CSC 336 - Fall 2021 - Assignment 2

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1 Question 1.

1.1 (a)

Table 1: Results

n	max intensity	min intensity	sum intensities	condition number
4	36.607	0.893	134.821	13.964
8	36.603	0.005	136.593	30.000
16	36.603	0.000	136.603	62.000
32	36.603	0.000	136.603	126.000

Figure 1: matrix A, $n = 8$

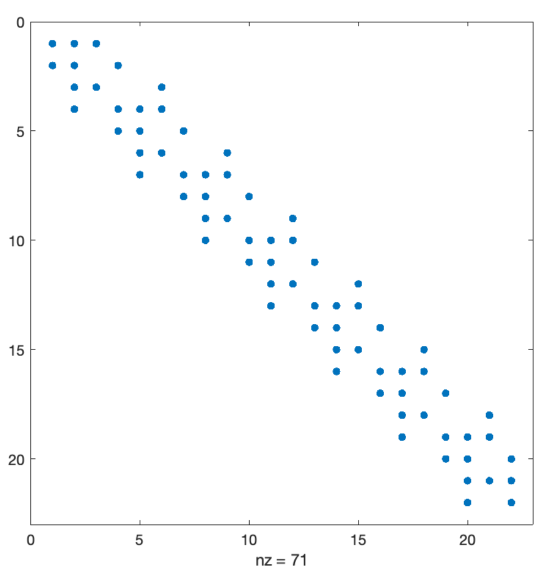


Figure 2: matrix L, $n = 8$

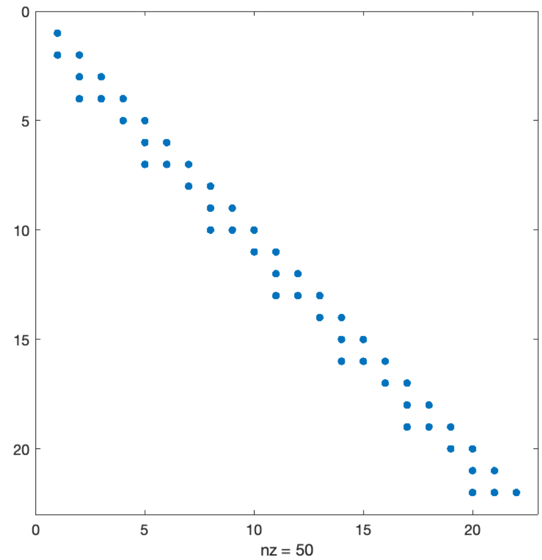


Figure 3: matrix U, $n = 8$

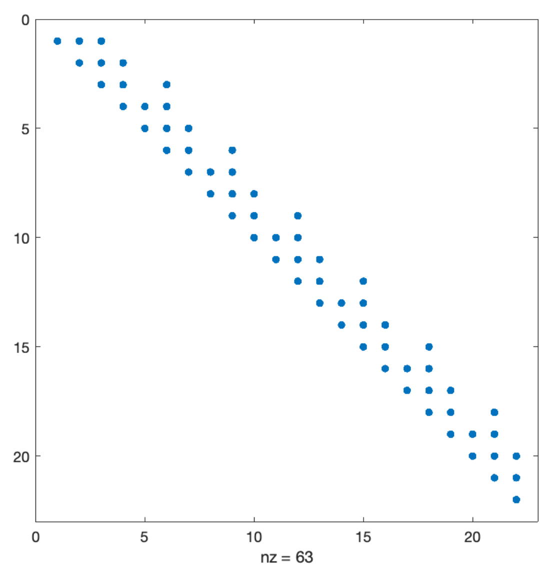
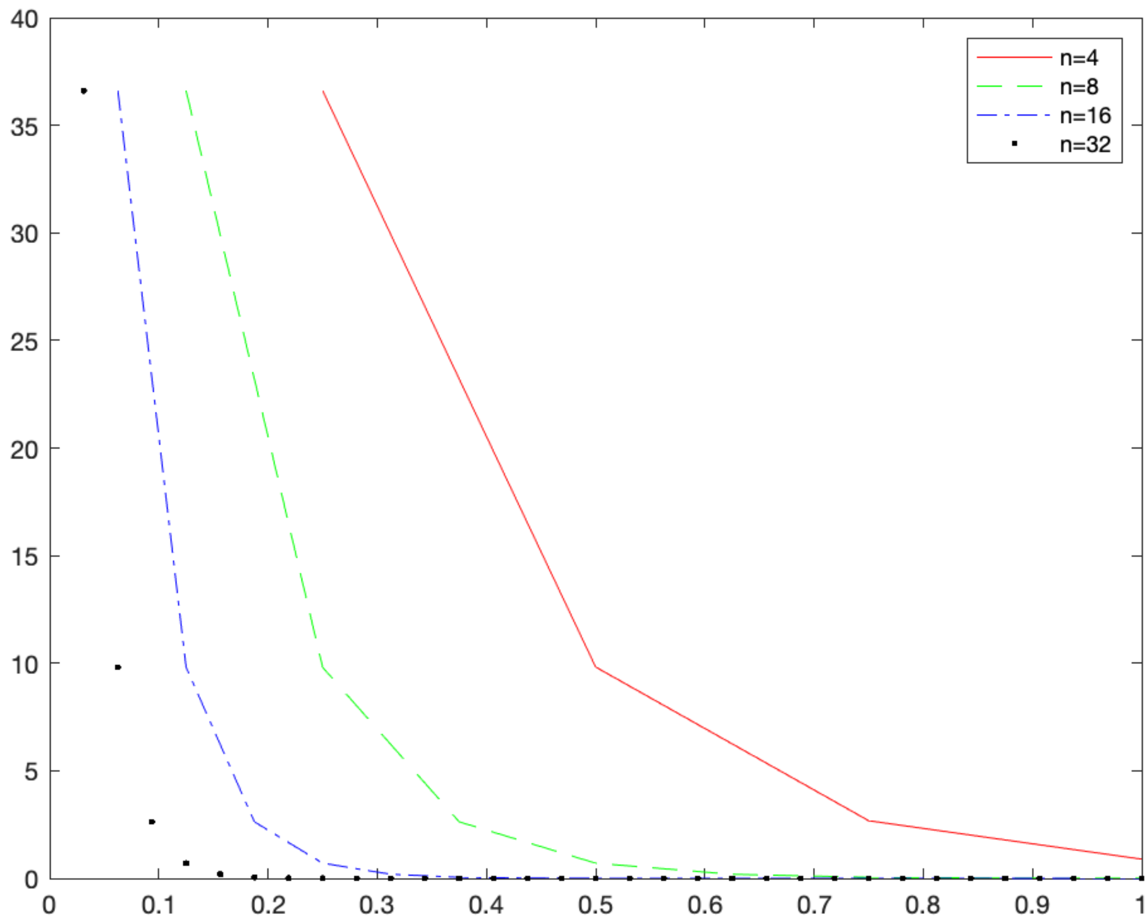


Figure 4: Top Line Intensities



MATLAB script:

```
index = 0;
ni = [4 8 16 32];
for n = ni
    N = 3*n-2;
    A = speye(N, N);
    % set loop equations
    A(1, 1:3) = [1 1 1];
    A(N, N-2:N) = [-1 0 2];
    for i = 4:3:(3*n-3)
        A(i, i-2:i+2) = [-1 0 1 1 1];
    end
    % set node equations
    for i = 1:3:(3*n-5)
```

```
        A(i+1, i:i+3) = [1 -1 0 -1];
