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Truncated Moment Formalism for Radiation Hydrodynamics in Numerical Relativity

Masaru SHIBATA,[1] Kenta KIUCHI,[1] Yu-ichiro SEKIGUCHI[1,2] and Yudai SUWA[1]

[1] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
[2] *Division of Theoretical Astronomy, National Astronomical Observatory of Japan, Mitaka 181-8588, Japan*

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A truncated moment formalism for general relativistic radiation hydrodynamics, based on Thorne's moment formalism, is derived. The fluid rest frame is chosen to be the fiducial frame for defining the radiation moments. Then, zeroth-, first-, and second-rank radiation moments are defined from the distribution function with a physically reasonable assumption for it in the optically thin and thick limits. The source terms are written, focusing specifically on the neutrino transfer and neglecting higher harmonic angular dependence of the reaction angle. Finally, basic equations for a truncated moment formalism for general relativistic radiation hydrodynamics in a closed covariant form are derived assuming a closure relation among the radiation stress tensor, energy density, and energy flux, and a variable Eddington factor, which works well.

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§1. Introduction

Radiation fields and their interaction with matter often play a crucial role in many astrophysical contexts. For example, the critical roles of photon pressure during proto-star and massive-star formation and of neutrino heating and cooling in supernova core collapse and explosion are well-known among many other phenomena. To theoretically clarify these phenomena, it is necessary to solve hydrodynamic equations as well as radiation transfer equations. For strictly handling the radiation transfer, it is necessary to numerically solve the Boltzmann equation, taking into account the absorption, emission, and scattering terms. However, this equation has 3+3+1 dimensional form (3 dimensions in real and phase spaces, respectively, and 1 dimension in time), and furthermore, the time scale for the interaction between matter and radiation is often shorter than the dynamical time scale of the system. Thus, in the current computational resources, it is not feasible to perform a well-resolved numerical simulation with a sufficient grid resolution. A certain approximate method incorporating key features of radiation effects is often required in numerical astrophysics. In particular, no useful formalism for multi-dimensional simulation in general relativity has been well developed (but see Refs. 1)–3)). Note that in spherical symmetry, this equation is simplified to a 1+2+1 dimensional form as formulated in Ref. 4). Simulations with similar formalism were performed in Ref. 5), and subsequently, sophisticated simulations including the state-of-the-art microphysics were achieved, e.g., in Refs. 6) and 7). However, the effort has been paid only to the spherical symmetric simulations. In this paper, we derive an approximate formalism of radiation hydrodynamics in general relativity, in which a numerical simulation will be feasible capturing the physically important ingredients.

Historically, a popular method for approximate radiation hydrodynamics is a flux-limited diffusion (FLD) method.[8] In this method, the radiation flux density is in general assumed to be described by the radiation energy density, and resulting evolution equation for the radiation energy density becomes a diffusion-type equation in the optically thick region (cf. §5). In this case, the propagation speed of characteristics may be larger than the speed of light, although in general relativity, the causality must not be violated. Another drawback of the FLD scheme is associated with the presence of constraint equations (Hamiltonian and momentum constraints) in the initial value problem of general relativity: In numerical relativity for multi-dimensional problems, we usually solve the evolution equations of Einstein's equation and matter equations self-consistently. As a result, the constraints are satisfied within a numerical error. However, in the case that we do not solve the energy and momentum equations for the radiation field self-consistently, the constraints are violated. In the FLD method, one solves an equation only for the radiation energy density component, and hence, the constraints will be violated in general.

Truncated moment formalisms have also been proposed for an approximate solution of radiation hydrodynamics[1),2)] (see also Ref. 9) for an application). In this approach, one derives a set of covariant equations for multi-pole moments defined from the distribution function of radiation. Then, assuming that higher-order moments may be neglected and imposing closure relations, a closed covariant form of basic equations is derived. With an appropriate choice of the closure relation, the causal relation can be preserved, and furthermore, a solution of the radiation transfer in the optically thick and thin limits can be derived from the resulting equations. In this paper, we derive a truncated moment formalism in general relativity following the covariant formalism developed by Thorne.[2)] In addition, we derive a closed coordinate-independent formalism including the absorption, emission, and collision terms, focusing specifically on neutrino transfer in high-density and high-temperature medium.

The paper is organized as follows: In §2, we review the covariant moment formalism derived by Thorne.[2)] In §3, a truncated moment formalism is presented, assuming a physically reasonable specific form for the distribution function. In §4, source terms of the moment formalism are written in terms only of the radiation field variables employed in our truncated moment formalism, focusing specifically on neutrino transfer. In §5, approximate solutions for the radiation fields in the optically thick limit are derived. In §6, we propose a closure relation among the radiation scalar, vector, and tensor. We also derive the characteristic propagation speeds of the radiation field in the optically thick and thin limits for the chosen closure relation. In §7, hydrodynamic equations coupled with the radiation fields are derived. In §8, radiation hydrodynamic equations in a slow-motion limit (usually referred to as Newtonian radiation hydrodynamic equations) are derived. Section 9 is devoted to a summary. Throughout this paper, Greek (α, β, $\gamma \cdots$) and Latin (i, j, $k \cdots$) subscripts denote the spacetime and space components, except for ν which always denotes the angular frequency of radiation (which never be the subscript of space or time). x^μ always denotes spacetime coordinates. We assume to use the Cartesian coordinates as the spatial coordinates x^i for simplicity. Unless otherwise stated, the units of $c = 1 = h$ are used, where c is the speed of light and h the Planck constant. k_b denotes the Boltzmann constant.

§2. Moment formalism of Thorne

First, we review Thorne's moment formalism.[2)] In the first step, he defines an unprojected moment of massless particles associated with a moving medium as

$$M_{(\nu)}^{\alpha_1\alpha_2\cdots\alpha_k}(x^\beta) = \int \frac{f(p'^\alpha, x^\beta)\delta(\nu - \nu')}{\nu'^{k-2}} p'^{\alpha_1} p'^{\alpha_2} \cdots p'^{\alpha_k} dV'_p, \tag{2·1}$$

where f is the distribution function of the relevant radiation, $\nu' = -u_\mu p'^\mu$ the frequency of the radiation in the rest-frame of the medium (i.e, in the rest-frame of the fiducial observer) with u^μ being medium's four velocity, p^μ the four-momentum of the radiation, and dV_p the invariant integration element on the light cone. k, here, is positive integer, 1, 2, $\cdots$. As pointed out by Thorne,[2)] the choice of the fiducial observer is crucial when deriving a good truncated formalism from his moment formalism. In the following, the fluid, coupled with the radiation, is chosen as the medium.[2),10),11)] Namely, the frequency, ν, in $M_{(\nu)}^{\alpha_1\alpha_2\cdots\alpha_k}$ always denote the frequency *measured in the rest-frame of the fluid* throughout this paper. This choice is crucially helpful when computing the source terms of the radiation equations.

We note that it is possible to choose any fiducial frame in the moment formalism. However, we have to keep in mind that for a truncated moment formalism in a closed form, it is necessary to assume a closure relation which is determined by a physically reasonable assumption. In the dense medium, radiation is strongly coupled to the matter field. This implies that at the zeroth order, the radiation is in equilibrium with the medium, and radiation flow (measured by an observer comoving with the matter) is a small correction. To reproduce this feature in the closure relation, the best method seems to choose the fluid rest frame as the fiducial frame.

We also note the following: As a result of our choice of the fiducial frame, the argument frequency in the distribution function is always *the frequency measured in the fluid rest frame*. By contrast, the argument variable should be in general the frequency in the laboratory frame (although any frame can be taken), if one fully solves the Boltzmann equation that the distribution function obeys.

The Boltzmann equation is written in the form[2)]

$$\frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha} + \frac{dp^i}{d\tau}\frac{\partial f}{\partial p^i} = (-p^\alpha u_\alpha) S(p^\mu, x^\mu, f), \tag{2·2}$$

where S denotes a source term and τ the affine parameter of a trajectory of radiation particles. In any orthonormal frame, the invariant integration element is given by[10)]

$$dV_p = \frac{d\hat{p}^1 d\hat{p}^2 d\hat{p}^3}{\hat{p}^0}, \tag{2·3}$$

where $\hat{p}^\alpha$ is the four-momentum of the radiation in the local orthonormal frame. In the local rest frame of an observer comoving with the fluid,

$$dV_p = \nu d\nu d\Omega, \tag{2·4}$$

where $\int d\Omega$ denotes integrations over solid angle on an unit sphere.

For the following, we write p^α in the form

$$\frac{dx^\alpha}{d\tau} = p^\alpha = \nu(u^\alpha + \ell^\alpha), \tag{2·5}$$

where ℓ^α is a unit normal four-vector orthogonal to u^α; $\ell_\alpha \ell^\alpha = 1$ and $u_\alpha \ell^\alpha = 0$. Using this decomposition of p^α, Eq. (2·1) is rewritten to give

$$M_{(\nu)}^{\alpha_1\alpha_2\cdots\alpha_k} = \nu^3 \int f(\nu, \Omega, x^\mu)(u^{\alpha_1} + \ell^{\alpha_1})(u^{\alpha_2} + \ell^{\alpha_2})\cdots(u^{\alpha_k} + \ell^{\alpha_k})d\Omega. \tag{2·6}$$

Here, the angular dependence is included in ℓ^α and the following relations hold,

$$\begin{aligned}&\int d\Omega \ell^\alpha = 0 = \int d\Omega \ell^\alpha \ell^\beta \ell^\gamma, \qquad \frac{1}{4\pi}\int d\Omega \ell^\alpha \ell^\beta = \frac{1}{3}h^{\alpha\beta}, \\ &\frac{1}{4\pi}\int d\Omega \ell^\alpha \ell^\beta \ell^\gamma \ell^\delta = \frac{1}{15}\Big(h^{\alpha\beta}h^{\gamma\delta} + h^{\alpha\gamma}h^{\beta\delta} + h^{\alpha\delta}h^{\beta\gamma}\Big).\end{aligned} \tag{2·7}$$

$h_{\alpha\beta}$ is the projection operator defined by

$$h_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta. \tag{2·8}$$

Following Thorne,[2] we denote $M_{(\nu)}^{\alpha_1\alpha_2\cdots\alpha_k}$ by $M_{(\nu)}^{A_k}$. Taking the covariant derivatives of $M_{(\nu)}^{A_k\beta}$, we obtain a covariant equation with respect to real-space coordinates[2]

$$\nabla_\beta M_{(\nu)}^{A_k\beta} - \frac{\partial}{\partial\nu}(\nu M_{(\nu)}^{A_k\beta\gamma}\nabla_\gamma u_\beta) - (k-1)M_{(\nu)}^{A_k\beta\gamma}\nabla_\gamma u_\beta = S_{(\nu)}^{A_k}, \tag{2·9}$$

where ∇_α denotes the covariant derivative associated with the spacetime metric $g_{\alpha\beta}$, and

$$S_{(\nu)}^{A_k} = \nu^3 \int S(\nu, \Omega, x^\mu, f)(u^{\alpha_1} + \ell^{\alpha_1})(u^{\alpha_2} + \ell^{\alpha_2})\cdots(u^{\alpha_k} + \ell^{\alpha_k})d\Omega. \tag{2·10}$$

Here, the spacetime derivative is taken holding ν and the frequency derivative is taken holding spacetime location. It should be noted that Eq. (2·9) has a coordinate-independent form as stressed by Thorne.[2] Also, the following relation is worthy to note:

$$M_{(\nu)}^{A_k\beta}u_\beta = -M_{(\nu)}^{A_k}. \tag{2·11}$$

Thus, the rank-$(k+1)$ equations include the lower-rank equations.

Since the frequency, ν, in Eq. (2·9) denotes the frequency *observed in a fluid-rest frame*, not in the laboratory frame, $M_{(\nu)}^{A_k}$ is not directly related to the spectrum observed in the laboratory frame. However, if the fluid is assumed to be at rest in a distant zone far away from a radiation source where we observe the spectrum, the radiation moments in the fluid-rest frame agree with those in the laboratory frame. We suppose that the present formalism will be used for the system that this assumption holds, e.g., supernova stellar core collapse and merger of compact objects. Thus, it is possible to directly compute the radiation spectrum from $M_{(\nu)}^{A_k}$, if we estimate it for $r \to \infty$ (cf. Appendix A for an example).

Integrating Eq. (2·9) by ν, we obtain (for each species of the radiation component)

$$\nabla_\beta M^{A_k\beta} - (k-1) M^{A_k\beta\gamma}\nabla_\gamma u_\beta = S^{A_k}, \tag{2·12}$$

where

$$M^{A_k} = \int_0^\infty d\nu M_{(\nu)}^{A_k} \quad \text{and} \quad S^{A_k} = \int_0^\infty d\nu S_{(\nu)}^{A_k}. \tag{2·13}$$

Equation (2·12) is essentially the same as the moment formalism derived by Anderson and Spiegel.[1] We note that the second-rank tensor $M^{\alpha\beta}$ is equal to the energy-momentum tensor for one of the radiation components.

In the following, we analyze only the second-rank part of Eq. (2·9), truncating the higher-rank parts (in §5, we partly use the third-rank equation for deriving a solution in the absence of closure relation). In the next section, we develop such formalism.

§3. Truncated moment formalism

First of all, we define the following moments:

$$J_{(\nu)} := \nu^3 \int f(\nu, \Omega, x^\mu) d\Omega, \tag{3·1}$$

$$H_{(\nu)}^{\alpha} := \nu^3 \int \ell^\alpha f(\nu, \Omega, x^\mu) d\Omega, \tag{3·2}$$

$$L_{(\nu)}^{\alpha\beta} := \nu^3 \int \ell^\alpha \ell^\beta f(\nu, \Omega, x^\mu) d\Omega, \tag{3·3}$$

$$N_{(\nu)}^{\alpha\beta\gamma} := \nu^3 \int \ell^\alpha \ell^\beta \ell^\gamma f(\nu, \Omega, x^\mu) d\Omega. \tag{3·4}$$

Here, all these integrals are assumed to be performed in the local rest frame comoving with the fluid, and ν denotes the angular frequency of radiation measured in this local rest frame. The second- and third-rank moments are denoted by

$$M_{(\nu)}^{\alpha\beta} = J_{(\nu)} u^\alpha u^\beta + H_{(\nu)}^{\alpha} u^\beta + H_{(\nu)}^{\beta} u^\alpha + L_{(\nu)}^{\alpha\beta}, \tag{3·5}$$

$$\begin{aligned} M_{(\nu)}^{\alpha\beta\gamma} = {} & J_{(\nu)} u^\alpha u^\beta u^\gamma + H_{(\nu)}^{\alpha} u^\beta u^\gamma + H_{(\nu)}^{\beta} u^\alpha u^\gamma + H_{(\nu)}^{\gamma} u^\alpha u^\beta \\ & + L_{(\nu)}^{\alpha\beta} u^\gamma + L_{(\nu)}^{\alpha\gamma} u^\beta + L_{(\nu)}^{\beta\gamma} u^\alpha + N_{(\nu)}^{\alpha\beta\gamma}, \end{aligned} \tag{3·6}$$

and the total stress-energy tensor for the radiation is

$$T^{\alpha\beta}_{\rm rad} = \sum \int_0^\infty d\nu M^{\alpha\beta}_{(\nu)}, \tag{3·7}$$

where the summation denotes summing up for all the species of the radiation.

