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An Introduction to
q-Difference Equations

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To Flaschka, Keldisch and Gosta B

Contents

Introduction

Studies on q-difference equations appeared already at the beginning of the last century in intensive works especially by F H Jackson [31], R D Carmichael [21], T E Mason [45], C R Adams [5], W J Trjitzinsky [52] and other authors such us Poincare, Picard, Ramanunjan. Unfortunately, from the years thirty up to the beginning of the eighties, only nonsignificant interest in the area were observed. Since years eighties [29], an intensive and somewhat surprising interest in the subject reappeared in many areas of mathematics and applications including mainly new difference calculus and orthogonal polynomials, q-combinatorics, q-arithmetics, integrable systems and variational q-calculus. However, though the abundance of specialized scientific publications and a relative classicality of the subject, a lack of popularized publications in the form of books accessible to a big public including under and upper graduated students is very sensitive. This book is intended to participate to the bridging of this gap.

It is to be understood that the choice of approach to be followed in the book as well as that of material to be treated in most of chapters are mainly dictated by the center of interest of the author. However, in preparing the present text, our underlying motivation does'nt consists in any kind of specialization but in our wish of making available a most possibly coherent and self contained material, that should appear very useful for graduate students and beginning researchers in the area itself or in its applications.

The first fourth chapters are concerned in an introduction to q-difference equations while the subsequent chapters are concerned in applications to orthogonal polynomials and mathematical control theory respectively.

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Chapter 1

Elements of q-difference calculus

1.1 Introduction

Following [42, 44], mathematical analysis can be considered on special lattices:
-The constant $x(s) = cte$,
-The uniform $x(s) = s$,
-The q-uniform $x(s) = q^s$,
-The q-nonuniform $x(s) = (q^s + q^{-s})/2$, $s \in \mathbf{Z}$, $0 < |q| < 1$, the subjacent theory being founded on the corresponding divided difference derivative [41, 42]:

$$\mathcal{D}f(x(s)) = \frac{f(x(s+\frac{1}{2}))-f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2})-x(s-\frac{1}{2})}. \tag{1.1}$$

The basic property of this derivative is that it sends a polynomial of degree n to a polynomial of degree $n-1$. In this connection it seems to be the most general one having this vital characteristic. When $x(s)$ is given by the first three lattices, the corresponding divided derivative gives respectively

$$Df(x) = \frac{d}{dx}f(x) \tag{1.2}$$

$$\Delta_{\frac{1}{2}}f(x) = \Delta f(t) = f(t+1) - f(t) = (e^{\frac{d}{dt}} - 1)f(t); \;\; t = x - \tfrac{1}{2} \tag{1.3}$$

$$D_{q^{\frac{1}{2}}}f(x) = D_q f(t) = \frac{f(qt)-f(t)}{qt-t} = \frac{e^{(q-1)t\frac{d}{dt}}-1}{qt-t}f(t); \;\; t = q^{-\frac{1}{2}}x. \tag{1.4}$$

When $x(s)$ is given by the latest lattice, the corresponding derivative is usually referred to as the *Askey-Wilson* first order divided difference operator [7] that one can write:

$$\mathcal{D}f(x(z)) = \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)}, \ x(z) = \frac{z+z^{-1}}{2}, \ z = q^s \tag{1.5}$$

This book is concerned in studies of q-difference equations that is q-functional equations of the form

$$F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) = 0, \quad x \in \mathbf{C} \tag{1.6}$$

where D_q is the derivative in (1.4), the so-called Jackson derivative [31],

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \ x(s) = q^s, s \in \mathbf{Z}. \tag{1.7}$$

The functional equations implying the first two derivatives in (1.2)-(1.3) correspond respectively to the very classical popularized differential (continuous) and difference equations while those implying the Askey-Wilson derivative in (1.5) is essentially at its embryonic state, except for numerous applications in orthogonal polynomials theory [43, 46] and a few applications in others area (see, e.g., [6, 9, 8]).
In this connection, our book is concerned in a fairly developed matter of mathematical analysis on lattices.
It is worth to be noted that the q-difference equations theory considered in this book is a special case of the general q-functional equations

$$F(x, y(x), y(qx), \ldots, y(xq^k)) = 0, \quad x \in \mathbf{C} \tag{1.8}$$

(studied e.g., in [16, 49]), since in our case, x belongs necessary to the q-uniform lattice q^s, $s \in \mathbf{Z}$. In this book, for concreteness, it will be understood, unless the contrary is noted, that q is real and $0 < q < 1$.
Hence our lattice reads

$$\begin{aligned} T = [0 = q^{\infty}, \ldots, q^{s+1}, q^s, \ldots, q^2, q, q^0 = 1, \\ q^{-1}, q^{-2}, \ldots, q^{-s}, q^{-s-1}, \ldots, q^{-\infty} = \infty]. \end{aligned} \tag{1.9}$$

This is clearly a geometric progression with a proportion equals to q. For this reason, q-difference equations are some times referred to as geometric difference equations [29].
Examples of geometric variables can be found in any area of life or social sciences (here q may be ≥ 1).

1. Suppose given the simplest model of evolution of species

$$p(s+1) = rp(s);\ p(0) = p_0 \tag{1.10}$$

where $p(s)$ is the population of the species at the period s and r is the constant rate of change. (1.10) gives

$$p(s) = r^s p_0. \tag{1.11}$$

$p(s)$ is clearly a geometric variable with $q = r$, $p_0 = 1$.

2. In economy or epidemiology, we can consider a certain quantity that increases or decreases at every period s, in a rate equal to r. We get the difference equation

$$q(s+1) = (1+r)q(s);\ q(0) = q_0 \tag{1.12}$$

so that

$$q(s) = (1+r)^s q_0. \tag{1.13}$$

which is a geometric variable with $q = 1 + r$ and $q_0 = 1$.

3. In a national economy, let $R(s)$, $I(s)$, $C(s)$ and $G(s)$ be respectively the national income, the investment, the consumer expenditure and the government expenditure in a given period s. We have [25]

$$R(s+2) - \alpha(1+\beta)R(s+1) + \alpha\beta R(s) = 1, \tag{1.14}$$

with the assumptions that

$$C(s) = \alpha R(s-1),\ \alpha > 0, \tag{1.15}$$

$$I(s) = \beta[C(s) - C(s-1)]\,\beta > 0 \tag{1.16}$$

and $G(s) = const$. The general solution of (1.14) reads

$$R(s) = c_1\lambda_1^s + c_2\lambda_2^s + c_3. \tag{1.17}$$

where λ_1 and λ_2 are roots of

$$\lambda^2 - \alpha(1+\beta)\lambda + \alpha\beta = 1. \tag{1.18}$$

We can clearly get geometric variables by convenient choices of the constants c_1, c_2, and c_3. In the particular case when $\alpha\beta = 1$, we can take $\lambda_1 = q$ and $\lambda_2 = 1/q$ and (1.17) is nothing else than a form of the *q-nonuniform* variable noted in the beginning of this section.

4. Consider the amortization of a loan that is the process by which a loan is repaid by a sequence of periodic payments, each of which is part payment of interest and part payment to reduce the outstanding principal. Let $p(s)$ represent the outstanding principal after the sth constant payment T and, suppose that the interest charges compound at the rate r per payment period. In this case, we have the equation

$$p(s+1) = (1+r)p(s) - T;\ p(0) = p_0 \tag{1.19}$$

which solution reads

$$p(s) = (p_0 - \frac{T}{r})(1+r)^s + \frac{T}{r}. \tag{1.20}$$

This is also a generalized geometric variable and it can be found in any phenomenon with similar evolution process.

1.2 q-Hypergeometric Series

When dealing with q-difference equations, arise naturally series solutions of the type

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{1.21}$$

Among them, are of particular interest these for which

$$\frac{c_{n+1}}{c_n} \tag{1.22}$$

is a rational function in q^n. If for example

$$\frac{c_{n+1}}{c_n} = \frac{\prod_{i=1}^{r}(\alpha_i - q^{-n})}{\prod_{i=1}^{s}(\beta_i - q^{-n})(q - q^{-n})}, \tag{1.23}$$

such series are seen to have the form

$$\begin{aligned} &{}_r\varphi_s \left(\begin{array}{cccc} \alpha_1, & \alpha_2, & \dots, & \alpha_r \\ \beta_1, & \beta_2, & \dots, & \beta_s \end{array} \middle| q; z \right) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1;q)_k(\alpha_2;q)_k \dots (\alpha_r;q)_k}{(\beta_1;q)_k(\beta_2;q)_k \dots (\beta_s;q)_k} \left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{1+s-r} \frac{z^k}{(q;q)_k}, \end{aligned} \tag{1.24}$$

where $(a_1,\ldots,a_p;q)_k := (a_1;q)_k\ldots(a_p;q)_k$, $(a;q)_0=1$, $(a;q)_k=(1-a)(1-aq)(1-aq^2)\ldots(1-aq^{k-1}),\ k=1,2\ldots$. These series are referred to as the q-(basic)hypergeometric series [27]. Since $\lim_{q\to1}\frac{(q^a;q)_k}{(1-q)^k}=(a)_k$, we have

$$\lim_{q\to 1}\,{}_r\varphi_s\left(\begin{array}{cccc} q^{\alpha_1}, & q^{\alpha_2}, & \ldots, & q^{\alpha_r} \\ q^{\beta_1}, & q^{\beta_2}, & \ldots, & q^{\beta_s}\end{array}\middle| q;(q-1)^{1+s-r}z\right)$$
$$= {}_rF_s\left(\begin{array}{cccc} \alpha_1, & \alpha_2, & \ldots, & \alpha_r \\ \beta_1, & \beta_2, & \ldots, & \beta_s\end{array}\middle| z\right), \tag{1.25}$$

where

$${}_rF_s\left(\begin{array}{cccc} \alpha_1, & \alpha_2, & \ldots, & \alpha_r \\ \beta_1, & \beta_2, & \ldots, & \beta_s\end{array}\middle| z\right)$$
$$= \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\ldots(\alpha_r)_k}{(\beta_1)_k(\beta_2)_k\ldots(\beta_s)_k}\frac{z^k}{k!}, \tag{1.26}$$

where $(a_1,\ldots,a_p)_k := (a_1)_k\ldots(a_p)_k$, $(a)_0=1$, $(a)_k=a(a+1)\ldots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}$, series referred to as hypergeometric series. As well for the generalized hypergeometric series as for the basic ones , the radius of convergence is given by

$$\rho_c=\begin{cases}\infty, & r<s+1\\ 1, & r=s+1\\ 0, & r>s+1.\end{cases} \tag{1.27}$$

Take for example the simplest q-difference equation

$$D_q y(x)=y(x). \tag{1.28}$$

Its solution reads

$$y(x)=\sum_{n=0}^{\infty}\frac{((1-q)x)^n}{(q;q)_n} ={}_1\varphi_0(0;-;q,(1-q)x), \tag{1.29}$$

a q-version of the $exp(x)$ function [34] (see also section 2.1 below).

1.3 q-Derivation and q-integration

Basic formulae for the q-derivation and q-integration are concerned, similarly to the differential or difference situations.

Derivative and integral

We define the q-derivative also referred to as the Jackson derivative [31] as follows

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}. \tag{1.30}$$

This derivative sends naturally a polynomial of degree n in a polynomial of degree $n-1$, since $D_q x^k = \frac{q^k - 1}{q-1} x^{k-1}$ and if $p(x) = \sum_{k=0}^{n} a_k x^k$, then

$$D_q p(x) = \sum_{k=0}^{n-1} a_{k+1} \frac{q^{k+1} - 1}{q - 1} x^k. \tag{1.31}$$

Together with the question of the q-derivative, arises naturally that of the q-primitive or q-indefinite integral of a given function. This is equivalent to solving the following simplest q-difference equation in g with known f

$$D_q g(x) = f(x). \tag{1.32}$$

Detailing (1.32) gives

$$\frac{1 - E_q}{(1-q)x} g(x) = f(x), \;\; E_q h(x) = h(qx) \tag{1.33}$$

or

$$g(x) = (1 - E_q)^{-1}[(1-q)xf(x)] = (1-q) \sum_{i=0}^{\infty} E_q^i [xf(x)], \tag{1.34}$$

or

$$g(x) = (1-q)x \sum_{i=0}^{\infty} q^i f(q^i x). \tag{1.35}$$

The preceding calculus is clearly valid only if the series in the rhs of (1.35) is convergent. To say that if the series in the rhs of (1.35) is convergent, the function in the rhs of that equality is a certain primitive of $f(x)$, namely that primitive that vanishes at $x = x_0 = 0$. Hence we can write

$$\int_0^x f(x) d_q x = (1-q)x \sum_{i=0}^{\infty} q^i f(q^i x). \tag{1.36}$$

It is easily seen that the expression on the rhs of (1.36) is a Riemann integral sum of the function f on $[0,x]$, $x \neq \infty$, where the segmentation is given by the geometric lattice q^s, $s = 0,\ldots,\infty$. This means that if $f(x)$ is Riemann integrable (RI) around $x_0 = 0$, then we can naturally give its primitive as in (1.36) which is defined for $x \in T$, $x \neq \infty$.
However, if the function $f(x)$ is not RI around $x_0 = 0$ but is RI around $x_0 = \infty$, then one will find a primitive of $f(x)$ under the form

$$\begin{aligned} \textstyle\int_\infty^x f(x)d_qx &= (1-q^{-1})x \textstyle\sum_{i=0}^\infty q^{-i} f(q^{-1-i}x), \\ &= (q-1)x \textstyle\sum_{i=1}^\infty q^{-i} f(q^{-i}x) \end{aligned} \tag{1.37}$$

which is defined on the lattice T in (1.9) except for $x = 0$. Furthermore, if the function $f(x)$ is not RI neither around $x_0 = 0$, nor around $x_0 = \infty$, but RI around some $x_0 = c = q^d$, $d \in \mathbf{Z}$, then the primitive of $f(x)$ reads

$$\begin{aligned} \textstyle\int_c^x f(x)d_qx &= (q-1)\textstyle\sum_{i=d}^{s-1} q^i f(q^i),\ \ c = q^d \geq x = q^s \\ &= (q-1)\textstyle\sum_{t=c}^{q^{-1}x} t f(t). \end{aligned} \tag{1.38}$$

For example taking $c = q^0 = 1$, we get

$$\textstyle\int_c^x f(x)d_qx = (q-1)\textstyle\sum_{i=0}^{s-1} q^i f(q^i). \tag{1.39}$$

Note that the integral in (1.36) is clearly a particular case of the more general integral

$$\int_a^x f(t)d_qt =^{def} (x-a)(1-q)\sum_0^\infty q^i f(a + q^i(x-a)). \tag{1.40}$$

where we set $a = 0$ to obtain (1.36).
Next we define the definite integral as

$$\begin{aligned} \textstyle\int_a^b f(x)d_qx &= (1-q)\textstyle\sum_{i=\alpha}^{\beta} q^i f(q^i),\ \ b = q^\alpha \geq a = q^{\beta+1} \\ &= (1-q)\textstyle\sum_{x=q^{-1}a}^{b} x f(x). \end{aligned} \tag{1.41}$$

If the function $f(x)$ is RI around $x_0 = 0$, (1.41) can be written another way:

$$\textstyle\int_a^b f(x)d_qx = [\int_0^b - \int_0^a] f(x)d_qx. \tag{1.42}$$

Clearly, if the function $f(x)$ is differentiable on the point x, the q-derivative in (1.30) tends to the ordinary derivative in the classical analysis when q tends to 1. Identically, if the function $f(x)$ is RI on the concerned intervals, the integrals in (1.36), (1.37), (1.39) and (1.40) tend to the Riemann integrals of $f(x)$ on the corresponding intervals when q tends to 1. Moreover, one easily remarks that the q-integral admits the general properties of Riemann integral on finite or infinite intervals.

Example 1. Evaluate $\int x^\alpha d_q x$

Solution. One distinguishes

a) $\alpha > -1$: $f(x) = x^\alpha$ is RI around $x_0 = 0$. Hence,

$$\int_0^x x^\alpha d_q x = (1-q)x \sum_0^\infty q^i q^{\alpha i} x^\alpha$$
$$= \frac{1-q}{1-q^{\alpha+1}} x^{\alpha+1} \rightsquigarrow \frac{x^{\alpha+1}}{\alpha+1} = \int_0^x x^\alpha dx,\ q \rightsquigarrow 1. \tag{1.43}$$

b) $\alpha < -1$: $f(x) = x^\alpha$ is not RI around $x_0 = 0$ but is RI around $x_0 = \infty$. Hence, using (1.37), one has $\int_\infty^x x^\alpha dx = (q-1)x \sum_0^\infty q^{-(i+1)} x^\alpha q^{-(i+1)\alpha} = \frac{1-q}{1-q^{\alpha+1}} x^{\alpha+1} \rightsquigarrow \frac{x^{\alpha+1}}{\alpha+1} = \int_0^x x^\alpha dx$, $q \rightsquigarrow 1$.

c) $\alpha = -1$: In this case, the function $f(x) = \frac{1}{x}$ is not RI neither around $x_0 = 0$ nor around $x_0 = \infty$. Hence the formulas (1.36) and (1.37) don't work. However using (1.39), one gets $\int_1^x \frac{d_q x}{x} = (q-1)\sum_{i=0}^{s-1}(1) = (q-1)s = \frac{q-1}{lnq} \ln x \rightsquigarrow \ln x = \int_1^x \frac{dx}{x}$, $q \rightsquigarrow 1$.

It follows in particular from a) that the indefinite integral of a polynomial of degree n is a polynomial of degree $n+1$.

Example 2. Evaluate $\int_0^\infty f(x) d_q x$ for a function f RI on $[0, \infty]$.

Solution. Considering (1.36) and (1.37) with $x = 1$, we have

$$\int_0^\infty f(x) d_q x = \int_0^1 f(x) d_q x + \int_1^\infty f(x) d_q x$$
$$= (1-q)\sum_0^\infty q^i f(q^i) + (1-q)\sum_1^\infty q^{-i} f(q^{-i})$$
$$= (1-q)\sum_{-\infty}^\infty q^i f(q^i) \tag{1.44}$$

Note that the last expression in (1.44) is a Riemann integral sum of f on $[0, \infty]$ with the segmentation in (1.9).

Derivative of a product

$$D_q(fg)(x) = g(qx) D_q f(x) + f(x) D_q g(x)$$
$$= f(qx) D_q g(x) + g(x) D_q f(x). \tag{1.45}$$

Derivative of a ratio

$$D_q\left(\frac{f}{g}\right)(x) = \frac{g(x)D_q f(x) - f(x) D_q g(x)}{g(qx)g(x)}. \tag{1.46}$$

Chain rule

$$D_q(f(g))(x) = \frac{f(g(qx))-f(g(x))}{g(qx)-g(x)}.\frac{g(qx)-g(x)}{qx-x}$$
$$= D_{q,g}f(g).D_{q,x}g(x) \tag{1.47}$$

Derivative of the inverse function

Let $y = f(x)$. In that case, $x = f^{-1}(y)$ where f^{-1} is the inverse function to f. Applying the q-derivative on each side of the equality, one gets

$$1 = D_q x = D_q f^{-1}(y) = \frac{f^{-1}(y(qx))-f^{-1}(y(x))}{y(qx)-y(x)}.\frac{y(qx)-y(x)}{qx-x}$$
$$= D_{q,y}f^{-1}(y).D_{q,x}y(x). \tag{1.48}$$

Consequently

$$D_{q,y}f^{-1}(y) = \frac{1}{D_{q,x}y}. \tag{1.49}$$

Fundamental principles of the q-analysis

(i)

$$D_q[\int_a^x f(x)d_qx] = D_q\{(1-q)[x\sum_0^\infty q^i f(q^i x) - a\sum_0^\infty q^i f(q^i a)]\}$$
$$= (1-q)x[\sum_0^\infty q^i f(q^i x) - \sum_0^\infty q^{i+1} f(q^{i+1}x]/[(1-q)x] = f(x) \tag{1.50}$$

(ii)

$$\int_a^x D_q f(x)d_qx = \int_a^x \frac{f(qx)-f(x)}{qx-x}d_qx$$
$$= (1-q)x\sum_0^\infty q^i \frac{f(q^i x)-f(q^{i+1}x)}{(1-q)xq^i} - (1-q)a\sum_0^\infty q^i \frac{f(q^i a)-f(q^{i+1}a)}{(1-q)aq^i}$$
$$= f(x) - f(a). \tag{1.51}$$

Clearly, (1.51) is a *q-version of the Newton-Leibniz formula*

Integration by parts

Consider the equality

$$f(x)D_q g(x) = D_q(fg) - g(qx)D_q f(x). \tag{1.52}$$

Let $h(x) = f(x)g(x)$. We have

$$\begin{aligned} \textstyle\int_a^b D_q h d_q x &= \textstyle\sum_0^\infty (h(q^i b) - h(q^{i+1}b)) - \sum_0^\infty (h(q^i a) - h(q^{i+1}a)) \\ &= h(b) - h(a) \end{aligned}$$

Hence

$$\textstyle\int_a^b f(x) D_q g(x) d_q x = [fg]_a^b - \int_a^b g(qx) D_q f(x) d_q x \tag{1.53}$$

Clearly, when $q \rightsquigarrow 1$, the formulae (1.30)-(1.53) converge to the corresponding formulae of the continuous analysis.

1.4 Exercises

1. Prove that

$$D_q^n(fg)(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q D_q^k(f)(xq^{n-k}) D_q^{n-k}(g)(x) \tag{1.54}$$

$(q - Leinitz formula)$ where

$$\left(\begin{array}{c} n \\ k \end{array} \right)_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \tag{1.55}$$

and evaluate successively $D_q^i(fg)(x)$ $i = 1, 2, \ldots, n$.

2. Evaluate explicitly the operators A and B such that

a)

$$D_q^i(fy) = [A(f)]y,\ i = 1, 2, \ldots, n \tag{1.56}$$

b)

$$\prod_{i=1}^{n} (D_q - a_i)(fy) = [B(f)]y. \tag{1.57}$$

3. Prove the reciprocal formulae

a)

$$D_q^n(f)(x) =$$
$$(q-1)^{-n}x^{-n}q^{-\binom{n}{2}}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}f(q^{n-k}x) \tag{1.58}$$

b)

$$f(q^n x) = \sum_{k=0}^{n}(q-1)^k x^k q^{\binom{k}{2}}\binom{n}{k}_q D_q^n(f)(x). \tag{1.59}$$

4. Write formally the solution y of

$$y(qx) - ay(x) = f(x). \tag{1.60}$$

5. Integrate by parts

$$\int_a^b p(x)f(x)d_qx;\ p(x) = ax^2 + bx + c;\ f(x) = \ln x. \tag{1.61}$$

6. Let $f(x) = x^m$; $m > 0$. Calculate

$$\int_a^b x^m d_qx \tag{1.62}$$

a) By definition,

b) Using the q-Newton-Leibniz formula.

7. Let $g(x) = cx^k$ and $f(x)$ a given function. Prove that $D_q(f\circ g)(x) = (D_{q^k}(f))(g(x))D_q(g)(x)$.

8. Prove that if $q^p = 1$ and p is prime, then $D_q^p(f) = 0$.

9. Prove that if p is a polynomial, then

$$\begin{aligned}[D_q x - qxD_q]p(x) &= p(x)\\ [D_q x - xD_q]p(x) &= p(qx)\end{aligned} \tag{1.63}$$

and

$$\begin{aligned}[D_q x^k - q^k x^k D_q]p(x) &= \{k\}_q x^{k-1} p(x)\\ [D_q x^k - x^k D_q]p(x) &= \{k\}_q x^{k-1} p(qx)\end{aligned} \tag{1.64}$$

where $\{k\}_q = \sum_{k=1}^{n} q^{k-1}$, $\{0\}_q = 0$.

10. Prove the q-Pascal identity

$$\begin{aligned}\begin{pmatrix} n+1 \\ k \end{pmatrix}_q &= \begin{pmatrix} n \\ k \end{pmatrix}_q q^k + \begin{pmatrix} n \\ k-1 \end{pmatrix}_q\\ &= \begin{pmatrix} n \\ k \end{pmatrix}_q + \begin{pmatrix} n \\ k-1 \end{pmatrix}_q q^{n+1-k}\end{aligned} \tag{1.65}$$

and the equivalent dual identities

$$\begin{aligned}D_q^k - q^k x D_q^k &= \{k\}_q D_q^{k-1}\\ D_q^k - xD_q^k &= E_q \{k\}_q D_q^{k-1}.\end{aligned} \tag{1.66}$$

11. Let

$$P_n(x,y) = (x-y)(x-qy)\dots(x-q^{n-1}y). \tag{1.67}$$

Prove that

a)

$$D_{q,x} P_n(x,y) = [n]_q P_{n-1}(x,y), \tag{1.68}$$

b)

$$D_{q,y} P_n(x,y) = -[n]_q P_{n-1}(x,qy). \tag{1.69}$$

Chapter 2

q-Difference equations of first order

By a q-difference equation of first order, one can understand an equation of the form

$$f(x, y(x), D_q y(x)) = 0, \tag{2.1}$$

but also an equation of the form

$$g(x, y(x), y(qx)) = 0. \tag{2.2}$$

The difference between (2.1) and (2.2) is that the former is first order in the operator D_q, while the later is first order in E_q, with $E_q f(x) = f(qx)$. Clearly, from an equation of the type (2.1), one can derive an equivalent equation of the type (2.2) and conversely. However, for their apparent adaptability in discretization of differential equations, we will consider in this book mainly equations of type (2.1) instead of equations of type (2.2).
Although there is no general analytical method for solving general q-difference equations of first order, some of their special cases can be solved explicitly. This is the cases of linear q-difference equations and equations transformable to them, as we shall see in the following sections.

2.1 Linear q-difference equations of first order

Consider the q-difference equation

$$D_q y(x) = a(x)y(qx) + b(x). \tag{2.3}$$

This is a first order nonconstant coefficients linear non homogenous q-difference equation. Its study is clearly equivalent to that of

$$D_q y(x) = a(x)y(x) + b(x). \tag{2.4}$$

Indeed, (2.3) is equivalent to ,

$$D_q y(x) = \tilde{a}(x)y(x) + \tilde{b}(x). \tag{2.5}$$

where

$$\tilde{a}(x) = a(qx);\ \tilde{b}(x) = b(qx), \tag{2.6}$$

in that sense that (2.5) can be obtained from (2.3) by replacing x by $q^{-1}x$ and then q by q^{-1} and vice versa.
Consider for example the equation (2.3). The corresponding homogenous equation reads

$$D_q y(x) = a(x)y(qx). \tag{2.7}$$

Detailing the D_q derivative in (2.7), the equation reads

$$y(x) = [1 + (1-q)xa(x)]y(qx). \tag{2.8}$$

Repeating the recurrence relation in (2.8) N times, one gets

$$\begin{aligned} y(x) &= y(x_0)\textstyle\prod_{t=q^{-1}x_0}^{x}[1+(1-q)ta(t)] \\ &= y(q^N x)\textstyle\prod_{i=0}^{N-1}[1+(1-q)xq^i a(q^i x)]. \end{aligned} \tag{2.9}$$

If $N \rightsquigarrow \infty$, with $0 < q < 1$, then $q^N \rightsquigarrow 0$, and one obtains

$$y(x) = y(0)\textstyle\prod_{i=0}^{\infty}[1+(1-q)q^i xa(q^i x)]. \tag{2.10}$$

Example. Suppose that $a(x) = \frac{q^k-1}{q-1}.\frac{1}{q^k x-1}$, $k \in \mathbf{N}$. Clearly, we have the solution $y(x) = y(0)\prod_{i=0}^{\infty}[1+(1-q)q^i xa(q^i x)] = y(0)\prod_0^{k-1}(1-q^i x) =^{def} y(0)(x;q)_k$.
Consider next the non homogenous equation (2.3). According to the method of "variation of constants", let

$$y(x) = c(x)y_0(x) \tag{2.11}$$

be its solution where $y_0(x)$ is the solution of the corresponding homogenous equation (2.7) and $c(x)$ is an unknown function to be determined. Loading (2.11) in (2.3), and solving the obtained equation, one obtains

$$c(x) = \int_{x_0}^{x} y_0^{-1}(t)b(t)d_qt + c \tag{2.12}$$

Hence the general solution of (2.3) reads

$$y(x) = y_0(x)c + \int_{x_0}^{x} y_0(x)y_0^{-1}(t)b(t)d_qt \tag{2.13}$$

with $c = y_0^{-1}(x_0)y(x_0)$. Taking $x_0 = 0$, we get respectively

$$c(x) = (1-q)x \textstyle\sum_0^\infty q^i y_0^{-1}(q^i x)b(q^i x) + c \tag{2.14}$$

and

$$y(x) = y_0(x)c + (1-q)x \textstyle\sum_0^\infty q^i y_0(x) y_0^{-1}(q^i x)b(q^i x). \tag{2.15}$$

Note that, when applied to the equation (2.4), the method of undetermined constants leads to the solution

$$y(x) = y_0(x)c + \int_{x_0}^{x} y_0(x)y_0^{-1}(qt)b(t)d_qt \tag{2.16}$$

or

$$y(x) = y_0(x)c + (1-q)x \textstyle\sum_0^\infty q^i y_0(x) y_0^{-1}(q^{i+1} x)b(q^i x). \tag{2.17}$$

for $x_0 = 0$.

We now observe that the solutions in (2.9) or (2.10) will remain formal as long as we will not succeed to calculate the related product explicitly, a task which is far from being elementary. However, in certain situations, the coefficient $a(x)$ could suggest a particular method of resolution. When for example, $a(x)$ is a polynomial in x, we are suggested to search the solution in form of series, as show the following few simple cases:

Case 1. Equations of the form

$$D_q y(x) = ay(x), \tag{2.18}$$

with a, some constant. To solve such an equation, we rewrite it as

$$y(qx) = [1 + (q-1)xa]y(x) \tag{2.19}$$

and search the solution under the form

$$y(x) = \sum_0^\infty c_n x^n. \tag{2.20}$$

Loading (2.20) in (2.19), one obtains

$$c_n = \left(\prod_{k=1}^n \frac{1-q}{1-q^k}\right)a^n \quad . \tag{2.21}$$

In view of the fact that $[k]_q =^{def} \frac{1-q^k}{1-q} \rightsquigarrow k$, $q \rightsquigarrow 1$, one can write (2.21) as

$$c_n = c_0 \frac{a^n}{[n]_q!}, \tag{2.22}$$

where $[n]_q! =^{def} \prod_{k=1}^n \frac{1-q^k}{1-q}$. Hence the solution in (2.20) is a q-version of the exponential function $c_0 exp(ax)$:

$$y_q(x) = c_0 e_q^{ax} = c_0 \sum_{n=0}^\infty \frac{a^n}{[n]_q!} x^n. \tag{2.23}$$

Case 2. Similarly, an equation of the form

$$D_q y(x) = ay(qx), \tag{2.24}$$

or equivalently

$$y(x) = [1 + (1-q)xa]y(qx), \tag{2.25}$$

has a solution of the form

$$y_{q^{-1}}(x) = c_0 e_{q^{-1}}^{ax} = c_0 \sum_0^\infty \frac{a^n}{[n]_{q^{-1}}!} x^n, \tag{2.26}$$

where $[n]_{q^{-1}}!$ is obtained from $[n]_q!$ by replacing q by q^{-1}.
The functions e_q^x and $e_{q^{-1}}^x$ are clearly q-versions of the usual exponential function e^x. A natural question that arises here consists in finding their respective inverse q-functions. The answer to this question can be easily found using the following

Theorem 2.1.1 *If*

$$D_q y = a(x)y(x)$$
$$D_q z = -a(x)z(qx) \tag{2.27}$$
$$y(x_0)z(x_0) = 1 \tag{2.28}$$

then

$$y(x)z(x) = 1 \tag{2.29}$$

Proof. We have $D_q(zy) = z(qx)D_qy(x) + D_qz(x).y(x) = z(qx)a(x)y(x) - z(qx)a(x)y(x) = 0$. Hence $y(x)z(x) = cte$. Use of (2.28) gives (2.29).

