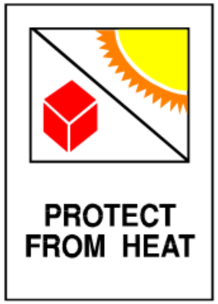
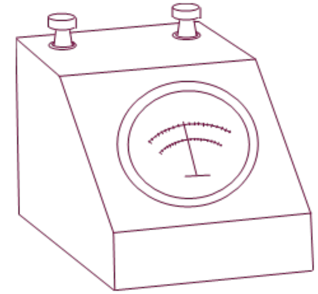
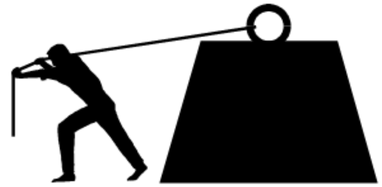




Chapter 7 Physical Reliability



Covariate Models

Static & Dynamic Models

Physics of Failure Models

C. Ebeling, *Intro to Reliability & Maintainability Engineering*, 2^{nd} *ed.* Waveland Press, *Inc.*

Basic Premise of Chapter 7

- Factors other than time alone influence failures
  - Factors may include external stresses and forces such as heat, pressure, contamination, vibration, and power surges.
  - Factors may also include internal design characteristics such as strength of material, geometry, and operating use.
  - Factors may also include the manufacturing environment such as production lot, manufacturing specs, quality control criteria, day of the week, and shift.
- Individual units will differ among these factors

Covariate Models

- One or more of the failure distribution parameters is a function of explanatory or covariate variables
- Usually a physical cause and effect
  - If cause and effect exist, then can use covariates to **control** reliability
  - Otherwise, can only use covariates to **predict** reliability.
  - Use design of experiments to establish cause and effect

General Covariate Model

$$\alpha(\mathbf{x}) = f(x_1, x_2 \ldots, x_n)$$

$\alpha(\mathbf{x})$ may be a mean, median, percentile, etc.

and the x_i may be a stress such as voltage, temperature pressure, or humidity.

The Effect of Stress

If x_1 and x_2 are two stress levels where $x_2 > x_1$, then for all positive values of t,

$$F_2(t; x_2) > F_1(t; x_1)$$

A more severe stress accelerates the time to failure.

Proportional Hazards Models

Has the property that individual component hazard rate functions are proportional to each other.

$$\lambda(t\,|\,\mathbf{x}) = \lambda_0(t)\, f(x_1, x_2 \ldots, x_n)$$

$$\frac{\lambda(t\,|\,\mathbf{x}_1)}{\lambda(t\,|\,\mathbf{x}_2)} = \frac{\lambda_0(t)\, f(x_1^1, x_2^1 \ldots, x_n^1)}{\lambda_0(t)\, f(x_1^2, x_2^2 \ldots, x_n^2)} = \frac{f(x_1^1, x_2^1 \ldots, x_n^1)}{f(x_1^2, x_2^2 \ldots, x_n^2)}$$

Not a function of time; therefore can extrapolate in stress without regard to time.

Exponential Distribution

Has a proportional hazard rate since $\lambda(t)$ is a constant and $\lambda_0(t) = 1$

linear model: $$\lambda(\mathbf{x}) = \sum_{i=0}^{k} a_i x_i$$

where the a_i are unknown parameters to be estimated and $x_0 = 1$.

multiplicative model: $$\lambda(\mathbf{x}) = \prod_{i=0}^{k} a_i x_i^{b_i}$$

Example – Exponential Multiplicative Model

Failure rate of ball bearings:

$$\lambda = \lambda_b \left(\frac{L_a}{L_s}\right)^y \left(\frac{A_e}{.006}\right)^{2.36} \left(\frac{\nu_0}{\nu_l}\right)^{.54} \left(\frac{C_l}{60}\right)^{.67} \left(\frac{M_b}{M_f}\right) C_w$$

b = base failure rate of a bearing per 10^6 hours of operation
L_a = actual radial load in pounds
L_s = specification radial load in pounds
y = 3.33 for roller bearings; 3.0 for ball bearings
A_e = alignment error in radians
v_0 = specification lubricant viscosity (lb-min/in2)
v_l = operating lubricant viscosity (lb-min/in2)
C_l = actual contamination level in micrograms/meter3
M_b = material factor, base material in PSI (yield strength)
M_f = material factor, operating material in PSI (yield strength)
C_w = water contamination factor (leakage of water into the oil lubricant)