In our truncated formalism, (i) we formally define the zeroth-, first-, second- and third-rank moments from the distribution function, and (ii) we solve the evolution equations only for the zeroth- and first-rank moments. For the optically thick region, this is approximately equivalent to assuming that the degree of anisotropy of the distribution function in the fluid local rest frame is weak and that the distribution function is approximated by

$$f(\nu, \Omega, x^\mu) = f_0(\nu, x^\mu) + f_1^\alpha(\nu, x^\mu)\ell_\alpha + f_2^{\alpha\beta}(\nu, x^\mu)\ell_\alpha\ell_\beta. \tag{3·8}$$

Here, f_0, f_1^α, and $f_2^{\alpha\beta}$ do not depend on the propagation angle of radiation in the fluid local rest frame and $f_2^{\alpha\beta}$ is a traceless tensor with respect to $h_{\alpha\beta}$ (i.e., $f_2^{\alpha\beta}h_{\alpha\beta} = 0$). We assume that $|f_0|$ is much larger than the absolute magnitude of f_1^α and $f_2^{\alpha\beta}$. For the expansion of Eq. (3·8), we obtain

$$J_{(\nu)} = 4\pi\nu^3 f_0, \tag{3·9}$$

$$H^{\alpha}_{(\nu)} = \frac{4\pi}{3}\nu^3 f_1^\alpha, \tag{3·10}$$

$$L^{\alpha\beta}_{(\nu)} = \frac{4\pi}{3}\nu^3\Big(f_0 h^{\alpha\beta} + \frac{2}{5}f_2^{\alpha\beta}\Big) = \frac{1}{3}J_{(\nu)}h^{\alpha\beta} + \frac{8\pi}{15}\nu^3 f_2^{\alpha\beta}, \tag{3·11}$$

$$N^{\alpha\beta\gamma}_{(\nu)} = \frac{1}{5}\Big(H^{\alpha}_{(\nu)}h^{\beta\gamma} + H^{\beta}_{(\nu)}h^{\alpha\gamma} + H^{\gamma}_{(\nu)}h^{\alpha\beta}\Big), \tag{3·12}$$

where we used the relations (2·7). Thus, f_0 and f_1^α are directly related to $J_{(\nu)}$ and $H^{\alpha}_{(\nu)}$, and $f_2^{\alpha\beta}$ to the traceless part of $L^{\alpha\beta}_{(\nu)}$, respectively. Because of the truncated expansion for $f(\nu, \Omega, x^\mu)$, we naturally obtain a closure relation for $N^{\alpha\beta\gamma}_{(\nu)}$.

For the optically thin limit, by contrast, we should first give a physical assumption *in the laboratory frame* because the radiation does not interact with matter. We employ the assumptions that the radiation should propagate with the speed of light and the radiation flow at each spacetime point should be pointed to a null direction. The former assumption then implies that the radiation flow is pointed to a null direction in any frame (although the spacetime coordinate basis changes). Thus, for such region, the distribution function may be written in the form (see §6.1 for details)

$$f(\nu, \Omega, x^\mu) = 4\pi f_{\rm f}(\nu, x^\mu)\delta(\Omega - \Omega_{\rm f}), \tag{3·13}$$

where $\Omega_{\rm f}$ denotes the flow direction in the fluid rest frame, and $f_{\rm f}(\nu, x^\mu)$ is the partial distribution function of $\Omega = \Omega_{\rm f}$. Then, the radiation moments are calculated to give

$$J_{(\nu)} = 4\pi\nu^3 f_{\rm f}, \tag{3·14}$$

$$H^{\alpha}_{(\nu)} = 4\pi\nu^3 f_{\rm f}\ell_{\rm f}^\alpha, \tag{3·15}$$

$$L^{\alpha\beta}_{(\nu)} = 4\pi\nu^3 f_{\rm f}\ell_{\rm f}^\alpha\ell_{\rm f}^\beta, \tag{3·16}$$

$$N^{\alpha\beta\gamma}_{(\nu)} = 4\pi\nu^3 f_{\rm f}\ell_{\rm f}^\alpha\ell_{\rm f}^\beta\ell_{\rm f}^\gamma, \tag{3·17}$$

where $\ell_{\rm f}^{\alpha}$ denotes the unit vector of the flow direction (observed in the fluid-rest frame). In Appendix A, we illustrate that the assumption of (3·14)–(3·17) would be appropriate for providing the radiation field solution in the optically thin-limit medium.

The equations for $J_{(\nu)}$ and $H_{(\nu)}^{\ \alpha}$ are derived from the second-rank part of Eq. (2·9) as

$$\nabla_\beta M_{(\nu)}^{\ \alpha\beta} - \frac{\partial}{\partial\nu}(\nu M_{(\nu)}^{\ \alpha\beta\gamma}\nabla_\gamma u_\beta) = S_{(\nu)}^{\ \alpha}, \tag{3·18}$$

where

$$\begin{aligned} M_{(\nu)}^{\ \alpha\beta\gamma}\nabla_\beta u_\gamma &= (H_{(\nu)}^{\ \gamma}u^\alpha u^\beta + L_{(\nu)}^{\ \alpha\gamma}u^\beta + L_{(\nu)}^{\ \beta\gamma}u^\alpha + N_{(\nu)}^{\ \alpha\beta\gamma})\nabla_\beta u_\gamma \\ &= \left(H_{(\nu)}^{\ \gamma}u^\alpha + L_{(\nu)}^{\ \alpha\gamma}\right)a_\gamma + \left(L_{(\nu)}^{\ \beta\gamma}u^\alpha + N_{(\nu)}^{\ \alpha\beta\gamma}\right)\Sigma_{\beta\gamma}. \end{aligned} \tag{3·19}$$

The acceleration a^α and the shear $\Sigma_{\alpha\beta}$ are defined by

$$a^\alpha := u^\beta\nabla_\beta u^\alpha, \tag{3·20}$$

$$\Sigma_{\alpha\beta} := \frac{1}{2}h_\alpha^{\ \gamma}h_\beta^{\ \delta}\Big[\nabla_\gamma u_\delta + \nabla_\delta u_\gamma\Big]. \tag{3·21}$$

To obtain a closed set of the equations, we have to determine $L_{(\nu)}^{\ \alpha\beta}$ [$N_{(\nu)}^{\ \alpha\beta\gamma}$ is given by Eq. (3·12) or (3·17)]. Instead of solving the equation for this, which may be derived from the moment equation of third rank, we will assume a closure relation for it; an artificial (but physically reasonable) relation between $L_{(\nu)}^{\ \alpha\beta}$ and $(J_{(\nu)}, H_{(\nu)}^{\ \alpha})$ will be assumed (see §6).

Substituting Eq. (3·5) into Eq. (3·18), the evolution equations for $J_{(\nu)}$ and $H_{(\nu)}^{\ \alpha}$ are obtained as

$$\nabla_\alpha Q_{(\nu)}^{\ \alpha} + Q_{(\nu)}^{\ \alpha\beta}\nabla_\beta u_\alpha - \frac{\partial}{\partial\nu}[\nu(Q_{(\nu)}^{\ \alpha\beta}\nabla_\beta u_\alpha)] = -S_{(\nu)}^{\ \alpha}u_\alpha, \tag{3·22}$$

$$h_{k\alpha}\Big[\nabla_\beta Q_{(\nu)}^{\ \alpha\beta} + Q_{(\nu)}^{\ \beta}\nabla_\beta u^\alpha - \frac{\partial}{\partial\nu}[\nu(L_{(\nu)}^{\ \alpha\gamma}u^\beta + N_{(\nu)}^{\ \alpha\beta\gamma})\nabla_\beta u_\gamma]\Big] = h_{k\alpha}S_{(\nu)}^{\ \alpha}, \tag{3·23}$$

where

$$Q_{(\nu)}^{\ \alpha} := -M_{(\nu)}^{\ \alpha\beta}u_\beta = J_{(\nu)}u^\alpha + H_{(\nu)}^{\ \alpha}, \tag{3·24}$$

$$Q_{(\nu)}^{\ \alpha\beta} := h^\alpha_{\ \gamma}M_{(\nu)}^{\ \gamma\beta} = H_{(\nu)}^{\ \alpha}u^\beta + L_{(\nu)}^{\ \alpha\beta}. \tag{3·25}$$

The frequency-integrated equations are

$$\nabla_\alpha Q^\alpha + Q^{\alpha\beta}\nabla_\beta u_\alpha = -S^\alpha u_\alpha, \tag{3·26}$$

$$h_{k\alpha}(\nabla_\beta Q^{\alpha\beta} + Q^\beta\nabla_\beta u^\alpha) = h_{k\alpha}S^\alpha, \tag{3·27}$$

where

$$Q^\alpha := \int_0^\infty d\nu Q_{(\nu)}^{\ \alpha}, \quad Q^{\alpha\beta} := \int_0^\infty d\nu Q_{(\nu)}^{\ \alpha\beta}. \tag{3·28}$$

Thus, the equations are not in the conservation form; the reason is that Q^0 and Q^{k0} are not conservative quantities even in the absence of the source terms.

Instead of using Eq. (3·5), $M_{(\nu)}^{\alpha\beta}$ may be written by

$$M_{(\nu)}^{\alpha\beta} = E_{(\nu)} n^\alpha n^\beta + F_{(\nu)}^{\alpha} n^\beta + F_{(\nu)}^{\beta} n^\alpha + P_{(\nu)}^{\alpha\beta}, \tag{3·29}$$

where n^α is a unit vector orthogonal to the spacelike hypersurface. $E_{(\nu)}$, $F_{(\nu)}^{\alpha}$, and $P_{(\nu)}^{\alpha\beta}$ may be regarded as radiation fields measured in the laboratory frame. We note again that the meaning of the frequency, ν, is unchanged; it is the frequency observed in the *fluid rest frame*. To obtain the quantities fully defined in the laboratory frame, we need the transformation of ν to the frequency measured in the laboratory frame. However, in the moment formalism, we do not consider such transformation, as already mentioned.

We however assume that $u^\mu = n^\mu$ in the far region with $r \to \infty$ as mentioned in §2. Our primary purpose is to develop an approximate formalism which can be used for simulation of stellar core collapse and merger of binary compact objects. For such a purpose, this assumption is acceptable. Thus, for $r \to \infty$, we suppose that $E_{(\nu)} = J_{(\nu)}$, $F_{(\nu)}^{\alpha} = H_{(\nu)}^{\alpha}$, and $P_{(\nu)}^{\alpha\beta} = L_{(\nu)}^{\alpha\beta}$, and ν agrees with the frequency measured in the laboratory frame. Therefore, if E_ν is extracted in a distant zone in the numerical simulation, we can obtain the spectrum of the radiation.

In the 3+1 formulation of general relativity,

$$n^\alpha = \Big(\frac{1}{\alpha}, -\frac{\beta^k}{\alpha}\Big), \tag{3·30}$$

where α is the lapse function and β^k the shift vector. Then, $E_{(\nu)}$, $F_{(\nu)}^{\alpha}$, and $P_{(\nu)}^{\alpha\beta}$ are defined by

$$E_{(\nu)} = M_{(\nu)}^{\alpha\beta} n_\alpha n_\beta, \quad F_{(\nu)}^{i} = -M_{(\nu)}^{\alpha\beta} n_\alpha \gamma_\beta^{\ i}, \quad P_{(\nu)}^{ij} = M_{(\nu)}^{\alpha\beta} \gamma_\alpha^{\ i} \gamma_\beta^{\ j}, \tag{3·31}$$

where $\gamma_{\alpha\beta}$ is the three metric

$$\gamma_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta. \tag{3·32}$$

Because $F_{(\nu)}^{\alpha} n_\alpha = P_{(\nu)}^{\alpha\beta} n_\alpha = 0$, we have the relations $F_{(\nu)}^{0} = P_{(\nu)}^{0\alpha} = 0$.

Here, we consider a formalism in which $E_{(\nu)}$ and $F_{(\nu)}^{k}$ are evolved, and $P_{(\nu)}^{ij}$ is determined by a closure relation. Then, $J_{(\nu)}$ and $H_{(\nu)}^{\alpha}$ are determined by

$$J_{(\nu)} = E_{(\nu)} w^2 - 2F_{(\nu)}^{k} w u_k + P_{(\nu)}^{ij} u_i u_j, \tag{3·33}$$

$$H_{(\nu)}^{\alpha} = (E_{(\nu)} w - F_{(\nu)}^{k} u_k) h^\alpha_{\ \beta} n^\beta + w h^\alpha_{\ \beta} F_{(\nu)}^{\beta} - h^\alpha_{\ i} u_j P_{(\nu)}^{ij}, \tag{3·34}$$

where $w = \alpha u^0$. We note $h^\alpha_{\ \beta} n^\beta = n^\alpha - w u^\alpha$ and $n_\alpha h^{\alpha\beta} \gamma_{\beta k} = -w u_k$. For later convenience, we give relations

$$Q_{(\nu)}^{\alpha} n_\alpha = -J_{(\nu)} w + H_{(\nu)}^{\alpha} n_\alpha = -E_{(\nu)} w + F_{(\nu)}^{k} u_k, \tag{3·35}$$

$$Q_{(\nu)}^{\alpha} \gamma_{\alpha i} = J_{(\nu)} u_i + H_{(\nu)i} = w F_{(\nu)i} - P_{(\nu)i}^{\ k} u_k. \tag{3·36}$$

As mentioned before, it is natural to assume that $u_i = 0$ ($w = 1/\alpha \approx 1$) for the distant zone far from the radiation source. Then, both the asymptotic power spectrum densities, $E_{(\nu)}$ and $J_{(\nu)}$, agree with each other, because the frequency ν agrees with that in the laboratory frame.

