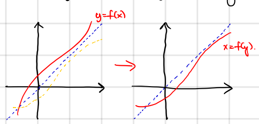
§6.1 Integration with respect to y.

Functions in y

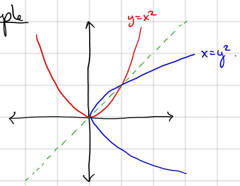
Sometimes it is convenient to express x as a function of y,

e.g. $x = f(y)$.

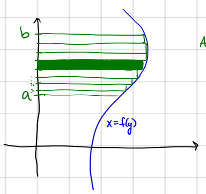
Interchanging x and y flips the graph along $x=y$.



Example



Integrating functions in y



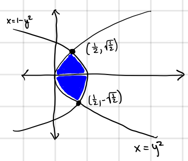
Area $\approx f(y_1)\Delta y_1 + f(y_2)\Delta y_2 + \dots + f(y_n)\Delta y_n$

where $\Delta y_i = y_i - y_{i-1}$.

As the partition $a=y_0<y_1<\dots<y_n=b$ gets finer and finer, this sum converges to

$$\int_a^b f(y)\,dy.$$

Example



$1-y^2 = y^2$

$1 = 2y^2$

$y = \pm\sqrt{\frac{1}{2}}$

As an integral in x, this area is given by

$$\int_0^{1/2} [\sqrt{x} - (-\sqrt{x})]\,dx + \int_{1/2}^{1} [\sqrt{1-x} - (-\sqrt{1-x})]\,dx$$

$$= 2\int_0^{1/2} \sqrt{x}\,dx + 2\int_{1/2}^{1} \sqrt{1-x}\,dx \qquad \begin{matrix} u = 1-x \\ du = -dx \end{matrix}$$

$$= 2\int_0^{1/2} \sqrt{x}\,dx + 2\int_{1/2}^{0} \sqrt{u}\,(-du)$$

$$= 4\int_0^{1/2} \sqrt{x}\,dx$$

$$= 4\left[\frac{2}{3}x^{3/2}\right]_0^{1/2}$$

$$= \frac{8}{3}\left(\frac{1}{2}\right)^{3/2}$$

$$= \frac{4}{3}\sqrt{\frac{1}{2}}$$

As an integral in y, this area is

$$\int_{-\sqrt{1/2}}^{\sqrt{1/2}} [(1-y^2) - y^2]\,dy = \int_{-\sqrt{1/2}}^{\sqrt{1/2}} [1-2y^2]\,dy$$

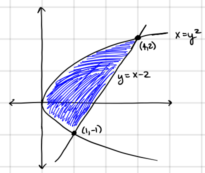
$$= \left(y - \frac{2}{3}y^3\right)\Big|_{-\sqrt{1/2}}^{\sqrt{1/2}}$$

$$= \left[\sqrt{\frac{1}{2}} - \frac{2}{3}\left(\frac{1}{2}\right)^{3/2}\right] - \left[-\sqrt{\frac{1}{2}} + \frac{2}{3}\left(\frac{1}{2}\right)^{3/2}\right]$$

$$= 2\sqrt{\frac{1}{2}} - \frac{2}{3}\sqrt{\frac{1}{2}}$$

$$= \frac{4}{3}\sqrt{\frac{1}{2}}$$

Example



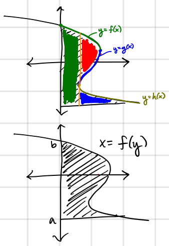
Calculating the area in 2 different ways, we see that

$$\int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 [\sqrt{x} - (x-2)]\,dx = \int_{-1}^{2} [(y+2) - y^2]\,dy$$

Calculating, we find they are both equal to $\frac{9}{2}$.

In the previous examples, much easier to express the area as an integral in y.

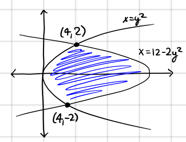
Example



As an integral in x, need 3 separate definite integrals to evaluate area.

As an integral in y, need only 1.

Example



$12-2y^2 = y^2$

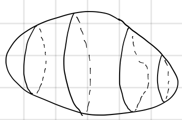
$12 = 3y^2$

$4 = y^2$

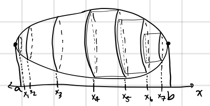
$y = \pm 2.$

§6.2 Volume by Cross-Sections

We have a solid S



To find the volume, we take slices



Each slice is approximated by a cylinder.

