M1 matter-radiation coupling

1 Introduction

The purpose of this document is to understand how to compute the M1 (energy-integrated) source terms defining radiation-matter coupling (Eq. (5) in *Radice et al. (2022)*) starting from the corresponding spectral quantities (see e.g. *Shibata et al. (2011)*). To better distinguish the energy-integrated equations from the spectral ones, in the text we will label the former in blue and the latter in red.

2 M1 source terms

Following *Shibata et al. (2011)*, we can decompose the four-momentum p^α of the radiation as:

$$\frac{dx^\alpha}{d\tau} = p^\alpha = \nu(u^\alpha + l^\alpha), \tag{1}$$

where ν is the frequency measured in the *fluid rest frame*, u^α is the medium four-velocity and l^α is a unit normal four-vector orthogonal to u^α, encoding the angular dependence of the momentum ($l_\alpha l^\alpha = l^2 = 1$ and $u_\alpha l^\alpha = 0$). Remember that for the angular integration the following relations hold:

$$\int d\Omega\, l^\alpha = 0 = \int d\Omega\, l^\alpha l^\beta l^\gamma, \tag{2}$$

$$\int d\Omega l^\alpha l^\beta = \frac{4\pi}{3} h^{\alpha\beta}, \tag{3}$$

where $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$ is the projection operator.
Let's now define the first two radiation moments in the truncated moment formalism considering the notation in *Shibata et al. (2011)*:

$$J_{(\nu)} \equiv \nu^3 \int f(\nu, \Omega, x^\mu) d\Omega, \tag{4}$$

$$H^\alpha_{(\nu)} \equiv \nu^3 \int l^\alpha f(\nu, \Omega, x^\mu) d\Omega, \tag{5}$$

where the cubic power in front is the product of the energy ν of a single radiation particle with a factor ν^2 coming from the future phase space integration (see below). Please keep in mind that the integrals in Eq. (4) and Eq. (5) are assumed to be performed in the local rest frame comoving with the fluid.

The energy-dependent source terms for the second-rank radiation field equations, defined in the local rest frame of the fluid, can be written as:

$$S^\alpha_{(\nu)} = \nu^3 \int B_{(\nu)}(\Omega, x^\mu)(u^\alpha + l^\alpha) d\Omega. \tag{6}$$

where $B_{(\nu)}$ is the so-called collision integral and is a function of the radiation energy ν. Following Eq. (5) in *Radice et al. (2022)*, the energy-integrated version of Eq. (6) is:

$$S^\alpha \equiv \int_0^\infty d\nu\, S^\alpha_{(\nu)} = (\eta - \kappa'_a J)u^\alpha - (\kappa''_a + \kappa_s)H^\alpha, \tag{7}$$

where η is the neutrino emissivity and κ'_a, κ''_a and κ_s are the neutrino absorption and scattering coefficients. Please note that κ'_a and κ''_a are in principle two different coefficients. J and H^α are the energy-integrated versions of Eq. (4) and Eq. (5), i.e.:

$$J \equiv \int_0^\infty d\nu\, J_{(\nu)}, \tag{8}$$

$$H^\alpha \equiv \int_0^\infty d\nu\, H^\alpha_{(\nu)}. \tag{9}$$

The idea now is to express the different terms in Eq. (7) in terms of the corresponding spectral quantities by integrating explicitely Eq. (6). Thus let's see the shape of the collision integral in the case of neutrino absorption and emission and for isoenergetic neutrino scattering.

Neutrino absorption and emission

In the case of absorption and emission of neutrinos by nucleons and heavy nuclei, the collision integral is:

$$\begin{aligned} B_{(\nu)} &= j_{(\nu)}[1 - f(\nu,\Omega,x^\mu)] - \frac{f(\nu,\Omega,x^\mu)}{\lambda_{(\nu)}} \\ &= j_{(\nu)} - \underbrace{(j_{(\nu)} + \lambda^{-1}_{(\nu)})}_{\kappa_{(\nu)}} f(\nu,\Omega,x^\mu), \end{aligned} \tag{10}$$