The evolution equations for $E_{(\nu)}$ and $F_{(\nu)i}$ are written in the conservative forms as

$$\begin{aligned}
&\partial_t(\sqrt{\gamma}E_{(\nu)}) + \partial_j[\sqrt{\gamma}(\alpha F_{(\nu)}^{\ j} - \beta^j E_{(\nu)})] + \frac{\partial}{\partial\nu}\Big(\nu\alpha\sqrt{\gamma}n_\alpha M_{(\nu)}^{\ \alpha\beta\gamma}\nabla_\gamma u_\beta\Big) \\
&\qquad = \alpha\sqrt{\gamma}[P_{(\nu)}^{\ ij}K_{ij} - F_{(\nu)}^{\ j}\partial_j \ln\alpha - S_{(\nu)}^{\ \alpha} n_\alpha], \qquad (3\cdot37)\\
&\partial_t(\sqrt{\gamma}F_{(\nu)i}) + \partial_j[\sqrt{\gamma}(\alpha P_{(\nu)i}^{\ \ \ j} - \beta^j F_{(\nu)i})] - \frac{\partial}{\partial\nu}\Big(\nu\alpha\sqrt{\gamma}\gamma_{i\alpha} M_{(\nu)}^{\ \alpha\beta\gamma}\nabla_\gamma u_\beta\Big) \\
&\qquad = \sqrt{\gamma}\Big[-E_{(\nu)}\partial_i\alpha + F_{(\nu)k}\partial_i\beta^k + \frac{\alpha}{2}P_{(\nu)}^{\ jk}\partial_i\gamma_{jk} + \alpha S_{(\nu)}^{\ \alpha}\gamma_{i\alpha}\Big], \qquad (3\cdot38)
\end{aligned}$$

where γ is the determinant of γ_{ij} and K_{ij} the extrinsic curvature.

The frequency-integrated equations are

$$\begin{aligned}
&\partial_t(\sqrt{\gamma}E) + \partial_j[\sqrt{\gamma}(\alpha F^j - \beta^j E)] \\
&\qquad = \alpha\sqrt{\gamma}[P^{ij}K_{ij} - F^j\partial_j\ln\alpha - S^\alpha n_\alpha], \qquad (3\cdot39)\\
&\partial_t(\sqrt{\gamma}F_i) + \partial_j[\sqrt{\gamma}(\alpha P_i^{\ j} - \beta^j F_i)] \\
&\qquad = \sqrt{\gamma}\Big[-E\partial_i\alpha + F_k\partial_i\beta^k + \frac{\alpha}{2}P^{jk}\partial_i\gamma_{jk} + \alpha S^\alpha\gamma_{i\alpha}\Big], \qquad (3\cdot40)
\end{aligned}$$

where

$$E := \int_0^\infty d\nu E_{(\nu)}, \quad F_j := \int_0^\infty d\nu F_{(\nu)j}, \quad P^{ij} := \int_0^\infty d\nu P_{(\nu)}^{\ ij}. \qquad (3\cdot41)$$

Equations (3·39) and (3·40) have fully conservative forms, because E and F_i are the conservative quantities in the absence of the source terms and gravitational fields. Thus in the numerical simulation, it will be better to adopt basic equations based on Eqs. (3·37) and (3·38).

§4. Source terms

The source terms for the second-rank radiation field equations, $S_{(\nu)}^{\ \alpha}$, have to be written in terms of the radiation moments $(J_{(\nu)}, H_{(\nu)}^{\ \alpha}, L_{(\nu)}^{\ \alpha\beta})$. $S_{(\nu)}^{\ \alpha}$ is formally written as

$$S_{(\nu)}^{\ \alpha} = \nu^3 \int B_{(\nu)}(\Omega, x^\mu)(u^\alpha + \ell^\alpha)d\Omega, \qquad (4\cdot1)$$

where $B_{(\nu)}$ is the so-called collision integral. In the following, we assume that $S_{(\nu)}$ and $B_{(\nu)}$ are written as a function of the phase-space coordinate defined in the local rest frame of the fluid. The real coordinate, x^μ, is arbitrarily chosen.

We derive the source terms focusing on the neutrino transfer in a high-density and high-temperature medium. We show that under a reasonable and often-used assumption (anisotropy of the collision integral is small), the source terms are totally written in terms of $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\ \alpha\beta}$.

4.1. *Neutrino absorption and emission*

First, we consider the absorption and emission of neutrinos by nucleons and heavy nuclei such as $n+\nu_e \leftrightarrow p+e^-$, $p+\bar{\nu}_e \leftrightarrow n+e^+$, and $(Z,A)+\nu_e \leftrightarrow (Z-1,A)+e^-$, where n, p, $e^\mp$, ν_e ($\bar{\nu}_e$), and (Z,A) denote neutrons, protons, electrons (positrons), electron neutrinos (anti neutrinos), and heavy nuclei, respectively. For these cases, the collision integral is written in the form[12),13)]

$$B_{(\nu)} = j_{(\nu)}[1 - f(\nu, \Omega, x^\mu)] - \frac{f(\nu, \Omega, x^\mu)}{\lambda_{(\nu)}}, \tag{4·2}$$

where $j_{(\nu)}$ denotes the emissivity, $\lambda_{(\nu)}$ is the neutrino absorption mean free path, and $f(\nu, \Omega, x^\mu)$ denotes the distribution function of relevant neutrinos (in the following, we omit the argument x^μ in f). $j_{(\nu)}$ and $\lambda_{(\nu)}$ are quantities independent of the neutrino propagation angle, Ω.

The integral of Eq. (4·1) is easily performed to give

$$\begin{aligned} S_{(\nu)}^{\ \alpha} &= 4\pi j_{(\nu)}\nu^3 u^\alpha - \Big(J_{(\nu)}u^\alpha + H_{(\nu)}^{\ \alpha}\Big)(j_{(\nu)} + \lambda_{(\nu)}^{-1}) \\ &= (j_{(\nu)} + \lambda_{(\nu)}^{-1})\Big[(J_{(\nu)}^{\rm eq} - J_{(\nu)})u^\alpha - H_{(\nu)}^{\ \alpha}\Big], \end{aligned} \tag{4·3}$$

where

$$J_{(\nu)}^{\rm eq} := 4\pi\nu^3 \frac{j_{(\nu)}}{j_{(\nu)} + \lambda_{(\nu)}^{-1}} = 4\pi\nu^3 f^{\rm eq}(\nu), \tag{4·4}$$

and $f^{\rm eq}(\nu)$ is the equilibrium distribution function. For neutrinos (fermions),

$$f^{\rm eq}(\nu) = \frac{1}{e^{(h\nu - \mu_{\rm c})/k_{\rm b}T} + 1}, \tag{4·5}$$

where $\mu_{\rm c}$ and T are the chemical potential and the temperature for the corresponding species of neutrinos which are in thermal equilibrium with matter. For the following, we define the opacity as

$$\kappa_{(\nu)} := j_{(\nu)} + \lambda_{(\nu)}^{-1}, \tag{4·6}$$

and thus,

$$S_{(\nu)}^{\ \alpha} = \kappa_{(\nu)}\Big[(J_{(\nu)}^{\rm eq} - J_{(\nu)})u^\alpha - H_{(\nu)}^{\ \alpha}\Big]. \tag{4·7}$$

4.2. *Neutrino-electron scattering*

Neutrinos are scattered by electrons, nucleons, and heavy nuclei. In the case of electron scattering, the collision integral is generally written as[12),13)]

$$\begin{aligned} B_{(\nu)} = \int \nu'^2 d\nu' d\Omega' [&f(\nu', \Omega')\{1 - f(\nu, \Omega)\}R^{\rm in}(\nu, \nu', \omega) \\ &- f(\nu, \Omega)\{1 - f(\nu', \Omega')\}R^{\rm out}(\nu, \nu', \omega)], \end{aligned} \tag{4·8}$$

where ω is the cosine of the scattering angle, and $R^{\rm in}$ and $R^{\rm out}$ are the scattering kernels. Following Refs. 12) and 13), we approximate these kernels by taking the terms up to the linear order in ω, i.e.,

$$R^{\rm in}(\nu,\nu',\omega) = R_0^{\rm in}(\nu,\nu') + R_1^{\rm in}(\nu,\nu')\omega, \tag{4·9}$$
$$R^{\rm out}(\nu,\nu',\omega) = R_0^{\rm out}(\nu,\nu') + R_1^{\rm out}(\nu,\nu')\omega. \tag{4·10}$$

ω is related to the angular part of the momentum-space coordinates $\Omega = (\theta,\varphi)$ and $\Omega' = (\theta',\varphi')$ of the ingoing and outgoing neutrinos by

$$\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'), \tag{4·11}$$

and thus, we have the following relations:

$$\int \omega d\Omega = 0 = \int \omega\ell^\alpha\ell^\beta d\Omega, \quad \int \omega\ell^\alpha d\Omega = \frac{4\pi}{3}\ell'^\alpha. \tag{4·12}$$

Consequently, the collision integral is written as

$$\begin{aligned} B_{(\nu)} = 4\pi\int \nu'^2 d\nu' \Big[&\{1-f(\nu,\Omega)\}\Big\{f_0(\nu')R_0^{\rm in}(\nu,\nu') + \frac{1}{3}f_1^\alpha(\nu')\ell_\alpha R_1^{\rm in}(\nu,\nu')\Big\} \\ &-f(\nu,\Omega)\Big\{\{(1-f_0(\nu')\}R_0^{\rm out}(\nu,\nu') - \frac{1}{3}f_1^\alpha(\nu')\ell_\alpha R_1^{\rm out}(\nu,\nu')\Big\}\Big], \end{aligned} \tag{4·13}$$

and the source term is

$$\begin{aligned} S_{(\nu)}^{\ \alpha} = \int \frac{d\nu'}{\nu'}\Big[&\{(4\pi\nu^3 - J_{(\nu)})u^\alpha - H_{(\nu)}^{\ \alpha}\}J_{(\nu')}R_0^{\rm in}(\nu,\nu') \\ &+ \frac{H_{(\nu')}^{\ \alpha}}{3}\Big\{(4\pi\nu^3 - J_{(\nu)})R_1^{\rm in}(\nu,\nu') + J_{(\nu)}R_1^{\rm out}(\nu,\nu')\Big\} \\ &- (h_{\gamma\sigma}H_{(\nu)}^{\ \gamma}H_{(\nu')}^{\ \sigma}u^\alpha + \tilde{L}_{(\nu)\beta}^{\ \alpha}H_{(\nu')}^{\ \beta})\{R_1^{\rm in}(\nu,\nu') - R_1^{\rm out}(\nu,\nu')\} \\ &- (J_{(\nu)}u^\alpha + H_{(\nu)}^{\ \alpha})(4\pi\nu'^3 - J_{(\nu')})R_0^{\rm out}(\nu,\nu')\Big], \end{aligned} \tag{4·14}$$

where $\tilde{L}_{(\nu)}^{\ \alpha\beta}$ is the traceless part of $L_{(\nu)}^{\ \alpha\beta}$:

$$\tilde{L}_{(\nu)}^{\ \alpha\beta} := L_{(\nu)}^{\ \alpha\beta} - \frac{1}{3}J_{(\nu)}h^{\alpha\beta}. \tag{4·15}$$

Because of the approximation in which the terms up to the linear order in ω for the scattering kernel are taken into account, the source term is totally written in terms of $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\ \alpha\beta}$.

4.3. *Pair production*

For the thermal neutrino pair production and pair annihilation, the collision integral has the following form:[12),13)]

$$\begin{aligned} B_{(\nu)} = \int \nu'^2 d\nu' d\Omega' \big[&\{1-f(\nu,\Omega)\}\{1-\bar{f}(\nu',\Omega')\}R^{\rm pro}(\nu,\nu',\omega) \\ &- f(\nu,\Omega)\bar{f}(\nu',\Omega')R^{\rm ann}(\nu,\nu',\omega)\big], \end{aligned} \tag{4·16}$$

where f and $\bar{f}$ are the distribution functions of neutrinos and anti-neutrinos, respectively, and $R^{\rm pro}$ and $R^{\rm ann}$ are the integration kernels for pair production and annihilation, respectively. This integral can be performed in the same manner as in the neutrino-electron scattering: For the expansion up to the linear order in ω,

$$R^{\rm pro}(\nu,\nu',\omega)=R_0^{\rm pro}(\nu,\nu')+R_1^{\rm pro}(\nu,\nu')\omega, \tag{4·17}$$
$$R^{\rm ann}(\nu,\nu',\omega)=R_0^{\rm ann}(\nu,\nu')+R_1^{\rm ann}(\nu,\nu')\omega, \tag{4·18}$$

we obtain

$$\begin{aligned}S_{(\nu)}^{\ \alpha}=\int\frac{d\nu'}{\nu'}\Big[&-\{(J_{(\nu)}-4\pi\nu^3)u^\alpha+H_{(\nu)}^{\ \alpha}\}(4\pi\nu'^3-\bar{J}_{(\nu')})R_0^{\rm pro}(\nu,\nu')\\&-\frac{\bar{H}_{(\nu')}^{\ \alpha}}{3}\Big\{(4\pi\nu^3-J_{(\nu)})R_1^{\rm pro}(\nu,\nu')+J_{(\nu)}R_1^{\rm ann}(\nu,\nu')\Big\}\\&+(h_{\gamma\sigma}H_{(\nu)}^{\ \gamma}\bar{H}_{(\nu')}^{\ \sigma}u^\alpha+\tilde{L}_{(\nu)\beta}^{\ \alpha}\bar{H}_{(\nu')}^{\ \beta})[R_1^{\rm pro}(\nu,\nu')-R_1^{\rm ann}(\nu,\nu')]\\&-(J_{(\nu)}u^\alpha+H_{(\nu)}^{\ \alpha})\bar{J}_{(\nu')}R_0^{\rm ann}(\nu,\nu')\Big],\end{aligned} \tag{4·19}$$

where the quantities with bar denote the radiation moments for anti-neutrinos. Again, the source term is totally written in terms of $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\ \alpha\beta}$.

4.4. *Isoenergy neutrino scattering*

In the neutrino scattering with nucleons and heavy nuclei, the energy exchange may be assumed to be zero. In such isoenergetic neutrino scattering, the collision integral is written as[12),13)]

$$\begin{aligned}B_{(\nu)}&=\nu^2\int d\Omega'[\{1-f(\nu,\Omega)\}f(\nu,\Omega')-f(\nu,\Omega)\{1-f(\nu,\Omega')\}]R^{\rm iso}(\nu,\omega)\\&=\nu^2\int d\Omega'[f(\nu,\Omega')-f(\nu,\Omega)]R^{\rm iso}(\nu,\omega),\end{aligned} \tag{4·20}$$

where ν denotes the angular frequency of the ingoing and outgoing neutrinos. Following Refs. 12) and 13), we again approximate the kernel $R^{\rm iso}(\nu,\omega)$ by taking the terms up to the linear order in ω as

$$R^{\rm iso}(\nu,\omega)=R_0^{\rm iso}(\nu)+\omega R_1^{\rm iso}(\nu). \tag{4·21}$$

Then,

$$B_{(\nu)}=4\pi\nu^2\Big[\frac{1}{3}f_1^\alpha\ell_\alpha R_1^{\rm iso}(\nu)-\Big((f_1^\alpha\ell_\alpha+f_2^{\alpha\beta}\ell_\alpha\ell_\beta)R_0^{\rm iso}(\nu)\Big)\Big], \tag{4·22}$$

and thus,

$$S_{(\nu)}^{\ \alpha}=-\kappa_{(\nu)}^{\rm iso}H_{(\nu)}^{\ \alpha}, \tag{4·23}$$

where

$$\kappa_{(\nu)}^{\rm iso}=4\pi\nu^2\Big[R_0^{\rm iso}(\nu)-\frac{1}{3}R_1^{\rm iso}(\nu)\Big]. \tag{4·24}$$

Therefore, the source term depends only on the radiation flux (not on $J_{(\nu)}$), as clearly shown in Ref. 13) [cf. Eq. (A.46) of Ref. 13)].

