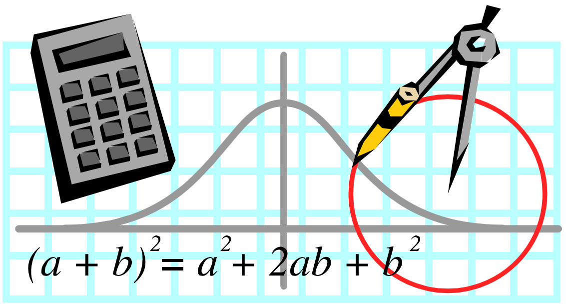
Chapter 15
Identifying Failure & Repair Distributions

Parameter Estimation

- maximum likelihood estimator



Maximum Likelihood Estimation (MLE)

Find estimates for the distribution parameters which will maximize the probability of obtaining the observed sample times.

$$\text{Max } f(t_1)\, f(t_2) \,.\,.\,.\, f(t_r)$$

Why MLE's????

1. MLE's are invariant: $y = h(\theta)$ then $\hat{y} = h(\hat{\theta})$

2. MLE's are Consistent:

$$as\ n \to \infty,\ \ \hat{\theta} \to \theta$$

3. MLE's are (best) asymptotically normal:

$$\sigma^2_{\hat{\theta}} \le \sigma^2_{\tilde{\theta}}$$

4. Required for certain tests such as the Chi-Square GOF test.
5. Has an intuitive appeal.
6. Can accommodate censored data

MLE - Geometric Distribution

Let X = a discrete random variable, the number of trials necessary to obtain the first failure. Assume the probability of a failure remains a constant p and each trial is independent, then:

$$\text{Prob}\{X=x\} = f(x) = (1-p)^{x-1}\, p, \quad x = 1, 2, \ldots, n$$

$$f_{x_1,\ldots,x_n}(x_1, x_2, \ldots, x_n) = f(x_1) f(x_2) \ldots f(x_n)$$

$$= (1-p)^{x_1-1}\, p (1-p)^{x_2-1}\, p \ldots (1-p)^{x_n-1}\, p$$

$$= p^n (1-p)^{\sum_{i=1}^{n} (x_i - 1)}$$

Geometric Distribution

$$\text{Max} \quad g(p) = p^{n}(1-p)^{\sum_{i=1}^{n}(x_i-1)}$$

$$0 \le p \le 1$$

$$\ln g(p) = n \ln p + \left[\sum_{i=1}^{n}(x_i - 1)\right] \ln(1-p) = 0$$

$$\frac{n}{p} + \frac{\sum_{i=1}^{n}(x_i - 1)}{1-p}(-1) = 0$$

$$\boxed{\hat{p} = \frac{n}{n + \sum_{i=1}^{n}(x_i - 1)} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

Example 15.13

The following data was collected on the number of production runs which resulted in a failure which stopped the production line: 5, 8, 2, 10, 7, 1, 2, 5. Therefore, X = the number of production runs necessary to obtain a failure.

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{8}{40} = .2$$

$$Prob[X = x] = f(x) = .8^{(x-1)}\ (.2)$$

$$\text{Mean} = 1/p = 40/8 = 5$$

$$\Pr\{X = 3\} = .8^2\ (0.2) = 0.128$$

Likelihood Function

$$L(\theta_1,\ldots,\theta_k) = \prod_{i=1}^{n} f(t_i|\theta_1,\ldots,\theta_k)$$

maximize the log of the likelihood function:

$$\frac{\partial \ln L(\theta_1,\ldots,\theta_k)}{\partial \theta_i} = 0 \;;\; i = 1,2,\ldots k$$

for Type I (right censored) data:

$$L(\theta_1,\ldots,\theta_k) = \prod_{i=1}^{r} f(t_i|\theta_1,\ldots,\theta_k)\left[R(t_*)\right]^{n-r}$$

for Type II, t_r

Exponential MLE - Type II data

$$f(t_j) = \lambda \, e^{-\lambda t_j}, \quad j = 1, 2, \ldots r$$

$$P[\, T_j > t_r \text{ for all } j > r \,] = \left(e^{-\lambda t_r}\right)^{n-r}$$

$$L(t_1, \ldots, t_r) = \prod_{j=1}^{r} \lambda e^{-\lambda t_j} \left(e^{-\lambda t_r}\right)^{n-r}$$

$$= \lambda^r \exp\left\{ -\lambda \sum_{j=1}^{r} t_j - \lambda(n-r)\, t_r \right\}$$

