

Ordered fragments of first-order logic

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The satisfiability problem

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- ▶ An important example of an interesting decidable fragment of FO is the two-variable logic FO^2 (every sentence can contain at most two variables). This logic is decidable because it has the following *bounded model property*: if $\varphi \in \text{FO}^2$ is satisfiable, then it has a model of size at most $2^{|\varphi|}$.

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- ▶ **Main idea:** Restrict the order in which variables can be quantified, the way variables can be permuted in atomic formulas and the manner in which boolean combinations of formulas can be formed.
- ▶ **This talk:** We will go through the syntax of the two most well-known ordered fragments (ordered logic, fluted logic) and their complexities. In addition, we will take a brief look at some recent results on the complexities of their variants (with respect to the satisfiability problem).

Definition

Let $\bar{v}_\omega = (v_1, v_2, \dots)$ be an infinite sequence of variables and let τ be a vocabulary. For every $k \in \mathbb{N}$ we define sets $OL^k[\tau]$ as follows.

1. Let $R \in \tau$ be an k -ary relational symbol and consider the prefix

$$(v_1, \dots, v_k)$$

of \bar{v}_ω . Now $R(v_1, \dots, v_k) \in OL^k[\tau]$.

2. If $\varphi, \psi \in OL^k[\tau]$, then $\neg\varphi, (\varphi \wedge \psi) \in OL^k[\tau]$.

3. If $\varphi \in OL^{k+1}[\tau]$, then $\exists v_{k+1}\varphi \in OL^k[\tau]$.

Finally we define $OL[\tau] := \bigcup_k OL^k[\tau]$.

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Example

$\forall v_1 (\neg P(v_1) \wedge \exists v_2 R(v_1, v_2))$ is a sentence of $OL[\{P, R\}]$, while $\exists v_1 \exists v_2 R(v_2, v_1)$, $\exists v_2 \exists v_1 R(v_1, v_2)$ and $\exists v_1 \exists v_2 (P(v_2) \wedge R(v_1, v_2))$ are not.

Complexity of ordered logic

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Theorem (Herzig, J.)

The satisfiability problem of OL is PSPACE-complete.

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Let $\bar{v}_\omega = (v_1, v_2, \dots)$ be an infinite sequence of variables and let τ be a vocabulary. For every $k \in \mathbb{N}$, we define sets $\text{FL}^k[\tau]$ as follows.

1. Let $R \in \tau$ be an n -ary relation symbol and consider the subsequence

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of \bar{v}_ω . Now $R(v_{k-n+1}, \dots, v_k) \in \text{FL}^k[\tau]$.

2. For every $\varphi, \psi \in \text{FL}^k[\tau]$, we have that $\neg\varphi, (\varphi \wedge \psi) \in \text{FL}^k[\tau]$.
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Note that $\text{OL} \subseteq \text{FL}$.

Example

$\exists v_1 \exists v_2 (P(v_2) \wedge R(v_1, v_2))$ is a sentence of $\text{FL}[\{P, R\}]$, while
 $\exists v_1 \exists v_2 \exists v_3 (R(v_1, v_2) \wedge R(v_2, v_3))$ is not.

Complexity of fluted logic

- FL also has a bounded model property: if $\varphi \in \text{FL}$ has a model, then it has one of size at most

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for some constant k .

- ▶ **Main idea of proof:** For each $k \geq 2$ and $\varphi \in \text{FL}^{(k+1)}$, there exists $\varphi' \in \text{FL}^k$ so that φ has a model iff φ' has and if φ' has a model of size N , then φ has a model of size (roughly) 2^N .

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Theorem (Pratt-Hartmann, Swast, Tendera)

The satisfiability problem of FL is TOWER-complete.

Algebraic view on ordered fragments

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- ▶ OL can be seen as consisting of three (relational) algebraic operators: complement \neg , intersection \cap and projection \exists .
- ▶ Similarly, FL consists of \neg, \exists and the so-called suffix intersection $\dot{\cap}$ (allows one to compute intersections of relations that have different arities).
- ▶ This point of view suggests naturally several syntactical variants of, say, OL and FL (simply add or remove algebraic operators).

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Brief summary of recent complexity results

- ▶ Adding (restricted) use of equality either to OL or FL does not affect complexity.
- ▶ Adding a swap operator (swap the last two elements in every tuple) increases complexity significantly: OL becomes NEXPTIME-complete while FL becomes undecidable.
- ▶ Replacing \exists with one-dimensional quantification (select at most the first element from every tuple) decreases complexity: OL becomes NP-complete while FL becomes NEXPTIME-complete. One-dimensional FL remains NEXPTIME-complete even in the presence of equality and swap.

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- ▶ How much can we extend the expressive power of the fluted logic while preserving its decidability?

Thanks! :-)