§5. Optically thick limit

In this section, we derive a solution of the radiation moment equations for the limit that the radiation is optically thick to absorption and scattering with matter fields. For neutrinos, such a limit may be realized only for a special phenomena, such as stellar core collapse and merger of binary neutron stars, in which an extreme state of high density and high temperature is likely to be realized. However, it is quite useful to derive the solution for the idealized situation and to confirm that the derived solution is physical, for validating a new formalism.

In the following, we analyze the radiation moment equations derived in the previous sections, taking into account only the neutrino absorption, emission, and isoenergy scattering for simplicity, for which the source terms of the collision integral are written by Eqs. (4·3) and (4·23). Then, the source terms of Eqs. (3·22) and (3·23) are in the form

$$-S_{(\nu)}^{\ \alpha} u_\alpha = \kappa_{(\nu)}(J_{(\nu)}^{\rm eq} - J_{(\nu)}), \tag{5·1}$$

$$h_{k\alpha} S_{(\nu)}^{\ \alpha} = -\tilde{\kappa}_{(\nu)} H_{(\nu)k}, \tag{5·2}$$

where

$$\tilde{\kappa}_{(\nu)} = \kappa_{(\nu)} + \kappa_{(\nu)}^{\rm iso}. \tag{5·3}$$

We define a mean free path as the inverse of the opacity

$$l_{(\nu)} := \kappa_{(\nu)}^{-1}, \tag{5·4}$$

which measures the optical thickness of neutrinos: In the optically thick limit, $l_{(\nu)}$ is much smaller than a characteristic length scale of system (e.g., a stellar radius or $\rho/|\ell^\mu \nabla_\mu \rho|$), and the radiation fields may be expanded by $l_{(\nu)}$ assuming that it is sufficiently small[1] (this is the so-called Thomas approximation[1,2]). Assuming that $\tilde{\kappa}_{(\nu)}^{-1}$ is also of order $l_{(\nu)}$, we can expand the radiation moments as

$$J_{(\nu)} = J_{(\nu)}^{\rm eq} + l_{(\nu)} J_{(\nu)}^{(1)} + O(l_{(\nu)}^2), \tag{5·5}$$

$$H_{(\nu)}^{\ \alpha} = \tilde{l}_{(\nu)} H_{(\nu)}^{(1)\alpha} + O(l_{(\nu)}^2), \tag{5·6}$$

$$L_{(\nu)}^{\ \alpha\beta} = \frac{1}{3} J_{(\nu)} h^{\alpha\beta} + l_{(\nu)} L_{(\nu)}^{(1)\alpha\beta} + O(l_{(\nu)}^2), \tag{5·7}$$

$$N_{(\nu)}^{\ \alpha\beta\gamma} = \frac{\tilde{l}_{(\nu)}}{5}\Big(H_{(\nu)}^{(1)\alpha} h^{\beta\gamma} + H_{(\nu)}^{(1)\beta} h^{\alpha\gamma} + H_{(\nu)}^{(1)\gamma} h^{\alpha\beta}\Big) + O(l_{(\nu)}^2), \tag{5·8}$$

where $\tilde{l}_{(\nu)} = \tilde{\kappa}_{(\nu)}^{-1}$. In the following, we determine $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\ \alpha\beta}$ up to the first order in $l_{(\nu)}$ ($\tilde{l}_{(\nu)}$). For this purpose (specifically for deriving $L_{(\nu)}^{(1)\alpha\beta}$), it is necessary to analyze not only the second-rank moment equations but also the third-rank moment equations

$$\nabla_\gamma M_{(\nu)}^{\ \alpha\beta\gamma} - M_{(\nu)}^{\ \alpha\beta\gamma\delta} \nabla_\delta u_\gamma - \frac{\partial}{\partial \nu}(\nu M_{(\nu)}^{\ \alpha\beta\gamma\delta} \nabla_\delta u_\gamma) = S_{(\nu)}^{\ \alpha\beta}. \tag{5·9}$$

Thus, we have to take into account the fourth-rank moment as

$$U_{(\nu)}^{\alpha\beta\gamma\delta} := \nu^3 \int \ell^\alpha \ell^\beta \ell^\gamma \ell^\delta f(\nu, \Omega, x^\mu) d\Omega, \tag{5·10}$$

and expand up to the zeroth order in $l_{(\nu)}$ as

$$U_{(\nu)}^{\alpha\beta\gamma\delta} = \frac{1}{15} J_{(\nu)}^{\rm eq} (h^{\alpha\beta} h^{\delta\gamma} + h^{\alpha\delta} h^{\beta\gamma} + h^{\alpha\gamma} h^{\beta\delta}) + O(l_{(\nu)}). \tag{5·11}$$

In addition, we have to calculate the second-rank moment of the source term which is

$$\begin{aligned} S_{(\nu)}^{\alpha\beta} =& \frac{1}{l_{(\nu)}} \Big[(J_{(\nu)}^{\rm eq} - J_{(\nu)}) \Big(u^\alpha u^\beta + \frac{1}{3} h^{\alpha\beta} \Big) - \frac{l_{(\nu)}}{\tilde{l}_{(\nu)}} (H_{(\nu)}^{\ \alpha} u^\beta + H_{(\nu)}^{\ \beta} u^\alpha) \\ & - \frac{l_{(\nu)}}{2\bar{l}_{(\nu)}} (\tilde{L}_{(\nu)}^{\ \alpha\gamma} h_\gamma^{\ \beta} + \tilde{L}_{(\nu)}^{\ \beta\gamma} h_\gamma^{\ \alpha}) \Big], \end{aligned} \tag{5·12}$$

where $\bar{l}_{(\nu)}^{-1} = \bar{\kappa}_{(\nu)} = \kappa_{(\nu)} + 4\pi\nu^2 R_0^{\rm iso}$.

The expanded solutions for the radiation moments are determined from the expanded equations for Eqs. (3·22) and (3·23) in each order in $l_{(\nu)}$. Their zeroth-order equations give the first-order solutions

$$J_{(\nu)}^{(1)} = -\frac{1}{3} \Big[3u^\alpha \nabla_\alpha J_{(\nu)}^{\rm eq} + \Big(4J_{(\nu)}^{\rm eq} - \frac{\partial}{\partial\nu} (\nu J_{(\nu)}^{\rm eq}) \Big) \nabla_\alpha u^\alpha \Big], \tag{5·13}$$

$$H_{(\nu)}^{(1)\alpha} = -\frac{1}{3} h^{\alpha\beta} \Big[\nabla_\beta J_{(\nu)}^{\rm eq} + \Big(4J_{(\nu)}^{\rm eq} - \frac{\partial}{\partial\nu} (\nu J_{(\nu)}^{\rm eq}) \Big) u^\gamma \nabla_\gamma u_\beta \Big], \tag{5·14}$$

and $L_{(\nu)}^{(1)\alpha\beta}$ is derived from the zeroth-order part of the third-rank moment equation. Taking into account that this term is traceless and its component is perpendicular to u^α, we obtain

$$L_{(\nu)}^{(1)\alpha\beta} = -\frac{\bar{l}_{(\nu)}}{15 l_{(\nu)}} \Big[4J_{(\nu)}^{\rm eq} - \frac{\partial}{\partial\nu} (\nu J_{(\nu)}^{\rm eq}) \Big] \sigma^{\alpha\beta}, \tag{5·15}$$

where

$$\sigma^{\alpha\beta} := h^{\alpha\gamma} h^{\beta\delta} \Big[\nabla_\gamma u_\delta + \nabla_\delta u_\gamma - \frac{2}{3} h_{\gamma\delta} \nabla_\sigma u^\sigma \Big]. \tag{5·16}$$

For the frequency-integrated case, all these solutions agree with those in Ref. 1). The physical meaning of the first-order correction for the frequency-dependent equations (except for the diffusion effect associated with the term $\nabla_\alpha J_{(\nu)}^{\rm eq}$) is essentially the same as that for the frequency-integrated case: The first-order corrections of $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\alpha\beta}$ are associated with the fluid expansion ($\Theta = \nabla_\alpha u^\alpha$), the fluid acceleration ($a_\beta = u^\alpha \nabla_\alpha u_\beta$), and the fluid shear ($\sigma_{\alpha\beta}$).

We note that from the derived first-order solutions, the second-order solutions for $J_{(\nu)}$ and $H_{(\nu)}^{\ \alpha}$ are easily derived from Eqs. (3·22) and (3·23):

$$J_{(\nu)}^{(2)} = -\nabla_\alpha (J_{(\nu)}^{(1)} u^\alpha) - \frac{\tilde{l}_{(\nu)}}{l_{(\nu)}} \nabla_\alpha H_{(\nu)}^{(1)\alpha}$$

$$+\nu\frac{\partial}{\partial\nu}\Big[\frac{1}{3}J^{(1)}_{(\nu)}\Theta+\frac{\tilde{l}_{(\nu)}}{l_{(\nu)}}H^{(1)\alpha}_{(\nu)}a_\alpha+\frac{1}{2}L^{(1)\alpha\beta}_{(\nu)}\sigma_{\alpha\beta}\Big], \tag{5·17}$$

$$H^{(2)\alpha}_{(\nu)}=-h^\alpha{}_\gamma\Big[\nabla_\beta\Big(H^{(1)\gamma}_{(\nu)}u^\beta+\frac{l_{(\nu)}}{\tilde{l}_{(\nu)}}L^{(1)\beta\gamma}_{(\nu)}\Big)+H^{(1)\beta}_{(\nu)}\nabla_\beta u^\gamma+\frac{l_{(\nu)}}{\tilde{3}l_{(\nu)}}\Big(\nabla^\gamma J^{(1)}_{(\nu)}+4J^{(1)}_{(\nu)}a^\gamma\Big)$$
$$-\frac{\partial}{\partial\nu}\Big\{\nu\Big(\frac{l_{(\nu)}}{3\tilde{l}_{(\nu)}}J^{(1)}_{(\nu)}a^\gamma+\frac{l_{(\nu)}}{\tilde{l}_{(\nu)}}L^{(1)\beta\gamma}_{(\nu)}a_\beta+\frac{1}{3}H^{(1)\gamma}_{(\nu)}\Theta+\frac{1}{5}H^{(1)\beta}_{(\nu)}\sigma_\beta{}^\gamma\Big)\Big\}\Big], \tag{5·18}$$

where $J^{(2)}_{(\nu)}$ and $H^{(2)\alpha}_{(\nu)}$ are the coefficients of $l^2_{(\nu)}$ and $\tilde{l}^2_{(\nu)}$, respectively. For clarifying the order and for simplicity, we here assume that $l_{(\nu)}$ and $\tilde{l}_{(\nu)}$ are constants.

It is interesting to note that for the frequency-integrated case,[1] the magnitude of the first-order terms is proportional to $J^{\rm eq}$ where for each species of neutrinos

$$J^{\rm eq}=\int_0^\infty d\nu J^{\rm eq}_{(\nu)}, \tag{5·19}$$

For the frequency-dependent equations, the magnitude depends universally on

$$J^{\rm eq}_{(\nu)}-\frac{1}{4}\frac{\partial}{\partial\nu}(\nu J^{\rm eq}_{(\nu)})=-\frac{4\pi\nu^4}{4}\frac{\partial f^{\rm eq}(\nu)}{\partial\nu}$$
$$=\frac{h\nu}{4k_{\rm b}T}J^{\rm eq}_{(\nu)}(1-f^{\rm eq}(\nu)). \tag{5·20}$$

Because of the presence of a factor $1-f^{\rm eq}(\nu)$ and $f^{\rm eq}(\nu)$, these first-order corrections play a role only by neutrinos of energy around $\mu_{\rm c}-k_{\rm b}T\lesssim h\nu\lesssim\mu_{\rm c}+k_{\rm b}T$, i.e., near the Fermi surface, if the corresponding species of neutrinos is degenerate. This is a reasonable consequence and characteristic property for fermions.[*)]

The first-order solutions for $J_{(\nu)}$, $H^\alpha_{(\nu)}$, and $L^{\alpha\beta}_{(\nu)}$ may be used to constitute a diffusion equation from which the first-order solutions are produced. Substituting the first-order solution into Eq. (3·22) with replacement of $J^{\rm eq}_{(\nu)}$ to $J_{(\nu)}$ gives

$$\nabla_\alpha(J_{(\nu)}u^\alpha)-\frac{1}{3}\nabla_\alpha\Big[\tilde{l}_{(\nu)}\Big(h^{\alpha\beta}\nabla_\beta J_{(\nu)}+4\tilde{J}_{(\nu)}a^\alpha\Big)\Big]$$
$$-\frac{\nu}{3}\frac{\partial}{\partial\nu}\Big[J_{(\nu)}\Theta-\tilde{l}_{(\nu)}\Big(a^\alpha\nabla_\alpha J_{(\nu)}+4\tilde{J}_{(\nu)}a^\alpha a_\alpha\Big)-\frac{8}{5}l_{(\nu)}\tilde{J}_{(\nu)}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\Big]$$
$$=\kappa_{(\nu)}(J^{\rm eq}_{(\nu)}-J_{(\nu)}), \tag{5·21}$$

where

$$\tilde{J}_{(\nu)}=J_{(\nu)}-\frac{1}{4}\frac{\partial}{\partial\nu}(\nu J_{(\nu)}). \tag{5·22}$$

Thus in general, the diffusion equation is modified by the acceleration and shear motion of the fluid and by neutrinos, if we do not assume the slow motion of the fluid.

[*)] We note that only electron neutrinos can be degenerate in general. For antielectron neutrinos, $\mu_{\rm c}<0$ and for muon and tau neutrinos, $\mu_{\rm c}=0$, when these neutrinos are in thermal equilibrium with matter. Thus, these are not degenerate in general.

We note that in an FLD approximation, the first-order solution for $H_{(\nu)}^{\ \alpha}$ is modified as

$$H_{(\nu)}^{\ \alpha} = -\frac{\tilde{l}_{(\nu)}}{3+\tilde{l}_{(\nu)}J_{(\nu)}^{-1}u^{\gamma}\nabla_{\gamma}J_{(\nu)}}h^{\alpha\beta}\Big[\nabla_{\beta}J_{(\nu)} + \Big(4J_{(\nu)} - \frac{\partial}{\partial\nu}(\nu J_{(\nu)})\Big)u^{\gamma}\nabla_{\gamma}u_{\beta}\Big], \tag{5·23}$$

and is then substituted in Eq. (3·22). With this prescription, the equation for $J_{(\nu)}$ reduces to a wave equation with the characteristic speed $\sim c$ for the case that $\tilde{l}_{(\nu)}$ is much longer than a characteristic length scale of the system.

