Neutrino rates - Documentation

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Contents

1 Notation and useful comments

The relativistic chemical potential of a given particle species is defined as:

$$\mu_i = m_i c^2 + \mu_i^0,$$

where m_i is the mass of the particle and μ_i^0 is typically referred to as the non-relativistic part of the chemical potential. For the tables containing the opacity results, we will stick to the following convention:

- $\hat{\mu} \equiv \mu_n^0 - \mu_p^0$: difference between the non-relativistic part of neutron and proton chemical potentials
- μ_e: relativistic chemical potential of electrons (including rest mass contribution)

In this documentation we will assume $h = c = 1$, unless otherwise specified.

2 Boltzmann transport equation

Under the assumption of spherical symmetry TODO: Check this, the Boltzmann transport equation (BTE) for a given neutrino type reads (Eq. (A1) in [BR85]):

$$\frac{d}{dt}[f_\nu(E_\nu)] = B_{\mathrm{AE}}[f_\nu] + B_{\mathrm{NES}}[f_\nu] + B_{\mathrm{IS}}[f_\nu] + B_{\mathrm{TP}}[f_\nu, \tilde{f}_\nu], \tag{1}$$

where $f = f(t, r, \mu, E_\nu)$ is the neutrino distribution function and E_ν is the neutrino energy. The terms on the right-hand side of Eq. (1) are the neutrino source terms, constructed under the assumption that matter constituents are not spatially correlated. Respectively, B_{AE} corresponds to reactions involving absorption and emission of neutrinos, B_{NES} corresponds to neutrino-electron scattering, B_{IS} corresponds to isoenergetic neutrino scattering (on protons, neutrons or nuclei, for which the energy exchange is assumed to be zero) and B_{TP} corresponds to thermal production and absorption of neutrino-antineutrino pairs.

3 Neutrino-matter interactions

3.1 (Anti)neutrino absorption and emission

3.1.1 Electron neutrino absorption on free neutrons

$$\nu_e + n \rightleftarrows e^- + p$$

The source term in Eq. (1) related to the absorption and emission of neutrino reads (Eq. (A5) in [BR85]):

$$B_{\mathrm{AE}}[f_{\nu_e}] = j(E_\nu)(1 - f_{\nu_e}(E_\nu)) - \kappa(E_\nu)\, f_{\nu_e}(E_\nu). \tag{2}$$

where j is the so-called emissivity and κ is the opacity (proportional to the inverse mean free path $1/\lambda$). By integrating Eq. (1) over the electron neutrino phase space, on the left-hand side we find the total time derivative of the electron-neutrino number density:

$$\int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} \frac{d}{dt} f_{\nu_e}(E_\nu) = \frac{d}{dt} \underbrace{\int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} f_{\nu_e}(E_\nu)}_{n_{\nu_e}} = \frac{d\, n_{\nu_e}}{dt}, \tag{3}$$

while for the emission/absorption term on the right-hand side we obtain:

$$\int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} B_{\mathrm{AE}}[f_{\nu_e}] = \int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} \left[j(E_\nu)(1 - f_{\nu_e}(E_\nu)) - \kappa(E_\nu)\, f_{\nu_e}(E_\nu)\right]. \tag{4}$$

Eq. (4) corresponds also to the net collision term for the process $\nu_e n \leftrightarrow e^- p$ defined in Burrows *et al.* (2006), namely Eq. (2) in [B06]:

$$\begin{aligned}\mathcal{C}_{\nu_e n \leftrightarrow e^- p} = \int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} \int \frac{d^3\mathbf{p}_e}{(2\pi)^3} \int \frac{d^3\mathbf{p}_n}{(2\pi)^3} \int \frac{d^3\mathbf{p}_p}{(2\pi)^3} \left(\sum_s |\mathcal{M}_{\mathrm{B06}}|^2\right) \times \\ \times\, \Xi(\nu_e n \leftrightarrow e^- p)(2\pi)^4 \delta^4(p_{\nu_e} + p_n - p_p - p_e),\end{aligned} \tag{5}$$

where $\sum_s |\mathcal{M}_{\mathrm{B06}}|^2$ is the square modulus of the matrix element $\mathcal{M}$ (summed over the possible final states and average over the initial ones?) while Ξ contains the distribution function of the particles in the initial state and the Pauli blocking factors for the particles in the final state for the direct and the inverse reaction, respectively:

$$\Xi(\nu_e n \leftrightarrow e^- p) = f_e\, f_p\, (1 - f_n)\, (1 - f_{\nu_e}) - f_{\nu_e}\, f_n\, (1 - f_e)\, (1 - f_p). \tag{6}$$

By substituting the two terms of Eq. (6) in Eq. (5) we can distinguish the two separate contributions of the neutrino emitting and absorbing reactions to the net collision term:

$$
\begin{aligned}
\mathcal{C}_{\nu_e n \leftrightarrow e^- p} = \int \frac{d^3\mathbf{p}_\nu}{(2\pi)^3} \Bigg[(1-f_{\nu_e}) \int \frac{d^3\mathbf{p}_e}{(2\pi)^3} \int \frac{d^3\mathbf{p}_n}{(2\pi)^3} \int \frac{d^3\mathbf{p}_p}{(2\pi)^3} \left(\sum_s |\mathcal{M}_{\mathrm{B06}}|^2 \right) \times \\
\times \, (2\pi)^4 \delta^4(p_{\nu_e} + p_n - p_p - p_e) f_e \, f_p \, (1-f_n) + \\
+ \, f_{\nu_e} \int \frac{d^3\mathbf{p}_e}{(2\pi)^3} \int \frac{d^3\mathbf{p}_n}{(2\pi)^3} \int \frac{d^3\mathbf{p}_p}{(2\pi)^3} \left(\sum_s |\mathcal{M}_{\mathrm{B06}}|^2 \right) \times \\
\times \, (2\pi)^4 \delta^4(p_{\nu_e} + p_n - p_p - p_e) f_n \, (1-f_e) \, (1-f_p) \Bigg].
\end{aligned}
\tag{7}
$$

By comparing Eq. (4) with Eq. (7), we find how to compute the electron neutrino emissivity and opacity for the reaction $\nu_e n \leftrightarrow e^- p$:

$$
\begin{Bmatrix} j(E_\nu) \\ 1/\lambda(E_\nu) \end{Bmatrix} = \int \frac{d^3\mathbf{p}_e}{(2\pi)^3} \int \frac{d^3\mathbf{p}_n}{(2\pi)^3} \int \frac{d^3\mathbf{p}_p}{(2\pi)^3} \left(\sum_s |\mathcal{M}_{\mathrm{B06}}|^2 \right) (2\pi)^4 \delta^4(p_\nu + p_n - p_e - p_p) \begin{Bmatrix} f_e \, f_p \, (1-f_n) \\ f_n \, (1-f_e) \, (1-f_p) \end{Bmatrix}.
\tag{8}
$$