Exponential Multiplicative Model

$$\lambda(\mathbf{x}) = \prod_{i=0}^{k} e^{a_i x_i} = e^{\sum_{i=0}^{k} a_i x_i}$$

$$R(t) = e^{-\lambda(\mathbf{x})t}$$

Use multiple linear regression to estimate the a_i

$$\ln[\lambda(\mathbf{x})] = \sum_{i=0}^{k} a_i x_i$$

Example - Exponential Multiplicative Model

From MILHDBK 217: monolithic microelectronic device:

$$g(x) = .11e^{.916+.0005638\,x_1 x_2}$$

where x_1 = operating voltage
x_2 = worst-case junction temperature

Example - Parametric Model

$$\text{MTBM} = 34.104 + .0009853 \times \textit{engine weight} - .31223\sqrt{\textit{engine weight}}$$

where MTBM = mean time (flying hours) between maintenance of an aircraft engine.

Weibull Case

$$\theta(\mathbf{x}) = e^{\sum_{i=0}^{k} a_i x_i} \quad \text{and} \quad R(t) = e^{-\left(\frac{t}{\theta(\mathbf{x})}\right)^{\beta}}$$

$$\lambda(t \mid \mathbf{x}) = \frac{\beta t^{\beta-1}}{\theta(\mathbf{x})^{\beta}} = \beta t^{\beta-1} \left\{ \exp\left[\sum_{i=1}^{k} a_i x_i \right] \right\}^{-\beta}$$

Proportional Hazard Rate

$$\lambda(t \mid \mathbf{x}) = \lambda_0(t)\left[\frac{1}{\theta(\mathbf{x})}\right]$$

$$\lambda_0(t) = \beta t^{\beta-1}$$

$$\frac{\lambda(t|\mathbf{x}_1)}{\lambda(t|\mathbf{x}_2)} = \left[\frac{\theta(\mathbf{x}_2)}{\theta(\mathbf{x}_1)}\right]^{\beta}$$

Weibull - Example

Time to failure of a motor is Weibull with a shape parameter of 1.5 and $\theta(x) = e^{23.2 - .134x}$

where x = load placed on a motor. Find the .95 design life if a motor is to have a load of 115. What if the load is reduced to 100?

Solution: $\theta(115) = 2416.3$

and $t_{.95} = 2416.3\,(-\ln .95)^{.6667} = 333.5$ hr.

$\theta(100) = 18033.7$

and $t_{.95} = 18{,}033.7\,(-\ln .95)^{.6667} = 2489.3$ hr.

Location-Scale Models

$$\mu(\mathbf{x}) = \sum_{i=0}^{k} a_i x_i$$

$$Y = \mu(\mathbf{x}) + \sigma z$$

where z has a specified probability distribution such as the normal with zero mean and variance of 1. Therefore Y is normal with mean u(**x**) and variance σ^2.

Lognormal Case

Set $T = e^Y$; if T is lognormal, then Y is normal with some mean $u(\mathbf{x})$ and standard deviation s, and

$$R(t) = 1 - \Phi\left(\frac{\ln t - \sum_{i=0}^{k} a_i x_i}{s} \right)$$

T is lognormal with shape parameter s and $t_{med} = e^{u(\mathbf{x})}$

Example - Lognormal

Time to failure of electrical connector is lognormal with s = .73 and

u(x) = -3.86 + .1213x_1 + .2886x_2 where x_1 = operating temperature and x_2 = number of electrical contacts

A connector used in a PC will operate at 80^0 C. and has 16 contact pins.

$$u(80,16) = -3.86 + 0.1213(80) + 0.2886(16) = 10.46$$

$$R(5000) = 1 - \Phi\left(\frac{\ln(5000) - 10.46}{.73}\right) = 1 - \Phi\left(-2.66\right) = .996$$

$$t_{\text{med}} = e^{10.46} = 34,891.55 \text{ hr.}$$

Static Models

Stress versus Strength

Stress is any load which may produce a failure. A failure occurs if the stress exceeds the strength. The strength is the highest stress value that the system can endure without failing.

