Chapter 12 – Part II
Data Collection and Empirical Methods

- Ungrouped Censored Data
- Grouped Censored Data
- Static Life Estimation
- Non-parametric confidence intervals

Singly Censored Data

Twenty units are placed on test for 72 hours with the following failures times recorded: 1.5, 3.2, 11.7, 26.4, 39.1, 56.0, 61.3

$$\hat{R}(t_i) = 1 - i/(20+1)$$

TIME	RELIABILITY	CUM PROB (CDF)
0	1	0
1.5	.952381	4.761904E-02
3.2	.9047619	9.523809E-02
11.7	.8571429	.1428571
26.4	.8095238	.1904762
39.1	.7619048	.2380952
56	.7142857	.2857143
61.3	.6666667	.3333333

Cannot compute a mean or variance

Ungrouped Censored Data

A sample consists of a set of ordered failure times plus censored times:

$$t_1,\ t_2,\ t_3^+,\ \ldots\ t_i,\ t_{i+1}^+,\ \ldots,\ t_n$$

Method	complete data
Product Limit Estimator	$\hat{R}(t_i) = 1 - i/(n+1)$
Kaplan-Meier	$\hat{R}(t_i) = 1 - i/n$
Rank adjustment	

Product Limit Estimator (PLE)

$$\hat{R}(t_{i-1}) = \frac{n+2-i}{n+1}$$ and $$\frac{\hat{R}(t_i)}{\hat{R}(t_{i-1})} = \frac{n+1-i}{n+2-i}$$

then $$\hat{R}(t_i) = \frac{n+1-i}{n+2-i}\hat{R}(t_{i-1})$$

however $$\hat{R}(t_i^+) = \hat{R}(t_{i-1})$$

$$\hat{R}(t_i) = \left(\frac{n+1-i}{n+2-i}\right)^{\delta_i}\hat{R}(t_{i-1})$$

where

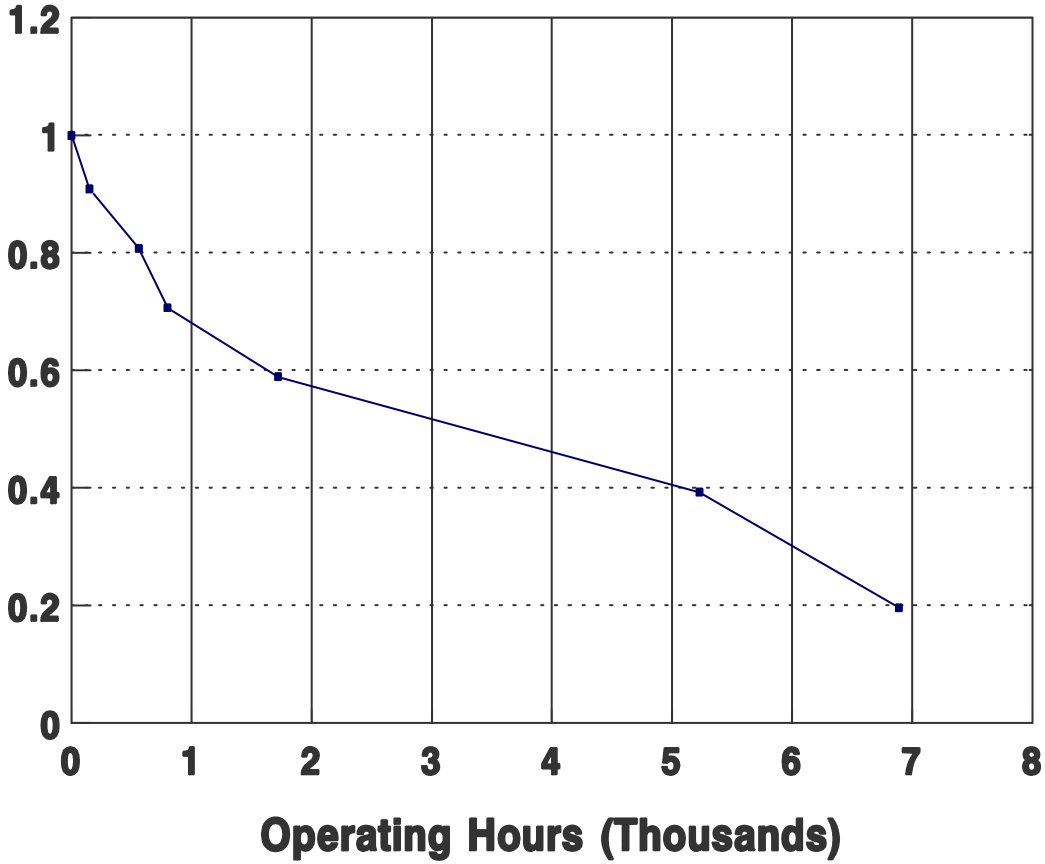
$$\delta_i = \begin{cases} 1 \text{ if failure occurs at time } t_i \\ 0 \text{ if censoring occurs at time } t_i \end{cases}$$