Exponential MLE - Type II data

$$L = \lambda^r \exp\left\{-\lambda \sum_{j=1}^{r} t_j - \lambda(n-r)\, t_r\right\}$$

$$\ln L = r \ln \lambda - \lambda \sum_{j=1}^{r} t_j - \lambda(n-r)\, t_r$$

$$\frac{d \ln L}{d\lambda} = \frac{r}{\lambda} - \sum_{j=1}^{r} t_j - (n-r)t_r = 0$$

$$\boxed{\hat{\lambda} = \frac{r}{\sum_{j=1}^{r} t_j + (n-r)t_r} = \frac{r}{T}}$$

Type I, use t_*

Total Time on Test - CFR

n = nbr on test
r = nbr failures
k = nbr multiply censors
t_i = failure time
t_i^+ = censor time
t_* = test time (Type I)
t_r = test timr (Type II)

MTTF = T / r

Complete: $\sum_{i=1}^{n} t_i;\ r = n$

Type I: $\sum_{i=1}^{n} t_i + (n-r)t_*$

Type II: $\sum_{i=1}^{n} t_i + (n-r)t_r$

Type I multiply: $\sum_{i=1}^{n} t_i^+ + (n-r-k)t_*$

Type II multiply: $\sum_{i=1}^{n} t_i^+ + (n-r-k)t_r$

Type I replacement: nt_*

Type II replacement: nt_r

Weibull MLE - Type II Data

$$L(\theta,\beta)=\prod_{i=1}^{r} f(t)R(t_r)^{n-r}=\left[\prod_{i=1}^{r}\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta-1}e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right]\left[e^{-\left(\frac{t_r}{\theta}\right)^{\beta}}\right]^{n-r}$$

$$\ln L = r\ln\beta-\beta r\ln\theta+\sum_{i=1}^{r}(\beta-1)\ln t_i-\sum_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{\beta}-(n-r)\left(\frac{t_r}{\theta}\right)^{\beta}$$

$$\frac{\partial\ln L}{\partial\theta}=-\beta r+\frac{\beta}{\theta^{\beta}}\sum_{i=1}^{r}t_i^{\ \beta}+\frac{(n-r)\beta}{\theta^{\beta}}t_r^{\ \beta}=0$$

$$\frac{\partial\ln L}{\partial\beta}=\frac{r}{\beta}+\sum_{i=1}^{r}\ln t_i-r\ln\theta+r\ln\theta-\frac{r\sum_{i=1}^{r}t_i^{\ \beta}+(n-r)t_r^{\ \beta}\ln t_r}{\sum_{i=1}^{r}t_i^{\ \beta}+(n-r)t_r^{\ \beta}}=0$$

Weibull MLE - singly censored

$$g(\hat{\beta}) = \frac{\sum_{i=1}^{r} t_i^{\hat{\beta}} \ln t_i + (n-r) t_s^{\hat{\beta}}}{\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r) t_s^{\hat{\beta}}} - \frac{1}{\hat{\beta}} - \frac{1}{r} \sum_{i=1}^{r} \ln t_i = 0$$

$$\hat{\theta} = \left\{ \frac{1}{r} \left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r) t_s^{\hat{\beta}} \right] \right\}^{\frac{1}{\hat{\beta}}}$$

$$\text{where } t_s = \begin{cases} 1 & \text{for complete data} \\ t_* & \text{for type I data} \\ t_r & \text{for type II data} \end{cases}$$

