504 HOMEWORK

1. Problem Set 1

1.1. **Problem 4.1.18.**

Proof. A function is *strictly increasing* if whenever $x < y$, we have $f(x) > f(y)$. Suppose $x < y$. By the Mean Value Theorem, $f(y) - f(x) = f'(\xi)(y - x)$ for some ξ in the interval (x, y). By hypothesis, $f'(\xi) < 0$ and $y - x > 0$ so that $f(y) - f(x) < 0$ and so f is increasing. □

1.2. **Problem 4.1.19.**

Proof. Let f and g be differentiable on (a, b) and suppose c is in this interval such that the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are zero. Also, suppose that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists.

The main feature of this proof is the Generalized Mean Value Theorem. Since f and g are differentiable on (a, b) and c is a point in this interval, f and g are both continuous at c, so the limits above can be replaced by the values of f and g at c: $f(c) = g(c) = 0$. So write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)}.$$

By the Generalized MVT, for each x, there is a point ξ (which depends on x) between c and x such that

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)}.$$

Taking the limit as $x \to c$ on the left hand side, since ξ is strictly between x and c, this is equivalent to taking $\xi \to c$ on the right hand side. Thus,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{\xi\to c} \frac{f'(\xi)}{g'(\xi)}$$

which is exactly what you wanted (after replacing ξ with the dummy variable x again). □

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One question that may be concerning is why we can replace $x \to c$ by $\xi \to c$ since ξ is actually a function of x. The only reason that we can do this is because ξ is a *continuous* function of x at the point c (this may be the only point where it is continuous) and also because ξ is never equal to c. So, just to write it out in detail, I should have said that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(\xi(x))}{g'(\xi(x))}$$

and then use continuity of ξ at c to see that taking the limit $x \to c$ is no different than taking the limit $\xi \to c$.

2. Problem Set 2

2.1. **Problem 4.1.24.**

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is C^1 on (a, b). Let $[c, d] \subset (a, b)$.

The two main theorems used here are the Mean Value Theorem and the fact that continuous functions on compact sets (closed and bounded intervals) are bounded.

For any $x, y \in [c, d]$, there is a ξ strictly between x and y such that $f(y)-f(x) = f'(\xi)(y - x)$. This gives us a bound, but only for the specific points x and y. But we notice that since f is C^1, f' is continuous on $[c, d]$, hence is bounded, say for some constant $M \geq 0$ we have $|f'(\xi)| \leq M$ for all ξ in $[c, d]$. In particular, for any x, y in $[c, d]$ and ξ chosen as before,

$$f(y) - f(x) = f'(\xi)(y - x) \leq |f'(\xi)| \cdot (y - x) \leq M(y - x).$$

□

2.2. **Problem 4.1.26.**

2.3. **Problem 4.1.28.**

3. Problem Set 3

3.1. **Problem 4.2.10.**

3.2. **Problem 4.3.8.**

Proof. Let

$$f(x, y) = \begin{cases} 0, & \text{if } (x, y) = (0, 0) \\ \frac{x^3}{x^2+y^2}, & \text{otherwise} \end{cases}.$$

To show that f is not differentiable at $(0, 0)$, we use the theorem that says that a function is totally differentiable if and only if it's partial derivatives exist and are continuous. By showing that either or both of the partial derivatives $D_1 f$ and

$D_2 f$ of f are not continuous at $(0,0)$, this will imply that f is not differentiable at $(0,0)$.

Thus, we must calculate both $D_1 f(0,0)$ and $D_1 f(x,y)$ for $(x,y) \neq (0,0)$ and similarly for $D_2 f$ (or not, we just need one of these to be discontinuous).

I'll calculate $D_1 f(0,0)$, then $D_2 f(0,0)$ is similar:

$$\begin{aligned} D_1 f(0,0) &= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t-0} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2+0} - 0}{t-0} \\ &= 1. \end{aligned}$$

You should get that $D_2 f(0,0) = 0$. If $(x,y) \neq 0$, then we can use standard rules for differentiating to calculate $D_1 f(x,y)$ and $D_2 f(x,y)$. It's found that

$$D_1 f(x,y) = \frac{3x^2}{x^2+y^2} - 2x\frac{x^3}{(x^2+y^2)^2}.$$

One thing that jumps out is that $D_1 f(0,y) = 0$, that is, along the y-axis, the $D_1 f$ is exactly 0. Thus, this function converges to 0 as $(x,y) \to 0$ a long this line. But we already showed that

$$D_1 f(0,0) = 1 \neq 0 = \lim_{y\to 0} D_1 f(0,y)$$

and so $D_1 f$ is not continuous at $(0,0)$. Thus, f is not differentiable at $(0,0)$. □

Part ii) is similar, so I'm not going to write it up. Just let me know if you have any questions about it.

