Unit 1: Derivatives

A. What is a derivative?

- Geometric interpretation
- Physical interpretation
- Important for any measurement (economics, political science, finance, physics, etc.)

B. How to differentiate *any* function you know.

- For example: $\frac{d}{dx}\left(e^{x \arctan x}\right)$. We will discuss what a derivative is today. Figuring out how to differentiate any function is the subject of the first two weeks of this course.

Lecture 1: Derivatives, Slope, Velocity, and Rate of Change

Geometric Viewpoint on Derivatives

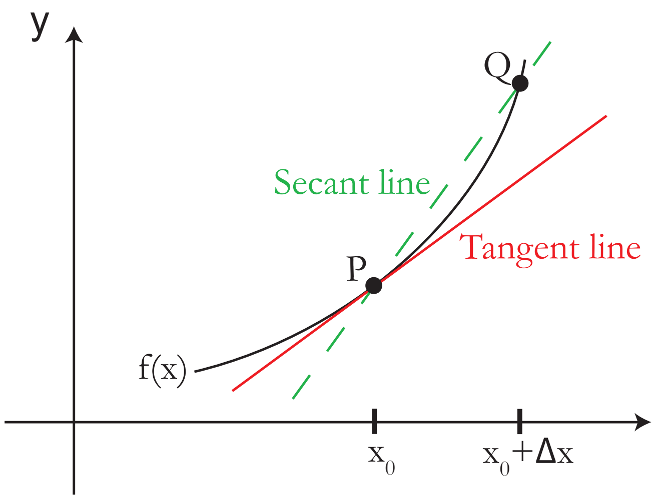

Figure 1: A function with secant and tangent lines

The derivative is the slope of the line tangent to the graph of $f(x)$. But what is a tangent line, exactly?

- It is NOT just a line that meets the graph at one point.
- It is the *limit* of the secant line (a line drawn between two points on the graph) as the distance between the two points goes to zero.

Geometric definition of the derivative:

Limit of slopes of secant lines PQ as $Q \to P$ (P fixed). The slope of $\overline{PQ}$:

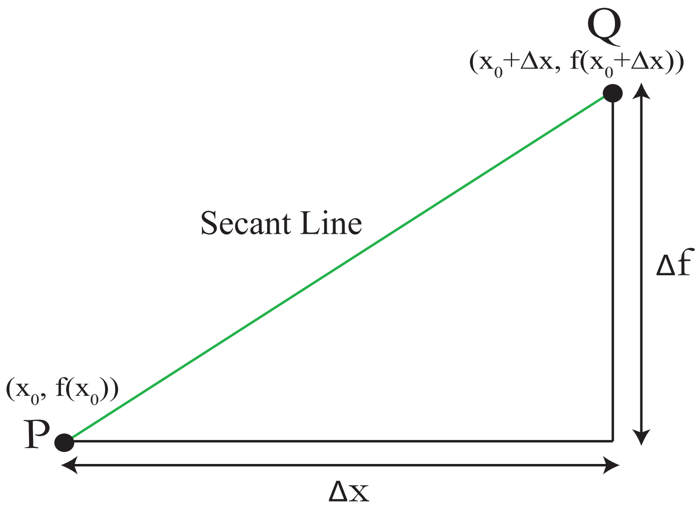

Figure 2: Geometric definition of the derivative

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \underbrace{\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}_{\text{"difference quotient"}} = \underbrace{f'(x_0)}_{\text{"derivative of } f \text{ at } x_0\text{"}}$$

Example 1. $f(x) = \dfrac{1}{x}$

One thing to keep in mind when working with derivatives: it may be tempting to plug in $\Delta x = 0$ right away. If you do this, however, you will always end up with $\dfrac{\Delta f}{\Delta x} = \dfrac{0}{0}$. You will always need to do some cancellation to get at the answer.

$$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x} = \frac{1}{\Delta x}\left[\frac{x_0 - (x_0 + \Delta x)}{(x_0 + \Delta x)x_0}\right] = \frac{1}{\Delta x}\left[\frac{-\Delta x}{(x_0 + \Delta x)x_0}\right] = \frac{-1}{(x_0 + \Delta x)x_0}$$

Taking the limit as $\Delta x \to 0$,

$$\lim_{\Delta x \to 0} \frac{-1}{(x_0 + \Delta x)x_0} = \frac{-1}{x_0^2}$$

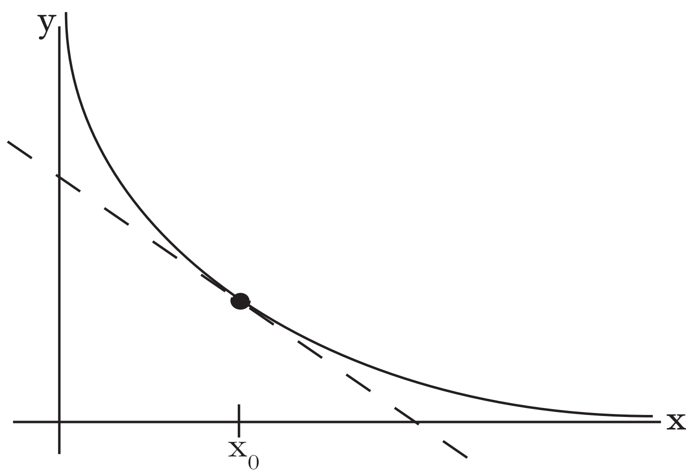

Figure 3: Graph of $\frac{1}{x}$

Hence,

$$f'(x_0) = \frac{-1}{x_0^2}$$

Notice that $f'(x_0)$ is negative — as is the slope of the tangent line on the graph above.

Finding the tangent line.

Write the equation for the tangent line at the point (x_0, y_0) using the equation for a line, which you all learned in high school algebra:

$$y - y_0 = f'(x_0)(x - x_0)$$

Plug in $y_0 = f(x_0) = \dfrac{1}{x_0}$ and $f'(x_0) = \dfrac{-1}{x_0^2}$ to get:

$$y - \frac{1}{x_0} = \frac{-1}{x_0^2}(x - x_0)$$

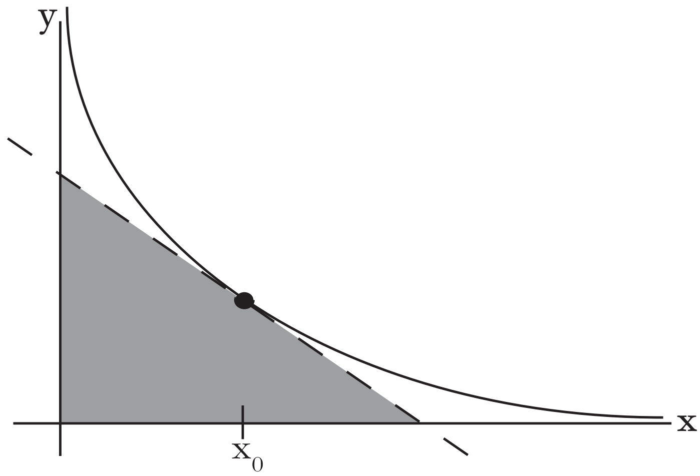

Figure 4: Graph of $\frac{1}{x}$

Just for fun, let's compute the area of the triangle that the tangent line forms with the x- and y-axes (see the shaded region in Fig. 4).

First calculate the x-intercept of this tangent line. The x-intercept is where $y = 0$. Plug $y = 0$ into the equation for this tangent line to get:

$$\begin{aligned} 0 - \frac{1}{x_0} &= \frac{-1}{x_0^2}(x - x_0) \\ \frac{-1}{x_0} &= \frac{-1}{x_0^2}x + \frac{1}{x_0} \\ \frac{1}{x_0^2}x &= \frac{2}{x_0} \\ x &= x_0^2(\frac{2}{x_0}) \ = \ 2x_0 \end{aligned}$$

So, the x-intercept of this tangent line is at $x = 2x_0$.

Next we claim that the y-intercept is at $y = 2y_0$. Since $y = \frac{1}{x}$ and $x = \frac{1}{y}$ are identical equations, the graph is symmetric when x and y are exchanged. By symmetry, then, the y-intercept is at $y = 2y_0$. If you don't trust reasoning with symmetry, you may follow the same chain of algebraic reasoning that we used in finding the x-intercept. (Remember, the y-intercept is where $x = 0$.)

Finally,

$$\text{Area} = \frac{1}{2}(2\mathrm{y}_0)(2\mathrm{x}_0) = 2\mathrm{x}_0\mathrm{y}_0 = 2\mathrm{x}_0(\frac{1}{\mathrm{x}_0}) = 2 \text{ (see Fig. 5)}$$

Curiously, the area of the triangle is *always* 2, no matter where on the graph we draw the tangent line.

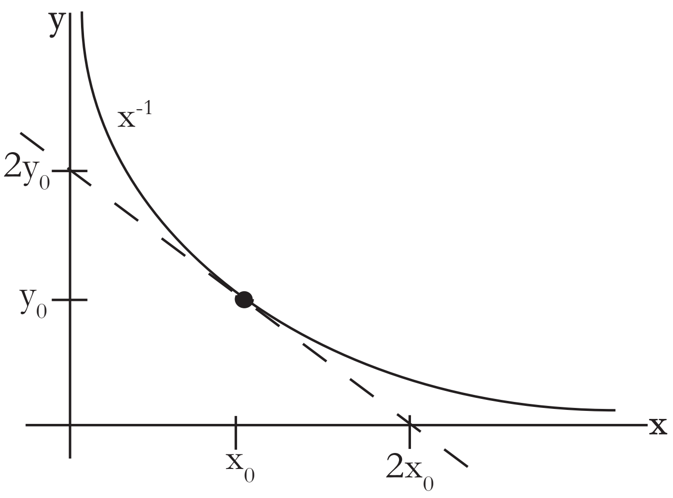

Figure 5: Graph of $\frac{1}{x}$

Notations

Calculus, rather like English or any other language, was developed by several people. As a result, just as there are many ways to express the same thing, there are many notations for the derivative.

Since $y = f(x)$, it's natural to write

$$\Delta y = \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

We say "Delta y" or "Delta f" or the "change in y".

If we divide both sides by $\Delta x = x - x_0$, we get two expressions for the difference quotient:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x}$$

Taking the limit as $\Delta x \to 0$, we get

$$\begin{aligned} \frac{\Delta y}{\Delta x} &\to \frac{dy}{dx} \text{ (Leibniz' notation)} \\ \frac{\Delta f}{\Delta x} &\to f'(x_0) \text{ (Newton's notation)} \end{aligned}$$

When you use Leibniz' notation, you have to remember where you're evaluating the derivative — in the example above, at $x = x_0$.

Other, equally valid notations for the derivative of a function f include

$$\frac{df}{dx}, f', \text{ and } Df$$

Example 2. $f(x) = x^n$ where $n = 1, 2, 3...$

What is $\frac{d}{dx}x^n$?

To find it, plug $y = f(x)$ into the definition of the difference quotient.

$$\frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

(From here on, we replace x_0 with x, so as to have less writing to do.) Since

$$(x + \Delta x)^n = (x + \Delta x)(x + \Delta x)...(x + \Delta x) \quad \text{n times}$$

We can rewrite this as

$$x^n + n(\Delta x)x^{n-1} + O\left((\Delta x)^2\right)$$

$\mathrm{O}(\Delta x)^2$ is shorthand for "all of the terms with $(\Delta x)^2$, $(\Delta x)^3$, and so on up to $(\Delta x)^n$." (This is part of what is known as the binomial theorem; see your textbook for details.)

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{x^n + n(\Delta x)(x^{n-1}) + \mathrm{O}(\Delta x)^2 - x^n}{\Delta x} = nx^{n-1} + \mathrm{O}(\Delta x)$$

Take the limit:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

Therefore,

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

This result extends to polynomials. For example,

$$\frac{d}{dx}(x^2 + 3x^{10}) = 2x + 30x^9$$

Physical Interpretation of Derivatives

You can think of the derivative as representing a rate of change (speed is one example of this).