Example 12.6

The following failure and censor times (in operating hours) were recorded on 10 turbine vanes. Censoring was a result of failure modes other than fatigue or wear out. Determine an empirical reliability curve. 150, 340+, 560, 800, 1130+, 1720, 2470+, 4210+, 5230, 6890

i	t_i	(11-i)/(12-i)	$\hat{R}(t_i)$
1	150	10/11	R(150) = 10/11x1 = .9090
2	340+	9/10	
3	560	8/9	R(560) = 8/9 x.9090= .8081
4	800	7/8	R(800) = 7/8 x.8081= .7071
5	1130+	6/7	
6	1720	5/6	R(1720) = 5/6 x.7071=.5892
7	2470+	4/5	
8	4210+	3/4	
9	5230	2/3	R(5230)= 2/3 x.5892= .3928
10	6890	1/2	R(6890)= 1/2 x..3928=.1964

Example 12.6



Kaplan-Meier PLE

Begin with with $R(t_0=0) = 1$

$$R(t_i) = R(t_i \mid t_{i-1})\ R(t_{i-1})$$

$$= (n_{i-1} - 1) / n_{i-1} \quad R(t_{i-1})$$

$$= (1 - 1 / n_{i-1}) \quad R(t_{i-1})$$

Kaplan-Meier PLE

$$\hat{R}(t) = \prod_{[j:t_j \le t]} \left(1 - \frac{1}{n_j}\right)$$

For $0 \le t < t_1$, $R(t) = 1$

$$\hat{Var}[\hat{R}(t)] = \hat{R}(t)^2 \sum_{[j:t_j < t]} \frac{1}{n_j(n_j - 1)}$$

Example 12.7

j	t_j	n_j	$1-1/n_j$	$\hat{R}(t_j + 0)$	Std Dev
1	150	10	9/10	R(150) = 9/10 x 1.0 = .90	.095
2	340+				
3	560	8	7/8	R(560) = 7/8 x.90 = .7875	.134
4	800	7	6/7	R(800) = 6/7 x.7875= .675	.155
5	1130+				
6	1720	5	4/5	R(1720) = 4/5 x.675 =.54	.173
7	2470+				
8	4210+				
9	5230	2	1/2	R(5230)= 1/2 x.54= .27	.210
10	6890	1	0	R(6890)= 0 x .3928= 0	

Rank Adjustment Method

$$\textit{rank increment} = \frac{(n+1) - (\textit{previous rank order})}{1 + \textit{nbr of units beyond present censored unit}}$$

$$i_{ti} = i_{ti-1} + \text{rank increment}$$

$$\hat{R}(t_i) = 1 - \frac{i_{t_i} - 0.3}{n + 0.4}$$

Example 12.8

i	t_i	Increment	Adj Rank (i)	$\hat{R}(t_i)$
1	150		1	.933
2	340+			
3	560	(11-1)/(1+8) = 1.111	1+1.111 = 2.111	.826
4	800		2.111 + 1.111 = 3.222	.719
5	1130+			
6	1720	(11-3.222)/(1+5) = 1.2963	3.222+1.2963 = 4.518	.594
7	2470+			
8	4210+			
9	5230	(11-4.518)/(1+2)= 2.16	4.518 + 2.160 = 6.679	.387
10	6890		6.679 + 2.160 = 8.839	.179