where $j_{(\nu)}$ denotes the energy-dependent emissivity, $\lambda_{(\nu)}$ is the energy-dependent absorption mean free path and $f(\nu,\Omega,x^\mu)$ denotes the distribution function of neutrinos (in the following we will omit the argument x^μ in f to simplify a bit the notation). Please remember that $j_{(\nu)}$ and $\lambda_{(\nu)}$ do not depend on the angular variables, they are a function only of the neutrino energy, i.e. $j_{(\nu)} = j_{(\nu)}(\nu,\cancel{\Omega})$ and $\lambda_{(\nu)} = \lambda_{(\nu)}(\nu,\cancel{\Omega})$. Eq. (10) is rewritten (see the second line) in order to express an equivalent collision integral for non-fermionic particles by defining a *stimulated absorption* with effective opacity $\kappa_{(\nu)} = j_{(\nu)} + \lambda^{-1}_{(\nu)}$. By inserting Eq. (10) into Eq. (6), one finds the following expression:

$$\begin{aligned} S^\alpha_{(\nu)} &= \nu^3 \int d\Omega\, \left[j_{(\nu)} - \kappa_{(\nu)}\, f(\nu,\Omega)\right](u^\alpha + l^\alpha) \\ &= \nu^3 \int d\Omega\, \left[j_{(\nu)} - \kappa_{(\nu)} f(\nu,\Omega)\right] u^\alpha + \cancel{\nu^3 \int d\Omega\, j_{(\nu)} l^\alpha} - \nu^3 \int d\Omega\, \kappa_{(\nu)} f(\nu,\Omega) l^\alpha \\ &= \left[4\pi\nu^3 j_{(\nu)} - \kappa_{(\nu)}\nu^3 \int d\Omega\, f(\nu,\Omega)\right] u^\alpha - \kappa_{(\nu)}\nu^3 \int d\Omega\, f(\nu,\Omega) l^\alpha \\ &= \left[4\pi\nu^3 j_{(\nu)} - \kappa_{(\nu)} J_{(\nu)}\right] u^\alpha - \kappa_{(\nu)} H^\alpha_{(\nu)}, \end{aligned} \tag{11}$$

where we exploited Eq. (2), Eq. (4) and Eq. (5) to simplify the expression. Let's now integrate Eq. (11) with respect to the neutrino energy in the fluid rest frame:

$$\begin{aligned} S^\alpha &= \int_0^{+\infty} d\nu\, S^\alpha_{(\nu)} \\ &= \left[\int_0^{+\infty} d\nu\, 4\pi\nu^3 j_{(\nu)} - \int_0^{+\infty} d\nu\, \kappa_{(\nu)}\nu^3 \int d\Omega\, f(\nu,\Omega)\right] u^\alpha - \int_0^{+\infty} d\nu\, \kappa_{(\nu)}\nu^3 \int d\Omega\, f(\nu,\Omega) l^\alpha \} \\ &= \left[\int_0^{+\infty} d\nu\, 4\pi\nu^3 j_{(\nu)} - \int_0^{+\infty} d\nu\, \kappa_{(\nu)} J_{(\nu)}\right] u^\alpha - \int_0^{+\infty} d\nu\, \kappa_{(\nu)} H^\alpha_{(\nu)} \end{aligned} \tag{12}$$

By comparing now Eq. (12) with Eq. (7), we obtain the following expressions for the neutrino emissivity and for the absorption coefficients:

$$\eta = \int_0^{+\infty} d\nu\, 4\pi\nu^3 j_{(\nu)}, \tag{13}$$

$$\kappa'_a = \frac{\int d\nu\, \kappa_{(\nu)} J_{(\nu)}}{J} = \frac{\int d\nu\, \kappa_{(\nu)}\nu^3 \int d\Omega f(\nu,\Omega)}{\int d\nu\, \nu^3 \int d\Omega f(\nu,\Omega)}, \tag{14}$$

$$\kappa''_a = \frac{\sqrt{\tilde{H}^\alpha \tilde{H}_\alpha}}{H} = \frac{\sqrt{\int d\nu\, \kappa_{(\nu)} H^\alpha_{(\nu)} \int d\nu'\, \kappa_{(\nu')} H_{(\nu')\,\alpha}}}{H}, \tag{15}$$