If $A(x)$ is the cross-sectional area of S, i.e. the area of the section obtained by intersecting S by a plane perpendicular to the x-axis through the point x, then the volume of the ith slice is approximately $A(x_i)\,\Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.

Thus, the volume of S is approximately

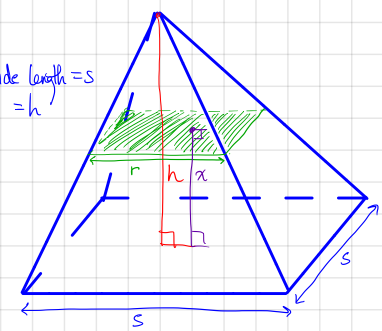
$$A(x_1)\Delta x_1 + A(x_2)\Delta x_2 + \dots + A(x_n)\Delta x_n.$$

Taking finer and finer partitions $\mathcal{P} = \{a = x_0 < x_1 < \dots < x_n = b\}$, we get that

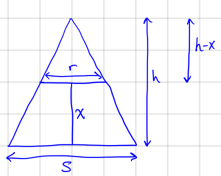
$$\boxed{\text{Volume}(S) = \int_a^b A(x)\,dx}$$

Example (Volume of Square Pyramid)

base side length $= s$
height $= h$



If we intersect the square pyramid by a plane parallel to the base, at a height x above the base, then the cross-section is a square of side length r and area $A(x)$.



By similar triangles,

$$\frac{s}{h} = \frac{r}{h-x}$$

$$r = \frac{s}{h}(h-x).$$

Thus, $A(x) = r^2 = \frac{s^2}{h^2}(h-x)^2$. Therefore,

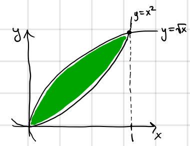
$$\text{Volume} = \int_0^h A(x)\,dx = \frac{s^2}{h^2}\int_0^h (h-x)^2\,dx$$

$$= \frac{s^2}{h^2}\left[-\frac{1}{3}(h-x)^3\right]_0^h$$

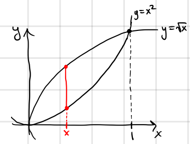
$$= \frac{s^2}{h^2}\left[\frac{1}{3}h^3\right]$$

$$\boxed{\text{Volume} = \frac{1}{3}s^2 h}$$

Example Let S be the solid with base the region



And whose cross-sections by planes perpendicular to x-axis are squares.



The side-length of each square cross-section is $\sqrt{x} - x^2$ so the cross-sectional area is

$$A(x) = (\sqrt{x} - x^2)^2$$

The volume of this solid is

$$V = \int_0^1 (\sqrt{x} - x^2)^2\,dx$$

$$= \int_0^1 (x - 2x^{5/2} + x^4)\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{4}{7}x^{7/2} + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{1}{2} - \frac{4}{7} + \frac{1}{5}$$

$$= \frac{35}{70} - \frac{40}{70} + \frac{14}{70}$$

$$= \boxed{\frac{9}{70}}$$

Example Same example as above but cross-sections are semi-circles with diameter lying on xy-plane.

The diameter of each cross-section is $\sqrt{x} - x^2$

The radius is $\frac{1}{2}[\sqrt{x} - x^2]$.

The area is $A(x) = \frac{\pi}{4}[\sqrt{x} - x^2]^2$.

The volume is $V = \int_0^1 A(x)\,dx$

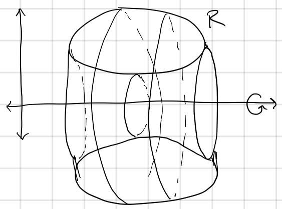
$$= \frac{\pi}{4}\int_0^1 [\sqrt{x} - x^2]^2\,dx$$

$$= \frac{\pi}{4} \cdot \frac{9}{70}$$

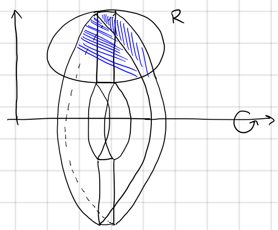
$$= \boxed{\frac{9\pi}{280}}$$

§6.2 Washer Method

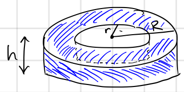
Take a region R and revolve it around the x-axis.