§6. Closure relations

In the truncated moment formalism derived in §3, we proposed to solve the equations for $E_{(\nu)}$ and $F_{(\nu)}^{\ i}$ but not to solve that for $P_{(\nu)}^{\ ij}$, which is assumed to be determined in terms of $E_{(\nu)}$ and $F_{(\nu)}^{\ i}$. In this section, we propose a physically reasonable closure relation.

6.1. *Optically thin case*

In the limit that the optical depth is zero, the emission, absorption, and scattering are negligible. When the source term of the radiation field equations can be neglected, the radiation freely propagates, and the radiation moments should obey a wave equation with no source.

One example for such region is the asymptotically flat region, far from the radiation source where curved spacetime effects as well as hydrodynamic effects play a tiny role (e.g., we may consider $u^{\mu} \approx n^{\mu}$, $J_{(\nu)} \approx E_{(\nu)}$, and $H_{(\nu)}^{\ \alpha} \approx F_{(\nu)}^{\ \alpha}$ as already mentioned in §§2 and 3). Thus, any closure relation assumed has to satisfy at least the equations in the flat spacetime.

For the flat spacetime, we obtain the equation for $F_{(\nu)}^{\ j}$ from Eq. (3·37)

$$\partial_j(\sqrt{\eta}F_{(\nu)}^{\ j}) = 0, \tag{6·1}$$

where η is the determinant of the flat three metric η_{ij}. This provides a reasonable solution of $F_{(\nu)}^{\ j}$ for the spatial infinity; for the spherically symmetric flow, $F_{(\nu)}^{\ r} \propto r^{-2}$, and for the plane symmetric flow, $F_{(\nu)}^{i}$ = constant for the flow direction. On the other hand, Eq. (3·38) gives

$$\partial_k(\sqrt{\eta}P_{(\nu)\ j}^{\ k}) = \frac{\sqrt{\eta}}{2}P_{(\nu)}^{\ ik}\partial_j\eta_{ik}. \tag{6·2}$$

For an appropriate solution of $E_{(\nu)}$, the following closure relation is the first candidate (and is that we finally choose):

$$P_{(\nu)}^{\ \alpha\beta} = E_{(\nu)}\frac{F_{(\nu)}^{\ \alpha}F_{(\nu)}^{\ \beta}}{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}. \tag{6·3}$$

This choice satisfies Eq. (6·2) in the asymptotically flat region. This is regarded as a general relativistic extension of the so-called M1 closure[14),15)] for the optically thin-limit region. In this case, $E_{(\nu)} \propto r^{-2}$ for the spherical symmetric flow and $E_{(\nu)}$ = constant for the plane symmetric flow, and hence, the reasonable condition is guaranteed. Furthermore, $P_{(\nu)}^{\ jk}\gamma_{jk}$ is equal to $E_{(\nu)}$ in this condition, guaranteeing the necessary condition for the radiation fields, $g_{\alpha\beta}T^{\alpha\beta}_{\rm rad} = 0$.

It should be noted that this choice *with no modification* may not be accepted in general relativity, because two of the characteristic speeds of the wave equations for $E_{(\nu)}$ and $F_{(\nu)}^{\ k}$ may exceed the speed of light (see §6.4 and Appendix B). The pure choice of this closure relation is allowed only for

$$E_{(\nu)} = \sqrt{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}, \tag{6·4}$$

in which the characteristic speed is guaranteed to be equal to the speed of light. This is guaranteed in the optically thin limit. However, for $E_{(\nu)} > \sqrt{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}$ which may often happen in a not-completely free streaming region, the characteristic speed may exceed the speed of light. This implies that an appropriate modification in the grey region is required in the choice of this closure relation to satisfy Eq. (6·4) (see §6.3 for a candidate choice of variable Eddington factor in the grey region and Appendix C for a satisfactory test result).

Another possible candidate is

$$P_{(\nu)}^{\ \alpha\beta} = \frac{F_{(\nu)}^{\ \alpha}F_{(\nu)}^{\ \beta}}{\sqrt{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}}. \tag{6·5}$$

This choice also satisfies Eq. (6·2) in the asymptotic region. With Eq. (6·5), Eq. (6·2) for the spherical and plane-symmetric stationary flows is written as

$$\sqrt{\eta}F_{(\nu)}^{\ k}\partial_k\hat{n}^i = 0, \tag{6·6}$$

where $\hat{n}^k$ is a unit vector parallel to $F_{(\nu)}^{\ k}$

$$\hat{n}^k := \frac{F_{(\nu)}^{\ k}}{\sqrt{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}}. \tag{6·7}$$

For the spherical and plane-symmetric stationary flows, the condition (6·6) is guaranteed, and hence, the closure relation (6·5) is acceptable.

With this choice, the characteristic speed of the equation for $F_{(\nu)i}$ is approximately the speed of light in the asymptotically flat region (see §6.4). However, $P_{(\nu)}^{\ jk}\gamma_{jk}$ is not a priori guaranteed to be equal to $E_{(\nu)}$; for this condition to be satisfied, $\sqrt{\gamma_{ij}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}$ has to be equal to $E_{(\nu)}$ but the condition will not be satisfied in the near zone, i.e., near the emission source (cf. Appendix B). Only in the far zone, this condition seems to be followed from Eq. (3·37) because the radiation propagates with the speed of light as $|F_{(\nu)}^{\ k}| \sim E_{(\nu)}$. Because of this reason, we choose Eq. (6·3) for the closure relation.

It is reasonable to suppose that Eqs. (6·3) and (6·4) are satisfied in the optically thin limit. Because we have $F^\alpha_{(\nu)} = E_{(\nu)} f^\alpha$ where f^α is a unit spatial vector, $f_\alpha f^\alpha = 1$, and orthogonal to n^α, $n^\alpha f_\alpha = 0$. $J_{(\nu)}$, $H_{(\nu)}^{\ \alpha}$, and $L_{(\nu)}^{\ \alpha\beta}$ are rewritten by

$$J_{(\nu)} = M_{(\nu)}^{\ \alpha\beta} u_\alpha u_\beta = E_{(\nu)} (u^\alpha q_\alpha)^2, \tag{6·8}$$

$$H_{(\nu)}^{\ \alpha} = -M_{(\nu)}^{\ \beta\gamma} u_\beta h_\gamma^{\ \alpha} = -E_{(\nu)} u^\beta q_\beta h_\gamma^{\ \alpha} q^\gamma, \tag{6·9}$$

$$L_{(\nu)}^{\ \alpha\beta} = M_{(\nu)}^{\ \gamma\delta} h_\gamma^{\ \alpha} h_\delta^{\ \beta} = E_{(\nu)} h_\gamma^{\ \alpha} h_\delta^{\ \beta} q^\gamma q^\delta, \tag{6·10}$$

where $q^\alpha = n^\alpha + f^\alpha$ is a null vector. Defining $4\pi\nu^3 f_\mathrm{f} \equiv E_{(\nu)}(u^\alpha q_\alpha)^2$ and $l_\mathrm{f}^\alpha \equiv -h_\beta^{\ \alpha} q^\beta/(u^\mu q_\mu)$, we obtain Eqs. (3·14)–(3·16), and find it reasonable to assume Eq. (3·13) as the distribution function in the optically thin limit. It is easy to confirm that $l_\mathrm{f}^\alpha l_{\mathrm{f}\alpha} = 1$ for the above definition.

Finally, we have to give a closure relation for the third-rank moment. We propose to employ

$$N_{(\nu)}^{\ \alpha\beta\gamma} = \frac{J_{(\nu)} H_{(\nu)}^{\ \alpha} H_{(\nu)}^{\ \beta} H_{(\nu)}^{\ \gamma}}{(h_{\alpha\beta} H_{(\nu)}^{\ \alpha} H_{(\nu)}^{\ \beta})^{3/2}}, \tag{6·11}$$

or

$$N_{(\nu)}^{\ \alpha\beta\gamma} = \frac{H_{(\nu)}^{\ \alpha} H_{(\nu)}^{\ \beta} H_{(\nu)}^{\ \gamma}}{h_{\alpha\beta} H_{(\nu)}^{\ \alpha} H_{(\nu)}^{\ \beta}}. \tag{6·12}$$

Here $H_{(\nu)}^{\ \alpha}$ is related to $E_{(\nu)}$, $F_{(\nu)}^{\ i}$, and $P_{(\nu)}^{\ ij}$ by Eq. (3·34). Associated with the choice of Eq. (6·3), we choose Eq. (6·11).

6.2. *Optically thick case*

As shown in §5, in the optically thick limit with $l_{(\nu)} \to 0$ (or $\kappa_{(\nu)} \to \infty$), $L_{(\nu)}^{\ \alpha\beta}$ is written as

$$L_{(\nu)}^{\ \alpha\beta} = \frac{1}{3} h^{\alpha\beta} J_{(\nu)} - \frac{4}{15} \bar{l}_{(\nu)} \sigma^{\alpha\beta} \Big[J_{(\nu)} - \frac{1}{4} \frac{\partial}{\partial\nu} (\nu J_{(\nu)}) \Big] + O(l_{(\nu)}^2). \tag{6·13}$$

The correction term of $O(l_{(\nu)})$ is associated with the so-called radiation viscosity. With this prescription, we can incorporate the first-order effect of $L_{(\nu)}^{\ \alpha\beta}$ in $l_{(\nu)}$ without solving the third-rank moment equation. (Note that $N_{(\nu)}^{\ \alpha\beta\gamma}$ should be given by Eq. (3·12) in the present formalism.) For the case that the velocity of the medium is much smaller than the speed of light, $|v^i| \ll c$, the term of $O(l_{(\nu)})$ may be neglected. This term will play a role for the medium moving around a black hole, such as in black hole accretion disks.

For numerical computation, we have to transform the relation of Eq. (6·13) to the relation of $P_{(\nu)}^{\ ij}$ as a function of $E_{(\nu)}$ and $F_{(\nu)}^{\ i}$. For this purpose, we omit the term, $\partial(\nu J_{(\nu)})/\partial\nu$, for simplicity. The reason is that in its presence, $P_{(\nu)}^{\ ij}$ is not written by $E_{(\nu)}$ and $F_{(\nu)}^{\ i}$ in a straightforward manner (although it is possible to do in an approximate manner). For the frequency-integrated case, this term vanishes, and hence, we may say that the radiation viscosity effect is taken into account in a frequency-averaged way. However, in this treatment, the low-energy neutrinos, which should not contribute to the radiation viscosity for degenerate neutrinos, may incorrectly play a role. To avoid this unphysical contribution, it will be appropriate to artificially reduce $\bar{l}_{(\nu)}$ to zero for $h\nu \lesssim \mu_{\rm c} - k_{\rm b}T$ when treating degenerate neutrinos.

Assuming that Eq. (6·13) holds with the omission of the third term, we have the relations

$$E_{(\nu)} = \Big[\frac{4w^2-1}{3} - \sigma_0\Big] J_{(\nu)} + 2H_{(\nu)j}V^j, \tag{6·14}$$

$$F_{(\nu)i} = \Big[\frac{4}{3}wu_i + \sigma_i\Big] J_{(\nu)} + wH_{(\nu)i} + \frac{u_i}{w}H_{(\nu)j}V^j, \tag{6·15}$$

where $V^i = \gamma^{ij}u_j$ $(V_i = u_i)$, and

$$\sigma_0 = \frac{4\bar{l}_{(\nu)}}{15}\sigma^{\alpha\beta}n_\alpha n_\beta, \quad \sigma_i = \frac{4\bar{l}_{(\nu)}}{15}\sigma^{\alpha\beta}n_\alpha\gamma_{\beta i}. \tag{6·16}$$

Also, we used $H_{(\nu)\alpha}u^\alpha = 0$ and $H_{(\nu)}^{\ 0} = (\alpha w)^{-1}H_{(\nu)i}V^i$. Equations (6·14) and (6·15) constitute simultaneous equations for $J_{(\nu)}$ and $H_{(\nu)i}$. Inverting them yields

$$J_{(\nu)} = \Big[\frac{2w^2+1}{3} + \sigma_0\Big]^{-1}\Big[(2w^2-1)E_{(\nu)} - 2wF_{(\nu)}^{\ k}u_k\Big], \tag{6·17}$$

$$\begin{aligned} H_{(\nu)i} = \frac{1}{w}F_{(\nu)i} + \frac{1}{w(2w^2+1+3\sigma_0)}\Big[&-[4w^3u_i + 3(2w^2-1)\sigma_i + 3\sigma_0 wu_i]E_{(\nu)} \\ &+[(4w^2+1)u_i + 6w\sigma_i + 3\sigma_0 u_i]F_{(\nu)}^{\ k}u_k\Big]. \end{aligned} \tag{6·18}$$

Note that $F_{(\nu)}^{\ k}u_k = F_{(\nu)k}V^k$ but $H_{(\nu)k}V^k \neq H_{(\nu)}^{\ k}u_k$; $H_{(\nu)}^{\ k} = (\gamma^{kl} - \beta^k\gamma^{lm}u_m/\alpha w)H_{(\nu)l}$. Also $w\sigma_0 = -\sigma_i V^i$. Then, $P_{(\nu)}^{\ ij}$ is given by

$$P_{(\nu)}^{\ ij} = J_{(\nu)}\Big[\frac{\gamma^{ij} + 4V^iV^j}{3} - \frac{4\bar{l}_{(\nu)}}{15}\sigma^{kl}\gamma_k^{\ i}\gamma_l^{\ j}\Big] + H_{(\nu)}^{\ i}V^j + H_{(\nu)}^{\ j}V^i, \tag{6·19}$$

where $J_{(\nu)}$ and $H_{(\nu)}^{\ k}(=\gamma^k_{\ \mu}H_{(\nu)}^{\ \mu})$ are given by Eqs. (6·17) and (6·18). With this closure relation for $P_{(\nu)}^{\ ij}$, the necessary condition for the radiation fields, $g_{\alpha\beta}T^{\alpha\beta}_{\rm rad} = 0$, is guaranteed to be satisfied.

We note that with the closure relation (6·19), the first-order term in $l_{(\nu)}$ may be accidentally larger than the zeroth-order term for a high value of σ_{ij}. Thus, it may be necessary to change the definition of $\bar{l}_{(\nu)}$ as

$$\bar{l}_{(\nu)} = \min\Big[\frac{1}{\bar{\kappa}_{(\nu)}}, C_\sigma\Big(\frac{V^ku_k}{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}\Big)^{1/2}\Big], \tag{6·20}$$

where C_σ is a coefficient smaller than unity.