where $\tilde{H}^\alpha \equiv \int d\nu \kappa_{(\nu)} H^\alpha_{(\nu)} = \int d\nu \kappa_{(\nu)} \nu^3 \int d\Omega f(\nu,\Omega) l^\alpha$ and $H \equiv \sqrt{H^2} = \sqrt{H^\alpha H_\alpha}$. Eq. (15) is obtained by computing $(\kappa''_a)^2 g_{\alpha\beta} H^\alpha H^\beta$ and then inverting it to find κ''_a. Please note that the relation $H^2 = J^2$ (which would result by taking $l_\alpha l^\alpha = 1$ in Eq. (15)) is not valid in general, it holds only in the case in which the flux is peaked in a given direction.

Isoenergetic neutrino scattering

In the case of neutrino scattering with nucleons and heavy nuclei, the energy exchange may be assumed to be zero (*isoenergetic neutrino scattering*). In this approximation, the collision integral is written as:

$$\begin{aligned} B_{(\nu)} &= \nu^2 \int d\Omega' \Big\{ \left[1 - f(\nu,\Omega)\right] f(\nu,\Omega') - f(\nu,\Omega)\left[1 - f(\nu,\Omega')\right] R^{\rm iso}(\nu,\omega) \\ &= \nu^2 \int d\Omega' \left[f(\nu,\Omega') - f(\nu,\Omega)\right] R^{\rm iso}(\nu,\omega), \end{aligned} \tag{16}$$

where ω is the cosine of the scattering angle, ν is the angular frequency of the ingoing and outgoing neutrinos and $R^{\rm iso}$ is the isoenergetic scattering kernel. We expand the angular dependence of the scattering kernel by keeping the terms up to the linear order in ω:

$$R^{\rm iso}(\nu,\omega) = R_0^{\rm iso}(\nu) + \omega R_1^{\rm iso}(\nu). \tag{17}$$

Please keep in mind that ω is related to the angular part of the momentum-space coordinates $\Omega = (\theta, \varphi)$ and $\Omega' = (\theta', \varphi')$ of the ingoing and outgoing neutrino by:

$$\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'). \tag{18}$$

Assuming to consider neutrino transfer in high-density and high-temperature medium, we can exploit the expansion of f in the limit of small anisotropy of the collision integral (see Eq. (31) below) to perform the angular integration in Eq. (16):

$$\begin{aligned} B_{(\nu)} &= \nu^2 \int d\Omega' \left[\cancel{f_0(\nu)} + f_1^\beta(\nu) l'_\beta - \cancel{f_0(\nu)} - f_1^\beta l_\beta(\nu)\right]\left[R_0^{\rm iso}(\nu) + \omega R_1^{\rm iso}(\nu)\right] \\ ,&= 4\pi\nu^2 f_1^\beta(\nu) l_\beta \left[\frac{R^{\rm iso}(\nu)}{3} - R_0^{\rm iso}(\nu)\right], \end{aligned} \tag{19}$$

where we have exploited the following relations:

$$\int \omega d\Omega = 0 = \int \omega l^\alpha l^\beta d\Omega, \qquad \int \omega l^\alpha d\Omega = \frac{4\pi}{3} l'^\alpha. \tag{20}$$

Consequently, the spectral source term for isoenergetic neutrino scattering is written as:

$$\begin{aligned} S^\alpha_{(\nu)} &= \nu^3 \int d\Omega\, 4\pi\nu^2 f_1^\beta(\nu) l_\beta \left[\frac{R^{\rm iso}(\nu)}{3} - R_0^{\rm iso}(\nu)\right](\cancel{u^\alpha} + l^\alpha) \\ &= 4\pi\nu^2\left[\frac{R^{\rm iso}(\nu)}{3} - R_0^{\rm iso}(\nu)\right] \underbrace{\frac{4\pi}{3}\nu^3 f_1^\alpha(\nu)}_{H^\alpha_{(\nu)}} = 4\pi\nu^2\left[\frac{R^{\rm iso}(\nu)}{3} - R_0^{\rm iso}(\nu)\right] H^\alpha_{(\nu)}, \end{aligned} \tag{21}$$

