_{h\to0} \frac{(h+7) - (7)}{h(\sqrt{h+7} + \sqrt{7})}$$
   $$= \lim_{h\to0} \frac{1}{\sqrt{h+7} + \sqrt{7}}$$
   $$= \boxed{\frac{1}{2\sqrt{7}}}$$

   (d) $\dfrac{x-1}{x+2}$ at $x = 0$

   $$f'(0) = \lim_{h\to0} \frac{\frac{h-1}{h+2} - \left(-\frac{1}{2}\right)}{h}$$
   $$= \lim_{h\to0} \frac{\left(\frac{2(h-1) + (h+2)}{2(h+2)}\right)}{h}$$
   $$= \lim_{h\to0} \frac{3h}{2h(h+2)}$$
   $$= \lim_{h\to0} \frac{3}{2(h+2)}$$
   $$= \boxed{\frac{3}{4}}$$

2. Differentiate the following functions.

(a) $f(x) = \dfrac{x^5}{7} - \dfrac{x^2}{3} - \dfrac{1}{x^2}$

$$\boxed{f'(x) = \frac{5}{7}x^4 - \frac{2}{3}x + \frac{2}{x^3}}$$

(b) $f(x) = \dfrac{x^2 - 2x - 3}{x^3 + 2x - 1}$

$$f'(x) = \frac{(2x-2)(x^3+2x-1) - (x^2-2x-3)(3x^2+2)}{(x^3+2x-1)^2}$$

$$= \frac{2x^4 + 4x^2 - 2x - 2x^3 - 4x + 2 - 3x^4 - 2x^2 + 6x^2 + 4x + 9x^2 + 6}{(x^3+2x-1)^2}$$

$$= \boxed{\frac{-x^4 - 2x^3 + 17x^2 - 2x + 8}{(x^3+2x-1)^2}}$$

(c) $f(x) = (x^4 - 2x^3 + 5x^2 - 2x + 1)(x^5 - x^4 + 4x^3 - 3x^2 + 1)$

$$\boxed{f'(x) = (4x^3 - 6x^2 + 10x - 2)(x^5 - x^4 + 4x^3 - 3x^2 + 1) + (x^4 - 2x^3 + 5x^2 - 2x + 1)(5x^4 - 4x^3 + 12x^2 - 6x)}$$

(d) $f(x) = \dfrac{1 + \cos^2(x)}{1 + \sin^2(x)}$

$$f'(x) = \frac{(-2\cos(x)\sin(x))(1+\sin^2(x)) - (1+\cos^2(x))(2\sin(x)\cos(x))}{(1+\sin^2(x))^2}$$

$$= \boxed{\frac{-4\cos(x)\sin(x) - 2\cos(x)\sin^3(x) - 2\sin(x)\cos^3(x)}{(1+\sin^2(x))^2}}$$

(e) $f(x) = \cot(x)(1 - \sec(x))$

$$f'(x) = -\csc^2(x)(1 - \sec(x)) + \cot(x)(-\sec(x)\tan(x))$$

$$= \boxed{-\csc^2(x) + \sec(x)\csc^2(x) - \sec(x)}$$

$\cot(x)\tan(x) = 1.$

(f) $f(x) = \csc(x) + \tan(x)\sin(x)$

$$f'(x) = -\csc(x)\cot(x) + [\sec^2(x)\sin(x) + \tan(x)\cos(x)]$$

$$= \boxed{-\csc(x)\cot(x) + \sec^2(x)\sin(x) + \sin(x)}$$

(g) $f(x) = (x^3 - 1)^{17}$

$$f'(x) = 17(x^3-1)^{16} \cdot (3x^2)$$

$$= \boxed{51x^2(x^3-1)^{16}}$$

(h) $f(x) = \big(\cos(x) - \sin(x)\big)^{12}$

$$f'(x) = 12(\cos(x) - \sin(x))^{11} \cdot (-\sin(x) - \cos(x))$$

$$= \boxed{-12(\sin(x) + \cos(x))(\cos(x) - \sin(x))^{11}}$$

(i) $f(x) = \cos^7(1 - x^2)$

$$f'(x) = (-2x) \cdot (-\sin(1-x^2)) \cdot (7\cos^6(1-x^2))$$

$$= \boxed{14x\sin(1-x^2)\cos^6(1-x^2)}$$

3. Find the equation of the tangent line to the graph of the following function at the given point.

(a) $f(x) = x^2 - 3x + 4$ at $(0, 4)$

Point - Slope Form : $y - y_0 = m(x - x_0)$

$(x_0, y_0) = (0, 4)$

$m = f'(0) = 2(0) - 3 = -3$

$y - 4 = -3(x - 0)$

$\boxed{y = -3x + 4}$

(b) $f(x) = \cos(x) + 2x$ at $\left(\frac{\pi}{4}, \frac{\sqrt{2} + \pi}{2}\right)$