6.3. *Grey zone*

For a solution of the radiation fields in the optically grey zone, in general, it is necessary to fully solve the radiation transfer equation in general relativity. However, it is not possible in the framework of truncated moment formalism and far beyond the scope of this paper. We propose an approximate method which is essentially the same as the variable Eddington factor method.[14] In this prescription, $P^{ij}_{(\nu)}$ is given by

$$P^{ij}_{(\nu)} = \frac{3\chi - 1}{2}(P^{ij}_{(\nu)})_{\rm thin} + \frac{3(1-\chi)}{2}(P^{ij}_{(\nu)})_{\rm thick}, \tag{6·21}$$

where χ is the so-called variable Eddington factor, which is $\chi = 1/3$ in the optically thick limit and $\chi = 1$ in the optically thin limit. Following Ref. 14), we choose that χ is a function of $\bar{F}$, for which in general relativity, the candidates are

$$\bar{F} := \left(\frac{\gamma_{ij}F^{i}_{(\nu)}F^{j}_{(\nu)}}{E^2_{(\nu)}}\right)^{1/2}, \tag{6·22}$$

and

$$\bar{F} := \left(\frac{h_{\alpha\beta}H^{\alpha}_{(\nu)}H^{\beta}_{(\nu)}}{J^2_{(\nu)}}\right)^{1/2}. \tag{6·23}$$

For the optically thick and thin limits, $\bar{F} = 0$ and $\bar{F} = 1$, respectively. For giving a correct value of $\bar{F}$ in the optically thick limit, Eq. (6·23) should be chosen because $H^{\alpha}_{(\nu)}$ should be zero in the comoving frame; if the fluid has a large uniform velocity, the value of $\bar{F}$ in Eq. (6·22) would be highly different from zero even in an optically thick medium. For giving a correct value of $\bar{F}$ in the optically thin limit, both Eqs. (6·22) and (6·23) can be chosen, because in such a limit, $M^{\alpha\beta}_{(\nu)}$ is proportional to $J_{(\nu)}p^{\alpha}p^{\beta}$ (p^{α} is a null vector) and $\bar{F} = 1$ for the null fluid in both definitions (see §3). For this reason, we choose Eq. (6·23) for $\bar{F}$.

With the choice of (6·23), $\bar{F}$ obeys an algebraic equation for a given set of $E_{(\nu)}$ and $F^{j}_{(\nu)}$. This can be written in the form

$$\bar{F}^2 = \frac{h_{\alpha\gamma}M^{\alpha\beta}_{(\nu)}u_{\beta}M^{\gamma\sigma}_{(\nu)}u_{\sigma}}{M^{\alpha\beta}_{(\nu)}u_{\alpha}u_{\beta}}, \tag{6·24}$$

where for $M^{\alpha\beta}_{(\nu)}$, Eq. (3·29) is used with Eq. (6·21). In numerical simulation, we have to solve this equation numerically.

Livermore proposed several functions for $\chi(\bar{F})$, e.g.,

$$\chi = \frac{3 + 4\bar{F}^2}{5 + 2\sqrt{4 - 3\bar{F}^2}}. \tag{6·25}$$

In Appendix C, we employ Eq. (6·25), and show that it is likely to work well. However, it should be kept in mind that it might not be the best one and better closure relations should be further explored.

For completeness, we have to provide $N_{(\nu)}^{\alpha\beta\gamma}$. We propose to employ

$$N_{(\nu)}^{\alpha\beta\gamma} = \frac{3\chi - 1}{2}(N_{(\nu)}^{\alpha\beta\gamma})_{\rm thin} + \frac{3(1-\chi)}{2}(N_{(\nu)}^{\alpha\beta\gamma})_{\rm thick}, \tag{6·26}$$

where $(N_{(\nu)}^{\alpha\beta\gamma})_{\rm thin}$ and $(N_{(\nu)}^{\alpha\beta\gamma})_{\rm thick}$ are given by Eqs. (6·11) and (3·12), respectively.

6.4. *Characteristic speed*

For numerical computation with conservation schemes, it is necessary to know characteristic speeds. Furthermore, the analysis of the characteristic speed is helpful to check whether the proposed closure relation is acceptable or not (i.e., it is smaller than the speed of light).

The characteristic speed of the radiation moment equations is computed from the Jacobian matrix (e.g., Refs. 16)–18)). For the conservative variables $E_{(\nu)}$ and $F_{(\nu)}^{i}$, the Jacobian matrix for the x direction is (in the following, we omit the subscript ν)

$$A_{ab} = \begin{bmatrix} -\beta^x & \alpha\gamma^{xx} & \alpha\gamma^{xy} & \alpha\gamma^{xz} \\ \alpha\dfrac{\partial P^x_{\ x}}{\partial E} & -\beta^x + \alpha\dfrac{\partial P^x_{\ x}}{\partial F_x} & \alpha\dfrac{\partial P^x_{\ x}}{\partial F_y} & \alpha\dfrac{\partial P^x_{\ x}}{\partial F_z} \\ \alpha\dfrac{\partial P^x_{\ y}}{\partial E} & \alpha\dfrac{\partial P^x_{\ y}}{\partial F_x} & -\beta^x + \alpha\dfrac{\partial P^x_{\ y}}{\partial F_y} & \alpha\dfrac{\partial P^x_{\ y}}{\partial F_z} \\ \alpha\dfrac{\partial P^x_{\ z}}{\partial E} & \alpha\dfrac{\partial P^x_{\ z}}{\partial F_x} & \alpha\dfrac{\partial P^x_{\ z}}{\partial F_y} & -\beta^x + \alpha\dfrac{\partial P^x_{\ z}}{\partial F_z} \end{bmatrix}. \tag{6·27}$$

The characteristic speeds are computed from

$$\det(A_{ab} - \lambda I_{ab}) = 0, \tag{6·28}$$

where I_{ab} is the unit matrix.

For the optically thin case with the closure relation (6·3),

$$\lambda = -\beta^x \pm \alpha\frac{F^x}{\sqrt{F_k F^k}} \quad \text{and} \quad \lambda = -\beta^x + \alpha E\frac{F^x}{F_k F^k} \quad \text{(double)}, \tag{6·29}$$

and with the closure relation (6·5),

$$\lambda = -\beta^x \quad \text{and} \quad \lambda = -\beta^x + \alpha\frac{F^x}{\sqrt{F_k F^k}} \quad \text{(triple)}. \tag{6·30}$$

As we already pointed out, $|\lambda|$ can exceed the speed of light for the closure relation (6·3) if the opacity is not zero limit; $|EF^x/F_kF^k|$ may exceed the speed of light. Hence, it is not allowed to be employed without appropriate choice of the variable Eddington factor for the optically grey zone. By contrast, with the closure relation (6·5), the characteristic speed is smaller than the speed of light (but in this case, the tracefree condition for the stress-energy tensor is not satisfied in general, as already mentioned).

For the optically thick limit with $l_{(\nu)} = 0$,

$$\lambda = -\beta^x + \frac{2w^2 p \pm \sqrt{\alpha^2 \gamma^{xx}(2w^2+1) - 2w^2 p^2}}{2w^2+1}$$
$$\text{and} \quad \lambda = -\beta^x + p \quad \text{(double)}, \tag{6·31}$$

where $p = \alpha V^x / w$. For $w = 1$ $(p = 0)$, the first one reduces to

$$\lambda = -\beta^x \pm \alpha \sqrt{\frac{\gamma^{xx}}{3}}, \tag{6·32}$$

and thus, we obtain a well-known characteristic speed for the radiation fluid in the diffusion limit ($\sim 1/\sqrt{3}$). For $w \to \infty$ $(p \to \alpha/\sqrt{\gamma_{xx}} < 1)$,

$$\lambda \to -\beta^x + p, \tag{6·33}$$

and thus, λ approaches to a local light speed (but never exceeds it). For $l_{(\nu)} \neq 0$, the characteristic speed is modified in a complicated form. However as far as $l_{(\nu)}$ is small, this effect is not important.

The general formula for the closure relation (6·3) is written in the following manner. For i-direction,

$$\lambda = -\beta^i \pm \alpha \frac{F^i}{\sqrt{F_k F^k}} \quad \text{and} \quad \lambda = -\beta^i + \alpha E \frac{F^i}{F_k F^k} \quad \text{(double)}, \tag{6·34}$$

for the optically thin case, and

$$\lambda = -\beta^i + \frac{2w^2 p^i \pm \sqrt{\alpha^2 \gamma^{ii}(2w^2+1) - 2(wp^i)^2}}{2w^2+1},$$
$$\text{and} \quad \lambda = -\beta^i + p^i \quad \text{(double)}, \tag{6·35}$$

for the optically thick case with $p^i = \alpha V^i / w$ and $l_{(\nu)} = 0$.

§7. Hydrodynamics

A conservative form of the hydrodynamic equations is derived from

$$\nabla_\alpha (\rho u^\alpha) = 0, \tag{7·1}$$
$$\gamma_{\beta i} \nabla_\alpha (T^{\alpha\beta}_{\rm fluid} + T^{\alpha\beta}_{\rm rad}) = 0, \tag{7·2}$$
$$n_\beta \nabla_\alpha (T^{\alpha\beta}_{\rm fluid} + T^{\alpha\beta}_{\rm rad}) = 0. \tag{7·3}$$

The first, second, and third equations are the continuity, Euler, and energy equations, respectively. For the perfect fluid,

$$T^{\alpha\beta}_{\rm fluid} = \rho h u^\alpha u^\beta + P g^{\alpha\beta}, \tag{7·4}$$

where h is the specific enthalpy defined by $1 + \varepsilon + P/\rho$, and ε and P are the specific internal energy and pressure.

For the neutrino-radiation hydrodynamics, it is further necessary to solve the continuity equation for leptons or electrons: The continuity equation for electrons is written in the form

$$\nabla_\alpha(\rho Y_e u^\alpha) = \rho Q_e, \tag{7·5}$$

where Y_e denotes the electron number per nucleon and Q_e the electron generation rate, determined by the electron capture and neutrino capture by nucleons or nuclei.

The explicit forms for these equations are

$$\partial_t(\sqrt{\gamma}\rho w) + \partial_j(\sqrt{\gamma}\rho w v^j) = 0, \tag{7·6}$$

$$\begin{aligned}&\partial_t(\sqrt{\gamma} j_i) + \partial_j[\sqrt{\gamma}(j_i v^j + \alpha P \delta_i{}^j)] \\ &\qquad = \sqrt{\gamma}\Big[-\rho_{\rm h}\partial_i\alpha + j_k \partial_i \beta^k + \frac{\alpha}{2} S^{jk}\partial_i\gamma_{jk} - \alpha S^\alpha \gamma_{\alpha i}\Big],\end{aligned} \tag{7·7}$$

$$\begin{aligned}&\partial_t(\sqrt{\gamma}\rho_{\rm h}) + \partial_j[\sqrt{\gamma}\{\rho_{\rm h} v^j + P(v^j + \beta^j)\}] \\ &\qquad = \alpha\sqrt{\gamma}[S^{ij}K_{ij} - \gamma^{ik} j_i \partial_k \ln\alpha + S^\alpha n_\alpha],\end{aligned} \tag{7·8}$$

$$\partial_t(\sqrt{\gamma}\rho w Y_e) + \partial_j(\sqrt{\gamma}\rho w Y_e v^j) = \rho Q_e \alpha\sqrt{\gamma}, \tag{7·9}$$

where $v^i = u^i/u^0$, and

$$j_i = -T^{\alpha\beta}_{\rm fluid} n_\alpha \gamma_{\beta i} = \rho w h u_i, \tag{7·10}$$

$$\rho_{\rm h} = T^{\alpha\beta}_{\rm fluid} n_\alpha n_\beta = \rho h w^2 - P, \tag{7·11}$$

$$S^{jk} = T^{\alpha\beta}_{\rm fluid}\gamma_\alpha{}^j \gamma_\beta{}^k = \rho h V^j V^k + P\gamma^{jk}. \tag{7·12}$$

As in the equations for the radiation moments $(E_{(\nu)}, F_{(\nu)i})$, the Euler and energy equations have conservative forms. We note that extension to the magnetohydrodynamic equations is straightforward (e.g., Ref. 19)).

To guarantee the conservation of total momentum and energy, it may be useful to solve

$$\begin{aligned}&\partial_t[\sqrt{\gamma}(j_i + F_i)] + \partial_j[\sqrt{\gamma}\{j_i v^j + \alpha(P\delta_i{}^j + P_i{}^j) - \beta^j F_i\}] \\ &\quad = \sqrt{\gamma}\Big[-(\rho_{\rm h} + E)\partial_i\alpha + (j_k + F_k)\partial_i\beta^k + \frac{\alpha}{2}(S^{jk} + P^{jk})\partial_i\gamma_{jk}\Big],\end{aligned} \tag{7·13}$$

$$\begin{aligned}&\partial_t[\sqrt{\gamma}(\rho_{\rm h} + E)] + \partial_j[\sqrt{\gamma}\{\rho_{\rm h} v^j + P(v^j + \beta^j) + \alpha F^j - \beta^j E\}] \\ &\qquad = \alpha\sqrt{\gamma}[(S^{ij} + P^{ij})K_{ij} - \gamma^{ik}(j_i + F_i)\partial_k \ln\alpha].\end{aligned} \tag{7·14}$$

We note that E, F_k, and P^{ij} here denote the sum of the contribution from all the neutrino species.

§8. Slow-motion limit

Here, we derive the radiation hydrodynamics equations in the case that (i) the spacetime is flat and (ii) the typical velocity of the matter field is much smaller than the speed of light. These approximations are often used in the radiation hydrodynamics with Newtonian gravity. Here, several additional words are necessary to clarify the condition (ii). First, we denote the typical time and length scales for the variation of the matter field by T and L, respectively, and the velocity by V. Then, the order of T is equal to L/V, and the acceleration and shear of the matter are of order $V/T \sim V^2/L$. In the Newtonian approximation for the radiation hydrodynamics, we take into account all the terms associated with the first order in V relative to the lowest-order term, but neglect the terms more than second order in V; terms of $O(V^2)$ such as acceleration and $(\partial_i v_j)^2$ are neglected. Because the Newtonian potential is the quantity of order V^2, we also neglect the contribution by this in the radiation moment equations.

