CSC 420 - Fall 2021 - Assignment 1

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In solving the questions in this assignment, I worked together with my classmate [Haining Tan, 1004153364]. I confirm that I have written the solutions/code/report in my own words.

1 Question 1.

$$\begin{aligned} h(n) * x(n) &= \sum_{s=-\infty}^{\infty} h(n-s)x(s) && \text{definition of convolution} \\ &= \sum_{s=-\infty}^{\infty} T[\delta(n-s)]x(s) && \delta \text{ is the impulse function} \\ &= T[\sum_{s=-\infty}^{\infty} \delta(n-s)x(s)] && T \text{ is linear} \end{aligned} \tag{1}$$

Since $\delta(n) = 0$ for every $n \neq 0$, the sum equals to the value at $s = n$, that is,

$$\begin{aligned} h(n) * x(n) &= (1) \\ &= T[x(n)] \end{aligned}$$

Showed as required.

2 Question 2.

Let u be $[a_n, a_{n-1}, \ldots, a_1, a_0]$, v be $[b_m, b_{m-1}, \ldots, b_1, b_0]$,
where $a_i, b_j \in \mathbb{R}, i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\}$.
The production of the polynomials of u, v representing is

$$\begin{aligned} (\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} b_j x^j) &= \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j x^i x^j \\ &= \sum_{s=0}^{n+m}[(\sum_{t=0}^{s} a_t b_{s-t})x^s] && \text{replace } i, j \text{ with } s, t \text{ using the idea of convolution} \end{aligned}$$

whose vector representation (simply take coefficients for every term) is

$$[a_n, \ldots a_0] * [b_m, \ldots, b_0] = u * v$$

Showed as required.

3 Question 3.

By definition of Laplacian pyramid,

$$L_k = I_k - U(I_{k+1}) \tag{2}$$

where $k \in \mathbb{N}$, and U is the up-sample operator.
Note U is linear because it does the inverse operation of down-sampling without Gaussian blurring.
Equation (1) can be written as

$$I_k = L_k + U(I_{k+1})$$

So

$$\begin{aligned}
I_0 &= L_0 + U(I_1) \\
&= L_0 + U[L_1 + U(I_2)] \\
&= L_0 + U(L_1) + U^2(I_2) && \text{By } U \text{ is linear} \\
&= \dots \\
&= L_0 + U(L_1) + U^2(L_2) + \dots + U^{n-1}(L_{n-1}) + U^n(I_n) \\
&= U^n(I_n) + \sum_{k=0}^{n-1} U^k(L_k) && \text{closed-form expression we want}
\end{aligned}$$

where U^0 represents the "do nothing" operator.
Therefore, the minimum information required is I_n. That is, the 1 pixel.
We reconstruct by up-sampling it n times and up-sampling every L_k, i.e. Laplacian pyramid of level k, k times. And adding all of them together.

4 Question 4.

Let r be an arbitrary direction, say we can rotate direction x by angle θ to get r. And r' is the unit direction vector orthogonal to r.

$$\begin{pmatrix} r \\ r' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The rotation matrix gives

$$r = x\cos\theta - y\sin\theta \tag{3}$$

$$r' = x\sin\theta + y\cos\theta \tag{4}$$

So by Chain rule,

$$
\begin{aligned}
\frac{\partial I}{\partial x} &= \frac{\partial I}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial I}{\partial r'}\frac{\partial r'}{\partial x} \\
&= \cos\theta \frac{\partial I}{\partial r} + \sin\theta \frac{\partial I}{\partial r'} \qquad (5) \\
\frac{\partial^2 I}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial I}{\partial x} \\
&= \cos\theta \frac{\partial}{\partial x}\frac{\partial I}{\partial r} + \sin\theta \frac{\partial}{\partial x}\frac{\partial I}{\partial r'} \qquad (6)
\end{aligned}
$$

By Clairaut's Theorem,

$$
\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial I}{\partial r} &= \frac{\partial}{\partial r}\frac{\partial I}{\partial x} \\
&= \frac{\partial}{\partial r}[\cos\theta \frac{\partial I}{\partial r} + \sin\theta \frac{\partial I}{\partial r'}] \qquad (7) \\
\frac{\partial}{\partial x}\frac{\partial I}{\partial r'} &= \frac{\partial}{\partial r'}\frac{\partial I}{\partial x} \\
&= \frac{\partial}{\partial r'}[\cos\theta \frac{\partial I}{\partial r} + \sin\theta \frac{\partial I}{\partial r'}] \qquad (8)
\end{aligned}
$$

Plug (7)(8) back into (6),

$$
\frac{\partial^2 I}{\partial x^2} = \cos^2 \frac{\partial^2 I}{\partial r^2} + 2\cos\theta\sin\theta \frac{\partial^2 I}{\partial r\,\partial r'} + \sin^2\theta \frac{\partial^2 I}{\partial r'^2} \qquad (9)
$$

Repeat the same process for the second order partial derivative to y, get

$$
\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial I}{\partial r} &= (-\sin\theta)\frac{\partial}{\partial r}\frac{\partial I}{\partial r} + \cos\theta \frac{\partial}{\partial r}\frac{\partial I}{\partial r'} \\
\frac{\partial}{\partial y}\frac{\partial I}{\partial r'} &= (-\sin\theta)\frac{\partial}{\partial r'}\frac{\partial I}{\partial r} + \cos\theta \frac{\partial}{\partial r'}\frac{\partial I}{\partial r'} \\
\frac{\partial^2 I}{\partial y^2} &= \sin^2\theta \frac{\partial^2 I}{\partial r^2} - 2\sin\theta\cos\theta \frac{\partial^2 I}{\partial r\,\partial r'} + \cos^2\theta \frac{\partial^2 I}{\partial r'^2} \qquad (10)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\nabla^2 I &= I_{xx} + I_{yy} \\
&= (9) + (10) \\
&= (\sin^2\theta + \cos^2\theta)\frac{\partial^2 I}{\partial r^2} + (\sin^2\theta + \cos^2\theta)\frac{\partial^2 I}{\partial r'^2} \\
&= \frac{\partial^2 I}{\partial r^2} + \frac{\partial^2 I}{\partial r'^2}
\end{aligned}
$$

The Laplacian is rotation invariant, showed as required.