where we took advantage of Eq. (2), Eq. (3) and of the expression of $H^\alpha_{(\nu)}$ in the small anisotropy limit (see Eq. (33)). Thus, the energy-integrated source term becomes:

$$S^\alpha = \int_0^{+\infty} d\nu\, S^\alpha_{(\nu)} = \int_0^{+\infty} d\nu\, 4\pi\nu^2\left[\frac{R^{\rm iso}(\nu)}{3} - R_0^{\rm iso}(\nu)\right] H^\alpha_{(\nu)}. \tag{22}$$

By comparing Eq. (22) with Eq. (7), we infer the following expression to compute the scattering coefficient κ_s:

$$\begin{aligned} \kappa_s &= \frac{\sqrt{\int d\nu\, 4\pi\nu^2 [R^{\rm iso}(\nu)/3 - R_0^{\rm iso}(\nu)] H^\alpha_{(\nu)} \int d\nu'\, 4\pi\nu'^2 [R^{\rm iso}(\nu')/3 - R_0^{\rm iso}(\nu')] H_{(\nu')\alpha}}}{H} \\ &= \frac{\sqrt{\int d\nu\, 4\pi\nu^2 [R^{\rm iso}(\nu)/3 - R_0^{\rm iso}(\nu)] \nu^3 \int d\Omega f(\nu,\Omega) l^\alpha \int d\nu'\, 4\pi\nu'^2 [R^{\rm iso}(\nu')/3 - R_0^{\rm iso}(\nu')] \nu'^3 \int d\Omega' f(\nu',\Omega') l'_\alpha}}{H}, \end{aligned} \tag{23}$$

While the way of computing Eq. (13) is conceptually clear (one has just to perform a numerical integration over the neutrino energy ν), the calculation of Eq. (14), Eq. (15) and Eq. (23) is trickier since it requires the knowledge of the unknown distribution function $f(\nu, \Omega)$. Under some approximations, one can reconstruct f starting from the energy-integrated quantities evolved by the numerical M1 code, but to do that we need some additional information. Therefore, let's discuss before the equations governing the evolution of the neutrino number density.

3 Neutrino number density

Following Eq. (19) in *Radice et al. (2022)*, one can define for each neutrino species the number density in the fluid frame as:

$$n = -N^\alpha u_\alpha, \tag{24}$$

where N^α is the neutrino number current. The latter satisfies the following continuity equation:

$$\nabla_\alpha N^\alpha = \sqrt{-g}(\eta^0 - \kappa_a^0 n), \tag{25}$$

where g is the determinant of the space-time metric and η^0 and κ_a^0 are the neutrino number emission and absorption coefficients, respectively. However Eq. (25) is not closed. *Radice et al. (2022)* adopts the following closure relation, assuming that the neutrino number and energy flux are aligned:

$$N^\alpha = n\, f^\alpha = n\left(u^\alpha + \frac{H^\alpha}{J}\right), \tag{26}$$

Please keep in mind that in general Eq. (26) is just an approximation for energy-integrated fluxes, it would be exact only in the particular case where neutrino had just a single energy. In the 3+1 form Eq. (25) becomes (see also *Foucart et al. (2016)*):

$$\partial_t(\sqrt{\gamma} n\Gamma) + \partial_i(\alpha\sqrt{\gamma} n f^i) = \alpha\sqrt{\gamma}(\eta^0 - \kappa_a^0 n), \tag{27}$$

where

$$\Gamma = \alpha f^0 = W - \frac{1}{J}H^\alpha n_\alpha, \qquad f^i = W\left(v_i - \frac{\beta^i}{\alpha}\right) + \frac{H^i}{J}, \tag{28}$$

with $W = -u^\alpha n_\alpha$ (fluid Lorentz factor) and $v^\alpha = \gamma^\alpha_\beta u^\beta$ (γ_{ik} is the three metric). At this stage let's make a brief consideration. One could think to obtain an equivalent spectral equation for the neutrino number density just by taking the spectral equation for the energy density and dividing the radiation moments and the source terms by the neutrino energy. However, since the latter equation is typically written in the laboratory frame (in order to have a conservative form), one should divide it by the neutrino energy in the same frame. This would give raise to additional terms in the equation, since the transformed energy no longer commutes with time and space derivatives on the left-hand side of the equation.

