

# MAD and MED families in computability

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# On the cardinality of the continuum

## Continuum Hypothesis (CH)

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CH is consistent with ZFC.

CH holds in the constructible universe  $L$  (Gödel 1940).

$\neg$ CH is consistent with ZFC.

CH fails in some forcing extensions  $V[G]$  (Cohen 1963).

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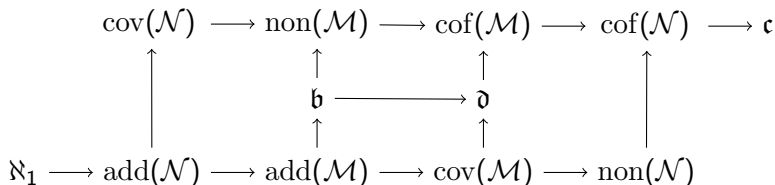
The bounding number,  $\mathfrak{b}$ , is the least cardinality of such an unbounded family.

## Observation: $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{c}$

${}^\omega\omega$  is unbounded, so  $\mathfrak{b} \leq \mathfrak{c}$ . If  $\{f_i\}_{i \in \omega}$  is a countable family, then  $n \mapsto \max_{i \leq n} f_i(n)$  dominates all of its elements, so  $\mathfrak{b} \geq \aleph_1$ .

# Cichoń's diagram

A popular topic in this direction has been cardinal characteristics associated with the null and meagre ideals.



**Figure:** Cichoń's diagram;  $\kappa \rightarrow \lambda$  means that ZFC proves  $\kappa \leq \lambda$ .

In celebrated work, it has been shown that these cardinals can be pairwise distinct simultaneously (Goldstern et al. 2022).

# MAD families

Certain families of sets can give interesting cardinal characteristics. One of the most well researched is the following notion:

## Definition:

A family of infinite sets of natural numbers is called *almost disjoint* (AD) if any two of its elements have finite intersection.

Such a family is *maximal almost disjoint* (MAD) if there is no AD family containing it as a proper subset.

The characteristic  $\mathfrak{a}$  is the least cardinality of a MAD family.

# MED families

The notion of MAD families is easy to generalise to other spaces.  
The case of  ${}^\omega\omega$  is particularly interesting.

## Definition:

A family of natural number functions is called *eventually different* (ED) if any two of its elements agree only finitely often.

Such a family is *maximal eventually different* (MED) if there is no ED family containing it as a proper subset.

$\mathfrak{a}_e$  is the least cardinality of a maximal eventually different (MED) family.

# Remarks on MAD and MED families.

MAD and MED families are interesting for reasons other than their cardinalities:

- No MAD family is analytic (Mathias 1977).
- There is a coanalytic MAD family in  $L$  (Miller 1989).
- It is consistent for there to be no MAD families, even with dependent choice (Horowitz and Shelah 2019).

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- It is consistent for there to be no MAD families, even with dependent choice (Horowitz and Shelah 2019).
- There is a Borel MED family (Horowitz and Shelah 2024).  
MED families can be simpler than any MAD family.
- We can even get an effectively closed ( $\Pi_1^0$ ) MED family (Schrittesser 2018).

# Cardinal characteristics and computability

- Rupprecht studied computability theoretic analogues of the cardinals in Cichoń's diagram in, developing a framework of **highness classes** (Rupprecht 2010).
- Brendle et al. built on Rupprecht's work, making a lot of the content more accessible and posing several problems (Brendle et al. 2015).
- Lempp et al. consider a new framework of **mass problems** as analogues of cardinal characteristics (Lempp et al. 2023).

This analogy is more flexible and suitable for other characteristics outside of Cichoń's diagram.

# Mass problems

A mass problem is the problem of computing an element of some nonempty class of sets or functions.

We identify the mass problem with the set of such solutions.

For example, define:

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as the mass problem for computing a dominating function.

We can obtain a natural reducibility and degree structure on mass problems:

## Definition:

Let  $\mathcal{C}$  and  $\mathcal{D}$  be mass problems.

We say  $\mathcal{C}$  is Muchnik-reducible to  $\mathcal{D}$ , written  $\mathcal{C} \leq_w \mathcal{D}$ , if every member of  $\mathcal{D}$  computes a member of  $\mathcal{C}$ .

# Encoding sequences

We want a way of studying families as computability theoretic objects.

The following allows us to encode countably many functions or sets as a single object:

## Definition:

Given a function  $f$ , we define the column  $f^{[e]} = x \mapsto f(\langle e, x \rangle)$ . Similarly, for a set  $F$  define  $F^{[e]} = \{x : \langle e, x \rangle \in F\}$ .

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Given a collection of countable functions  $C$ , we will encode it as a function  $f$  where  $C = \{f^{[e]}\}_{e \in \omega}$ .

We will often write this as  $f = \langle f_e \rangle_{e \in \omega}$ .

