Encoding NP-hard problems

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The purpose of this note is to give several reductions from NP-hard graph problems to the validity problem of positive first-order logic FO^+. We will start by giving reductions for the following problems.

1. Hamiltonian circuit. The task is to determine whether a given graph contains a cycle that "visits" each vertex exactly ones.
2. Clique set. The task is to determine whether a given graph contains a set of k-vertices so that every two members of the set are adjacent.
3. Vertex cover. The task is to determine whether a given graph contains a set of k-vertices so that for every edge of the graph, at least one of its endpoints belongs to this set.

First we will introduce some notation. Let $G = (V, E)$ be the input graph, where $V = \{v_1, ..., v_n\}$. We will use $\Sigma(G)$ to denote the following set of atomic formulas.

$$\{c_i \neq c_j \mid 1 \leq i < j \leq n\} \cup \{c_i E c_j \mid (v_i, v_j) \in E\} \cup \{\neg c_i E c_j \mid (c_i, c_j) \notin E\}.$$

Notice that we are using E to denote both a binary relational symbol E as well as the edge set. Now we will associate to each of the above problems a sentence ϕ of FO^+ with the property that $\Sigma(G) \models \phi$ if and only if $\Sigma(G)$ is a positive instance of the problem.

1. In the case of Hamiltonian circuit the sentence is
$$\exists x_1 ... \exists x_n (\bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq j \leq n} c_i = x_j \wedge \bigwedge_{1 \leq i < n} x_i E x_{i+1} \wedge x_n E x_1)$$
2. In the case of clique set the sentence is
$$\exists x_1 ... \exists x_k \bigwedge_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq k} x_i E x_j$$
3. In the case of vertex cover the sentence is
$$\exists x_1 ... \exists x_k (\bigwedge_{1 \leq \ell \leq k} \bigvee_{1 \leq i \leq n} x_\ell = c_i \wedge \bigwedge_{c_i E c_j} \bigvee_{1 \leq \ell \leq k} (x_\ell = c_i \vee x_\ell = c_j))$$