In order to find out how to compute η^0 and κ_a^0 in terms of the corresponding spectral quantities we can recall what we have seen before. Indeed, the spectral source term for the neutrino number equations is nothing but Eq. (6) divided by the neutrino energy itself, with a factor ν^2 instead of ν^3 in front. Exploiting this relation, we can infer the neutrino number emission and absorption coefficients just by analogy with Eq. (13) and Eq. (14):

$$\eta^0 = \int_0^{+\infty} d\nu\, 4\pi\nu^2 j_{(\nu)}, \tag{29}$$

$$\kappa_a^0 = \frac{\int d\nu\, \kappa_{(\nu)}\nu^2 \int d\Omega f(\nu,\Omega)}{n} = \frac{\int d\nu\, \kappa_{(\nu)}\nu^2 \int d\Omega f(\nu,\Omega)}{\int d\nu\, \nu^2 \int d\Omega f(\nu,\Omega)} \tag{30}$$

At this point we have all the ingredients to define an approximated recipe to reconstruct the neutrino distribution function $f = f(\nu,\Omega,x^\mu)$ needed for the computation of the energy-integrated source terms. Let's see this separately for optically thick and optically thin conditions.

4 Optically thick limit

In the optically thick limit, one can assume that the degree of anisotropy of the distribution function in the fluid local rest frame is weak. Therefore, we can expand in series the angular dependence of f neglecting second- and higher-order terms:

$$f(\nu,\Omega,x^\mu) = f_0(\nu,x^\mu) + f_1^\alpha(\nu,x^\mu) l_\alpha, \tag{31}$$

where $|f_0|$ is much larger than the magnitude of f_1^α. Please note that the coefficients of the expansion, f_0 and f_1^α, do not depend on the radiation propagation angle. By inserting Eq. (31) into Eq. (4) and Eq. (5), one obtains respectively:

$$J_{(\nu)} = 4\pi\nu^3 f_0(\nu), \tag{32}$$

$$H^\alpha_{(\nu)} = \frac{4\pi}{3}\nu^3 f_1^\alpha(\nu). \tag{33}$$

The corresponding energy-integrated quantities are (recall that $n = \int d\nu J_{(\nu)}/\nu$):

$$n = \int_0^{+\infty} d\nu\, 4\pi\nu^2 f_0(\nu), \tag{34}$$

$$J = \int_0^{+\infty} d\nu\, 4\pi\nu^3 f_0(\nu), \tag{35}$$

$$H^\alpha = \int_0^{+\infty} d\nu\, \frac{4\pi}{3}\nu^3 f_1^\alpha(\nu). \tag{36}$$

At this point one can make a parametric ansatz about the functional form of f_0 and f_1^α and then constrain the unknown parameters by exploiting Eq. (34), Eq. (35) and Eq. (36). In particular, for f_0 we can take a Fermi-Dirac like distribution:

$$f_0(\nu;C) = \frac{1}{\exp{(A\nu - B)} + 1}, \tag{37}$$

and find the values of the coefficients $C = \{A, B\}$ by imposing Eq. (34) and Eq. (35):

$$(n, J) \Longrightarrow f_0(\nu;C). \tag{38}$$

One can verify the physical consistency of this procedure by checking that for very-high optical depths, f_0 matches the equilibrium (i.e. Fermi-Dirac) distribution. Once we know f_0, the first absorption coefficient κ_a' and the number absorption coefficient κ_a^0 can be calculated in the following way:

$$\kappa_a' = \frac{\int d\nu\, \kappa_{(\nu)}\nu^3 4\pi f_0(\nu,C)}{J} = \frac{\int d\nu\, \kappa_{(\nu)}\nu^3 f_0(\nu,C)}{\int d\nu\, \nu^3 f_0(\nu,C)}, \tag{39}$$

$$\kappa_a^0 = \frac{\int d\nu\, \kappa_{(\nu)}\nu^2 4\pi f_0(\nu,C)}{n} = \frac{\int d\nu\, \kappa_{(\nu)}\nu^2 f_0(\nu,C)}{\int d\nu\, \nu^2 f_0(\nu,C)}. \tag{40}$$