Please note that Eq. (8) is the equivalent of Eq. (C4) in [BR85], it is just written in a different form. Indeed, Eq. (C4) in [BR85] reads:

$$
\begin{Bmatrix} j(E_\nu) \\ 1/\lambda^{(a)}(E_\nu) \end{Bmatrix} = \int \frac{d^3p_p}{(2\pi)^3} \int \frac{d^3p_n}{(2\pi)^3} \int \frac{d^3p_e}{(2\pi)^3} \begin{Bmatrix} 2f_p(E_p)\,[1-f_n(E_n)]\,2f_e(E_e)\, r(p_p + p_e \to p_n + p_\nu) \\ [1-f_p(E_p)]\,2f_n(E_n)\,[1-f_e(E_e)]\, r(p_n + p_\nu \to p_p + p_e) \end{Bmatrix},
\tag{9}
$$

where the rate $r(p_n + p_\nu \to p_p + p_e)$ is given by:

$$
r(p_n + p_\nu \to p_p + p_e) = \frac{1}{(2E_\nu)\,(2E_e)\,(2E_p)\,(2E_n)} (2\pi)^4 \delta^4(p_\nu + p_n - p_e - p_p) \left(\sum_s |\mathcal{M}_{\mathrm{BR85}}|^2 \right).
\tag{10}
$$

Note also that, in order to have correspondence between Eq. (8) and Eq. (9), the matrix element $\mathcal{M}$ must be defined in [BR85] and [B06] following two different notations/conventions. TODO: Check this and also form of Eq. (10)

Let's now try to compute the opacity (inverse mean free path) for the absorption of electron neutrinos on free neutrons starting from Eq. (9). Neglecting the weak magnetism contribution, the matrix element $\mathcal{M}$ describing the reaction is (let's stick to [BR85] convention for $\mathcal{M}$ from now on):

$$
\mathcal{M} = \frac{G}{\sqrt{2}} \left[\bar{u}_p(p_p) \gamma^\mu (g_V - g_A \gamma_5) u_n(p_n) \right] \left[\bar{u}_e(p_e) \gamma_\mu (1-\gamma_5) u_\nu(p_\nu) \right],
\tag{11}
$$

where G^2 is the Fermi constant $(G^2/(\hbar c)^4 = 5.18 \times 10^{-44}\,\mathrm{MeV}^{-2}\,\mathrm{cm}^2)$, $g_V = 1$ and $g_A = 1.23$ (constant, form factors in the zero momentum transfer limit). The square modulus of Eq. (11), summed over the possible final states and averaged over the initial states, is consequently equal to:

$$
\left(\sum_s |\mathcal{M}|^2 \right) = 16\,G^2 \left[(g_V + g_A)^2 \, p_p \cdot p_e \, p_n \cdot p_\nu + (g_V - g_A)^2 \, p_p \cdot p_\nu \, p_n \cdot p_e - (g_V^2 - g_A^2)\, M_n \, M_p \, p_e \cdot p_\nu \right].
\tag{12}
$$

By inserting Eq. (12) in Eq. (10), we find that the reaction rate r is equal to:

$$
\begin{aligned}
r(p_n + p_\nu \to p_p + p_e) = & \frac{G^2}{E_\nu \, E_e \, E_p \, E_n} (2\pi)^4 \delta^4(p_\nu + p_n - p_e - p_p) \times \\
& \times \left[(g_V + g_A)^2 \, p_p \cdot p_e \, p_n \cdot p_\nu + (g_V - g_A)^2 \, p_p \cdot p_\nu \, p_n \cdot p_e - (g_V^2 - g_A^2)\, M_n \, M_p \, p_e \cdot p_\nu \right] = \\
& = G^2 (2\pi)^4 \delta^4(p_\nu + p_n - p_e - p_p) \times \\
& \times \Bigg[(g_V + g_A)^2 \, \frac{E_p \, E_e - \bar{p}_p \bar{p}_e \cos\theta_{p,e}}{E_p \, E_e} \, \frac{E_\nu \, E_n - E_\nu \, \bar{p}_n \cos\theta_{n,\nu}}{E_\nu \, E_n} \\
& + \, (g_V - g_A)^2 \, \frac{E_\nu \, E_p - E_\nu \bar{p}_p \cos\theta_{p,\nu}}{E_\nu \, E_p} \, \frac{E_e \, E_n - \bar{p}_e \, \bar{p}_n \cos\theta_{n,e}}{E_e \, E_n} + \\
& + \, (g_V^2 - g_A^2) \, \frac{M_n}{E_n} \, \frac{M_p}{E_p} \, \frac{E_\nu \, E_e - E_\nu \, \bar{p}_e \cos\theta_{e,\nu}}{E_\nu \, E_e} \Bigg].
\end{aligned}
\tag{13}
$$

In the approximation where nucleons are treated as **non relativistic** particles ($|\bar{p}_p| \ll M_n$ and $|\bar{p}_n| \ll M_p$, as done in [BR85] and [B06]), we can neglect all the contributions that are of order $\bar{p}_i/M_i$ $(i = p, n)$ or higher than that, i.e.:

$$\begin{cases} \dfrac{\bar{p}}{E} = \dfrac{\bar{p}}{\sqrt{M^2+\bar{p}^2}} = \dfrac{\bar{p}}{M}\dfrac{1}{\sqrt{1+\dfrac{\bar{p}^2}{M^2}}} \simeq \dfrac{\bar{p}}{M}\left(1-\dfrac{1}{2}\dfrac{\bar{p}^2}{M^2}\right) \sim \mathcal{O}\left(\dfrac{\bar{p}}{M}\right), \\ \dfrac{M}{E} = \dfrac{M}{\sqrt{M^2+\bar{p}^2}} = \dfrac{1}{\sqrt{1+\dfrac{\bar{p}^2}{M^2}}} \simeq \left(1-\dfrac{1}{2}\dfrac{\bar{p}^2}{M^2}\right) \simeq 1. \end{cases} \tag{14}$$

In this approximation the reaction rate can be therefore simplified to:

$$\begin{aligned} r(p_n+p_\nu \to p_p+p_e) = & G^2(2\pi)^4\delta^4(p_\nu+p_n-p_e-p_p)\ \Bigg[(g_V+g_A)^2\left(1-\cancel{\frac{\bar{p}_p\bar{p}_e}{E_pE_e}}\cos\theta_{p,e}\right)\left(1-\cancel{\frac{\bar{p}_n}{E_n}}\cos\theta_{n,\nu}\right)+ \\ & (g_V-g_A)^2\left(1-\cancel{\frac{\bar{p}_p}{E_p}}\cos\theta_{p,\nu}\right)\left(1-\cancel{\frac{\bar{p}_e\bar{p}_n}{E_eE_n}}\cos\theta_{n,e}\right)+ \\ & (g_V^2-g_A^2)\frac{M_n}{E_n}\frac{M_p}{E_p}\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\Bigg] = \\ = & G^2(2\pi)^4\delta^4(p_\nu+p_n-p_e-p_p)\left[(g_V+g_A)^2+(g_V-g_A)^2-(g_V^2-g_A^2)\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\right] = \\ = & G^2(2\pi)^4\delta^4(p_\nu+p_n-p_e-p_p)\left[g_V^2\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)+g_A^2\left(3-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\right] \end{aligned} \tag{15}$$

and the opacity becomes:

$$\begin{aligned} \frac{1}{\lambda^{(a)}(E_\nu)} = & G^2\int\frac{d^3\mathbf{p}_e}{(2\pi)^3}[1-f_e(E_e)]\left[g_V^2\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)+g_A^2\left(3-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\right]\times \\ & \times 2\int\frac{d^3\mathbf{p}_p}{(2\pi)^3}\int\frac{d^3\mathbf{p}_n}{\cancel{(2\pi)^3}}(2\pi)^{\cancel{4}}\delta^4(p_\nu+p_n-p_e-p_p)[1-f_p(E_p)]\,f_n(E_n). \end{aligned} \tag{16}$$

