Ordered fragments of first-order logic

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The satisfiability problem

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- An important example of an interesting decidable fragment of FO is the two-variable logic FO^2 (every sentence can contain at most two variables). This logic is decidable because it has the following *bounded model property*: if $\varphi \in \mathrm{FO}^2$ is satisfiable, then it has a model of size at most $2^{|\varphi|}$.

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- **Main idea:** Restrict the order in which variables can be quantified, the way variables can be permuted in atomic formulas and the manner in which boolean combinations of formulas can be formed.
- **This talk:** We will go through the syntax of the two most well-known ordered fragments (ordered logic, fluted logic) and their complexities. In addition, we will take a brief look at some recent results on the complexities of their variants (with respect to the satisfiability problem).

Syntax of ordered logic

Definition

Let $\overline{v}_\omega = (v_1, v_2, \dots)$ be an infinite sequence of variables and let τ be a vocabulary. For every $k \in \mathbb{N}$ we define sets $\mathrm{OL}^k[\tau]$ as follows.

1. Let $R \in \tau$ be an k-ary relational symbol and consider the prefix
$$(v_1, \dots, v_k)$$
of $\overline{v}_\omega$. Now $R(v_1, \dots, v_k) \in \mathrm{OL}^k[\tau]$.
2. If $\varphi, \psi \in \mathrm{OL}^k[\tau]$, then $\neg\varphi, (\varphi \wedge \psi) \in \mathrm{OL}^k[\tau]$.
3. If $\varphi \in \mathrm{OL}^{k+1}[\tau]$, then $\exists v_{k+1}\varphi \in \mathrm{OL}^k[\tau]$.

Finally we define $\mathrm{OL}[\tau] := \bigcup_k \mathrm{OL}^k[\tau]$.

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Finally we define $\mathrm{OL}[\tau] := \bigcup_k \mathrm{OL}^k[\tau]$.

Example

$\forall v_1(\neg P(v_1) \wedge \exists v_2 R(v_1, v_2))$ is a sentence of $\mathrm{OL}[\{P, R\}]$, while $\exists v_1 \exists v_2 R(v_2, v_1)$, $\exists v_2 \exists v_1 R(v_1, v_2)$ and $\exists v_1 \exists v_2 (P(v_2) \wedge R(v_1, v_2))$ are not.

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Theorem (Herzig, J.)

The satisfiability problem of OL *is* PSPACE*-complete.*

Syntax of fluted logic

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1. Let $R \in \tau$ be an n-ary relation symbol and consider the subsequence
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of $\overline{v}_\omega$. Now $R(v_{k-n+1}, \ldots, v_k) \in \mathrm{FL}^k[\tau]$.
2. For every $\varphi, \psi \in \mathrm{FL}^k[\tau]$, we have that $\neg\varphi, (\varphi \wedge \psi) \in \mathrm{FL}^k[\tau]$.
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Complexity of fluted logic

- FL also has a bounded model property: if $\varphi \in \mathrm{FL}$ has a model, then it has one of size at most
$$\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{|\varphi|^k\text{-times}},$$
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- **Main idea of proof**: For each $k \geq 2$ and $\varphi \in \mathrm{FL}^{(k+1)}$, there exists $\varphi' \in \mathrm{FL}^k$ so that φ has a model iff φ' has and if φ' has a model of size N, then φ has a model of size (roughly) 2^N.

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Theorem (Pratt-Hartmann, Swast, Tendera)

The satisfiability problem of FL *is* $\textsc{Tower}$*-complete.*

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- Similarly, FL consists of $\neg, \exists$ and the so-called suffix intersection $\dot{\cap}$ (allows one to compute intersections of relations that have different arities).
- This point of view suggests naturally several syntactical variants of, say, OL and FL (simply add or remove algebraic operators).

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- Adding (restricted) use of equality either to OL or FL does not affect complexity.
- Adding a swap operator (swap the last two elements in every tuple) increases complexity significantly: OL becomes $\mathrm{NE{\small XP}T{\small IME}}$-complete while FL becomes undecidable.
- Replacing $\exists$ with one-dimensional quantification (select at most the first element from every tuple) decreases complexity: OL becomes NP-complete while FL becomes $\mathrm{NE{\small XP}T{\small IME}}$-complete. FL remains $\mathrm{NE{\small XP}T{\small IME}}$-complete even in the presence of equality and swap.

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Thanks! :-)