Note that the first-order term in the f-expansion does not contribute to κ_a' or κ_a^0 since it vanishes during the angular integration (see Eq. (2)).

For the four-components of f_1^α the situation is a bit trickier since there are no physical assumptions that motivate the choice of a given function over another. Here we suggest two different possibilities:

(1) Assume that $f_1^\alpha = 0\ \forall\alpha$. In the absence of a physically-motivated choice, the simplest thing to do is to assume they are identically equal to zero. In this way the distribution function becomes isotropic, since it is given just by the zeroth-order coefficient of the expansion:

$$f(\nu,\Omega,x^\mu) \simeq f_0(\nu,x^\mu). \tag{41}$$

As a consequence, the second absorption coefficient vanishes because of Eq. (2):

$$\kappa_a'' = 0. \tag{42}$$

(2) In order to have a non-vanishing κ_a'', we have to consider a different functional form. The easy choice is taking them constant, $f_1^\alpha = \xi^\alpha$, however this would lead to a divergent energy integration in the computation of H^α. Therefore, we multiply them with a damping function of energy, e.g. an exponential:

$$f_1^\alpha(\nu) = e^{-\nu}\,\xi^\alpha \tag{43}$$

This choice is physically motivated by the fact that the last neutrinos to reach equilibrium are those with the smallest energies, so we expect anisotropic deviations to become smaller and smaller for increasing neutrino energies. The amplitudes ξ^α can be found by inverting Eq. (36):

$$\xi^\alpha = \frac{H^\alpha}{\int d\nu \frac{4\pi}{3}\nu^3 e^{-\nu}}. \tag{44}$$

Another possibility is choosing a polynomial, $f_1^\alpha = \nu^{-\gamma}\xi^\alpha$ (with $\gamma \geq 4$ in order to have convergent integrals, e.g. $f_1^\alpha = \nu^{-5}\xi^\alpha$), however this would alter also the low-energy part of the integral, differently from the exponential case. By choosing the functional form defined in Eq. (43), the second absorption coefficient becomes:

$$\begin{aligned} \kappa_a'' &= \frac{\sqrt{\left(\int d\nu \kappa_{(\nu)}\nu^3\,\xi^\alpha e^{-\nu}\int d\Omega l_\alpha l_\beta\right)\left(\int d\nu' \kappa_{(\nu')}\nu'^3 \xi_\alpha e^{-\nu'}\int d\Omega' l'^\alpha l'^\beta\right)}}{H} = \\ &= \sqrt{\frac{\left(\int d\nu \kappa_{(\nu)}\nu^3\,\xi^\alpha e^{-\nu}\int d\Omega l_\alpha l_\beta\right)\left(\int d\nu' \kappa_{(\nu')}\dots l'^\beta\right)}{\left(\int d\nu \nu^3\,\xi^\alpha e^{-\nu}\int d\Omega l_\alpha l_\beta\right)\left(\int d\nu' \nu'^3 \dots l'^\beta\right)}}. \end{aligned} \tag{45}$$

Please note that in Eq. (45) the angular integration of the zeroth-order term of the f-expansion vanishes because of Eq. (2).