Multiply censored comparison

Failure time	PLE	Kaplan -Meier	Rank Adj
150	.909	.90	.933
340+			
560	.808	.788	.826
800	.707	.675	.719
1130+			
1720	.589	.54	.594
2470+			
4210+			
5230	.393	.27	.387
6890	.196	0	.179

Example 12.9 - multiply censoring

3 components
in series

Unit	Failed Component	Failure Time
#1	C1	352 hrs
#2	C2	521
#3	C1	177
#4	C1	67
#5	C3	411
#6	C2	125
#7	C1	139
#8	C1	587
#9	C3	211
#10	C1	379

Example 12.9 - multiply censoring

	TIME	FACTOR	Component 1 RELIABILITY
1	67	.9090909	.9090909
2	125 +	1	
3	139	.8888889	.8080809
4	177	.875	.7070708
5	211 +	1	
6	352	.8333333	.5892256
7	379	.8	.4713805
8	411 +	1	
9	521 +	1	
10	587	.5	.2356903

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Grouped Censored Data Life Tables

Assume that the failure and censor times have been grouped into k+1 intervals of the form $[t_{i-1}, t_i)$, for $i = 1, 2, \ldots, k+1$; where $t_0 = 0$ and $t_{k+1} = \infty$.

Grouped Censored Data Life Tables

F_i = the number of failures in the ith interval
C_i = the number of censors in the ith interval
H_i = the number at risk at time t_{i-1}
where $H_i = H_{i-1} - F_{i-1} - C_{i-1}$ and $H'_i = H_i - C_i/2$
Then F_i/H'_i = conditional probability of a failure in the ith interval given survival to time t_{i-1}
and $p_i = 1-F_i/H'_i$ = conditional probability of surviving the ith interval given survival to time t_{i-1}

$$\hat{R}_i = \left[1 - \frac{F_i}{H_{i'}}\right] x\, \hat{R}_{i-1}$$

Grouped Censored Data Life Tables

Interval	Number Failures	Number Censored	At Risk	Adj at Risk	Prob Survival	Reliability
$[t_{i-1}, t_i)$	F_i	C_i	H_i	H'_i	p_i	R_i

reliability data

$H'_i = H_i - C_i/2$

$H_i = H_{i-1} - F_{i-1} - C_{i-1}$

$p_i = 1 - F_i / H'_i$

$$\hat{VAR}(\hat{R}_i) = \hat{R}_i^2 \sum_{k=1}^{i} \frac{1 - p_k}{H_{k'} \, p_k} \longleftrightarrow \hat{R}_i = p_i \times \hat{R}_{i-1}$$

Example 12.10

Construct a life table for the engine of a fleet of 200 single engine aircraft having the following annual failures and removals (censors).

Year	Failures	Removals
1981	5	0
1982	10	1
1983	12	5
1984	8	2
1985	10	0
1986	15	6
1987	9	3
1988	8	1
1989	4	0
1990	3	1

Example 12.10

YEAR	Fi	Ci	Hi	H'i	pi	Ri	Std Dev
1	5	0	200	200	.975	.975	.011
2	10	1	195	194.5	.949	.925	.019
3	12	5	184	181.5	.934	.864	.024
4	8	2	167	166	.952	.822	.027
5	10	0	157	157	.936	.770	.030
6	15	6	147	144	.896	.690	.033
7	9	3	126	124.5	.928	.640	.035
8	8	1	114	113.5	.930	.595	.036
9	4	0	105	105	.962	.572	.036
10	3	1	101	100.5	.970	.555	.036

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Static Life Estimation

- n units at risk for a time t_0 with r failures observed
- If an event of short duration is observed, t_0 may be omitted and the point reliability estimate is based simply on the number of failures r resulting from the application of n static loads
- A point estimate for the reliability is given by

$$\hat{R}(t_0) = 1 - \frac{r}{n}$$

Static Life Estimation – interval estimation

From the binomial distribution:

$$\sum_{i=0}^{r} \binom{n}{i} (1-R_L)^i (R_L)^{n-i} = \alpha / 2$$

$$\sum_{i=r}^{n} \binom{n}{i} (1-R_U)^i (R_U)^{n-i} = \alpha / 2$$

$\Pr\{R_L \le R(t_0) \le R_U\} = 1 - \alpha$, and we are $100(1 - \alpha)$ percent confident that that the population static reliability falls between R_L and R_U