        if i ~= 3*n-5
            A(i+2, i+1:i+5) = [1 -1 0 0 1];
        else
            A(N-1, N-2:N) = [1 -1 1];
        end
    end
    % generate RHS vector
    b = zeros(N, 1);
    b(1) = 100;
    % solve the system
    x = A\b;
    % store info for plotting
    index = index + 1;
    for i = 1:n
        t(i, index) = x(i * 3 - 2);
    end
    % output
    fprintf('n = %d\n', n);
    fprintf('Max Intensity: %.3f\n', max(x));
    fprintf('Min Intensity: %.3f\n', min(x));
    fprintf('Sum of Intensities: %.3f\n', sum(x));
    fprintf('Condition Number: %.3f\n', condest(A));
    if n == 8
        [L, U] = lu(A);
        figure(1);
        spy(A);
        figure(2);
        spy(L);
        figure(3);
        spy(U);
    end
end
figure(4);
plot([1:ni(1)]/ni(1), t(1:ni(1), 1), 'r-', ...
[1:ni(2)]/ni(2), t(1:ni(2), 2), 'g--', ...
[1:ni(3)]/ni(3), t(1:ni(3), 3), 'b-.', ...
[1:ni(4)]/ni(4), t(1:ni(4), 4), 'k.');
legend('n=4', 'n=8', 'n=16', 'n=32')
```

