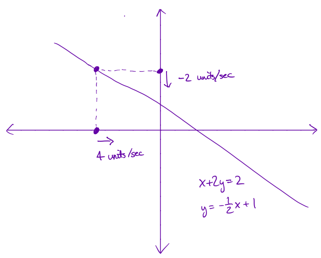
Handout

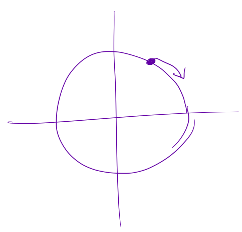
10 December 2018

1. A point moves along the line $x + 2y = 2$. Find the rate of change of the y-coordinate, given that the x-coordinate is increasing at a rate of 4 units per second.



$$x+2y=2$$
$$\frac{dx}{dt}+2\frac{dy}{dt}=0$$
$$4+2\frac{dy}{dt}=0$$
$$\boxed{\frac{dy}{dt}=-2}$$

2. A particle is moving in the circular orbit $x^2 + y^2 = 25$. As it passes through the point $(3, 4)$, its y-coordinate is decreasing at the rate of 2 units per second. At what rate is the x-coordinate changing?

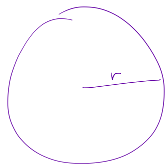


$$x^2+y^2=25$$
$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$
$$2(3)\frac{dx}{dt}+2(4)(-2)=0$$
$$6\frac{dx}{dt}-8=0 \implies \boxed{\frac{dx}{dt}=\frac{4}{3}\text{ units/sec}}$$

3. At a certain instant the side of an equilateral triangle is α centimeters long and increasing at the rate of k centimeters per minute. How fast is the area increasing?

4. A boat is held by a bow line that is wound about a bollard 6 feet higher than the bow of the boat. If the boat is drifting away at the rate of 8 feet per minute, how fast is the line unwinding when the bow is 30 feet from the bollard.

5. A spherical snowball is melting in such a manner that its radius is changing at a constant rate, decreasing from 16 cm to 10 cm in 30 minutes. How fast is the volume of the snowball changing when the radius is 12 cm?



$V = \frac{4\pi}{3} r^3$

$\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt}$

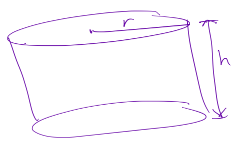
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{10-16}{30} = \frac{-6}{30}$ cm/s $= -\frac{1}{5}$ cm/s

$$\frac{dV}{dt} = 4\pi (12)^2 \left(-\frac{1}{5}\right)$$

$$\boxed{\frac{dV}{dt} = -\frac{576\pi}{5} \text{ cm}^3/\text{s}}$$

6. The diameter and height of a right circular cylinder are found at a certain instant to be 10 cm and 20 cm, respectively. If the diameter is increasing at the rate of 1 cm per second, what change in height will keep the volume constant?



diameter = 10 cm

$\Rightarrow$ radius = r = 5 cm

$V = \pi r^2 h$

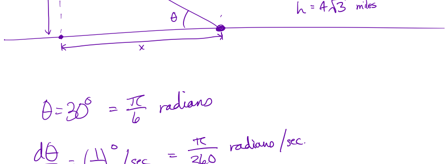
$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$

$0 = 2\pi(5)(20)(1) + \pi(5)^2 \frac{dh}{dt}$

$0 = 200\pi + 25\pi \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = -8 \text{ cm/s}}$$

7. An airplane is flying at constant speed and altitude on a line that will take it directly over a radar station on the ground. At the instant the plane is 12 miles from the station, it is noted that the plane's angle of elevation is 30° and is increasing at the rate of 0.5° per second. Give the speed of the plane in miles per hour.



$\tan\theta = \frac{h}{x}$

$h = x\tan\theta$ is constant

$= 12\tan(30°)$

$= 12 \cdot \frac{\sqrt{3}}{3}$

$h = 4\sqrt{3}$ miles

$\theta = 30° = \frac{\pi}{6}$ radians

$\frac{d\theta}{dt} = \left(\frac{1}{2}\right)^\circ/\text{sec} = \frac{\pi}{360}$ radians/sec.

$\tan\theta = \frac{4\sqrt{3}}{x}$

$\sec^2\theta \frac{d\theta}{dt} = -\frac{4\sqrt{3}}{x^2}\frac{dx}{dt}$

When $\theta = \frac{\pi}{6}$, $\frac{d\theta}{dt} = \frac{\pi}{360}$, and $x = 12$, we have

$\left(\frac{2}{\sqrt{3}}\right)^2 \left(\frac{\pi}{360}\right) = -\frac{4\sqrt{3}}{144}\frac{dx}{dt}$

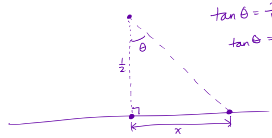
$\frac{4}{3} \cdot \frac{\pi}{360} \cdot \left(-\frac{144}{4\sqrt{3}}\right) = \frac{dx}{dt}$

$\frac{dx}{dt} = -\frac{144\pi}{1080\sqrt{3}} = -\frac{2\pi}{15\sqrt{3}}$

The airplane's speed is $\boxed{\frac{2\pi}{15\sqrt{3}} \text{ miles/sec}}$

8. Two cars, car A traveling east at 30 mph and car B traveling north at 22.5 mph, are heading toward an intersection I. At what rate is the angle IAB changing when cars A and B are 300 feet and 400 feet, respectively, from the intersection?

