Función Gamma

Sea z un número real positivo cualquiera, o un real negativo no entero, entonces se define a la función Gamma de la siguiente forma:

$$\boxed{\Gamma(z) = \int_0^{\infty} x^{z-1} \cdot e^{-x}\, dx}$$

Además, si $z \in \mathbb{Z}^+$, entonces $\Gamma(z) = (z-1)!$, por lo que se puede interpretar a la función gamma como una extensión del factorial a los números reales.

Calculemos $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\, dx = \int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\, dx = \int_0^{\infty} \frac{e^{-u^2}}{\cancel{u}}\, 2\cancel{u}\, du = 2 \cdot \underbrace{\int_0^{\infty} e^{-u^2}\, du}_{I}$$

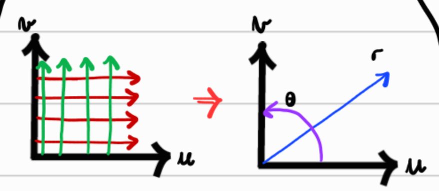
$x = u^2$
$dx = 2u\, du$

$$I = \int_0^{\infty} \underbrace{e^{-u^2}}_{>0,\ \forall u}\, du \Rightarrow I^2 = \left(\int_0^{\infty} e^{-u^2} du\right)\left(\int_0^{\infty} e^{-v^2} dv\right) = \int_0^{\infty}\int_0^{\infty} e^{-u^2} \cdot e^{-v^2}\, dv\, du$$

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)}\, dv\, du$$

Cambio a coordenadas polares:

$$\begin{cases} u = r\cos\theta \\ v = r\,\mathrm{sen}\,\theta \end{cases}$$



$u \in [0, +\infty[$
$v \in [0, +\infty[$

$r \in [0, +\infty[$
$\theta \in [0, \frac{\pi}{2}]$

$$u^2 + v^2 = r^2$$

Jacobiano:

$$J(r,\theta)=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\,\mathrm{sen}\,\theta\\ \mathrm{sen}\,\theta & r\cos\theta\end{vmatrix}=\underbrace{r\cos^2\theta+r\,\mathrm{sen}^2\theta}_{r(\cos^2\theta+\mathrm{sen}^2\theta)}=r$$

Entonces:

$$I^2=\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,dv\,du=\int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta=\underbrace{\left(\int_0^{\pi/2}d\theta\right)}_{\pi/2}\left(\int_0^\infty e^{-r^2}r\,dr\right)$$

$$=\frac{\pi}{2}\cdot\left(-\frac{1}{2}\int_0^\infty e^{-r^2}(-2r\,dr)\right)=-\frac{\pi}{4}\cdot\left(e^{-r^2}\right)\Big]_0^\infty=-\frac{\pi}{4}\cdot(0-1)=\frac{\pi}{4}$$

Luego:

$$I^2=\frac{\pi}{4}\Rightarrow I=\sqrt{\frac{\pi}{4}}=\frac{\sqrt{\pi}}{2}$$

Por lo tanto, volviendo al inicio, tenemos:

$$\Gamma\left(\tfrac{1}{2}\right)=2I=\cancel{2}\cdot\frac{\sqrt{\pi}}{\cancel{2}}=\sqrt{\pi}\quad\rightarrow\quad\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

Propiedad importante:

$$\Gamma(z+1)=z\cdot\Gamma(z);\quad\forall\ z\in\mathrm{Dom}_{\mathbb{R}}(\Gamma)$$

Demostración:

$$\Gamma(z+1)=\int_0^\infty \underbrace{x^z}_{u}\underbrace{e^{-x}dx}_{dv}=x^z\cdot(-e^{-x})\Big]_0^\infty-\int_0^\infty -e^{-x}zx^{z-1}\,dx$$

$$=\underbrace{0^z}_{0}\cdot\underbrace{e^{-0}}_{1}-\lim_{x\to\infty}\left(\frac{x^z}{e^x}\right)+z\cdot\overbrace{\int_0^\infty x^{z-1}e^{-x}\,dx}^{\Gamma(z)}=z\cdot\Gamma(z)$$

Así, por ejemplo:

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\left(\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\right) = \frac{3}{4}\cdot\underbrace{\Gamma\left(\frac{1}{2}\right)}_{=\sqrt{\pi}} = \frac{3}{4}\sqrt{\pi}$$

Distribución Gamma

Sea X una variable aleatoria continua, diremos que X se ajusta a una distribución gamma de parámetros $\alpha, \lambda \in \mathbb{R}^+$ (lo que denotaremos como "$X \sim \text{Gamma}(\alpha, \lambda)$"), si su función de densidad viene dada por:

$$f(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

Si $\alpha = 1$, entonces para cualquier $\lambda \in \mathbb{R}^+$, se cumple que:

$$f(x) = \frac{\lambda\cdot(\lambda x)^{1-1}\cdot e^{-\lambda x}}{\Gamma(1)} = \frac{\lambda\cdot\cancel{(\lambda x)^{0}}^{\,1}\cdot e^{-\lambda x}}{\cancel{0!}^{\,1}} = \lambda e^{-\lambda x} \sim \mathcal{E}xp(\lambda)$$

Es decir, la distribución exponencial es un caso particular de la distribución gamma, de modo que:

$$\mathcal{E}xp(\lambda) \equiv \text{Gamma}(1, \lambda)$$

¿Cuál es la esperanza de la distribución Gamma?

