## § 7.2 Natural Logarithm

Properties

1) $\ln(1) = 0$
2) $\ln(ab) = \ln(a) + \ln(b)$
3) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
4) $\ln\left(a^{p/q}\right) = \frac{p}{q}\ln(a)$

The range of $\ln(x)$ is $(-\infty,\infty)$. Therefore, by IVT, there is some $e$ such that $\ln(e) = 1$.

The value of $e \approx 2.718281828459$

Example Let $f(x) = \ln|x|$ for $x \neq 0$. Then
$$f'(x) = \frac{1}{x} \text{ for } x \neq 0.$$

Rewritten in terms of integration $\int \frac{1}{x}\,dx = \ln|x| + C$.

## Integration of Trig Functions

Theorem
1) $\int \tan x\,dx = -\ln|\cos x| + C$
2) $\int \cot x\,dx = \ln|\sin x| + C$
3) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
4) $\int \csc x\,dx = \ln|\csc x + \cot x| + C$

Proof 1) Let $f(x) = -\ln|\cos x|$. Then
$$f'(x) = -\frac{1}{\cos x}\cdot(-\sin x) = \frac{\sin x}{\cos x} = \tan x.$$

Therefore, $\int \tan x\,dx = -\ln|\cos x| + C$.

2) $$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx \xrightarrow[du = \cos x\,dx]{u = \sin x} \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

3) $$\int \sec x\,dx = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$
$$\xrightarrow[du = (\sec x\tan x + \sec^2 x)\,dx]{u = \sec x + \tan x} \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

Example 1) $$\int \frac{\sin x}{2+\cos x}\,dx \xrightarrow[du = -\sin x\,dx]{u = 2+\cos x} \int \frac{-du}{u} = -\ln|u| + C = \boxed{-\ln|2+\cos x| + C}$$

2) $$\int \frac{1}{x\ln x}\,dx \xrightarrow[du = \frac{1}{x}dx]{u = \ln x} \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\ln(x)| + C}$$

3) $$\int (1+\sec x)^2\,dx = \int (1 + 2\sec x + \sec^2 x)\,dx = \boxed{x + 2\ln|\sec x + \tan x| + \tan x + C}$$

## Logarithmic Differentiation

Problem: Find $\frac{d}{dx}\left[g_1(x)\cdots g_n(x)\right]$.

Solution: Let $g(x) = g_1(x)\cdots g_n(x)$.
$$\ln g(x) = \ln g_1(x) + \cdots + \ln g_n(x).$$
$$\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}$$
$$\boxed{g'(x) = g(x)\left[\frac{g_1'(x)}{g_1(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}\right]}$$

Example 1) Let $g(x) = x(x-1)(x-2)(x-3)$.
Then
$$g'(x) = x(x-1)(x-2)(x-3)\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$

## § 7.4 Exponential

Definition. The inverse of $\ln(x)$ is denoted $\exp(x)$.

Recall that $e$ was a number such that $\ln(e) = 1$.

If $\frac{p}{q}$ is rational, then
$$\ln\left(e^{p/q}\right) = \frac{p}{q}\ln(e) = 1.$$

Therefore, $e^{p/q} = \exp\left(\frac{p}{q}\right)$ since they both solve the equation
$$\ln(x) = \frac{p}{q}.$$

If $x$ is not rational, we don't know what $e^x$ means, but we can define $e^x = \exp(x)$.

For this reason, $\exp(x) = e^x$ is called the exponential function.

Properties of $e^x$

1) $e^0 = 1$
2) The domain of $e^x$ is $(-\infty,\infty)$.
   The range of $e^x$ is $(0,\infty)$, so $e^x > 0$ for all $x$.
3) $e^{\ln x} = x$ for all $x > 0$.
4) $e^{a+b} = e^a \cdot e^b$ for all $a, b$
5) $\frac{d}{dx}[e^x] = e^x$
6) $\int e^x\,dx = e^x + C$

Proof 1) Since $\ln(1) = 0$, $e^0 = 1$.

2) The domain of $e^x$ is the range of $\ln(x)$ and the range of $e^x$ is the domain of $\ln(x)$.

3) This is a property of inverse functions.

4) Let $c$ and $d$ be positive real numbers.