Let's now consider the **elastic approximation** for capture reactions, in which the momentum transfer between leptons and nucleons does not lead to a change in absolute momentum of the nucleon, hence $\bar{p}_n = \bar{p}_p$. This is a reasonable approximation if the energy of the lepton is considerably smaller than the mass of the nucleons. In this approximation the delta function can be split into:

$$\delta^4(p_\nu+p_n-p_e-p_p) = \delta^3(\bar{p}_p-\bar{p}_n)\,\delta(E_\nu+E_n-E_p-E_e)$$

and the opacity reduces to:

$$\begin{aligned} \frac{1}{\lambda^{(a)}(E_\nu)} = & 2\pi\,G^2\int\frac{d^3\mathbf{p}_e}{(2\pi)^3}[1-f_e(E_e)]\left[g_V^2\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)+g_A^2\left(3-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\right]\times \\ & \times\int\frac{2\,d^3\mathbf{p}_p}{(2\pi)^3}\int\cancel{d^3\mathbf{p}_n\delta^3(\bar{p}_p-\bar{p}_n)}\,\delta(E_\nu+E_n-E_p-E_e)[1-f_p(E_p)]\,f_n(E_n) = \\ = & 2\pi\,G^2\int\frac{d^3\mathbf{p}_e}{(2\pi)^3}[1-f_e(E_e)]\left[g_V^2\left(1-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)+g_A^2\left(3-\frac{\bar{p}_e}{E_e}\cos\theta_{e,\nu}\right)\right]\times \\ & \times\quad\delta(E_\nu+\underbrace{M_n-M_p}_{\Delta_{np}}-E_e)\int\frac{2\,d^3\mathbf{p}}{(2\pi)^3}[1-f_p(E)]\,f_n(E). \end{aligned} \tag{17}$$

The integral over nucleon phase space in Eq. (17) can be easily solved in the approximation of non-relativistic nucleons ($E = p^2/2M$) assuming $M_n \simeq M_p \equiv M$:

$$\begin{aligned}\eta_{np} &= \int \frac{2\,d^3\mathbf{p}}{(2\pi)^3}\, f_n(E)\,[1-f_p(E)] = \\ &= \int \frac{2\,d^3\mathbf{p}}{(2\pi)^3}\, f_n(E)\, f_p(E) \exp[\beta(E-\mu_p^0)] \times \frac{\exp[\beta(\mu_p^0-\mu_n^0)]-1}{\exp[\beta(\mu_p^0-\mu_n^0)]-1} = \\ &= \int \frac{2\,d^3\mathbf{p}}{(2\pi)^3}\, f_n(E)\, f_p(E) \frac{\exp[\beta(E-\mu_n^0)] - \exp[\beta(E-\mu_p^0)]}{\exp[\beta(\mu_p^0-\mu_n^0)]-1} = \\ &= \frac{1}{\exp[\beta(\mu_p^0-\mu_n^0)]-1} \int \frac{2\,d^3\mathbf{p}}{(2\pi)^3} \big(f_p(E)\,[1-f_n(E)] - f_n(E)\,[1-f_p(E)]\big) = \\ &= \frac{1}{\exp[\beta(\mu_p^0-\mu_n^0)]-1} \int \frac{2\,d^3\mathbf{p}}{(2\pi)^3} \big(f_p(E) - f_n(E)\big) = \frac{n_p - n_n}{\exp[\beta\hat{\mu}]-1},\end{aligned} \tag{18}$$

where we have defined $\hat{\mu} \equiv \mu_n^0 - \mu_p^0$ as the difference between the non-relativistic part of neutron and proton chemical potentials. The calculation above makes use of the following property:

$$1 - f(E) = \frac{\exp[\beta(E-\mu^0)]}{\exp[\beta(E-\mu^0)]+1} = f(E)\,\exp[\beta(E-\mu^0)]. \tag{19}$$

Considering now the reference frame in which the electron neutrino travels along the z-direction ($\cos\theta_{e,\nu} \to \cos\theta$) and performing the integral over

$$d^3\mathbf{p}_e = d\phi\, d\cos\theta\, d\bar{p}_e\, \bar{p}_e^2 = 2\pi\, d\cos\theta\, dE_e E_e\, \bar{p}_e = 2\pi\, d\cos\theta\, dE_e E_e^2 \sqrt{1 - \frac{m_e^2}{E_e^2}},$$

one finally ends up with the following expression for the opacity of electron neutrinos **in the elastic approximation for non relativistic nucleons**:

$$\begin{aligned}\frac{1}{\lambda^{(a)}(E_\nu)} &= 2\pi G^2\, \eta_{np} \int \frac{d^3 p_e}{(2\pi)^3}\, [1-f_e(E_e)]\, \delta(E_\nu + \Delta_{np} - E_e) \left[g_V^2\left(1 - \frac{\bar{p}_e}{E_e}\cos\theta\right) + g_A^2\left(3 - \frac{\bar{p}_e}{E_e}\cos\theta\right)\right] = \\ &= 2\pi G^2\, \eta_{np} \frac{2\pi}{(2\pi)^3} \int_{m_e}^{+\infty} dE_e\, E_e^2 \sqrt{1-\frac{m_e^2}{E_e^2}}\, [1-f_e(E_e)]\delta(E_\nu+\Delta_{np}-E_e) \times \\ &\qquad \times \int_{-1}^{+1} d\cos\theta \left[g_V^2\left(1 - \frac{\bar{p}_e}{E_e}\cancel{\cos\theta}\right) + g_A^2\left(3 - \frac{\bar{p}_e}{E_e}\cancel{\cos\theta}\right)\right] = \\ &= \frac{G^2}{\cancel{2}\pi}\, \eta_{np}\, \cancel{2}\,(g_V^2 + 3g_A^2) \int_{m_e}^{+\infty} dE_e\, E_e^2 \sqrt{1-\frac{m_e^2}{E_e^2}}\, [1-f_e(E_e)]\, \delta(E_\nu+\Delta_{np}-E_e).\end{aligned} \tag{20}$$

$$\boxed{\frac{1}{\lambda^{(a)}(E_\nu)} = \frac{G^2}{\pi}\, \eta_{np}\, (g_V^2 + 3g_A^2)\, (E_\nu + \Delta_{np})^2 \sqrt{1 - \frac{m_e^2}{(E_\nu+\Delta_{np})^2}}\, [1 - f_e(E_\nu + \Delta_{np})] \qquad (21)}$$

TODO: Include weak magnetism in Eq. (21) TODO: Write relation with emissivity

Stimulated absorption

The factor $\Xi(\nu_e n \leftrightarrow e^- p)$ entering the collision term (Eq. (6)) can be written, after some algebraic manipulations (e.g. Eq. (19)), as:

$$
\begin{aligned}
\Xi(\nu_e n \leftrightarrow e^- p) &= f_e\, f_p\,(1-f_n)\,(1-f_{\nu_e}) - f_{\nu_e}\, f_n\,(1-f_e)\,(1-f_p) = \\
&= f_n\,(1-f_e)\,(1-f_p)\left[\frac{f_e}{1-f_e}\frac{f_p}{1-f_p}\frac{1-f_n}{f_n}(1-f_{\nu_e}) - f_{\nu_e}\right] = \\
&= f_n\,(1-f_e)\,(1-f_p)\left[\frac{\exp\left[\beta(E_n-\mu_n)\right]}{\exp\left[\beta(E_e-\mu_e)\right]\,\exp\left[\beta(E_p-\mu_p)\right]}(1-f_{\nu_e}) - f_{\nu_e}\right] = \\
&= f_n\,(1-f_e)\,(1-f_p)\left[\frac{1}{\exp[\beta(\underbrace{E_n-E_e-E_p}_{E_\nu^{\rm eq}}) - (\underbrace{\mu_e-\mu_n+\mu_p}_{\mu_{\nu_e}^{\rm eq}})]}(1-f_{\nu_e}) - f_{\nu_e}\right] = \\
&= f_n\,(1-f_e)\,(1-f_p)\left[\frac{1}{\exp\left[\beta(E_\nu^{\rm eq}-\mu_{\nu_e}^{\rm eq})\right]}(1-f_{\nu_e}) - f_{\nu_e}\right] = \\
&= f_n\,(1-f_e)\,(1-f_p)\left[\frac{f_{\nu_e}^{\rm eq}}{1-f_{\nu_e}^{\rm eq}}(1-f_{\nu_e}) - f_{\nu_e}\right] = \\
&= \frac{f_n\,(1-f_e)\,(1-f_p)}{1-f_{\nu_e}^{\rm eq}}\left[f_{\nu_e}^{\rm eq} - f_{\nu_e}\right],
\end{aligned}
\tag{22}
$$

where

$$
f_{\nu_e}^{\rm eq} = \frac{1}{\exp\left[\beta(E_\nu^{\rm eq}-\mu_\nu^{\rm eq})\right]+1}.
$$

Using the formalism of [B06], the net emission of a neutrino is given by the difference between the emissivity and the absorption of the medium:

$$
\mathcal{J}_{\rm net} = \eta_\nu - \kappa_a I_\nu, \tag{23}
$$

where κ_a is the so-called absorptive opacity and $\mathcal{I}_\nu$ the specific intensity (directly proportional to the neutrino distribution function). All absorption processes involving fermions are inhibited by Pauli blocking due to final-state occupancy. Hence, η_ν in Eq. (23) includes the blocking term $(1-f_{\nu_e})$. Please note that Eq. (23) is analogous to Eq. (2) in the formalism depicted above and recall that Eq. (5) (the one containing the Ξ factor) is obtained by integrating Eq. (2) over the neutrino phase space. Therefore, by looking at the shape of Eq. (22), one can rewrite Eq. (23) in a way that makes explicit that I_ν is manifestly driven to B_ν, the equilibrium intensity (stimulated absorption):

$$
\mathcal{J}_{\rm net} = \frac{\kappa_a}{1-f_\nu^{\rm eq}}(B_\nu - I_\nu) = \kappa_a^*(B_\nu - I_\nu). \tag{24}
$$

From Eq. (24) it is evident that the stimulated absorption correction to κ_a reads:

$$
\kappa_a^* = \frac{\kappa_a}{1-f_{\nu_e}^{\rm eq}}. \tag{25}
$$

Burrows *et al.* (2006) calculation

The cross section per baryon for ν_e neutrino absorption on free neutrons is equal to:

$$
\sigma_{\nu_e n}^a = \sigma_0\left(\frac{1+3g_A^2}{4}\right)\left(\frac{E_\nu+\Delta_{np}}{m_e c^2}\right)^2\left[1-\left(\frac{m_e c^2}{E_\nu+\Delta_{np}}\right)\right]^{1/2} W_M, \tag{26}
$$

where $g_A \sim -1.23$ is the axial-vector coupling constant, $\Delta_{np} = m_n c^2 - m_p c^2$ and $E_e = E_\nu + \Delta_{np}$, assuming that the electron carries away all of the kinetic energy. σ_0 is a convenient reference neutrino cross section:

$$
\sigma_0 = \frac{4G_F^2\cos^2\theta_C (m_e c^2)^2}{\pi(\hbar c)^4} \simeq 1.705\times 10^{-44}\,\mathrm{cm}^2,
$$

where $G_F \simeq 1.436\times 10^{-49}\,\mathrm{erg\,cm}^{-3}$ is the Fermi weak coupling constant. TODO: Compare this with [BR85] value. The weak magnetism correction in Eq. (26) is taken from [V84] and for small neutrino energies is equal to:

$$
W_M = 1 + 1.1\frac{E_\nu}{m_n c^2}.
$$

We note that Eq. (26) (Eq. (10) of [B06]) resembles Eq. (21) (Eq. (C13) of [BR85]) except for the absence of the electron blocking factor and of the term coming from the integration over the nucleon phase space (η_{np}). This similarity suggests us that Eq. (26) is also derived in the **elastic approximation** assuming **non-relativistic nucleons**, even if not explicitely stated in [B06].

As written in Section 3.1 of [B06], in order to calculate κ_a^*, $\sigma^a_{\nu_e n}$ must be multiplied by the stimulated absorption correction, $1/(1-f_\nu^{\rm eq})$ (not necessary if one wants to compute directly κ_a), the final-state electron blocking term and the number density of target particles ($n_n = X_n \rho N_A$ TODO: Check units):

$$\kappa_a^* = \sigma^a_{\nu_e n} X_n \rho N_A (1-f_\nu^{\rm eq}) \qquad\qquad \kappa_a = (1-f_\nu^{\rm eq})\,\kappa_a^*, \tag{27}$$

where N_A is the Avogadro's number, X_n is the neutron fraction and ρ is the total mass density (see also Section B.1.2 of [TH02], where $(1-f_e)$ is already included in the definition of $\sigma^a_{\nu_e n}$). Please note that Eq. (27) holds only when nucleons are non-degenerate. At high densities nucleons are no longer non-degenerate and $X_n\rho N_A$ should be replaced by the η_{np} factor (Eq. (18)), which takes into account the effects of final state nucleon blocking. By doing this, one ends up with the same result for the opacity as the one obtained following [BR85] derivation (Eq. (21)).

3.1.2 Electron antineutrino absorption on free protons

TODO: Write paragraph on antineutrino absorption

3.2 Isoenergetic neutrino scattering

The source term associated with neutrino scattering on nucleons (or nuclei) appearing on the right-hand side of Eq. (1) has the following form:

$$\begin{aligned} B_{\rm IS}[f_\nu] &= (1-f_\nu)\,\tilde\eta_\nu^{\rm scatt} - f_\nu\,\tilde\chi_\nu^{\rm scatt} = \\ &= [1-f_\nu(\mu,E_\nu)] \int \frac{d^3\mathbf{p}'_\nu}{c(2\pi\hbar c^3)}\, f_\nu(\mu',E'_\nu)\, R^{\rm in}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) + \\ &\quad - f_\nu(\mu,E_\nu) \int \frac{d^3\mathbf{p}'_\nu}{c(2\pi\hbar c^3)}\, [1-f_\nu(\mu',E'_\nu)]\, R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta), \end{aligned} \tag{28}$$

where $\tilde\eta_\nu^{\rm scatt}$ and $\tilde\chi_\nu^{\rm scatt}$ are respectively the emissivity and the opacity associated with neutrino scattering on nucleons (nuclei):

$$\begin{Bmatrix} \tilde\eta_\nu^{\rm scatt} \\ \tilde\chi_\nu^{\rm scatt} \end{Bmatrix} \equiv \int \frac{d^3\mathbf{p}'_\nu}{c(2\pi\hbar c^3)} \begin{Bmatrix} f_\nu(\mu',E'_\nu)\, R^{\rm in}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) \\ [1-f_\nu(\mu',E'_\nu)]\, R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) \end{Bmatrix}. \tag{29}$$