Corollary 2.1.1 *The functions e_q^x and $e_{q^{-1}}^x$ satisfy*

$$e_q^x e_{q^{-1}}^{-x} = 1. \tag{2.30}$$

Similar q-versions of the $exp(x)$ and its inverse can be found considering the following

Theorem 2.1.2 *[34] Let*

$$\sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k} =_1 \varphi_0(a;-;q,x);\ |x| < 1 \tag{2.31}$$

be the q-binomial series. We have

$$\textstyle\sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k} = \frac{(ax;q)_\infty}{(x;q)_\infty} \tag{2.32}$$

where $(\alpha;q)_\infty = \prod_1^\infty (1 - q^{k-1}\alpha)$.

Proof. Let $h_a(x)$ be the series in the lhs of (2.32). Then, one easily verifies that $(1-x)h_a(x) = (1-ax)h_a(qx)$, or equivalently $h_a(x) = \frac{1-ax}{1-x}h_a(qx)$. Which leads recursively to $h_a(x) = \frac{(az;q)_\infty}{(z;q)_\infty}$ and the theorem is proved.

Corollary 2.1.2 *[34] Let* $\tilde{e}_q(x) = \sum_{k=o}^{\infty} \frac{x^k}{(q;q)_k} =_1 \varphi_0(0;-;q,x)$, $|x| < 1$ *and* $E_q(x) = \sum_{k=o}^{\infty} \frac{q^{k(k-1)/2}x^k}{(q;q)_k} =_0 \varphi_0(-;-;q,-x)$, $x \in T$. *We have that*

$$\tilde{e}_q(x)E_q(-x) = 1. \tag{2.33}$$

Proof. Loading $a = 0$ in (2.32), one obtains that $\tilde{e}_q(x) = \frac{1}{(x;q)_\infty}$, $|x| < 1$. On the other side, replacing in the same identity, $x = x/a$ and letting $a \rightsquigarrow \infty$, one gets $E_q(x) = (-x;q)_\infty$, and the corollary follows.
Note that

$$\lim_{q\to 1} \tilde{e}_q((1-q)x) = \lim_{q\to 1} E_q((1-q)x) = e^x. \tag{2.34}$$

Hence $\tilde{e}_q((1-q)x)$ and $E_q((1-q)x)$ are q-versions of the ordinary $exp(x)$ function. It is interesting to remark that

$$e_q^x = \tilde{e}_q((1-q)x); e_{q^{-1}}^x = E_q((1-q)x). \tag{2.35}$$

Case 3. Equations of the form

$$D_q y(x) = ay(x) + b. \tag{2.36}$$

According to the case 1 and the method of undetermined coefficients with $x_0 = 0$, its solution reads

$$\begin{aligned} y(x) &= e_q^{ax}[y(0) + b\int_0^x e_{q^{-1}}^{-aqt} d_q t] \\ &= e_q^{ax}[y(0) - \frac{b}{a} e_{q^{-1}}^{-ax} + \frac{b}{a}]. \end{aligned} \tag{2.37}$$

Hence using (2.30), we get

$$y(x) = e_q^{ax} y(0) - \frac{b}{a} + \frac{b}{a} e_q^{ax}. \tag{2.38}$$

Case 4. Equations of the form

$$D_q y(x) = ay(qx) + b. \tag{2.39}$$

Here also, according to the case 2 and the method of undetermined coefficients with $x_0 = 0$, its solution reads

$$\begin{aligned} y(x) &= e_{q^{-1}}^{ax}[y(0) + b\int_0^x e_q^{-at} d_q t] \\ &= e_{q^{-1}}^{ax}[y(0) - \frac{b}{a} e_q^{-ax} + \frac{b}{a}], \end{aligned} \tag{2.40}$$

and using (2.30), we get

$$y(x) = e_{q^{-1}}^{ax} y(0) - \frac{b}{a} + \frac{b}{a} e_{q^{-1}}^{ax}. \tag{2.41}$$

Case 5. Equations of the form

$$D_q y(x) = \alpha x y(x). \tag{2.42}$$

Searching the solution under the form

$$y(x) = \textstyle\sum_0^\infty c_n x^n, \tag{2.43}$$

on gets

$$c_{2n} = \alpha \frac{c_{2n-2}}{\frac{1-q^{2n}}{1-q}} = \alpha^n \frac{c_0}{\frac{1-q^{2n}}{1-q}\frac{1-q^{2n-2}}{1-q}\cdots\frac{1-q^2}{1-q}.1}$$
$$= \alpha^n \frac{c_0}{[2n]_q!!};\ n = 1, 2, \ldots \tag{2.44}$$

where

$$[2n]_q!! =^{def} \frac{1-q^{2n}}{1-q}\frac{1-q^{2n-2}}{1-q}\cdots\frac{1-q^2}{1-q}.1; \tag{2.45}$$

and $c_{2n+1} = 0$, $n = 0, 1, 2, \ldots$. Its is easily seen that $[2n]_q!! = [n]_q!(2)_q^n$ where $(2)_q^n =^{def} (1+q)(1+q^2)\ldots(1+q^n)$ and that $\lim_{q\to 1}[2n]_q!! = (2n)!! =^{def} (2n)(2n-2)(2n-4)\ldots .2.1 = 2^n n! = \lim_{q\to 1}[n]_q!(2)_q^n$. Hence the solution of (2.42) reads

$$y_q(x) = c_0 \mathcal{E}_q^{\frac{\alpha x^2}{2}} \tag{2.46}$$

where

$$\mathcal{E}_q^{\frac{\alpha x^2}{2}} = \sum_0^\infty \frac{\alpha^n x^{2n}}{[n]_q!(2)_q^n} \tag{2.47}$$

is a q-version of the function $e^{\frac{\alpha x^2}{2}}$ (see another q-version of this function in [24]).

2.2 Nonlinear q-difference equations transformable into linear equations

Here, we consider nonlinear q-difference equations of type (2.1) or (2.2) transformable in linear equations.

Case 1. Riccati type equations:

$$D_q y(x) = a(x)y(qx) + b(x)y(x)y(qx). \tag{2.48}$$

To solve this equation, we set $y(x) = 1/z(x)$ and obtain

$$D_q z(x) = -[a(x)z(x) + b(x)]. \tag{2.49}$$

Example. Solve the equation

$$y(qx)y(x) - y(qx) + y(x) = 0. \tag{2.50}$$

Letting $y(x) = 1/z(x)$, it gives $z(x) = z(qx) + 1$ which solution is $z(x) = -\ln x/\ln q$. Hence $y(x) = -\ln q/\ln x$.

Case 2. Homogenous equations of the form

$$f(\tfrac{D_q y(x)}{y(x)}, x) = 0. \tag{2.51}$$

They can be transformed into linear equations in $z(x)$ with $z(x) = \frac{D_q y(x)}{y(x)}$.

Example. Solve the equation

$$[D_q y(x)]^2 - 2y(x)D_q y(x) - 3[y(x)]^2 = 0. \tag{2.52}$$

We have

$$[\frac{D_q y(x)}{y(x)}]^2 - 2[\frac{D_q y(x)}{y(x)}] - 3 = 0, \tag{2.53}$$

or $z^2(x) - 2z(x) - 3 = 0$, $z(x) = \frac{D_q y(x)}{y(x)}$. This gives $z(x) = 3$ and $z(x) = 1$, or $y(x) = ce_q^{3x}$ and $y(x) = ce_q^x$, respectively.

Case 3. Equations of the form

$$[y(qx)]^{c_1}[y(x)]^{c_2} = g(x), \tag{2.54}$$

c_1 and c_2, some constants. In that case, we apply the ln function and get

$$c_1 \ln(y(qx)) + c_2 \ln(y(x)) = \ln(g(x)) \tag{2.55}$$

and set $z(x) = \ln(y(x))$ to obtain

$$c_1 z(qx) + c_2 z(x) = \ln(g(x)) \tag{2.56}$$

Example. Contemplate the equation

$$[y(x)^2]/y(qx) = e^{x^2}. \tag{2.57}$$

Applying the ln function, one gets

$$2z(x) - z(qx) = x^2,\ z(x) = \ln y(x) \tag{2.58}$$

The solution of the homogenous equation is $z(x) = x^{\ln 2/\ln q}$. The particular solution can be found by inverting the operator in the lhs $1 - \frac{1}{2}E_q$, to get

$$z(x) = (1 - \tfrac{1}{2}E_q)^{-1}x^2/2 = \tfrac{x^2}{2}\textstyle\sum_0^\infty 2^{-i}q^{2i} = \tfrac{x^2}{2-q^2}. \tag{2.59}$$

Hence the solution of (2.58) reads $z(x) = cx^{\ln 2/\ln q} + \frac{x^2}{2-q^2}$. Consequently,

$$\begin{aligned} y(x) &= exp(cx^{\frac{\ln 2}{\ln q}} + \tfrac{x^2}{2-q^2}) \\ &= exp(c2^{\frac{x}{\ln q}} + \tfrac{x^2}{2-q^2}). \end{aligned} \tag{2.60}$$

2.3 Exercises

1. Let be defined the following q-versions of the $cos(x)$, $\sin(x)$, $cosh(x)$ and $\sinh(x)$ functions

$$\begin{aligned} \cos_q(x) &= \frac{e_q^{ix} + e_q^{-ix}}{2};\ \sin_q(x) = \frac{e_q^{ix} - e_q^{-ix}}{2i} \\ \cosh_q(x) &= \frac{e_q^{x} + e_q^{-x}}{2};\ \sinh_q(x) = \frac{e_q^{x} - e_q^{-x}}{2}, \end{aligned} \tag{2.61}$$

$$\begin{aligned} \cos_{q^{-1}}(x) &= \frac{e_{q^{-1}}^{ix} + e_{q^{-1}}^{-ix}}{2};\ \sin_{q^{-1}}(x) = \frac{e_{q^{-1}}^{ix} - e_{q^{-1}}^{-ix}}{2i} \\ \cosh_{q^{-1}}(x) &= \frac{e_{q^{-1}}^{x} + e_{q^{-1}}^{-x}}{2};\ \sinh_{q^{-1}}(x) = \frac{e_{q^{-1}}^{x} - e_{q^{-1}}^{-x}}{2}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} \cos_{qq^{-1}}(x) &= \frac{e_q^{ix} + e_{q^{-1}}^{-ix}}{2};\ \sin_{qq^{-1}}(x) = \frac{e_q^{ix} - e_{q^{-1}}^{-ix}}{2i} \\ \cosh_{qq^{-1}}(x) &= \frac{e_q^{x} + e_{q^{-1}}^{-x}}{2};\ \sinh_{qq^{-1}}(x) = \frac{e_q^{x} - e_{q^{-1}}^{-x}}{2}. \end{aligned} \tag{2.63}$$

Prove that

a) $D_q \cos_q(x) = -\sin_q(x)$,

b) $D_q \sin_q(x) = \cos_q(x)$,

c) $D_q \cosh_q(x) = \sinh_q(x)$,

d) $D_q \sinh_q(x) = \cosh_q(x)$,

e) $\cos^2_{qq^{-1}}(x) + \sin^2_{qq^{-1}}(x) = 1$,

f) $\cosh^2_{qq^{-1}}(x) - \sinh^2_{qq^{-1}}(x) = 1$,

g) $\cos_q(x) cos_{q^{-1}}(x) + \sin_q(x) \sin_{q^{-1}}(x) = 1$,

h) $\cosh_q(x) cosh_{q^{-1}}(x) - \sinh_q(x) \sinh_{q^{-1}}(x) = 1$.

2. Find the general solution of

$$(D_q y(x))^2 - 2y(x)D_q y(x) - 3y^2(x) = 0 \tag{2.64}$$

3. Solve

a) $D_q y(x) = xy(x)$

b) $(D_q y(x))^2 - (2 + x)y(x)D_q y(x) - 2xy^2(x) = 0$

4. Solve

a) $D_q y(x) = ay(x) + x^2$

b) $D_q y(x) = ay(x) + e_q^x$

c) $D_q y(x) = p_2(x)y(x)$; $p_2(x) = ax^2 + bx + c$.

5. Let $f(x) = \sum_{n=-\infty}^{+\infty} A_n x^n$ and put

$$f[x \pm y]_q = \sum_{n=-\infty}^{+\infty} A_n x^n (\mp \frac{y}{x}; q)_n. \tag{2.65}$$

Prove that

$$\begin{aligned} e_q(x)e_{q^{-1}}(y) &= e_q[x+y]_q \\ \frac{e_q(y)}{e_q(x)} &= e_q[y-x]_q. \end{aligned} \tag{2.66}$$

Chapter 3

Systems of linear q-difference equations

In this chapter, we are concerned in systems of linear q-Difference equations of first order. The methods of solving such equations are in big parts similar to that of solving linear *scalar* first order q-difference equations discussed in the section 2.1. The general theory however is more rich since the space state is now multidimensional.

3.1 General theory

Consider the system of linear q-difference equations

$$D_q y(x) = A(x)y(qx) + b(x), \tag{3.1}$$

where

$$\begin{gathered} y(x) = (\eta_1(x), \dots, \eta_k(x))^t,\ b(x) = (b_1(x), \dots, b_k(x))^t \in \mathbf{R}^k, \\ A(x) = (a_{i,j}(x))_{i,j=1}^k, \end{gathered} \tag{3.2}$$

such that the matrix $I + (1-q)xA(x)$ is nonsingular. The latter requirement can naturally be achieved by taking q sufficiently close to 1. Remark that the notation $D_q Z(x)$, where $Z(x)$ is a vector or a matrix means the vector or the matrix for which the elements are the q-derivatives of the elements of the concerned vector or matrix.
As in the scalar case, the system (3.1) is equivalent to the following

$$D_q y(x) = \tilde{A}(x)y(x) + \tilde{b}(x) \tag{3.3}$$

where $\tilde{A}(x) = A(qx)$ and $\tilde{b}(x) = b(qx)$, in the sense that one can be obtained from the other by first replacing x by $q^{-1}x$ and then q by q^{-1}. But for question of convenience, we consider the system (3.1) here.
The system (3.1) is said to be *non homogenous* and *non autonomous* since respectively, the independent term (also called *input* or *external force*) is not vanishing, and its coefficients are dependent on x.
Thus, the corresponding to (3.1) homogenous equation is

$$D_q y(x) = A(x)y(qx), \tag{3.4}$$

As in the case of scalar equation, one can rewrite the equations (3.1) and (3.4) respectively in the recurrent forms

$$y(x) = [I + (1-q)xA(x)]y(qx) + (1-q)xb(x). \tag{3.5}$$

and

$$y(x) = [I + (1-q)xA(x)]y(qx). \tag{3.6}$$

Consider first the homogenous equation (3.4). According to (3.6), the solution of this system reads:

$$y_0(x) = (\prod_{t=q^{-1}x_0}^{x}[I + (1-q)tA(t)])y(x_0). \tag{3.7}$$

Taking $x_0 = 0$, (3.7) gives

$$y(x) = (\prod_0^\infty [I + (1-q)q^i xA(q^i x)])y(0). \tag{3.8}$$

From this, one deduces the following

Theorem 3.1.1 *For any vector $v_0 \in \mathbf{R}^k$, there exists a unique solution of (3.4) satisfying $y(x_0) = v_0$.*

Let $y_1(x), \ldots, y_k(x)$ be a system of k vectors in $\mathbf{R}^k$ and let $Y(x)$ be the $k.k$ matrix which columns are constituted by the vectors $y_1(x), \ldots, y_k(x)$. The following proposition is easily verified.

Theorem 3.1.2 *The matrix $\Phi(x)$ is a solution of the homogenous system (3.4) iff every vector of the set $\{y_1(x), \ldots, y_k(x)\}$ is.*

From theorems (3.1.1) and (3.1.2) follows clearly the

Theorem 3.1.3 *For any kxk-matrix V_0, there exists a unique matrix solution of (3.4) satisfying $Y(x_0) = V_0$.*

Considering (3.6), such solution reads

$$Y(x) = (\prod_{t=q^{-1}x_0}^{x}[I + (1-q)tA(t)])V_0, \tag{3.9}$$

or for $x_0 = 0$

$$Y(x) = (\prod_0^\infty [I + (1-q)q^i xA(q^i x)])V_0. \tag{3.10}$$

Theorem 3.1.4 *Let Y and Z be such that*

$$D_qY(x) = A(x)Y(x)$$
$$D_qZ(x) = -Z(qx)A(x) \tag{3.11}$$
$$Y(x_0)Z(x_0) = I \tag{3.12}$$

then

$$Y(x)Z(x) = I. \tag{3.13}$$

where I is the unit matrix.

Proof. $D_q(Z(x)Y(x)) = Z(qx).D_qY(x) + D_qZ(x).Y(x) = Z(qx).Ay(x) - Z(qx).AY(x) = 0$ i.e. $ZY = const$ and by (3.12), we get $ZY = I$.
Similarly, one easily proves the following

Theorem 3.1.5 *Let Y and Z be such that*

$$D_qY(x) = A(x)Y(qx)$$
$$D_qZ(x) = -Z(x)A(x) \tag{3.14}$$
$$Y(x_0)Z(x_0) = I \tag{3.15}$$

then

$$Y(x)Z(x) = I. \tag{3.16}$$

The following corollaries are direct consequences of the preceding theorem

Corollary 3.1.1 *The matrices $Y(x)$ and $Z(x)$ in (3.12)-(3.16) are mutually inverse.*

Corollary 3.1.2 *The matrix solution of*

$$D_qY(x) = A(x)Y(x) \tag{3.17}$$

is nonsingular iff it is nonsingular for $x = x_0$.

Definition 3.1.1 *A set of k linear independent solutions* $\{y_1(x), \ldots, y_k(x)\}$ *of (3.4) is said to be a fundamental system of solutions. The corresponding matrix* $Y(x)$*, which is clearly nonsingular, is also said to be a fundamental matrix of the system.*

Theorem 3.1.6 *Any system of linear q-difference equations like (3.4) admits always a fundamental system of solutions or equivalently a fundamental matrix.*

Proof. Consider then a system of k linear independent in $\mathbf{R}^k$ vectors $v_1, \ldots, v_k$, and following (3.1.1), let $\{y_1(x), \ldots, y_k(x)\}$ be solutions of (3.4) satisfying $y_i(x_0) = v_i$, $i = 1, \ldots, k$. This means that the set of solutions $y_i(x)$, $i = 1, \ldots, k$ are linear independent on the point $x = x_0$, or equivalently, the corresponding matrix $Y(x)$ is nonsingular for $x = x_0$. By the corollary 3.1.2, this means that $Y(x)$ is nonsingular for every x and the corresponding system of solutions $\{y_1(x), \ldots, y_k(x)\}$ is fundamental which proves the theorem.

Corollary 3.1.3 *The space of solutions of the homogenous system (3.4) is a k-dimensional linear space.*

Consider now the non homogenous equation (3.1). Suppose that the matrix $Y(x)$ is a fundamental matrix for the corresponding homogenous system (3.4). In that case, similarly to the scalar case, the method of variation of constants suggests to search the general solution for (3.1) under the form $y(x) = Y(x)C(x)$, where $C(x)$ is an unknown k-dimensional vector. The result reads

$$C(x) = C + \int_{x_0}^{x} Y^{-1}(t)b(t)d_qt \tag{3.18}$$

and

$$y(x) = Y(x)C + \int_{x_0}^{x} Y(x)Y^{-1}(t)b(t)d_qt \tag{3.19}$$

where $C = Y^{-1}(x_0)y(x_0)$, or equivalently

$$y(x) = \Phi(x, x_0)y(x_0) + \int_{x_0}^{x} \Phi(x, t)b(t)d_qt \tag{3.20}$$

with

$$\Phi(x, y) = Y(x)Y^{-1}(t), \tag{3.21}$$

the *q-transition matrix*. In the controllability theory (see chapter 7), one writes (3.20) in the convenient form

$$y(x) = \Phi(x, x_0)[y(x_0) + \int_{x_0}^{x} \Phi(x_0, t)b(t)d_qt]. \tag{3.22}$$

When $x_0 = 0$, (3.18), (3.19), (3.20) and (3.22) take the forms

$$C(x) = C + (1-q)x \sum_0^\infty q^i Y^{-1}(q^i x)b(q^i x), \tag{3.23}$$

$$y(x) = Y(x)C + (1-q)x \sum_0^\infty q^i Y(x)Y^{-1}(q^i x)b(q^i x), \tag{3.24}$$

$$y(x) = \Phi(x, 0)y(0) + (1-q)x \sum_0^\infty q^i \Phi(x, xq^i)b(q^i x), \tag{3.25}$$

and

$$y(x) = \Phi(x, 0)[y(0) + (1-q)x \sum_0^\infty q^i \Phi(0, xq^i)b(q^i x)]. \tag{3.26}$$

The function

$$y_p(x) = \int_{x_0}^{x} \Phi(x, t)b(t)d_qt \tag{3.27}$$

is a particular solution of (3.1). Hence we have the following

Theorem 3.1.7 *The general solution of the non homogenous q-difference equation (3.1) is a sum of its particular and the general solution of the corresponding homogenous equation (3.4).*

3.2 Autonomous systems

Let distinguish the following most interesting cases.

Case1. Equations of the form

$$D_q y(x) = Ay(qx), \tag{3.28}$$

where now, A is independent of x. The equation is first reduced to the following

$$y(x) = [I + (1-q)xA]y(qx). \tag{3.29}$$

which solution reads

$$y(x) = (\prod_0^\infty [I + (1-q)xq^i A])y(0)$$
$$= \sum_0^\infty C_n x^n, \tag{3.30}$$

where C_n is now a k-dimensional vector and is given by

$$C_n = \frac{A^n}{[n]_{q^{-1}}!} C_0. \tag{3.31}$$

In other words,

$$y_{q^{-1}}(x) = C_0 exp_{q^{-1}}(Ax) \tag{3.32}$$

according to the notation of section 2.1.

Case 2. Equations of the form

$$D_q y(x) = Ay(x). \tag{3.33}$$

According to the same notation, its solution reads

$$y(x) = \sum_0^\infty C_n x^n \tag{3.34}$$

where

$$C_n = \frac{A^n}{[n]_q!} C_0, \tag{3.35}$$

that is

$$y_q(x) = C_0 exp_q(Ax). \tag{3.36}$$

Let note that the evaluation of A^n in (3.31) or in (3.35) can be done following the *Putzer algorithm* (see e.g. [25]), while the evaluation of the functions in (3.32) and (3.36) can be done following *Sylvester's formula* or using the minimal polynomial of the matrix A (see e.g. [15]).

Remark 3.2.1 *According to theorem 3.1.1, and the preceding, it follows that*

$$exp_q(Ax) exp_{q^{-1}}(-Ax) = I. \tag{3.37}$$

As a consequence, the transition matrix takes the form:

$$\Phi(x,t) = exp_q(Ax) exp_{q^{-1}}(-At). \tag{3.38}$$

3.3 Exercises

1. Given the non homogenous linear system

$$D_q y(x) = A(x)y(x) + b(x). \tag{3.39}$$

a) Write down the general solution of the corresponding homogenous equation

$$D_q y(x) = A(x)y(x), \tag{3.40}$$

b) Prove that its general solution reads

$$y(x) = \Phi(x, x_0)[y(x_0) + \int_{x_0}^{x} \Phi(x_0, qt)b(t)d_q t], \tag{3.41}$$

with

$$\Phi(x, y) = Y(x).Y^{-1}(y), \tag{3.42}$$

$Y(x)$ being the fundamental matrix of (3.40).

2. For a $2x2$-matrix A and a 2-vector b, solve

a) $D_q y(x) = Ay(x) + b$;

b) $D_q y(x) = Ay(x) + bx^2$.

Chapter 4

Linear q-difference equations of higher order

4.1 General theory

Consider the equation

$$[D^k_{q^{-1}} + a_1(x)D^{k-1}_{q^{-1}} + \ldots + a_{k-1}(x)D_{q^{-1}} + a_k(x)]y(x) = g(x). \quad (4.1)$$

It is said to be a k-order nonconstant coefficients linear non homogenous q-difference equation of order k. The corresponding homogenous equation reads

$$[D^k_{q^{-1}} + a_1(x)D^{k-1}_{q^{-1}} + \ldots + a_{k-1}(x)D_{q^{-1}} + a_k(x)]y(x) = 0. \quad (4.2)$$

The general theory of a scalar equation such as (4.1) is reduced to the general theory for a system of equations such as (3.1). The reason for this is that an equation such as (4.1) can be reduced to a system such as (3.1). Indeed, supposing

$$z_1(x) = y(x);\ z_2(x) = D_{q^{-1}}y(x); \ldots ; z_k(x) = D^{k-1}_{q^{-1}}y(x), \quad (4.3)$$

we obtain the system

$$\begin{aligned}
D_{q^{-1}}z_1(x) &= z_2(x),\\
D_{q^{-1}}z_2(x) &= z_3(x),\\
&\ldots\\
D_{q^{-1}}z_{k-1}(x) &= z_k(x)\\
D_{q^{-1}}z_k(x) &= -(a_1(x)z_k(x) + \ldots + a_k(x)z_1(x)) + g(x) \quad (4.4)
\end{aligned}$$

In matrices terms, we have

$$D_q z(x) = A(x) z(qx) + G(x) \tag{4.5}$$

where $z(x) = (z_1(x), \ldots, z_k(x))^t$,

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & 1 \\ -a_k(qx), & . & . & . & . & -a_1(qx) \end{pmatrix} \tag{4.6}$$

and $G(x) = (0, \ldots, 0, g(qx))^t$. So, from (4.4), it follows that the existence of a unique solution of (4.1) under the initial constraints $y(x_0) = y_0$, $D_{q^{-1}} y(x_0) = y_1$, …, $D_{q^{-1}}^{k-1} y(x_0) = y_{k-1}$, is equivalent to the existence of a unique solution of (4.5) under the constraints $(z_1(x_0), \ldots, z_k(x_0))^t = (y_0, \ldots, y_{k-1})^t$. As a consequence, the existence of a fundamental system of solutions $y_1(x), \ldots, y_k(x)$ of (4.2) is equivalent to the existence of a fundamental system $(y_1(x), D_{q^{-1}} y_1(x), \ldots, D_{q^{-1}}^{k-1} y_1(x))^t$, …, $(y_k(x), D_{q^{-1}} y_k(x), \ldots, D_{q^{-1}}^{k-1} y_k(x))^t$ of the homogenous part of(4.5)

$$D_q z(x) = A(x) z(qx), \tag{4.7}$$

with the fundamental matrix

$$\Phi(x) = \begin{pmatrix} y_1(x) & y_2(x) & \ldots & y_k(x) \\ D_{q^{-1}} y_1(x) & D_{q^{-1}} y_2(x) & \ldots & D_{q^{-1}} y_k(x) \\ . & . & . & . \\ D_{q^{-1}}^{k-1} y_1(x) & D_{q^{-1}}^{k-1} y_2(x) & \ldots & D_{q^{-1}}^{k-1} y_k(x) \end{pmatrix}. \tag{4.8}$$

Indeed, if

$$\textstyle\sum_{i=1}^k \alpha_i y_i(x) = 0 \tag{4.9}$$

then

$$\begin{gathered} \textstyle\sum_{i=1}^k \alpha_i D_{q^{-1}} y_i(x) = 0 \\ \ldots \\ \textstyle\sum_{i=1}^k \alpha_i D_{q^{-1}}^{k-1} y_i(x) = 0 \end{gathered} \tag{4.10}$$

or

$$\Phi(x) \alpha = 0 \tag{4.11}$$

where $\Phi(x)$ is in (4.8) and $\alpha = (\alpha_1, \ldots, \alpha_k)^t$. Hence the system $y_i, i = 1, \ldots, k$ is linear independent iff the matrix $\Phi(x)$ in (4.8) is non singular. The matrix $\Phi(x)$ can naturally be called the q-Wronskian or q-Casaratian of the equation (4.2), correspondingly to the continuous or discrete cases. Consider now the question of deriving the solution of the non homogenous (4.1). If $y_1(x), \ldots, y_k(x)$ is a fundamental system of solution of the homogenous equation (4.2), corresponding to the fundamental matrix $\Phi(x)$, then according to the general theory of q-difference systems, the general solution of (4.5) reads

$$z(x) = \Phi(x).C(x) \tag{4.12}$$

with $C(x) = (C_1(x), \ldots, C_k(x))^t$ and

$$C(x) = C + (1-q)x \textstyle\sum_0^\infty q^i \Phi^{-1}(q^i x) G(q^i x). \tag{4.13}$$

and the general solution of (4.1) reads

$$y(x) = z_1(x) = \textstyle\sum_{i=1}^k c_i(x) y_i(x). \tag{4.14}$$

4.2 Linear q-difference equations with constant coefficients

Consider now the equations (4.1) and (4.2) in the cases when the coefficients a_i are not dependent of x. We have

$$[D^k_{q^{-1}} + a_1 D^{k-1}_{q^{-1}} + \ldots + a_{k-1} D_{q^{-1}} + a_k] y(x) = g(x) \tag{4.15}$$

and

$$[D^k_{q^{-1}} + a_1 D^{k-1}_{q^{-1}} + \ldots + a_{k-1} D_{q^{-1}} + a_k] y(x) = 0. \tag{4.16}$$

In this case, the equations can be solved explicitly whether or not it is true for the corresponding algebraic equation. Consider first the equation $D_{q^{-1}} y(x) = \lambda y(x)$. According to the treatment of the chapter 1, its solution reads $y(x) = exp_{q^{-1}}(\lambda x)$. Loading this function in (4.16), one obtains the following algebraic equation in λ called the characteristic equation of (4.16):

$$\lambda^k + a_1 \lambda^{k-1} + \ldots + a_{k-1}\lambda + a_k = 0. \tag{4.17}$$

Here we distinguish two cases:

Theorem 4.2.1 *(i) If the equation (4.17) has k distinct roots, λ_1, λ_2, ..., λ_k, then, the equation (4.16) admits as k linear independent solutions the functions $y_i(x) = exp_{q^{-1}}(\lambda_i x)$, $i = 1, \dots, k$.*
(ii) If some of the roots of the characteristic equation are not distinct, then in that case also, the equation (4.16) admits k linear independent solutions. If for example a given root λ admits a multiplicity equal to m, so the corresponding independent solutions need to be searched among functions of the form

$$y(x) = \textstyle\sum_{n=0}^{\infty} c_n x^n \tag{4.18}$$

where the coefficients c_n satisfies

$$\textstyle\sum_{i=0}^{m} \left[\left(\begin{array}{c} i \\ m \end{array} \right) (-\lambda)^{m-i} \left(\prod_{k=0}^{i-1} \frac{1-q^{-(n+i)+k}}{1-q^{-1}} \right) \right] c_{n+i} = 0 \tag{4.19}$$

a difference homogenous equation of order m.