On Halloween, MIT students have a tradition of dropping pumpkins from the roof of this building, which is about 400 feet high.

The equation of motion for objects near the earth's surface (which we will just accept for now) implies that the height above the ground y of the pumpkin is:

$$y = 400 - 16t^2$$

The average speed of the pumpkin (difference quotient) $= \frac{\Delta y}{\Delta t} = \frac{\text{distance travelled}}{\text{time elapsed}}$

When the pumpkin hits the ground, $y = 0$,

$$400 - 16t^2 = 0$$

Solve to find $t = 5$. Thus it takes 5 seconds for the pumpkin to reach the ground.

$$\text{Average} \quad \text{speed} = \frac{400 \text{ ft}}{5 \text{ sec}} = 80 \text{ ft/s}$$

A spectator is probably more interested in how fast the pumpkin is going when it slams into the ground. To find the instantaneous velocity at $t = 5$, let's evaluate y':

$$y' = -32t = (-32)(5) = -160 \text{ ft/s} \quad (\text{about } 110\,\text{mph})$$

y' is negative because the pumpkin's y-coordinate is decreasing: it is moving downward.

Lecture 2: Limits, Continuity, and Trigonometric Limits

More about the "rate of change" interpretation of the derivative

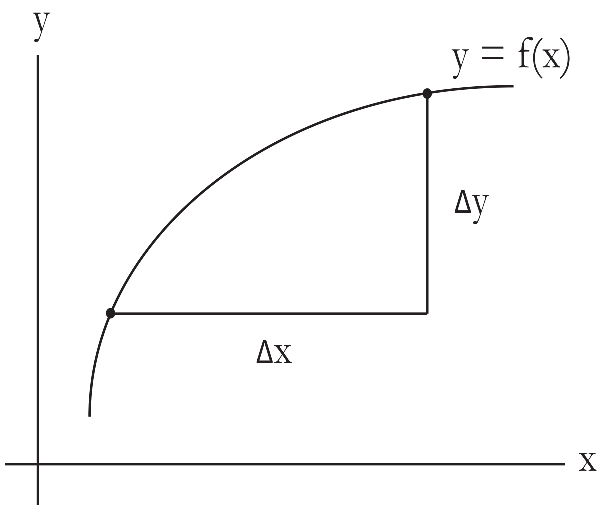

Figure 1: Graph of a generic function, with Δx and Δy marked on the graph

$$\frac{\Delta y}{\Delta x} \quad \to \quad \frac{dy}{dx} \text{ as } \Delta x \to 0$$

Average rate of change $\to$ Instantaneous rate of change

Examples

1. $q =$ charge $\quad \dfrac{dq}{dt} =$ electrical current
2. $s =$ distance $\quad \dfrac{ds}{dt} =$ speed
3. $T =$ temperature $\quad \dfrac{dT}{dx} =$ temperature gradient

4. Sensitivity of measurements: An example is carried out on Problem Set 1. In GPS, radio signals give us h up to a certain measurement error (See Fig. 2 and Fig. 3). The question is how accurately can we measure L. To decide, we find $\frac{\Delta L}{\Delta h}$. In other words, these variables are related to each other. We want to find how a change in one variable affects the other variable.

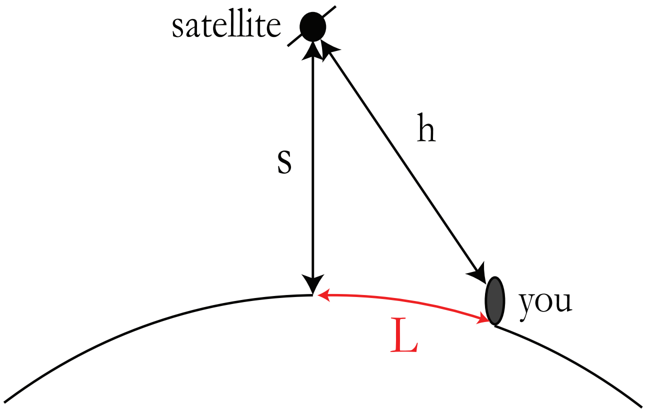

Figure 2: The Global Positioning System Problem (GPS)

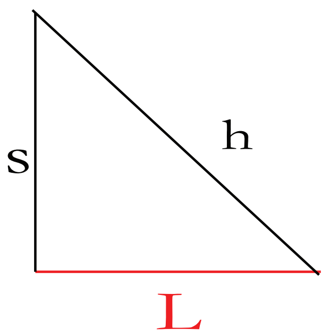

Figure 3: On problem set 1, you will look at this simplified "flat earth" model

Limits and Continuity

Easy Limits

$$\lim_{x\to 3} \frac{x^2+x}{x+1} = \frac{3^2+3}{3+1} = \frac{12}{4} = 3$$

With an easy limit, you can get a meaningful answer just by plugging in the limiting value.

Remember,

$$\lim_{x\to x_0} \frac{\Delta f}{\Delta x} = \lim_{x\to x_0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

is never an easy limit, because the denominator $\Delta x = 0$ is not allowed. (The limit $x \to x_0$ is computed under the implicit assumption that $x \neq x_0$.)

Continuity

We say $f(x)$ is continuous at x_0 when

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Pictures

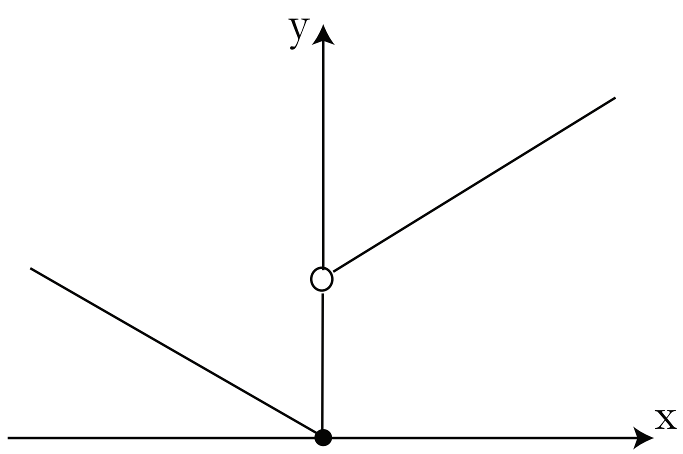

Figure 4: Graph of the discontinuous function listed below

$$f(x) = \begin{cases} x+1 & x > 0 \\ -x & x \geq 0 \end{cases}$$

This *discontinuous* function is seen in Fig. 4. For $x > 0$,

$$\lim_{x \to 0} f(x) = 1$$

but $f(0) = 0$. (One can also say, f is continuous from the left at 0, not the right.)

1. Removable Discontinuity

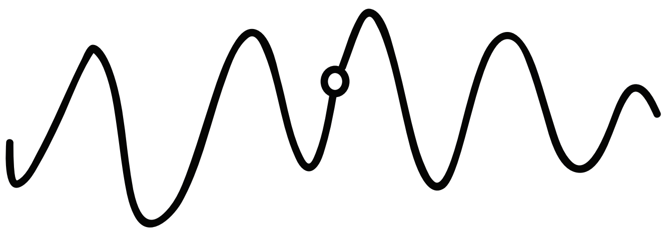

Figure 5: A removable discontinuity: function is continuous everywhere, except for one point

Definition of removable discontinuity

Right-hand limit: $\lim\limits_{x \to x_0^+} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x > x_0$.

Left-hand limit: $\lim\limits_{x \to x_0^-} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x < x_0$.

If $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x)$ but this is not $f(x_0)$, or if $f(x_0)$ is undefined, we say the discontinuity is *removable*.

For example, $\dfrac{\sin(x)}{x}$ is defined for $x \neq 0$. We will see later how to evaluate the limit as $x \to 0$.

2. **Jump Discontinuity**

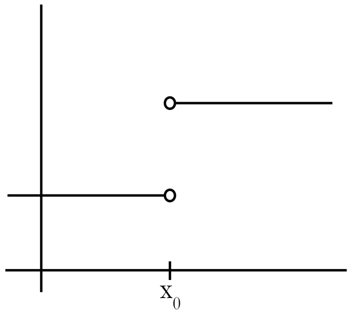

Figure 6: An example of a jump discontinuity

$\lim_{x \to x_0^+}$ for $(x < x_0)$ exists, and $\lim_{x \to x_0^-}$ for $(x > x_0)$ also exists, but they are NOT equal.

3. **Infinite Discontinuity**

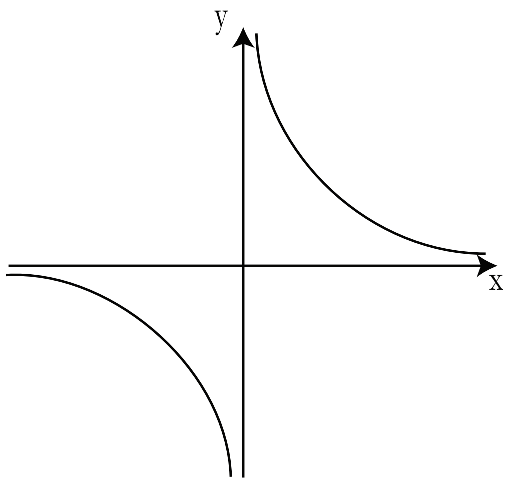

Figure 7: An example of an infinite discontinuity: $\frac{1}{x}$

Right-hand limit: $\lim_{x \to 0^+} \frac{1}{x} = \infty$; Left-hand limit: $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

4. **Other (ugly) discontinuities**

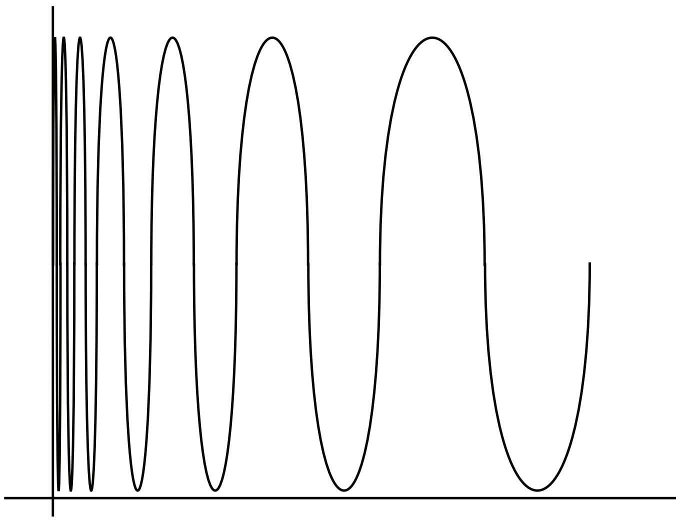

Figure 8: An example of an ugly discontinuity: a function that oscillates a lot as it approaches the origin

This function doesn't even go to $\pm\infty$ — it doesn't make sense to say it goes to anything. For something like this, we say the limit does not exist.

Picturing the derivative

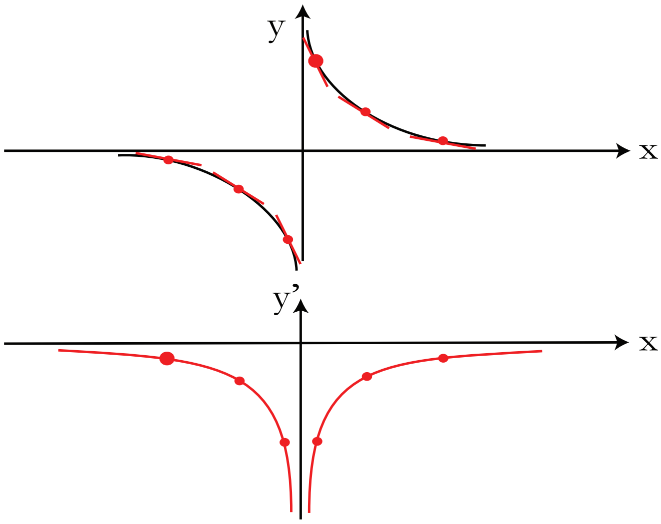

Figure 9: Top: graph of $f(x) = \frac{1}{x}$ and Bottom: graph of $f'(x) = -\frac{1}{x^2}$

Notice that the graph of $f(x)$ does NOT look like the graph of $f'(x)$! (You might also notice that $f(x)$ is an odd function, while $f'(x)$ is an even function. The derivative of an odd function is always even, and vice versa.)