Strength = f(material properties, design configuration or geometry, dimensions)

Case 1: Random Stress and Constant Strength
Case 2: Constant Stress and Random Strength
Case 3: Random Stress and Random Strength

The Random Variables

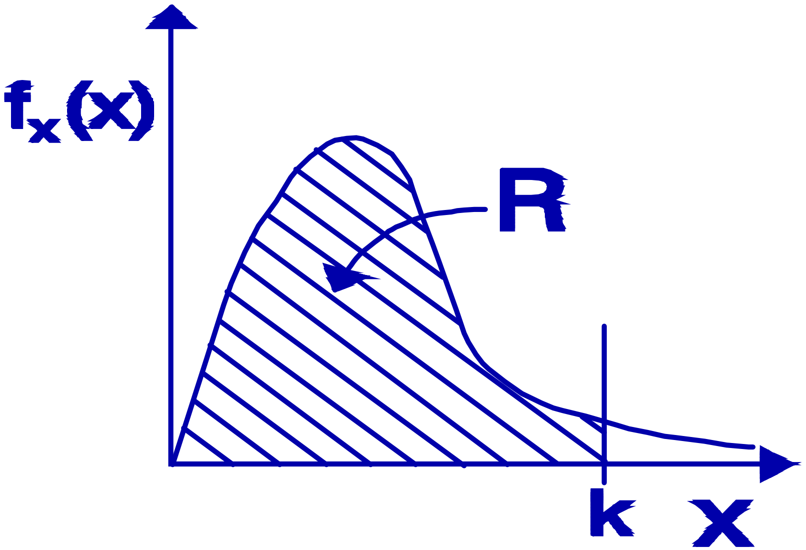
Let X = a continuous random variable, the stress placed on a system with

$$P(X \le x) = F_x(x) = \int_0^x f_{x'}(x')dx'$$

Y = a random variable, the strength (capacity) of the system with

$$P(Y \le y) = F_y(y) = \int_0^y f_{y'}(y')dy'$$

Case 1: Random Stress and Constant Strength



$$R = \int_0^k f_x(x)dx = F_x(k)$$

Case 1 Example

Wind velocity

Location	Mean velocity (mph)	Coefficient of variation
Baltimore, MD	55.9	.12
Rochester, NY	53.5	.10
Detroit, MI	48.9	.14
St. Louis, MO	47.4	.16
Austin, TX	45.1	.12
Tucson, AZ	51.4	.17
Sacramento, CA	46.0	.22

Maximum Annual Value

Case 1 Example

I designed this building to withstand winds up to 70 mph.

Baltimore: Mean = 55.9, k = .12

Assuming a normal distribution:

$$k = \frac{\sigma}{\mu}; \ \ \sigma = \mu k = 55.9(.12) = 6.708$$

$$R = \Phi\left(\frac{70 - 55.9}{6.71}\right) = \Phi\left(2.10\right) = .98214$$

Why the normal?

Bonus! Bonus! Bonus!

Maximum Extreme Value Distribution

- the distribution of the largest value of x

$$F_x(x) = \exp\left[-e^{-(x-\mu)/\alpha}\right];\ -\infty < x < \infty,\ \alpha > 0$$

α is a scale parameter and μ is a location parameter – the mode

$$E[X] = \mu + \alpha\gamma \text{ where } \gamma = .577215665$$

$$\sigma_x = \frac{\alpha\pi}{\sqrt{6}}$$

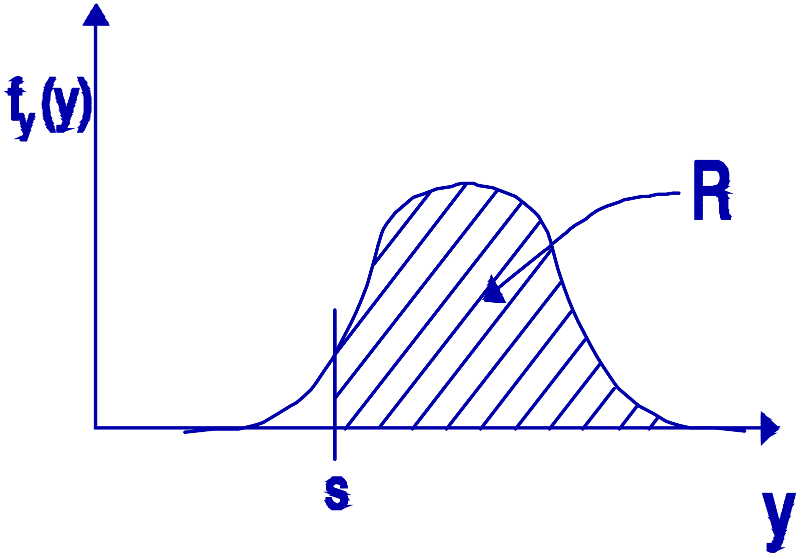
New Solution...