5 Optically thin limit

On the other hand, in the optically thin limit the radiation propagates with the speed of light with a flow that is pointed to a null direction in any frame. For such region, the distribution function may be written in the form:

$$f(\nu, \Omega, x^\mu) = 4\pi f_{\mathrm{f}}(\nu, x^\mu)\delta(\Omega - \Omega_{\mathrm{f}}), \tag{46}$$

where Ω_{f} denotes the flow direction in the fluid rest frame and $f_{\mathrm{f}}(\nu, x^\mu)$ is the partial distribution function of $\Omega = \Omega_f$. In this limit, the radiation moments are given by:

$$J_{(\nu)} = 4\pi\nu^3 f_{\mathrm{f}}(\nu), \tag{47}$$

$$H^\alpha_{(\nu)} = 4\pi\nu^3 f_{\mathrm{f}}(\nu) l^\alpha_{\mathrm{f}} = l^\alpha_{\mathrm{f}} J_{(\nu)}, \tag{48}$$

where l^α_{f} denotes the unit vector of the flow direction (observed in the fluid-rest frame). The corresponding energy-integrated quantities are:

$$n = \int_0^{+\infty} d\nu\, 4\pi\nu^2 f_{\mathrm{f}}(\nu), \tag{49}$$

$$J = \int_0^{+\infty} d\nu\, 4\pi\nu^3 f_{\mathrm{f}}(\nu), \tag{50}$$

$$H^\alpha = \int_0^{+\infty} d\nu\, 4\pi\nu^3 f_{\mathrm{f}}(\nu) l^\alpha_{\mathrm{f}} = l^\alpha_{\mathrm{f}} J. \tag{51}$$

By squaring Eq. (51), one obtains the following relation between the radiation energy density J and the radiation flux H^α:

$$H^2 = H^\alpha H_\alpha = \underbrace{l^\alpha_{\mathrm{f}} l_{\mathrm{f}\alpha}}_{l^2_{\mathrm{f}}=1} J^2 = J^2. \tag{52}$$

At this stage we follow a similar procedure to the one seen for the optically thick limit, i.e. using Eq. (49) and Eq. (50) we fix the parameters entering the ansatz proposed for the functional shape of f_{f}. Following *paper*, we model the magnitude of the distribution function as a Maxwell-like distribution (multiplied by a polynomial):

$$f_{\mathrm{f}}(\nu; C) = \nu^A e^{-B\nu}, \tag{53}$$

where the coefficients $C = \{A, B\}$ are fixed by the values of n, J:

$$(n, J) \Longrightarrow f_{\mathrm{f}}(\nu; C). \tag{54}$$

An alternative approach would consist in taking a Fermi-Dirac like distribution:

$$f_{\mathrm{f}}(\nu; C) = \frac{1}{\exp{(A\nu - B)} + 1}, \tag{55}$$

however, this should be equivalent to Eq. (53) in the limit of optically thin conditions, i.e. $\nu/(k_B T) \gtrsim 3-4 \gg 1$ and $\mu_\nu = 0$. After having determined f_{f}, the four components of l^α_{f} can be simply found by inverting Eq. (51):

$$l^\alpha_{\mathrm{f}} = \frac{H^\alpha}{J} = \frac{H^\alpha}{\int d\nu\, 4\pi\nu^3 f_{\mathrm{f}}(\nu; C)}. \tag{56}$$

Therefore, the first and second absorption coefficients can be calculated as (recall that $l^\alpha_{\mathrm{f}} l_{\mathrm{f}\alpha} = 1$ and $H^\alpha H_\alpha = J^2$):

$$\kappa'_a = \frac{\int d\nu\, \kappa_{(\nu)} \nu^3 4\pi f_{\mathrm{f}}(\nu; C)}{J} = \frac{\int d\nu\, \kappa_{(\nu)} \nu^3 f_{\mathrm{f}}(\nu; C)}{\int d\nu\, \nu^3 f_{\mathrm{f}}(\nu; C)} = \kappa''_a. \tag{57}$$

Therefore in the optically thin limit the two absorption coefficients are equivalent. The neutrino number absorption coefficient is instead:

$$\kappa^0_a = \frac{\int d\nu\, \kappa_{(\nu)} \nu^2 4\pi f_{\mathrm{f}}(\nu; C)}{n} = \frac{\int d\nu\, \kappa_{(\nu)} \nu^2 f_{\mathrm{f}}(\nu; C)}{\int d\nu\, \nu^2 f_{\mathrm{f}}(\nu; C)}. \tag{58}$$