Pumpkin Drop, Part II
This time, someone throws a pumpkin over the tallest building on campus.

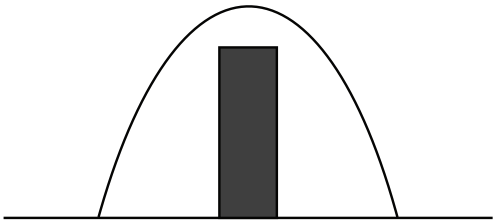

Figure 10: $y = 400 - 16t^2$, $-5 \le t \le 5$

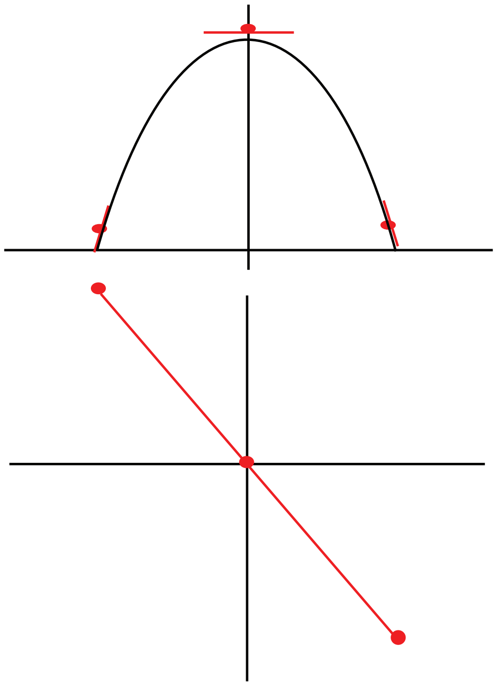

Figure 11: Top: graph of $y(t) = 400 - 16t^2$. Bottom: the derivative, $y'(t)$

Two Trig Limits

Note: In the expressions below, θ is in radians— NOT degrees!

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1; \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0}$$

Here is a geometric proof for the first limit:

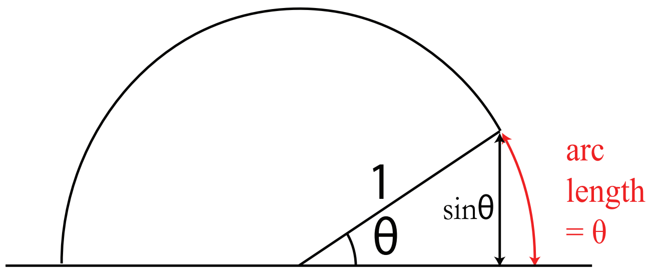

Figure 12: A circle of radius 1 with an arc of angle θ

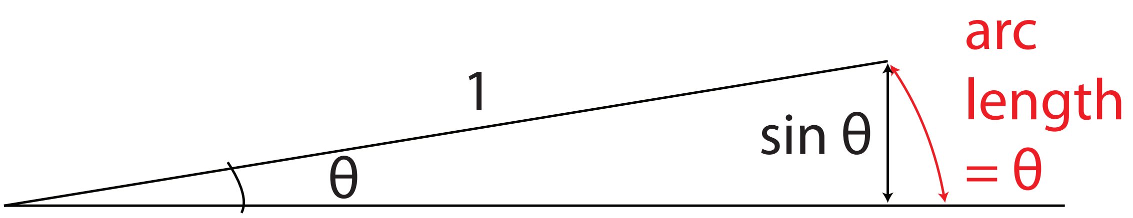

Figure 13: The sector in Fig. 12 as θ becomes very small

Imagine what happens to the picture as θ gets very small (see Fig. 13). As $\theta \to 0$, we see that $\dfrac{\sin\theta}{\theta} \to 1$.

What about the second limit involving cosine?

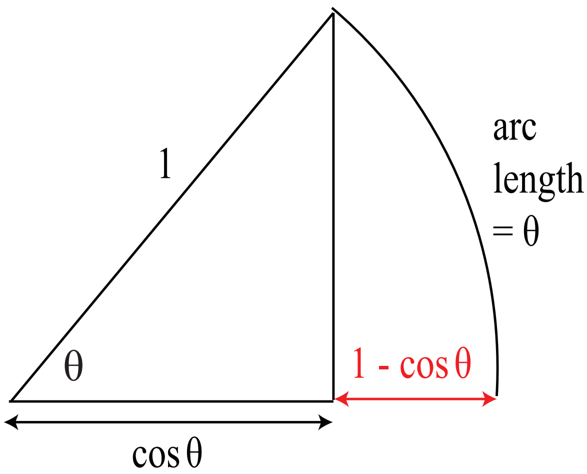

Figure 14: Same picture as Fig. 12 except that the horizontal distance between the edge of the triangle and the perimeter of the circle is marked

From Fig. 15 we can see that as $\theta \to 0$, the length $1 - \cos\theta$ of the short segment gets much smaller than the vertical distance θ along the arc. Hence, $\dfrac{1 - \cos\theta}{\theta} \to 0$.

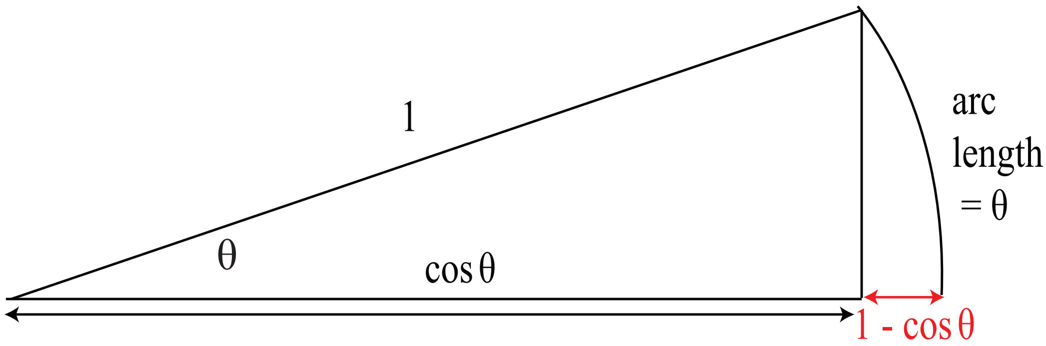

Figure 15: The sector in Fig. 14 as θ becomes very small

We end this lecture with a theorem that will help us to compute more derivatives next time.

Theorem: Differentiable Implies Continuous.
If f is differentiable at x_0, then f is continuous at x_0.

Proof: $\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right](x-x_0) = f'(x_0)\cdot 0 = 0.$

Remember: you can never divide by zero! The first step was to multiply by $\frac{x-x_0}{x-x_0}$. It looks as if this is illegal because when $x = x_0$, we are multiplying by $\frac{0}{0}$. But when computing the limit as $x \to x_0$ we always assume $x \neq x_0$. In other words $x - x_0 \neq 0$. So the proof is valid.

Lecture 3
Derivatives of Products, Quotients, Sine, and Cosine

Derivative Formulas

There are two kinds of derivative formulas:

1. Specific Examples: $\frac{d}{dx}x^n$ or $\frac{d}{dx}\left(\frac{1}{x}\right)$
2. General Examples: $(u+v)' = u' + v'$ and $(cu) = cu'$ (where c is a constant)

A notational convention we will use today is:

$$(u+v)(x) = u(x) + v(x); \quad uv(x) = u(x)v(x)$$

Proof of $(u+v) = u' + v'$. (General)

Start by using the definition of the derivative.

$$\begin{aligned}
(u+v)'(x) &= \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\} \\
(u+v)'(x) &= u'(x) + v'(x)
\end{aligned}$$

Follow the same procedure to prove that $(cu)' = cu'$.

Derivatives of $\sin x$ and $\cos x$. (Specific)

Last time, we computed

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= 1 \\
\frac{d}{dx}(\sin x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\sin(0+\Delta x) - \sin(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \\
\frac{d}{dx}(\cos x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\cos(0+\Delta x) - \cos(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0
\end{aligned}$$

So, we know the value of $\frac{d}{dx}\sin x$ and of $\frac{d}{dx}\cos x$ at $x = 0$. Let us find these for arbitrary x.

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Recall:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

So,

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \lim_{\Delta x \to 0} \cos x \left(\frac{\sin \Delta x}{\Delta x} \right)
\end{aligned}$$

Since $\dfrac{\cos \Delta x - 1}{\Delta x} \to 0$ and that $\dfrac{\sin \Delta x}{\Delta x} \to 1$, the equation above simplifies to

$$\frac{d}{dx}\sin x = \cos x$$

A similar calculation gives

$$\frac{d}{dx}\cos x = -\sin x$$

Product formula (General)

$$(uv)' = u'v + uv'$$

Proof:

$$(uv)' = \lim_{\Delta x \to 0} \frac{(uv)(x+\Delta x) - (uv)(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

Now obviously,

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

so adding that to the numerator won't change anything.

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We can re-arrange that expression to get

$$(uv)' = \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x) - u(x)}{\Delta x} \right) v(x) + u(x+\Delta x) \left(\frac{v(x+\Delta x) - v(x)}{\Delta x} \right)$$

Remember, the limit of a sum is the sum of the limits.

$$\left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \right] v(x) + \lim_{\Delta x \to 0} \left(u(x+\Delta x) \left[\frac{v(x+\Delta x) - v(x)}{\Delta x} \right] \right)$$

$$(uv)' = u'(x)v(x) + u(x)v'(x)$$

Note: we also used the fact that

$$\lim_{\Delta x \to 0} u(x+\Delta x) = u(x) \qquad \text{(true because } u \text{ is continuous)}$$

This proof of the product rule assumes that u and v have derivatives, which implies both functions are continuous.

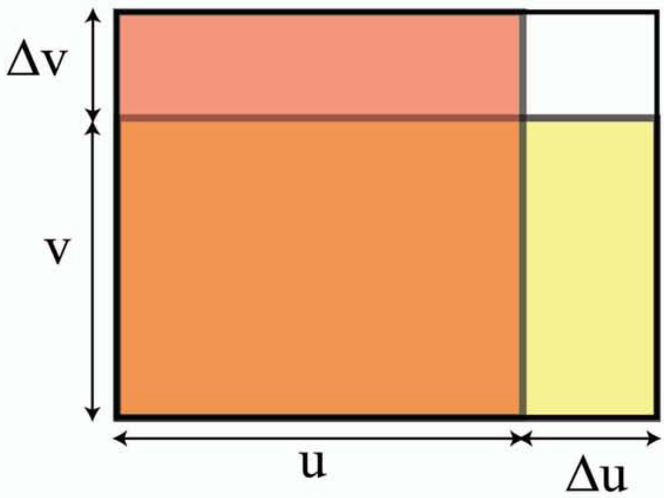

Figure 1: A graphical "proof" of the product rule

An intuitive justification:

We want to find the difference in area between the large rectangle and the smaller, inner rectangle. The inner (orange) rectangle has area uv. Define Δu, the change in u, by

$$\Delta u = u(x + \Delta x) - u(x)$$

We also abbreviate $u = u(x)$, so that $u(x + \Delta x) = u + \Delta u$, and, similarly, $v(x + \Delta x) = v + \Delta v$. Therefore the area of the largest rectangle is $(u + \Delta u)(v + \Delta v)$.

If you let v increase and keep u constant, you add the area shaded in red. If you let u increase and keep v constant, you add the area shaded in yellow. The sum of areas of the red and yellow rectangles is:

$$[u(v + \Delta v) - uv] + [v(u + \Delta u) - uv] = u\Delta v + v\Delta u$$

If Δu and Δv are small, then $(\Delta u)(\Delta v) \approx 0$, that is, the area of the white rectangle is very small. Therefore the difference in area between the largest rectangle and the orange rectangle is approximately the same as the sum of areas of the red and yellow rectangles. Thus we have:

$$[(u + \Delta u)(v + \Delta v) - uv] \approx u\Delta v + v\Delta u$$

(Divide by Δx and let $\Delta x \to 0$ to finish the argument.)