$R^{\rm in}_{\rm IS}(E_\nu,E'_\nu,\cos\theta)$ is the scattering kernel for scattering into the bin (E_ν,μ) from any bin (E'_ν,μ') while $R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta)$ is the scattering kernel for scattering out of the bin (E_ν,μ) to any bin (E'_ν,μ'). The two scattering kernels are related through detailed balance in the following way:

$$R^{\rm in}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) = e^{-\beta\omega} R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) \qquad\qquad \omega \equiv E_\nu - E'_\nu. \tag{30}$$

Therefore, we need deal only with $R^{\rm out}_{\rm IS}$. θ instead is the angle between the incident and emergent neutrino and is given by (ϕ is the azimuthal angle of one neutrino relative to the other):

$$\cos\theta = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos\phi. \tag{31}$$

One generally approximates the angular dependence of the scattering kernel with a truncated Legendre series:

$$R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) = \sum_{l=0}^{\infty} \frac{2l+1}{2}\Phi_l(E_\nu,E'_\nu)\,P_l(\cos\theta) \simeq \frac{1}{2}\Phi_0(E_\nu,E'_\nu) + \frac{3}{2}\Phi_1(E_\nu,E'_\nu)\cos\theta, \tag{32}$$

where Φ_l are the coefficients of the expansion:

$$\phi_l(E_\nu,E'_\nu) = \int_{-1}^{+1} d\cos\theta\, R^{out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta)\,P_l(\cos\theta). \tag{33}$$

By substituting Eq. (30) and Eq. (32) into Eq. (28), we find:

$$\begin{aligned} B_{\rm IS}[f_\nu] &= [1-f_\nu(\mu,E_\nu)] \int_0^{+\infty} \frac{dE'_\nu\,{E'_\nu}^2}{c(2\pi\hbar c^3)} e^{-\beta\omega} \int_{-1}^{+1} d\mu'\, f_\nu(\mu',E'_\nu) \int_0^{2\pi} d\phi \left[\frac{1}{2}\Phi_0(E_\nu,E'_\nu) + \frac{3}{2}\Phi_1(E_\nu,E'_\nu)\cos\theta\right] + \\ &\quad - f_\nu(\mu,E_\nu) \int_0^{+\infty} \frac{dE'_\nu\,{E'_\nu}^2}{c(2\pi\hbar c^3)} \int_{-1}^{+1} d\mu'\, [1-f_\nu(\mu',E'_\nu)] \int_0^{2\pi} d\phi \left[\frac{1}{2}\Phi_0(E_\nu,E'_\nu) + \frac{3}{2}\Phi_1(E_\nu,E'_\nu)\cos\theta\right] \end{aligned} \tag{34}$$

Let's now express $\cos\theta$ as in Eq. (31) and integrate over angular variables:

$$
\begin{aligned}
B_{\rm IS}[f_\nu] =& [1-f_\nu(\mu,E_\nu)] \int_0^{+\infty} \frac{dE'_\nu\, {E'_\nu}^2}{c(2\pi\hbar c^3)} e^{-\beta\omega} \int_{-1}^{+1} d\mu'\, f_\nu(\mu',E'_\nu) \int_0^{2\pi} d\phi \left[\frac{1}{2}\Phi_0 + \frac{3}{2}\Phi_1\big(\mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cancel{\cos\phi}\big)\right] + \\
& - f_\nu(\mu,E_\nu) \int_0^{+\infty} \frac{dE'_\nu\, {E'_\nu}^2}{c(2\pi\hbar c^3)} \int_{-1}^{+1} d\mu'\, [1-f_\nu(\mu',E'_\nu)] \int_0^{2\pi} d\phi \left[\frac{1}{2}\Phi_0 + \frac{3}{2}\Phi_1\big(\mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cancel{\cos\phi}\big)\right] = \\
& = [1-f_\nu(\mu,E_\nu)] \frac{2\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega} \int_{-1}^{+1} d\mu'\, f_\nu(\mu',E'_\nu) \left[\frac{1}{2}\Phi_0 + \frac{3}{2}\Phi_1\mu\mu'\right] + \\
& - f_\nu(\mu,E_\nu) \frac{2\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 \int_{-1}^{+1} d\mu'\, [1-f_\nu(\mu',E'_\nu)] \left[\frac{1}{2}\Phi_0 + \frac{3}{2}\Phi_1\mu\mu'\right] = \\
& = [1-f_\nu(\mu,E_\nu)] \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega} \left[\frac{1}{2}\Phi_0 \tilde{J}'_\nu + \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right] + \\
& - f_\nu(\mu,E_\nu) \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 \left[\frac{1}{2}\Phi_0 (1-\tilde{J}'_\nu) - \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right],
\end{aligned}
\tag{35}
$$

where we have defined:

$$
\tilde{J}_\nu \equiv \frac{1}{2}\int_{-1}^{+1} d\mu\, f_\nu(\mu,E_\nu) \qquad \tilde{H}_\nu \equiv \frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\, f_\nu(\mu,E_\nu) \qquad \tilde{P}_\nu \equiv \frac{3}{2}\int_{-1}^{+1} d\mu\,\mu^2\, f_\nu(\mu,E_\nu). \tag{36}
$$

The contribution to the **zeroth moment transport equation** (integrate the source term over the zenith angle of the unprimed neutrino) would therefore be:

$$
\begin{aligned}
\frac{1}{2}\int_{-1}^{+1} d\mu\, B_{\rm IS}[f_\nu] =& \frac{1}{2}\int_{-1}^{+1} d\mu\, [1-f_\nu(\mu,E_\nu)] \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega} \left[\frac{1}{2}\Phi_0 \tilde{J}'_\nu + \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right] + \\
& - \frac{1}{2}\int_{-1}^{+1} d\mu\, f_\nu(\mu,E_\nu) \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 \left[\frac{1}{2}\Phi_0 (1-\tilde{J}'_\nu) - \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right] = \\
& = \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega} \left[\frac{1}{2}\Phi_0 (1-\tilde{J}_\nu)\tilde{J}'_\nu - \frac{3}{2}\Phi_1\tilde{H}_\nu\tilde{H}'_\nu\right] + \\
& - \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 \left[\frac{1}{2}\Phi_0 \tilde{J}_\nu(1-\tilde{J}'_\nu) - \frac{3}{2}\Phi_1\tilde{H}_\nu\tilde{H}'_\nu\right] = \\
& = \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 \left[\frac{1}{2}\Phi_0 \big[e^{-\beta\omega}(1-\tilde{J}_\nu)\tilde{J}'_\nu - \tilde{J}_\nu(1-\tilde{J}'_\nu)\big] - \frac{3}{2}\Phi_1\tilde{H}_\nu\tilde{H}'_\nu(e^{-\beta\omega}-1)\right].
\end{aligned}
\tag{37}
$$

In the isoenergetic approximation (zero energy exchange between neutrino and nucleon/nucleus, $\Longrightarrow \omega = 0$), we can write the scattering kernel as:

$$
R^{\rm in}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) = R^{\rm out}_{\rm IS}(E_\nu,E'_\nu,\cos\theta) \equiv R^0_{\rm IS}(E_\nu,E'_\nu,\cos\theta)\,\delta(E_\nu - E'_\nu) \tag{38}
$$

and define an analogous Legendre expansion for the isoenergetic part of the scattering kernel:

$$
R^0_{\rm IS}(E_\nu,E'_\nu,\cos\theta) = \frac{1}{2}\tilde{\Phi}_0(E_\nu,E'_\nu) + \frac{3}{2}\tilde{\Phi}_1(E_\nu,E'_\nu) \tag{39}
$$

where:

$$
\begin{cases}
\Phi_0(E_\nu,E'_\nu) = \tilde{\Phi}_0(E_\nu,E'_\nu)\,\delta(E_\nu - E'_\nu) \\
\Phi_1(E_\nu,E'_\nu) = \tilde{\Phi}_1(E_\nu,E'_\nu)\,\delta(E_\nu - E'_\nu).
\end{cases}
\tag{40}
$$

