CSC 420 - Fall 2021 - Assignment 3

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In solving the questions in this assignment, I worked together with my classmate [Haining Tan, 1004153364]. I confirm that I have written the solutions/code/report in my own words.

Question 1.

1.1

Let us say white has intensity of value 255 and black has intensity 0.
Thus we can ignore all the black pixels in computing the magnitude of the response of the Laplacian filter.
Define

$$\begin{aligned} S &= \{v \in \mathbb{R}^2 : |v| > \frac{D}{2}\} \\ \phi(r,\theta) &= (r\cos\theta, r\sin\theta) \end{aligned}$$

Then

$$\begin{aligned} \text{response magnitude} &= \int_S 255\nabla^2 G(x,y,\sigma)\,dx\,dy \\ &= \int_{\frac{D}{2}}^{\infty}\int_0^{2\pi} 255\nabla^2 G(r\cos\theta, r\sin\theta,\sigma)|\det D\phi(r,\theta)|\,d\theta\,dr \\ &= \int_{\frac{D}{2}}^{\infty}\int_0^{2\pi} \frac{255}{\pi\sigma^4}(\frac{r^2}{2\sigma^2}-1)e^{-\frac{r^2}{2\sigma^2}}r\,d\theta\,dr \\ &= 255\int_{\frac{D}{2}}^{\infty}\frac{2r}{\sigma^4}(\frac{r^2}{2\sigma^2}-1)e^{-\frac{r^2}{2\sigma^2}}\,dr \\ &= 255\int_{\frac{D}{2}}^{\infty}\frac{r^3}{\sigma^6}e^{-\frac{r^2}{2\sigma^2}}\,dr - 255\int_{\frac{D}{2}}^{\infty}\frac{2r}{\sigma^4}e^{-\frac{r^2}{2\sigma^2}}\,dr \end{aligned}$$

The first integral:

$$
\begin{aligned}
255\int_{\frac{D}{2}}^{\infty}\frac{r^3}{\sigma^6}e^{-\frac{r^2}{2\sigma^2}}\,dr &= 255\frac{1}{\sigma^4}\int_{\frac{D}{2}}^{\infty}\frac{r^2}{2\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\,dr^2 \\
&= 255\frac{1}{\sigma^4}\int_{\frac{D^2}{8\sigma^2}}^{\infty} te^{-t}\,d(2\sigma^2 t) \qquad\qquad (t=\frac{r^2}{2\sigma^2}) \\
&= 255\frac{1}{\sigma^4}\cdot 2\sigma^2\int_{\frac{D^2}{8\sigma^2}}^{\infty} te^{-t}\,dt \\
&= 255\frac{2}{\sigma^2}(-te^{-t}-e^{-t})|_{t=\frac{D^2}{8\sigma^2}}^{t=\infty} \\
&= 255\frac{2}{\sigma^2}(\frac{D^2}{8\sigma^2}e^{-\frac{D^2}{8\sigma^2}}+e^{-\frac{D^2}{8\sigma^2}}) \\
&= 255(\frac{D^2}{4\sigma^4}e^{-\frac{D^2}{8\sigma^2}}+\frac{2}{\sigma^2}e^{-\frac{D^2}{8\sigma^2}})
\end{aligned}
$$

The second integral:

$$
\begin{aligned}
-255\int_{\frac{D}{2}}^{\infty}\frac{2r}{\sigma^4}e^{-\frac{r^2}{2\sigma^2}}\,dr &= 255\frac{2}{\sigma^2}\cdot e^{-\frac{r^2}{2\sigma^2}}|_{\frac{D}{2}}^{\infty} \\
&= -255\frac{2}{\sigma^2}e^{-\frac{D^2}{8\sigma^2}}
\end{aligned}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\sigma}\text{response magnitude} = -255e^{-\frac{D^2}{8\sigma^2}}[-\frac{D^2}{\sigma^5}+\frac{D^2}{4\sigma^4}\frac{D^2}{4\sigma^3}]
$$

Set the derivative to 0, solve the equation and get

$$
\begin{aligned}
\sigma^2 &= \frac{D^2}{16} \\
\sigma^* &= \frac{D}{4}
\end{aligned}
$$

The scale $\frac{D}{4}$ maximises the magnitude of response.

1.2

Similarly as 1.1,

$$
\begin{aligned}
\text{response magnitude} &= \int_0^{\frac{D}{2}}\int_0^{2\pi} 255\nabla^2 G(r\cos\theta, r\sin\theta,\sigma)|\det D\phi(r,\theta)|\,d\theta\,dr \\
&= 255\frac{2}{\sigma^2}(-te^{-t}-e^{-t})|_{t=0}^{t=\frac{D^2}{8\sigma^2}} + 255\frac{2}{\sigma^2}\cdot e^{-\frac{r^2}{2\sigma^2}}|_{r=0}^{r=\frac{D}{2}} \\
&= -255\frac{D^2}{4\sigma^4}e^{-\frac{D^2}{8\sigma^2}}
\end{aligned}
$$

Set its derivative to 0,

$$\frac{D^2}{\sigma^5} - \frac{D^4}{16\sigma^7} = 0$$
$$\sigma = \frac{D}{4}$$

We should use $\sigma = \frac{D}{4}$.

1.3

See the answer to this question in the **.ipynb** file.

Question 2.

2.1

solve

$$\det N - \lambda I = 0$$
$$\det \begin{pmatrix} I_x^2 - \lambda & I_x I_y \\ I_x I_y & I_y^2 - \lambda \end{pmatrix} = 0$$
$$I_x^2 I_y^2 - \lambda(I_x^2 + I_y^2) + \lambda^2 - I_x^2 I_y^2 = 0$$
$$\lambda^2 = \lambda(I_x^2 + I_y^2)$$
$$\lambda_1 = 0, \lambda_2 = I_x^2 + I_y^2$$

2.2

Want to show M is positive semi-definite. Let $z = (z_1, z_2)^T$ be arbitrary.
We will show $z^T M z \geq 0$.

$$z^T M z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} [\sum_x \sum_y w(x,y) \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}] \begin{pmatrix} z_1 & z_2 \end{pmatrix}$$
$$= \sum_x \sum_y w(x,y)[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \end{pmatrix}]$$
$$= \sum_x \sum_y w(x,y)[z^T N z]$$

By 2.1, because $\lambda_1, \lambda_2 \geq 0$, we know N is positive semi-definite so that $z^T N z \geq 0$.
Also the window function $w(x,y) \geq 0$. Therefore,

$$z^T M z = \sum_x \sum_y w(x,y)[z^T N z]$$
$$\geq 0$$

M is positive semi-definite.