$$E[X] = \mu + \alpha\gamma = 55.9;\ \sigma_x = \frac{\alpha\pi}{\sqrt{6}} = 6.708$$

$$solving:\ \mu = 52.88,\ \alpha = 5.23$$

$$F_x(70) = \exp\left[-e^{-(70-\mu)/\alpha}\right] = \exp\left[-e^{-(70-52.88)/5.23}\right] = .9628$$

Case 2: Constant Stress and Random Strength

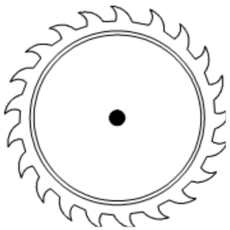


$$R = P(Y \geq s) = \int_s^\infty f_y(y)dy = 1 - F_y(s)$$

Case 2 Example

Material Properties

	Yield Stress k(lb/in^2)		Ultimate Stress, k(lb/in^2)	
Material	Mean	Std Deviation	Mean	Std Deviation
Aluminum alloy - forgings	63.0	2.23	70.0	1.89
Aluminum alloy - sheet	50.1	2.85	63.6	2.51
Titanium Alloy	130.6	7.2	135.5	6.7
Carbon steel (hot rolled sheet)	35.7	.80	48.3	.52
Carbon steel (hot rolled rnd bars)	49.5	5.36	86.2	3.92
High strength structural steel	49.6	3.69	76.9	2.06
Type 202 Stainless steel	49.9	1.32	99.7	2.71
Type 301 stainless steel	166.8	9.37	191.2	5.82
Malleable cast iron	34.9	1.47	53.4	2.68



Example Case 2

	Yield Stress k(lb/in^2)		Ultimate Stress, k(lb/in^2)	
Material	Mean	Std Deviation	Mean	Std Deviation
Aluminum alloy - forgings	63.0	2.23	70.0	1.89

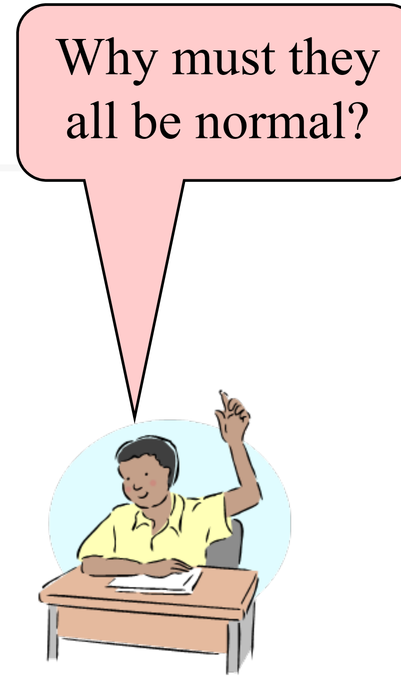
Observe that I am applying a constant load of 65 pounds per square inch.

Load = s = 65 lb/in^2

Assume normal:

$$R = 1 - \Phi\left(\frac{65-70}{1.89}\right) = 1 - \Phi(-2.645) = .99592$$

Case 2 Lognormal Example



$$m_y e^{s^2/2} = 70$$

$$m_y^2 e^{s^2}\left(e^{s^2} - 1\right) = 1.89^2$$

$$solving:$$

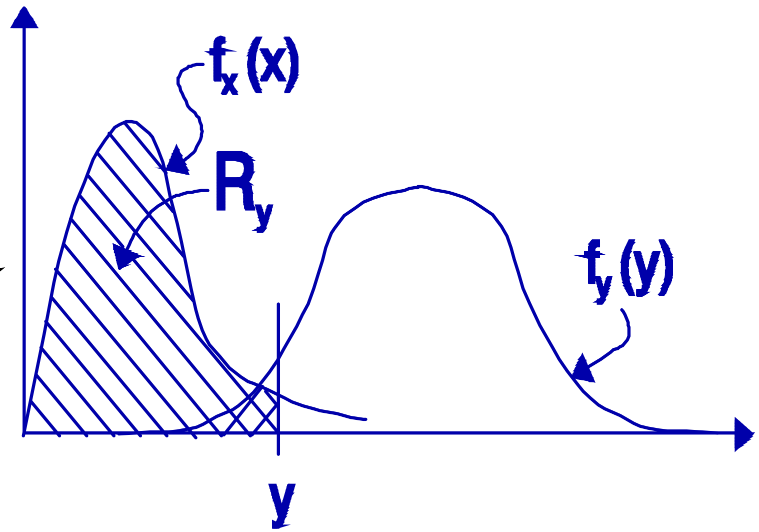
$$m_y^2 = \frac{70^2}{\left(e^{s^2/2}\right)^2};\quad \frac{70^2}{\left(e^{s^2/2}\right)^2} e^{s^2}\left(e^{s^2} - 1\right) = 1.89^2$$

$$s = \left[\ln\left(\frac{1.89^2}{70^2} + 1\right)\right]^{.5} = .026995$$

$$m_y = 69.974 \qquad R = 1 - \Phi\left(\frac{1}{.027}\ln\frac{65}{69.974}\right) = 1 - \Phi(-2.73) = .99683$$

Case 3: Random Stress and Random Strength

Interference Theory



$$R = Pr\{X \le Y\} = \int_0^\infty \left[\int_0^y f_x(x)dx\right] f_y(y)dy$$

$$= \int_0^\infty F_x(y)\, f_y(y)dy$$

Case 3 Example - X & Y Random

$$f_x(x) = \frac{200}{\left(x+10\right)^3};\ x \geq 0 \qquad f_y(y) = \frac{1}{10};\ 10 \leq y \leq 20$$

$$F_x(x) = \int_0^x \frac{200}{\left(x'+10\right)^3} dx' = \left.\frac{200}{-2\left(x'+10\right)^2}\right|_0^x = 1 - \frac{100}{\left(x+10\right)^2}$$

$$R = \int_0^\infty \left[\int_0^y f_x(x)\, dx\right] f_y(y)\, dy = \int_0^\infty \left[F_x(y)\right] f_y(y)\, dy$$

Continuing Case 3 Example

$$R = 1 \cdot F_x(10) + \int_{10}^{20} \left[\int_{10}^{y} f_x(x)dx \right] f_y(y)\,dy + 0 \cdot \left[1 - F_x(20)\right]$$

$$R = 1 - \frac{100}{(10+10)^2} + \int_{10}^{20} \left[\int_{10}^{y} \frac{200}{(x+10)^3} dx \right] f_y(y)\,dy + 0$$

$$= \left(1 - \frac{1}{4}\right) + \int_{10}^{20} \left[\frac{200}{-2(x+10)^2} \right]_{10}^{y} \frac{1}{10} dy + 0$$

$$R = \frac{3}{4} + \frac{1}{10} \int_{10}^{20} \left(\frac{1}{4} - \frac{100}{(y+10)^2} \right) dy = \frac{3}{4} + \frac{1}{10} \left[\frac{y}{4} + \frac{100}{(y+10)} \right]_{10}^{20} = .75 + .083 = .833$$

Exponential Case 3

$$f_x(x) = \frac{1}{\mu_x} e^{-\frac{x}{\mu_x}} \quad and \quad f_y(y) = \frac{1}{\mu_y} e^{-\frac{y}{\mu_y}}$$

$$R = \int_0^\infty \left[\int_0^y \frac{1}{\mu_x} e^{-\frac{x}{\mu_x}} dx \right] \frac{1}{\mu_y} e^{-\frac{y}{\mu_y}} dy$$

$$= \int_0^\infty \left[1 - e^{-\frac{y}{\mu_x}} \right] \frac{1}{\mu_y} e^{-\frac{y}{\mu_y}} dy$$

$$= \int_0^\infty \frac{1}{\mu_y} e^{-\frac{y}{\mu_y}} dy - \int_0^\infty \frac{1}{\mu_y} e^{-y\left(\frac{1}{\mu_x} + \frac{1}{\mu_y}\right)} dy$$