1.2 (b)

Lower bandwidth is 2. Upper bandwidth is 3. So the total bandwidth is 5, for any n.
Reason: For the lower bandwidth, consider $i = 4, 7, \cdots$, i.e. all the middle loops.
The coefficient of x_{i-2} in the corresponding equation is -1. In other words, $A_{i,i-2} = -1 \neq 0$.
Thus j has to be smaller than $i - 2$ at least to make $A_{i,j} = 0$.
For the upper bandwidth, consider equations when $i = 3, 6, \cdots$ (bottom nodes except the rightmost one).
Notice the coefficient of x_{i+3} is 1 in these equations. Hence $A_{i,i+3} = 1$ for $i = 3, 6, \cdots$

So j has to be greater than $i+3$ to make $A_{i,j}=0$.
Above equations give lower bandwidth is at least 2 and upper bandwidth is at least 3.
We can verify all other non-zeros entries are contained in the $(2,3)$-band easily.

The number of non-zero entries in A is $(10n-9)$
Reason: There are n loops in total.
The leftmost one involves 3 non-zero entries, and the rightmost one involves 2.
The remaining general loop equations contain 4 non-zero entries.
Hence the loops lead to $3+2+4(n-2)=4n-3$ non-zero entries.
Similarly, there are $2n-2$ nodes.
Every node equation has 3 non-zero elements. As a result, there are $6n-6$ non-zero elements.
Therefore, the number of non-zero entries in A is $4n-3+6n-6=10n-9$.

$P=I$ is a valid permutation matrix, where I is the identity matrix.
Reason: Notice every element in the main diagonal has magnitude 1 in A. Also all non-zero entries are equal to 1 or -1.
Since every element above it in the same column ($A_{x-i,x}$ for any i) has the different sign with the diagonal entry, during the elimination, it will increase the magnitude of the diagonal entry.
As a result, for every k, the magnitude of $A_{k,k}$ will be larger or equal to all the other entries in the same column. Which implies the indentity matrix I is a valid choice for P_k.
Therefore, $P=I$ is a valid permutation matrix.
To verify the form of P:
Use the MATLAB function

```
[L, U, P] = lu(A);
```

and verify the value of P

L:
lower bandwidth: 2, upper bandwidth: 0, total bandwidth: 2
U:
lower bandwidth: 0, upper bandwidth: 3, total bandwidth: 3
Reason: First, it is trivial that the upper bandwidth of L and the lower bandwidth of U must be 0 by definition.
Then, because $A=LU$ and A has lower and upper bandwidth of value 2, 3 respectively.
The L and U factors preserve the lower and upper bandwidths of A, i.e. L, U has lower, upper bandwidth of 2, 3 respectively.
This can be proved by fewer 0s are introduced to each column during the row elimination. Also no new 0 is added to the upper triangle when we modify downwards.
The number of non-zero entries in L is $7n-5$; in U is $9n-9$
Reason: for L, every loop (excluding the leftmost one) equation will introduce a non-zero entry in $L_{x,x-2}$ for some x.
So there are $(n-1)$ such non-zero elements of the form $L_{x,x-2}$. Besides, every element in the diagonal

(N entries) and 'first lower band' ($L_{x,x-1}$ for some x, $N-1$ entries) is non-zero.
The total number is $(n-1)+N+(N-1) = n+3n-2+3n-3 = 7n-5$.
Similarly, every right node equation brings a non-zero element in $U_{x,x+3}$ for some x while the element $U_{x,x+2}$ in the same line is 0 after LU.
So the number of non-zero entries in U equals to the total number of elements in the diagonal, first-band and second-band.
That is, there are $N+(N-1)+(N-2) = 3(3n-2)-3 = 9n-9$.

The condition number of A increases as n increases, based on the results.

The values of the intensities of the top line decreases (quickly at the beginning) and goes to 0, as we go from left to right based on results.

According to the results, maximum value are almost the same (slight numerical difference exists for $n=4$) for any n.
The minimum intensity decreases as n grows and its value equals to 0 for large n.
The sum of intensities are almost the same and converges to 136.603.