Quotient formula (General)

To calculate the derivative of u/v, we use the notations Δu and Δv above. Thus,

$$\begin{aligned}
\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)} &= \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} \\
&= \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)v} \quad \text{(common denominator)} \\
&= \frac{(\Delta u)v - u(\Delta v)}{(v+\Delta v)v} \quad \text{(cancel } uv - uv\text{)}
\end{aligned}$$

Hence,

$$\frac{1}{\Delta x}\left(\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right) = \frac{(\frac{\Delta u}{\Delta x})v - u(\frac{\Delta v}{\Delta x})}{(v+\Delta v)v} \longrightarrow \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2} \qquad \text{as } \Delta x \to 0$$

Therefore,

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

.

Lecture 4
Chain Rule, and Higher Derivatives

Chain Rule

We've got general procedures for differentiating expressions with addition, subtraction, and multiplication. What about composition?

Example 1. $y = f(x) = \sin x, x = g(t) = t^2$.
So, $y = f(g(t)) = \sin(t^2)$. To find $\frac{dy}{dt}$, write

$t_0 = t_0$	$t = t_0 + \Delta t$
$x_0 = g(t_0)$	$x = x_0 + \Delta x$
$y_0 = f(x_0)$	$y = y_0 + \Delta y$

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

As $\Delta t \to 0$, $\Delta x \to 0$ too, because of continuity. So we get:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \leftarrow \textbf{The Chain Rule!}$$

In the example, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dx} = \cos x$.

$$\text{So,} \quad \frac{d}{dt}\left(\sin(t^2)\right) = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = (\cos x)(2t) = (2t)\left(\cos(t^2)\right)$$

Another notation for the chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t))g'(t) \qquad \left(\text{or} \quad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)\right)$$

Example 1. (continued) Composition of functions $f(x) = \sin x$ and $g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = \sin(x^2)$$
$$(g \circ f)(x) = g(f(x)) = \sin^2(x)$$

Note: $f \circ g \neq g \circ f$. *Not Commutative!*

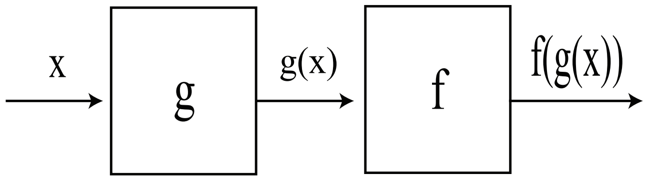

Figure 1: Composition of functions: $f \circ g(x) = f(g(x))$

Example 2. $\dfrac{d}{dx}\cos\left(\dfrac{1}{x}\right) = ?$

Let $u = \dfrac{1}{x}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
\frac{dy}{du} &= -\sin(u); \qquad \frac{du}{dx} = -\frac{1}{x^2} \\
\frac{dy}{dx} &= \frac{\sin(u)}{x^2} = (-\sin u)\left(\frac{-1}{x^2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2}
\end{aligned}$$

Example 3. $\dfrac{d}{dx}\left(x^{-n}\right) = ?$

There are two ways to proceed. $x^{-n} = \left(\dfrac{1}{x}\right)^n$, or $x^{-n} = \dfrac{1}{x^n}$

1. $\dfrac{d}{dx}\left(x^{-n}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x}\right)^n = n\left(\dfrac{1}{x}\right)^{n-1}\left(\dfrac{-1}{x^2}\right) = -nx^{-(n-1)}x^{-2} = -nx^{-n-1}$
2. $\dfrac{d}{dx}\left(x^{-n}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = nx^{n-1}\left(\dfrac{-1}{x^{2n}}\right) = -nx^{-n-1}$ (Think of x^n as u)

Higher Derivatives

Higher derivatives are derivatives of derivatives. For instance, if $g = f'$, then $h = g'$ is the second derivative of f. We write $h = (f')' = f''$.

Notations

$f'(x)$	Df	$\frac{df}{dx}$
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$
$f^{(n)}(x)$	$D^n f$	$\frac{d^n f}{dx^n}$

Higher derivatives are pretty straightforward —- just keep taking the derivative!

Example. $D^n x^n = ?$
Start small and look for a pattern.

$$\begin{aligned}
Dx &= 1 \\
D^2x^2 &= D(2x) = 2 \quad (= 1 \cdot 2) \\
D^3x^3 &= D^2(3x^2) = D(6x) = 6 \quad (= 1 \cdot 2 \cdot 3) \\
D^4x^4 &= D^3(4x^3) = D^2(12x^2) = D(24x) = 24 \quad (= 1 \cdot 2 \cdot 3 \cdot 4) \\
D^n x^n &= n! \leftarrow \text{we guess, based on the pattern we're seeing here.}
\end{aligned}$$

The notation $n!$ is called "n factorial" and defined by $n! = n(n-1)\cdots 2 \cdot 1$

Proof by Induction: We've already checked the base case ($n = 1$).

Induction step: Suppose we know $D^n x^n = n!$ (n^{th} case). Show it holds for the $(n+1)^{st}$ case.

$$\begin{aligned}
D^{n+1}x^{n+1} &= D^n\left(Dx^{n+1}\right) = D^n\left((n+1)x^n\right) = (n+1)D^n x^n = (n+1)(n!) \\
D^{n+1}x^{n+1} &= (n+1)!
\end{aligned}$$

Proved!

Lecture 11: Max/Min Problems

Example 1. $y = \dfrac{\ln x}{x}$ (same function as in last lecture)

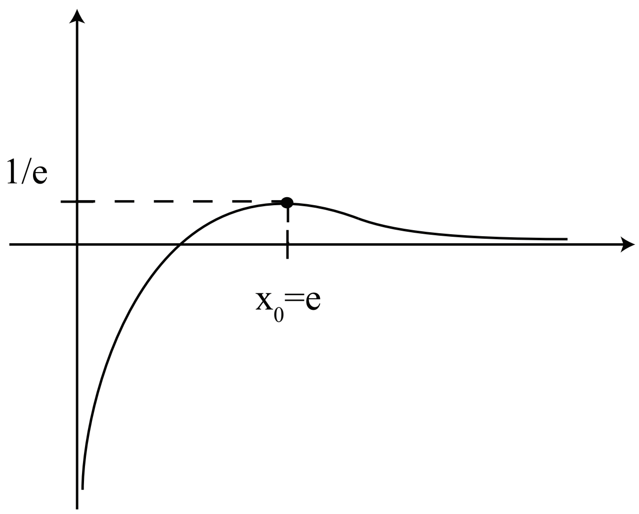

Figure 1: Graph of $y = \dfrac{\ln x}{x}$.

- What is the maximum value? Answer: $y = \dfrac{1}{e}$.
- Where (or at what point) is the maximum achieved? Answer: $x = e$. (See Fig. 1).)

Beware: Some people will ask "What is the maximum?". The answer is *not* e. You will get so used to finding the critical point $x = e$, the main calculus step, that you will forget to find the maximum value $y = \dfrac{1}{e}$. Both the critical point $x = e$ and critical value $y = \dfrac{1}{e}$ are important. Together, they form the point of the graph $(e, \dfrac{1}{e})$ where it turns around.

Example 2. Find the max and the min of the function in Fig. 2

Answer: If you've already graphed the function, it's obvious where the maximum and minimum values are. The point is to find the maximum and minimum without sketching the whole graph.

Idea: Look for the max and min among the critical points and endpoints.You can see from Fig. 2 that we only need to compare the heights or y-values corresponding to endpoints and critical points. (Watch out for discontinuities!)

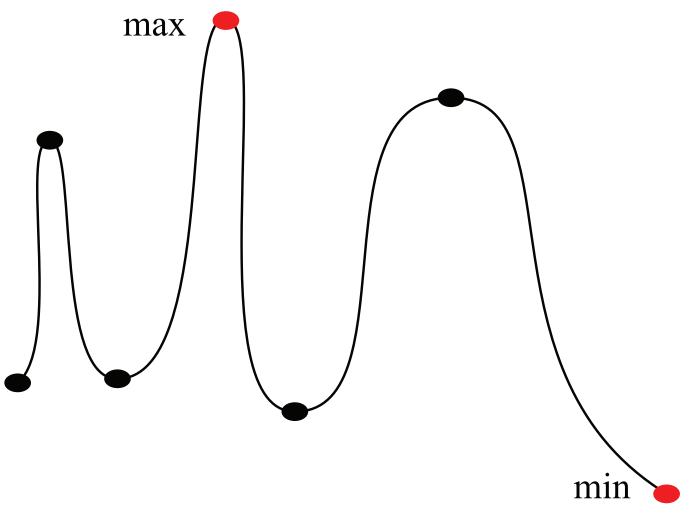

Figure 2: Search for max and min among critical points and endpoints

Example 3. Find the open-topped can with the least surface area enclosing a fixed volume, V.

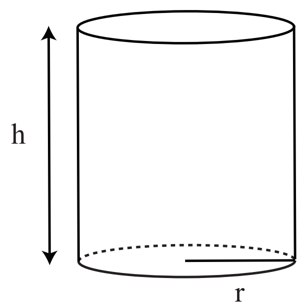

Figure 3: Open-topped can.

1. Draw the picture.
2. Figure out what variables to use. (In this case, r, h, V and surface area, S.)
3. Figure out what the constraints are in the problem, and express them using a formula. In this example, the constraint is

$$V = \pi r^2 h = \text{constant}$$

   We're also looking for the surface area. So we need the formula for that, too:

$$S = \pi r^2 + (2\pi r)h$$

   Now, in symbols, the problem is to minimize S with V constant.

4. Use the constraint equation to express everything in terms of r (and the constant V).

$$h = \frac{V}{2\pi r}; \quad S = \pi r^2 + (2\pi r)\left(\frac{V}{\pi r^2}\right)$$

5. Find the critical points (solve $dS/dr = 0$), as well as the endpoints. S will achieve its max and min at one of these places.

$$\frac{dS}{dr} = 2\pi r - \frac{2V}{r^2} = 0 \implies \pi r^3 - V = 0 \implies r^3 = \frac{V}{\pi} \implies r = \left(\frac{V}{\pi}\right)^{1/3}$$

We're not done yet. We've still got to evaluate S at the endpoints: $r = 0$ and "$r = \infty$".

$$S = \pi r^2 + \frac{2V}{r}, \quad 0 \leq r < \infty$$

As $r \to 0$, the second term, $\frac{2}{r}$, goes to infinity, so $S \to \infty$. As $r \to \infty$, the first term πr^2 goes to infinity, so $S \to \infty$. Since $S = +\infty$ at each end, the minimum is achieved at the critical point $r = (V/\pi)^{1/3}$, not at either endpoint.

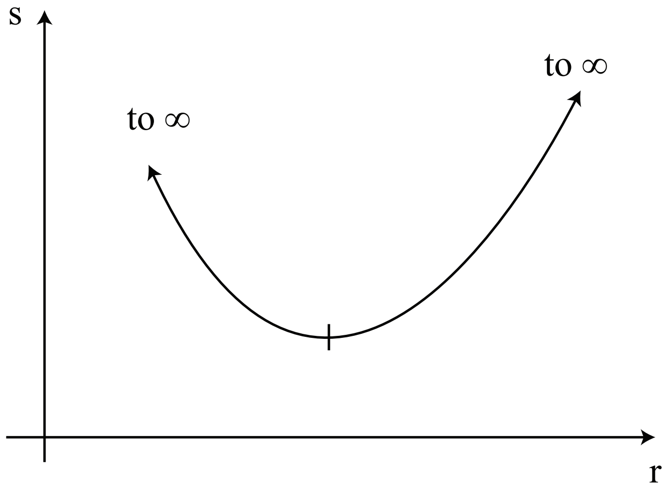

Figure 4: Graph of S

We're <u>still</u> not done. We want to find the minimum value of the surface area, S, and the values of h.

$$r = \left(\frac{V}{\pi}\right)^{1/3}; \quad h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi}\left(\frac{V}{\pi}\right)^{-2/3} = \left(\frac{V}{\pi}\right)^{1/3}$$

$$S = \pi r^2 + 2\frac{V}{r} = \pi\left(\frac{V}{\pi}\right)^{2/3} + 2V\left(\frac{V}{\pi}\right)^{1/3} = 3\pi^{-1/3}V^{2/3}$$

Finally, another, often better, way of answering that question is to find the proportions of the can. In other words, what is $\frac{h}{r}$? Answer: $\frac{h}{r} = \frac{(V/\pi)^{1/3}}{(V/\pi)^{1/3}} = 1$.