Exponential Case 3 (continued)

$$R = \int_0^\infty \frac{1}{\mu_y} e^{-\frac{y}{\mu_y}} dy - \int_0^\infty \frac{1}{\mu_y} e^{-y\left(\frac{1}{\mu_x}+\frac{1}{\mu_y}\right)} dy$$

$$= 1 - \frac{1}{\mu_y}\left[\frac{e^{-y\left(\frac{1}{\mu_x}+\frac{1}{\mu_y}\right)}}{-\left[\frac{1}{\mu_x}+\frac{1}{\mu_y}\right]}\right]\Bigg|_0^\infty$$

$$= 1 - \frac{1}{\mu_y}\frac{\mu_x \mu_y}{\mu_x + \mu_y} = 1 - \frac{\mu_x}{\mu_x + \mu_y} = \frac{\mu_y}{\mu_x + \mu_y} = \frac{1}{1 + \frac{\mu_x}{\mu_y}}$$

Exponential Case 3

u_y / u_x (safety factory)	u_x / u_y	Reliability $\dfrac{1}{1+\dfrac{\mu_x}{\mu_y}}$
1.0	1.0	0.50
1.11	0.9	0.53
1.25	0.8	0.56
1.43	0.7	0.59
1.67	0.6	0.63
2.0	0.5	0.67
2.5	0.4	0.71
3.33	0.3	0.77
5	0.2	0.83
10	0.1	0.91

Static Reliability -Normal Case

Let X = a normal stress with mean u_x and std deviation σ_x
and Y = a normal strength with mean u_y and std deviation σ_y.

Then R = P(Y >= X) = P(Y-X >= 0) = P(W >= 0)

where W = Y-X, E[W] = E[Y-X] = u_y - u_x and,

Var[W] = Var[Y-X] = $\sigma_y^2 + \sigma_x^2$

Y and X are independent

Static Reliability -Normal Case

$$R = P(W \geq 0) = P\left(\frac{W - \mu_W}{\sigma_W} \geq \frac{-(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right)$$

$$= P\left(\frac{W - \mu_W}{\sigma_W} \leq \frac{(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}} \right) = \Phi\left(\frac{\mu_y - \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}} \right)$$

Static Reliability -Lognormal Case

X = a lognormal stress with median m_x and shape parameter s_x
Y = a lognormal strength with median m_y and shape parameter s_y.

Then R = P(Y >= X) = P(Y/X >= 1)

Let W = ln(Y/X) = ln Y - ln X, then W is normal with mean,

$W = \ln(m_y/m_x)$ and, $\mathrm{Var}[W] = s_y^2 + s_x^2$ (Y and X are independent)

Static Reliability -Lognormal Case

$$R = P(W \geq \ln 1) = P(W \geq 0) = P\left(\frac{0-\mu_W}{\sigma_W} \geq \frac{0-\mu_W}{\sigma_W}\right)$$

$$= P\left(\frac{W-\mu_W}{\sigma_W} \leq \frac{\mu_W}{\sigma_W}\right) = \Phi\left(\frac{\ln\left(\frac{m_y}{m_x}\right)}{\sqrt{s_y^2+s_x^2}}\right)$$

Summary - Static Reliability and the Theoretical Distributions

Distribution	Constant Strength k	Constant Stress s	Random Stress and strength
Exponential	$R = 1 - \exp\left(-\frac{k}{\mu_x}\right)$	$R = \exp\left(-\frac{s}{\mu_y}\right)$	$R = \frac{\mu_y}{\mu_x + \mu_y}$
Weibull	$R = 1 - \exp\left(-\frac{k}{\theta_x}\right)^{B_x}$	$R = \exp\left(-\frac{s}{\theta_y}\right)^{B_y}$	solve numerically
Normal	$R = \Phi\left(\frac{k - \mu_x}{\sigma_x}\right)$	$R = 1 - \Phi\left(\frac{s - \mu_y}{\sigma_y}\right)$	$R = \Phi\left(\frac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right)$
Lognormal	$R = \Phi\left(\frac{1}{s_x} \ln \frac{k}{m_x}\right)$	$R = 1 - \Phi\left(\frac{1}{s_y} \ln \frac{s}{m_y}\right)$	$R = \Phi\left(\frac{\ln\left(m_y / m_x\right)}{\sqrt{s_x^2 + s_y^2}}\right)$