Static Life Estimation – interval estimation

From the relationship between the binomial and F distributions:

$$R_L = \left[1 + F_1\left(\frac{r+1}{n-r}\right)\right]^{-1}$$

$$R_U = \frac{F_2}{F_2 + r/(n-r+1)}$$

$$F_1 = F_{\alpha/2,2r+2,2n-2r} \qquad F_2 = F_{\alpha/2,2n-2r+2,2r}$$

$F_{\alpha/2,n_1,n_2}$ is a value from the F-distribution having n_1 and n_2 degrees of freedom and having an upper-tail probability of $\alpha/2$.

Example 12.12

It is desired to estimate the launch reliability of a booster rocket used to launch communication satellites into orbit. Twenty launches have been completed to date with one failure observed. Compute a 90 percent confidence interval for the rocket launch reliability.

Solution. With n = 20 and r = 1,

$$R = 1 - \frac{1}{20} = 0.95$$

$$F_1 = F_{.05,4,38} = 2.62$$

$$F_2 = F_{.05,40,2} = 19.47$$

$$R_L = \frac{1}{1 + (2.62)(2/19)} = 0.7838$$

$$R_U = \frac{19.47}{19.47 + 1/(20 - 1 + 1)} = 0.9974$$

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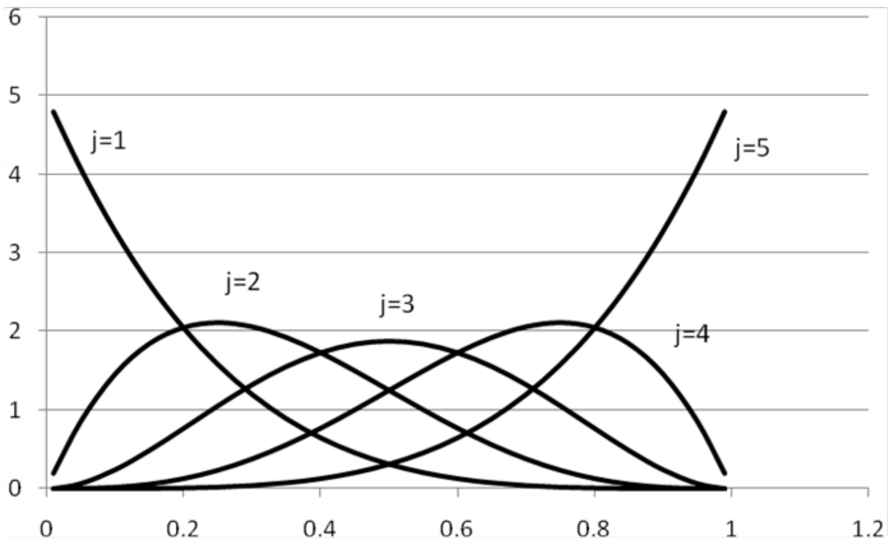
Nonparametric Confidence Intervals

Define Y_j to be a random variable, the fraction in a sample of size n that fail prior to t_j.



$$g(y_j) = \frac{n!}{(j-1)!(n-j)!} y_j^{\,j-1}(1-y_j)^{n-j}\;;\; 0 \le y_j \le 1;\; j = 1,\ldots,n$$

Nonparametric Confidence Intervals

If a 100(1-α) percent confidence interval is desired for the fraction failing prior to the j[th] failure time, then define L_j and U_j so that

$$\int_{0}^{L_j} g\left(y_j\right) dy_j = \alpha / 2 \text{ and } \int_{U_j}^{1} g\left(y_j\right) dy_j = \alpha / 2$$

Therefore $\Pr\{ L_j \leq \mathrm{Y}_j \leq U_j \} = 1 - \alpha$.