Then, the equations for the radiation moments defined in the fluid comoving frame, $(J_{(\nu)},\ H_{(\nu)}^{\ i})$, are

$$\partial_t J_{(\nu)} + \partial_i (J_{(\nu)} v^i + H_{(\nu)}^{\ i}) - \nu \frac{\partial (L_{(\nu)}^{\ ij} \partial_i v_j)}{\partial \nu} = \kappa_{(\nu)} (J_{(\nu)}^{\rm eq} - J_{(\nu)}), \tag{8·1}$$

$$\partial_t H_{(\nu)}^{\ i} + \partial_j (H_{(\nu)}^{\ i} v^j + H_{(\nu)}^{\ j} v^i + L_{(\nu)}^{\ ij}) - v^i \partial_j H_{(\nu)}^{\ j} - \frac{\partial}{\partial \nu}(\nu N_{(\nu)}^{\ ijk} \partial_j v_k) = -\tilde{\kappa}_{(\nu)} H_{(\nu)}^{\ i}, \tag{8·2}$$

where we set $u^0 = 1 + O(V^2)$, $u_0 = -1 + O(V^2)$, $u^i = u_i = v^i = v_i$, $a^k := u^\mu \nabla_\mu u^k = O(V^2)$, and

$$\begin{aligned} N_{(\nu)}^{\ ijk} \partial_j v_k = {}& \frac{3\chi - 1}{2} \frac{J_{(\nu)} H_{(\nu)}^{\ i} H_{(\nu)}^{\ j} H_{(\nu)}^{\ k}}{(H_{(\nu)}^{\ l} H_{(\nu)l})^{3/2}} \partial_j v_k \\ & + \frac{3(1-\chi)}{10} \Big(H_{(\nu)}^{\ i} \partial_j v^j + H_{(\nu)}^{\ j} \partial_j v^i + H_{(\nu)}^{\ j} \partial_i v_j \Big), \end{aligned} \tag{8·3}$$

$$L_{(\nu)}^{\ ij} = \frac{3\chi - 1}{2} \frac{J_{(\nu)} H_{(\nu)}^{\ i} H_{(\nu)}^{\ j}}{H_{(\nu)}^{\ k} H_{(\nu)k}} + \frac{1-\chi}{2} J_{(\nu)} \delta^{ij}. \tag{8·4}$$

For simplicity, we here take into account only the neutrino emission, absorption, and isoenergy scattering. The derived forms of Eqs. (8·1) and (8·2) agree with those in the standard textbooks (e.g., Ref. 8)). On the other hand, the equations for the radiation moments defined in a laboratory frame, $(E_{(\nu)},\ F_{(\nu)i})$, are

$$\partial_t E_{(\nu)} + \partial_i F_{(\nu)}^{\ i} - \frac{\partial (\nu L_{(\nu)}^{\ ij} \partial_i v_j)}{\partial \nu} = \kappa_{(\nu)} (J_{(\nu)}^{\rm eq} - E_{(\nu)} + F_{(\nu)}^{\ i} v_i), \tag{8·5}$$

$$\begin{aligned} & \partial_t F_{(\nu)}^{\ i} + \partial_j P_{(\nu)}^{\ ij} - \frac{\partial}{\partial \nu}(\nu N_{(\nu)}^{\ ijk} \partial_j v_k) \\ & \qquad = -\tilde{\kappa}_{(\nu)} (F_{(\nu)}^{\ i} - P_{(\nu)}^{\ ik} v_k) + [\kappa_{(\nu)} J_{(\nu)}^{\rm eq} + (\tilde{\kappa}_{(\nu)} - \kappa_{(\nu)}) E_{(\nu)}] v^i, \end{aligned} \tag{8·6}$$

where

$$N_{(\nu)}^{\ ijk} \partial_j v_k = \frac{3\chi - 1}{2} \frac{E_{(\nu)} F_{(\nu)}^{\ i} F_{(\nu)}^{\ j} F_{(\nu)}^{\ k}}{(F_{(\nu)}^{\ l} F_{(\nu)l})^{3/2}} \partial_j v_k$$

$$+\frac{3(1-\chi)}{10}\Big(F_{(\nu)}^{\ i}\partial_j v^j + F_{(\nu)}^{\ j}\partial_j v^i + F_{(\nu)}^{\ j}\partial_i v_j\Big), \qquad (8\cdot7)$$

$$L_{(\nu)}^{\ ij} = \frac{3\chi-1}{2}\frac{E_{(\nu)}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}{F_{(\nu)}^{\ k}F_{(\nu)k}} + \frac{1-\chi}{2}E_{(\nu)}\delta^{ij}, \qquad (8\cdot8)$$

$$P_{(\nu)}^{\ ij} = \frac{3\chi-1}{2}\frac{E_{(\nu)}F_{(\nu)}^{\ i}F_{(\nu)}^{\ j}}{F_{(\nu)}^{\ k}F_{(\nu)k}} + \frac{3(1-\chi)}{2}\Big(\frac{E_{(\nu)}}{3}\delta^{ij} + F_{(\nu)}^{\ i}v^j + F_{(\nu)}^{\ j}v^i - \frac{2}{3}\delta^{ij}F_{(\nu)}^{\ k}v_k\Big), \qquad (8\cdot9)$$

and we used

$$J_{(\nu)} = E_{(\nu)} - 2F_{(\nu)}^{\ k}v_k, \qquad (8\cdot10)$$

$$H_{(\nu)}^{\ k} = -E_{(\nu)}v^k + F_{(\nu)}^{\ k} - P_{(\nu)}^{\ kl}v_l. \qquad (8\cdot11)$$

Again, we note that ν is the frequency in the *fluid rest frame* (not in the laboratory frame). As expected, Eqs. (8·5) and (8·6) have a conservative form.

The hydrodynamic equations are

$$\partial_t\rho + \partial_j(\rho v^j) = 0, \qquad (8\cdot12)$$

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v^j + P\delta_i^{\ j}) = -\rho\partial_i\phi_{\rm N} + \int d\nu\Big(\tilde\kappa_{(\nu)}(F_{(\nu)i} - P_{(\nu)ik}v^k) - [\kappa_{(\nu)}J_{(\nu)}^{\rm eq} + (\tilde\kappa_{(\nu)} - \kappa_{(\nu)})E_{(\nu)}]v_i\Big), \qquad (8\cdot13)$$

$$\partial_t\Big[\rho\Big(\varepsilon + \frac{v^2}{2}\Big)\Big] + \partial_j\Big[\Big(\rho\varepsilon + P + \frac{1}{2}\rho v^2\Big)v^j\Big] = -\rho v^i\partial_i\phi_{\rm N} - \int d\nu\kappa_{(\nu)}(J_{(\nu)}^{\rm eq} - E_{(\nu)} + F_{(\nu)}^{\ i}v_i), \qquad (8\cdot14)$$

$$\partial_t(\rho Y_e) + \partial_j(\rho Y_e v^j) = \rho Q_e, \qquad (8\cdot15)$$

where $\phi_{\rm N}$ is the Newtonian potential. We note that when taking the Newtonian limit of general relativistic hydrodynamics equations, the conservative rest-mass density $\rho w\sqrt{\gamma}$ is replaced to ρ. The total energy equation is written as

$$\partial_t\Big[\rho\Big(\varepsilon + \frac{v^2}{2}\Big) + E\Big] + \partial_j\Big[\Big(\rho\varepsilon + P + \frac{1}{2}\rho v^2\Big)v^j + F^j\Big] = -\rho v^i\partial_i\phi_{\rm N}. \qquad (8\cdot16)$$

§9. Summary

We derived a truncated moment formalism for general relativistic radiation hydrodynamics modifying Thorne's original formalism.[2)] The equations for the radiation field are written for the variables defined in the laboratory frame as well as in the fluid local rest frame, although the argument angular frequency for the radiation moments is always the frequency measured in the fluid local rest frame. In the former case, the equations are written in a conservative form (for $E_{(\nu)}$ and $F_{(\nu)i}$) and essentially the same as those for the hydrodynamic equations in general relativity. Thus, they seem to be useful for a well-resolved numerical simulation.

The source terms are written, focusing on the neutrino transfer in the assumption that anisotropy of the scattering kernel is small. Then, a formalism for the radiation hydrodynamics in numerical relativity is derived in a closed form, assuming a physically reasonable closure relation among the radiation stress tensor, energy density, and energy flux. As long as the radiation field is not extremely anisotropic in the fluid rest frame (in the optically thick medium), the employed approximation should work well. One merit in the present formalism is that we do not have to perform any coordinate transformation when computing the source term, because the angular frequency for the radiation field is defined in the fluid local rest frame.

The derived equations constitute wave equations for the radiation field. The closure relation and variable Eddington factor are appropriately chosen so that the characteristic speed is smaller than the speed of light in the free-streaming and grey regions. We also notice that (i) for the derivation of the basic equations for the radiation field, we do not assume that the fluid velocity is much smaller than the speed of light, and (ii) with the chosen closure relation, the effect associated with the fluid motion (fluid expansion, acceleration, and shear) may be taken into account. Thus, the derived formalism can be employed for the radiation field associated with a fast motion, e.g., a fluid moving in the vicinity of a black hole.

In this formalism, we need to solve 3+1+1 equations (3 is space, 1 is time, and frequency space) of $(E_{(\nu)}, F_{(\nu)i})$ or 3+1 equations of (E, F_i) for the radiation part. For both cases, the equations for these variables are written in a conservative form, and hence, conservation of mass and moments is likely to be well achieved in the formalism with these quantities. For the 3+1+1 case, the equations, including absorption, emission, and scattering term for neutrinos, are written in the closed form, and hence, we do not have to assume anything further. For the 3+1 case, computational costs will be saved significantly, but we have to impose several additional conditions when performing the frequency integral for the source term: We have to assume certain functions for $J_{(\nu)}$ and $H^{\alpha}_{(\nu)}$ (e.g., Ref. 15)), for which a physically appropriate assumption is required.

The truncated moment formalism may be a starting point for upgrading the current leakage scheme for general relativistic radiation hydrodynamics (e.g., Ref. 21) for a review). The leakage scheme is often used for phenomenologically incorporating radiation cooling and for a relatively inexpensive radiation hydrodynamic simulation. The method is usually quite phenomenological: One first determines optically thick and thin regions, respectively, using a rather approximate prescription. Then, for the optically thick region, one assumes that the radiation escapes in a diffusive manner and for the optically thin region, the radiation escapes freely. In general relativistic leakage schemes,[21)] one incorporates the cooling effect on the right-hand side of the hydrodynamics equations ($\nabla_\alpha T^{\alpha\beta} = -S^\beta$), and in addition, an equation for radiation four-vector field is evolved for the optically thin region. Namely, the basic equations are quite similar to those derived in this paper. What is different is on the treatment for the source term $S_{(\nu)}^{A_k}$; in the leakage scheme proposed so far, the source term is determined in a quite phenomenological manner. If $S_{(\nu)}^{A_k}$ is approximated in a more strict manner starting from the basic equations derived in this paper, it will be possible to derive a better and well-funded leakage scheme. Furthermore, it will be possible to incorporate the frequency-dependent effect as well as neutrino heating. Such work is left for the future.

Finally, the truncated moment formalism derived here may be used for the transfer of photons by exchanging the source terms (if we may assume that anisotropy of the scattering kernel is small). For example, this formalism will be useful for studying an accretion flow in the vicinity of stellar-mass and supermassive black holes in general relativity.

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Appendix A
—— *Stationary Radiation in the Spherical Dilute Medium* ——

In this appendix, we show the solution of a radiation field in the Bondi flow[20] composed of a dilute medium (i.e., optically thin approximation is assumed to work) and illustrate that gravitational redshift and Doppler effects are appropriately taken into account in the radiation spectrum observed at infinity in our formalism.

As discussed in §3, in the optically thin region (where $S_{(\nu)}^{A_k}=0$), the radiation moments may be written by

$$H_{(\nu)}^{\alpha}=J_{(\nu)}\ell^{\alpha},\quad L_{(\nu)}^{\alpha\beta}=J_{(\nu)}\ell^{\alpha}\ell^{\beta},\tag{A·1}$$

where ℓ^{α} denotes a unit spatial vector for which the spatial component is composed only of the radial component. Substitution of these relations into Eq. (3·22) with $S_{(\nu)}^{\alpha}=0$ yields

$$\nabla_{\alpha}(J_{(\nu)}l^{\alpha})-\nu\frac{\partial J_{(\nu)}}{\partial\nu}\ell^{\alpha}l^{\beta}\nabla_{\beta}u_{\alpha}=0,\tag{A·2}$$

where $l^{\alpha}:=u^{\alpha}+\ell^{\alpha}$ is a null vector. Substitution, in addition, of $N_{(\nu)}^{\alpha\beta\gamma}=J_{(\nu)}\ell^{\alpha}\ell^{\beta}\ell^{\gamma}$ into Eq. (3·23) yields the same equation as Eq. (A·2).

For a solution of the four velocity, u^{α}, we here consider a stationary spherical accretion flow (Bondi accretion flow) in the spacetime of a spherical black hole of mass M. We choose the line element in the Kerr-Schild coordinates,

$$ds^2=-\Big(1-\frac{2GM}{r}\Big)d\bar{t}^2+\frac{4M}{r}d\bar{t}dr+\Big(1+\frac{2GM}{r}\Big)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2),\tag{A·3}$$

in which the coordinate singularity at $r=2GM$ does not give any messy problem. We note, however, that an analytic solution is also easily derived in the Schwarzschild coordinates.

We denote the infall velocity by $u^r = -u(r) < 0$ (cf. for the wind solution $u^r > 0$), and note the relations $\ell_t = u$ and $\ell_r = u^t$, where $-u_t = u^t(1 - 2GM/r) + 2GMu/r = \sqrt{1 + u^2 - 2GM/r}$. Thus, $l^r = -u + \sqrt{1 + u^2 - 2GM/r} > 0$, and $l^t = l^r(r + 2GM)/(r - 2GM)$. Using these relations, we finally reach

$$\ell^\alpha l^\beta \nabla_\beta u_\alpha = \frac{dl^r}{dr}. \tag{A·4}$$

Substitution of this relation into Eq. (A·2) gives

$$l^r \frac{\partial J_{(\nu)}}{\partial r} + \frac{J_{(\nu)}}{r^2} \frac{d(r^2 l^r)}{dr} = \nu \frac{\partial J_{(\nu)}}{\partial \nu} \frac{dl^r}{dr}, \tag{A·5}$$

and assuming that $J_{(\nu)} > 0$ and $l^r > 0$, we obtain

$$\frac{\partial y_{(\nu)}}{\partial r} = \nu \frac{\partial y_{(\nu)}}{\partial \nu} \frac{d \ln l^r}{dr}, \tag{A·6}$$

where $y_{(\nu)} := J_{(\nu)} l^r r^2$. For the case that l^r is a monotonically increasing function of r (this is the case for the typical problem), Eq. (A·6) is rewritten to give

$$\frac{\partial y_{(\nu)}}{\partial \ln l^r} = \frac{\partial y_{(\nu)}}{\partial \ln \nu}. \tag{A·7}$$

Thus, $y_{(\nu)}(r)$ constitutes a wave equation for the arguments $(\ln l^r, \ln \nu)$, and therefore, the general solution can be derived as

$$y_{(\nu)} = F(l^r \nu), \quad \text{or} \quad J_{(\nu)} = \frac{F(l^r \nu)}{l^r r^2}, \tag{A·8}$$

where $F(x)$ is an arbitrary function of x. Here, $l^r \to 1$ for $r \to \infty$. Namely, at infinity, the observed spectrum is

$$J_{(\nu)} = \frac{F(\nu)}{r^2}. \tag{A·9}$$

On the other hand, for the finite value of r, l^r is smaller than unity. This implies that the radiation spectrum should be homogeneously (irrespectively of the value of ν) shifted to the lower frequency side during outgoing propagation. The redshift factor is given by l^r; an observed radiation with frequency ν at infinity is originally emitted at a finite radius, $r_{\rm emit}$, with frequency $\nu_{\rm emit} = \nu / l^r(r_{\rm emit}) > \nu$.