$y - y_0 = m(x - x_0)$ $(x_0, y_0) = \left(\frac{\pi}{4}, \frac{\sqrt{2}+\pi}{2}\right)$

$m = f'\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) + 2 = 2 - \frac{\sqrt{2}}{2}$

$y - \frac{\sqrt{2}+\pi}{2} = \left(2 - \frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right)$

$y = \left(2 - \frac{\sqrt{2}}{2}\right)x - \frac{\pi}{4}\left(2 - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\pi}{2}$

$y = \left(2 - \frac{\sqrt{2}}{2}\right)x - \frac{\pi}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{4} + \frac{\sqrt{2}}{2} + \frac{\pi}{2}$

$\boxed{y = \left(2 - \frac{\sqrt{2}}{2}\right)x + \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right)}$

(c) $\dfrac{1}{x - 2}$ at $(3, 1)$

$y - y_0 = m(x - x_0)$

$x_0 = 3$

$y_0 = 1$

$m = f'(3) = -\frac{1}{(3-2)^2}$

$= -1$

$y - 1 = -(x - 3)$

$\boxed{y = -x + 4}$

4. Calculate

(a) $\dfrac{d^2}{dx^2}\Big[(2x+3)\cos(x)\Big]$

$\frac{d}{dx}\left[(2x+3)\cos(x)\right] = 2\cos(x) - (2x+3)\sin(x)$

$\boxed{\frac{d^2}{dx^2}\left[(2x+3)\cos(x)\right] = -2\sin(x) - 2\sin(x) - (2x+3)\cos(x)}$

(b) $\dfrac{d^3}{dx^3}\Big[x^3 - 2x^2 + x + 5\Big]$

$\frac{d}{dx}\left[x^3-2x^2+x+5\right] = 3x^2 - 4x + 1$

$\frac{d^2}{dx^2}\left[x^3-2x^2+x+5\right] = 6x - 4$

$\boxed{\frac{d^3}{dx^3}\left[x^3-2x^2+x+5\right] = 6}$

(c) $\dfrac{d^2}{dx^2}\Big[\csc(x)\Big]$

$\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$

$\frac{d^2}{dx^2}\left[\csc(x)\right] = -\left[-\csc(x)\cot^2(x) - \csc^3(x)\right]$

$= \boxed{\csc(x)\left[\cot^2(x) + \csc^2(x)\right]}$

(d) $\dfrac{d^4}{dx^4}\Big[\cos(x)\Big]$

$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$

$\frac{d^2}{dx^2}\left[\cos(x)\right] = -\cos(x)$

$\frac{d^3}{dx^3}\left[\cos(x)\right] = \sin(x)$

$\frac{d^4}{dx^4}\left[\cos(x)\right] = \boxed{\cos(x).}$

5. A bucket which is in the shape of an inverted cone. The height of the bucket is 10 m and the radius of its base is 5 m. The bucket is being filled with ~~paint~~ water at a rate of 0.25 m^3/s. Find the rate at which the height of the water is rising when there is 1 m^3 of water in the bucket.

Given a cone of height h with base radius r, its volume is given by $V = \frac{1}{3}\pi r^2 h$.

Hint: You will need to use similar triangles to find a relationship between the base radius and height of the water in the bucket ($r = h/2$).

We want $\frac{dh}{dt}$.

To find this, we can use $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$.

We know $\frac{dV}{dt} = 0.25\ m^3/s$.