Proof. The first part of the theorem is proved straightforwardly. To prove the second part, it suffices also to load (4.18) in the following auxiliary equation

$$(D_{q^{-1}} - \lambda)^m y(x) = 0 \tag{4.20}$$

Note finally that the particular solution of (4.15) can be obtained by the method of variation of constants as in the case of non constant coefficients. The equation (4.1) admits another interesting particular cases. Consider for example the case when all the coefficients $a_i(x)$ have the form $a_i(x) = x^i d_i$ $i = 0, \dots, k$, where the d_i are constants. After simplifying, the equation reads

$$\textstyle\sum_{i=0}^{k} b_i y(q^i x) = g(x), \tag{4.21}$$

for some constants b_i, $i = 0, \dots, k$. The homogenous version naturally reads

$$\textstyle\sum_{i=0}^{k} b_i y(q^i x) = 0, \tag{4.22}$$

To solve it, one first solves the equation $y(qx) = \lambda y(x)$. Its solution was seen in the first chapter to be

$$y(x) = x^{\frac{\ln \lambda}{\ln q}}. \tag{4.23}$$

Loading (4.23) in (4.22), one gets a k-order algebraic equation called characteristic equation for (4.22)

$$\sum_{i=0}^{k} b_i \lambda^i = 0. \tag{4.24}$$

Here as well two possible situations arise, as shows the following

Theorem 4.2.2 *(i) If the characteristic equation (4.24) has k distinct roots, λ_i, $i = 1, \ldots, k$. In that case, (4.22) admits k linear independent solutions $y_i(x) = x^{\frac{\ln \lambda_i}{\ln q}}$, $i = 1, \ldots, k$.*
(ii) If some roots are multiple, then the equation (4.22) admits as well k linear independent solutions. If a root say λ is m-iple, then to it correspond m solutions reading $y_i(x) = x^{\frac{\ln \lambda}{\ln q}} \dfrac{\ln^i x}{\ln^i q}$, where $i = 0, \ldots, m-1$.

Proof. The proof of the first part of the theorem is straightforward. For the second part, it suffices to proof that

$$(E_q - \lambda)^m x^{\frac{\ln \lambda}{\ln q}} \frac{\ln^i x}{\ln^i q} = 0;\; 0 \le i \le m-1. \tag{4.25}$$

Indeed,

$$\begin{aligned} &(E_q - \lambda)^m x^{\frac{\ln \lambda}{\ln q}} \frac{\ln^i x}{\ln^i q} \\ &= \lambda^n (E_q - 1)^m \frac{\ln^i x}{\ln^i q} = 0;\; 0 \le i \le m-1. \end{aligned} \tag{4.26}$$

4.3 Nonlinear q-difference equations transformable into linear equations of higher order

As in the case of first order, some nonlinear q-difference equations are transformable into linear ones.

Case 1. q-Difference equations of the form

$$\prod_{i=0}^{k} [D_{q^{-1}}^i y(x)]_i^r = 0 \tag{4.27}$$

or equivalently

$$\prod_{i=0}^{k} [y(q^i x)]_i^r = 0. \tag{4.28}$$

It is made linear by applying the ln function on the lhs of the equality.

Case 2. The Riccati type equation:

$$a_0(x)D_q y(x) = b_0(x)y(x)y(qx) + c_0(x)\frac{y(x)+y(qx)}{2} + d_0(x). \tag{4.29}$$

This equation can be written in the following homographic form

$$y(qx) = \tfrac{a(x)y(x)+b(x)}{c(x)y(x)+d(x)}. \tag{4.30}$$

To make (4.30) linear, it suffices to suppose

$$z(qx)/z(x) = c(x)y(x) + d(x). \tag{4.31}$$

The resulting second order linear q-difference equation reads

$$\begin{aligned}&[c(x)]z(q^2x) + [-c(x)d(qx) - c(qx)a(x)]z(qx)\\ &+[c(qx)a(x)d(x) - c(qx)b(x)c(x)]z(x) = 0.\end{aligned} \tag{4.32}$$

4.4 Linear q-difference equations of second order

As in the case of differential or difference equations, linear second order q-difference equations are of particular interest in the theory and applications of q-difference equations. Examples of such applications can be found in the chapter 6 in connection with orthogonal polynomials. We can write a general linear q-difference equation of second order in the form:

$$a_0(x)D_q^2 y + a_1(x)D_q y + a_2(x)y = b(x). \tag{4.33}$$

This is a non homogenous equation while the corresponding homogenous one reads

$$a_0(x)D_q^2 y + a_1(x)D_q y + a_2(x)y = 0. \tag{4.34}$$

However, in some investigations, it is appropriate to consider the equations of the forms

$$a_0(x)D_{q^{-1}}D_q y + a_1(x)D_q y + a_2(x)y = b(x) \tag{4.35}$$

and

$$a_0(x)D_{q^{-1}}D_q y + a_1(x)D_q y + a_2(x)y = 0 \tag{4.36}$$

respectively. As it appeared in the theory of general higher order linear q-difference equations, the essential part of the study of (4.33) or (4.35) consists in the one done for (4.34) or (4.36). Hence, the crux of the matter in this section will concern the equation (4.34). Here we consider particularly some questions of solvability and orthogonality of solutions of (4.34) or (4.36).

Solvability

For the questions to be treated here, we can consider quite simply the normalized form of (4.34):

$$D_q^2 y + a_1(x) D_q y + a_2(x) y = 0. \tag{4.37}$$

1. According to the general theory of linear q-difference equations, the equation (4.37) admits two linear independent solutions forming a fundamental system of solutions. But when the coefficients a_1 and a_2 are not constant, there is generally no way for finding in quadratures these solutions. However, when one solution of the equation is known, this allows as in general, to decrease by one the degree of the equation and consequently to find the second solution. Indeed, let $y_1 = y_1(x)$ be one of the solutions. The second solution will be searched under the form:

$$y_2 = z(x) y_1(x) \tag{4.38}$$

where $z(x)$ is an unknown function to be determined. Loading (4.38) in (4.37) and taking in account the fact that y_1 is a solution of (4.37), one obtains the following equation for z:

$$y_1(q^2x) D_q^2 z + \{a_1 y_1(qx) + D_q[y_1(qx)] + [D_q y_1](qx)\} D_q z = 0. \tag{4.39}$$

Letting

$$D_q z = t(x) \tag{4.40}$$

one obtains a first order q-difference equation for $t(x)$, which can naturally be solved explicitly in quadratures.

2. Another way of finding the second solution y_2, once one solution y_1 of (4.37) is known, consists in using a q-version of the so called in differential equations theory Liouville-Ostrogradsky formula. To establish the formula, let write (4.37) in the form

$$y(q^2x) + \tilde{a}_1(x) y(qx) + \tilde{a}_2(x) y(x) = 0, \tag{4.41}$$

where $\tilde{a}_1(x) = a_1(q-1)xq - q - 1$ and $\tilde{a}_2(x) = a_2(q-1)^2x^2q - a_1(q-1)xq + q$. Since y_1 and y_2 are solutions of (4.41), we have

$$\begin{pmatrix} y(x) & y_1(x) & y_2(x) \\ y(qx) & y_1(qx) & y_2(qx) \\ y(q^2x) & y_1(q^2x) & y_2(q^2x) \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{a}_1(x) \\ \tilde{a}_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.42}$$

$$\Leftrightarrow \begin{vmatrix} y(x) & y_1(x) & y_2(x) \\ y(qx) & y_1(qx) & y_2(qx) \\ y(q^2x) & y_1(q^2x) & y_2(q^2x) \end{vmatrix} = 0 \tag{4.43}$$

Developing the determinant by the first column and comparing the resulting equation with (4.37) gives

$$V(qx) = \tilde{a}_2(x)V(x), \tag{4.44}$$

with $V(x) = y_1(x)y_2(qx) - y_2(x)y_1(qx)$. Let

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ D_qy_1(x) & D_qy_2(x) \end{vmatrix} = y_1(x)D_qy_2(x) - y_2(x)D_qy_1(x). \tag{4.45}$$

We have

$$V(x) = (q-1)xW(x). \tag{4.46}$$

Using (4.44), we get

$$D_qW(x) = [(q-1)xa_2(x) - a_1]W(x). \tag{4.47}$$

Clearly, for $q \rightsquigarrow 1$, (4.47) tends to the Liouville-Ostrogradsky formula in the differential calculus. Hence we can refer to (4.47) as q-Liouville-Ostrogradsky formula. On the other side if in (4.47), y_1 is known, the second solution y_2 satisfies a first order q-difference equation and consequently can be found in quadratures.

3. *Solutions in series* . Contemplating the nature of the coefficients $a_i, i = 0, 1, 2,$ in (4.34), one can guess which kind of solutions is involved. For example, if the related coefficients in (4.34) are polynomials, so one can expect that the solutions are of polynomial type. This case will be discussed in details in the chapter 6. Consider here the situation when the coefficients in (4.34) are "analytic functions" at the origin i.e. they can be developed in entire powers series:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n; \quad g(x) = \sum_{n=0}^{\infty} g_n x^n. \tag{4.48}$$

and attempt to prove that the equation admits analytic solutions. We have the following

Theorem 4.4.1 *The second order q-difference equation*

$$D_q^2 y + f(x)D_q + g(x)y = 0 \tag{4.49}$$

with $f(x)$ and $g(x)$ analytic functions say at the origin, admits two linear independent analytic solutions at the origin.

Proof. As $f(x)$ and $g(x)$ are analytic at the origin, they can be developed in entire powers series:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n; \quad g(x) = \sum_{n=0}^{\infty} g_n x^n. \tag{4.50}$$

As the solutions are expected to be analytic at the origin, we write

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{4.51}$$

Loading (4.50) and (4.51) in (4.49) and equating the coefficients to zero, one gets

$$a_{n+2} \frac{q^{n+2}-1}{q-1} \frac{q^{n+1}-1}{q-1} =$$

$$-\sum_{k=0}^{n} (f_{n-k} a_{k+1} \frac{q^{k+1}-1}{q-1} + g_{n-k} a_k), \quad n = 0, 1, 2, \ldots \tag{4.52}$$

This equation allows to determine the coefficients a_n, $n = 2, 3 \ldots$ in (4.51). The coefficients a_0 and a_1 being arbitrarily, they can be chosen so that the corresponding solutions be linear independent, and the theorem is proved.

4. *Constant coefficients.* As noted for the higher order case, second order linear q-difference equations can be solved explicitly. Consider the equation (4.37) with a_1 and a_2 constant in x:

$$D_q^2 y + a_1 D_q y + a_2 y = 0. \tag{4.53}$$

Suppose that the function $e_q(\lambda x)$ is a solution of (4.53). We get

$$\lambda^2 + a_1 \lambda + a_2 = 0, \tag{4.54}$$

which is said to be the characteristic equation of (4.53). Let a and b be the roots of (4.54). This means that (4.53) can be written as

$$(D_q - b)(D_q - a)y = (D_q - a)(D_q - b)y = 0. \tag{4.55}$$

There is here two possibilities:
Case 1. The roots are distinct. In this case, the two independent solutions of (4.53) read clearly

$$y_1(x) = e_q(ax); \quad y_2(x) = e_q(bx) \tag{4.56}$$

Case 2. The equation (4.54) has a double root and we need to solve

$$(D_q - a)^2 y = D_q^2 + a_1 D_q y + a_2 y = 0, \tag{4.57}$$

with $a_1 = -2a$ and $a_2 = a^2$. Here the first solution reads clearly $y_1 = e_q(ax)$. Let search next the second solution under the form $y_2 = z(x)y_1(x)$. Loading this expression in (4.57), one gets

$$y_1(qx)D_q^2 z(x) + [a_1 y_1(qx) + D_q y_1(qx)]D_q z(x) + D_q[y_1(qx)]D_q z(qx) = 0 \tag{4.58}$$

Letting $D_q z(x) = t(x)$, one gets

$$y_1(qx)D_q t(x) + [a_1 y_1(qx) + D_q y_1(qx)]t(x) + D_q[y_1(qx)]t(qx) = 0. \tag{4.59}$$

Considering the value of $y_1(x)$ and taking in account the fact that $D_q e_q(\lambda x) = \lambda e_q(\lambda x)$, (4.59) simplifies in

$$D_q t(x) + [a_1 + a]t(x) + aqt(qx) = 0 \tag{4.60}$$

or

$$D_q t(x) - at(x) + aqt(qx) = 0. \tag{4.61}$$

Searching the solution of (4.61) under the form

$$t(x) = \sum_{n=0}^{\infty} t_n x^n \tag{4.62}$$

we get the recurrence equation for the coefficients

$$\frac{q^{n+1}}{q-1} t_{n+1} + a(q^{n+1} - 1)t_n = 0 \tag{4.63}$$

which solution reads

$$t_n = (1-q)^n a^n t_0. \tag{4.64}$$

Consider next the equation

$$D_q z(x) = t(x) = t_0 \sum_0^\infty (1-q)^n a^n x^n \tag{4.65}$$

and letting $z(x) = \sum_0^\infty z_n x^n$, one gets

$$z_{n+1} \frac{q^{n+1}-1}{q-1} = t_0 (1-q)^n a^n \tag{4.66}$$

or

$$z_n = t_0 \frac{((1-q)a)^n}{a(1-q^n)} = t_0 \frac{(1-q)^{n-1} a^{n-1}}{1+q+q^2+\ldots+q^{n-1}}, n = 1, 2 \ldots \tag{4.67}$$

z_0 being arbitrarily. Hence

$$y_2 = z(x) y_1(x) \tag{4.68}$$

with

$$z(x) = z_0 + t_0 x + t_0 (1-q) \sum_2^\infty \frac{(1-q)^{n-2} a^{n-1}}{1+q+q^2+\ldots+q^{n-1}} x^n \tag{4.69}$$

It is worth remarking that for $q \rightsquigarrow 1$, $z(x) \rightsquigarrow z_0 + t_0 x$ and y_1 and y_2 take the form of solutions of a constant coefficients second order differential equation whose characteristic equation has a double root.

Classification of Singularities. Solutions in the neighborhood of a regular singularity

1. $z = z_0$, *finite.* Consider the equation

$$D_q^2 y(x) + f(x) D_q y(x) + g(x) y(x) = 0 \tag{4.70}$$

where $x \in T$ and let x_0 be a given point in T. If the functions $f(z)$ and $g(z)$ are analytic at the point $x = x_0$, then x_0 is said to be an *ordinary point* of the q-difference equation (4.70). On the other side, if x_0 is not an ordinary point but the functions $(x-x_0)f(x)$ and $(x-x_0)^2 g(x)$ are analytic at $x = x_0$, then $x = x_0$ is said to be a *regular singular point* for the equation (4.70). If $x = x_0$ is a pole for both $f(x)$ and $g(x)$, and l is the least integer such that $(x-x_0)^l f(x)$ and $(x-x_0)^{2l} g(x)$ are both analytic at $x = x_0$, then $x = x_0$ is said to have a *singularity of rank* $l-1$. Thus for example, a regular singular

point is of rank zero. If either $f(x)$ or $g(x)$ has an essential singularity at $x = x_0$, then $x = x_0$ is said to have a singularity of infinite rank.
Consider now the question of solvability of (4.70) in series around a regular singular point $x = x_0$. For simplicity, we let $x_0 = 0$. We remember from the preceding section that if $x = 0$ is an ordinary point, then the equation (4.70) admits a pair of independent solutions given in form of series. We now extend this result in the situation when $x = 0$ is a regular singular point. Clearly, when $x = 0$ is a regular singular point, (4.70) may be written as

$$D_q^2 y(x) + \frac{f(x)}{x} D_q y(x) + \frac{g(x)}{x^2} y(x) = 0 \tag{4.71}$$

where $f(x)$ and $g(x)$ are analytic functions at $x = 0$. Let $f(x) = \sum_{k=0}^{\infty} f_k x^k$ and $g(x) = \sum_{k=0}^{\infty} g_k x^k$ and try the following form for the solution

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha} \tag{4.72}$$

Loading these expressions in (4.71), and equating the resulting coefficient to zero, we get

$$Q(q^{\alpha+k}) a_k = -\sum_{j=0}^{k-1} (f_{k-j} \frac{q^{\alpha+j} - 1}{q - 1} + g_{k-j}) a_j \tag{4.73}$$

$$k = 1, 2, \dots \tag{4.74}$$

where

$$Q(q^{\alpha}) = \frac{q^{\alpha} - 1}{q - 1} \frac{q^{\alpha-1} - 1}{q - 1} + f_0 \frac{q^{\alpha} - 1}{q - 1} + g_0. \tag{4.75}$$

The equation

$$E(\alpha) = Q(q^{\alpha}) = 0 \tag{4.76}$$

is called the *indicial equation* and its roots *indices* or *exponents*. From (4.74) it appears that if the roots of the indicial equation are distinct and do not differ by an integer, then one gets two distinct sequences of coefficients a_k, corresponding to two distinct solutions of (4.71) in form (4.72). In other cases, only one solution of this type is available, unless the rhs of (4.74) vanishes at the same value of the positive integer k for which $E(\alpha + k) = Q(q^{\alpha+k}) = 0$.

2. $z_0 = \infty$. Consider again the equation (4.70). To precise that the q-derivative is performed along the variable x, we will write D_{qx} instead of D_q. Thus we write (4.70) as

$$D_{qx}^2 y(x) + f(x) D_{qx} y(x) + g(x) y(x) = 0 \tag{4.77}$$

To discuss the character of the point at the infinity, we need to make the transformation $x = 1/t$. At the same time the parameter q is replaced by q^{-1}. Considering the relation

$$D_{qx} w(x) = -qt^2 D_{qt} w(1/t) \tag{4.78}$$

the equation (4.70) becomes

$$D_{qt}^2 y(1/t) + p(t) D_{qt} y(1/t) + q(t) y(1/t) = 0 \tag{4.79}$$

where

$$p(t) = \frac{q+1}{q^2 t} - \frac{1}{q^3 t^2} f(1/t); \quad q(t) = \frac{1}{q^4 t^4} g(1/t). \tag{4.80}$$

Hence we have

(1) $z = \infty$ is an ordinary point for the equation (4.77) iff the point $t = 0$ is an ordinary point for the equation (4.79), i.e. the functions $p(t)$ and $q(t)$ are analytic at the point $t = 0$ or equivalently the functions $\frac{q+1}{q} x - \frac{x^2}{q^3} f(x)$ and $\frac{x^4}{q^4} g(x)$ are analytic at the point $x = \infty$.

(2) $z = \infty$ is a regular singular point for the equation (4.77) iff the point $t = 0$ is a regular singular point for (4.79) that is the functions

$$tp(t) = \frac{q+1}{q^2} - \frac{1}{q^3 t} f(1/t); \quad t^2 q(t) = \frac{1}{q^4 t^2} g(1/t) \tag{4.81}$$

are analytic at the point $t = 0$ or equivalently $x f(x)$ and $x^2 g(x)$ are analytic at $x = \infty$. This suggests that

$$f(x) = \frac{1}{x} \sum_{k=0}^{\infty} f_{-k} x^{-k} \tag{4.82}$$

$$g(x) = \frac{1}{x^2} \sum_{k=0}^{\infty} g_{-k} x^{-k}. \tag{4.83}$$

Loading (4.82) and (4.83) in (4.77), we get (4.74) with k and α replaced by $-k$ and $-\alpha$ respectively. For $f(x) = \frac{f_0}{x}$; $g(x) = \frac{g_0}{x^2}$; $y(x) = x^{-\alpha}$, we get

$$\frac{q^{-\alpha}-1}{q-1}\frac{q^{-\alpha-1}-1}{q-1} + \frac{q^{-\alpha}-1}{q-1}f_0 + g_0 = 0 \tag{4.84}$$

also called *indicial* equation.

3. *Example.* Consider for example the *q-hypergeometric equation*[34]

$$x(q^c - q^{a+b+1}x)D_q^2 y(x) + [\frac{1-q^c}{1-q} - (q^b\frac{1-q^a}{1-q} + q^a\frac{1-q^{b+1}}{1-q})x]D_q y(x) \\ - \frac{1-q^a}{1-q}\frac{1-q^b}{1-q}y(x) = 0. \tag{4.85}$$

This is a q-version of the classical hypergeometric differential equation

$$x(1-x)y''(x) + (c - (a+b+1)z)y'(x) - aby(x) = 0 \tag{4.86}$$

having as particular solutions

$$y_1(x) =_2 F_1(a,b;c;x); \quad y_2(x) = x_2^{1-c}F_1(a-c+1, b-c+1; 2-c; x). \tag{4.87}$$

The equation (4.85) can clearly be written in the form (4.70) with

$$f(x) = \frac{(1-q^c) - (q^b(1-q^a) + q^a(1-q^{b+1}))x}{(1-q)x(q^c - q^{a+b+1}x)}$$
$$g(x) = \frac{(q^a-1)(1-q^b)}{(1-q)^2 x(q^c - q^{a+b+1}x)}. \tag{4.88}$$

The equation (4.85) has three regular singularities at the points $x = 0$, $x = q^{c-a-b-1}$ and $x = \infty$ corresponding to the regular singularities $x = 0$, $x = 1$ and $x = \infty$ of (4.86) .
Considering the indicial equation (4.76), that one can write

$$[\alpha]_q^2 - (1 - qf_0)[\alpha]_q + qg_0 = 0, \tag{4.89}$$

where

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \tag{4.90}$$

we have

(1) For $x = 0$:

$$f_0 = -(q^{-c} - 1)/(-1 + q); \quad g_0 = 0 \tag{4.91}$$

and

$$[\alpha]_{q,1} = 0; \quad [\alpha]_{q,2} = (-1 + q^{-c+1})/(-1 + q). \tag{4.92}$$

(2) For $x = q^{c-a-b-1}$:

$$f_0 = -(-q^{-c+1} + q^{-a} - 1 + q^{-b})/(q(-1 + q)); \quad g_0 = 0 \tag{4.93}$$

and

$$[\alpha]_{q,1} = 0; \quad [\alpha]_{q,2} = -(q^{-c+1} - q^{-a} + 2 - q - q^{-b})/(-1 + q). \tag{4.94}$$

For the regular singular point at $x = \infty$, we consider the indicial equation (4.84), that one can write

$$[-\alpha]_q^2 - (1 - qf_0)[-\alpha]_q + qg_0 = 0, \tag{4.95}$$

where

$$[-\alpha]_q = \frac{1 - q^{-\alpha}}{1 - q}, \tag{4.96}$$

for

$$\begin{aligned} f_0 &= -(q^{-a} - 1 + q^{-b} - q)/(q(-1 + q)); \\ g_0 &= -(-q^{-a-b} + q^{-a} + q^{-b} - 1)/((-1 + q)^2 q) \end{aligned} \tag{4.97}$$

and obtain

$$\begin{aligned} [-\alpha]_{q,1} = 1/2(2q - q^2 - q^{-a} + 1 - q^{-b} \\ -sqrt((-2q^2 - 4q^3 + q^4 - 4q^{-a+1} + 6q^{-a+2} + 4q \\ -4q^{-b+1} + 6q^{-b+2} + q^{-2a} - 2q^{-a} + 2q^{-a-b} \\ +1 - 2q^{-b} + q^{-2b} - 4q^{2-a-b})/(q^2(-1 + q)^2))q \\ +sqrt((-2q^2 - 4q^3 + q^4 - 4q^{-a+1} + 6q^{-a+2} + 4q - 4q^{-b+1} \\ +6q^{-b+2} + q^{-2a} - 2q^{-a} + 2q^{-a-b} \\ +1 - 2q^{-b} + q^{-2b} \\ -4q^{2-a-b})/(q^2(-1 + q)^2))q^2)/(q^2(-1 + q)) \end{aligned} \tag{4.98}$$

and

$$\begin{aligned}
[-\alpha]_{q,2} &= -1/2(-2q + q^2 + q^{-a} - 1 + q^{-b} \\
&-sqrt((-2q^2 - 4q^3 + q^4 - 4q^{-a+1} + 6q^{-a+2} \\
&+4q - 4q^{-b+1} + 6q^{-b+2} + q^{-2a} - 2q^{-a} + 2q^{-a-b} \\
&+1 - 2q^{-b} + q^{-2b} - 4q^{2-a-b})/(q^2(-1+q)^2))q \\
&+sqrt((-2q^2 - 4q^3 + q^4 - 4q^{-a+1} + 6q^{-a+2} + 4q - 4q^{-b+1} \\
&+6q^{-b+2} + q^{-2a} - 2q^{-a} + 2q^{-a-b} \\
&+1 - 2q^{-b} + q^{-2b} \\
&-4q^{2-a-b})/(q^2(-1+q)^2))q^2)/(q^2(-1+q)).
\end{aligned} \tag{4.99}$$

Consider for example the simplest case when the regular singularity is located at $x = 0$. Searching the solution under the form

$$y(x) = \sum_{s=0}^{\infty} c_s x^s \tag{4.100}$$

we get

$$\frac{c_{s+1}}{c_s} = \frac{(1 - q^{a+\alpha+s})(1 - q^{b+\alpha+s})}{(1 - q^{c+\alpha+s})(1 - q^{1+\alpha+s})} \tag{4.101}$$

which lead to the particular solutions of (4.85)

$$y_1(x) = {}_2\phi_1(q^a, q^b; q^c; q, x), \;\; y_2(x) = x^{1-c}{}_2\phi_1(q^{1+a-c}, q^{1+b-c}; q^{2-c}; q, x), \tag{4.102}$$

(q-analogues of (4.87)), corresponding to the roots of the indicial equations $[\alpha]_{q,1} = 0$ and $[\alpha]_{q,2} = [1 - c]$ respectively.

Orthogonality

For this purpose, we write the second order linear q-difference equation in the form

$$a_0(x) D_{q^{-1}} D_q y + a_1(x) D_q y + a_2(x) y = 0. \tag{4.103}$$

This equation can be written as

$$A(x)y(x) = [u(x)E_q + v(x) + w(x)E_q^{-1}]y(x) = \lambda y(x), \tag{4.104}$$

where $E_qf(x) = f(qx)$ and $E_q^{-1}f(x) = E_{q^{-1}}f(x) = f(x/q)$. Let $y_n(x)$ and $y_m(x)$ be two sequences of its egenfunctions corresponding to two distinct sequences of eigenvalues λ_n and λ_m respectively. We will search the conditions under which $y_n(x)$ and $y_m(x)$ are orthogonal. The usual receipt is to find the conditions under which

$$(A(x)y_n(x), y_m(x))_{\rho(x)} = (y_n(x), A(x)y_m(x))_{\rho(x)} \tag{4.105}$$

where $(f(x), g(x))_{\rho(x)}$ stands for the q-discrete weighted inner product:

$$(f(x), g(x))_{\rho(x)} =^{def} \int_a^b f(x)g(x)\rho(x)d_qx. \tag{4.106}$$

Substracting member with member the equalities

$$(Ay_n)y_m = \lambda_n y_n y_m \tag{4.107}$$

and

$$(Ay_m)y_n = \lambda_m y_n y_m \tag{4.108}$$

one gets

$$(\lambda_n - \lambda_m)y_n y_m = (Ay_n)y_m - (Ay_m)y_n. \tag{4.109}$$

Multiplying next the two members of the equality by $\rho(x)$ and q-integrating from 0 to x, we obtain

$$\begin{gathered}(\lambda_n - \lambda_m)(1-q)x\sum_{i=0}^{\infty} q^i y_n(xq^i)y_m(xq^i)\rho(xq^i) \\ = (1-q)x\sum_{i=0}^{\infty} q^i[(Ay_n)(xq^i)y_m(xq^i) - (Ay_m)(xq^i)y_n(xq^i)]\rho(xq^i).\end{gathered} \tag{4.110}$$

Simplifications give

$$\begin{gathered}(\lambda_n - \lambda_m)(1-q)x\sum_{i=0}^{\infty} q^i y_n(xq^i)y_m(xq^i)\rho(xq^i) \\ = (1-q)xq^{-1}u(xq^{-1})\rho(xq^{-1})[y_n(xq^{-1})y_m(x) - y_n(x)y_m(xq^{-1})],\end{gathered} \tag{4.111}$$

under the constraint defining the q-discrete weight $\rho(x)$:

$$\frac{\rho(qx)}{\rho(x)} = \frac{u(x)}{qw(qx)}. \tag{4.112}$$

As a consequence we get

$$\begin{gathered}(\lambda_n - \lambda_m)\int_a^b y_n(x)y_m(x)\rho(x)d_qx \\ = (1-q)xq^{-1}u(xq^{-1})\rho(xq^{-1})[y_n(xq^{-1})y_m(x) - y_n(x)y_m(xq^{-1})]\mid_a^b\end{gathered} \tag{4.113}$$

From where we obtain the condition of q-orthogonality for y_n and y_m, $n \neq m$:

$$u(xq^{-1})\rho(xq^{-1})[y_n(xq^{-1})y_m(x) - y_n(x)y_m(xq^{-1})]\mid_a^b = 0. \tag{4.114}$$

4.5 Exercises

1. Prove that

a) the functions $cos_q(x)$, $sin_q(x)$ are solutions of

$$\begin{aligned} D_q^2 y(x) + y(x) = 0, \\ D_q^4 y(x) + y(x) = 0, \end{aligned} \tag{4.115}$$

b) the functions $cosh_q(x)$, $sinh_q(x)$ are solutions of

$$\begin{aligned} D_q^2 y(x) - y(x) = 0, \\ D_q^4 y(x) - y(x) = 0, \end{aligned} \tag{4.116}$$

c) the functions $cos_{q^{-1}}(x)$, $sin_{q^{-1}}(x)$ are solutions of

$$\begin{aligned} D_{q^{-1}}^2 y(x) + y(x) = 0, \\ D_{q^{-1}}^4 y(x) + y(x) = 0, \end{aligned} \tag{4.117}$$

d) the functions $cosh_{q^{-1}}(x)$, $sinh_{q^{-1}}(x)$ are solutions of

$$\begin{aligned} D_{q^{-1}}^2 y(x) - y(x) = 0, \\ D_{q^{-1}}^4 y(x) - y(x) = 0, \end{aligned} \tag{4.118}$$

e)the functions $cos_{qq^{-1}}(x)$, $sin_{qq^{-1}}(x)$ are solutions of

$$(D_q + 1)(D_{q^{-1}} + 1)y(x) = 0, \tag{4.119}$$

f) the functions $cosh_{qq^{-1}}(x)$, $sinh_{qq^{-1}}(x)$ are solutions of

$$(D_q - 1)(D_{q^{-1}} - 1)y(x) = 0. \tag{4.120}$$

2. Solve

a) $D_q^2 y(x) = ay(x) + b$,

b) $D_q^2 y(x) - 6D_q y(x) + 6y(x) = x^2$

c)

$$y(q^{-1}x) = \frac{ay(x)+b}{cy(x)+d}. \tag{4.121}$$

3. Find two q-operators $A = D_q^3 + u_1(x)D_q^2 + u_2(x)D_q + u_3(x)$ and $B = D_q^2 + v_1(x)D_q + v_2(x)$ such that their q-commutation

$$[A, B] = A.B - qB.A \tag{4.122}$$

is an operator of degree zero in D_q.