# The mass problem $\mathcal{B}$

Now we can define mass problems corresponding to our cardinal characteristics:

## Definition:

A family  $\langle f_e \rangle_{e \in \omega}$  of computable functions is called *unbounded* if there is no computable function  $h$  which dominates each column  $f_e$ .

Let  $\mathcal{B}$  be the mass problem of all unbounded families of computable functions.

# The mass problems $\text{MaxAD}_{\text{rec}}$ and $\text{MaxED}_{\text{rec}}$

Definition: (Lempp et al. 2023)

A family  $\langle G_e \rangle_{e \in \omega}$  of computable sets is called *MAD* if it is AD and for any computable set  $H$  there is some  $e$  such that  $H \cap G_e$  is infinite.

Let  $\text{MaxAD}_{\text{rec}}$  be the mass problem of MAD sequences.

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Definition:

A sequence  $\langle g_e \rangle_{e \in \omega}$  of computable functions is called *MED* if it is ED and for any computable function  $h$  there is some  $e$  such that  $\{x : h(x) = g_e(x)\}$  is infinite.

Let  $\text{MaxED}_{\text{rec}}$  be the mass problem of MED sequences.

# Remarks about $\text{MaxAD}_{\text{rec}}$ and $\text{MaxED}_{\text{rec}}$

- Every MED sequence is unbounded: for every  $h$  computable there is some member infinitely often equal to  $h + 1$ .  
So  $\mathcal{B} \leq_w \text{MaxED}_{\text{rec}}$  by the trivial reduction.
- There is a MAD family below every noncomputable c.e. set (Lempp et al. 2023).  
A similar construction shows that there is a MED family below every noncomputable c.e. set.
- Lempp et al. also show that every nonlow set computes a MAD family.
- In contrast, every MED sequence has hyperimmune degree and there are nonlow hyperimmune-free degrees.  
This also shows that  $\text{MaxED}_{\text{rec}} \not\leq_w \text{MaxAD}_{\text{rec}}$ .

# Is every MED family above a MAD family?

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It seems hopeless, but trying to use the combinatorial properties of MED sequences is the wrong approach. We show  $\text{MaxAD}_{\text{rec}} \leq_w \mathcal{B}$ .

# Nice unbounded sequences of functions

It is helpful to observe that we can require that an unbounded sequence of functions has some additional properties, without increasing the computational complexity.

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Given an unbounded sequence  $f = \langle f_e \rangle_{e \in \omega} \in \mathcal{B}$ , there is a sequence  $\hat{f} = \langle \hat{f}_e \rangle_{e \in \omega}$  such that:

- $\hat{f} \leq_T f$  and  $\hat{f} \in \mathcal{B}$ ,
- for each indices  $k < e$  we have  $\hat{f}_k < \hat{f}_e$ ,
- each function  $\hat{f}_e$  is increasing.

In other words, each unbounded sequence computes an unbounded sequence which is increasing in both variables.

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Define

$$\hat{f}_e(x) = \sum_{k \leq e} \sum_{y \leq x} (1 + f_k(y)).$$

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We will define a new sequence of functions  $\hat{f} = \langle \hat{f}_e \rangle_{e \in \omega}$  below  $f$ .  
Let  $\hat{f}_0 = f_0$ . For each  $e$  and  $x$  define

$$N_{e,x} = \max\{f_{e+1}(x), \hat{f}_e(x), \hat{f}_{e+1}(k) : k < x\}.$$

and

$$\hat{f}_{e+1}(x) = \min\{y \in \text{ran}(\hat{f}_e) : \text{ran}(\hat{f}_e) \cap (N_{e,x}, y) \neq \emptyset\}.$$

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We get that:

- For each  $e$  we have  $f_e < \hat{f}_e$ , so  $\hat{f}$  is unbounded,
- each  $\hat{f}_e$  is computable and increasing, hence each  $\text{ran}(\hat{f}_e)$  is computable,
- for indices  $e < k$  we have  $\text{ran}(\hat{f}_k) \subseteq \text{ran}(\hat{f}_e)$  and the difference is infinite.

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Suppose  $R$  is an infinite computable set, and  $p$  enumerates  $R$ . Choose  $e$  such that  $\hat{f}_e(x)$  is infinitely often greater than  $p(2x)$ . For such  $x$ , we have

$$|\text{ran}(\hat{f}_e) \cap [0, p(2x)]| < x,$$










$$|\text{ran}(p) \cap [0, p(2x)]| = 2x.$$

As  $\text{ran}(p) = R$  we can observe that  $R \setminus \text{ran}(\hat{f}_e)$  is infinite. A pigeonhole argument shows that there is some  $k \leq e$  such that  $R \cap G_k$  is infinite.

# Unanswered questions

- Which Muchnik relations can be reversed?  
In particular, can we separate the mass problem of MAD families and that of towers of functions?
- There is a  $\Pi_1^0$  MED family whose computable members are MED for computable functions.  
Is there a  $\Pi_1^0$ -class whose computable members are MAD for computable sets?

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