Newton-Raphson Method

$$\hat{\beta}_{j+1} = \hat{\beta}_j - \frac{g(\hat{\beta}_j)}{g'(\hat{\beta}_j)}$$

where $$g'(x) = \frac{d\,g(x)}{dx}$$

Normal & Lognormal MLEs complete data

NORMAL

$$\hat{\mu} = \bar{x}$$

$$\hat{\sigma}^2 = \frac{(n-1)s^2}{n}$$

recall:

$$s^2 = \sum_{i=1}^{n} \frac{(t_i - \widehat{MTTF})^2}{n-1}$$

LOGNORMAL

$$\hat{\mu} = \sum_{i=1}^{n} \frac{\ln t_i}{n}$$

$$\hat{t}_{MED} = e^{\hat{\mu}}$$

$$\hat{s} = \sqrt{\frac{\sum_{i=1}^{n} (\ln t_i - \hat{\mu})^2}{n}}$$

Example 15.16

Ex. 15.8: 47.1, 84.8, 151.9, 122.5, 218.2, 99.6, 59.8, 138.8, 213.5, 53.4,102.4, 100.8, 230.1, 104.6, 61.5, 122.1, 186.2, 498.4, 77.0, 78.7,112.3, 44.0, 151.3, 151.3, 222.8

L-S estimates: $\mathbf{t_{med} = 116}$ **and** $\mathbf{s = .613}$

μ = (ln 44 + ln 47.1 + … + ln 498.4)/25 = 118.8/25 = 4.752

$t_{med} = e^{4.753} = 115.932$

$s^2 = [(\ln 44 - 4.752)^2 + (\ln 47.1 - 4.752)^2 +$

$\ldots + (\ln 498.4 - 4.752)^2]/25 = .31798$

s = .5639.

MLE with Multiply Censored Data

prob of failure occurring at time t_i

prob of failure occurring after time t_i^+

$$L(\theta) = \prod_{i \varepsilon F} f(t_i;\theta) \prod_{i \varepsilon C} R(t_i^+;\theta)$$

F = set of indices for failure times

C = set of indices for censored times (including singly censored times)

MLE Exponential - multiply censored data

$$L(\lambda) = \prod_{i \varepsilon F} \lambda\, e^{-\lambda t_i} \prod_{i \varepsilon C} e^{-\lambda t_i^+} = \lambda^r e^{-\lambda \sum_{i \varepsilon F} t_i} \quad e^{-\lambda \sum_{i \varepsilon C} t_i^+}$$

$$\ln L(\lambda) = r \ln \lambda - \lambda \sum_{i \varepsilon F} t_i - \lambda \sum_{i \varepsilon C} t_i^+$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{r}{\lambda} - \sum_{i \varepsilon F} t_i - \sum_{i \varepsilon C} t_i^+ = 0$$

$$\boxed{\hat{\lambda} = \frac{r}{\sum_{i \varepsilon F} t_i + \sum_{i \varepsilon C} t_i^+}}$$

T

MLE Weibull multiply censored data

$$L(\theta,\beta)=\prod_F f(t_i)\prod_C R(t_i)=\left[\prod_F \frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta-1} e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right]\prod_C\left[e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right]$$

$$\ln L=\sum_F\left[\ln\beta-\beta\ln\theta+(\beta-1)\ln t_i-\left(\frac{t_i}{\theta}\right)^{\beta}\right]-\sum_C\left(\frac{t_i}{\theta}\right)^{\beta}$$

$$\frac{\partial L}{\partial\theta}=\sum_F\left[\frac{-\beta}{\theta}+\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta}\right]+\sum_C\left[\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta}\right]=0$$

$$\frac{\partial L}{\partial\beta}=\sum_F\frac{1}{\beta}+\ln\left(\frac{t_i}{\theta}\right)-\left(\frac{t_i}{\theta}\right)^{\beta}\ln\left(\frac{t_i}{\theta}\right)-\sum_C\left(\frac{t_i}{\theta}\right)^{\beta}\ln\left(\frac{t_i}{\theta}\right)=0$$