4. Prove that the functions $sin_{q^{-1}}(x)$ and $cos_{q^{-1}}(x)$ are solutions of

$$(q-1)^2 D_q^2 y(x) + qy(q^2 x) = 0 \tag{4.123}$$

and that the functions $sin_q(x)$ and $cos_q(x)$ solve

$$(q-1)^2 D_q^2 y(x) + y(x) = 0. \tag{4.124}$$

Chapter 5

q-Laplace transform

Laplace transform of an exponential type function $f(x)$ is given by

$$F(p) = \mathcal{L}\{f(x)\} = \int_0^{+\infty} e^{-px} f(x)dx,\ p = a + ib \in \mathbf{C}. \tag{5.1}$$

and plays a major role in pure and applied analysis, specially in solving differential equations. If we consider $f(x)$ as a function of a discrete variable i.e. $t \in \mathbf{Z}$, then the transformation (5.1) reads

$$F(z) = Z\{f(x)\} = \sum_{j=0}^{+\infty} f(j) z^{-j},\ z = e^{-p}. \tag{5.2}$$

It is referred to as *Z transform* and plays similar role in difference analysis as Laplace transform in continuous analysis, specially in solving difference equations.

In this chapter, we are concerned in a q-version of the Laplace transform (5.1) which is expected to play similar role in q-difference analysis as Laplace transform in continuous analysis or Z transform in difference analysis, specially in solving q-difference equations.

Studies of q-versions of Laplace transform go back up to Hahn [30]. Researches in the area were then pursued in many works by W H Abdi [1, 2, 3, 4] and more recently in [36]. However, there is very significant difference between the approach in the latter work and the previous ones. Indeed, in the studies by Hahn and Abdi, the q-version of the Laplace transform consists in choosing a q-version of the exponential function e^{-px} and then replace the integral in (5.1) by the corresponding q-integral. On the other side, in work of Lenzi and coauthors, the q-version of the Laplace transform consists in replacing simply the e^{-px} function in (5.1) by its certain q-deformation.

In this chapter, we are clearly concerned in the first of the two approaches. However, contrarily to Hahn or Abdi, it seemed to us natural to take $e_{q^{-1}}^{-px}$, as the q-version of e^{-px}, since $e_{q^{-1}}^{-px}$ is the exact inverse of e_q^{px}, as e^{-px} is the exact inverse of e^{px}. More exactly, for a given function $f(x)$ on the lattice (1.9), we define its q-Laplace transform as the function

$$F(p) = \mathcal{L}_q\{f(x)\} = \int_0^{+\infty} e_{q^{-1}}^{-px} f(x) d_q x,\ p = s + i\sigma \in \mathbf{C}. \tag{5.3}$$

and we denote $f(x) \rightleftharpoons_q F(p)$. Here, $f(x)$ is referred to as the *q-original* of $F(p)$, while $F(p)$ is referred to as the *q-image* of $f(x)$ by the q-Laplace transform operation. In the following paragraphes, we study its basic properties and apply it to certain q-difference equations.

5.1 Properties of the q-Laplace transform

1. *Linearity.* Clearly

$$\mathcal{L}_q\{\alpha f(x) + \beta g(x)\} = \alpha \mathcal{L}_q\{f(x)\} + \beta \mathcal{L}_q\{g(x)\} \tag{5.4}$$

2. *Scaling.* Let

$$F(p) = \int_0^{+\infty} e_{q^{-1}}^{-px} f(x) d_q x. \tag{5.5}$$

We get

$$\begin{aligned} F(\frac{p}{\alpha}) &= \int_0^{+\infty} e_{q^{-1}}^{-\frac{p}{\alpha}x} f(x) d_q x \\ &= \alpha \int_0^{+\infty} e_{q^{-1}}^{-px} f(\alpha x) d_q x. \end{aligned} \tag{5.6}$$

Or

$$\begin{aligned} \frac{1}{\alpha} F(\frac{p}{\alpha}) &= \int_0^{+\infty} e_{q^{-1}}^{-\frac{p}{\alpha}x} f(x) d_q x \\ &= \int_0^{+\infty} e_{q^{-1}}^{-px} f(\alpha x) d_q x. \end{aligned} \tag{5.7}$$

Hence

$$f(\alpha x) \rightleftharpoons_q \frac{1}{\alpha} F(\frac{p}{\alpha}). \tag{5.8}$$

3. *Attenuation*, or *Substitution.* We have

$$\begin{aligned} F(p-p_0) &= \int_0^{+\infty} e_{q^{-1}}^{-(p-p_0)x} f(x) d_q x \\ &= \int_0^{+\infty} e_{q^{-1}}^{-px} [e_{q^{-1}}^{-px+p_0x} e_q^{px} f(x)] d_q x \\ &= \mathcal{L}_q \{ e_{q^{-1}}^{-px+p_0x} e_q^{px} f(x) \}. \end{aligned} \tag{5.9}$$

Hence, *formally*

$$e_{q^{-1}}^{-px+p_0x} e_q^{px} f(x) \rightleftharpoons_q F(p-p_0). \tag{5.10}$$

4. *Transform of derivatives.* Let

$$f(x) \rightleftharpoons_q F(p). \tag{5.11}$$

We have

$$\begin{aligned} G(p) &= \int_0^{+\infty} e_{q^{-1}}^{-px} [D_q f(x)] d_q x \\ &= [e_{q^{-1}}^{-px} f(x)]_0^{+\infty} + p \int_0^{\infty} f(qx) e_{q^{-1}}^{-pqx} d_q x \\ &= -f(0) + p \int_0^{\infty} f(qx) e_{q^{-1}}^{-pqx} d_q x \\ &= -f(0) + pq^{-1} \int_0^{\infty} f(x) e_{q^{-1}}^{-px} d_q x \\ &= \frac{p}{q} F(p) - f(0), \end{aligned} \tag{5.12}$$

where we used the q-integration by parts and the change of variable $x := xq^{-1}$. Thus

$$D_q f(x) \rightleftharpoons_q \frac{p}{q} F(p) - f(0). \tag{5.13}$$

As a consequence, we get

$$\begin{aligned} D_q^2 f(x) &\rightleftharpoons_q \frac{p}{q} [\frac{p}{q} F(p) - f(0)] - D_q f(0) \\ &= \frac{p^2}{q^2} F(p) - \frac{p}{q} f(0) - D_q f(0), \end{aligned} \tag{5.14}$$

$$D_q^3 f(x) \rightleftharpoons_q \frac{p^3}{q^3}F(p) - \frac{p^2}{q^2}f(0) - \frac{p}{q}D_q f(0) - D_q^2 f(0), \tag{5.15}$$

$$\dots\dots\dots \tag{5.16}$$

$$D_q^n f(x) \rightleftharpoons_q \frac{p^n}{q^n}F(p) - \frac{p^{n-1}}{q^{n-1}}f(0) - \frac{p^{n-2}}{q^{n-2}}D_q f(0) - \dots - D_q^{n-1}f(0). \tag{5.17}$$

Hence

$$D_q^n f(x) \rightleftharpoons_q \frac{p^n}{q^n}F(p) - \sum_{j=0}^{n-1}(\frac{p}{q})^{n-1-j}D_q^j f(0). \tag{5.18}$$

5. *Derivative of transforms.* Again, let

$$F(p) = \int_0^{+\infty} e_{q^{-1}}^{-px} f(x) d_q x. \tag{5.19}$$

Calculate

$$D_{q,p} e_{q^{-1}}^{-px} = -x e_{q^{-1}}^{-pqx}, \tag{5.20}$$

$$D_{q,p}^2 e_{q^{-1}}^{-px} = (-x)(-qx) e_{q^{-1}}^{-pq^2 x}, \tag{5.21}$$

$$D_{q,p}^3 e_{q^{-1}}^{-px} = (-x)(-qx)(-q^2 x) e_{q^{-1}}^{-pq^3 x}, \tag{5.22}$$

$$\dots\ \dots\ \dots \tag{5.23}$$

$$\begin{aligned} D_{q,p}^n e_{q^{-1}}^{-px} &= (-x)(-qx)(-q^2 x)\dots(-q^{n-1}x) e_{q^{-1}}^{-pq^n x} \\ &= (-x)^n q^{\frac{n}{2}(n-1)} e_{q^{-1}}^{-pq^n x}. \end{aligned} \tag{5.24}$$

Thus

$$D_{q,p}^n F(p) = \int_0^{+\infty} (-x)^n q^{\frac{n}{2}(n-1)} e_{q^{-1}}^{-pq^n x} f(x) d_q x \tag{5.25}$$

Using the summation form of the integral and then making the replacing $x := xq^{-n}$ gives

$$
\begin{aligned}
D^n_{q,p}F(p) &= (1-q)\sum_{x=0}^{+\infty} x(-x)^n q^{\frac{n}{2}(n-1)} e_{q^{-1}}^{-pq^n x} f(x) \\
&= q^{-n}(1-q)\sum_{x=0}^{+\infty} x e_{q^{-1}}^{-px}(-xq^{-n})^n q^{\frac{n}{2}(n-1)} f(xq^{-n}) \\
&= q^{-n}\int_0^{+\infty} e_{q^{-1}}^{-px}(-xq^{-n})^n q^{\frac{n}{2}(n-1)} f(xq^{-n})d_qx \\
&= \int_0^{+\infty} e_{q^{-1}}^{-px}[(-x)^n q^{\frac{-n}{2}(n+3)} f(xq^{-n})]d_qx \\
&= \mathcal{L}_q\{(-x)^n q^{\frac{-n}{2}(n+3)} f(xq^{-n})\}. \qquad (5.26)
\end{aligned}
$$

Hence

$$
(-x)^n q^{\frac{-n}{2}(n+3)} f(xq^{-n}) \rightleftharpoons_q D^n_{q,p}F(P). \qquad (5.27)
$$

6. *Transform of integrals.* We have

$$
\begin{aligned}
\mathcal{L}_q\{\int_0^x f(x)d_qx\} &= \int_0^\infty e_{q^{-1}}^{-px}[\int_0^x f(x)d_qx]d_qx \\
&= -\frac{q}{p}\int_0^\infty [\int_0^x f(x)d_qx][D_q e_{q^{-1}}^{-\frac{p}{q}x}]d_qx \\
&= -\frac{q}{p}[(\int_0^x f(x)d_qx)e_{q^{-1}}^{-\frac{p}{q}x}]_0^\infty + \frac{q}{p}\int_0^\infty e_{q^{-1}}^{-px} f(x)d_qx \\
&= \frac{q}{p}\int_0^\infty e_{q^{-1}}^{-px} f(x)d_qx = q\frac{F(p)}{p}. \qquad (5.28)
\end{aligned}
$$

Hence

$$
\int_0^x f(x)d_qx \rightleftharpoons_q q\frac{F(p)}{p} \qquad (5.29)
$$

7. *Integral of transforms.* q-Integrating (5.5) both sides from p to ∞, and interchanging the integrals (supposing that the conditions for this are satisfied), we get

$$
\int_p^\infty F(p)d_qp = \int_0^\infty (\int_p^\infty e_{q^{-1}}^{-px} d_qp) f(x)d_qx
$$

$$= q\int_0^\infty \frac{e_{q^{-1}}^{-\frac{p}{q}x}}{x} f(x)d_qx$$

$$= q\int_0^\infty e_{q^{-1}}^{-px}\frac{f(qx)}{x}d_qx = q\mathcal{L}_q\{\frac{f(qx)}{x}\}. \tag{5.30}$$

Hence

$$\int_p^\infty F(p)d_qp \rightleftharpoons_q q\frac{f(qx)}{x}. \tag{5.31}$$

This formula is especially useful for the calculus of infinite integrals. Indeed, letting $p \to 0$ in (5.31), we get the useful formula

$$\int_0^\infty F(p)d_qp = q\int_0^\infty \frac{f(qx)}{x}d_qx = q\int_0^\infty \frac{f(x)}{x}d_qx. \tag{5.32}$$

8. *Product of transforms.*

Let define the *q-convolution product* between f and g as

$$f(x) *_q g(x) =^{def} \int_0^x f(x)\tilde{g}(x-\tau)d_qx \tag{5.33}$$

where the relation between $g(x)$ and $\tilde{g}(x)$ is to be determined latter, in order that be fulfilled the condition

$$f(x) *_q g(x) \rightleftharpoons_q \mathcal{L}_q\{f(x)\}\mathcal{L}_q\{g(x)\} = F(p)G(p). \tag{5.34}$$

We have

$$\begin{aligned} f(x) *_q g(x) &\rightleftharpoons_q \int_0^\infty e_{q^{-1}}^{-px}(\int_0^x f(t)\tilde{g}(x-t)d_qt)d_qx \\ &= \int_0^\infty\int_0^x e_{q^{-1}}^{-px} f(x)\tilde{g}(x-t)d_qtd_qx \\ &= \int_0^\infty\int_{qt}^\infty e_{q^{-1}}^{-px} f(t)\tilde{g}(x-t)d_qxd_qt \\ &= \int_0^\infty f(t)(\int_{qt}^\infty e_{q^{-1}}^{-px}\tilde{g}(x-t)d_qx)d_qt. \end{aligned} \tag{5.35}$$

We calculate

$$I_1 = \int_{qt}^\infty e_{q^{-1}}^{-px}\tilde{g}(x-t)d_qx$$

$$= (1-q)\sum_{x=qt}^{\infty} q^{-1}xe_{q^{-1}}^{-pq^{-1}x}\tilde{g}(q^{-1}x-t)$$

$$= (1-q)\sum_{x=t}^{\infty} xe_{q^{-1}}^{-px}\tilde{g}(x-t)$$

$$= (1-q)\sum_{r=0}^{\infty}(r+t)e_{q^{-1}}^{-p(r+t)}\tilde{g}(r)$$

$$= (1-q)\sum_{r=0}^{\infty} r[\frac{r+t}{r}e_{q^{-1}}^{-p(r+t)}\tilde{g}(r)]$$

$$= \int_{r=0}^{\infty}\frac{r+t}{r}e_{q^{-1}}^{-p(r+t)}\tilde{g}(r)d_q r$$

$$= e_{q^{-1}}^{-pt}\int_{r=0}^{\infty} e_{q^{-1}}^{-pr}[\frac{r+t}{r}e_{q^{-1}}^{-p(r+t)}e_q^{pt}e_q^{pr}\tilde{g}(r)]d_q r \tag{5.36}$$

Thus, if we set

$$\tilde{g}(r) = \frac{r}{r+t}e_q^{p(r+t)}e_{q^{-1}}^{-pt}e_{q^{-1}}^{-pr}g(r), \tag{5.37}$$

or

$$\tilde{g}(x-t) = \frac{x-t}{x}e_q^{px}e_{q^{-1}}^{-pt}e_{q^{-1}}^{-p(x-t)}g(x-t), \tag{5.38}$$

we get

$$f(x) *_q g(x) \rightleftharpoons_q (\int_0^{\infty} e_{q^{-1}}^{-pt}f(t)d_q t)(\int_0^{\infty} e_{q^{-1}}^{-pr}g(r)d_q r)$$
$$= F(p)G(p). \tag{5.39}$$

In other words, if we define *formally* the q-convolution product between f and g as

$$f(x) *_q g(x) =^{def} \int_0^x f(x)\frac{x-t}{x}e_q^{px}e_{q^{-1}}^{-pt}e_{q^{-1}}^{-p(x-t)}g(x-t)d_q x, \tag{5.40}$$

we get

$$f(x) *_q g(x) \rightleftharpoons_q F(p)G(p). \tag{5.41}$$

5.2 q-Laplace transforms of some elementary functions

1. $f(x) = 1$. We have

$$F(p) = \int_0^\infty e_{q^{-1}}^{-px} d_q x = -\frac{q}{p}\int_0^\infty D_q e_{q^{-1}}^{-\frac{p}{q}x} d_q x$$
$$= -\frac{q}{p}[e_{q^{-1}}^{-\frac{p}{q}x}]_0^\infty = \frac{q}{p} \tag{5.42}$$

2. $f(x) = x$. We calculate

$$F(p) = \int_0^\infty e_{q^{-1}}^{-px} x d_q x = -\frac{q}{p}\int_0^\infty x D_q e_{q^{-1}}^{-\frac{p}{q}x} d_q x$$
$$= -\frac{q}{p}\{[x e_{q^{-1}}^{-\frac{p}{q}x}]_0^\infty - \int_0^\infty e_{q^{-1}}^{-px} d_q x\}$$
$$= \frac{q}{p}\int_0^\infty e_{q^{-1}}^{-px} d_q x = \frac{q^2}{p^2}. \tag{5.43}$$

3. $f(x) = x^n$. Contemplate the formula (5.27) and consider the case when $f(t) = 1$. Then, using (5.42), we obtain the following

$$(-x)^n \rightleftharpoons_q q^{\frac{n}{2}(n+3)} D_{q,p}^n(\frac{q}{p}). \tag{5.44}$$

Next, using iteratively the fact that

$$D_q p^{-k} = \frac{q^{-k}-1}{q-1} p^{-(k+1)},\ k = 1, 2, \ldots, \tag{5.45}$$

one easily finds that

$$D_q^n p^{-1} = (-1)^n q^{-\frac{n}{2}(n+1)} [n]_q! p^{-(n+1)}. \tag{5.46}$$

Finally (5.44) and (5.46) leads to

$$x^n \rightleftharpoons_q [n]_q! (\frac{q}{p})^{n+1}. \tag{5.47}$$

4. $f(x) = \delta(x - x_0) = \{^{1,\ x=x_0=q^{s_0}}_{0,\ x\neq x_0}$. We have

$$F(p) = \int_0^\infty e_{q^{-1}}^{-px}\delta(x - x_0)d_qx$$
$$= (1-q)\sum_{i=-\infty}^{\infty} q^i e_{q^{-1}}^{-pq^i}\delta(q^i - q^{s_0})$$
$$= (1-q)q^{s_0}e_{q^{-1}}^{-pq^{s_0}} = (1-q)x_0 e_{q^{-1}}^{-px_0}. \tag{5.48}$$

5. $f(x) = e_q^{ax}$. Since $e_q^{ax} = \sum_{n=0}^{\infty}\frac{a^n x^n}{[n]_q!}$, we get

$$F(p) = \sum_0^\infty \frac{a^n}{[n]_q!}\mathcal{L}_q\{x^n\} = \frac{q}{p}\sum_0^\infty (\frac{qa}{p})^n$$
$$= \frac{q}{p - qa}, \tag{5.49}$$

where we used (5.47).

6. $f(x) = e_{q^{-1}}^{ax}$. Since

$$e_{q^{-1}}^{ax} = \sum_{n=0}^{\infty}\frac{a^n x^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} q^{\frac{n}{2}(n-1)}\frac{a^n x^n}{[n]_q!}, \tag{5.50}$$

we have

$$F(p) = \sum_0^\infty q^{\frac{n}{2}(n-1)}\frac{a^n}{[n]_q!}\mathcal{L}_q\{x^n\} = \frac{q}{p}\sum_0^\infty q^{\frac{n}{2}(n-1)}(\frac{qa}{p})^n. \tag{5.51}$$

7. $f(x) = cos_q wx = \frac{e_q^{iwx} + e_q^{-iwx}}{2}$. Now

$$F(p) = \mathcal{L}_q\{cos_q wx\} = \frac{\mathcal{L}_q\{e_q^{iwx}\} + \mathcal{L}_q\{e_q^{-iwx}\}}{2}$$
$$= \frac{qp}{p^2 + q^2w^2} \tag{5.52}$$

8. $f(x) = sin_q wx = \frac{e_q^{iwx} - e_q^{-iwx}}{2i}$. We get

$$F(p) = \mathcal{L}_q\{sin_q wx\} = \frac{\mathcal{L}_q\{e_q^{iwx}\} - \mathcal{L}_q\{e_q^{-iwx}\}}{2i}$$
$$= \frac{q^2w}{p^2 + w^2}. \tag{5.53}$$

9. $f(x) = cosh_q wx = \frac{e_q^{wx} + e_q^{-wx}}{2}$. We have

$$F(p) = \mathcal{L}_q\{cosh_q wx\} = \frac{\mathcal{L}_q\{e_q^{wx}\} + \mathcal{L}_q\{e_q^{-wx}\}}{2}$$
$$= \frac{qp}{p^2 - q^2 w^2}. \tag{5.54}$$

10. $f(x) = sinh_q wx = \frac{e_q^{wx} - e_q^{-wx}}{2}$. We get

$$F(p) = \mathcal{L}_q\{sin_q wx\} = \frac{\mathcal{L}_q\{e_q^{wx}\} - \mathcal{L}_q\{e_q^{-wx}\}}{2}$$
$$= \frac{q^2 w}{p^2 - w^2}. \tag{5.55}$$

11. $f(x) = \sum_0^\infty a_n x^n$. We get

$$F(p) = \sum_0^\infty a_n \mathcal{L}_q\{x^n\} = \frac{q}{p}\sum_0^\infty a_n [n]_q! (\frac{q}{p})^n. \tag{5.56}$$

5.3 Inverse q-Laplace transform

In most of the cases, the search of the q-original of a given q-image is performed using the results of the transformation of basic elementary functions combined with the application of the properties of the q-Laplace transform. In other cases, it is useful to refer to the so called first or second theorems of development.

Theorem 5.3.1 *(First theorem of development) If the q-image of the unknown q-original can be developed in an integer series of powers of $\frac{1}{p}$, i.e.*

$$F(p) = \sum_{j=0}^\infty a_j p^{-j-1} \tag{5.57}$$

(this series is convergent to $F(p)$ for $|p| > R$, where $R = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \neq \infty$.), then the q-original $f(x)$ is given by the formula

$$f(x) = \sum_{j=0}^\infty \frac{a_j}{q^{j+1}[j]_q!} x^j \tag{5.58}$$

the series being convergent for every value of x.

Proof. Note first that if a function $g(x)$ is given by the power series $g(x) = \sum_{j=0}^{\infty} c_j x^j$, then clearly $c_j = \frac{D_q^j g(0)}{[j]_q!}$. Hence, if $F(p) = \sum_{j=0}^{\infty} a_j p^{-j-1}$, then $a_j = \frac{D_q^j [p^{-1}F(p^{-1})]_{p=0}}{[j]_q!}$. Next, denoting the inverse of the q-Laplace transform by $f(t) = \mathcal{L}_q^{-1}\{F(p)\}$, we calculate

$$\begin{aligned} f(t) = \mathcal{L}_q^{-1}\{F(p)\} &= \sum_0^{\infty} \frac{D_q^j [p^{-1}F(p^{-1})]_{p=0}}{[j]_q!} \mathcal{L}_q^{-1}\{p^{-j-1}\} \\ &= \sum_0^{\infty} \frac{D_q^j [p^{-1}F(p^{-1})]_{p=0}}{[j]_q!} \left(\frac{x^j}{[j]_q! q^{j+1}}\right) \\ = \sum_0^{\infty} \frac{D_q^j [p^{-1}F(p^{-1})]_{p=0}}{[j]_q^2! q^{j+1}} x^j &= \sum_{j=0}^{\infty} \frac{a_j}{q^{j+1}[j]_q!} x^j \end{aligned} \tag{5.59}$$

and the theorem is proved.

Example. Using the first theorem of development, find the inverse of $F(p) = \frac{1}{p-p_0}$.

Solution. We have $F(p) = \sum_{j=0}^{\infty} p_0^j p^{-j-1}$. Hence $f(x) = \sum_{j=0}^{\infty} \frac{p_0^j}{[j]_q! q^{j+1}} x^j = \frac{1}{q} \sum_{j=0}^{\infty} \frac{(p_0 q^{-1} x)^j}{[j]_q!} = q^{-1} e_q^{p_0 q^{-1} x}$.

The *second theorem of development* gives the possibility of determining the q-original of a q-image that is a rational function of p:

$$F(p) = \frac{u(p)}{v(p)}, \tag{5.60}$$

where $u(p)$ and $v(p)$ are polynomial functions of p of degree m and k ($m < k$), respectively. If the development of the function $v(p)$ in simple factors has the form

$$v(p) = \prod_{i=1}^{r} (p - p_i)^{k_i}, \ \left(\sum_{i=1}^{r} k_i = k\right), \tag{5.61}$$

then it is known that $F(p)$ can be developed in sum of simple fractions of the form

$$\frac{A}{(p - p_0)^n} \tag{5.62}$$

where A is a constant, p_0 a root of $v(p)$ and n is $\leq$ its algebraic multiplicity. Hence, to handle the inversion of a function of type (5.60), it suffices to handle that of functions of type

$$\frac{1}{(p-p_0)^n} \tag{5.63}$$

If all the roots of $v(p)$ are simple, then $n \equiv 1$ and the problem is reduced to the inversion of

$$\frac{1}{p-p_0}. \tag{5.64}$$

In this case, one quickly thinks about (5.49) and gets

$$q^{-1}e_q^{p_0 q^{-1}t} \rightleftharpoons_q \frac{1}{p-p_0}. \tag{5.65}$$

On the other side, if some of the roots of $v(p)$ are not simple, one should have to deal with the inversion of (5.63) with $n > 1$. In this case, one can also attempt to use the combining of (5.39) and (5.10) which *formally* gives

$$\frac{1}{(p-p_0)^{n+1}} \rightleftharpoons_q \frac{e_{q^{-1}}^{-pt+p_0 t} e_q^{pt} t^n}{q^{n+1}[n]_q!}. \tag{5.66}$$

However, considering even the simplest case of $n = 0$ in (5.66), we have

$$e_q^{p_0 q^{-1}x} \rightleftharpoons_q \frac{q}{p-p_0} \rightleftharpoons_q e_{q^{-1}}^{-pt+p_0 t} e_q^{pt} \tag{5.67}$$

or equivalently

$$e_q^{ax} \rightleftharpoons_q \frac{q}{p-qa} \rightleftharpoons_q e_{q^{-1}}^{-pt+qat} e_q^{pt} \tag{5.68}$$

and one should be quickly disenchanted noting that the right hand side of (5.68) is not a solution of the q-difference equation

$$D_q y(x) = ay(x) \tag{5.69}$$

which its self leads to the left hand side of (5.68).

To handle a little better the difficulty, we must push father our thinking in the q-world. For that we need remark that as the function

$$\frac{(-1)^n n!}{(p-p_0)^n} \tag{5.70}$$

is obtained from (5.64) by deriving it n times, the q-version of (5.70) should be determined by q-deriving (5.64) n times also. This gives

$$D_q^n \frac{1}{p-p_0} = \frac{(-1)^n [n]_q!}{\prod_{i=0}^n (pq^i - p_0)} \tag{5.71}$$

Next, putting (5.65) in 5.27) and using 5.71), we get

$$\frac{1}{\prod_{i=0}^n (pq^i - p_0)} \rightleftharpoons_q \frac{t^n q^{-\frac{n}{2}(n+3)} e_q^{p_0 q^{-n-1} t}}{q[n]_q!} \tag{5.72}$$

But, $\prod_{i=0}^n (pq^i - p_0)$, tough it is a natural q-deformation of $(p-p_0)^n$, it has no multiple roots. Hence, we can summarize the thinking as follows: (i) If all the roots $v(p)$, $F(p) = \frac{u(p)}{v(p)}$, $degu(p) < degv(p)$, are simple, so we can either use (5.72) if the roots are in the form $q^{-i}p_0$, $i = \bar{1,n}$, or develop $F(p)$ in sum of simple fractions and use (5.65), (ii) If some roots of $v(p)$ are multiple, so one should resort to the first theorem of development.

5.4 Application of q-Laplace transform to certain q-difference equations

As Laplace transform and Z-transform are largely applied in solving differential and difference equations respectively, the q-Laplace transform is expected to play the same role but now in q-difference equations. The principle lying behind is always the same:

1. Given a k-order linear constant coefficients q-difference equation, with initial conditions

$$\begin{aligned} a_0 D_q^k y(x) + a_1 D_q^{k-1} y(x) + \ldots + a_{k-1} D_q y(x) + a_k y(x) = b(x), \\ y(0) = y_0, D_q y(0) = y_1, \ldots, D^{k-1} y(0) = y_{k-1}, \end{aligned} \tag{5.73}$$

we apply the q-Laplace transform on both sides of the equation, algebraically solve for $Y(p) \rightleftharpoons_q y(x)$ and then carefully use the inverse q-Laplace transform to find the unknown function $y(x)$.

Consider for example the second order case:

$$\begin{aligned} a_0 D_q^2 y(x) + a_1 D_q y(x) + \ldots + a_2 y(x) = b(x) \\ y(0) = y_0, D_q y(0) = y_1. \end{aligned} \tag{5.74}$$

Suppose $y(x) \rightleftharpoons_q Y(p)$, $f(x) \rightleftharpoons_q B(p)$. Next, using (5.18), one gets

$$D_q y(x) \rightleftharpoons_q \frac{p}{q} Y(p) - y(0),$$
$$D_q^2 y(x) \rightleftharpoons_q (\frac{p}{q})^2 Y(p) - \frac{p}{q} y(0) - D_q y(0) \tag{5.75}$$

Loading (5.75) in (5.74), one gets

$$Y(p) = \frac{B(p) + a_0 y_0 \frac{p}{q} + (a_0 y_1 + a_1 y_0)}{a_0 (\frac{p}{q})^2 + a_1 \frac{p}{q} + a_2}. \tag{5.76}$$

The remaining task consists in finding the explicit version of $y(t) = \mathcal{L}_q^{-1}\{Y(p)\}$, i.e. that not containing the parameter p (for example the lhs of (5.68) instead of its rhs, when solving the first order q-difference equation (5.69)) which is the expected solution of (5.74).

