Math 15200 Homework

Due Wednesday, January 16, 2019

1. Let $f(x) = \sin(x^2)$ and consider the partition
$$\mathcal{P} = \left\{0, \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \ldots, \frac{10}{11}, 1\right\}$$
of $[0, 1]$. Calculate the upper sum $U(f, \mathcal{P})$ and the lower sum $L(f, \mathcal{P})$.

2. Use Problem 1 to show that
$$\frac{1}{4} \leq \int_0^1 \sin(x^2)\, dx \leq \frac{4}{10}.$$

1. $f(x) = \sin(x^2)$

$\mathcal{P} = \{0, \frac{1}{11}, \frac{2}{11}, \ldots, \frac{10}{11}, 1\}.$

$$U(f,\mathcal{P}) = \sin\left(\left(\tfrac{1}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{2}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{3}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{4}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{5}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{6}{11}\right)^2\right)\cdot\tfrac{1}{11}$$
$$+ \sin\left(\left(\tfrac{7}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{8}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{9}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{10}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(1^2\right) + \tfrac{1}{11}$$
$$= \boxed{0.349}$$

$$L(f,\mathcal{P}) = \sin\left(\left(\tfrac{0}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{1}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{2}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{3}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{4}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{5}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{6}{11}\right)^2\right)\cdot\tfrac{1}{11}$$
$$+ \sin\left(\left(\tfrac{7}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{8}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{9}{11}\right)^2\right)\cdot\tfrac{1}{11} + \sin\left(\left(\tfrac{10}{11}\right)^2\right)\cdot\tfrac{1}{11}$$
$$= \boxed{0.273}$$

2. We know that for any function f continuous on $[a,b]$ that
$$L(f,\mathcal{P}) \leq \int_a^b f(x)\,dx \leq U(f,\mathcal{P})$$

Therefore,
$$\frac{1}{4} < 0.273 \leq \int_0^1 \sin(x^2)\,dx \leq 0.349 < \frac{4}{10}$$

So $\quad \frac{1}{4} < \int_0^1 \sin(x^2)\,dx < \frac{4}{10}.$

Homework 2

Due: 30 January 2019

1. Suppose h is a function satisfying
$$h''(x) = 0$$
and
$$h'(0) = h(0) = 1.$$
What is the function $h(x)$?
Hint: Recall that the fundamental theorem of calculus says that
$$\int_a^b f'(t)\,dt = f(b) - f(a)$$
for any differentiable function f.

If $h''(x)=0$ then by the FTC,

$$0 = \int_0^x h''(t)\,dt = h'(x) - h'(0) = h'(x) - 1$$

because $h''(t)=0$ for all t

by Fundamental Theorem of Calculus.

Therefore, $h'(x) = 1$ for all x and

$$x = \int_0^x dt = \int_0^x h'(t)\,dt = h(x) - h(0) = h(x) - 1$$

$h'(t)=1$ for all t

FTC.

Thus, $\boxed{h(x) = x+1.}$

2. If $f(x)$ is an even function, show that
$$F(x) = \int_0^x f(t)\,dt$$
is an odd function.

Want to show that $F(-x) = -F(x)$. That is,

$$\int_0^{-x} f(t)\,dt = -\int_0^x f(t)\,dt.$$

Start with $\int_0^{-x} f(t)dt$. Use u-substitution with $u=-x$. $du = -dx$.

Then. $$\int_0^{-x} f(t)\,dt = \int_0^x f(-u)\,(-du)$$
$$= -\int_0^x f(-u)\,du$$

$f(-u) = f(u)$

$$= -\int_0^x f(u)\,du$$

u is a dummy variable

$$= -\int_0^x f(t)\,dt.$$

Therefore,

$$\int_0^{-x} f(t)\,dt = -\int_0^x f(t)\,dt$$

and $F(-x) = -F(x)$

So $F(x)$ is odd.

Homework 3 and 4: Math 15200

1. (10 points) **Due February 11.** Define a function $A(x) = \displaystyle\int_1^x \frac{\ln(t)\,dt}{t^2+1}$. Prove that

$$A\left(\frac{1}{x}\right) = A(x).$$

2. (10 points) **Due February 15.** Find the volume of the solid obtained by revolving the region bounded by $y = 1 - x^2$ and $y = 2x$ about the x-axis.

   Hint: Note that this region does not lie completely above or below the x-axis, so there will be some overlap. Instead of directly using the washer method, try determining what the cross-sections look like.

HW3

1. Method 1 $\quad A(x) = \displaystyle\int_1^x \frac{\ln(t)\,dt}{t^2+1}$

Want to show that $A(\frac{1}{x}) = A(x)$, that is,

$$\int_1^{1/x} \frac{\ln(t)}{t^2+1}\,dt = \int_1^x \frac{\ln(t)}{t^2+1}\,dt.$$

Let's start with left side.

$$A(\tfrac{1}{x}) = \int_1^{1/x} \frac{\ln(t)}{t^2+1}\,dt \xrightarrow[du = -\frac{1}{t^2}dt]{u=\frac{1}{t}} \int_1^x \frac{\ln(\frac{1}{u})}{(\frac{1}{u})^2+1}\cdot\left(-\frac{1}{u^2}\,du\right)$$

$$= \int_1^x \frac{-\ln(u)}{(\frac{1}{u})^2+1}\left(-\frac{1}{u^2}\,du\right)$$

$$= \int_1^x \frac{\ln(u)}{u^2+1}\,du$$

$$= \int_1^x \frac{\ln(t)}{t^2+1}\,dt$$

$$= A(x).$$

Method 2 (due to EW).

We have $\dfrac{d}{dx}\left[A(\tfrac{1}{x})\right] = -\frac{1}{x^2}A'(\tfrac{1}{x}) \overset{FTC}{=} -\frac{1}{x^2}\left[\dfrac{\ln(\frac{1}{x})}{(\frac{1}{x})^2+1}\right] = \dfrac{\ln(x)}{x^2+1}$

Also, $\dfrac{d}{dx}\left[A(x)\right] \overset{FTC}{=} \dfrac{\ln(x)}{x^2+1}$.

FTC mean "Fundamental Theorem of Calculus"

Thus, $\dfrac{d}{dx}\left[A(\tfrac{1}{x}) - A(x)\right] = 0 \implies A(\tfrac{1}{x}) - A(x) = C = \text{constant}.$

To find this constant, we plug in $x=1$ to get $C = A(\frac{1}{1}) - A(1) = 0$

Thus, $A(\frac{1}{x}) - A(x) = 0$ and $A(\frac{1}{x}) = A(x)$.