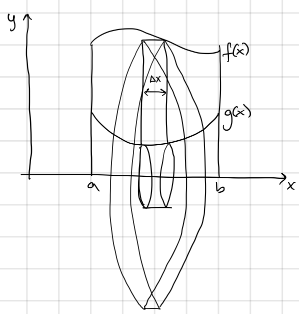
Vertical rectangles revolve into "washers".



Volume of Washer $= \pi(R^2-r^2)h$



Suppose

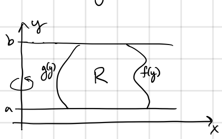


This washer has volume $\pi\left[f(x)^2-g(x)^2\right]\Delta x$.

Thus the volume of the solid S obtained by revolving R around x-axis is

$$\boxed{V=\int_a^b \pi\left[f(x)^2-g(x)^2\right]dx}$$

If you have a region R



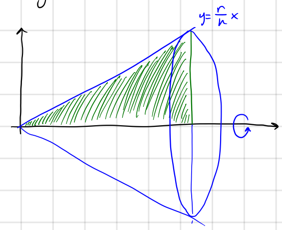
and the solid S is obtained by revolving around the y-axis, then the volume is

$$\boxed{V=\int_a^b \pi\left[f(y)^2-g(y)^2\right]dy}$$

Example (Volume of a Cone)

Let r = radius of base

h = height.



The cone S is obtained by revolving $y=\frac{r}{h}x$, $0\le x\le h$ about x-axis.

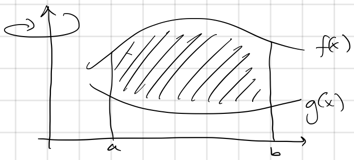
$$\text{Volume} = \pi\int_0^h\left(\frac{r}{h}x\right)^2dx = \pi\frac{r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{r^2\pi}{h^2}\cdot\frac{1}{3}x^3\Big|_0^h$$

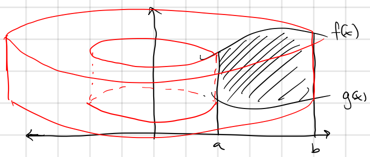
$$= \boxed{\frac{1}{3}\pi r^2 h.}$$

§6.3 Shell Method

Consider the following region R



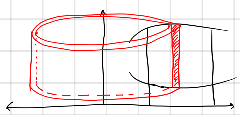
Revolve this region around y-axis



Obtain a solid S.

What is the volume of this solid?

The solid S can be approximated by shells



Let R = outer radius
r = inner radius
h = height

Then $V_{shell} = \pi R^2 h - \pi r^2 h$
$= \pi (R+r)\underbrace{(R-r)}_{\text{thickness of wall}} h$

If we choose a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, then the dimensions of the shells are

$R = x_{i+1}$
$r = x_i$
$h = f(x_i) - g(x_i)$

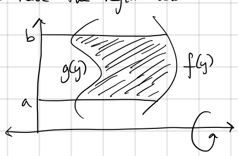
So the volume of the ith shell is approximately (assuming $\Delta x_i := x_{i+1} - x_i$ is small)

$V_i = 2\pi x_i \left[f(x_{i+1}) - g(x_i) \right] \Delta x_i$

Summing over these volumes, and taking the limit as the partitions tend to 0, gives

$$\boxed{\text{Volume}(S) = \int_a^b 2\pi x \left[f(x) - g(x) \right] dx}$$

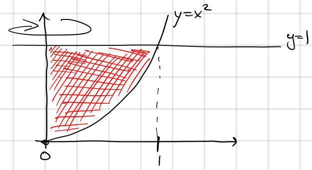
If we have the region below



The solid of revolution about the x-axis is

$$\boxed{V = \int_a^b 2\pi y \left[f(y) - g(y) \right] dy}$$

Example Find the volume of the solid obtained by revolving the region about the y-axis



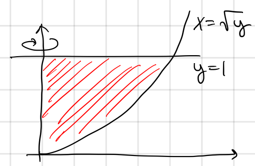
$V = 2\pi \int_0^1 x \left[1 - x^2 \right] dx$