Example 4. Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

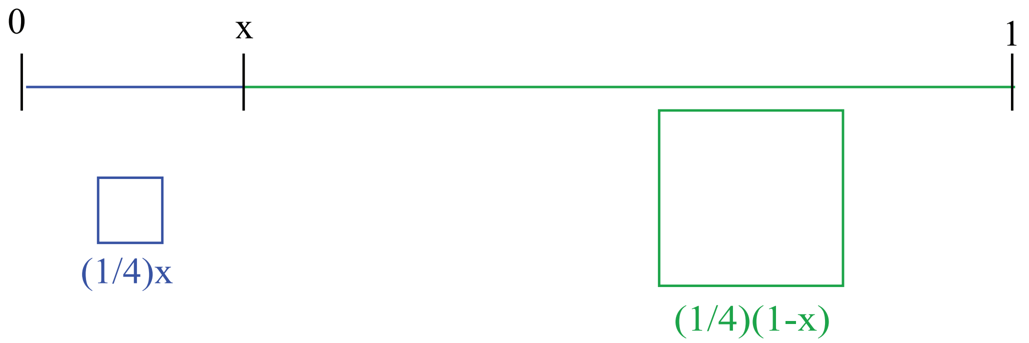

Figure 5: Illustration for Example 5.

The first square will have sides of length $\frac{x}{4}$. Its area will be $\frac{x^2}{16}$. The second square will have sides of length $\frac{1-x}{4}$. Its area will be $\left(\frac{1-x}{4}\right)^2$. The total area is then

$$
\begin{aligned}
A &= \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\
A' &= \frac{2x}{16} + \frac{2(1-x)}{16}(-1) = \frac{x}{8} - \frac{1}{8} + \frac{x}{8} = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2}
\end{aligned}
$$

So, one extreme value of the area is

$$A = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}$$

We're not done yet, though. We still need to check the endpoints! At $x = 0$,

$$A = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}$$

At $x = 1$,

$$A = \left(\frac{1}{4}\right)^2 + 0^2 = \frac{1}{16}$$

By checking the endpoints in Fig. 6, we see that the *minimum* area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or 1. The maximum corresponds to using the whole length of wire for one square.

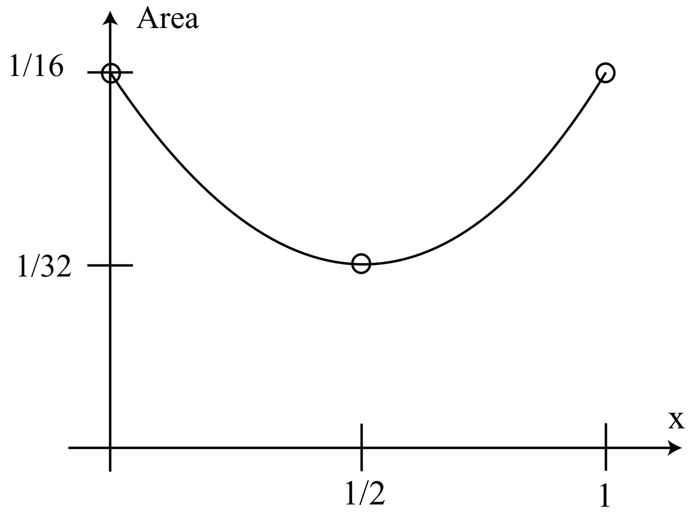

Figure 6: Graph of the area function.

Moral: Don't forget endpoints. If you only look at critical points you may find the worst answer, rather than the best one.

Lecture 12: Related Rates

Example 1. Police are 30 feet from the side of the road. Their radar sees your car approaching at 80 feet per second when your car is 50 feet away from the radar gun. The speed limit is 65 miles per hour (which translates to 95 feet per second). Are you speeding?

First, draw a diagram of the setup (as in Fig. 1):

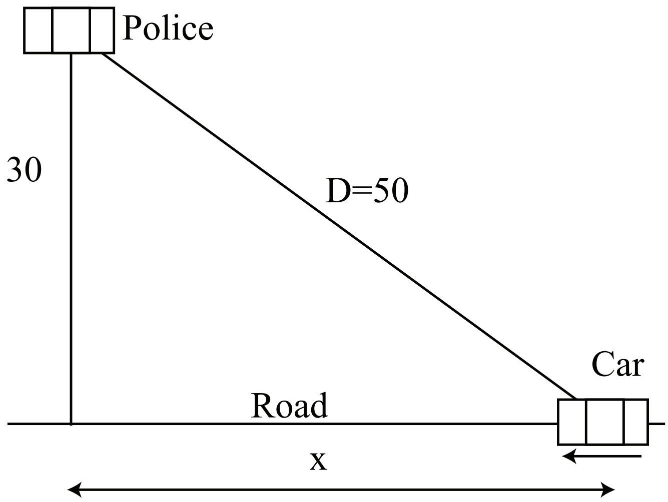

Figure 1: Illustration of example 1: triangle with the police, the car, the road, D and x labelled.

Next, give the variables names. The important thing to figure out is which variables are changing.

At $D = 50$, $x = 40$. (We know this because it's a 3-4-5 right triangle.) In addition, $\frac{dD}{dt} = D' = -80$. D' is negative because the car is moving in the $-x$ direction. Don't plug in the value for D yet! D is changing, and it depends on x.

The Pythagorean theorem says

$$30^2 + x^2 = D^2$$

Differentiate this equation with respect to time (implicit differentiation:

$$\frac{d}{dt}\left(30^2 + x^2 = D^2\right) \implies 2xx' = 2DD' \implies x' = \frac{2DD'}{2x}$$

Now, plug in the instantaneous numerical values:

$$x' = \frac{50}{40}(-80) = -100\frac{\text{feet}}{\text{s}}$$

This exceeds the speed limit of 95 feet per second; you are, in fact, speeding.

There is another, longer, way of solving this problem. Start with

$$D = \sqrt{30^2 + x^2} = (30^2 + x^2)^{1/2}$$

$$\frac{d}{dt}D = \frac{1}{2}(30^2 + x^2)^{-1/2}(2x\frac{dx}{dt})$$

Plug in the values:

$$-80 = \frac{1}{2}(30^2 + 40^2)^{-1/2}(2)(40)\frac{dx}{dt}$$

and solve to find

$$\frac{dx}{dt} = -100\frac{\text{feet}}{\text{s}}$$

(A third strategy is to differentiate $x = \sqrt{D^2 - 30^2}$). It is easiest to differentiate the equation in its simplest algebraic form $30^2 + x^2 = D^2$, our first approach.

The general strategy for these types of problems is:

1. Draw a picture. Set up variables and equations.
2. Take derivatives.
3. Plug in the given values. Don't plug the values in until *after* taking the derivatives.

Example 2. Consider a conical tank. Its radius at the top is 4 feet, and it's 10 feet high. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is 5 feet high?

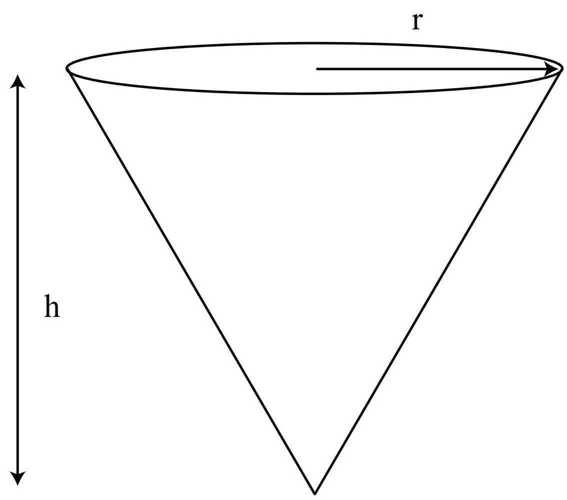

Figure 2: Illustration of example 2: inverted cone water tank.

From Fig. 2), the volume of the tank is given by

$$V = \frac{1}{3}\pi r^2 h$$

The key here is to draw the two-dimensional cross-section. We use the letters r and h to represent the variable radius and height of the water at any level. We can find the relationship between r and h from Fig. 3) using similar triangles.

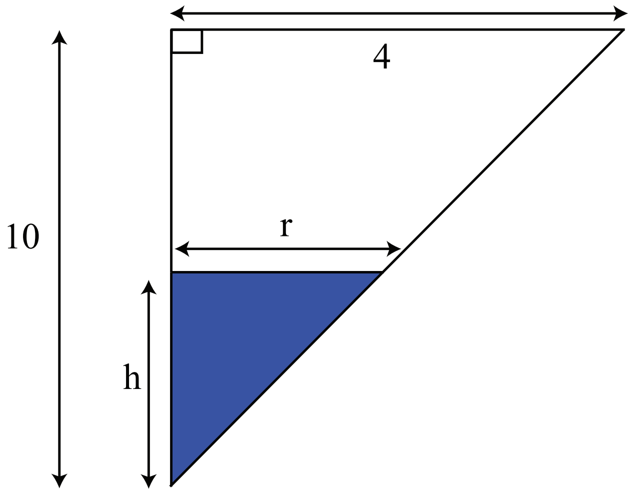

Figure 3: Relating r and h.

From Fig. 3), we see that

$$\frac{r}{h} = \frac{4}{10}$$

or, in other words,

$$r = \frac{2}{5}h$$

Plug this expression for r back into V to get

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{3(25)}\pi h^3$$

$$\frac{dV}{dt} = V' = \frac{4}{25}\pi h^2 h'$$

Now, plug in the numbers ($\frac{dV}{dt} = 2$, $h = 5$):

$$2 = \left(\frac{4}{25}\right)\pi(5)^2 h'$$

$$h' = \frac{1}{2\pi}$$

Related rates also arise on Problem Set 3 (Fig. 4). There's a part II margin of error problem involving a satellite, where you're asked to find $\dfrac{\Delta L}{\Delta h}$.

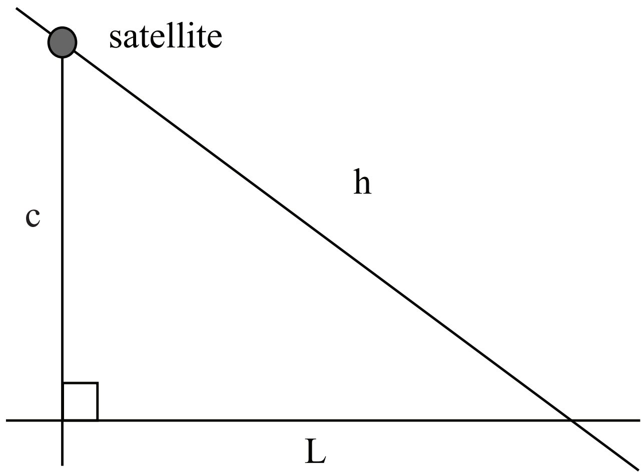

Figure 4: Illustration of the satellite problem.

$$\begin{aligned} L^2 + c^2 &= h^2 \\ 2LL' &= 2hh' \\ \text{Hence,}\quad \frac{\Delta L}{\Delta h} \approx \frac{L'}{h'} &= \frac{h}{L} \end{aligned}$$

There is also a parabolic mirror problem based on similar ideas (Fig. 5).

