Midterm 1

Math 15100

Instructor: Harris

October 28, 2019

Name: ______________________________

Instructions:

1. Print your name clearly at the top of each page.
2. This midterm has 6 questions for a total of 100 points. The value of each part of each question is stated.
3. Please **NEATLY** show your work in the space provided. Partial credit will be awarded for partial or incomplete solutions.
4. **Show ALL of your work.** If you need more space, you can use the final page, which is blank, or request additional paper.

Good Luck!

Question	Points	Score
1	15	
2	15	
3	20	
4	20	
5	15	
6	15	
Total:	100	

1. (15 points) Consider the following inequality:

$$3 - 4\sin^2(x) \geq 0$$

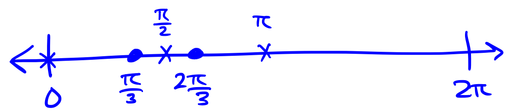
for $x \in [0, 2\pi]$. Find the set of solutions (12 points). Write your answer in interval notation (3 points).

$$f(x) = 3 - 4\sin^2(x) \geq 0$$

$$3 - 4\sin^2(x) = 0$$

$$\sin(x) = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$



Test points: 1) $[0, \frac{\pi}{3}]$ ✓

$$f(0) = 3 - 4(0) = 3 \geq 0$$

2) $[\frac{\pi}{3}, \frac{2\pi}{3}]$ ✗

$$f(\tfrac{\pi}{2}) = 3 - 4 = -1 < 0$$

3) $[\frac{2\pi}{3}, 2\pi]$ ✓

$$f(\pi) = 3 - 4\sin^2(\pi) = 3 \geq 0$$

$$\boxed{\text{Solutions: } [0, \tfrac{\pi}{3}] \cup [\tfrac{2\pi}{3}, 2\pi]}$$

2. (15 points) Use Mathematical Induction to prove that for every integer $n \geq 1$,

$$(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + (4 \cdot 5) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Base Case: If $n=1$, then left hand side is $1 \cdot 2 = 2$ and the right-hand side is $\frac{(1)(2)(3)}{3} = 2$.

Inductive Step: Suppose

$$(1\cdot 2) + (2 \cdot 3) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

Then

$$\begin{aligned}(1\cdot 2)+(2\cdot 3)+\cdots+k(k+1)+(k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} \\ &= (k+1)(k+2)\frac{k+3}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3}\end{aligned}$$

3. (20 points) Use the $\epsilon - \delta$ definition for limits to prove that

$$\lim_{x \to 1}(x^2 - x - 5) = -5.$$

Let $\varepsilon > 0$. Pick $\delta = \min\{1, \frac{\delta}{2}\}$.

If $|x-1| < \delta$, then

$$\begin{aligned}|(x^2 - x - 5) - (-5)| &= |x^2 - x| \\ &= |x| \cdot |x-1| \\ &< \delta \cdot |x| \\ &= \delta \cdot |(x-1)+1| \\ &\leq \delta \cdot (|x-1|+1) \\ &< \delta \cdot (\delta + 1) \\ &\leq 2\delta \\ &\leq \varepsilon\end{aligned}$$

4. Evaluate the following limits, if they exist. If they do not exist, explain why.

(a) (5 points) $\lim_{x\to 1}\left(x^3 - \pi x^2 + 1\right)$

$= 1^3 - \pi(1)^2 + 1$

$= 2 - \pi$

(b) (5 points) $\lim_{x\to 3}\frac{|x-3|}{x-3}$

DNE since

$\lim_{x\to 3^-}\frac{|x-3|}{x-3} = \lim_{x\to 3^-}\frac{-(x-3)}{x-3} = -1$

$\lim_{x\to 3^+}\frac{|x-3|}{x-3} = \lim_{x\to 3^+}\frac{x-3}{x-3} = 1$.

The LH and RH limits are not equal, so limit DNE.

(c) (5 points) $\lim_{x\to -5}\left(\frac{x^2+6x+5}{2x^2+8x-10}\right)$

$= \lim_{x\to -5}\frac{(x+5)(x+1)}{2(x+5)(x-1)}$

$= \lim_{x\to -5}\frac{x+1}{2(x-1)} = \frac{-5+1}{2(-5-1)} = \frac{-4}{2(-6)} = \boxed{\frac{1}{3}}$.

(d) (5 points) $\lim_{x\to -1}\left(\frac{x^2+x}{x^2+2x+1}\right) = \lim_{x\to -1}\frac{x(x+1)}{(x+1)^2}$

$= \lim_{x\to -1}\frac{x}{x+1}$

DNE since denominator is 0 and numerator is $\neq 0$.

Now, we can say: if $x < -1$, then $\frac{x}{x+1} > 0$
if $-1 < x < 0$, then $\frac{x}{x+1} < 0$

Thus, $\lim_{x\to -1^-}\frac{x}{x+1} = \infty$ and $\lim_{x\to -1^+}\frac{x}{x+1} = -\infty$. Therefore, the limit does not exist.

5. The following statements are false. Explain why or give counterexamples.

   (a) (5 points) If

$$\lim_{x\to 1} f(x) = 2 \quad \text{and} \quad \lim_{x\to 0} g(x) = 1,$$

   then

$$\lim_{x\to 0} f(g(x)) = 2.$$

False. Counterexample: $g(x) = 1$ for all x

$$f(x) = \begin{cases} 2, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

Then $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 0} g(x) = 1$, but $\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} f(1) = 0 \neq 2$.

   (b) (5 points) If $\lim_{x\to 0} f(x)$ does not exist and $\lim_{x\to 0} g(x)$ does not exist, then

$$\lim_{x\to 0} \big(f(x) + g(x)\big) \text{ does not exist.}$$

False. Counterexample: $f(x) = \frac{1}{x}$

$g(x) = 1 - \frac{1}{x}$.

   (c) (5 points) If f and g are not continuous at 0, then $f + g$ is not continuous at 0.

False. Counterexample: $f(x) = \begin{cases} 1, & x = 0 \\ 1/x, & x \neq 0 \end{cases}$

$$g(x) = \begin{cases} 0, & x = 0 \\ 1 - 1/x, & x \neq 0. \end{cases}$$

Then $(f+g)(x) = 1$ for all x, so $f+g$ is continuous at $x=0$ but $f(x)$ and $g(x)$ are not continuous at $x=0$.

6. (15 points) Find values for a and b such that the function given by

$$f(x) = \begin{cases} \dfrac{x^2 - x + a}{x - 3}, & x > 3 \\ 2x - b, & x \leq 3 \end{cases}$$

is continuous at $x = 3$. Explain your reasoning.

We have $f(3) = \lim_{x \to 3^-} f(x) = 6 - b$.

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{x^2 - x + a}{x - 3}$$

For this right-hand limit to exist, we must have $9 - 3 + a = 0 \Rightarrow 6 + a = 0$

$$\Rightarrow \boxed{a = -6}$$

Thus, $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \dfrac{x^2 - x - 6}{x - 3}$

$$= \lim_{x \to 3^+} \frac{(x-3)(x+2)}{x-3}$$

$$= 5$$

Thus, $6 - b = 5 \Rightarrow \boxed{b = 1}$

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