2 Question 2.

The linear system can be written as

$$Ax = \begin{pmatrix} 0.03 & 58.9 \\ 5.31 & -6.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59.2 \\ 47.0 \end{pmatrix}$$

2.1 (a)

$$\begin{aligned} A = \begin{pmatrix} 0.03 & 58.9 \\ 5.31 & -6.1 \end{pmatrix} &\xrightarrow[k=1]{elim} \begin{pmatrix} 0.03 & 58.9 \\ 177 & -6.1 - 58.9 \times 177 \end{pmatrix} \\ &= \begin{pmatrix} 0.03 & 58.9 \\ 177 & -10400 \end{pmatrix} \end{aligned}$$

, by $fl(-58.9 \times 177) = -10400$, $fl(-10400 - 6.1) = -10400$
so

$$L = \begin{pmatrix} 1 & 0 \\ 177 & 1 \end{pmatrix}, U = \begin{pmatrix} 0.03 & 58.9 \\ 0 & -10400 \end{pmatrix}$$

To solve $Ax = LUx = b$, solve $Ly = b$ with forward substitution first,

$$Ly = \begin{pmatrix} 1 & 0 \\ 177 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 59.2 \\ 47.0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 59.2 \\ y_2 = 47 - 59.2 \times 177 = -10500 \end{cases}$$

Then solve $Ux = y$ with backward substitution,

$$U\hat{x} = \begin{pmatrix} 0.03 & 58.9 \\ 0 & -10400 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59.2 \\ -10500 \end{pmatrix} \Rightarrow \begin{cases} x_1 = (59.2 - 1.01 \times 58.9)/0.03 = -10 \\ x_2 = 1.01 \end{cases}$$

The computed solution $\hat{x}$, to the system is $(-10, 1.01)^T$
Relative error in the infinity norm is

$$\frac{\|x - \hat{x}\|_\infty}{\|x\|_\infty} = \frac{20}{10} = 2$$

2.2 (b)

$$A = \begin{pmatrix} 0.03 & 58.9 \\ 5.31 & -6.1 \end{pmatrix} \xrightarrow[k=1]{piv} \begin{pmatrix} 5.31 & -6.1 \\ 0.03 & 58.9 \end{pmatrix} \qquad , P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow[k=1]{elim} \left(\begin{array}{c:c} 5.31 & -6.1 \\ \hdashline 5.65 \times 10^{-3} & 58.9 + 6.1 \times 5.65 \times 10^{-3} \end{array}\right) = \left(\begin{array}{c:c} 5.31 & -6.1 \\ \hdashline 5.65 \times 10^{-3} & 58.9 \end{array}\right)$$

hence

$$L = \begin{pmatrix} 1 & 0 \\ 5.65 \times 10^{-3} & 1 \end{pmatrix}, U = \begin{pmatrix} 5.31 & -6.1 \\ 0 & 58.9 \end{pmatrix}$$

Since $PA = LU$, $Ax = b \Rightarrow PAx = LUx = Pb$,
solve $Ly = Pb$ by forward substitution,

$$Ly = \begin{pmatrix} 1 & 0 \\ 5.65 \times 10^{-3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 47.0 \\ 59.2 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 47.0 \\ y_2 = 59.2 - 47 \times 5.65 \times 10^{-3} = 58.9 \end{cases}$$

Then solve $Ux = y$ by backward substitution,

$$U\hat{x} = \begin{pmatrix} 5.31 & -6.1 \\ 0 & 58.9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 47.0 \\ 58.9 \end{pmatrix} \Rightarrow \begin{cases} x_1 = (47 + 6.1 \times 1)/5.31 = 10 \\ x_2 = 1 \end{cases}$$

The computed solution to the system is $(10, 1)^T$
Relative error in the infinity norm is 0 because $\hat{x} = x$.
That is, the result we get equals to the exact solution.

2.3 (c)

$$A = \begin{pmatrix} 0.03 & 58.9 \\ 5.31 & -6.1 \end{pmatrix} \xrightarrow[k=1,Q]{complete.piv} \begin{pmatrix} 58.9 & 0.03 \\ -6.1 & 5.31 \end{pmatrix} \qquad , Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow[k=1]{elim} \left(\begin{array}{c:c} 58.9 & 0.03 \\ \hdashline -0.104 & 5.31 + 0.03 \times 0.104 \end{array}\right) = \left(\begin{array}{c:c} 58.9 & 0.03 \\ \hdashline -0.104 & 5.31 \end{array}\right)$$

hence

$$L = \begin{pmatrix} 1 & 0 \\ -0.104 & 1 \end{pmatrix}, U = \begin{pmatrix} 58.9 & 0.03 \\ 0 & 5.31 \end{pmatrix}$$