By inserting Eq. (40) into Eq. (37) and integrating over dE'_ν, we can rewrite the source for the zeroth moment of the transport equation as:

$$
\begin{aligned}
\frac{1}{2}\int_{-1}^{+1} d\mu\, B_{\rm IS}[f_\nu] =& \frac{4\pi}{c(2\pi\hbar c^3)} \int_0^{+\infty} \cancel{dE'_\nu}\, {E'_\nu}^2 \left[\frac{1}{2}\tilde{\Phi}_0 \big[e^{-\beta\omega}(1-\tilde{J}_\nu)\tilde{J}'_\nu - \tilde{J}_\nu(1-\tilde{J}'_\nu)\big] + \right. \\
& \left. - \frac{3}{2}\tilde{\Phi}_1\tilde{H}_\nu\tilde{H}'_\nu(e^{-\beta\omega}-1)\right] \cancel{\delta(E_\nu - E'_\nu)} = \\
& = \frac{4\pi}{c(2\pi\hbar c^3)} E_\nu^2 \left[\frac{1}{2}\tilde{\Phi}_0(\tilde{J}'_\nu - \tilde{J}_\nu)\right].
\end{aligned}
\tag{41}
$$

If we expand also the neutrino occupation probability in a Legendre series keeping only the first two terms:

$$f_\nu(\mu, E_\nu) = \Psi^{(0)}(E_\nu) + \mu\Psi^{(1)}(E_\nu), \tag{42}$$

the angular integrals $\tilde{J}_\nu$ and $\tilde{H}_\nu$ become:

$$\tilde{J}_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu f_\nu(\mu, E_\nu) \simeq \frac{1}{2}\int_{-1}^{+1} d\mu\big[\Psi^{(0)}(E_\nu) + \cancel{\mu\Psi^{(1)}(E_\nu)}\big] = \Psi^{(0)}(E_\nu), \tag{43}$$

$$\tilde{H}_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\, f_\nu(\mu, E_\nu) \simeq \frac{1}{2}\int_{-1}^{+1} d\mu\,\mu\,\big[\cancel{\Psi^{(0)}(E_\nu)} + \mu\Psi^{(1)}(E_\nu)\big] = \frac{1}{3}\Psi^{(1)}(E_\nu). \tag{44}$$

By doing this the contribution of neutrino scattering on nucleons to the zeroth order equation vanishes:

$$\frac{1}{2}\int_{-1}^{+1} d\mu\, B_{\rm IS}[f_\nu] = \frac{4\pi}{c(2\pi\hbar c^3)}\, E_\nu^2\Big[\frac{1}{2}\tilde{\Phi}_0\underbrace{(\tilde{J}'_\nu - \tilde{J}_\nu)}_{\Psi^{(0)}-\Psi^{(0)}=0}\Big] = 0. \tag{45}$$

The contribution to the **first order transport equation** (multiply the source term by the cosine of the zenith angle of the unprimed neutrino and integrate over it) is instead:

$$\begin{aligned}
\frac{3}{2}\int_{-1}^{+1} d\mu\,\mu\, B_{\rm IS}[f_\nu] &= \frac{3}{2}\int_{-1}^{+1} d\mu\,\mu\,[\cancel{1} - f_\nu(\mu, E_\nu)]\frac{4\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega}\left[\frac{1}{2}\Phi_0\tilde{J}'_\nu + \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right] + \\
&\quad -\frac{3}{2}\int_{-1}^{+1} d\mu\,\mu\, f_\nu(\mu, E_\nu)\frac{4\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} dE'_\nu\, {E'_\nu}^2\left[\frac{1}{2}\Phi_0(1-\tilde{J}'_\nu) - \frac{3}{2}\Phi_1\mu\tilde{H}'_\nu\right] = \\
&= \frac{4\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} dE'_\nu\, {E'_\nu}^2 e^{-\beta\omega}\left[\frac{1}{2}\Phi_0(-3\tilde{H}_\nu)\tilde{J}'_\nu + \frac{3}{2}\Phi_1(1-\tilde{P}_\nu)\tilde{H}'_\nu\right] + \\
&\quad -\frac{4\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} dE'_\nu\, {E'_\nu}^2\left[\frac{1}{2}\Phi_0(3\tilde{H}_\nu)(1-\tilde{J}'_\nu) - \frac{3}{2}\Phi_1\tilde{P}_\nu\tilde{H}'_\nu\right] = \\
&= \frac{2\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} dE'_\nu\, {E'_\nu}^2\Big[-3\Phi_0\tilde{H}_\nu\big[e^{-\beta\omega}\tilde{J}'_\nu + (1-\tilde{J}'_\nu)\big] + 3\Phi_1\tilde{H}'_\nu\big[e^{-\beta\omega}(1-\tilde{P}_\nu) - \tilde{P}_\nu\big]\Big].
\end{aligned} \tag{46}$$

In the isoenergetic approximation Eq. (46) simplifies to:

$$\begin{aligned}
\frac{3}{2}\int_{-1}^{+1} d\mu\,\mu\, B_{\rm IS}[f_\nu] &= \frac{2\pi}{c(2\pi\hbar c^3)}\int_0^{+\infty} \cancel{dE'_\nu}\, {E'_\nu}^2\Big[-3\tilde{\Phi}_0\tilde{H}_\nu\big[e^{-\beta\omega}\tilde{J}'_\nu + (1-\tilde{J}'_\nu)\big] + \\
&\qquad + 3\tilde{\Phi}_1\tilde{H}'_\nu\big[e^{-\beta\omega}(1-\tilde{P}_\nu) - \tilde{P}_\nu\big]\Big]\,\delta(\cancel{E_\nu - E'_\nu}) = \\
&= \frac{2\pi}{c(2\pi\hbar c^3)}E_\nu^2\Big[-3\tilde{\Phi}_0\tilde{H}_\nu\big[\cancel{\tilde{J}'_\nu} + (1-\cancel{\tilde{J}'_\nu})\big] + 3\tilde{\Phi}_1\tilde{H}'_\nu\big[(1-\cancel{\tilde{P}_\nu}) - \cancel{\tilde{P}_\nu}\big]\Big] = \\
&= \frac{2\pi}{c(2\pi\hbar c^3)}E_\nu^2\big[3\tilde{\Phi}_1\tilde{H}'_\nu - 3\tilde{\Phi}_0\tilde{H}_\nu\big]
\end{aligned} \tag{47}$$

Expanding the neutrino distribution function in a truncated Legendre series (Eq. (42)), we get in the end:

$$\frac{3}{2}\int_{-1}^{+1} d\mu\,\mu\, B_{\rm IS}[f_\nu] = \frac{2\pi}{c(2\pi\hbar c^3)}E_\nu^2\left[3\tilde{\Phi}_1\frac{\Psi^{(1)}}{3} - 3\tilde{\Phi}_0\frac{\Psi^{(1)}}{3}\right] = \frac{2\pi}{c(2\pi\hbar c^3)}E_\nu^2(\tilde{\Phi}_1 - \tilde{\Phi}_0)\Psi^{(1)}. \tag{48}$$