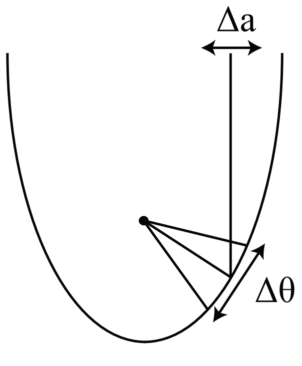

Figure 5: Illustration of the parabolic mirror problem.

Here, you want to find either $\frac{\Delta a}{\Delta \theta}$ or $\frac{\Delta \theta}{\Delta a}$. This type of sensitivity of measurement problem matters in every measurement problem, for instance predicting whether asteroids will hit Earth.

Lecture 14: Mean Value Theorem and Inequalities

Mean-Value Theorem

The Mean-Value Theorem (MVT) is the underpinning of calculus. It says:

> If f is differentiable on $a < x < b$, and continuous on $a \le x \le b$, then
> $$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{(for some } c,\ a < c < b)$$

Here, $\dfrac{f(b) - f(a)}{b - a}$ is the slope of a secant line, while $f'(c)$ is the slope of a tangent line.

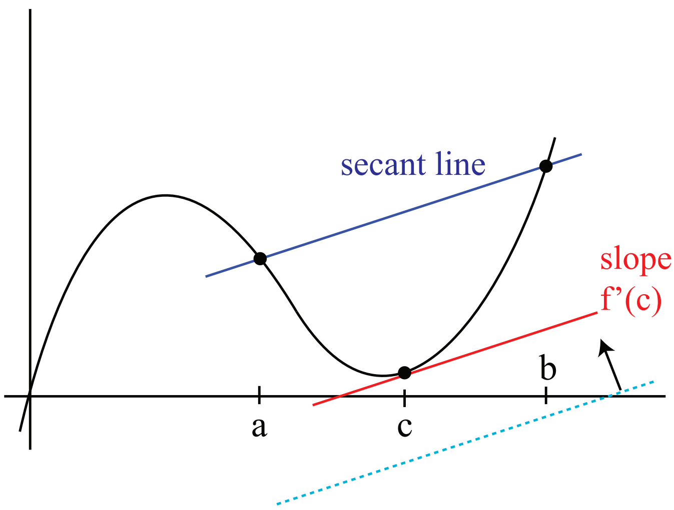

Figure 1: Illustration of the Mean Value Theorem.

Geometric Proof: Take (dotted) lines parallel to the secant line, as in Fig. 1 and shift them up from below the graph until one of them first touches the graph. Alternatively, one may have to start with a dotted line above the graph and move it down until it touches.

If the function isn't differentiable, this approach goes wrong. For instance, it breaks down for the function $f(x) = |x|$. The dotted line always touches the graph first at $x = 0$, no matter what its slope is, and $f'(0)$ is undefined (see Fig. 2).

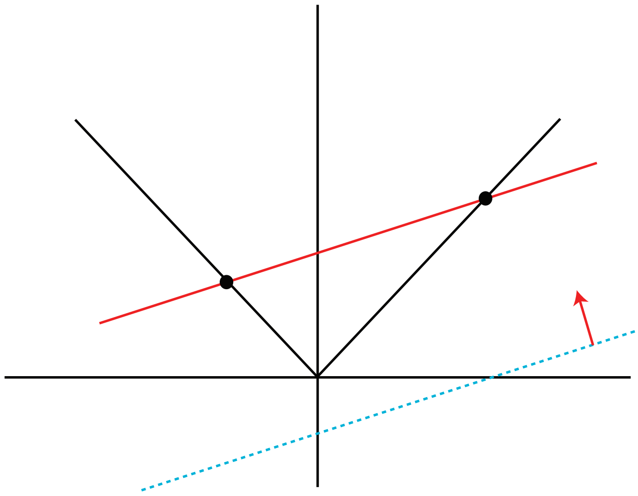

Figure 2: Graph of $y = |x|$, with secant line. (MVT goes wrong.)

Interpretation of the Mean Value Theorem

You travel from Boston to Chicago (which we'll assume is a 1,000 mile trip) in exactly 3 hours. At some time in between the two cities, you must have been going at exactly $\frac{1000}{3}$mph.

$f(t) =$ position, measured as the distance from Boston.

$$f(3) = 1000, \quad f(0) = 0, \quad a = 0, \text{ and b} = 3.$$

$$\frac{1000}{3} = \frac{f(b) - f(a)}{3} = f'(c)$$

where $f'(c)$ is your speed at some time, c.

Versions of the Mean Value Theorem

There is a second way of writing the MVT:

$$\begin{aligned} f(b) - f(a) &= f'(c)(b - a) \\ f(b) &= f(a) + f'(c)(b - a) \quad \text{(for some } c, a < c < b) \end{aligned}$$

There is also a third way of writing the MVT: change the name of b to x.

$$\boxed{f(x) = f(a) + f'(c)(x - a) \quad \text{for some } c, a < c < x}$$

The theorem does not say what c is. It depends on f, a, and x.

This version of the MVT should be compared with linear approximation (see Fig. 3).

$$f(x) \approx f(a) + f'(a)(x - a) \quad x \text{ near } a$$

The tangent line in the linear approximation has a definite slope $f'(a)$. by contrast formula is an exact formula. It conceals its lack of specificity in the slope $f'(c)$, which could be the slope of f at any point between a and x.

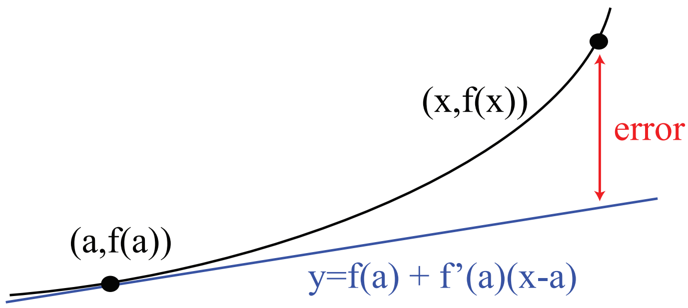

Figure 3: MVT vs. Linear Approximation.

Uses of the Mean Value Theorem.

Key conclusions: (The conclusions from the MVT are theoretical)

1. If $f'(x) > 0$, then f is increasing.
2. If $f'(x) < 0$, then f is decreasing.
3. If $f'(x) = 0$ all x, then f is constant.

Definition of increasing/decreasing:
Increasing means $a < b \Rightarrow f(a) < f(b)$. Decreasing means $a < b \implies f(a) < f(b)$.

Proofs:
Proof of 1:

$$\begin{aligned} a &< b \\ f(b) &= f(a) + f'(c)(b-a) \end{aligned}$$

Because $f'(c)$ and $(b-a)$ are both positive,

$$f(b) = f(a) + f'(c)(b-a) > f(a)$$

(The proof of 2 is omitted because it is similar to the proof of 1)

Proof of 3:

$$f(b) = f(a) + f'(c)(b-a) = f(a) + 0(b-a) = f(a)$$

Conclusions 1,2, and 3 seem obvious, but let me persuade you that they are not. Think back to the definition of the derivative. It involves infinitesimals. It's not a sure thing that these infinitesimals have anything to do with the non-infinitesimal behavior of the function.

Inequalities

The fundamental property $f' > 0 \implies f$ is increasing can be used to deduce many other inequalities.

Example. e^x

1. $e^x > 0$
2. $e^x > 1$ for $x > 0$
3. $e^x > 1 + x$

Proofs. We will take property 1 ($e^x > 0$) for granted. Proofs of the other two properties follow:

Proof of 2: Define $f_1(x) = e^x - 1$. Then, $f_1(0) = e^0 - 1 = 0$, and $f_1'(x) = e^x > 0$. (This last assertion is from step 1). Hence, $f_1(x)$ is increasing, so $f(x) > f(0)$ for $x > 0$. That is:

$$e^x > 1 \text{ for } x > 0$$

.

Proof of 3: Let $f_2(x) = e^x - (1 + x)$.

$$f_2'(x) = e^x - 1 = f_1(x) > 0 \quad (\text{if } x > 0).$$

Hence, $f_2(x) > 0$ for $x > 0$. In other words,

$$e^x > 1 + x$$

Similarly, $e^x > 1 + x + \dfrac{x^2}{2}$ (proved using $f_3(x) = e^x - (1 + x + \dfrac{x^2}{2})$). One can keep on going: $e^x > 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!}$ for $x > 0$. Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad (\text{an infinite sum})$$

We will be discussing this when we get to Taylor series near the end of the course.

Lecture 15: Differentials and Antiderivatives

Differentials

New notation:

$$\boxed{dy = f'(x)dx} \quad (y = f(x))$$

Both dy and $f'(x)dx$ are called *differentials*. You can think of

$$\frac{dy}{dx} = f'(x)$$

as a quotient of differentials. One way this is used is for linear approximations.

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Example 1. Approximate $65^{1/3}$

Method 1 (review of linear approximation method)

$$\begin{aligned} f(x) &= x^{1/3} \\ f'(x) &= \frac{1}{3}x^{-2/3} \\ f(x) &\approx f(a) + f'(a)(x-a) \\ x^{1/3} &\approx a^{1/3} + \frac{1}{3}a^{-2/3}(x-a) \end{aligned}$$

A good base point is $a = 64$, because $64^{1/3} = 4$.

Let $x = 65$.

$$65^{1/3} = 64^{1/3} + \frac{1}{3}64^{-2/3}(65-64) = 4 + \frac{1}{3}\left(\frac{1}{16}\right)(1) = 4 + \frac{1}{48} \approx 4.02$$

Similarly,

$$(64.1)^{1/3} \approx 4 + \frac{1}{480}$$

Method 2 (review)

$$65^{1/3} = (64+1)^{1/3} = [64(1+\frac{1}{64})]^{1/3} = 64^{1/3}[1+\frac{1}{64}]^{1/3} = 4\left[1+\frac{1}{64}\right]^{1/3}$$

Next, use the approximation $(1+x)^r \approx 1 + rx$ with $r = \frac{1}{3}$ and $x = \frac{1}{64}$.

$$65^{1/3} \approx 4(1 + \frac{1}{3}(\frac{1}{64})) = 4 + \frac{1}{48}$$

This is the same result that we got from Method 1.

Method 3 (with differential notation)

$$\begin{aligned} y &= x^{1/3}|_{x=64} = 4 \\ dy &= \frac{1}{3}x^{-2/3}dx|_{x=64} = \frac{1}{3}\left(\frac{1}{16}\right)dx = \frac{1}{48}dx \end{aligned}$$

We want $dx = 1$, since $(x + dx) = 65$. $dy = \frac{1}{48}$ when $dx = 1$.

$$(65)^{1/3} = 4 + \frac{1}{48}$$

What underlies all three of these methods is

$$\begin{aligned} y &= x^{1/3} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-2/3}|_{x=64} \end{aligned}$$

Anti-derivatives

$F(x) = \int f(x)dx$ means that F is the antiderivative of f.

Other ways of saying this are:

$$F'(x) = f(x) \quad \text{or,} \quad dF = f(x)dx$$

Examples:

1. $\int \sin x dx = -\cos x + c$ where c is any constant.
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
3. $\int \frac{dx}{x} = \ln|x| + c$ (This takes care of the exceptional case $n = -1$ in 2.)
4. $\int \sec^2 x dx = \tan x + c$
5. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ (where $\sin^{-1} x$ denotes "inverse sin" or arcsin, and not $\frac{1}{\sin x}$)
6. $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$

Proof of Property 2: The absolute value $|x|$ gives the correct answer for both positive and negative x. We will double check this now for the case $x < 0$:

$$\begin{aligned} \ln|x| &= \ln(-x) \\ \frac{d}{dx}\ln(-x) &= \left(\frac{d}{du}\ln(u)\right)\frac{du}{dx} \quad \text{where } u = -x. \\ \frac{d}{dx}\ln(-x) &= \frac{1}{u}(-1) = \frac{1}{-x}(-1) = \frac{1}{x} \end{aligned}$$

Uniqueness of the antiderivative up to an additive constant.