2. Given a constant coefficients linear system of q-difference equations of the form

$$D_q y(x) = Ay(x) + b(x),$$
$$y(0) = y_0, \tag{5.77}$$

where A is a $k.k.$ matrix, $y(x)$ and $b(x)$, k-vectors. Applying the q-Laplace transform on both sides leads to

$$\frac{p}{q} Y(p) - y(0) = AY(p) + B(p), \tag{5.78}$$

where we used the rule that if $z(x) = (z_1(x), \ldots, z_k(x))$, then $\mathcal{L}_q\{z(x)\}$=$(\mathcal{L}_q\{z_1(x)\}, \ldots, \mathcal{L}_q\{z_k(x)\})$. From (5.78), we get

$$Y(p) = (\frac{p}{q} I - A)^{-1}(y_0 + B(p)), \tag{5.79}$$

which gives $y(t) = \mathcal{L}_q^{-1}\{Y(p)\}$.

Example. Using the q-Laplace transform, solve the equations

a)

$$D_q^2 y(x) + y(x) = 0,\ y(0) = 1, D_q y(0) = 0 \tag{5.80}$$

b)

$$D_q^2 y(x) - y(x) = 0,\ y(0) = 0, D_q y(0) = 1 \tag{5.81}$$

Solution. a) Using (5.76) and the data in (5.80), we get $Y(p) = \frac{p}{p^2+q^2}$, which by (5.52) with $w = 1$, gives $y(t) = cos_q(x)$.
b) Similarly, using (5.76) and the data in (5.84), we get $Y(p) = \frac{q^2}{p^2-q^2}$, which by (5.55) with $w = 1$, gives $y(t) = sinh_q(x)$.

5.5 Exercises

1. Find the q-original of

a) $F(p) = \frac{p+1}{p(p-1)(p-2)(p-3)}$,

b) $F(p) = \frac{a}{\prod_{i=0}^{5}(pq^i - a)}$,

c) $F(p) = \frac{2p+3}{p(p^2+1)}$,

d) $F(p) = \frac{1}{p(p^2+1)(p^2+4)}$.

2. Using the q-Laplace transform, solve the equations

a)

$$D_q^2 y(x) = ay(x) + b,\ y(0) = 1, D_q y(0) = 0 \tag{5.82}$$

b)

$$D_q^2 y(x) - 3D_q y(x) + 2y(x) = 0,\ y(0) = 0, D_q y(0) = 1 \tag{5.83}$$

c)

$$(q-1)^2 D_q^2 y(x) + y(x) = 0,\ y(0) = 1, D_q y(0) = 2. \tag{5.84}$$

3. Find the q-image of

a)

$$D_q y(x) + y(x) + \int_0^x y(t) d_q t;$$
$$y(0) = 1, y(t) \rightleftharpoons_q Y(p) \tag{5.85}$$

b)

$$D_q y(x) - \int_0^x y(t) d_q t;$$
$$y(0) = 0, y(t) \rightleftharpoons_q Y(p). \tag{5.86}$$

4. Solve

a) The q-difference equation

$$D_q^2 y(x) + D_q y(x) - 2y(x) = e^{-x};$$
$$y(0) = 0, D_q y(0) = 1 \tag{5.87}$$

b) The q-integral equation

$$y(x) = \int_0^x y(t) d_q t + 1. \tag{5.88}$$

5. Solve the system of q-difference equations

$$\begin{cases} D_q x(t) &= & x(t) + 2y(t) \\ D_q y(t) &= & 2x(t) + y(t) + 1 \end{cases} \tag{5.89}$$

6. Solve the q-difference equation with variable coefficients

$$(x^2 + a_0^2) D_q^2 y(x) + a_1 x D_q y(x) + a_2 y(x) = b(x); \tag{5.90}$$
$$y(0) = D_q y(0) = 0. \tag{5.91}$$

Chapter 6

q-Difference orthogonal polynomials

As in the case of differential and difference equations, orthogonal polynomials are probably the most beautiful and applicable solutions of q-difference equations. The main method of deriving or transforming polynomial solutions of q-difference equations are special versions of the famous factorization method also known as Darboux transformation [22]. In this chapter, we will focused on polynomial solutions of the linear second order q-difference equations. In the first section we first use the factorization method to obtain (polynomial) solutions of the equations. In the second we show how to use the factorization method for transforming a solvable linear equation into a new one. In each section the general theory is illustrated by the case of the hypergeometric q-difference equations.

6.1 The factorization method for the solvability of q-difference equations

6.1.1 The general theory

Consider the general second order q-difference eigenvalue equation

$$[u(x)E_q + v(x) + w(x)E_q^{-1}]y_n(x) = \lambda_n y_n(x), \tag{6.1}$$

where $v(x) = -(u(x) + w(x))$. Our objective is to study the solvability of such an equation. Here, a type [14, 13, 10] factorization method will be used.

First, write (6.1) under the form

$$Ly_n(x) = [a(x)E_q + b(x) + c(x)E_q^{-1}]y_n(x) = \lambda(n)\theta(x)y_n(x), \qquad (6.2)$$

where

$$a(x) = \theta(x)u(x); b(x) = \theta(x)v(x); c(x) = \theta(x)w(x) \qquad (6.3)$$

for some $\theta(x) \neq 0$. Consider next the operator

$$H(x,n) = E_q[\rho(L - \lambda\theta)\rho^{-1}] = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx), \qquad (6.4)$$

where

$$\rho(qx)/\rho(x) = a(x); d(x) = a(x/q)c(x). \qquad (6.5)$$

So the eigenvalue equation (6.2) is "equivalent" to the equation

$$H(x,n)y_n(x) = 0, \qquad (6.6)$$

in the sense that if $y_n(x)$ is a solution of (6.2), then $\rho(x)y_n(x)$ is a solution of (6.6) and conversely if $y_n(x)$ is a solution of (6.6), then $\rho^{-1}(x)y_n(x)$ is a solution of (6.2).

Consider now for H, the following type of factorization

$$\begin{aligned} H(x,n) - \mu(n) &= (E_q + g(x,n))(E_q + f(x,n)), \\ H(x,n+1) - \mu(n) &= (E_q + f(x,n))(E_q + g(x,n)), \end{aligned} \qquad (6.7)$$

for some functions $f(x,n)$, $g(x,n)$, and constants (in x) $\lambda(n)$, $\mu(n)$. Consider next the eigenvalue equation

$$\tilde{L}\tilde{y}_n(x) = [g(x,-1)E_q - b(x) + f(x/q,-1)E_q^{-1}]y_n(x) = -\lambda(n)\theta(x)\tilde{y}_n(x)(6.8)$$

and the operator

$$\tilde{H}(x,n) = E_q[\tilde{\rho}(\tilde{L} - \lambda\theta)\tilde{\rho}^{-1}] = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + \tilde{d}(qx), \qquad (6.9)$$

where

$$\tilde{\rho}(qx)/\tilde{\rho}(x) = -g(x,-1); \tilde{d}(x) = g(x/q,-1)f(x/q,-1). \qquad (6.10)$$

It is easily seen in this case also that the eigenvalue equation (6.8) is "equivalent" to the equation

$$\tilde{H}(x,n)\tilde{y}_n(x) = 0. \qquad (6.11)$$

Consider also for $\tilde{H}$, the factorization

$$\begin{aligned}\tilde{H}(x,n)-\tilde{\mu}(n) &= (E_q+g(x,n))(E_q+f(x,n)),\\ \tilde{H}(x,n+1)-\tilde{\mu}(n) &= (E_q+f(x,n))(E_q+g(x,n)),\end{aligned}\tag{6.12}$$

with $\tilde{\mu}(n)=\mu(n)-\mu(-1)$, and some $f(x,n)$, $g(x,n)$, and constants (in x) $\lambda(n)$, $\mu(n)$ as in (6.7). We can now give the main statement of this section

Theorem 6.1.1 *Suppose that*
There exist functions $f(x,n)$, $g(x,n)$, constants (in x) $\lambda(n)$, $\mu(n)$, for which H admits the factorization (6.7) with f and g such that

$$f(x,n)-g(x,n-1)=c_1(n)x+c_2(n),\tag{6.13}$$

$c_1(n)\neq 0,\ \infty$.
In that case, the following situations hold:
(i) The eigenvalue equation (6.8) admits a sequence of polynomial solutions satisfying the difference relations

$$\begin{aligned}\tilde{y}_{n+1}(x) &= (-g(x,-1)E_q+f(x,n))\tilde{y}_n(x)\\ -\tilde{\mu}(n-1)\tilde{y}_{n-1}(x) &= (-g(x,-1)E_q+g(x,n-1))\tilde{y}_n(x),\ \ n=0,1,2\ldots\end{aligned}\tag{6.14}$$

and the three-term recurrence relations (TTRR)

$$\tilde{y}_{n+1}(x)+\tilde{\mu}(n-1)\tilde{y}_{n-1}(x)=(c_1(n)x+c_2(n))\tilde{y}_n(x),\tag{6.15}$$

$$\tilde{y}_0=1,\ \ \tilde{y}_1=c_1(0)x+c_2(0).\tag{6.16}$$

(ii) The eigenvalue equation (6.2) admits a sequence of eigenfunctions satisfying the difference relations

$$\begin{aligned}\psi_{n+1}(x) &= (a(x)E_q+f(x,n))\psi_n(x),\\ -\mu(n-1)\psi_{n-1}(x) &= (a(x)E_q+g(x,n-1))\psi_n(x),\ \ n=0,1,2,\ldots\end{aligned}\tag{6.17}$$

and the TTRR

$$\psi_{n+1}(x)+\mu(n-1)\psi_{n-1}(x)=(c_1(n)x+c_2(n))\psi_n(x),\quad n=0,1,2,\ldots.\tag{6.18}$$

(iii) If $\mu(-1)=0$, the equations (6.8) and (6.2) as well as their solutions in (i) and (ii), become identical .

Proof.
(i)Note first that from the relations in (6.7) follow in particular the equations

$$f(x,n)g(x,n) = d(qx) - \mu(n), \tag{6.19}$$
$$f(qx,n) + g(x,n) = b(qx) - \theta(qx)\lambda(n), \tag{6.20}$$

with

$$f(qx,n+1) + g(x,n+1) = f(x,n) + g(qx,n), \tag{6.21}$$
$$f(x,n+1)g(x,n+1) = f(x,n)g(x,n) + \mu(n) - \mu(n+1), \tag{6.22}$$

or equivalently the equations (6.19) and (6.20) together with the q-difference equation

$$\Delta_q(f(x,n) - g(x,n)) = (\lambda(n+1) - \lambda(n))\theta(x);\ \Delta_q = E_q - 1. \tag{6.23}$$

Remark next that from (6.4), (6.9), (6.10) and (6.19) (with $n = -1$), it follows that $H = \tilde{H} + \mu(-1)$. Hence from (6.7) follows (6.12). On the other side, from (6.12) follows the interconnection relations

$$\begin{aligned} \tilde{H}(x,n+1)(E_q + f(x,n)) &= (E_q + f(x,n))\tilde{H}(x,n), \\ \tilde{H}(x,n)(E_q + g(x,n)) &= (E_q + g(x,n))\tilde{H}(x,n+1), \end{aligned} \tag{6.24}$$

from which one deduces a sequence of solutions of (6.12) satisfying

$$\begin{aligned} \phi_{n+1}(x) &= (E_q + f(x,n))\phi_n(x), \\ -\tilde{\mu}(n-1)\phi_{n-1}(x) &= (E_q + g(x,n-1))\phi_n(x),\ n = 0,1,2,\ldots \end{aligned} \tag{6.25}$$

On the other side from (6.20) (with $n = 0$) and (6.21) (with $n = -1$) follows that $\tilde{y}_0 = 1$ is a solution of (6.8) with $n = 0$. Hence from (6.9) and (6.10) follows that $\tilde{\rho}(x)$ is a solution of (6.12) with $n = 0$. Hence from (6.25) follows (6.14) and consequently (6.15). To obtain the remaining relation which is the second equality in (6.16), one needs only consider (6.13) (with $n = 0$) and the first relation in (6.14) (with $n = 0$).
(ii) To obtain (6.17) and then obviously (6.18), one needs to use interconnection relations for H similar to the ones in (6.24) for $\tilde{H}$.
(iii) This is a direct consequence of the fact that if $\mu(-1) = 0$ then $\tilde{H} = H$.

Note that if the polynomials $\tilde{y}_n(x)$ satisfy (6.15), then their normalized monique forms $P_n = \tilde{y}_n(x)/\varrho(n)$ where

$$\varrho(n+1)/\varrho(n) = c_1(n) \tag{6.26}$$

satisfies

$$P_{n+1} + a_n^2 P_{n-1} = (x - b_n) P_n \tag{6.27}$$

where

$$a_n^2 = \frac{\mu(n-1)}{c_1(n)c_1(n-1)}; \qquad b_n = -c_2(n)/c_1(n). \tag{6.28}$$

6.1.2 The hypergeometric q-difference equation

Consider the hypergeometric q-difference equation

$$[\sigma(x) D_{q^{-1}} D_q + \tau(x) D_q] y_n(x) = \lambda_n y_n(x), \tag{6.29}$$

where $\sigma(x) = \sigma_0 x^2 + \sigma_1 x + \sigma_2$, $\tau(x) = \tau_0 x + \tau_1$, $\tau_0 \neq 0$. This equation may be written as in (6.2):

$$[a(x) E_q + b(x) + c(x) E_q^{-1}] y_n(x) = \theta(x) \lambda(n) y_n(x) \tag{6.30}$$

with

$$\begin{aligned} a(x) &= (\sigma_0 + (1 - 1/q)\tau_0)x + \sigma_1 + (1 - 1/q)\tau_1 + \sigma_2/x; \\ c(x) &= q(\sigma_0 x + \sigma_1 + \sigma_2/x); b(x) = -(a(x) + c(x)); \\ \theta(x) &= (1 - 1/q)x. \end{aligned} \tag{6.31}$$

Theorem 6.1.2 *The operator*

$$H(x,n) = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx), \tag{6.32}$$

$d(x) = a(x/q)c(x)$, *admits a factorization of the type (6.7) with*

$$\begin{aligned} f(x,n) &= -\sigma_2/x - 1/2(-\tau_1 - qc_0(n) + \tau_1 q + q\sigma_1 + q^2\sigma_1)/q - (-\tau_0 \\ &\quad + q^2\sigma_0 + q\sigma_0 + \tau_0 q + \lambda(n)q - \lambda(n+1))x/(1+q); \\ g(x,n) &= -\sigma_2/x - 1/2(-\tau_1 - qc_0(n) + \tau_1 q + q\sigma_1 + q^2\sigma_1)/q - c_0(n) \\ &\quad + (-(-\tau_0 + q^2\sigma_0 + q\sigma_0 + \tau_0 q + \lambda(n)q - \lambda(n+1))/(1+q) \\ &\quad - \lambda(n+1) + \lambda(n))x; \end{aligned} \tag{6.33}$$

$$\begin{aligned} \mu(n) = 1/4(&-q^6\sigma_1^2 + 2\tau_1^2 q^2 - 8q^4\sigma_2\sigma_0 + 4q^3\sigma_2\tau_0 + c_0^2(n)q^4 - q^2\sigma_1^2 \\ &+ c_0^2(n)q^2 + 2\tau_1 q\sigma_1 - 4\tau_1 q^3\sigma_1 - \tau_1^2 - \tau_1^2 q^4 + 2q^4\sigma_1^2 + 2c_0^2(n)q^3 \\ &+ 4q^2\sigma_2\lambda(n) + 4q^2\sigma_2\tau_0 - 4q\tau_0\sigma_2 + 4q^2\sigma_0\sigma_2 + 4q^2\lambda(n+1)\sigma_2 - 4q^4\sigma_2\tau_0 \\ &- 4q^4\sigma_2\lambda(n+1) - 4q^4\lambda(n)\sigma_2 + 2\tau_1 q^5\sigma_1 + 4q^6\sigma_2\sigma_0)/(q^2(1+q)^2) \end{aligned} \tag{6.34}$$

where

$$
\begin{aligned}
c_0(n) = (&-2\tau_1 q^4\sigma_0 + q^3\sigma_1\lambda(n) + q^3\sigma_1\lambda(n+1) + 2\tau_1 q^3\sigma_0 + q^2\lambda(n)\tau_1 \\
&+2\tau_1 q^2\sigma_0 + \tau_1 q^2\lambda(n+1) + 2\tau_1 q^2\tau_0 - q\lambda(n+1)\sigma_1 - 2q\lambda(n+1)\tau_1 \\
&-2\tau_1 q\lambda(n) - q\sigma_1\lambda(n) - 4\tau_1 q\tau_0 - 2\tau_1 q\sigma_0 + \lambda(n+1)\tau_1 \\
&+\tau_1\lambda(n) + 2\tau_1\tau_0)/(q(1+q)(\lambda(n+1)-\lambda(n)))
\end{aligned} \tag{6.35}
$$

and

$$\lambda(n) = ((1-q)q^{-n} + q^2\sigma_0/k)(q^n q\sigma_0 + q^n\tau_0 q - kq - q^n\tau_0 + k)(q-1)^{-2} \tag{6.36}$$

where k is a free parameter.

Proof. The proof of the theorem consists in direct computations.

We will note that the functions f and g satisfy the condition (6.13). (6.36) can equivalently be written as

$$\lambda(n) = -[1 - tq^{-n}][\frac{q^2\sigma_0}{q-1} - (\frac{q\sigma_0}{q-1} + \tau_0)t^{-1}q^n], \tag{6.37}$$

where $t = \frac{q-1}{q^2\sigma_0}k$. Note finally that all the functions of the variable n (f, g, μ) are explicit functions in $\lambda(n)$ and $\lambda(n+1)$ but implicit in n.
Next, let f, g, μ and λ be given in the theorem 6.1.2. We have the following

Corollary 6.1.1 *(a) Type (6.8) equation admits a sequence of polynomial solutions satisfying type (6.14) and (6.15)-(6.16) relations.*
(b) For $t \neq 1$, we have $\mu(-1) \neq 0$ and $\lambda(0) \neq 0$. However equation (6.30) admits a sequence of eigenfunctions satisfying type (6.17) and (6.18) relations where $\mu(n) = \tilde{\mu}(n+r) = \tilde{\mu}(n) + \mu(-1)$, $t = q^r$, that is r-associated relations to the ones in (a).
(c) For $t \rightsquigarrow 1$, we obtain $\mu(-1) = \lambda(0) = 0$, $\mu(n) = \tilde{\mu}(n)$, $H = \tilde{H}$, $\tilde{\pounds} = -\pounds$ and the cases (a) and (b) become identical.

Example 1. The q-Hahn case.

In the q-Hahn case, we have

$$
\begin{aligned}
a(x) &= \alpha(x-1)(x\beta q - q^{-N})/(x); \\
b(x) &= (x^2 - xq^{-N} - x\alpha q + q^{-N+1}\alpha)/x
\end{aligned} \tag{6.38}
$$

and the formulas for $f(x,n), g(x,n), \mu(n), \lambda(n)$ for the factorization are obtained from the ones above by substituting

$$\sigma_0 = 1/q; \sigma_1 = -(q^{-N} + q\alpha)/q; \sigma_2 = q^{-N}\alpha; \tau_0 = (\alpha\beta q^2 - 1)/(q-1);$$
$$\tau_1 = -(\alpha\beta q^2 + q^{-N+1}\alpha - q^{-N} - q\alpha)/(q-1). \tag{6.39}$$

Example 2. The q-Big Jacobi case.

In the q-Big Jacobi case, we have

$$a(x) = aq(x-1)(bx-c)/x;$$
$$b(x) = (x-aq)(x-cq)/x \tag{6.40}$$

and the formulas for $f(x,n), g(x,n), \mu(n), \lambda(n)$ for the factorization are obtained from the ones above by substituting

$$\sigma_0 = 1/q; \sigma_1 = -(a+c); \sigma_2 = aqc; \tau_0 = (aq^2b - 1)/(q-1);$$
$$\tau_1 = (q(a+c) - aq^2(b+c))/(q-1). \tag{6.41}$$

The data above for the q-Hahn and q-Big Jacobi cases are clearly identical up to the correspondence: $a = \alpha,\ b = \beta,\ c = q^{-1-N}$.

6.1.3 The Askey-Wilson second order q-difference equation case.

Consider now the Askey-Wilson second order q-difference equation (the equation can also be written using the derivative in (1.5) (see e.g. [13])):

$$Ly_n(x) = [a(x)E_q - [a(x) + b(x)] + b(x)E_q^{-1}]y_n(x) = \lambda(n)\theta(x)y_n(x) \tag{6.42}$$

where

$$a(x) = \frac{a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2}{qx - x^{-1}}; b(x) = \frac{a_2x^{-2} + a_1x^{-1} + a_0 + a_{-1}x + a_{-2}x^2}{x - qx^{-1}}$$
$$a_{-2} = 1; a_{-1} = -(a+b+c+d); a_0 = ab + ac + ad + bc + bd + cd$$
$$a_1 = -(abc + abd + bcd + acd); a_2 = abcd; \theta(x) = x - x^{-1}. \tag{6.43}$$

The operator

$$H(x,n) = E_q^2 + (b(qx) - \lambda(n)\theta(qx))E_q + d(qx), \tag{6.44}$$

$d(x) = a(x/q)c(x)$, admits a factorization as the one in (6.7), with

$$f(x;n) = \tfrac{f_{-2}x^{-2}+f_{-1}x^{-1}+f_0+f_1x+f_2x^2}{qx-x^{-1}};$$
$$g(x;n) = \tfrac{(f_{-2}-\beta_{-1})x^{-2}+(f_{-1}-\beta_0)x^{-1}+f_0+(f_1+\beta_0 q)x+(f_2+\beta_1 q)x^2}{qx-x^{-1}}; \quad (6.45)$$

where

$$f_{-2}(n) = \tfrac{\lambda(n)-q\lambda(n+1)}{q^2-1} - \tfrac{q+a_2}{q^2+q}; f_2(n) = \tfrac{\lambda(n)q-\lambda(n+1)}{1-q^2}q^2 - \tfrac{q^2+qa_2}{q+1};$$
$$\beta_0(n) = \tfrac{1-q}{(\lambda(n)-\lambda(n+1))q^3}\{(2\tfrac{\lambda(n)q-\lambda(n+1)}{1-q^2}q^2 + \tfrac{\lambda(n+1)-\lambda(n)}{1-q}q^2;$$
$$-2\tfrac{q^2+qa_2}{1+q})(a_1 + qa_{-1}) + (2a_1q^2 + 2a_2a_{-1}q)\};$$
$$\beta_{-1} = \tfrac{\lambda(n+1)-\lambda(n)}{1-q}; \quad \beta_1 = q\beta_{-1};$$
$$f_{-1}(n) = \tfrac{\beta_0(n)}{2} - \tfrac{a_1+qA_{-1}}{2q}; f_1(n) = -\tfrac{q\beta_0(n)}{2} - \tfrac{a_1+qa_{-1}}{2};$$
$$f_0(n) = \tfrac{1}{q+q^2}\{q^2 - q^3 - a_0(q+q^2) + a_2(q-1) + q^2(\lambda(n)+\lambda(n+1))\};$$
$$(6.46)$$

while

$$\mu(n) = a_0 + a_1a_{-1}q^{-1} + a_0a_2q^{-2} + f_0(n)\beta_{-1}(n) + f_{-1}(n)\beta_0(n)$$
$$-2f_{-2}(n)f_0(n) - f_{-1}^2(n), \quad (6.47)$$

and

$$\lambda(n) = -(1-tq^{-n})(1-abcdt^{-1}q^{n-1}). \quad (6.48)$$

Here also, as one can verify, for $t = 1$, we have $\mu(-1) = \lambda(0) = 0$ (and $f(x/q,-1) = -c(x); g(x,-1) = -a(x)$), and the corresponding polynomials in (6.15)-(6.16) are of classical type. Taking $t = q^{-r}$, we obtain Laguerre-Hahn polynomials r-associated to classical polynomials. Otherwise (if such an exponential expression is not allowed for t), the corresponding polynomials are Laguerre-Hahn ones, not necessary r-associated to classical polynomials.

It is worth noting that these results are surely characteristic for the "classical" polynomials since they are valid not only for the q-hypergeometric and the Askey-Wilson second order q-difference equations but also for the difference hypergeometric ones (see [14]).

6.2 The factorization method for the transformation of q-difference equations

6.2.1 The general theory

Consider the general second-order q-difference operator

$$H(x) = u(x)E_q + v(x) + w(x)E_q^{-1}. \tag{6.49}$$

Suppose next that it is "bispectral" in the sense that it admits two sequences of distinct systems of eigenelements say (λ_n, y_n) and (γ_n, z_n):

$$\begin{aligned} Hy_n(x) &= \lambda_n y_n(x) \\ Hz_n(x) &= \gamma_n z_n(x), \quad n = 0, 1, \ldots . \end{aligned} \tag{6.50}$$

In that case, one can use one of the two eigenelements , say for example (γ_n, z_n), to transform H into another solvable operator $\tilde{H}$ in the following manner. Factorize H and define $\tilde{H}$ as follows,

$$\begin{aligned} H - \gamma_m &= L_m R_m \\ \tilde{H} - \gamma_m &= R_m L_m, \quad m = 0, 1, \ldots . \end{aligned} \tag{6.51}$$

where

$$\begin{aligned} R_m = 1 + f(x,m)E_q^{-1} \qquad & L_m = u(x)E_q + g(x,m) \\ f(x,m) = -\tfrac{z_m(x)}{z_m(x/q)} \qquad & g(x,m) = -w(x)\tfrac{z_m(x/q)}{z_m(x)}. \end{aligned} \tag{6.52}$$

It follows from (6.51) that the functions $\tilde{y}_n(x,m)$, $m, n = 0, 1, \ldots$ defined by

$$\begin{gathered} [u(x)E_q + g(x,m)]\tilde{y}_0(x,m) = 0, \\ \tilde{y}_n(x,m) = [1 + f(x,m)E_q^{-1}]y_{n-1}(x), \quad m = 0, 1, \ldots, n = 1, 2 \ldots, \end{gathered} \tag{6.53}$$

are eigenfunctions of $\tilde{H}(x,m)$ corresponding to the eigenvalues γ_m, λ_n, for $m = 0, 1, \ldots$,$n = 0, 1, \ldots$ respectively. We will refer here to H and y_n as the *transformable* operator and functions respectively, z_n as the *transformation* functions and finally $\tilde{H}$ and $\tilde{y}_n$ as the *transformed* operator and functions respectively. The point here is that if

$$\tfrac{y_n(x)}{y_n(x/q)} \neq \tfrac{z_n(x)}{z_n(x/q)} \tag{6.54}$$

so, for a fixed m, the transformed functions $\tilde{y}_n(x,m)$, $n=0,1,\ldots$ are non-trivial solutions of the transformed operator $\tilde{H}$. Moreover under some additional conditions, the transformed functions $\tilde{y}_n$ admit most of the mathematical properties of the transformable y_n, such as polynomial character, difference eigenvalue equations, closure and orthogonality, difference and recurrence relations, duality, transformability property [11]:

Difference equations

Clearly, the functions $\tilde{y}_n(x,m)$ satisfy the eigenvalue equation

$$\begin{aligned} \tilde{H}(x,m)\tilde{y}_0(x,m) &= \gamma_m \tilde{y}_0(x,m) \\ \tilde{H}(x,m)\tilde{y}_n(x,m) &= \lambda_{n-1}\tilde{y}_n(x,m), n=1,2,\ldots \end{aligned} \tag{6.55}$$

for

$$\tilde{H}(x,m) = u(x)E_q + \tilde{v}(x,m) + \tilde{w}(x,m)E_q^{-1} \tag{6.56}$$

where

$$\begin{aligned} \tilde{v}(x,m) &= g(x,m) + f(x,m)u(x/q) + \gamma_m \\ &= v(x) + f(x,m)u(x/q) - u(x)f(qx,m) \\ \tilde{w}(x,m) &= f(x,m)g(x/q,m) = w(x)\tfrac{g(x/q,m)}{g(x,m)}. \end{aligned} \tag{6.57}$$

Orthogonality, closure

Consider the functions $\rho(x)$ and $\tilde{\rho}(x)$ defined by

$$\begin{aligned} \tfrac{\rho^2(qx)}{\rho^2(x)} &= \tfrac{u(x)}{qf(qx,m)g(qx,m)} = \tfrac{u(x)}{qw(qx)}; \\ \tfrac{\tilde{\rho}^2(qx,m)}{\tilde{\rho}^2(x,m)} &= \tfrac{u(x)}{qf(qx,m)g(x,m)} = \tfrac{u(x)g(qx,m)}{qw(qx)g(x,m)}; \end{aligned} \tag{6.58}$$

Interesting relations exist between $\rho(x)$, $\tilde{\rho}(x,m)$ and $\tilde{y}_0(x,m)$. One has

$$\begin{aligned} \tilde{\rho}^2(x,m) &= \rho^2(x)g(x,m); \\ \tilde{y}_0(x,m) &= \tfrac{1}{\rho^2(x)g(x,m)z_m(x)(xq^{-\frac{1}{2}}-xq^{\frac{1}{2}})} \\ &= \tfrac{1}{\tilde{\rho}^2(x,m)z_m(x)(xq^{-\frac{1}{2}}-xq^{\frac{1}{2}})} \end{aligned} \tag{6.59}$$

Next, as it is easily seen, the similarity reductions $\rho H \rho^{-1}$ and $\tilde{\rho}\tilde{H}\tilde{\rho}^{-1}$ send H and $\tilde{H}$ respectively, in their formal symmetric form , that is like

$$qc(qx)E_q + b(x) + c(x)E_q^{-1} \tag{6.60}$$

or

$$a(x)E_q + b(x) + q^{-1}a(x/q)E_q^{-1}. \tag{6.61}$$

Denote by $\ell^2(q^\beta, q^\alpha; \rho^2)$ the linear space of q-discrete functions

$$\psi(x),\ x = q^\beta, q^{\beta-1}, \ldots, q^\alpha;\ \alpha, \beta \in \mathbf{Z} \cup \{-\infty, \infty\} \tag{6.62}$$

in which is defined a discrete-weighted inner product

$$\begin{aligned}(\psi, \phi)_{\rho^2} = (1-q) \textstyle\int_{q^\beta}^{q^\alpha} \psi(x)\phi(x)\rho^2(x)d_hx = \\ \textstyle\sum_\alpha^{\beta-1} q^i\psi(q^i)\phi(q^i)\rho^2(q^i)\end{aligned} \tag{6.63}$$

The similar space for $\tilde{\rho}^2$ will be denoted by $\tilde{\ell}^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$. Moreover, one easily verifies, using the q-summation by parts, that for ψ and ϕ satisfying boundary constraints

$$u(xq^{-1})\varrho(xq^{-1})[\psi(x)\phi(xq^{-1}) - \psi(xq^{-1})\phi(x)]\,|_{q^\alpha}^{q^\beta} = 0 \tag{6.64}$$

for ϱ^2 equals ρ^2 and $\tilde{\rho}^2$ respectively, we have

$$(H\psi, \phi)_{\rho^2} = (\psi, H\phi)_{\rho^2}; (\tilde{H}\psi, \phi)_{\tilde{\rho}^2} = (\psi, \tilde{H}\phi)_{\tilde{\rho}^2}. \tag{6.65}$$

Also, for ψ and ϕ satisfying

$$\begin{aligned}u(xq^{-1})\rho(xq^{-1})\phi(x)\psi(xq^{-1})\,|_{q^\alpha}^{q^\beta} \\ = 0\end{aligned} \tag{6.66}$$

we have

$$(\phi, R_m\psi)_{\tilde{\rho}^2} = (L_m\phi, \psi)_{\rho^2}. \tag{6.67}$$

The following theorem defers the properties of orthogonality and closure of $\{\tilde{y}_n\}_{n\geq 0}$ in $\tilde{\ell}^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$ to these of $\{y_n\}_{n\geq 0}$ in $\ell^2(q^\beta, q^\alpha; \rho^2)$.