Since $\ln(cd) = \ln(c) + \ln(d)$, we can take exponentials of both sides to get
$$e^{\ln(cd)} = e^{\ln c + \ln d}$$
$$cd = e^{\ln(c) + \ln(d)}$$
$$e^{\ln c}\cdot e^{\ln d} = e^{\ln(c)+\ln(d)}$$

Since $a = \ln c$ and $b = \ln d$ can take on any real value, we have
$$e^a \cdot e^b = e^{a+b} \quad \text{for all real } a, b.$$

5) Recall that if $f$ is one-to-one with inverse $f^{-1}$, then
$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

Using $f(x) = \ln x$ and $f^{-1}(x) = e^x$, we get
$$\frac{d}{dx}[e^x] = \frac{1}{1/e^x} = e^x.$$

6) Since $\frac{d}{dx}[e^x] = e^x$, the exponential is its own antiderivative, and we have
$$\int e^x\,dx = e^x + C.$$

Examples 1) $$\int \frac{e^x}{e^x+2}\,dx \xrightarrow[du = e^x dx]{u = e^x} \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|e^x+2| + C}$$

2) $$\int \frac{e^x}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^x + e^{-x}}\cdot\frac{e^x}{e^x}\,dx = \int \frac{e^{2x}}{e^{2x}+1}\,dx \xrightarrow[du = 2e^{2x}dx]{u = e^{2x}+1} \int \frac{\frac{1}{2}du}{u}$$
$$= \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|e^{2x}+1| + C}$$

# Exponential Growth and Decay

The exponential function is useful in describing the following scenario: Let $A(t)$ be some quantity depending on time $t$. Suppose that the rate of change $A'(t)$ is proportional to $A(t)$, i.e.

$$A'(t) = r\,A(t)$$

for some constant $r$.

Then $\boxed{A(t) = A(0)\cdot e^{rt}}$.

Examples 1) Let $P(t)$ = population at some time $t$. Then $P'(t) = k\,P(t)$ where $k$ is some constant.

2) (Radioactivity)

$$^{236}U \longrightarrow {}^{92}Kr + {}^{141}Br$$

Let $A(0)$ be starting amount of $^{236}U$.

Let $A(t)$ = amount after time $t$.

Fact The rate of fission is proportional to the amount of fissable material present.

Thus, there is a constant $k>0$ such that

$$A'(t) = -k\,A(t)$$

$$\Updownarrow$$

$$A(t) = A(0)\,e^{-kt}.$$

There is a time, denoted $\lambda$, such that half the material remains:

$$A(\lambda) = \tfrac{1}{2}A(0)$$

$$\Rightarrow e^{-k\lambda} = \tfrac{1}{2}$$

$$\boxed{\lambda = \frac{\ln 2}{k}}$$

This $\lambda$ is called the half life.

Theorem If $A(t)$ satisfies

$$A'(t) = kA(t)$$

then $A(t) = A(0)\,e^{kt}$.

Proof

$$A'(t) = k\,A(t)$$

$$\frac{A'(t)}{A(t)} = k$$

$$\int_0^t \frac{A'(z)}{A(z)}\,dz = \int_0^t k\,dz = kt$$

$$\ln|A(t)| = kt + C$$

$$A(t) = Ce^{kt}$$

Then $A(0) = Ce^{k(0)} = C$

so $A(t) = A(0)e^{kt}$.

Example You have 100 g of a radioactive substance. After 1 hr, there is 60 g of the substance left. How much is left after 1.5 hr?

$$A(t) = A(0)e^{-rt} \qquad A(0) = 100$$

$$A(1) = 100\,e^{-r} = 60$$

$$e^{-r} = \tfrac{3}{5}$$

$$-r = \ln\tfrac{3}{5}$$

$$r = \ln\tfrac{5}{3}.$$

Therefore, $A(t) = 100\left(\frac{5}{3}\right)^{-t}$

$$\boxed{A(1.5) = 100\,(5/3)^{-1.5}}$$

Example A population is 300 million. Ten years later, the population is 340 million. How long until it reaches 400 million?

$$P(t) = P(0)e^{kt} \qquad P(0) = 300$$

$$P(10) = 300\,e^{10k} = 340$$

$$e^{10k} = \tfrac{17}{15}$$

$$k = \tfrac{1}{10}\ln\left(\tfrac{17}{15}\right)$$

$$P(t) = 300\,e^{\frac{t}{10}\ln\left(\frac{17}{15}\right)} = 400$$

$$e^{t\ln(17/15)/10} = 4/3$$

$$\tfrac{1}{10}t\ln\left(\tfrac{17}{15}\right) = \ln\left(\tfrac{4}{3}\right)$$

$$\boxed{t = \frac{10\ln(4/3)}{\ln(17/15)}}$$