Since $AQ = LU$,

$$\begin{aligned} Ax = b &\Rightarrow AQQ^{-1}x = b \\ &\Rightarrow LUQ^{-1}x = b \end{aligned}$$

Solve $Ly = b$ by forward substitution,

$$Ly = \begin{pmatrix} 1 & 0 \\ -0.104 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 59.2 \\ 47.0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 59.2 \\ y_2 = 47 + 0.104 \times 59.2 = 53.2 \end{cases}$$

Then solve $U(Q^{-1}x) = y$ by backward substitution,

$$U(Q^{-1}x) = \begin{pmatrix} 58.9 & 0.03 \\ 0 & 5.31 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 59.2 \\ 53.2 \end{pmatrix} \Rightarrow \begin{cases} x_1 = (59.2 - 0.03 \times 10)/58.9 = 1 \\ x_2 = 10.0 \end{cases}$$

Since Q is square and orthogonal,

$$\begin{aligned} Q^{-1}x = Q^T x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 10 \end{pmatrix} && \Longrightarrow \\ \hat{x} = \begin{pmatrix} 10 \\ 1 \end{pmatrix} \end{aligned}$$

The computed solution to the system is $(10, 1)^T$
Relative error in the infinity norm is 0 because $\hat{x} = x$.
That is, the result we get equals to the exact solution.

3 Question 3.

3.1 (a)

Assume $(I - A)^{-1}$ does not exist for the contradiction. That is, $(I - A)$ is singular.
Hence there always exists $\vec{v} \neq 0$, s.t.

$$\begin{aligned} (I - A)v &= 0 && \Rightarrow \\ Iv &= Av && \text{which implies} \\ \frac{\|Av\|}{\|v\|} &= \frac{\|Iv\|}{\|v\|} \\ &= \frac{\|v\|}{\|v\|} \\ &= 1 \end{aligned}$$

Since v is non-zero, by the definition of the matrix norm,

$$\begin{aligned}\|A\| &= \max_{x\neq 0}\frac{\|Ax\|}{\|x\|}\\ &\geq \frac{\|Av\|}{\|v\|}\\ &= 1\end{aligned}$$

Which contradicts $\|A\| < 1$. Therefore, $(I-A)^{-1}$ exists.

3.2 (b)

Consider the system $(I-A)x = b$, where $\|b\| = 1$.
Clearly the solution $x = (I-A)^{-1}b$ because $(I-A)$ is non-singular.
We can treat the solution $x(b)$ as a function of b.
By Question 1 from Tutorial 5, we know

$$\begin{aligned}\|(I-A)^{-1}\| &= \max_{\|b\|=1}\|(I-A)^{-1}b\|\\ &= \max_{\|b\|=1}\|x\|\end{aligned}$$

Since

$$\begin{aligned}\|b\| = \|(I-A)x\| = \|x-Ax\| &= 1 && \Rightarrow\\ 1 = \|x-Ax\| &\geq \|x\| - \|Ax\| && \text{, by } \|Ax\| + \|x-Ax\| \geq \|x\|\\ &\geq \|x\| - \|A\|\cdot\|x\|\end{aligned}$$

Hence

$$\begin{aligned}&\|x\|\cdot(1-\|A\|) \leq 1\\ &\|x\| \leq \frac{1}{1-\|A\|} && \text{, by } \|A\| < 1\end{aligned}$$

Therefore,

$$\begin{aligned}\|(I-A)^{-1}\| &= \max_{\|b\|=1}\|x\|\\ &\leq \frac{1}{1-\|A\|}\end{aligned}$$

3.3 (c)

$$\begin{aligned}[(I-A)^{-1} - (I+A)](I-A) &= I - (I+A)(I-A)\\ &= I - I^2 + A^2\\ &= I - I + A^2\\ &= A^2\end{aligned}$$

Hence we know

$$(I - A)^{-1} - (I + A) = A^2(I - A)^{-1}$$

Therefore,

$$\begin{aligned}
\|(I - A)^{-1} - (I + A)\| &= \|A^2(I - A)^{-1}\| \\
&\leq \|A^2\| \cdot \|(I - A)^{-1}\| \\
&\leq \frac{\|A^2\|}{1 - \|A\|}
\end{aligned}$$

, by $\|(I - A)^{-1}\| \leq \frac{1}{1-\|A\|}$ showed in part (b).