Theorem 6.2.1 *If (6.66) is satisfied for $\psi = y_j$ and $\phi = \tilde{y}_i$, $i, j = 0, 1, \ldots$ then*
(i) From the orthogonality of $\{y_n\}_{n\geq 0}$ in $\ell^2(q^\beta, q^\alpha; \rho^2)$ follows that of $\{\tilde{y}_n\}_{n\geq 0}$ in $\tilde{\ell}^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$
(i) From the completeness of $\{y_n\}_{n\geq 0}$ in $\ell^2(q^\beta, q^\alpha; \rho^2)$ follows that of $\{\tilde{y}_n\}_{n\geq 0}$ in $\tilde{\ell}^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$

Proof. (i) In (6.67) take $\phi = R_m y_i$ and $\psi = y_j$. We obtain $(R_m y_i, R_m y_j)_{\tilde{\rho}^2} = (L_m R_m y_i, y_j)_{\rho^2} = (\gamma_m - \lambda_j)(y_i, y_j)_{\rho^2} = 0$. Hence $\tilde{y}_{i+1} = R_m y_i \perp \tilde{y}_{j+1} = R_m y_j$, $i, j = 0, 1, \ldots$. Taking $\phi = \tilde{y}_0$ and $\psi = y_j$, we have $(\tilde{y}_0, R_m) y_j)_{\tilde{\rho}^2} = (L_m \tilde{y}_0, y_j)_{\rho^2} = 0$, $j = 1, \ldots$ (since $L_m \tilde{y}_0 =^{def} 0$). Hence $\tilde{y}_0 \perp \tilde{y}_j$, $j = 0, 1 \ldots$. In sum $\tilde{y}_i \perp \tilde{y}_j$, $i, j = 0, 1 \ldots$.
(ii) Suppose that exists a certain y such that $y \perp \tilde{y}_j$, $\jmath = 1, \ldots$. Take $\phi = y$ and $\psi = \tilde{y}_j$. We get $0 = (y, \tilde{y}_j)_{\tilde{\rho}^2} = (y, R_m y_j)_{\tilde{\rho}^2} = (L_m y, y_j)_{\rho^2}$. From the closure of $\{y_j\}_{j \geq 0}$, it follows that $L_m y = 0$. In other words $y = \tilde{y}_0$. Hence the system $\{\tilde{y}_n\}_{n \geq 0}$ is closed $\tilde{\ell}^2(q^{\tilde{\beta}}, q^{\tilde{\alpha}}; \tilde{\rho}^2)$ and the theorem is completely proved.

Difference and recurrence relations

Suppose that the transformable functions satisfy the difference relations

$$\begin{aligned} \alpha_n y_{n+1} &= H_n^- y_n \\ \beta_n y_n &= H_n^+ y_{n+1}, \quad n = 0, 1, \ldots \end{aligned} \tag{6.68}$$

On the other side, from (6.51), one has

$$\begin{aligned} \tilde{y}_{n+1} &= R_m y_n \\ (\lambda_n - \gamma_m) y_n &= L_m \tilde{y}_{n+1}, \quad n = 0, 1, \ldots \end{aligned} \tag{6.69}$$

A combination of (6.68) and (6.69) leads to the following three-term difference relations for $\tilde{y}_n$, $n = 1, 2, \ldots$

$$\begin{aligned} \alpha_n (\lambda_n - \gamma_m) \tilde{y}_{n+2} &= R_m H_n^- L_m \tilde{y}_{n+1} \\ \beta_n (\lambda_{n+1} - \gamma_m) \tilde{y}_{n+1} &= R_m H_n^+ L_m \tilde{y}_{n+2}. \end{aligned} \tag{6.70}$$

Using the difference eigenvalue equation satisfied by the $\tilde{y}_n$ (see (6.55)) and the preceding relations, one can naturally reach first order difference relations connecting $\tilde{y}_n$, $n = 1, 2, \ldots$.
Suppose now that the transformable functions y_n satisfy a three-term recurrence relation

$$y_{n+1} + (b_n - x) y_n + a_n^2 y_{n-1} = 0, \tag{6.71}$$

so, using the first relation in (6.69), one shows that the transformed $\tilde{y}_n$, $n = 1, 2, \ldots$, satisfy the following five-term recurrence relation

$$\begin{aligned} &\tilde{y}_{n+4} + [b_{n+2} + b_{n+1} - x - x/q] \tilde{y}_{n+3} \\ &+ [(b_{n+1} - x)(b_{n+1} - x/q) + a_{n+1}^2 + a_{n+2}^2] \tilde{y}_{n+2} \\ &+ a_{n+1}^2 [b_{n+1} + b_n - x - x/q] \tilde{y}_{n+1} + a_{n+1}^2 a_n^2 \tilde{y}_n = 0. \end{aligned} \tag{6.72}$$

We remark however that the preceding relations do not include $\tilde{y}_0$. If for a_n^2 in (6.71), one has $a_0^2 = 0$, or if one suppose that $y_{-1} = 0$, so using the second relation in (6.69), one establishes the following difference-recurrence relations for the system of transformed functions $\tilde{y}_n$, $n = 0, 1, 2, \ldots$

$$\begin{aligned}(\lambda_{n-1} - \gamma_m)(\lambda_n - \gamma_m)L_m\tilde{y}_{n+2}\\ +(\lambda_{n-1} - \gamma_m)(\lambda_{n+1} - \gamma_m)(b_n - h(x))L_m\tilde{y}_{n+1}\\ +(\lambda_{n+1} - \gamma_m)(\lambda_n - \gamma_m)L_m\tilde{y}_n = 0.\end{aligned} \tag{6.73}$$

Duality

Suppose that the transformable functions $y_n(x) = y_n(q^s)$ are also explicit functions of n. In that case, one can consider the functions $\theta_s(n) = \tilde{y}_n(q^s, m)$ dual to the transformed $\tilde{y}_n(x, m)$, $n = 0, 1, \ldots$ defining

$$\begin{aligned}\theta_s(0) = \tilde{y}_0(q^s, m);\\ \theta_s(n) = R_m y_n(q^s) = y_n(q^s) + f(q^s, m)y_n(q^{s-1}),\ n = 1, 2, \ldots\end{aligned} \tag{6.74}$$

From (6.51), one finds that the functions $\theta_s(n)$ satisfy the three-term recurrence relation

$$\begin{aligned}\theta_{s+1}(n) + (\tilde{v}(q^s) - \delta_n)\theta_s(n) + \tilde{w}(q^s)u(q^{s-1})\theta_{s-1}(n) = 0,\\ \delta_0 = \gamma_m,\ \delta_n = \lambda_{n-1},\ n \geq 1.\end{aligned} \tag{6.75}$$

If $\tilde{w}(1)u(q^{-1}) = 0$, then the functions in (6.75) are up to a multiplication by $\theta_0(n)$, polynomials in δ_n of degree s. If $\tilde{v}(q^s)$ is real for $s \geq 0$ and $\tilde{w}(q^s)u(q^{s-1}) > 0$, $s > 0$, so the polynomials are naturally orthogonal with positive discrete weight (Favard theorem).

A typical example

Consider the eigenvalue equation

$$[a(x)E_q + b(x) + c(x)E_q^{-1}]\bar{y}_n(x) = \gamma_n\bar{y}_n(x). \tag{6.76}$$

where $b(x) = -(a(x) + c(x))$.
Consider the situation when $c(x)$ doesn't depend explicitly on q and $a(x) = dc(x)$, d, a constant (a similar reasoning should be developed considering

that $a(x)$ does not depend explicitly on q and c(x)=d a(x)). In that case (6.76) reads

$$[dc(x)E_q - (dc(x) + c(x)) + c(x)E_q^{-1}]\bar{y}_n(x,q) = \gamma_n(q)\bar{y}_n(x,q). \quad (6.77)$$

Substituting q by q^{-1} in (6.77), and performing a similarity reduction on the obtained operator in the left hand side, one gets

$$\begin{aligned}[dc(x)E_q - (dc(x) + c(x)) + c(x)E_q^{-1}]\pi(x)\bar{y}_n(x,q^{-1}) \\ = \gamma_n(1/q)\pi(x)\bar{y}_n(x,q^{-1}).\end{aligned} \quad (6.78)$$

where

$$\pi(qx)/\pi(x) = 1/d. \quad (6.79)$$

This means that the operator in the left hand side of (6.77) and (6.78) is "bispectral" with two distinct systems of eigenelements $(\gamma_n(q), \bar{y}_n(x,q))$ and $(\gamma_n(q^{-1}), \bar{z}_n(x,q))$ where $\bar{z}_n(x,q) = \pi(x)\bar{y}_n(x,q^{-1})$. Hence it can be transformed according to the scheme studied in the first subsection. But as one can see, if $a(x)$ is for example a polynomial, the functions $\bar{y}_n(x,q)$ are not in general orthogonal (the interval of orthogonality is empty). That is why we rewrite (6.77) and (6.78) in a more convenient form for the transformation. For that, supposing that $\lambda_n(q) \neq 0$, for $n > 0$ (this is generally the case for polynomial type of solutions), we define the functions $y_n(x,q)$ by

$$\begin{aligned}y_n(x,q) = \tfrac{1}{\gamma_{n+1}(q)}[dE_q - (d+1) + E_q^{-1}]\bar{y}_{n+1}(x,q) \\ n = 0, 1, \ldots\end{aligned} \quad (6.80)$$

As one can verify, the functions $y_n(x,q)$ are given by

$$y_n(x,q) = \tfrac{\bar{y}_{n+1}(x,q)}{c(x)}, \ n = 0, 1 \ldots \quad (6.81)$$

and satisfy the eigenvalue equation

$$[u(x)E_q + v(x) + w(x)E_q^{-1}]y_n(x,q) = \lambda_n(q)y_n(x,q). \quad (6.82)$$

where

$$\begin{aligned}u(x) = dc(qx); v(x) = -(d+1)c(x) - \gamma_1(q); \\ w(x) = c(x/q); \lambda_n(q) = \gamma_{n+1}(q) - \gamma_1(q).\end{aligned} \quad (6.83)$$

In particular, if $y_0(x,q) \equiv const$, then $v(x) = -(u(x)+w(x))$. Similarly, the functions

$$z_n(x,q) = \pi(x) y_n(x,q^{-1}) \tag{6.84}$$

satisfy the equation

$$[u(x)E_q + v(x) + w(x)E_q^{-1}] z_n(x,q) = \nu_n(q) z_n(x,q). \tag{6.85}$$

where $\nu_n(q) = \gamma_{n+1}(q^{-1}) - \gamma_1(q)$. Thus, the operator in the left hand side of (6.82) and (6.85) is "bispectral" and under additional boundary constraints, the functions $y_n(x,q)$ are orthogonal with the weight

$$\rho(x) = \tfrac{w(qx)}{x\pi(x)}. \tag{6.86}$$

Hence the considerations from the first subsection can be reported here.

6.2.2 The hypergeometric q-difference equation

The transformable and transformation functions.

Applying the preceding considerations to the q-hypergeometric case,

$$[\bar{a}(x)E_q + \bar{b}(x) + \bar{c}(x)E_q^{-1}]\bar{y}_n(x) = \gamma_n \bar{y}_n(x), \tag{6.87}$$

with

$$\begin{aligned} \bar{a}(x) &= [(\sigma_0 + (1-1/q)\tau_0)x^2 + (\sigma_1 + (1-1/q)\tau_1)x + \sigma_2]/x^2; \\ \bar{c}(x) &= [q(\sigma_0 x^2 + \sigma_1 x + \sigma_2)]/x^2; \bar{b}(x) = -(\bar{a}(x) + \bar{c}(x)), \end{aligned} \tag{6.88}$$

one is led to the following simple "bispectral" situation

$$\begin{aligned} &[u(x)E_q + v(x) + w(x)E_q^{-1}] y_n(x,q) = \lambda_n(q) y_n(x,q) \\ &[u(x)E_q + v(x) + w(x)E_q^{-1}] z_n(x,q) = \gamma_n(q) z_n(x,q), \end{aligned} \tag{6.89}$$

where

$$u(x) = -c(q^3x-1)/x; w(x) = -(xq-1)/x; v(x) = -(u(x)+w(x)) \tag{6.90}$$

$$\lambda_n = q^{1-n}(1-q^n)(cq^{2+n}-1); \gamma_n = q^{1-n}(q^{2+n}-1)(c-q^n) \tag{6.91}$$

and the functions $y_n(x,q)$ are a special case of the Little q-Jacobi $p_n(qx;c,q|q)$ or equivalently Big q-Jacobi $P_n(q^3x;q,c,0;q)$ polynomials. The transformation functions $z_n(x,q)$ being on the other side defined as in (6.79) and (6.84). For their use in the formulas (6.70) and (6.72), we give here for the polynomials $y_n(x,q)$ the difference relations (in literature, they are not given in this form) and recurrence ones. We have

$$\begin{aligned} c_1(n)y_{n+1}(x,q) &= [r(x)E_q + f_n(x)]y_n(x,q) \\ -a_{n+1}^2 c_1(n+1)c_1(n)y_n(x,q) &= [r(x)E_q + g_n(x)]y_{n+1}(x,q), \end{aligned} \tag{6.92}$$

$$y_{n+1}(x,q) + (b_n - x)y_n(x,q) + a_n^2 y_{n-1}(x,q) = 0 \tag{6.93}$$

where $(Q = q^n)$

$$c_1(n) = -\frac{cQ^2q^3+cQ^2q^2+Q^6c^3q^7-c^2Q^4q^6-c^2Q^4q^4-q-q^5c^2Q^4+q^4Q^2c}{Q(cQ^2q^3-1)(cQ^2q-1)} \tag{6.94}$$

$$b_n = \frac{(Q^2c^2q^2-Qcq^2+cQ^2q^2-2Qcq+c-Qc+1)Q}{(cQ^2q-1)(cQ^2q^3-1)q} \tag{6.95}$$

$$a_n^2 = \frac{Q^2(Qc-1)(Qcq-1)(-1+Q)(-1+qQ)c}{(-1+cQ^2)(cQ^2q-1)^2(-1+cQ^2q^2)q^3}; \quad r(x) = -c(q^3x-1) \tag{6.96}$$

$$f_n(x) = \frac{qx}{Q} - \frac{-1-c+Qcq^2+Qcq}{cQ^2q^3-1}; \quad g_n(x) = xcq^4Q - cqQ\frac{Qq^2+Qcq^2-q-1}{cQ^2q^3-1} \tag{6.97}$$

Note finally that the polynomials $y_n(x)$ are orthogonal on the interval $[0, q^{-\frac{5}{2}}]$ with respect to the weight $\rho(x)$ given by (6.86) where $w(x)$ is given by (6.90) and $\pi(x)$ by (6.79). As the interval of orthogonality is finite, they are also closed in the corresponding inner product space.

The transformed functions. For a given m, the properties of the transformed functions $\tilde{y}_n(x,m)$, $n = 0,1,\ldots$ are those derived in the first subsection of the current section: They satisfy type (6.55) difference equations, type (6.70), (6.72) and (6.73) difference and recurrence relations. And since the conditions of orthogonality of the theorem (6.2.1) are satisfied, they are orthogonal in the inner product space $\tilde{\ell}_2^2(0, q^{-\frac{5}{2}}; \tilde{\rho}^2)$ where $\tilde{\rho}^2$ is given by the formula in (6.59). For the closure, we have that the system $\tilde{y}_n(x,m)$, $n = 0,1,\ldots$ is closed in the space since the unique element $\tilde{y}_0(x,m)$ orthogonal to it in its totality is not quadratically integrable.

How seem the transformed objects? Let us note that, using simple procedures in Maple computation system for example, allows to evaluate explicitly any one of them at least for no very higher m and n (as long as the software and the computer capacities allow).
The case $m = 1$ illustrates the first non-classical situation for the transformed objects. As only in this case, the required volume to display the main data is admissible, we consider only this case here. The main data are ($m = 1$, $n = 0, 1, 2$):

$$f(x,1) = -\tfrac{(cq-q^4)x-c+q}{((c-q^3)x-c+q)c};\ g(x,1) = \tfrac{(xq-1)c(xc-q^3x-c+q)}{(xcq-c-xq^4+q)x} \tag{6.98}$$

$$\begin{aligned}\tilde{v}(x,1) = [(c^3q^4 - 2q^7c^2 + c^2q^2 + cq^{10} - 2cq^5 + q^8)x^3 + (-c^3q^4 - c^3q^3 \\ -qc^3 + q^7c^2 + q^6c^2 + 2q^5c^2 + c^2q^4 - 2qc^2 - 2cq^8 + cq^5 + 2cq^4 + q^3c \\ +cq^2 - q^8 - q^6 - q^5)x^2 + (c^3q^3 + qc^3 + c^3 - 3c^2q^4 - c^2q^3 + qc^2 + cq^5 \\ -q^3c - 3cq^2 + q^6 + q^3 + q^5)x - c^3 + qc^2 + cq^2 - q^3]/ \\ [((cq - q^4)x - c + q)((c - q^3)x - c + q)x]\end{aligned} \tag{6.99}$$

$$\tilde{w}(x,1) = -\tfrac{(xcq-c-xq^4+q)(x-1)(xc-q^3x-cq+q^2)}{(xc-q^3x-c+q)^2x} \tag{6.100}$$

$$\tilde{\rho}^2(x,1) = c_1\rho^2(x)g(x,1) = c_2\tfrac{q^2x-1}{x^3\pi(x)}\tfrac{(xq-1)(xc-q^3x-c+q)}{(xcq-c-xq^4+q)} \tag{6.101}$$

$$\begin{aligned}\tilde{y}_0(x,1) &= \tfrac{c_3}{\tilde{\rho}^2(x,1)\pi(x)y_1(x,1/q)(h(xq^{-\frac{1}{2}})-h(xq^{\frac{1}{2}}))} \\ &= c_4\tfrac{(xq-1)(xc-q^3x-c+q)}{(q^2x-1)(xcq-c-xq^4+q)^2}\end{aligned} \tag{6.102}$$

where $c_i,\ i = 1, \ldots, 4$ are some constants of integration,

$$\tilde{y}_1(x,1) = \tfrac{(-q+c)(xc-c-q^3x+1)}{(xc-q^3x-c+q)c} \tag{6.103}$$

$$\begin{aligned}\tilde{y}_2(x,1) = [q(c^2q^2x + c - c^2q^2 - c^2x^2q - q + xc^2q^4 - xc^3q \\ +q^3c^2x + cq^7x^2 - c^2q^4x^2 + c^2q^5x + c^3q^4x^2 - c^2q^7x^2 - xc^3q^4 \\ +x^2cq - q^5cx - xcq - q^3cx - xcq^4 + cq^2 + qc^3 - qc^2 + c^2x + xq^4 \\ -q^4x^2 + xq + xc^2q + cq^4x^2 - c^2 + cq - cq^2x - xc)]/ \\ [(q^3c - 1)(xc - q^3x - c + q)c]\end{aligned} \tag{6.104}$$

We will remark that if $w(x) \sim x^\alpha$ while $x \rightsquigarrow \infty$, so $\tilde{y}_0(x,m) \sim \frac{1}{x^{m+\alpha}}$ and $\tilde{y}_n(x,m) \sim x^{m+n-1}$, $n = 1, 2, \ldots$ (in our particular case, $\alpha = 0$ and $m = 1$).

6.3 Exercises

1. Using the corresponding difference and three term recurrence relations, write down the first five polynomials for the q-Hahn, q-Big Jacobi and Askey-Wilson cases (use a computer algebra system).

2. Find a special case of the Askey-Wilson second order q-difference equation that can be transformable following the factorization method given in section 6.2.

3. Find the interconnection between the factorization method given in section 6.2 and that given in [28].

4. Use (4.112) and (4.114) to find the weights and the intervals of orthogonality of the q-Hahn, q-Big Jacobi and Askey-Wilson polynomials.

Chapter 7

q-Difference linear control systems

7.1 Introduction

Linear control systems theory consists in study of controllability of linear systems, that is a set of well defined interconnected objects which interactions can be modeled by mathematically linear systems of divided difference functional equations. Thus a divided difference linear control system can be modeled as

$$(\mathcal{D}y)(x(s)) = A(x(s))y(x(s+\tfrac{1}{2})) + B(x(s))u(x(s)) \tag{7.1}$$

where y is a k-vector, A a $k.k$ matrix, B, a $k.m$ matrix, and u, a m-vector. The divided difference derivative and the variable $x = x(s)$, $s \in \mathbf{Z}^+$, are given the section 1.1. The vector y stands for the *state variable* of the system, describing the state of the system at a given time s, while u stands for the input or the external force constraining the system that is the resulting trajectory to adopt a predetermined behavior. Thus, u controls the system from which one says of controlled systems. The matrices A and B are intrinsic characterization or description of the system. In (7.1), the state of the system is described by k variables and the external forces act with m inputs.
When $x = x(s) = x_0$, or $x = x(s) = s$ (7.1) is an usual differential [15]

$$y'(x) = A(x)y(x) + B(x)u(x), \tag{7.2}$$

or difference[25]

$$y(x+1) = A(x)y(x) + B(x)u(x). \tag{7.3}$$

linear control system. In this book, we are concerned in the case when $x = x(s) = q^s$. In this case, (7.1) is a q-difference linear control system that one can write:

$$D_q y(x) = A(x)y(qx) + B(x)u(x). \tag{7.4}$$

Clearly, the differentiation between (7.2), (7.3), and (7.4) resides in how varies the time s and how vary the independent variable x at the time s. However, the idea acting behind the controllability concept remains the same: How to choose the input u so that to bring the state of the system from a given position to a predetermined second one.
In practice, it is often difficult even impossible to determine the state of a system its self because it is generally characterized by very numerous variables. Instead, one observes the out put of the system $z(x)$, characterized by a small number of variables. For example, to inquire of the health of his patient, the doctor collect some indicator data such as the blood pressure, the color of eyes, and so on. Hence, a mathematical model more suitable than (7.4) for the study of the systems controllability reads

$$\begin{aligned} D_q y(x) &= A(x)y(qx) + B(x)u(x) \\ z(x) &= C(x)y(x). \end{aligned} \tag{7.5}$$

with c, a $r.k$ matrix and z, a r-vector. In the subsequent sections, we will study the controllability and observability and the interconnection between these concepts and that of primality between polynomials.

7.2 Controllability

There are many versions of definition of the concept of controllability in mathematical control theory:The controllability of the state, controllability of the output, controllability at the origin, complete controllability and so on. The following definition consists in the complete controllability of the state system.

Definition 7.2.1 *The system (7.5) is said to be completely controllable (c.c.) if for any given value of $x = x_0 = q^{s_0}$, and any initial value of $y = y_0 = y(x_0)$, and any final value of $y = y_f$, there exists a finite value $x = x_1 = q^{s_1}$, and a control $u(x)$, $x_0 \le x \le x_1$ such that $y(x_1) = y_f$.*

According to (3.22), the solution of (7.5) reads

$$y(x) = \Phi(x, x_0)[y_0 + \int_{x_0}^{x} \Phi(x_0, t)B(t)u(t)d_qt] \tag{7.6}$$

Hence, the system is c.c. if for any q-discrete value x_0 and any values y_0 and y_f, there exists a finite q-discrete value x_1 and a q-discrete function $u(x)$, $x_0 \le x \le x_1$, such that

$$y_f = y(x_1) = \Phi(x_1, x_0)[y_0 + \int_{x_0}^{x_1} \Phi(x_0, t)B(t)u(t)d_qt] \tag{7.7}$$

Example. Is the scalar system

$$\begin{aligned} D_q y(x) &= ay(qx) + bu(x) \\ z(x) &= cy(x). \end{aligned} \tag{7.8}$$

c.c.?

Solution : The solution of the first order linear non homogenous q-difference equation reads (as the system is of constant coefficients, one can take $x_0 = 0$)

$$\begin{aligned} y(x) &= e_q^{ax}[\int_{t=0}^{x} e_{q^{-1}}^{-at} bu(t)d_qt] \\ &= e_q^{ax}[b(1-q)x \sum_{i=0}^{\infty} e_{q^{-1}}^{-aq^ix} bu(q^ix)] \quad . \end{aligned} \tag{7.9}$$

and clearly for any y_f there exists finite x_1 and a control $u(x)$ such that

$$\sum_{i=0}^{\infty} e_{q^{-1}}^{-aq^ix_1} u(q^ix_1) = e_{q^{-1}}^{-ax_1} y_f/[b(1-q)x_1] \quad . \tag{7.10}$$

Such a function can be defined for example as

$$u(x) = \{ \begin{array}{cc} 0, & x \ne x_1 \\ y_f/b, & x = x_1. \end{array} \tag{7.11}$$

As we shall see in the subsequent sections, even higher order linear q-difference equations are always c.c.
Suppose next that the transition matrix transfers $y(x_0)$ in $y_f = y(x_1)$. In this case we have

$$y(x_1) = \Phi(x_1, x_0)[y_0 + \int_{x_0}^{x_1} \Phi(x_0, t)B(t)u(t)d_qt] \tag{7.12}$$

$$\Leftrightarrow 0 = \Phi(x_1, x_0)[y_0 - \Phi(x_0, x_1)y_f + \int_{x_0}^{x_1} \Phi(x_0, t)B(t)u(t)d_qt] \tag{7.13}$$

This means that at the same interval of time, the state $y_0 - \Phi(x_0, x_1)y_f$ is transferred to 0. As y_0 is arbitrary, one can always suppose that $y_f = 0$.
Consider next the case when the matrices A, B and C are constant (we shall speak in such cases of "time constant systems"):

$$\begin{aligned} D_q y(x) &= Ay(qx) + Bu(x) \\ z(x) &= Cy(x). \end{aligned} \tag{7.14}$$

In this case,we get a simple but powerful criterion of c.c:

Theorem 7.2.1 *The system (7.14) is c.c. iff the controllability matrix*

$$U(A,B) = [B, AB, \dots, A^{k-1}B] \tag{7.15}$$

is of $rankk$.

Proof. *Necessity.*Suppose the contrary, i.e. $rankU < k$. It follows that there exists a k-dimensional row vector q such that

$$qB = 0, qAB = 0, \dots, \dots, qA^{k-1}B = 0 \tag{7.16}$$

Let (7.6) be the solution of (7.14) with $\Phi(x,t) = e_q^{At} e_{q^{-1}}^{-At}$. As the system is constant, we can take $x_0 = 0$. However, the system being c.c., there exists x_1:

$$\begin{aligned} 0 = y(x_1) &= \Phi(x_1, 0)[y_0 + \textstyle\int_0^{x_1} \Phi(0,t)B(t)u(t)d_qt] \\ &= e_q^{Ax}[y_0 + \textstyle\int_0^{x_1} e_{q^{-1}}^{-At} B(t)u(t)d_qt] \\ &\Leftrightarrow 0 = y_0 + \textstyle\int_0^{x_1} e_{q^{-1}}^{-At} B(t)u(t)d_qt. \end{aligned} \tag{7.17}$$

On the other side $e_{q^{-1}}^{-At}$ (as e^{At}) can be expressed as $r(A)$, where $r(\lambda)$ is a polynomial of degree $\leq k-1$. Hence $e_{q^{-1}}^{-Ax}Bu(t) = (r_0 I + r_1 A + \ldots + r_{k-1}A^{k-1})Bu(t) = (r_0 B + r_1 AB + \ldots + r_{k-1}A^{k-1}B)u(t)$. And consequently

$$0 = y_0 + \textstyle\int_0^{x_1} (r_0 I + r_1 A + \ldots + r_{k-1}A^{k-1})u(t)d_qt \tag{7.18}$$

Multiplying both side of the equality by q and considering (**??**), one obtains $qy_0 = 0$ which implies that $q = 0$, since y_0 is arbitrary, contradicting the fact that $rankU < k$.

Sufficiency. Suppose now that $rankU = k$ and show that the system is c.c. Consider again the equation

$$y(x_1) = \Phi(x_1, 0)[y_0 + \int_0^{x_1} \Phi(0, t)B(t)u(t)d_qt] \tag{7.19}$$

$$\Leftrightarrow e_{q^{-1}} - Axy(x_1) = y_0 + \int_0^{x_1} r(A)Bu(t)d_qt \tag{7.20}$$

$$= y_0 + Bs_0 + ABs_1 + \ldots + A^{k-1}Bs_{k-1};\ s_i = \int_{x_0}^{x_1} r_iud_qt \tag{7.21}$$

As $rankU = k$, there exists a solution $s = [s_0, \ldots, s_{k-1}]$ of the system $Us = -y_0 \Leftrightarrow y(x_1) = 0$, and the theorem is completely proved.
The following controllability criterion is valid not only for constant systems but also for varying ones. Moreover it gives an explicit expression for the control function $u(x)$.