Let's now compute explicitely the scattering kernel:

$$R_{\rm IS}^{\left\{ in \atop out \right\}}(E_\nu, E'_\nu, \cos\theta) = \int\frac{d^3\mathbf{p}_N}{(2\pi)^3}\int\frac{d^3\mathbf{p}'_N}{(2\pi)^3}\left\{\begin{matrix}[1-f_N(E_N)]\,2f_N(E'_N)\,r(p'_N + p'_\nu \to p_N + p_\nu)\\ 2f_N(E_N)\,[1-f_N(E'_N)]\,r(p_N + p_\nu \to p'_N + p'_\nu)\end{matrix}\right\}. \tag{49}$$

The matrix element $\mathcal{M}$ which describes the scattering reaction is:

$$\mathcal{M} = \frac{G}{\sqrt{2}}\big[\bar{u}_N(p'_N)\gamma^\mu(h_V^N - h_A^N\gamma_5)u_N(p_N)\big]\big[\bar{u}_\nu(p'_\nu)\gamma_\mu(1-\gamma_5)u_\nu(p_\nu)\big]. \tag{50}$$

The square modulus of Eq. (50), averaged over the initial states and summed over the possible final states, is equal to:

$$\left(\sum_s|\mathcal{M}|^2\right) = 16\,G^2\big[(h_V^N + h_A^N)^2\,(p_N\cdot p_\nu)\,(p'_N\cdot p'_\nu) + (h_V^N - h_A^N)^2\,(p_N\cdot p'_\nu)\,(p'_N\cdot p_\nu) - \big((h_V^N)^2 - (h_A^N)^2\big)\,M_N^2\,p_\nu\cdot p'_\nu\big]. \tag{51}$$

The reaction rate r, appearing on the right-hand side of Eq. (49), is consequently given by:

$$\begin{aligned}
r(p_N + p_\nu \to p'_N + p'_\nu) =& \frac{G^2}{E_\nu\, E'_\nu\, E_N\, E'_N}(2\pi)^4\delta^4(p_N + p_\nu - p'_N - p'_\nu)\times \\
&\times\big[(h_V^N + h_A^N)^2\,(p_N\cdot p_\nu)\,(p'_N\cdot p'_\nu) + (h_V^N - h_A^N)^2\,(p_N\cdot p'_\nu)\,(p'_N\cdot p_\nu) + \\
&-\big((h_V^N)^2 - (h_A^N)^2\big)\,M_N^2\,p_\nu\cdot p'_\nu\big] = \\
=& G^2(2\pi)^4\delta^4(p_N + p_\nu - p'_N - p'_\nu)\times \\
&\times\Bigg[(h_V^N - h_A^N)^2\,\frac{E_N\,E_\nu - \bar{p}_N E_\nu\cos\theta_{N,\nu}}{E_N\,E_\nu}\,\frac{E'_\nu\,E'_N - E'_\nu\,\bar{p}'_N\cos\theta_{N',\nu'}}{E'_\nu\,E'_N} \\
&+\ (h_V^N - h_A^N)^2\,\frac{E_N\,E'_\nu - E'_\nu\bar{p}_N\cos\theta_{N,\nu'}}{E'_\nu\,E_N}\,\frac{E_\nu\,E'_N - \ E_\nu\,\bar{p}'_N\cos\theta_{N',\nu}}{E_\nu\,E'_N} + \\
&+\ \big((h_V^N)^2 - (h_A^N)^2\big)\,\frac{M_N^2}{E_N\,E'_N}\,\frac{E_\nu\,E'_\nu - E_\nu\,E'_\nu\cos\theta_{\nu,\nu'}}{E_\nu\,E'_\nu}\Bigg].
\end{aligned} \tag{52}$$

Under the assumption of non relativistic nucleons ($\bar{p}_N/E_N \ll 1$, see Eq. (14)), the reaction rate simplifies to:

$$\begin{aligned}
r(p_N + p_\nu \to p'_N + p'_\nu) =& G^2(2\pi)^4\delta^4(p_N + p_\nu - p'_N - p'_\nu)\,\Bigg[(h_V^N + h_A^N)^2\left(1 - \cancel{\frac{\bar{p}_N}{E_N}}\cos\theta_{N,\nu}\right)\left(1 - \cancel{\frac{\bar{p}'_N}{E'_N}}\cos\theta_{N',\nu'}\right) + \\
&- (h_V^N - h_A^N)^2\left(1 - \cancel{\frac{\bar{p}_N}{E_N}}\cos\theta_{N,\nu'}\right)\left(1 - \cancel{\frac{\bar{p}'_N}{E'_n}}\cos\theta_{N',\nu}\right) + \\
&- \big((h_V^N)^2 - (h_A^N)^2\big)\,\frac{M_N^2}{E_N\,E'_N}\,(1 - \cos\theta_{\nu,\nu'})\Bigg] = \\
=& G^2(2\pi)^4\delta^4(p_N + p_\nu - p'_N - p'_\nu)\big[(h_V^N + h_A^N)^2 + (h_V^N - h_A^N)^2 - \big((h_V^N)^2 - (h_A^N)^2\big)(1 - \cos\theta)\big] = \\
=& G^2(2\pi)^4\delta^4(p_N + p_\nu - p'_N - p'_\nu)\big[(h_V^N)^2 + 3(h_A^N)^2 + \big((h_V^N)^2 - (h_A^N)^2\big)\cos\theta\big].
\end{aligned} \tag{53}$$

where we have renamed $\cos\theta_{\nu,\nu'}$ as $\cos\theta$, i.e. the cosine of the scattering angle θ. If we now substitute Eq. (53) into Eq. (49) we get:

$$\begin{aligned}
R_{\mathrm{IS}}^{\mathrm{out}}(E_\nu, E'_\nu, \cos\theta) =& \int\frac{d^3\mathbf{p}_N}{(2\pi)^3}\int\frac{d^3\mathbf{p}'_N}{(2\pi)^3}\,2f_N(E_N)\,[1 - f_N(E'_N)]\,r(p_N + p_\nu \to p'_N + p'_\nu) = \\
=& G^2(2\pi)^{\cancel{4}}\big[(h_V^N)^2 + 3(h_A^N)^2 + \big((h_V^N)^2 - (h_A^N)^2\big)\cos\theta\big]\times \\
&\times\int\frac{d^3\mathbf{p}_N}{(2\pi)^3}\int\frac{d^3\mathbf{p}'_N}{\cancel{(2\pi)^3}}\,2f_N(E_N)\,[1 - f_N(E'_N)]\,\delta^4(p_N + p_\nu - p'_N - p'_\nu).
\end{aligned} \tag{54}$$

In the **isoenergetic, zero momentum transfer approximation** ($E_\nu = E'_\nu$ and $\bar{p}_N = \bar{p}'_N$):

$$\delta^4(p_N + p_\nu - p'_N - p'_\nu) = \delta(E_\nu - E'_\nu)\,\delta^3(\bar{p}_N - \bar{p}'_N), \tag{55}$$

the scattering kernel becomes:

$$\begin{aligned}
R_{\mathrm{IS}}^{\mathrm{out}}(E_\nu, E'_\nu, \cos\theta) =& 2\pi G^2\,\delta(E_\nu - E'_\nu)\big[(h_V^N)^2 + 3(h_A^N)^2 + \big((h_V^N)^2 - (h_A^N)^2\big)\cos\theta\big]\times \\
&\times\int\frac{d^3\mathbf{p}_N}{(2\pi)^3}\underbrace{\int\cancel{d^3\mathbf{p}'_N}\,2f_N(E_N)\,[1 - f_N(E'_N)]\,\cancel{\delta^3(\bar{p}_N - \bar{p}'_N)}}_{\bar{p}_N = \bar{p}'_N \Longrightarrow E_N = E'_N} = \\
=& 2\pi G^2\,\delta(E_\nu - E'_\nu)\big[(h_V^N)^2 + 3(h_A^N)^2 + \big((h_V^N)^2 - (h_A^N)^2\big)\cos\theta\big]\times \\
&\times\int\frac{2\,d^3\mathbf{p}_N}{(2\pi)^3}\,f_N(E_N)\,[1 - f_N(E_N)].
\end{aligned} \tag{56}$$