Example 12.13

For a sample size of 5 and $\alpha = .20$, Lj, U_j, and the median of Y_j are

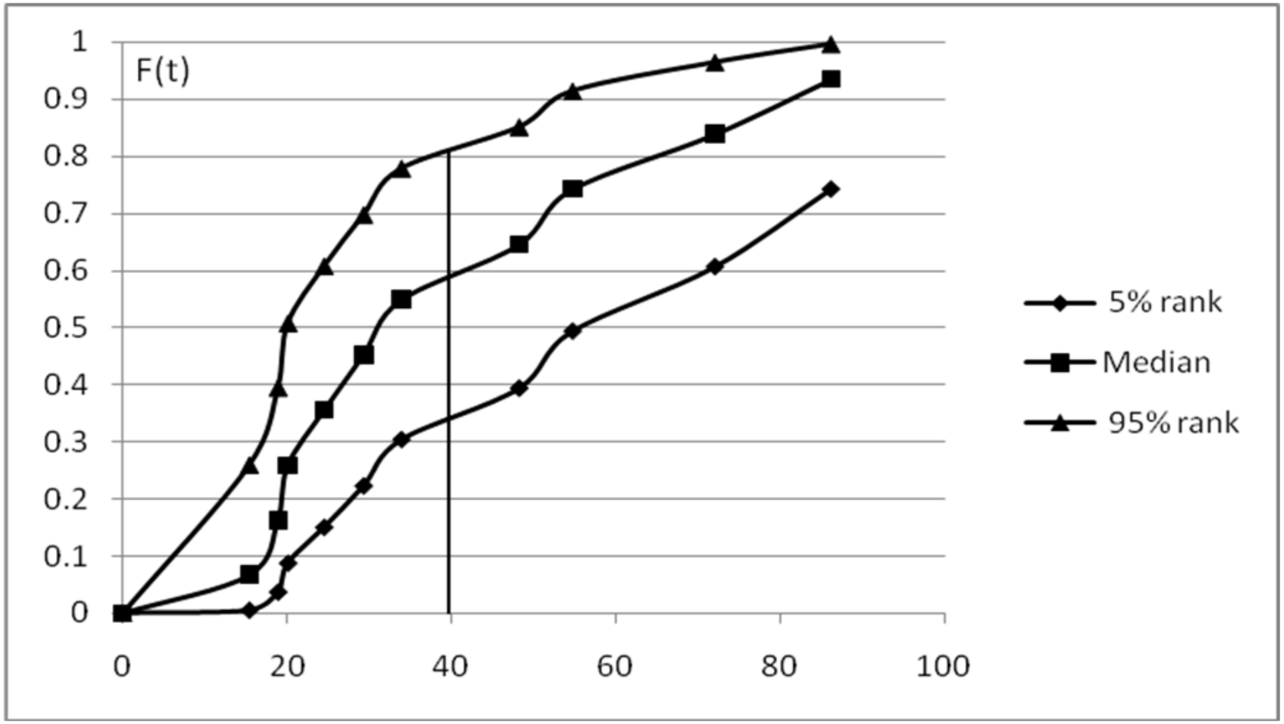
j	L_j	median	U_j
1	0.0209	0.1294	0.3690
2	0.1122	0.3138	0.5839
3	0.2466	0.5000	0.7534
4	0.4161	0.6862	0.8878
5	0.6310	0.8706	0.9791

Example 12.14

Based upon the failure times given in Example 12.2 with n = 10 and α = .10, the following plotting positions are computed:

failure times	5% rank	Median	95% rank
0	0	0	0
15.4	0.0051	0.0670	0.2589
18.9	0.0368	0.1623	0.3942
20.1	0.0873	0.2586	0.5069
24.5	0.1500	0.3551	**0.6066**
29.3	0.2224	0.4517	0.6965
33.9	0.3035	0.5483	0.7776
48.2	0.3934	0.6449	0.8500
54.7	0.4931	0.7414	0.9127
72	**0.6058**	0.8377	0.9632
86.1	0.7411	0.9330	0.9949

Example 12.14



Chapter 12 – Data Collection and Empirical Methods

- Ungrouped Complete Data
- Grouped Complete Data
- Ungrouped Censored Data
- Grouped Censored Data
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- Non-parametric confidence intervals