Equation (A·8) indeed captures gravitational redshift and Doppler effects. This is clearly found by taking the slow-motion and weak-gravitation approximation for l^r as

$$l^r \approx 1 - u + \frac{u^2}{2} - \frac{GM}{r}. \tag{A·10}$$

The first, second, and third terms denote the Doppler, second-order Doppler, and gravitational redshift effects, respectively.

Integration of $J_{(\nu)}$ by ν gives

$$J = \int d\nu J_{(\nu)} = \frac{L_0}{(l^r r)^2}, \tag{A·11}$$

where L_0 denotes a total flux

$$L_0 = \int d\nu F(\nu). \tag{A·12}$$

Thus $l^r r$ may be regarded as a luminosity distance. Note that for $r \to 2GM$, $l^r \to 0$. Thus, $J \to \infty$ at $r = 2GM$; the solution is similar to a solution in the flat spacetime in which a "point" source exists at origin (here which is located at $l^r r = 0$).

Finally, we point out that the solution derived here will be used as a test-bed problem for checking the reliability of a radiation hydrodynamic code based on the truncated moment formalism (note that with Eq. (A·1) the closure relation holds). We also note that the solution given here holds not only for the Bondi flow but also any solution in the stationary, spherically symmetric spacetime as long as l^r is a monotonic function of r.

Appendix B
—— *Radiation Flow in the Spherical Dilute Medium* ——

Next, we analyze a time-dependent spherically symmetric radiation flow in the Schwarzschild spacetime. Again, we assume that neutrinos propagate in the optically thin medium. The purpose of this section is to clarify a nature of the closure relations (6·3) and (6·5). For this, we ignore the frequency-dependent effects and analyze Eqs. (3·39) and (3·40). For the background metric, we again adopt Eq. (A·3). In this case, the necessary geometric quantities are

$$\begin{aligned}
&\alpha = \left(1 + \frac{2M}{r}\right)^{-1/2}, \quad \beta^r = \frac{2M}{r+2M}, \quad \gamma_{rr} = 1 + \frac{2M}{r}, \\
&\text{and} \quad K_{rr} = -\frac{2M(r+M)}{r^{5/2}(r+2M)^{1/2}},
\end{aligned} \tag{B·1}$$

where we use the units of $G = 1$ (or we may say that GM is replaced to M). Because of the spherical symmetry, we only need to consider the radial component of radiation moments, F^r. For the following, we define $F := F^r \gamma_{rr}^{1/2}$. Then the condition $g_{\mu\nu} T^{\mu\nu}_{\rm rad} = 0$ for the closure relations (6·3) and (6·5) is written as

$$E = |F|. \tag{B·2}$$

For the closure relation (6·3), the equations for E and F are

$$\dot{e} - \frac{2M}{r+2M} e' + \frac{r}{r+2M} f' + \frac{2M(2r+M)}{r(r+2M)^2} e + \frac{3M}{(r+2M)^2} f = 0, \tag{B·3}$$

$$\dot{f} - \frac{2M}{r+2M} f' + \frac{r}{r+2M} e' + \frac{2M(2r+M)}{r(r+2M)^2} f + \frac{3M}{(r+2M)^2} e = 0, \tag{B·4}$$

where $e = Er^2\gamma_{rr}^{1/2}$ and $f = Fr^2\gamma_{rr}^{1/2}$. The dot ($\dot{e}$) and dash ($e'$) denote $\partial_t e$ and $\partial_r e$, respectively. Defining $u_\pm = e \pm f$, we obtain two independent equations

$$\dot{u}_+ + \frac{r-2M}{r+2M}u'_+ + \frac{M(7r+2M)}{r(r+2M)^2}u_+ = 0, \tag{B·5}$$

$$\dot{u}_- - u'_- + \frac{M}{r(r+2M)}u_- = 0, \tag{B·6}$$

and $e = (u_+ + u_-)/2$ and $f = (u_+ - u_-)/2$. This implies that in the absence of u_+ or u_-, the condition (B·2) is satisfied, but in general, it is not. In particular, for the point which satisfies $f = 0$ ($u_+ = u_-$), one of the characteristic speed becomes infinity (see Eq. (6·29)).*) Thus this closure relation should be prohibited for such a situation (this is resolved in an appropriate choice of the variable Eddington factor shown in §6.3).

For the closure relation (6·5), the equations for e and f are

$$\dot{e} - \frac{2M}{r+2M}e' + \frac{r}{r+2M}f' + \frac{2M}{(r+2M)^2}e + \frac{3Mr+2M(r+M)s}{r(r+2M)^2}f = 0, \tag{B·7}$$

$$\dot{f} - \frac{2M-rs}{r+2M}f' + \frac{2M(M+2r+rs)}{r(r+2M)^2}f + \frac{M}{(r+2M)^2}e = 0, \tag{B·8}$$

where $s = 1$ (-1) for $F^r > 0$ (< 0). Defining $u = e - sf = (E - sF)r^2\gamma_{rr}^{1/2}$, we obtain

$$\dot{u} - \frac{2M}{r+2M}u' + \frac{2M-sM}{(r+2M)^2}u = 0, \tag{B·9}$$

$$\dot{f} - \frac{2M-rs}{r+2M}f' + \frac{M(2M+4r+3rs)}{r(r+2M)^2}f + \frac{M}{(r+2M)^2}u = 0. \tag{B·10}$$

Thus, there are also two components: One is determined by u which is a mode of physically zero characteristic speed because the coefficient of the transport term is equal to $-\beta^r$. The other is associated with f, which is an outgoing or ingoing mode and obeys the same equation as that of u_+ and u_- for $f > 0$ and $f < 0$, respectively. u is regarded as an unphysical mode because it is the measure of deviation from the condition (B·2); if $u = 0$ is satisfied, we can follow only the physical mode, but this will not be in general the case, in particular in the near zone. The important fact, however, is that u does not propagate outward and damps exponentially with time in the absence of the source term. This implies that in the zone distant from the source, u will be zero because the emission source should be zero there. Thus, in the distant optically thin zone, the condition (B·2) is likely to be satisfied.

It will be useful to give the solutions of Eqs. (B·5) and (B·6): The general solutions for these are written as

$$u_+ = \left[\frac{r(r+2M)^3}{(r-2M)^4}\right]^{1/2} g_+(t - r_*), \tag{B·11}$$

*) In the analysis of spherically symmetric flow here, the characteristic speeds are $(r-2M)/(r+2M)$ and -1. However, if we solve the equation in the Cartesian or cylindrical coordinates, the extra characteristic speed (6·29) appears.

$$u_- = \left[\frac{r}{r+2M}\right]^{1/2} g_-(t-r), \tag{B·12}$$

where $g_\pm$ are arbitrarily functions and r_* is a retarded time

$$r_* = \int dr \frac{r+2M}{r-2M} = r + 4M \ln\left(\frac{r-2M}{M}\right). \tag{B·13}$$

Appendix C
—— *Numerical Experiment for Free Evolution* ——

To confirm that the closure relation and variable Eddington factor (6·25) described in §6 work well in the optically thin medium, we numerically solve radiation field equations (3·39) and (3·40) on a Bondi flow of a Schwarzschild spacetime. The closure relation is written as

$$P^{ij} = \frac{3\chi-1}{2} E \frac{F^i F^j}{\gamma_{kl} F^k F^l} + \frac{3(1-\chi)}{2}\left(J \frac{\gamma^{ij} + 4V^i V^j}{3} + H^i V^j + H^j V^i\right), \tag{C·1}$$

where χ is assumed to be a function of $\bar{F} = |F|/E = \sqrt{\gamma_{kl} F^k F^l}/E$. The source terms are set to be zero ($S^\alpha = 0$) for simplicity. Numerical simulation was performed assuming the axial and equatorial plane symmetries. As in Appendices A and B, the Kerr-Schild coordinates are adopted, and the same Bondi solution as in Ref. 19) is employed. The basic equations are essentially the same as those solved in general relativistic hydrodynamic simulation. We employ the same scheme as used in Ref. 19) for a solution of E and F_k. Specifically, the transport term is handled using a Kurganov-Tadmor scheme[22)] with a piecewise parabolic reconstruction for the quantities of cell interfaces. The fourth-order Runge-Kutta method is employed for the time integration. The characteristic speed is not analytically computed for the general form of P_{ij}. Thus, we simply write it in the linear combination form

$$\lambda = \frac{3\chi-1}{2}\lambda_{\rm thin} + \frac{3(1-\chi)}{2}\lambda_{\rm thick}. \tag{C·2}$$

For the case that the relation, $E < |F|$, is accidentally realized at a point, we set $\chi = 1$, and $\lambda_{\rm thin}$ is limited to be smaller than unity.

With the time evolution, the radiation fields flow away from the computational domain. To handle this correctly, an outgoing boundary condition is imposed for the outer boundaries, and inside the radius $r \le 1.8M$, we artificially set $E = F_k = 0$.

First, we consider the solution for $E = |F|$ derived in Appendix B. In this case, $\bar{F}$ is always unity, and thus, $\chi = 1$ always holds. For the outgoing flow $F = E$, the solution is written as

$$E\gamma_{rr}^{1/2} = F\gamma_{rr}^{1/2} = \frac{1}{2}\left[\frac{(r+2M)^3}{r^3(r-2M)^4}\right]^{1/2} g_+(t-r_*), \tag{C·3}$$

and for the ingoing flow $F = -E$,

$$E\gamma_{rr}^{1/2} = -F\gamma_{rr}^{1/2} = \frac{1}{2}[r^3(r+2M)]^{-1/2} g_-(t-r). \tag{C·4}$$

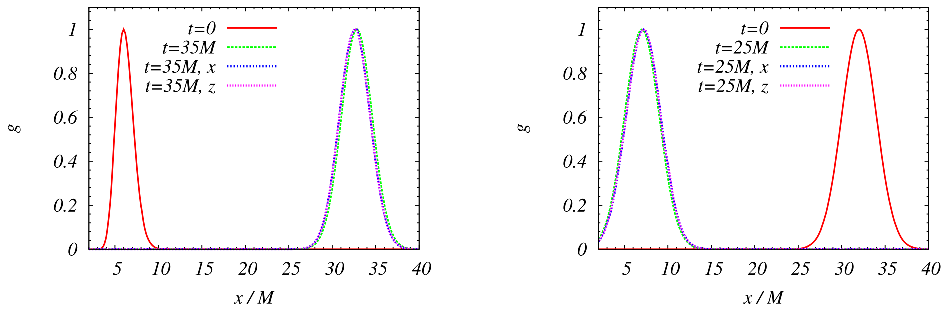

Fig. 1. Evolution of outgoing (left) and ingoing (right) solutions. For the outgoing solution, the profiles of $E\gamma_{rr}^{1/2}(r-2M)^2(1+2M/r)^{-3/2}$ at $t=0$ and $t=35M$ along the x and z axes are plotted. For the ingoing solution, the profiles of $E\gamma_{rr}^{1/2}r^{3/2}(r+2M)^{1/2}$ at $t=0$ and $t=25M$ along the x and z axes are plotted. The solid and dashed curves denote the numerical and exact solutions. The numerical solutions for x and z axes agree approximately.

We choose a form of a wave packet as

$$g_+(r_*) = \exp[-(r_* - r_{*0})^2/8M^2], \quad g_-(r) = \exp[-(r-r_0)^2/8M^2]. \tag{C·5}$$

Numerical simulations were performed for $r_{*0} = r_*(r=6M)$ and $r_0 = 32M$. The computational domain covers a region $[0:40M]$ both for x and z with a uniform grid spacing $0.1M$.

Figure 1 plots the evolution of $E\gamma_{rr}^{1/2}(r-2M)^2(1+2M/r)^{-3/2}$ for the outgoing solution (left) and $E\gamma_{rr}^{1/2}r^{3/2}(r+2M)^{1/2}$ for the ingoing solution (right). This shows that besides a small phase error, the numerical solutions reproduce the exact solutions.

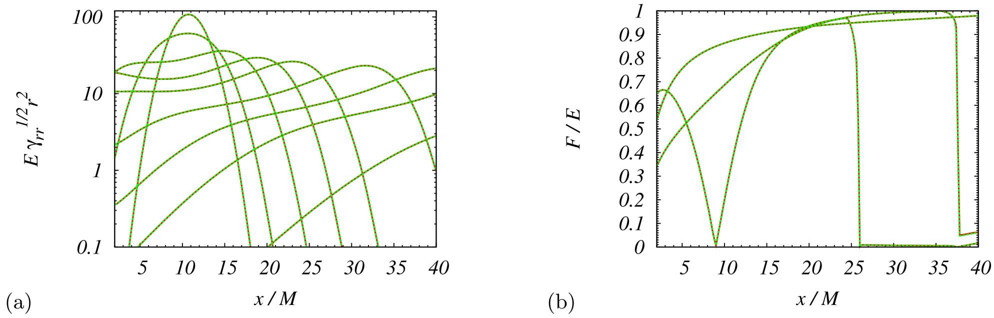

Fig. 2. (a) Evolution of the wave packet along the x and z axes for the initial condition (C·6) with $F=0$. The profiles are shown for $t/M = 0$, 5, 10, 15, 20, 30, 50, and 70. (b) F/E along the x and z axes for $t/M = 5$, 20, and 50. For both figures, the results for x and z axes agree approximately.

We also performed a simulation for $F=0$ at $t=0$ with

$$E\gamma_{rr}^{1/2} = \exp[-(r-10M)^2/8M^2]. \tag{C·6}$$

In this case, $\chi = 1/3$ at $t=0$, but with the time evolution, $|F|$ becomes nonzero and χ becomes larger than 1/3. Because the variable Eddington factor χ is varied with the evolution, we do not have the exact solution. The purpose is to test if our formalism allows a stable numerical solution.

Figure 2(a) plots the time evolution of the wave packet. Here we plot $E\gamma_{rr}^{1/2}r^2$ along x and z axes (two results approximately agree and cannot be distinguished in the figure). After the evolution starts, the wave packet is split into outgoing and ingoing parts. Both modes propagate smoothly with no trouble. Figure 2(b) plots the evolution of F/E. This is initially zero, but with the free propagation, it approaches to unity: At $t/M = 5$, 20, and 50, F/E is larger than 0.8 for $15 \lesssim x/M \lesssim 25$, $15 \lesssim x/M \lesssim /35$, and $x/M \gtrsim 5$, respectively. No problem is found for the propagation, and thus, as far as the numerical issues are concerned, the closure relation and variable Eddington factor employed here have no problem.

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