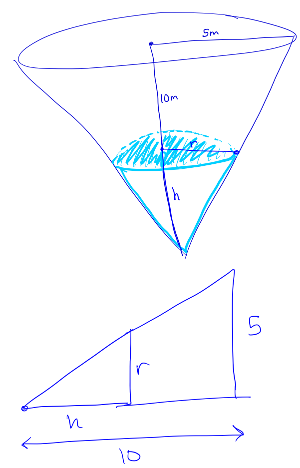
Therefore, we need $\frac{dV}{dh}$.

$$V = \frac{\pi}{3} r^2 h$$
$$= \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h \quad \left(\text{since } r = \frac{h}{2}\right)$$
$$= \frac{\pi}{12} h^3$$
$$\frac{dV}{dh} = \frac{\pi}{4} h^2$$

When $V = 1 m^3$, $\frac{\pi}{12} h^3 = 1 \Rightarrow h^3 = \frac{12}{\pi} \Rightarrow h = \left(\frac{12}{\pi}\right)^{1/3}$

Thus, $\frac{dV}{dh} = \frac{\pi}{4}\left(\frac{12}{\pi}\right)^{2/3}$ and

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$$
$$0.25 = \frac{\pi}{4}\left(\frac{12}{\pi}\right)^{2/3} \cdot \frac{dh}{dt}$$
$$\boxed{\frac{dh}{dt} = \frac{1}{\pi}\left(\frac{\pi}{12}\right)^{2/3} \ m/s}$$



$$\frac{r}{h} = \frac{5}{10}$$
$$r = \frac{h}{2}$$

6. A rubber band is stretched along the equator of a spherical balloon which is being filled at a rate of 3 cm^3/s. How fast is the rubber band stretching when the balloon contains 50 cm^3 of air?

The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and the circumference of a circle of radius r is $C = 2\pi r$.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$
$$= 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$
$$= \frac{3}{4\pi r^2}$$

When there is 50 cm^3 of air in the balloon, the radius is

$$50 = \frac{4}{3}\pi r^3$$
$$\frac{150}{4\pi} = r^3$$
$$r = \sqrt[3]{\frac{150}{4\pi}}$$

So $\frac{dr}{dt} = \frac{3}{4\pi} \cdot \left(\frac{150}{4\pi}\right)^{1/3}$
$$= \frac{3}{4\pi}\left(\frac{75}{2\pi}\right)^{1/3}$$

Thus,

$$C = 2\pi r$$
$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$
$$= 2\pi\left(\frac{3}{4\pi}\right)\left(\frac{75}{2\pi}\right)^{1/3}$$
$$\boxed{\frac{dC}{dt} = \frac{3}{2}\left(\frac{75}{2\pi}\right)^{1/3} \ cm/s}$$

7. Find the **equation of the tangent line** to the curve given by the following equation at the given point.

(a) $2x^3 + y^2 - 3xy = 0$ at $(1, 2)$

Implicit Differentiate

$$6x^2 + 2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$

$$(2y-3x)\frac{dy}{dx} + (6x^2-3y) = 0$$

$$\frac{dy}{dx} = \frac{3y-6x^2}{2y-3x}$$

Plug in $x=1$, $y=2$

$$\frac{dy}{dx} = \frac{3(2)-6(1)^2}{2(2)-3(1)} = \frac{6-6}{4-3} = 0$$

Point-Slope Form

$$y - y_0 = m(x-x_0)$$

$$y - 2 = 0\cdot(x-1)$$

$$\boxed{y = 2}$$

(b) $\cos(xy) = x^2 - 1$ at $\left(1, \dfrac{\pi}{2}\right)$.

Implicit Differentiate

$$\left(y + x\frac{dy}{dx}\right)\cdot(-\sin(xy)) = 2x$$

Solve for $\frac{dy}{dx}$

$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} = 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\sin(xy)}{-x\sin(xy)}}$$

Plug in $x=1$, $y=\frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{2(1) + \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)}{-(1)\sin\left(\frac{\pi}{2}\right)} = \frac{2+\frac{\pi}{2}}{-1} = -2-\frac{\pi}{2}$$

Point-Slope Form

$$y - y_0 = m(x-x_0)$$

$$y - \frac{\pi}{2} = -\left(2+\frac{\pi}{2}\right)(x-1)$$

$$y = -\left(2+\frac{\pi}{2}\right)x + \left(2+\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$\boxed{y = -\left(2+\frac{\pi}{2}\right)x + (2+\pi)}$$