If $F'(x) = f(x)$, and $G'(x) = f(x)$, then $G(x) = F(x) + c$ for some constant factor c.

Proof:

$$(G - F)' = f - f = 0$$

Recall that we proved as a corollary of the Mean Value Theorem that if a function has a derivative zero then it is constant. Hence $G(x) - F(x) = c$ (for some constant c). That is, $G(x) = F(x) + c$.

Method of substitution.

Example 1. $\int x^3(x^4+2)^5 dx$

Substitution:

$$u = x^4 + 2, \quad du = 4x^3 dx, \quad (x^4+2)^5 = u^5, \quad x^3 dx = \frac{1}{4} du$$

Hence,

$$\int x^3(x^4+2)^5 dx = \frac{1}{4}\int u^5 du = \frac{u^6}{4(6)} = \frac{u^6}{24} + c = \frac{1}{24}(x^4+2)^6 + c$$

Example 2. $\int \frac{x}{\sqrt{1+x^2}} dx$

Another way to find an anti-derivative is "advanced guessing." First write

$$\int \frac{x}{\sqrt{1+x^2}} dx = \int x(1+x^2)^{-1/2} dx$$

Guess: $(1+x^2)^{1/2}$. Check this.

$$\frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2}(2x) = x(1+x^2)^{-1/2}$$

Therefore,

$$\int x(1+x^2)^{-1/2} dx = (1+x^2)^{1/2} + c$$

Example 3. $\int e^{6x} dx$

Guess: e^{6x}. Check this:

$$\frac{d}{dx} e^{6x} = 6e^{6x}$$

Therefore,

$$\int e^{6x} dx = \frac{1}{6} e^{6x} + c$$

Example 4. $\int xe^{-x^2}dx$

Guess: e^{-x^2} Again, take the derivative to check:

$$\frac{d}{dx}e^{-x^2} = (-2x)(e^{-x^2})$$

Therefore,

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + c$$

Example 5. $\int \sin x \cos x dx = \frac{1}{2}\sin^2 x + c$

Another, equally acceptable answer is

$$\int \sin x \cos x dx = -\frac{1}{2}\cos^2 x + c$$

This seems like a contradiction, so let's check our answers:

$$\frac{d}{dx}\sin^2 x = (2\sin x)(\cos x)$$

and

$$\frac{d}{dx}\cos^2 x = (2\cos x)(-\sin x)$$

So both of these are correct. Here's how we resolve this apparent paradox: the difference between the two answers is a constant.

$$\frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2}\sin^2 x - \frac{1}{2} = \frac{1}{2}(\sin^2 x - 1) = \frac{1}{2}(-\cos^2 x) = -\frac{1}{2}\cos^2 x$$

The two answers are, in fact, equivalent. The constant c is shifted by $\frac{1}{2}$ from one answer to the other.

Example 6. $\int \frac{dx}{x\ln x}$ (We will assume $x > 0$.)

Let $u = \ln x$. This means $du = \frac{1}{x}dx$. Substitute these into the integral to get

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}du = \ln u + c = \ln(\ln(x)) + c$$

Lecture 18: Definite Integrals

Integrals are used to calculate cumulative totals, averages, areas.

Area under a curve: (See Figure 1.)

1. Divide region into rectangles
2. Add up area of rectangles
3. Take limit as rectangles become thin

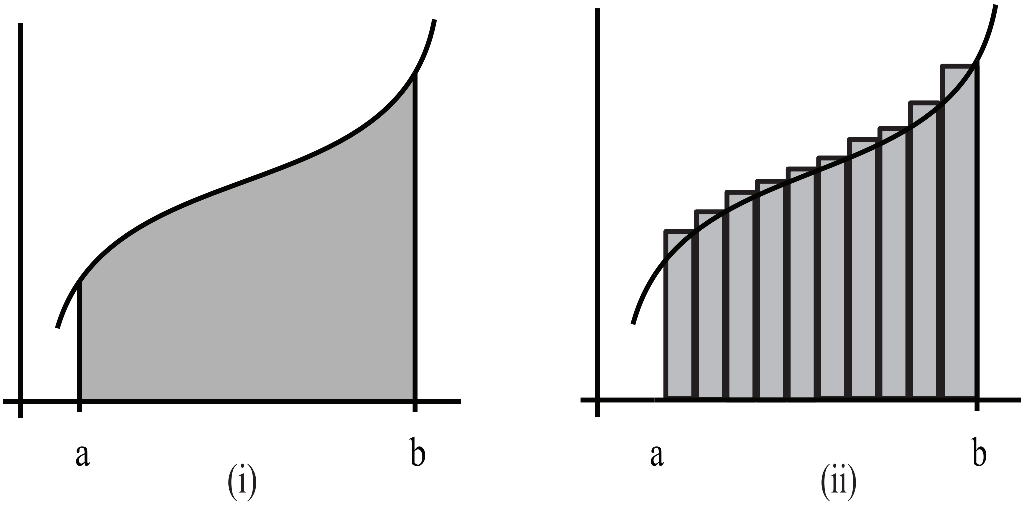

Figure 1: (i) Area under a curve; (ii) sum of areas under rectangles

Example 1. $f(x) = x^2$, $a = 0$, b arbitrary

1. Divide into n intervals
   Length $b/n =$ base of rectangle
2. Heights:
   - 1^{st}: $x = \dfrac{b}{n}$, height $= \left(\dfrac{b}{n}\right)^2$
   - 2^{nd}: $x = \dfrac{2b}{n}$, height $= \left(\dfrac{2b}{n}\right)^2$

Sum of areas of rectangles:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

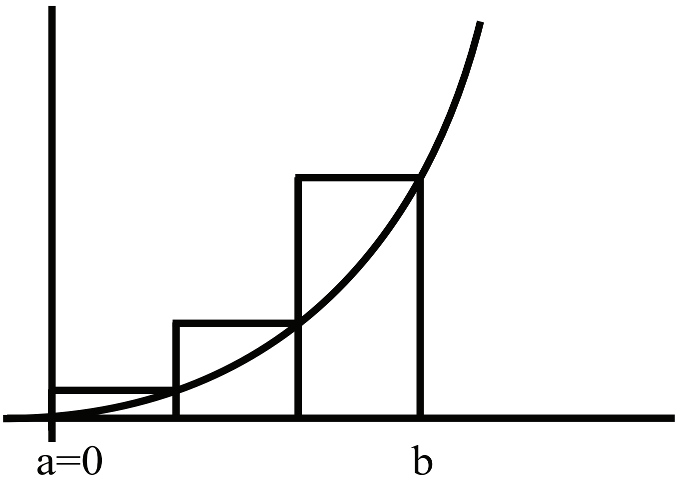

Figure 2: Area under $f(x) = x^2$ above $[0, b]$.

We will now estimate the sum using some 3-dimensional geometry.

Consider the staircase pyramid as pictured in Figure 3.

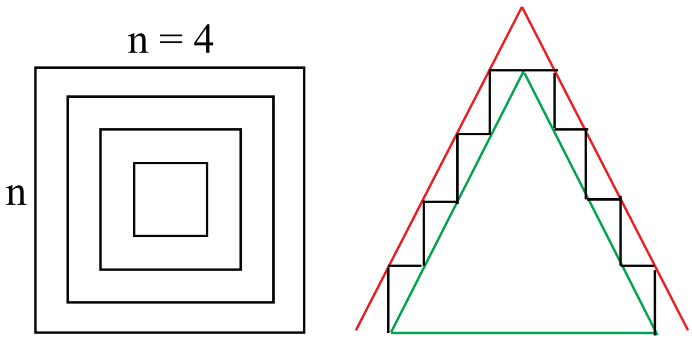

Figure 3: Staircase pyramid: left(top view) and right (side view)

1^{st} level: $n \times n$ bottom, represents volume n^2.
2^{nd} level: $(n-1) \times (n-1)$, represents volumne $(n-1)^2)$, etc.
Hence, the total volume of the staircase pyramid is $n^2 + (n-1)^2 + \cdots + 1$.

Next, the volume of the pyramid is greater than the volume of the inner prism:

$$1^2 + 2^2 + \cdots + n^2 > \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3}n^2 \cdot n = \frac{1}{3}n^3$$

and less than the volume of the outer prism:

$$1^2 + 2^2 + \cdots + n^2 < \frac{1}{3}(n+1)^2(n+1) = \frac{1}{3}(n+1)^3$$

In all,

$$\frac{1}{3} = \frac{\frac{1}{3}n^3}{n^3} < \frac{1^2 + 2^2 + \cdots + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3}$$

Therefore,

$$\lim_{n\to\infty} \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{1}{3}b^3,$$

and the area under x^2 from 0 to b is $\frac{b^3}{3}$.

Example 2. $f(x) = x$; area under x above $[0, b]$. Reasoning similar to Example 1, but easier, gives a sum of areas:

$$\frac{b^2}{n^2}(1 + 2 + 3 + \cdots + n) \to \frac{1}{2}b^2 \quad (\text{as } n \to \infty)$$

This is the area of the triangle in Figure 4.

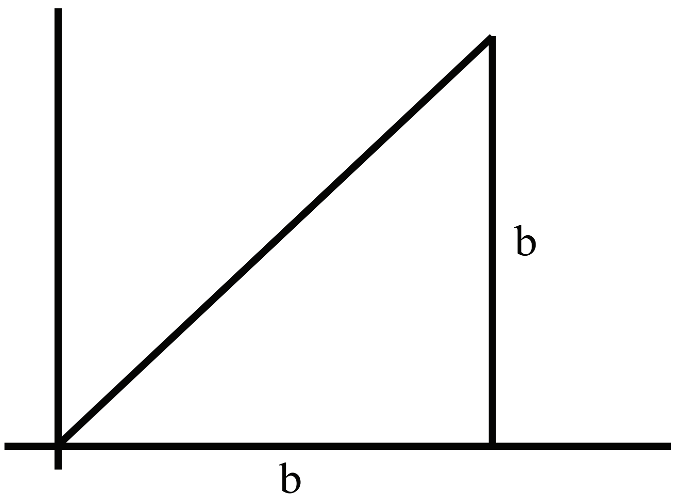

Figure 4: Area under $f(x) = x$ above $[0, b]$.

Pattern:

$$\frac{d}{db}\left(\frac{b^3}{3}\right) = b^2$$

$$\frac{d}{db}\left(\frac{b^2}{2}\right) = b$$

The area $A(b)$ under $f(x)$ should satisfy $A'(b) = f(b)$.

General Picture

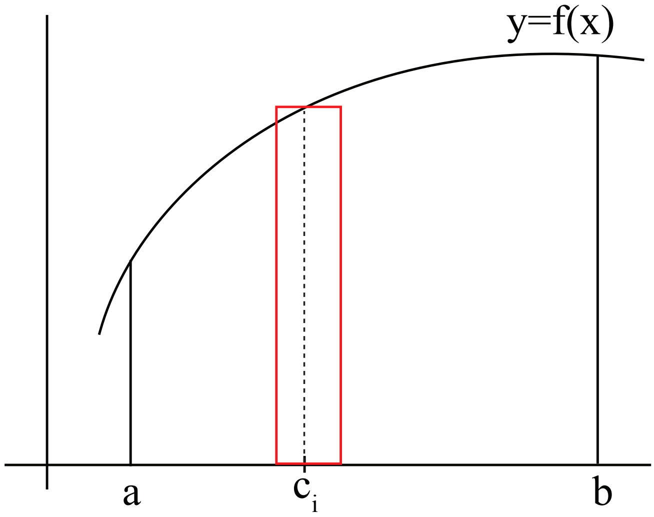

Figure 5: One rectangle from a Riemann Sum

- Divide into n equal pieces of length $= \Delta x = \dfrac{b-a}{n}$
- Pick any c_i in the interval; use $f(c_i)$ as the height of the rectangle
- Sum of areas: $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$

In summation notation: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x \quad \leftarrow$ called a *Riemann sum.*

Definition:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx \ \leftarrow \ \text{called a } \textit{definite integral}$$

This definite integral represents the area under the curve $y = f(x)$ above $[a, b]$.