$= 2\pi \int_0^1 \left[x - x^3 \right] dx$

$= 2\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^3 \right]_0^1$

$$\boxed{V = \frac{\pi}{2}}$$

Notice the we can also describe the region above as



Using the washer method, we find that

$\text{Volume} = \pi \int_0^1 \left[\sqrt{y} \right]^2 dy$

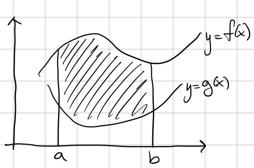
$= \pi \cdot \frac{1}{2} y^2 \Big|_0^1$

$$\boxed{V = \frac{\pi}{2}}$$

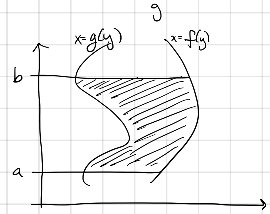
Just as above.

When do we know when to use washer method versus the shell method?

Define Type I region as one of the following form:



Define a Type II re ion as one of the following form:
g

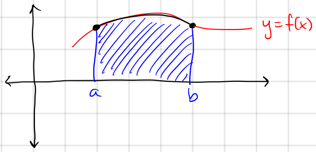


Which method should be used is detaild in the following chart.

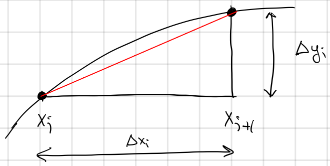
Type of Region / Axis of Rotation	Type I	Type II
x-axis	Washer	Shell
y-axis	Shell	Washer

Surface Area and Arc Length

Arclength Suppose I have a continuous function $f(x)$ on $[a,b]$.



What is the length of the arc of the graph of $f(x)$ from a to b?



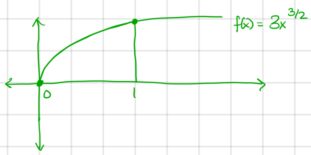
$$\text{Length} \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$$

Sum over i and take limit $\|P\| \to 0$, we get $\Delta x_i \to 0$ and so

$$\boxed{L = \int_a^b \sqrt{1+f'(x)^2}\, dx}$$

Example



What is the length of the blue arc?

$$f'(x) = \frac{9}{2} x^{1/2}$$

$$L = \int_0^1 \sqrt{1 + \frac{81}{4}x}\ dx$$

$$\xrightarrow[du = \frac{81}{4}dx]{u = 1+\frac{81}{4}x} \int_1^{\frac{85}{4}} \sqrt{u} \cdot \frac{4}{81}\, du$$

$$= \frac{4}{81}\left[\frac{2}{3}u^{3/2}\right]_1^{\frac{85}{4}}$$

$$= \frac{8}{243}\left[\left(\frac{85}{4}\right)^{3/2} - 1\right]$$

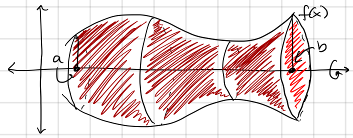
$$= \boxed{\frac{8}{243}\left[\frac{1}{8}(85)^{3/2} - 1\right]}$$

If $x = g(y)$ is rotated about y-axis, then arclength from $y=a$ to $y=b$ is

$$\boxed{L = \int_a^b \sqrt{1+[g'(y)]^2}\, dy}$$

Area of Surface of Revolution

Let $f(x)$ be continuous on $[a,b]$.



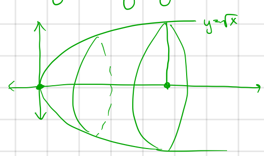
Rotate graph of $f(x)$ about x-axis.
Get a surface.

$$\boxed{\text{Surface Area} = A = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx}$$

If $x = g(y)$ is rotated about y-axis, then surface area from $y=a$ to $y=b$ is.

$$A = 2\pi \int_a^b g(y)\sqrt{1+[g'(y)]^2}\, dy$$

Example Find the surface area of the surface from $x=0$ to $x=4$, obtained by revolving $y=\sqrt{x}$ about x-axis.



$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$$A = 2\pi \int_0^4 \sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}}\ dx$$

$$= 2\pi \int_0^4 \sqrt{x + \frac{1}{4}}\ dx$$

$$\xrightarrow[du = dx]{u = x + \frac{1}{4}} 2\pi \int_{1/4}^{17/4} \sqrt{u}\, du$$

$$= 2\pi \cdot \frac{2}{3}u^{3/2}\Big|_{1/4}^{17/4}$$

$$= \boxed{\frac{4}{3}\pi\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]}$$