Theorem 7.2.2 *The system (7.5) is c.c. iff the k.k symmetric matrix*

$$U(x_0, x_1) = \int_{x_0}^{x_1} \Phi(x_0, t)B(t)B^T(t)\Phi(x_0, t)^T d_q(t) \tag{7.22}$$

is nonsingular. In the latter case, the control function is given by

$$u(x) = -B^T(x)\Phi(x_0, x)^T U^{-1}(x_0, x_1)[y_0 - \Phi(x_0, x)y_f]$$
$$x_0 \le x \le x_1 \tag{7.23}$$

and transfers $y_0 = y(x_0)$ to $y_f = y(x_1)$

Proof. *Necessity.* By contradiction: Suppose that the system is c.c. while the matrix $U(x_0, x_1)$ is singular. As $U(x_0, x_1)$ is symmetric, we have that for an arbitrary $k - vector\ \alpha$:

$$\alpha^T U\alpha = \int_{x_0}^{x_1} \phi^T(t, x_0)\phi(t, x_0)d_qt$$
$$= \int_{x_0}^{x_1} \parallel \phi \parallel^2 d_qt \ge 0 \tag{7.24}$$

where $\phi(x, x_0) = B^T(x)\Phi^T(x_0, x)\alpha$. Thus U is positive semi-definite. It remains to show that the inequality is rigorous. Suppose that there exists $\hat{\alpha} : \hat{\alpha}^T U\hat{\alpha} = 0$. In that case

$$\int_{x_0}^{x_1} \parallel \hat{\phi} \parallel^2 d_qt = 0;\ \hat{\phi} = B^T(x)\Phi^T(x_0, x)\hat{\alpha} \Leftrightarrow \parallel \hat{\phi} \parallel = 0 \Leftrightarrow \hat{\phi} = 0 \tag{7.25}$$

As the system is c.c. let $\hat{u}(x)$ be the control that transfers $y(x_0) = \hat{\alpha}$ in $y(x_1) = 0$. We have

$$\hat{\alpha} = -\int_{x_0}^{x_1} \Phi(x_0, t)B(t)\hat{u}(t)d_qt \tag{7.26}$$

Hence

$$\begin{aligned}&\| \hat{\alpha} \|^2\\ &= \hat{\alpha}^T\hat{\alpha} = -\int_{x_0}^{x_1} \hat{u}^T(t)B^T(t)\Phi^T(x_0,t)\hat{\alpha}d_qt = 0 \Leftrightarrow \hat{\alpha} = 0\end{aligned} \tag{7.27}$$

Thus U is positive definite hence it is nonsingular.
Sufficiency. If U is nonsingular, the control in (7.23) is defined and we need to show that it transfers $y_0 = y(x_0)$ to $y_f = y(x_1)$. Loading (7.23) in (7.7) gives

$$\begin{aligned}y(x_1) = \Phi(x,x_0)[y_0 - (\int_{x_0}^{x_1} \Phi(x_0,t)B(t)B^T(t)\Phi^T(x_0,t)d_qt)U^{-1}(x_0,x_1)(y_0 - \Phi(x_0,x_1)y_f)]\\ = \Phi(x,x_0)[y_0 - (y_0 - \Phi(x_0,x_1)y_f)] = y_f\end{aligned} \tag{7.28}$$

and the theorem is proved.
If the system is not c.c., for some y_0 and y_f, there can be or not a control $u(x)$ that joins them. The existence of such a connection control between two given states is given by the following

Theorem 7.2.3 *If for given (x_0, y_0) and (x_1, y_f), there exists a k-vector γ such that*

$$U(x_0,x_1)\gamma = y_0 - \Phi(x_0,x_1)y_f \tag{7.29}$$

then the control $u(x) = B^T(x)\Phi^T(x_0,x)\gamma$ transfers $y_0 = y(x_0)$ in $y_f = y(x_1)$.

proof. Loading $u(x)$ in (7.7) gives

$$\begin{aligned}y(x_1) = \Phi(x,x_0)[y_0 - (\int_{x_0}^{x_1} \Phi(x_0,t)B(t)B^T(t)\Phi^T(x_0,t)\gamma]\\ = \Phi(x,x_0)[y_0 - (y_0 - \Phi(x_0,x_1)y_f)] = y_f\end{aligned} \tag{7.30}$$

We now analyze the impact of the transformation of coordinates on the controllability propriety of a q-difference system. Let S be a $k.k$ nonsingular matrix and let.

$$\hat{y}(x) = Sy(x). \tag{7.31}$$

From

$$D_qy = A(x)y(qx) + B(x)u \tag{7.32}$$

we have $D_q\hat{y} = D_q(Sy) = SD_qy = S(Ay(qx) + Bu) = SAy(qx) + SBu$ that is $D_q\hat{y} = [SAS^{-1}]\hat{y}(qx) + [SB]u$ or

$$\begin{aligned}D_q\hat{y} = \hat{A}\hat{y}(qx) + \hat{B}u\\ \hat{A} = SAS^{-1};\ \hat{B} = SB\end{aligned} \tag{7.33}$$

The system (7.33) is said to be algebraically equivalent to (7.32). For algebraically equivalent systems we have the following property

Theorem 7.2.4 *If the $\Phi(x, x_0)$ is the state transition matrix for (7.32) then $\hat{\Phi}(x, x_0) = S\Phi(x, x_0)S^{-1}$is the one for (7.33).*

Proof. We need to prove that $D_q\hat{\Phi}(x, x_0) = \hat{A}\hat{\Phi}(qx, x_0)$ provided $D_q\Phi(x, x_0) = A\Phi(qx, x_0)$. In other words, we need to prove that $D_q[S\Phi(x, x_0)S^{-1}] = [SAS^{-1}]S\Phi(qx, x_0)S^{-1}$. The rhs equals $SA\Phi(qx, x_0)S^{-1}$ and the lhs is $[SD_q\Phi(x, x_0)]S^{-1} = SA\Phi(qx, x_0)S^{-1}$ and the theorem is proved.
Important for the sequel is the invariance of controllability propriety given by the following

Theorem 7.2.5 *If the system (7.32) is c.c. then (7.33) is also c.c.*

Proof. Loading the values of $\hat{B}$ and $\hat{\Phi}$ in

$$\hat{U}(x_0, x_1) = \int_{x_0}^{x_1} \hat{\Phi}(x_0, t)\hat{B}(t)\hat{B}^T(t)\hat{\Phi}^T(x_0, t) \tag{7.34}$$

we get

$$\hat{U}(x_0, x_1) = \int_{x_0}^{x_1} S\Phi(x_0, t)S^{-1}SB(t)B^T(t)S^T(S^{-1})^T\Phi^T(x_0, t)S^T d_qt \tag{7.35}$$

This means that

$$\hat{U}(x_0, x_1) = SU(x_0, x_1)S^T \tag{7.36}$$

Hence $\hat{U}(x_0, x_1)$ is nonsingular iff $U(x_0, x_1)$ is, and the theorem is proved.

7.2.1 Controllability canonical forms

Consider the following constant coefficients linear q-difference equation of order k

$$D^k_{q^{-1}}y(x) + a_1D^{k-1}_{q^{-1}}y(x) + \ldots + a_{k-1}D_{q^{-1}}y(x) + a_ky(x) = u(q^{-1}x). \tag{7.37}$$

By the change of dependent variables

$$z_1 = y,\ z_2 = D_{q^{-1}}y, \ldots, z_k = D^{k-1}_{q^{-1}}y, \tag{7.38}$$

write (7.37) in the matrix form

$$D_qz(x) = \hat{C}z(qx) + du(x), \tag{7.39}$$

with

$$\hat{C} = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & . & . & . & -a_1 \end{pmatrix}, \tag{7.40}$$

and

$$z = [z_1, \ldots, z_k]^T, d = [0, \ldots, 1]^T. \tag{7.41}$$

By the way, note that the matrix $\hat{C}$ has the same characteristic equation as the equation (7.39),

$$\lambda^k + a_1\lambda^{k-1} + \ldots + a_k = 0. \tag{7.42}$$

Thus, the question of controllability of the scalar q-difference equation (7.37) is reducible to that of the controllability of the linear system in canonical form (7.39). To inquire about the controllability of (7.39), we naturally refer to the theorem 7.2.1 and evaluate the rank of $U(\hat{C}, d)$. We have

$$U(\hat{C}, d) = [d, \hat{C}d, \ldots, \hat{C}^{k-1}d] \tag{7.43}$$

$$= \begin{pmatrix} 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & 0 & . & 1 & v_1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 1 & . & . & v_{k-3} \\ 0 & 1 & v_1 & . & . & v_{k-2} \\ 1 & v_1 & v_2 & . & . & v_{k-1} \end{pmatrix}, \tag{7.44}$$

where

$$v_j = -\sum_{i=0}^{j-1} a_{i+1} v_{j-i-1},$$
$$j = 1, 2, \ldots, k-1;\ v_0 = 1. \tag{7.45}$$

The matrix $U(\hat{C}, d)$ is in triangular form and clearly has rank k. Hence the system (7.39) is completely controllable. The matrix $\hat{C}$ in (7.40) is generally said to have a *companion form* and the system (7.39) with $\hat{C}$ and d given by

(7.40) and (7.41) is said to be in *controllability canonical form*.
Thus, any q-difference linear scalar equation of the form (7.39) is equivalent to a system in the controllable canonical form and consequently is necessary c.c. The converse is also valid that is to say if the system

$$D_q y(x) = Ay(qx) + bu(x), \tag{7.46}$$

with A a kxk-matrix, b a k-vector, is c.c., then it is algebraically equivalent to a system in controllability canonical form such as (7.39). To see this consider the kxk controllability matrix for (7.46): $U = [b, Ab, \ldots, A^{k-1}b]$. As the system is c.c., the matrix U is nonsingular and consequently invertible. Let write U^{-1} in terms of its rows as

$$U^{-1} = [w_1, \ldots, w_k]^T. \tag{7.47}$$

Next, consider the set $w_k, w_kA, \ldots, w_kA^{k-1}$ and show that it is linearly independent. For this suppose that for some constants $a_1, \ldots, a_k$, we have

$$a_1 w_k b + a_2 w_k Ab + \ldots + a_k w_k A^{k-1} b = 0. \tag{7.48}$$

Since $U^{-1}U = I$, we have $w_k b = w_k Ab = dots = w_k A^{k-2} b = 0$ and $w_k A^{k-1} b = 1$. Hence it follows from (7.48) that $a_k = 0$. One may repeat this procedure by multiplying (7.48) by A with $a_k = 0$ to conclude that $a_{k-1} = 0$. Continuing this procedure, one may show that $a_i = 0$ for $1 \leq i \leq k$. This proves that the vectors $w_k, w_kA, \ldots, w_kA^{k-1}$ are linearly independent. Hence the matrix

$$P = \begin{pmatrix} w_k \\ w_k A \\ . \\ . \\ . \\ w_k A^{k-1} \end{pmatrix} \tag{7.49}$$

is nonsingular. Next, define a change of variables for the system (7.46) by

$$\hat{y}(x) = Py(x) \tag{7.50}$$

to obtain

$$D_q \hat{y}(x) = \tilde{A}\hat{y}(qx) + \tilde{b}u(x), \tag{7.51}$$

with

$$\tilde{A} = PAP^{-1},\, \tilde{b} = Pb. \tag{7.52}$$

Clearly

$$\tilde{b} = Pb = (0, 0, \ldots, 0, 1)^T. \tag{7.53}$$

Next

$$\tilde{A} = PAP^{-1} = \begin{pmatrix} w_k A \\ w_k A^2 \\ . \\ . \\ . \\ w_k A^k \end{pmatrix} P^{-1}. \tag{7.54}$$

since $w_k A$ is the second row in P, it follows that

$$w_k A P^{-1} = (0, 1, 0, \ldots, 0). \tag{7.55}$$

Similarly

$$w_k A^2 P^{-1} = (0, 0, 1, \ldots, 0) \tag{7.56}$$

...

$$w_k A^{k-1} P^{-1} = (0, 0, \ldots, 1), \tag{7.57}$$

while

$$w_k A^k P^{-1} = (-p_k, -p_{k-1}, \ldots, -p_1), \tag{7.58}$$

with $-p_k, -p_{k-1}, \ldots, -p_1$, some constants. Thus

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ -p_k & -p_{k-1} & -p_{k-2} & . & . & . & -p_1 \end{pmatrix}, \tag{7.59}$$

with the same characteristic equation as A,

$$\lambda^k + p_1 \lambda^{k-1} + \ldots + p_k = 0. \tag{7.60}$$

The preceding leads to the following

Theorem 7.2.6 *The system*

$$D_q y(x) = Ay(qx) + bu(x), \tag{7.61}$$

is c.c. iff it is equivalent to a kth order q-difference equation of the form (7.37).

Another controllable canonical form is (see exercise 1, below).

$$D_q \hat{y}(x) = \tilde{A}\hat{y}(qx) + \tilde{b}u(x), \tag{7.62}$$

with

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & . & \dots & -p_k \\ 1 & 0 & 0 & . & \dots & -p_{k-1} \\ 0 & 1 & 0 & . & \dots & -p_{k-2} \\ . & . & . & . & . & . \\ 0 & . & 0 & 1 & 0 & -p_2 \\ 0 & . & . & . & 1 & -p_1 \end{pmatrix} \tag{7.63}$$

and

$$\tilde{b} = (1, 0 \dots, 0)^T. \tag{7.64}$$

It is a more popular form among engineers due to its simple derivative.

7.3 Observability

The concept of observability is closed related to that of controllability. Generally speaking, a system is completely observable iff the knowledge of the input and output suffices to determine the state of the system.

Definition 7.3.1 *The system (7.5) is completely observable (c.o.) if for any x_0, there exists a finite x_1 such that the knowledge of $z(x)$ and $u(x)$ for $x_0 \leq x \leq x_1$ suffice to determine $y_0 = y(x_0)$.*

similarly to the theorem 7.2.2, the basic observability criterion for time varying systems reads

Theorem 7.3.1 *The system (7.5) is c.o. iff the symmetric matrix*

$$V(x_0, x_1) = \int_{x_0}^{x_1} \Phi^T(t, x_0) C^T(t) C(t) \Phi(t, x_0) d_q t \tag{7.65}$$

is nonsingular. In latter case, we have

$$y_0 = V^{-1}(x_0, x_1) \int_{x_0}^{x_1} \Phi^T(t, x_0) C^T(t) z(t) d_q t \tag{7.66}$$

Proof. *Necessity.* The proof is similar to the corresponding one in theorem 7.2.2.
Sufficiency. Supposing that $u(x) \equiv 0$ (this does't decrease the generalities), $x_0 \leq x \leq x_1$, we have $y(x) = \Phi(x, x_0)y_0$. Hence $z(x) = C(x)y(x) = C(x)\Phi(x, x_0)y_0$. Multiplying on the left by $\Phi^T(x, x_0)C^T(x)$, we obtain

$$\begin{aligned} &\textstyle\int_{x_0}^{x_1} \Phi^T(t, x_0)C^T(t)z(t)d_qt \\ &= (\textstyle\int_{x_0}^{x_1} \Phi^T(t, x_0)C^T(t)C(t)\Phi(t, x_0)d_qt) \\ &= V(x_0, x_1)y_0 \end{aligned} \tag{7.67}$$

Thus if $V(x_0, x_1)$ is nonsingular, we have

$$y_0 = V^{-1}(x_0, x_1) \textstyle\int_{x_0}^{x_1} \Phi^T(t, x_0)C^T(t)z(t)d_qt \tag{7.68}$$

The controllability and observability are two concepts with distinct physically meanings but that are mathematically equivalent as shows the following *duality*

Theorem 7.3.2 *The system (7.5) is c.c. iff the dual system*

$$\begin{aligned} D_qy(x) &= -A^T(x)y(x) + C^T(x)u(x) \\ z(x) &= B^T(x)y(x) \end{aligned} \tag{7.69}$$

is c.o. and conversely.

Proof. Considering (7.5),(7.22),(7.65), and (7.69), we remark that it suffices to prove that if $D_q\Phi(x, x_0) = A(x)\Phi(qx, x_0)$ then $D_q\Phi^T(x_0, x) = -A^T(x)\Phi^T(x_0, qx)$. Indeed from the theorem 3.1.5 follows that if $\Phi(x, x_0)$ satisfies $D_qY(x) = A(x)Y(qx)$ then its inverse that is $\Phi(x_0, x)$ satisfies $D_qZ(x) = -Z(x)A(x)$. Carrying out the transpose on both sides, one gets the required equality.
This duality allows greatly to relate most of results in controllability and observability theories. In particular, the controllability criterion for time constant systems given in theorem 7.2.1 leads to the following one for observability:

Theorem 7.3.3 *The system (7.14) is c.o. iff the observability matrix*

$$V(A, C) = [C, CA, \ldots, CA^{k-1}]^T \tag{7.70}$$

has rank k.

7.3.1 Observability canonical forms

Consider the system

$$\begin{aligned} D_q y(x) &= Ay(qx) + bu(x), \\ z(x) &= cy(x) \end{aligned} \tag{7.71}$$

with A a constant kxk matrix, $b = (b_1, b_2, \ldots, b_k)^T$ and $c = (c_1, c_2, \ldots, c_k)$, and suppose that it is c.o. In subsection 7.2.1, we derived two canonical forms of (7.46) reading as (7.39) and (7.62). By exactly parallel procedures, we can obtain two *observability canonical forms* of (7.71). Both procedures are based on the nonsingularity of the observability matrix

$$V = [ccA \ldots cA^{k-1}]^T. \tag{7.72}$$

By the change of variables

$$\hat{y}(x) = Vy(x) \tag{7.73}$$

one obtains the first observability canonical form

$$\begin{aligned} D_q \hat{y}(x) &= \tilde{A}\hat{y}(qx) + \tilde{b}u(x), \\ z(x) &= \tilde{c}\hat{y}(x) \end{aligned} \tag{7.74}$$

with

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & . & . & . & -a_1 \end{pmatrix}, \tag{7.75}$$

and

$$\tilde{b} = Vb, \tilde{c} = (1, 0, \ldots, 0, 0). \tag{7.76}$$

The second observability canonical form of (7.71) reads as

$$\begin{aligned} D_q \hat{y}(x) &= \tilde{A}\hat{y}(qx) + \tilde{b}u(x), \\ z(x) &= \tilde{c}\hat{y}(x) \end{aligned} \tag{7.77}$$

with

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & . & \dots & -p_k \\ 1 & 0 & 0 & . & \dots & -p_{k-1} \\ 0 & 1 & 0 & . & \dots & -p_{k-2} \\ . & . & . & . & . & . \\ 0 & . & 0 & 1 & 0 & -p_2 \\ 0 & . & . & . & 1 & -p_1 \end{pmatrix} \tag{7.78}$$

and

$$\tilde{c} = (0, 0, \dots, 0, 1). \tag{7.79}$$

(see exercise 2, below).

7.4 Controllability and polynomials

Here we derive interesting interconnection between controllability (similar results can be obtained for observability) and polynomials primality. Consider the time constant system

$$\begin{aligned} D_q y(x) &= Ay(x) + bu(x) \\ z(x) &= Cy(x). \end{aligned} \tag{7.80}$$

with scalar input and suppose that A is in the "companion" form

$$A = \begin{pmatrix} 0 & 0 & 0 & . & \dots & -a_k \\ 1 & 0 & 0 & . & \dots & -a_{k-1} \\ 0 & 1 & 0 & . & \dots & -a_{k-2} \\ . & . & . & . & . & . \\ 0 & . & 0 & 1 & 0 & -a_2 \\ 0 & . & . & . & 1 & -a_1 \end{pmatrix} \tag{7.81}$$

We have the following

Theorem 7.4.1 *The system (7.80) is c.c. iff the polynomials $k(\lambda) = det(\lambda I - A) = \lambda^k + a_1\lambda^{k-1} + \ldots + a_k$ and $p(\lambda) = b_k\lambda^{k-1} + b_{k-1}\lambda^{k-2} + \ldots + b_1$ are relatively prime.*

Proof. We have that if $\lambda_1, \dots, \lambda_k$ are characteristic roots of A, so $p(\lambda_1), \dots, p(\lambda_k)$ are characteristic roots of $p(A)$. Hence $detp(A) = p(\lambda_1) \dots p(\lambda_k)$. Hence $p(A)$

is singular iff $p(\lambda)$ and $k(\lambda)$ have common roots. It remains to prove that $p(A) = U(A, b)$: Let e_i and f_i be the *ith* columns of I and $p(A)$ respectively. One easily verifies: $f_1 = [b_1, \ldots, b_k] = b$. Moreover $e_i = Ae_{i-1}$, $i = 2, \ldots, k$ and $f_i = p(A)e_i$. Hence $f_i = p(A)e_i = p(A)Ae_{i-1} = Ap(A)e_{i-1} = Af_{i-1}$, $i = 2, \ldots, k$. We get $f_i = A^{i-1}b$, $i = 2, \ldots, k$. In other words $p(A) = U(A, b)$, and the theorem is proved.

Remark 7.4.1 *In the particular case when $b = \tilde{b}$ in (7.64), then $p(\lambda) = 1$ and it is necessary relatively prime with $k(\lambda)$. Hence the complete controllability of (7.62) can be obtained as a corollary of the theorem 7.4.1.*

7.5 Exercises

1. Show that the change of variables $\hat{y} = Uy(x)$ in (7.61), where U is its controllability matrix, transforms it in (7.61).

2. Find a change of variable that transforms (7.71) in (7.77).

3. Derive an analog theory by considering not the system (7.5) but

$$\begin{aligned} D_q y(x) &= A(x)y(x) + B(x)u(x) \\ z(x) &= C(x)y(x). \end{aligned} \tag{7.82}$$

4. Show that the system

$$\begin{aligned} D_q y_1(x) &= ay_1(x) + by_2(x) \\ D_q y_1(x) &= cy_2(x) \end{aligned} \tag{7.83}$$

is not C.c.

5. Discuss the c.c. of the system

$$\begin{aligned} D_q y_1(x) &= ay_1(x) + by_2(x) \\ D_q y_1(x) &= cy_1(x) + dy_2(x) \end{aligned} \tag{7.84}$$

6. Contemplate the system

$$D_q y(x) = Ay(x) + bu(x). \tag{7.85}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}; \; b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.86}$$

a) Is it c.c.?

b) Find the control $u(x)$ and the time x_1, necessary to reach the state

$$\begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{7.87}$$

from

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7.88}$$

Chapter 8

q-Difference variational calculus

In this chapter we discuss some fundamental concepts of the variational calculus on the q-uniform lattice $x(s) = q^s$, such as the q-Euler equations and its applications to the isoperimetric and Lagrange problem and commutation equations [9]. Basically, we are concerned in the extremum problem for the following functional

$$\begin{aligned} J(y(x)) &= \int_a^b F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) d_q x \\ &=^{def} (1-q) \sum_{q^\alpha}^{q^\beta} x F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) \end{aligned} \tag{8.1}$$

under the boundary constraints

$$\begin{gathered} y(q^\alpha) = y(q^{\beta+1}) = c_0 \\ D_q y(q^\alpha) = D_q y(q^{\beta+1}) = c_1 \\ \ldots \\ D_q^{k-1} y(q^\alpha) = D_q^{k-1} y(q^{\beta+1}) = c_{k-1} \end{gathered} \tag{8.2}$$

where

$$a = q^{\beta+1} \leq b = q^\alpha \tag{8.3}$$

and the summation is performed by x on the set (we shall sometimes write simply $\sum_{q^\beta}^{q^\beta}$ or $\sum_L$)

$$L = \{q^\beta, q^{\beta-1}, \ldots, q^{\alpha+1}, q^\alpha\}, \quad 0 \leq \alpha < \beta \leq +\infty. \tag{8.4}$$

For $\alpha \rightsquigarrow 0$, $\beta \rightsquigarrow +\infty$, (8.1) and (8.2) read

$$\begin{aligned} &J(y(x)) = \int_0^1 F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) d_q x \\ &=^{def} (1-q) \sum_0^1 x F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) \end{aligned} \tag{8.5}$$

and

$$D_q^i y(0) = D_q^i y(1), \quad i = 0, \ldots, k-1 \tag{8.6}$$

respectively. If the function $\tilde{F}(x) = F(x, y(x), Dy(x), \ldots, D^k y(x))$ is Riemann-integrable on the interval $[0, 1]$, then it is easily seen that for $q \rightsquigarrow 1$, the q-integral in eq. (8.5) and the constraints in eq. (8.6) tends to the continuous integral

$$J(y(x)) = \int_0^1 F(x, y(x), Dy(x), \ldots, D^k y(x)) dx \tag{8.7}$$

where $Df(x) = \frac{d}{dx} f(x)$, and the boundary constraints

$$\begin{aligned} y(0) &= y(1) = c_0 \\ Dy(0) &= Dy(1) = c_1 \\ &\ldots \\ D^{k-1} y(0) &= D^{k-1} y(1) = c_{k-1} \end{aligned} \tag{8.8}$$

respectively. Hence the functional in eq. (8.5) can be considered as a natural q-version of the one in eq. (8.7).
Remark 1. By carrying out in (8.1) the linear change of variable

$$t(s) = a + x(s)(b-a) = a + q^s (b-a) \tag{8.9}$$

(a , b, finite for simplicity), we obtain a q-version of the integral obtained from (8.7) by the linear change of variable

$$t = a + x(b-a), \tag{8.10}$$

and both the two new integrals have now a and b as boundaries of integration. Clearly the converse to (8.9) and (8.10) transformations are also valid. Hence in that sense, there is no lost of generalities considering in this work integrals of type (8.5) or (8.7) or even the little bit more general integral in (8.1). This allows to avoid cumbersome treatments unessential in addition in the reasoning.

8.1 The q-Euler-Lagrange equation

We consider the q-integral functional,

$$J(y(x)) = (1-q)\sum_{q^\alpha}^{q^\beta} xF(x, y(x), D_q y(x), \ldots, D_q^k y(x)). \tag{8.11}$$

Here the function $F(x, y_0(x), \ldots, y_k(x))$ is defined on A as a function of x, together with its first partial derivatives relatively to all its arguments. Let E be the linear space of functions $y(x)$ ($q^\beta \le x \le q^\alpha$) in which is defined the norm

$$\|y\| = \max_{0\le i\le k}(\max_{x\in L} |D_q^i y(x)|) \tag{8.12}$$

and let E' be the linear manifold of functions belonging in E and satisfying to the constraints in (8.2). We study the extremum problem for the functional J, on the manifold E'. We first calculate the first variation of the functional J on the linear manifold E':

$$\begin{aligned} \delta J(y(x), h(x)) &= \tfrac{d}{dt} J(y(x) + th(x))|_{t=0} \\ = (1-q)\tfrac{d}{dt}\sum_{q^\alpha}^{q^\beta} &[xF(x, y(x) + th(x), \ldots, D_q^k y(x) + tD_q^k h(x))]|_{t=0} \\ = (1-q)\sum_{q^\alpha}^{q^\beta} &[\textstyle\sum_{i=0}^{k}[xF_i(x, y(x), D_q y(x), \ldots, D_q^k y(x))D_q^i h(x)] \end{aligned} \tag{8.13}$$

where

$$F_i = \tfrac{\partial F}{\partial y_i}\ (F = F(x, y_0, y_1, \ldots, y_k)),\ i = 0, \ldots, k. \tag{8.14}$$

The variation is dependent of an arbitrary function $h(x)$. Since the variation is performed on the linear manifold E', $h(x)$ is such that $y(x)+th(x)$ belongs also to the linear manifold E' and in particular satisfies the constraints (8.2). A direct consequence of this is that the function $h(x)$ satisfies the constraints:

$$\begin{aligned} h(q^\alpha) &= h(q^{\beta+1}) = 0 \\ D_q h(q^\alpha) &= D_q h(q^{\beta+1}) = 0 \\ &\cdots \\ D_q^{k-1} h(q^\alpha) &= D_q^{k-1} h(q^{\beta+1}) = 0 \end{aligned} \tag{8.15}$$

From the relation $D_q(fg)(x) = f(qx)D_q g(x) + g(x)D_q f(x)$, one obtains the formula of the q-integration by parts:

$$\begin{aligned} &(1-q)\sum_{q^\alpha}^{q^\beta} xf(qx)D_q g(x) = \\ &(1-q)\sum_{q^\alpha}^{q^\beta} xD_q(fg) - (1-q)\sum_{q^\alpha}^{q^\beta} xg(x)D_q f(x). \end{aligned} \tag{8.16}$$

Using (8.15), and (8.16), (8.13) gives

$$\begin{aligned} &\delta J(y(x), h(x)) = \\ &(1-q)\textstyle\sum_{q^\alpha}^{q^\beta} x[\sum_0^k (-1)^i q^{\frac{(i-1)}{2}i} D_q^i [F_i(q^{-i}x, y(q^{-i}x), D_q y(q^{-i}x), \ldots \\ &\ldots, D_q^k y(q^{-i}x)]]h(x) \end{aligned} \tag{8.17}$$

(Very important to distinguish $D_q f(kx)$ which means here $[D_q f](kx)$ with $D_q[f(kx)]$ meaning $D_q g(x)$ for $g(x) = f(kx)$). Next, it is necessary to note that the boundary constraints in eq. (8.15) are equivalents to the following

$$h(q^{\alpha+i}) = h(q^{\beta+1+i}) = 0, i = 0, 1, \ldots, k-1. \tag{8.18}$$

Consequently, (8.17) gives

$$\begin{aligned} &\delta J(y(x), h(x)) = \\ &(1-q)\textstyle\sum_{q^{\alpha+k}}^{q^\beta} x[\sum_0^k (-1)^i q^{\frac{(i-1)}{2}i} D_q^i [F_i(q^{-i}x, y(q^{-i}x), D_q y(q^{-i}x), \ldots \\ &\ldots, D_q^k y(q^{-i}x)]h(x). \end{aligned} \tag{8.19}$$

For deriving the corresponding q-Euler-Lagrange equation, we need the following lemma, which constitutes a q-version of what is called "fundamental lemma of variational calculus".

Lemma 8.1.1 *Consider the functional*

$$I(\hat{f}) = (1-q)\sum_B x\hat{f}(x)h(x) \tag{8.20}$$

where $B = \{q^r, q^{r+1}, \ldots, q^s\}$. *If* $I(\hat{f}) = 0$, *for all* h *defined on* B, *then* $\hat{f}(x) \equiv 0$ *on* B.