Example 3. (Integrals applied to quantity besides area.) Student borrows from parents.
$P =$ principal in dollars, $t =$ time in years, $r =$ interest rate (e.g., 6 % is $r = 0.06$/year).
After time t, you owe $P(1 + rt) = P + Prt$

The integral can be used to represent the total amount borrowed as follows. Consider a function $f(t)$, the "borrowing function" in dollars per year. For instance, if you borrow \$ 1000 /month, then $f(t) = 12,000$/year. Allow f to vary over time.

Say $\Delta t = 1/12$ year $= 1$ month.

$$t_i = i/12 \quad i = 1, \cdots, 12.$$

$f(t_i)$ is the borrowing rate during the i^{th} month so the amount borrowed is $f(t_i)\Delta t$. The total is

$$\sum_{i=1}^{12} f(t_i)\Delta t.$$

In the limit as $\Delta t \to 0$, we have

$$\int_0^1 f(t)dt$$

which represents the total borrowed in one year in dollars per year.

The integral can also be used to represent the total amount owed. The amount owed depends on the interest rate. You owe

$$f(t_i)(1 + r(1 - t_i))\Delta t$$

for the amount borrowed at time t_i. The total owed for borrowing at the end of the year is

$$\int_0^1 f(t)(1 + r(1 - t))dt$$

Lecture 30: Integration by Parts, Reduction Formulae

Integration by Parts

Remember the product rule:

$$(uv)' = u'v + uv'$$

We can rewrite that as

$$uv' = (uv)' - u'v$$

Integrate this to get the formula for integration by parts:

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example 1. $\displaystyle\int \tan^{-1} x\,dx$.

At first, it's not clear how integration by parts helps. Write

$$\int \tan^{-1} x\,dx = \int \tan^{-1} x(1 \cdot dx) = \int uv'\,dx$$

with

$$u = \tan^{-1} x \quad \text{and} \quad v' = 1.$$

Therefore,

$$v = x \quad \text{and} \quad u' = \frac{1}{1+x^2}$$

Plug all of these into the formula for integration by parts to get:

$$\int \tan^{-1} x\,dx = \int uv'\,dx = (\tan^{-1} x)x - \int \frac{1}{1+x^2}(x)dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

Alternative Approach to Integration by Parts

As above, the product rule:

$$(uv)' = u'v + uv'$$

can be rewritten as

$$uv' = (uv)' - u'v$$

This time, let's take the *definite* integral:

$$\int_a^b uv'\,dx = \int_a^b (uv)'\,dx - \int_a^b u'v\,dx$$

By the fundamental theorem of calculus, we can say

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

Another notation in the indefinite case is

$$\int u\,dv = uv - \int v\,du$$

This is the same because

$$dv = v'\,dx \implies uv'\,dx = u\,dv \quad \text{and} \quad du = u'\,dx \implies u'v\,dx = vu'\,dx = v\,du$$

Example 2. $\displaystyle\int (\ln x)dx$

$$u = \ln x;\ du = \frac{1}{x}dx \quad \text{and} \quad dv = dx;\ v = x$$

$$\int (\ln x)dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$$

We can also use "advanced guessing" to solve this problem. We know that the derivative of *something* equals $\ln x$:

$$\frac{d}{dx}(??) = \ln x$$

Let's try

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

That's almost it, but not quite. Let's repair this guess to get:

$$\frac{d}{dx}\left(x\ln x - x\right) = \ln x + 1 - 1 = \ln x$$

Reduction Formulas (Recurrence Formulas)

Example 3. $\displaystyle\int (\ln x)^n\,dx$

Let's try:

$$u = (\ln x)^n \implies u' = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$v' = dx;\ v = x$$

Plugging these into the formula for integration by parts gives us:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1}\cancelto{1}{x\left(\frac{1}{x}\right)}dx$$

Keep repeating integration by parts to get the full formula: $n \to (n-1) \to (n-2) \to (n-3) \to$ etc

Example 4. $\displaystyle\int x^n e^x\,dx$ Let's try:

$$u = x^n \implies u' = nx^{n-1};\quad v' = e^x \implies v = e^x$$

Putting these into the integration by parts formula gives us:

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$

Repeat, going from $n \to (n-1) \to (n-2) \to$ etc.

Bad news: If you change the integrals just a little bit, they become impossible to evaluate:

$$\int \left(\tan^{-1} x\right)^2 dx = \text{impossible}$$

$$\int \frac{e^x}{x} \, dx = \text{also impossible}$$

Good news: When you can't evaluate an integral, then

$$\int_1^2 \frac{e^x}{x} dx$$

is an *answer*, not a question. This *is* the solution– you don't have to integrate it!

The most important thing is setting up the integral! (Once you've done that, you can always evaluate it numerically on a computer.) So, why bother to evaluate integrals by hand, then? Because you often get families of related integrals, such as

$$F(a) = \int_1^\infty \frac{e^x}{x^a} dx$$

where you want to find how the answer depends on, say, a.

Arc Length

This is very useful to know for 18.02 (multi-variable calculus).

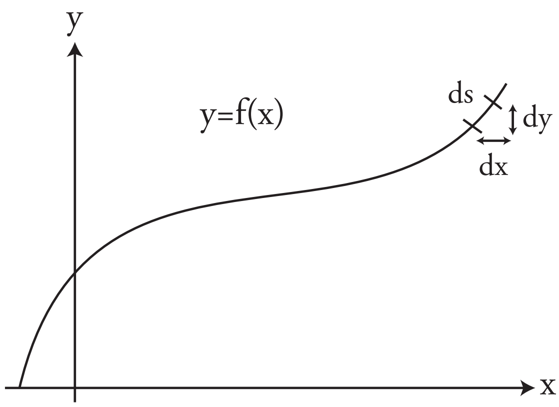

Figure 1: Infinitesimal Arc Length ds

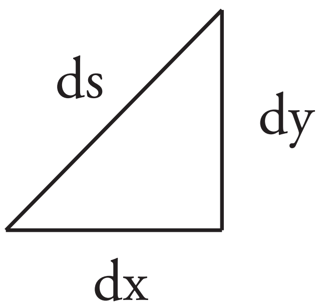

Figure 2: Zoom in on Figure 1 to see an approximate right triangle.

In Figures 1 and 2, s denotes arc length and ds = the infinitesmal of arc length.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}dx$$

Integrating with respect to ds finds the length of a curve between two points (see Figure 3). To find the length of the curve between P_0 and P_1, evaluate:

$$\int_{P_0}^{P_1} ds$$

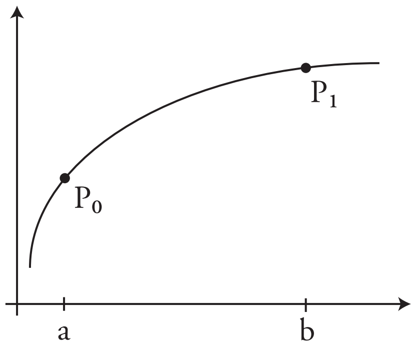

Figure 3: Find length of curve between P_0 and P_1.

We want to integrate with respect to x, not s, so we do the same algebra as above to find ds in terms of dx.

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore,

$$\int_{P_0}^{P_1} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 5: The Circle. $x^2 + y^2 = 1$ (see Figure 4).

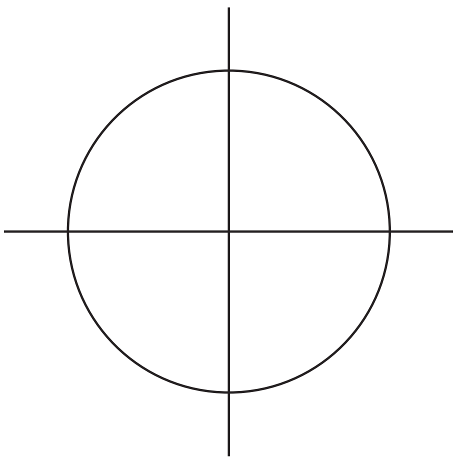

Figure 4: The circle in Example 1.

We want to find the length of the arc in Figure 5:

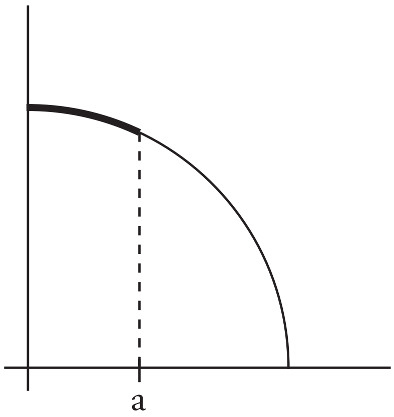

Figure 5: Arc length to be evaluated.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}dx$$

$$1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1+\frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$ds = \sqrt{\frac{1}{1-x^2}}dx$$

$$s = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^a = \sin^{-1} a - \sin^{-1} 0 = \sin^{-1} a$$

$$\sin s = a$$

This is illustrated in Figure 6.

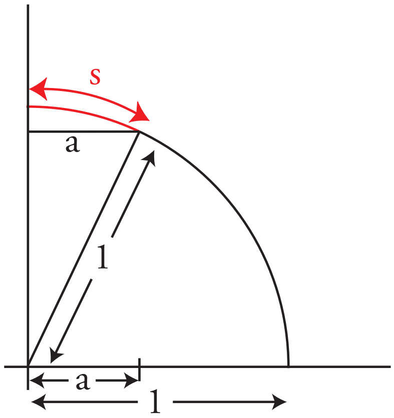

Figure 6: $s =$ angle in radians.

Parametric Equations

Example 6.

$$x = a\cos t$$

$$y = a\sin t$$

Ask yourself: what's constant? What's varying? Here, t is variable and a is constant.
Is there a relationship between x and y? Yes:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$

Extra information (besides the circle):
At $t = 0$,

$$x = a\cos 0 = a \quad \text{and} \quad y = a\sin 0 = 0$$

At $t = \frac{\pi}{2}$,

$$x = a\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y = a\sin\frac{\pi}{2} = a$$

Thus, for $0 \le t \le \pi/2$, a quarter circle is traced counter-clockwise (Figure 7).

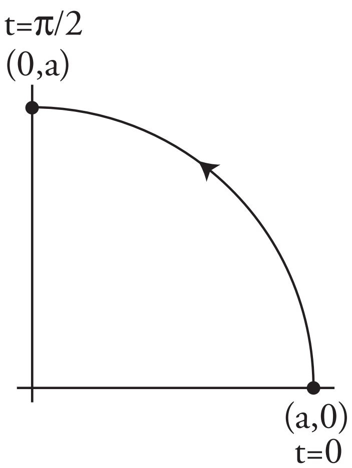

Figure 7: Example 6. $x = a\cos t,\ y = a\sin t;$ the particle is moving counterclockwise.

Example 7: The Ellipse See Figure 8.

$$x = 2\sin t; \quad y = \cos t$$

$$\frac{x^2}{4} + y^2 = 1 (\implies (2\sin t)^2/4 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1)$$

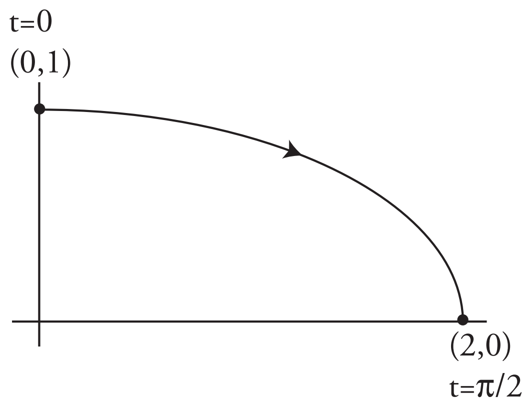

Figure 8: Ellipse: $x = 2\sin t,\ y = \cos t$ (traced clockwise).

Arclength ds for Example 6.

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\,dt)^2 + (a\cos t\,dt)^2} = \sqrt{(a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$$