Propulsion Academy

Exercise

Single Variable Calculus

Q1. The volume of a spherical balloon is decreasing at the instantaneous rate of -10cm3/sec, at the moment when its radius is 20 cm. At that moment, how rapidly is its radius decreasing?

Q2. Where are the following functions discontinuous?

a.

$$\frac{1+x^2}{1-x^2}$$

b.

$$\frac{d|x|}{dx}$$

Q3. A container in the shape of a right circular cone with vertex angle a right angle is partially filled with water.

- Suppose water is added at the rate of 3 cu.cm./sec. How fast is the water level rising when the height h = 2cm?
- Suppose instead no water is added, but water is being lost by evaporation. Show the level falls at a constant rate. (You will have to make a reasonable physical assumption about the rate of water loss-state it clearly.)

Q4. Evaluate

$$\int_0^1 \frac{x\,dx}{\sqrt{1+3x^2}}$$

Q5. A bank gives interest at a rate r, compounded continuously, so that an amount A_0 deposited grows after t years to an amount $A(t) = A_0 e^{rt}$.
You make a daily deposit at a constant annual rate k; in other words, over the time period δt you deposit $k\delta t$ dollars. Set up a definite integral (give reasoning) which tells how much is in your account at the end of a year. (Do not evaluate integral)

Q6. Find the volume of the solid obtained by rotating about the y-axis (that's the y-axis) the area under the graph of $y = e^x$ and over the interval $0 < x < 1$. (suggestion: use cylindrical shells).

Q7. K-mart is selling at half-price it's left-over pumpkins–thin orange plastic shell filled with half-priced Halloween candy.

A great pumpkin has the shape of the curve $x^2 + y^4 = 1$, rotated around the vertical axis, i.e., the y-axis.This curve is symmetric around the x-axis and y-axis – it looks something like a circle, but somewhat flatter at the top and bottom.

Using units in feet, how much cubic feet of candy will it take to fill the Great pumpkin. Give an exact answer and tell if 5 cubic feet would be enough.

Multi Variable Calculus

Q1. Find the tangent plane at $(1, 1, 1)$ to the surface $x^2 + 2y^2 + 2z^2 = 5$; give the equation in the form ax+by+cz=d and simplify the coefficients.

Q2. Let $w = x^2 - xy^3$ and $P = (2, 1)$

- Find the directional derivative $\frac{dw}{ds}$ at P in the direction of $A = 3i + 4j$.
- If you start at P and go at a distance of 0.01 in the direction of A, by approximately how much will w change?

Q3. Find the points on the plane $2x + y - z = 6$ which is closest to the origin, by using Lagrange multipliers.

Q4. Given a 3 x 3 matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{bmatrix}$$

- Let $a = 2$: show that $|A_2| = 0$
- Find the line of solutions to $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Suppose now that $a = 1$, and that $A_1^{-1} = \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix}$. Find p.

Q5. Let $f(x, y) = x + 4y + \frac{2}{xy}$

- Find the critical point(s) of $f(x, y)$
- Use the second derivative test to test the critical point(s) found in part(a).

Q6. $\mathbf{F}(x, y, z) = (y + y^2 z)\mathbf{i} + (x - z + 2xyz)\mathbf{j} + (-y + xy^2)\mathbf{k}$

- Show that $\mathbf{F}(x, y, z)$ is a gradient field using the derivative conditions.
- Find a potential function $f(x, y, z,)$ for $\mathbf{F}(x, y, z)$, using any systemic method. Show the method used and all work clearly.
- Find $\int_c F.dr$,

  where c is the straight line joining the points $(2, 2, 1)$ and $(1, -1, 2)$ (in that order), using as little computation as possible.

Q7. Let $w = f(x, y, z)$ with the constraint $g(x, y, z) = 3$. At the point $P : (0, 0, 0)$, we have $\delta f =< 1, 1, 2 >$ and $\delta g =< 2, -1, -1 >$, find the value at P of two quantities:

- $(\frac{\partial z}{\partial x})_y$
- $(\frac{\partial w}{\partial x})_x$

Linear Algebra

Q1. Let A be the 2 x 2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Compute the eigenvalues of matrix A^{19}.