Proof. As $I(\hat{f}) = 0$, $\forall h$ defined on B, we have that:

$$\begin{array}{rcl} q^r\hat{f}(q^r)h_1(q^r)+ & \ldots & +q^s\hat{f}(q^s)h_1(q^s) = 0 \\ q^r\hat{f}(q^r)h_2(q^r)+ & \ldots & +q^s\hat{f}(q^s)h_2(q^s) = 0 \\ & \ldots & \\ q^r\hat{f}(q^r)h_{s-r+1}(q^r)+ & \ldots & +q^s\hat{f}(q^s)h_{s-r+1}(q^s) = 0 \end{array} \tag{8.21}$$

for any choice of the $(s-r+1)^2$ numbers

$$a_{ij} = h_i(q^{j+r-1}), i, j = 1, \ldots, s-r+1. \tag{8.22}$$

This is a linear homogenous system with the matrix

$$(a_{ij})_{i,j=1}^{s-r+1} \tag{8.23}$$

and the vector $[T_j = q^{j+r-1}\hat{f}(q^{j+r-1})]_{j=1}^{s-r+1}$. Choosing the numbers

$$h_i(q^{j+r-1}), i, j = 1, \ldots, s-r+1 \tag{8.24}$$

in such a way that the corresponding matrix in (8.23) doesn't be singular, (8.21) gives $T_j = 0, j = 1, \ldots, s-r+1$ or equivalently, $\hat{f}(q^{j+r-1}) = 0, j = 1, \ldots, s-r+1$ which proves the lemma.
Next, remark that (8.19) is written under the form

$$\delta J(y(x), h(x)) = I(\hat{f}) = (1-q)\sum_{q^{\alpha+k}}^{q^{\beta}} x\hat{f}(x)h(x) \tag{8.25}$$

where $\hat{f}$ represents the expression within the external brackets. Hence the necessary condition for the extremum problem (8.1)-(8.4) can be written

$$I(\hat{f}) = 0 \tag{8.26}$$

and this for all $h(x)$ defined on

$$B = \{q^r, q^{r+1}, \ldots, q^s\}, \quad r = \alpha + k,\ \beta = s \tag{8.27}$$

By the fundamental lemma of the variational q-calculus (see Lemma 8.1.1), this leads to

$$\hat{f}(x) \equiv 0. \tag{8.28}$$

Thus the necessary condition for the extremum problem (8.1)-(8.4) reads

$$\begin{gathered}\sum_0^k (-1)^i q^{\frac{(i-1)}{2}i} D_q^i[F_i(q^{-i}x, y(q^{-i}x), D_q y(q^{-i}x), \ldots, D_q^k y(q^{-i}x)] \\ = 0, \\ D_q^i y(q^{\alpha}) = D_q^i y(q^{\beta+1}) = c_i,\ i = 0, \ldots, k-1.\end{gathered} \tag{8.29}$$

For $k = 1$ and $k = 2$, for example, we have respectively:

$$\begin{gathered}F_0(x, y(x), D_q y(x)) - D_q[F_1(q^{-1}x, y(q^{-1}x), D_q y(q^{-1}x))] = 0, \\ y(q^{\alpha}) = y(q^{\beta+1}) = c_0\end{gathered} \tag{8.30}$$

and

$$\begin{gathered}F_0(x, y(x), D_q y(x), D_q^2 y(x)) \\ -D_q[F_1(q^{-1}x, y(q^{-1}x), D_q y(q^{-1}x), D_q^2 y(q^{-1}x))] \\ +qD_q^2[F_2(q^{-2}x, y(q^{-2}x), D_q y(q^{-2}x), D_q^2 y(q^{-2}x))] = 0, \\ y(q^\alpha) = y(q^{\beta+1}) = c_0;\ D_q y(q^\alpha) = D_q y(q^{\beta+1}) = c_1\end{gathered} \tag{8.31}$$

Let us note that while the q-integral (8.1) tends to the continuous integral (8.7) for $q \rightsquigarrow 1$, $\alpha \rightsquigarrow 0$, $\beta \rightsquigarrow +\infty$, the q-equation in (8.29) tends to the corresponding to (8.7) differential Euler-Lagrange equation:

$$\begin{gathered}\sum_0^k (-1)^i D^i F_i(x, y(x), Dy(x), \ldots, D^k y(x)) = 0, \\ D^i y(0) = D^i y(1)) = c_i,\ i = 0, \ldots, k-1.\end{gathered} \tag{8.32}$$

That is why it is convenient to call (8.29), the *q-Euler-Lagrange equation* corresponding to the q-integral (8.1). The equation (8.29) is a q-difference equation of degree $2k$ which is in principle solved uniquely under the $2k$ boundary constraints.

Remark 2. If the functional in (8.11) is dependent of more that one variable i.e. $J = J(y_1, \ldots, y_n)$, then the necessary extremum condition leads to type (8.29) n q-Euler-Lagrange equations with y replaced by y_i, $i = 1, \ldots, n$.

8.2 Applications

8.2.1 On the continuous variational calculus

The direct application of the variational q-calculus is its application on the continuous (differential) variational calculus: Instead of solving the Euler-Lagrange equation (8.32) for finding the extremum of the functional (8.7), it suffices to solve the q-Euler-Lagrange equation (8.29) and then pass to the limit while $q \rightsquigarrow 1$. Remark that thought this can appear at the first glad as a contradiction (by the fact of the phenomenon of discretization), the variational q-calculus is a generalization of the continuous variational calculus due to the presence of the extra-parameter q (which may be physical, economical or another) in the first and its absence in the second.

Example. Suppose it is desirable to find the extremum of the integration functional

$$J(y(x)) = \int_0^1 (x^\nu y + \tfrac{1}{2}(Dy)^2)dx, \quad \nu > 0, \tag{8.33}$$

under the boundary constraints $y(0) = c$; $y(1) = \tilde{c}$. The q-version of the problem consists in finding the extremum of the q-integration functional

$$J(y(x)) = (1-q)\sum_0^1 x[x^\nu y + \tfrac{1}{2}(D_q y)^2], \nu > 0, \tag{8.34}$$

under the same boundary constraints. According to (8.30), the q-Euler-Lagrange equation of the latter problem reads:

$$x^\nu - D_q[D_q y(q^{-1}x)] = 0 \tag{8.35}$$

which solution is

$$y(x) = x^{\nu+2}[\tfrac{(1-q)^2 q^{\nu+1}}{(1-q^{\nu+1})(1-q^{\nu+2})}] + [y(1) - y(0) - \tfrac{(1-q)^2 q^{\nu+1}}{(1-q^{\nu+1})(1-q^{\nu+2})}]x \\ +y(0). \tag{8.36}$$

As it can be verified, for $q \rightsquigarrow 1$, the function in (8.36) tends to the function

$$y(x) = \tfrac{x^{\nu+2}}{(\nu+1)(\nu+2)} + [y(1) - y(0) - \tfrac{1}{(\nu+1)(\nu+2)}]x + y(0), \tag{8.37}$$

solution of the Euler-Lagrange equation of the functional in (8.33).

8.2.2 The q-isoperimetric problem

Suppose that it is required to find the extremum of the functional

$$J(y(x)) = (1-q)\sum_{q^\alpha}^{q^\beta} xf(x, y(x), D_q y(x), \ldots, D_q^k y(x)) \\ D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i,\ i = 0, 1, \ldots, k-1 \tag{8.38}$$

under the constraints

$$\tilde{J}_i(y(x)) = (1-q)\sum_{q^\alpha}^{q^\beta} xf^i(x, y(x), D_q y(x), \ldots, D_q^k y(x)) = C_i, \\ i = 1, \ldots, m. \tag{8.39}$$

To solve this problem we needs to consider the following generalities. Let $J(y)$ and $\tilde{J}_1(y), \ldots, \tilde{J}_m$ be some differentiable functionals on the normed space E, or on its manifold E'. We have the following theorem (see for ex. [33])

Theorem 8.2.1 *If a functional $J(y)$ attains its extremum in the point $\bar{y}$ under the additional conditions $\tilde{J}_i(y) = C_i$, $i = 1, \ldots, m$ and $\bar{y}$ is not a stationary point for any one of the functionals $\tilde{J}_i$ ($\delta\tilde{J}_i(\bar{y}, h) \neq 0, i = 1, \ldots, m,$ identically) while the functionals $\delta\tilde{J}_i$, ($i = 1, \ldots, m$) are linearly independent, then $\bar{y}$ is a stationary point for the functional $J - \sum_{i=1}^m \lambda_i \tilde{J}_i$ where the λ_i are some constants.*

Thus by this theorem, the necessary extremum condition for the functional J(y) under the additional constraints $\tilde{J}_i(y) = C_i$, $i = 1, \ldots, m$, verifying the conditions of the theorem (let us note that considering the formula (8.17), a type (8.11) functional i.e. satisfying the same definition conditions, is differentiable on E'), is given by the equation (8.29) with

$$F = f - \textstyle\sum_{i=1}^{m} \lambda_i f^i \tag{8.40}$$

It is a q-difference equation of order $2k$ containing m unknown parameters. It is in principle solved uniquely under the $2k$ boundary constraints and the additional m conditions.

Example. Suppose it required to solve the problem of finding the extremum of the q-integration functional

$$J(y(x)) = (1-q)\textstyle\sum_{q^\alpha}^{q^\beta} x[ax^2(D_q^2 y)^2 + b(D_q y)^2], \quad a, b > 0 \tag{8.41}$$

under the boundary constraints

$$D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i, \ i = 0, 1, \tag{8.42}$$

and an additional condition that $J_1(y(x)) = c$, c some constant, where J_1 is a q-integration functional given by

$$J_1(y(x)) = (1-q)\textstyle\sum_{q^\alpha}^{q^\beta} x^2 y. \tag{8.43}$$

According to the theorem 8.2.1, the problem is equivalent to that of finding the extremum of the q-integration functional

$$J(y(x)) = (1-q)\textstyle\sum_{q^\alpha}^{q^\beta} x[ax^2(D_q^2 y)^2 + b(D_q y)^2 - \lambda xy], \tag{8.44}$$

for some constant λ, under the same boundary constraints (8.42). The corresponding q-Euler-Lagrange equation reads

$$-\lambda x - 2bD_q[D_q y(q^{-1}x)] + 2aq^{-3}D_q^2[x^2 D_q^2 y(q^{-2}x)] = 0 \tag{8.45}$$

or equivalently after reduction and integration (c_1, c_2, constants of integration)

$$y(x) - [q(q-1)^2 b/a + q + 1]y(q^{-1}x) + qy(q^{-2}x) = \tfrac{(1-q)^2}{2a}(c_1 x + c_2 + \tfrac{\lambda x^3}{(q+1)(q^2+q+1)}). \tag{8.46}$$

This is a constant coefficients linear nonhomogeneous second-order q-difference equation which can be solved uniquely (under the constraints (8.42)) by methods similar to that of analogous differential or difference equations.

8.2.3 The q-Lagrange problem

Suppose now that it is required to find the extremum of the functional

$$J(y_1(x),\dots,y_n(x))$$
$$= (1-q)\sum_{q^\alpha}^{q^\beta} xf(x,y_1(x),\dots,y_n(x),D_qy_1(x),\dots,D_qy_n(x)) \quad (8.47)$$

under the constraints

$$f^i(x,y_1(x),\dots,y_n(x),D_qy_1(x),\dots,D_qy_n(x))=0,\ i=1,\dots,m; m<n,$$
$$y_i(q^\alpha)=y_i(q^{\beta+1})=c_i,\ i=1,\dots,n. \quad (8.48)$$

This problem can be transformed in the q-isoperimetric one as follows: First, multiply every *ith* equation in (8.48) by an arbitrary function $\lambda_i(x)$ defined as all the remaining on $L=\{q^\beta,\dots,q^\alpha\}$ and then apply the q-integration on L on the result:

$$\tilde{J}_i(y_1(x),\dots,y_n(x))$$
$$= (1-q)\sum_{q^\alpha}^{q^\beta} x\lambda_i(x)f^i(x,y_1(x),\dots,y_n(x),D_qy_1(x),\dots,D_qy_n(x))=0,$$
$$i=1,\dots,m \quad (8.49)$$

The remaining question is that of knowing if the two constraints (8.48) and (8.49) are equivalent. The answer is yes since obviously from (8.48) follows (8.49). Finally, it is by the fundamental lemma of the variational q-calculus (see Lemma 8.1.1) that (8.48) follows from (8.49).

Example. Suppose that the problem consists in finding the extremum of the functional

$$J(x(t),u(t)) = \tfrac{1}{2}(1-q)\sum_{q^\alpha}^{q^\beta} t[u^2(t)-x^2(t)] \quad (8.50)$$

under the constraints

$$D_q^2x=u$$
$$x(q^\alpha)=x(q^{\beta+1})=c; D_qx(q^\alpha)=D_qx(q^{\beta+1})=\tilde{c}. \quad (8.51)$$

The problem is equivalent to the q-Lagrange problem of finding the extremum of the functional

$$J(x(t),y(t),z(t)) = \tfrac{1}{2}(1-q)\sum_{q^\alpha}^{q^\beta} t[z^2(t)-x^2(t)] \quad (8.52)$$

under the constraints

$$D_q x = y; \quad D_q y = z$$
$$x(q^\alpha) = x(q^{\beta+1}) = c; \quad y(q^\alpha) = y(q^{\beta+1}) = \tilde{c}. \tag{8.53}$$

Hence the problem is equivalent to that of finding the extremum of the functional

$$J(x, y, z, \lambda_1, \lambda_2) = (1-q) \sum_{q^\alpha}^{q^\beta} tF(x(t), y(t), z(t), \lambda_1(t), \lambda_2(t)) \tag{8.54}$$

where

$$F(x(t), y(t), z(t), \lambda_1(t), \lambda_2(t))$$
$$= \frac{1}{2}(z^2(t) - x^2(t)) + \lambda_1(t)(D_q x(t) - y(t)) + \lambda_2(t)(D_q y(t) - z(t)) \tag{8.55}$$

under the boundary constraints

$$x(q^\alpha) = x(q^{\beta+1}) = c; \quad y(q^\alpha) = y(q^{\beta+1}) = \tilde{c}. \tag{8.56}$$

The corresponding q-Euler-Lagrange equations give

$$y(t) = D_q x(t); \; z(t) = \lambda_2(t) = D_q^2 x(t); \; \lambda_1(t) = -q^2 D_q^3[x(q^{-1}t)], \tag{8.57}$$
$$-x(t) + q^5 D_q^4[x(q^{-2}t)] = 0. \tag{8.58}$$

Hence it is sufficient to solve the equation (8.58). Searching its solution as an integer power series $x(t) = \sum_0^\infty C_n t^n$, one is led to the following fourth order difference equation for the coefficient c_n:

$$C_n = q^{2n-5}\left(\frac{1-q}{1-q^n}\right)\left(\frac{1-q}{1-q^{n-1}}\right)\left(\frac{1-q}{1-q^{n-2}}\right)\left(\frac{1-q}{1-q^{n-3}}\right)C_{n-4} \tag{8.59}$$

with the coefficients C_0, C_1, C_2, C_3 determined by the four boundary constraints (8.56).The solution of (8.59) reads

$$C_n = \prod_{i=n_c}^{n}\left(\frac{1-q}{1-q^i}\right) \prod_{i=1}^{\frac{n-n_c}{4}} q^{2(n_c+4i)-5} C_{n_c} \tag{8.60}$$

where $n \equiv n_c \; mod \; 4$, $0 \leq n_c \leq 3$.
To obtain the four basic elements for the space of solutions of (8.58), one can make the following four independent choices for the constants C_0, C_1, C_2, C_3: Choosing (a) $C_n = \frac{1}{n!}$ for $n = 0, \ldots, 3$ leads to $x(t) = \hat{e}_q^t$; (b) $C_n = \frac{(-1)^n}{n!}$ for $n = 0, \ldots, 3$ leads to $x(t) = \hat{e}_q^{-t}$; (c) $C_n = \frac{(-1)^{\frac{n}{2}}[(1)^n + (-1)^n]}{2n!}$ for $n = 0, \ldots, 3$

leads to $x(t) = cos_q t$; (d) $C_n = \frac{(-1)^{\frac{n-1}{2}}[(1)^n-(-1)^n]}{2n!}$ for $n = 0, \ldots, 3$ leads to $x(t) = sin_q t$.
The functions $\hat{e}_q^t, \hat{e}_q^{-t}, cos_q t$ and $sin_q t$ have in the integer power series, the indicated coefficients for $n = 0, \ldots, 3$ and the coefficients in (8.60) for $n > 3$. As it can be verified, for $q \rightsquigarrow 1$, these functions have as limits the functions $e^t, e^{-t}, cost$ and $sint$, respectively. The latter are nothing else than a basis of the space of solutions of a similar to (8.58) differential equation for the corresponding continuous problem.

8.2.4 A q-version of the commutation equations

Let $L = -D^2 + y(x)$, where $Df(x) = \frac{df(x)}{dx} = f'(x)$, be the Schrodinger operator and let A_m be a sequence of differential operators of order $2m+1, m = 0, 1, 2, \ldots$, which coefficients are arbitrary differential polynomials of the potential $y(x)$. By commutation equations, one understands the equations $[L, A_m] = LA_m - A_m L = 0$, in the coefficients of the operators. It is known since [18, 19] that for any $m, m = 0, 1, 2, \ldots$ there exists such an operator A_m of order $2m+1$, such that the operator $[L, A_m] = LA_m - A_m L$ is an operator of multiplication by a scalar function $f_m(y, y', y'', \ldots)$: $[L, A_m] = f_m(y, y', y'', \ldots)$. The corresponding commutation equations then read

$$[L, A_m] = f_m(y, y', y'', \ldots) = 0 \tag{8.61}$$

Its non-trivial solutions are elliptic or hyperelliptic (or their degenerate cases) functions for $m = 1$ and $m > 1$ respectively (see [18, 19]). Since years seventies of the last century (see for ex. [23], paragr. 30), it is known that the commutation equations (8.61) are equivalent to type (8.32) Euler-Lagrange equations for the functionals

$$J_m(y(x)) = \int_a^b L_m(y(x), y'(x), \ldots, y^{(k)}(x))dx \tag{8.62}$$

with L_m related to A_m in a known way (see for ex. [23]).
If $m = 1$ for example, $L_1(y, y') = y'^2/2 + y^3 + c_1 y^2 + c_2 y$, ($c_1, c_2$: constants), and the corresponding Euler-Lagrange equation (commutation equation) reads:

$$y'' = 3y^2 + 2c_1 y + c_2 \quad . \tag{8.63}$$

Up to a linear transformation $y \to c_3 y + c_4$, its solution is the well known Weierstrass function $\mathcal{P}(x)$.

Considering now the q-functional

$$J_m(y(x)) = (1-q)\sum_{q^\alpha}^{q^\beta} x L_m(y(x), D_q y(x), \ldots, D_q^k y(x)) \qquad (8.64)$$

we obtain that the corresponding to type (8.29) q-Euler-Lagrange equations are *q-versions of the commutation equations* (8.61). For example for $m = 1$, we have $L_1(y(x), D_q y(x)) = [D_q y]^2/2 + y^3 + c_1 y^2 + c_2 y$ and the corresponding q-Euler-Lagrange equation reads

$$3y^2 + 2c_1 y + c_2 - q D_q^2[y(q^{-1}x)] = 0 \qquad (8.65)$$

or equivalently

$$y(qx) = (q+1)y(x) + (qx-x)^2(3y^2(x) + 2c_1 y(x) + c_2) - qy(q^{-1}x) \qquad (8.66)$$

Obviously, the q-Euler-Lagrange equation(8.65) (or (8.66)) tends to the Euler-Lagrange one in (8.63), while $q \rightsquigarrow 1$. A particular solution ($c_1 = c_2 = 0$) of (8.66) is given by the function

$$y(x) = \tfrac{(1+q)(1+q+q^3)}{3q^2x^2}, \qquad (8.67)$$

a q-version of the degenerate case of the Weierstrass function $\mathcal{P}(x)$ (solution of (8.63)) while its periods tend to ∞. One will note that even without giving an analytical general resolution of this equation, its solution satisfying given boundary constraints, can be found recursively. Here is naturally the main advantage of the analysis on lattices.

8.3 Exercises

1. Determine the extremals of the functionals

$$J(y) = \int_0^1 [(D_q y)^2 - y^2] d_q x; \quad y(0) = 0 \qquad (8.68)$$

2. Find the extremals of the functional

$$J(y) = \int_0^1 [(D_q y)^2 + x^2] d_q x \qquad (8.69)$$

under the constraints

$$J(y) = \int_0^1 y^2 d_q x = 2; \quad y(0) = y(1) = 1 \qquad (8.70)$$

3. Find the extremals of the isoperimetric problems

$$J(y) = \int_{x_1}^{x_2} (D_q y)^2 d_q x; \ J(y) = \int_{x_1}^{x_2} y^2 d_q x = 0 \tag{8.71}$$

Chapter 9

q-Difference optimal control

In chapter 7, we were dealing with controllability problems, that is the problem of determining either exists a control that could transfers the trajectories from a given sate to another predetermined one. We were fully indifferent toward the quality of the control function. However, in many practical problems, one is interested not only by the existence of a control function but in an *optimal* control function, that is that control which among others constitutes the extremum element for a given functional.

9.1 The q-optimal control problem

Suppose that it is given a k-dimensional q-controlled system

$$\begin{aligned} D_q z(x) &= \tilde{f}^0(x, z(x), u(x)) \\ z(q^\alpha) &= z(q^\beta) = C \end{aligned} \tag{9.1}$$

and the q-functional of the form

$$\tilde{J}(z(x), u(x)) = (1-q) \textstyle\sum_{q^\alpha}^{q^\beta} x \tilde{f}(x, z(x), u(x)) \tag{9.2}$$

The optimal control problem consists in that among all admissible control functions $u(x)$, find that for which the corresponding solution of the q-boundary problem (9.1) is an extremum for the functional in (9.2). Thus following the q-Lagrange problem, our extremum problem consists in finding the extremum of the functional under the constraints below (remark that as there is no any derivative of $u(x)$, no boundary constraints for it are needed):

$$\hat{J}(y(x), u(x)) = (1-q) \textstyle\sum_{q^\alpha}^{q^\beta} x \{\tilde{f}(x, z, u) - \lambda(x)[\tilde{f}^0(x, z, u) - D_q z]\},$$

$$z(q^{\alpha}) = z(q^{\beta}) = C \tag{9.3}$$

According to (8.30), the corresponding q-Euler-Lagrange system reads

$$\begin{aligned}(\tilde{f}_z - \lambda(x)\tilde{f}^0_z) - D_q[\lambda(q^{-1}x)] = 0\\ \tilde{f}_u - \lambda(x)\tilde{f}^0_u = 0\end{aligned} \tag{9.4}$$

Combining (9.4) with the first eq. in (9.1), we conclude that the solution of the problem satisfies the system:

$$\begin{aligned}D_q z &= +H_{\lambda}\\ D_q[\lambda(q^{-1}x)] &= -H_z\\ 0 &= H_u\end{aligned} \tag{9.5}$$

where

$$H(x, z, \lambda, u) = -\tilde{f}(x, z, u) + \lambda(x)\tilde{f}^0(x, z, u) \tag{9.6}$$

Seen the similarities of the problem posed and the formula obtained (eqs.(9.5)-(9.6)), with their analogs in the continuous optimal control, one can say that we were dealing with a q-version of one of the version of the "*maximum principle*" [48]. Hence we can refer to H in (9.6) as the q-Hamilton-Pontriaguine function, (9.5) as the q-Hamilton-Pontriaguine system. Recall that the reference to L S Pontriaguine is linked to the "maximum principle" in [48], the one to Hamilton is linked to the fact that in the case of pure calculus of variation (the control function and system are not present explicitly), the Hamilton and Hamilton-Pontriaguine systems are equivalent (see the following subsection for the q-situation).

Example. $(q - Linear - quadraticproblem)$ Suppose that the problem is that of finding a control function $u(x)$ such that the corresponding solution of the controlled system

$$D_q y = -ay(x) + u(x), \quad a > 0 \tag{9.7}$$

satisfying the boundary conditions $y(q^{\alpha}) = y(q^{\beta+1}) = c$, is an extremum element for the q-integral functional (*q-quadratic cost functional*)

$$J(y(x), u(x) = \tfrac{1}{2}(1-q)\sum_{q^{\alpha}}^{q^{\beta}} x(y^2(x) + u^2(x)). \tag{9.8}$$

According to (9.5) and (9.6), the solution of the problem satisfies

$$\begin{aligned} D_q y &= H_\lambda \\ D_q[\lambda(q^{-1}x)] &= -H_y \\ H_u &= 0, \end{aligned} \tag{9.9}$$

where

$$H(y,\lambda,u) = -\frac{1}{2}(y^2+u^2) + (-ay+u)\lambda(x). \tag{9.10}$$

(9.9) and (9.10) give

$$\begin{aligned} D_q y &= -ay+u \\ D_q\lambda(x) &= qy(qx) + aq\lambda(qx) \\ \lambda &= u. \end{aligned} \tag{9.11}$$

In term of $y(x)$, this system can be simplified in the following

$$D_q^2 y(x) + aD_q y(x) = (a^2+1)qy(qx) + aqD_q y(qx). \tag{9.12}$$

Searching the solution of (9.12) under the form of an integer power series

$$y(x) = \textstyle\sum_0^\infty c_n x^n \tag{9.13}$$

one is led to a variable coefficient linear homogenous second-order difference equation for c_n:

$$c_n = a(q-1)c_{n-1} + q(a^2+1)\tfrac{(1-q)^2}{(1-q^{n-1})(1-q^n)}c_{n-2}. \tag{9.14}$$

This difference equation can naturally be solved recursively starting from the initial data c_0 and c_1.
However, even without solving it, we can search for what give the corresponding function in (9.13), in the the limiting case when $q \rightsquigarrow 1$. In (9.14), for $q \rightsquigarrow 1$, the factor of c_{n-1} give zero, while that of c_{n-2} give $\frac{a^2+1}{n(n-1)}$. Hence for $q \rightsquigarrow 1$, (9.14) give

$$c_n = \tfrac{a^2+1}{n(n-1)}c_{n-2}; \quad n = 2, \dots. \tag{9.15}$$

Choosing c_0 and c_1 (this equivalent to that choosing $y(q^\alpha)$ and $y(q^{\beta+1})$) as $c_0 = 1$ and $c_1 = \sqrt{a^2+1}$ or $c_1 = -\sqrt{a^2+1}$, (9.15) give as solutions $c_n = \frac{(a^2+1)^{\frac{n}{2}}}{n!}$ or $c_n = (-1)^n\frac{(a^2+1)^{\frac{n}{2}}}{n!}$ and the corresponding power series give $y(x) = exp(\sqrt{a^2+1}x)$ or $y(x) = exp(-\sqrt{a^2+1}x)$ respectively. As it can be verified, the latter are the solutions for $y(x)$ in the corresponding continuous problem.

9.2 Interconnection between the variational q-calculus, the q-optimal control and the q-Hamilton system

Here, we have the following

Theorem 9.2.1 *For the simplest functional*

$$J(y(x)) = (1-q)\sum_{q^\alpha}^{q^\beta} xF(y(x), D_q y(x)),$$
$$y(q^\alpha) = y(q^{\beta+1}) = c_0 \tag{9.16}$$

the q-Euler-Lagrange equation, the q-Hamilton-Pontriaguine and the q-Hamilton systems are equivalent.

Proof. We show this in three steps:
a)We first show how to obtain the q-Hamilton system from the q-Euler-Lagrange equation. For the functional in (9.16), the q-Euler-Lagrange equation reads

$$F_0(y(x), D_q y(x)) - D_q[F_1(y(q^{-1}x), D_q y(q^{-1}x))] = 0. \tag{9.17}$$

Letting

$$\lambda(x) = F_1(y(x), D_q y(x)), \tag{9.18}$$

and

$$H = -F + \lambda(x) D_q y, \tag{9.19}$$

then we get from (9.17),(9.18) and (9.19) the q-Hamilton system

$$D_q y = +H_\lambda(y(x), \lambda, D_q y)$$
$$D_q[\lambda(q^{-1}x)] = -H_y(y(x), \lambda, D_q y) \tag{9.20}$$

b) To get the q-Hamilton-Pontriaguine system from q-Hamilton system (9.20), it suffices to suppose $u(x) = D_q y(x)$ to be the control q-equation for the given initial non controlled extremum problem. In that case, (9.20) gives

$$D_q y = +H_\lambda(y(x), \lambda, u(x))$$
$$D_q[\lambda(q^{-1}x)] = -H_y(y(x), \lambda, u(x)) \tag{9.21}$$

with

$$H(y(x), \lambda(x), u(x)) = -F(y(x), u(x)) + \lambda(x)u(x), \tag{9.22}$$

the q-Hamilton-Pontriaguine function, and from (9.18) we get the third equation in (9.5):

$$H_u = 0. \tag{9.23}$$

c) We finally show how to obtain the q-Euler-Lagrange equation (9.17) from the q-Hamilton-Pontriaguine system (9.21), (9.22) and (9.23). From (9.22) and (9.23), we have

$$\lambda(x) = F_1(y(x), u(x)) = F_1(y(x), D_q y(x)), \tag{9.24}$$

while from (9.21) we get

$$D_q[\lambda(q^{-1}x)] = F_0(y(x), u(x)) = F_0(y(x), D_q y(x)). \tag{9.25}$$

Finally, (9.24) and (9.25) give the q-Euler-Lagrange equation (9.17), which proves the theorem.

9.3 Energy q-optimal control

Consider again the linear control system (7.5). In theorem 7.2.2 it was shown that under the condition of the theorem, the control function (7.23) transfers $y =_0$ to $y =_f$ in time $x_0 \leq x \leq x_1$. It is interesting to note that in fact that control is optimal in the sense that it minimizes the integral

$$\textstyle\int_{x_0}^{x_1} \parallel u(t) \parallel^2 d_q t = \int_{x_0}^{x_1} (u_1^2 + \ldots + u_m^2) d_q t, \tag{9.26}$$

seen as a measure of "control" energy involved.

Theorem 9.3.1 *If $\tilde{u}(x)$ is another control transferring $y = y_0 = y(x_0)$ to $y = y_f = y(x_1)$ then*

$$\textstyle\int_{x_0}^{x_1} \parallel \tilde{u} \parallel^2 > \int_{x_0}^{x_1} \parallel u \parallel^2 d_q t, \tag{9.27}$$

provided $\tilde{u} \neq u$

Proof. We have

$$y(x_1) = \Phi(x,x_0)[y_0 + \textstyle\int_{x_0}^{x_1} \Phi(x_0,x)B(t)u(t)] \tag{9.28}$$
$$y(x_1) = \Phi(x,x_0)[y_0 + \textstyle\int_{x_0}^{x_1} \Phi(x_0,x)B(t)\tilde{u}(t)] \tag{9.29}$$

Substracting members by members:

$$0 = \textstyle\int_{x_0}^{x_1} \Phi(x_0,t)B(t)[\tilde{u}(t) - u(t)] \tag{9.30}$$

Multiplying on the left by $[y_0 - \Phi(x_0,x_1)y_f]^T[U^{-1}(x_0,x_1)]^T$ and use the transpose of (7.23):

$$0 = \textstyle\int_{x_0}^{x_1} y_0 - \Phi(x_0,x_1)y_f]^T[U^{-1}(x_0,x_1)]^T\Phi(x_0,t)B(t)[\tilde{u}(t) - u(t)]d_qt$$
$$\Leftrightarrow 0 = \textstyle\int_{x_0}^{x_1} u^T[\tilde{u}(t) - u(t)]d_qt \tag{9.31}$$

We have next $\int_{x_0}^{x_1} \parallel \tilde{u}(t) - u(t) \parallel^2 d_qt = \int_{x_0}^{x_1}(\tilde{u}(t) - u(t))^T(\tilde{u}(t) - u(t))d_qt$ $= -\int_{x_0}^{x_1} \tilde{u}^T(\tilde{u}(t) - u(t))d_qt = \int_{x_0}^{x_1} \parallel \tilde{u} \parallel^2 d_qt - \int_{x_0}^{x_1} \parallel u \parallel^2 d_qt$. Hence $\int_{x_0}^{x_1} \parallel \tilde{u} \parallel^2 d_qt = \int_{x_0}^{x_1} \parallel u \parallel^2 d_qt + \int_{x_0}^{x_1} \parallel \tilde{u} - u \parallel^2 d_qt$, which proves the theorem.

9.4 Exercises

1. Given the example in section 8.2.1, write down and solve the equivalent Hamilton and Hamilton-Pontriaguine systems.

2. Given the example in section 9.1, write down and solve the equivalent q-Euler-Lagrange equation and Hamilton system.

3. Given the system

$$\begin{aligned} D_qx &= v \\ D_qv &= u \end{aligned} \tag{9.32}$$

describing the moving of a point in the plan x, v. Determine the control function $u(t)$ such that a point $A(x_0, v_0)$ reaches the position $B(0,0)$ in the smallest time under the condition $|u| \leq 1$.

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