The integral over nucleon phase space in Eq. (56) can be easily solved in the following way (take advantage of Eq. (19)):

$$\begin{aligned}\eta_{NN} &= \int \frac{2\,d^3\mathbf{p}_N}{(2\pi)^3} f_N(E_N)\,[1-f_N(E_N)] = \int \frac{2\,d^3\mathbf{p}_N}{(2\pi)^3}\, f_N^2(E_N)\, \exp\left[\beta(E_N-\mu_N)\right] = \\ &= \int \frac{2\,d^3\mathbf{p}_N}{(2\pi)^3}\, \frac{\exp\left[\beta(E_N-\mu_N)\right]}{(\exp\left[\beta(E_N-\mu_N)\right]+1)^2} = \int \frac{2\,d^3\mathbf{p}_N}{(2\pi)^3}\, \frac{\partial}{\partial\mu_N}\left[\frac{1}{\beta}\frac{1}{\exp\left[\beta(E_N-\mu_N)\right]+1}\right] = \\ &= \frac{1}{\beta}\frac{\partial}{\partial\mu_N} \underbrace{\int \frac{2\,d^3\mathbf{p}_N}{(2\pi)^3} f_N(E_N)}_{n_N} = \frac{1}{\beta}\frac{\partial n_N}{\partial\mu_N},\end{aligned} \tag{57}$$

$$\eta_{NN} = \begin{cases} n_N, & \text{non-degenerate N;} \\ \dfrac{3}{2}\dfrac{n_N}{\beta\,\mu_N}, & \text{degenerate N.} \end{cases} \tag{58}$$

The expression of η_{NN} for degenerate N comes from the thermodynamics of a degenerate ideal Fermi gas, i.e. TODO: Check this (relativistic or non relativistic chemical potential):

$$\frac{\partial n_N}{\partial\mu_N} = \frac{\partial n_N}{\partial p_F}\frac{\partial p_F}{\partial\mu_N} = \frac{3p_F^2}{\alpha^3}\frac{2m_N}{3p_F} = \frac{3}{2}\underbrace{\frac{p_F^3}{\alpha^2}}_{n_N}\underbrace{\frac{2m_N}{p_F^2}}_{\mu_N^0} = \frac{3}{2}\frac{n_N}{\mu_N^0}, \tag{59}$$

where:

$$p_F = \hbar\,(3\pi^2 n_N)^{1/3} = \alpha n_N^{1/3} \qquad\qquad \mu_N = m_N c^2 + \mu_N^0 = m_N c^2 + \frac{p_F^2}{2m_N}. \tag{60}$$

Therefore, the final result for the scattering kernel in the **isoenergetic approximation** with **non-relativistic nucleons** is the following:

$$\begin{aligned}R_{\mathrm{IS}}^{\mathrm{out}}(E_\nu, E'_\nu, \cos\theta) &= R_{\mathrm{IS}}^0(E_\nu, E'_\nu, \cos\theta)\,\delta(E_\nu - E'_\nu) \\ &= 2\pi G^2\,\eta_{NN}\left[(h_V^N)^2 + 3(h_A^N)^2 + \left((h_V^N)^2 - (h_A^N)^2\right)\cos\theta_{\nu,\nu'}\right]\delta(E_\nu - E'_\nu),\end{aligned} \tag{61}$$

where the constants h_V^N and h_A^N are equal to (θ_W is the Weinberg angle):

$$\begin{cases} h_V^p = \frac{1}{2} - 2\sin^2\theta_W \\ h_A^p = \frac{1}{2}g_A, \end{cases} \qquad\qquad \begin{cases} h_V^n = -\frac{1}{2} \\ h_A^n = -\frac{1}{2}g_A. \end{cases} \tag{62}$$

By comparing the Legendre expansion in Eq. (39) with Eq. (61), it is easy to identify the first two coefficients of the truncated series:

$$\begin{cases} \tilde{\Phi}_0(E_\nu, E'_\nu) = 4\pi G^2\,\eta_{NN}\left[(h_V^N)^2 + 3(h_A^N)^2\right] \\ \tilde{\Phi}_1(E_\nu, E'_\nu) = \frac{4\pi}{3} G^2\,\eta_{NN}\left[(h_V^N)^2 - (h_A^N)^2\right]. \end{cases} \tag{63}$$

Finally, let's check that in the isoenergetic approximation the scattering kernels $R_{\mathrm{IS}}^{\mathrm{in}}$ and $R_{\mathrm{IS}}^{\mathrm{out}}$ are equivalent. The scattering rates for the direct and the inverse reactions are related through the detailed balance relation:

$$(2s_i+1)(2s_j+1)\,r(p_i + p_j \to p'_k + p'_l) = (2s_k+1)(2s_l+1)\,r(p'_k + p'_l \to p_i + p_j). \tag{64}$$

Since the particles in the initial and final states are the same, the direct and inverse rates are equivalent:

$$r(p_N + p_\nu \to p'_N + p'_\nu) = r(p'_N + p'_\nu \to p_N + p_\nu). \tag{65}$$

Therefore, by repeating the same procedure described above, one ends up with the same result for the scattering kernel $R_{\mathrm{IS}}^{\mathrm{in}}(E_\nu, E'_\nu, \cos\theta)$, i.e.:

$$\begin{aligned}R_{\mathrm{IS}}^{\mathrm{in}} &= \int \frac{d^3\mathbf{p}_N}{(2\pi)^3}\int \frac{d^3\mathbf{p}'_N}{(2\pi)^3}[1-f_N(E_N)]\,2f_N(E'_N)\,r(p'_N + p'_\nu \to p_N + p_\nu) = \\ &= R_{\mathrm{IS}}^0\,\delta(E_\nu - E'_\nu) = 2\pi G^2\,\eta_{NN}\left[(h_V^N)^2 + 3(h_A^N)^2 + \left((h_V^N)^2 - (h_A^N)^2\right)\cos\theta\right]\delta(E_\nu - E'_\nu).\end{aligned} \tag{66}$$

TODO: Add refs to [B06] and [BR85] + add weak magnetism

4 Bibliography

List of references this documentation is basing on:

- [BR85]: S. W. Bruenn, *Stellar core collapse: Numerical model and infall epoch*, Astrophys. J. Suppl. **58** (1985), 771-841
- [TH02]: T. A. Thompson, *Topics in the theory of core-collapse supernovae* (PhD Thesis, The University of Arizona), ProQuest Dissertations and Theses (2002)
- [B06]: BA. Burrows, S. Reddy and T. A. Thompson, *Neutrino opacities in nuclear matter*, Nucl. Phys. A **777** (2006), 356-394
- [V84]: P. Vogel, *Analysis of the Anti-neutrino Capture on Protons*, Phys. Rev. D **29** (1984), 1918