Calculemos:

$$\mathbb{E}(X) = \int_0^\infty \frac{x\cdot\lambda\cdot(\lambda x)^{\alpha-1}\cdot e^{-\lambda x}}{\Gamma(\alpha)}\,dx = \int_0^\infty \frac{\lambda^\alpha x^\alpha e^{-\lambda x}}{\Gamma(\alpha)}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\int_0^\infty x^\alpha e^{-\lambda x}\,dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \cdot \int_0^\infty \frac{1}{\cancel{\lambda^\alpha}} (\lambda x)^\alpha \cdot e^{-\lambda x} dx \xrightarrow[du = \lambda dx]{u = \lambda x} = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty u^\alpha \cdot e^{-u} \frac{du}{\lambda} = \frac{1}{\lambda \cdot \Gamma(\alpha)} \cdot \underbrace{\int_0^\infty u^\alpha e^{-u} du}_{= \Gamma(\alpha+1)}$$

$$= \frac{1}{\lambda \cdot \Gamma(\alpha)} \cdot \underbrace{\Gamma(\alpha+1)}_{} = \frac{1}{\lambda \cdot \cancel{\Gamma(\alpha)}} \cdot \underbrace{\alpha \cdot \cancel{\Gamma(\alpha)}}_{} = \frac{\alpha}{\lambda}$$

Se demostró previamente que son equivalentes

¿Cuál es la FGM de esta distribución?

Calculemos:

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{\lambda (\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)} dx = \int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{(t-\lambda)x}}{\Gamma(\alpha)} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty x^{\alpha-1} \cdot e^{(t-\lambda)x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty \frac{1}{(\lambda-t)^{\alpha-1}} \left((\lambda-t)x\right)^{\alpha-1} \cdot e^{-(\lambda-t)x} \cdot \frac{1}{\lambda-t} \cdot (\lambda-t)\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{(\lambda-t)^\alpha} \cdot \int_0^\infty (\underbrace{(\lambda-t)x}_{u})^{\alpha-1} \cdot e^{-\overbrace{(\lambda-t)x}^{u}} \cdot \underbrace{(\lambda-t)\, dx}_{du} = \frac{\lambda^\alpha}{\Gamma(\alpha) \cdot (\lambda-t)^\alpha} \cdot \underbrace{\int_0^\infty u^{\alpha-1} \cdot e^{-u} du}_{= \Gamma(\alpha)}$$

Obs: Asumimos que $\lambda > t$

$$= \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)} \cdot (\lambda-t)^\alpha} \cdot \cancel{\Gamma(\alpha)} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$

¿Cuál será su varianza?

Recordemos que:

$$V(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \alpha_2 - \mathbb{E}(X)^2$$

Para calcular α_2, debemos hacer:

$$\alpha_2 = \left.\frac{d^2M}{dt^2}\right|_{t=0}$$

Entonces, procediendo:

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$

$$\Rightarrow \frac{dM}{dt} = \alpha \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \cdot \left(\frac{\lambda}{\lambda-t}\right)' = \alpha \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \cdot \frac{(-\lambda)\cdot(-1)}{(\lambda-t)^2} = \frac{\alpha}{\lambda-t} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}$$

$$\Rightarrow \frac{d^2M}{dt^2} = \left(\frac{\alpha}{\lambda-t}\right)' \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} + \frac{\alpha}{\lambda-t} \cdot \left(\left(\frac{\lambda}{\lambda-t}\right)^{\alpha}\right)' = \frac{-\alpha\cdot(-1)}{(\lambda-t)^2} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} + \frac{\alpha}{\lambda-t} \cdot \frac{\alpha}{\lambda-t} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}$$

$$= \frac{\alpha}{(\lambda-t)^2} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} + \frac{\alpha^2}{(\lambda-t)^2}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha} = \frac{\alpha}{(\lambda-t)^2} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot (1+\alpha)$$

$$\Rightarrow \left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{\alpha}{(\lambda-0)^2} \cdot \left(\frac{\lambda}{\lambda-0}\right)^{\alpha} \cdot (1+\alpha) = \frac{\alpha}{\lambda^2} \cdot \underbrace{1^{\alpha}}_{1} \cdot (1+\alpha) = \frac{\alpha(1+\alpha)}{\lambda^2} = \alpha_2$$

Por lo tanto:

$$V(X) = \alpha_2 - \mathbb{E}(X)^2 = \frac{\alpha(1+\alpha)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha + \cancel{\alpha^2} - \cancel{\alpha^2}}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

¿Cómo demostramos que la función de densidad $f(x)$ de la distribución gamma, es efectivamente una función de densidad?

$$f(x) = \begin{cases} \dfrac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \;; & \text{si } x \geq 0 \\ 0 & \text{en caso contrario.} \end{cases}$$

Análisis:

① $f(x) \geq 0$

$\dfrac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \geq 0$, ya que los parámetros $\alpha, \lambda \in \mathbb{R}^+$ por su naturaleza

Luego, como la función de densidad se define así para los $x \in [0,+\infty[$, en esencia se tiene un producto de puras cosas positivas.

② $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \cancelto{0}{\int_{-\infty}^{0} 0\, dx} + \int_{0}^{\infty} \frac{\lambda(\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)}\, dx = \frac{\lambda}{\Gamma(\alpha)} \cdot \int_{0}^{\infty} (\lambda x)^{\lambda-1} \cdot e^{-\lambda x}\, dx$$

$$= \frac{\cancel{\lambda}}{\Gamma(\alpha)} \cdot \int_{0}^{\infty} \underbrace{(\lambda x)}_{u}^{\alpha-1} \cdot e^{-\overset{u}{\lambda x}} \cdot \frac{1}{\cancel{\lambda}} \cdot \underbrace{\lambda\, dx}_{du} = \frac{1}{\Gamma(\alpha)} \cdot \overbrace{\int_{0}^{\infty} u^{\alpha-1} \cdot e^{-u}\, du}^{= \Gamma(\alpha)} = \frac{1}{\cancel{\Gamma(\alpha)}} \cdot \cancel{\Gamma(\alpha)} = 1$$