3.3. **Problem 4.3.13.**

Proof. Let $f(x,y) = \sqrt{|x|+|y|}$.

Since we see absolute values in the definition of f, the first thing you should do is to split it up into parts which are much easier to deal with (for example, much like you do when you differentiate $|x|$). Notice is that when you restrict to any one of the four quadrants:

(1) $x > 0, y > 0$,
(2) $x > 0, y < 0$,
(3) $x < 0, y < 0$,
(4) $x < 0, y > 0$,

we get pretty simple forms for what f is. In each of these quadrants, f has the form

(1) $f(x,y) = \sqrt{x+y}$
(2) $f(x,y) = \sqrt{x-y}$
(3) $f(x,y) = \sqrt{-x-y}$

(4) $f(x,y) = \sqrt{-x+y}$

Each of these is easily seen to be differentiable on the domains that I specified. So the only points to check differentiability are those with $x = 0$ or $y = 0$. We do this by showing that either $D_1 f$ or $D_2 f$ are discontinuous at each point on the axis. To check this, we need to proceed by cases. I will do one of them here.

For instance, take a point $(x, 0)$ on the x-axis with $x > 0$. Then, we take the derivatives of f in quadrants (1) and (2). Because $(x, 0)$ lies on the x-axis, we will want to check $D_2 f$ since this is the derivative in the vertical direction which moves between the two quadrants and really is our only hope to be discontinuous at $(x, 0)$. We get (where x was fixed before), by the standard rules of differentiation,

$$D_2 f(x,y) = \begin{cases} \frac{1}{2\sqrt{x+y}}, & \text{if } y > 0 \\ \frac{-1}{2\sqrt{x-y}}, & \text{if } y < 0 \end{cases}.$$

Notice, that if we approach $(x, 0)$ from above (taking $y \to 0$ on the top case), we get $\frac{1}{2\sqrt{x}}$ and if we approach from below (taking $y \to 0$ on the bottom case) we get $\frac{-1}{2\sqrt{x}}$. These are two different values, so we get that $D_2 f$ is discontinuous at $(x, 0)$, hence f is not differentiable at $(x, 0)$. Since $x > 0$ was arbitrary, f is not differentiable on the positive x-axis.

The other three cases are

(1) $(x, 0)$ where $x < 0$
(2) $(0, y)$ where $y > 0$ and
(3) $(0, y)$ where $y < 0$.

To show that f is not differentiable at $(0, 0)$, you just use the partial derivatives $D_i f$ of f in any quadrant and note that $D_i f$ tends to ∞ as we approach 0. $\square$

4. Problem Set 3

4.1. **Problem 4.5.10.** What is meant by $D_h^r f(x)$? So, $f : \mathbb{R}^n \to \mathbb{R}$ is a function and h is some unit vector. When we take the ordinary directional derivative, we get another function $D_h f$, taking x to $D_h f(x)$, on $\mathbb{R}^n$, which just answers how f changes along the line through a point x in the direction h. Since this is a function, we can take another directional derivative in the direction h: $D_h(D_h f)$, which we just write as $D_h^2 f$. We can do this r times and we arrive at $D_h^r f$. Evaluating this function at x gives $D_h^r f(x)$. This is one way to think about these directional derivatives.

Another, more geometric way is a lot more helpful in solving this problem. First, we are given a fixed point x. Let h be a unit vector, some direction. We then get a line through x in the direction h, which is parameterized in the variable t by

$$L(t) = (h_1 t + x_1, h_2 t + x_2, \ldots, h_n t + x_n),$$

where h_i are the coordinates of h and x_i are the coordinates of x. So, as t goes from $-\infty$ to ∞, the function $L(t)$ traces out a line through x in the direction h.

How does this relate to the problem? We can restrict f to this line. Originally, f is a function on $\mathbb{R}^n$, but restricting f to this line, parameterized by the time variable t makes f a function of one variable. Formally, we look at

$$f(L(t)) = f(h_1 t + x_1, h_2 t + x_2, \ldots, h_n t + x_n).$$

Something to note is that the derivative of this function, evaluated at $t = 0$, $(f \circ L)'(0)$ is exactly the directional derivative $D_h f(x)$ of f. We can keep going with this and say that the rth derivative at $t = 0$, $(f \circ L)^{(r)}(0)$, is equal to the rth directional derivative $D_h^r f(x)$.

Take a look at what the textbook defines the Taylor series, in multiple variables, as and plug in $D_h^r f(x) = (f \circ L)^{(r)}(0)$ and notice that it just becomes the Taylor series of $f \circ L$. This will likely make the problem much easier than calculating every third partial derivative of $f(x, y, z)$ (there should be 10 distinct ones).