Q2. Consider the following matrix A = $\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigenvalues of A are 4 and 8. Find x and y.

Q3. How many solutions does the following system of linear equations have?
$-x + 5y = -1$, $x - y = 2$, $x + 3y = 3$.

Q4. In the given matrix, one of the eigenvalues is 1. the eigenvectors corresponding to the eigenvalue 1 are

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ (Hint: use $AX - zX = 0$)

Q5. Suppose that the eigenvalues of matrix A are $1, 2, 4$. Compute A_T^{-1}.

Q6. Find LU decomposition of the following matrix
$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$.

Maths using Python

Q1. Create a null vector of size 10. (Hint: numpy)

Q2. Create a null vector of size 10 fill the fifth value which is 1.

Q3. Create a 8x8 matrix and fill it with a checkerboard pattern.

Q4. Create and Normalise a 5x5 random matrix.

Q5. Multiply a 5x3 matrix by a 3x2 matrix (Dot product).

Q6. Find out if the vector represented by $(-16, 3, 5)$ is orthogonal to $a = (1, -3, 5)$ and $b = (2, -1, 7)$.

Q7. Create a matrix A = $\begin{bmatrix} 1 & 5 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & -5 \end{bmatrix}$ and compute it's determinant and inverse. (Use native *numpy* function)

Q8. Solve for $x1$ and x_2
$x_1 + 3x_2 = 2$
$x_1 + 2x_2 = 0.$
(Hint : remember AX = B, check *linalg* in *numpy*).

Q9. Find critical points of the function $f(x) = x^3 - 2x^2 + x$ and use second derivative to find the maximum on interval $x \subset [0, 1]$. (Hint1: remember, first differentiation and second differentiation, hint2: *sympy* python library with use of $solve, dif$).

Q10. Maximise $f(x, y, z) = xy + yz$, subject to the constraints $x + 2y = 6$ and $x - 3z = 0$.
(Hint1 : set up the Lagrange equation,
$f(x, y, z) - \lambda(constraint1) - \mu(constraint2)$, Hint2 : Set partial derivatives to zero and solve the following set of equations, Hint3 : Use *solve*.)

Statistics using R

We recommend you to also learn R and practise following exercises in R. If you want to stick to python, feel free todo so.

Q1. Suppose there are twelve multiple choice questions in an English class quiz. Each question has five possible answers, and only one of them is correct. Find the probability of having four or less correct answers if a student attempts to answer every question at random. (Hint : dbinom/pbinom)

Q2. Suppose the mean checkout time of a supermarket cashier is three minutes. Find the probability of a customer checkout being completed by the cashier in less than two minutes (Hint : exponential distribution).

Q3. Find the 2.5th and 97.5th percentiles of the Student t distribution with 5 degrees of freedom.

Q4. Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. In a sample of 30 light bulbs, it was found that they only last 9,900 hours on average. Assume the population standard deviation is 120 hours. At .05 significance level, can we reject the claim by the manufacturer?

Q5. Suppose 60% of citizens voted in last election. 85 out of 148 people in a telephone survey said that they voted in current election. At 0.5 significance level, can we reject the null hypothesis that the proportion of voters in the population is above 60% this year?

Q6. Please use data *immer* from library $Mass$ in R to answer following

- Assuming that the data in *immer* follows the normal distribution, find the 95% confidence interval estimate of the difference between the mean barley yields between years 1931 and 1932.

Q7. Please use data *mtcars* data (help(mtcars)) in R to answer following

- Assuming that the data in mtcars follows the normal distribution, find the 95% confidence interval estimate of the difference between the mean gas mileage of manual and automatic transmissions.

Q8. Please use data *immer* from library $Mass$ in R to answer following

- Without assuming the data to have normal distribution, test at .05 significance level if the barley yields of 1931 and 1932 in data set immer have identical data distributions.