Mixed distributions: Exponential stress and gamma strength

$$f_x(x) = \frac{1}{\mu} e^{-x/\mu} \text{ and } f_y(y) = \frac{y^{\gamma-1} e^{-y/\alpha}}{\alpha^\gamma \Gamma(\gamma)}$$

$$F_x(y) = 1 - e^{-y/\mu} \text{ and } R = \int_0^\infty \left(1 - e^{-y/\mu}\right) \frac{y^{\gamma-1} e^{-y/\alpha}}{\alpha^\gamma \Gamma(\gamma)} dy$$

$$= \int_0^\infty \frac{y^{\gamma-1} e^{-y/\alpha}}{\alpha^\gamma \Gamma(\gamma)} dy - \frac{1}{\alpha^\gamma \Gamma(\gamma)} \int_0^\infty \left(e^{-y/\mu}\right) y^{\gamma-1} e^{-y/\alpha} dy = 1 - \frac{1}{\alpha^\gamma \Gamma(\gamma)} \int_0^\infty y^{\gamma-1} e^{-y(1/\alpha + 1/\mu)} dy$$

$$\text{Letting } k = \frac{\alpha\mu}{\mu + \alpha}, \quad R = 1 - \frac{k^\gamma}{\alpha^\gamma} \int_0^\infty \frac{y^{\gamma-1} e^{-y/k}}{k^\gamma \Gamma(\gamma)} dy = 1 - \left(\frac{k}{\alpha}\right)^\gamma = 1 - \left(\frac{\mu}{\mu + \alpha}\right)^\gamma$$

Gamma stress and exponential strength see Exercise 7.41

Example 7.12

The stress placed on a manufactured part has been measured in a lab to be exponentially distributed with a mean of 20 psi. Because of impurities in the manufacturing process, the strength of the part has been found to have a gamma distribution with a scale parameter of 70 psi and a shape parameter of 3.4. Therefore,

$$R = 1 - \left(\frac{\mu}{\mu+\alpha}\right)^{\gamma} = 1 - \left(\frac{20}{20+70}\right)^{3.4} = .994$$

Bonus Case 3 (no extra charge)

Normal strength and exponential stress

$$R = \Phi\left(\frac{\mu_y}{\sigma_y}\right) - e^{-.5\left(2\mu_y/\mu_x - \sigma_y^2/\mu_x^2\right)}\left[\Phi\left(\frac{\mu_y - \sigma_y^2/\mu_x}{\sigma_y}\right)\right]$$

exponential strength and normal stress

$$R = \Phi\left(-\frac{\mu_x}{\sigma_x}\right) - e^{-.5\left(2\mu_x/\mu_y - \sigma_x^2/\mu_y^2\right)}\left[\Phi\left(\frac{\mu_x - \sigma_x^2/\mu_y^2}{\sigma_x}\right)\right]$$

Reference: Kapur & Lamberson, *Reliability in Engineering Design*, Wiley 1977

Example of the bonus case

$$R = \Phi\left(\frac{\mu_y}{\sigma_y}\right) - e^{-.5\left(2\mu_y/\mu_x - \sigma_y^2/\mu_x^2\right)}\left[\Phi\left(\frac{\mu_y - \sigma_y^2/\mu_x}{\sigma_y}\right)\right]$$

$\text{strength:normal with } \mu_y = 100 \text{ psi and } \sigma_y = 10 \text{ psi}$

$\text{load is exponential with } u_x = 50 \text{ psi}$

$$R = \Phi\left(\frac{100}{10}\right) - e^{-.5\left(2(100)/50 - 100/50^2\right)}\left[\Phi\left(\frac{100 - 100/50}{10}\right)\right]$$

$$= 1 - e^{-1.98}\left[\Phi\left(9.8\right)\right] = 1 - .138 = .862$$

Summary

- Covariate Models
- Static Models
  - Case 1 random stress
  - Case 2 ransom strength
  - Case 3 random stress and strength

Coming soon in Chapter 7

- Dynamic Models
- Physics of Failure Models