9. A revolving searchlight 1/2 mile from a straight shoreline makes 1 revolution per minute. How fast is the light moving along the shore as it passes over a shore point 1 mile from the shore point nearest to the searchlight.



$\tan\theta = \frac{x}{1/2} = 2x$

$\tan\theta = 2x$

$\frac{d\theta}{dt} = 2\pi$ radians/sec

$\tan\theta = 2x$

$\sec^2\theta \frac{d\theta}{dt} = 2\frac{dx}{dt}$

$\frac{1}{2}(1+\tan^2\theta)\frac{d\theta}{dt} = \frac{dx}{dt}$

When $x = 1$ mile,

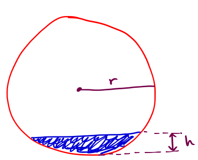
$\tan\theta = 2$ so

$\frac{dx}{dt} = \frac{1}{2}(1+2^2)(2\pi)$

$= \boxed{5\pi \text{ miles/minute}}$

10. A balloon containing 729π cm^3 of water is being blown up so that the radius of the balloon is changing at a rate of 2 cm/s. How fast is the height of the water level changing when the radius of the balloon is 12 cm? *Hint: the volume of a spherical segment is given by*

$$V = \pi r h^2 - \frac{1}{3}\pi h^3.$$



$r = 12$cm

$\frac{dr}{dt} = 2$cm/s

$V = \pi r h^2 - \frac{1}{3}\pi h^3$

$729\pi = \pi r h^2 - \frac{1}{3}\pi h^3$

$729 = rh^2 - \frac{1}{3}h^3$

When $r = 12$ cm,

$729 = 12h^2 - \frac{1}{3}h^3$

$\Rightarrow h = 9$ cm

Take $\frac{d}{dt}$:

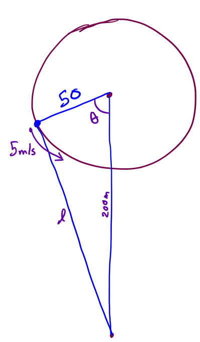
$0 = \frac{dr}{dt}h^2 + 2rh\frac{dh}{dt} - \frac{3}{2}h^2\frac{dh}{dt}$

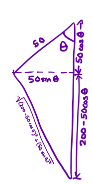
$0 = (2)(9)^2 + 2(12)(9)\frac{dh}{dt} - \frac{3}{2}(9)^2\frac{dh}{dt}$

$0 = 162 + (216 - \frac{243}{2})\frac{dh}{dt}$

$-162 = (\frac{189}{2})\frac{dh}{dt} \rightsquigarrow \frac{dh}{dt} = -\frac{2\cdot 162}{189} = \boxed{-\frac{12}{7} \text{ cm/s}}$

11. An athlete is running around a circular track of radius 50 meters at the rate of 5 m/s. A spectator is 200 m from the center of the track. How fast is the distance between the two changing when the runner is approaching the spectator and the distance between them is 200 m?





$\ell = \sqrt{(200 - 50\cos\theta)^2 + (50\sin\theta)^2}$

$\ell^2 = (200 - 50\cos\theta)^2 + (50\sin\theta)^2$

$2\ell\frac{d\ell}{dt} = 2(200 - 50\cos\theta)(\sin\theta)\frac{d\theta}{dt} + 2(50\sin\theta)\cos\theta\frac{d\theta}{dt}$

$2\ell\frac{d\ell}{dt} = [100(4 - \cos\theta)\sin\theta + 100\sin\theta\cos\theta]\frac{d\theta}{dt}$

$2\ell\frac{d\ell}{dt} = [400\sin\theta - 100\cos\theta\sin\theta + 100\sin\theta\cos\theta]\frac{d\theta}{dt}$

$2\ell\frac{d\ell}{dt} = 400\sin\theta\frac{d\theta}{dt}$

$\ell = 200$,

arc of circle $= s = r\theta$

$\frac{ds}{dt} = r\frac{d\theta}{dt}$ where r is constant $\Rightarrow \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{1}{50}(-5)$

$\frac{d\theta}{dt} = -\frac{1}{10}$

When $\ell = 200$m,

$40000 = (200 - 50\cos\theta)^2 + (50\sin\theta)^2$

$40000 = 40000 - 20000\cos\theta + \underbrace{2500\cos^2\theta + 2500\sin^2\theta}_{=2500}$

$0 = -20000\cos\theta + 2500$

$\cos\theta = \frac{2500}{20000} = \frac{25}{200} = \frac{1}{8} \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$

$\sin\theta = \sqrt{1 - \frac{1}{64}}$

$= \frac{\sqrt{63}}{8}$

$2\ell\frac{d\ell}{dt} = 400\sin\theta\frac{d\theta}{dt}$

$2(200)\frac{d\ell}{dt} = 400\frac{\sqrt{63}}{8}(-\frac{1}{10})$

$\frac{d\ell}{dt} = -\frac{\sqrt{63}}{80}$

$= -\frac{\sqrt{9\cdot 7}}{80}$

$\boxed{\frac{d\ell}{dt} = -\frac{3\sqrt{7}}{80}}$