MLE Weibull - multiply censored data

$$\sum_{i \varepsilon F} \frac{\ln t_i}{r} = \sum_{\text{all } i} t_i^{\beta} \ln t_i \sum_{\text{all } i} (t_i^{\beta})^{-1} - \frac{1}{\beta}$$

solve numerically

monotonically increasing RHS

$$\hat{\theta} = \left[\sum_{\text{all } i} \frac{t_i^{\hat{\beta}}}{r} \right]^{\frac{1}{\hat{\beta}}}$$

Example 15.17

Fifteen units were placed on test for 500 hours. The following failure times and censor times were observed prior to concluding the test:

34 136 145+ 154 189 200+ 286 287 334 353 380+

For the exponential,
T = 34+136+145+154+189+200+286+287+334+353+380+4(500) = 4498 and the MLE for the MTTF = T/r = 4498/8 = 562.25.

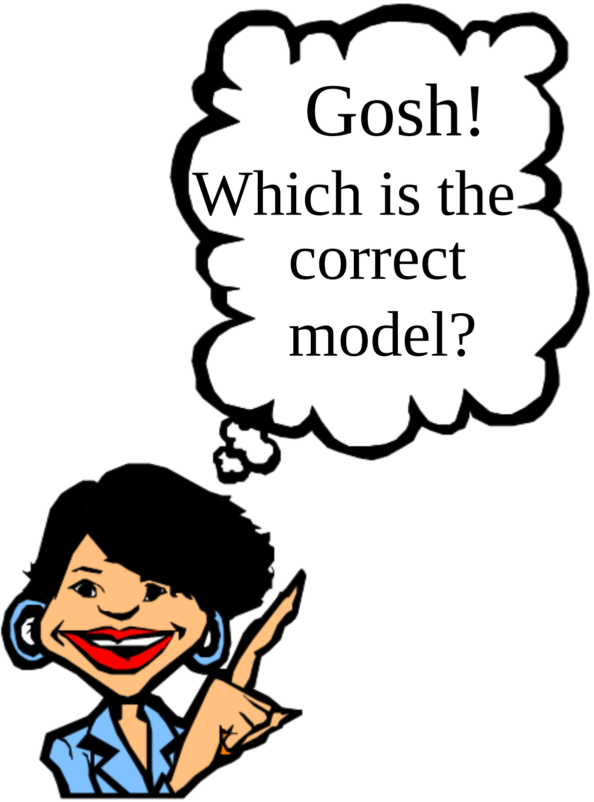
For the Weibull, the 4 units which had not failed by the end of the test are assigned censored times of 500 hours. The left hand side of MLE Eq. equals 5.21385. Beginning with β = .1 and increasing β in the right hand side by .01 until it exceeds 5.21, results in β = 1.43. Then θ = 491.

Example 15.17

exponential:

$$R(t) = e^{-t/562.25}$$

R(100) = .837



Weibull:

$$R(t) = e^{-\left(\frac{t}{491}\right)^{1.43}}$$

R(100) = .902

Normal Distribution - Censored Data

$$L(\mu,\sigma)=\prod_{i\varepsilon F} f(t_i)\prod_{i\varepsilon C} R(t_i^{+})$$

$$\ln L(\mu,\sigma)=\sum_{i=1}^{r}\ln\left[\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}}\right]$$

$$+\sum_{i=1}^{n-r}\ln\left[\int_t^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma}\;e^{-\frac{(t-\mu)^2}{2\sigma^2}}\;dt'\right]$$

Maximize using a
Numerical search algorithm

Minimum Extreme Value Distribution

Complete or right censored data
Likelihood function:

$$L(\alpha,\mu)=\prod_{i=1}^{r}\left(\frac{1}{\alpha}\right)e^{\frac{(t_i-\mu)}{\alpha}}e^{-e^{\frac{(t_i-\mu)}{\alpha}}}\prod_{i=}^{n-r}e^{-e^{\frac{\left(t_s^{+}-\mu\right)}{\alpha}}}$$

$$\ln L(\alpha,\mu)=-r\ln\alpha+\sum_{i=1}^{r}\frac{(t_i-\mu)}{\alpha}-\sum_{i=1}^{r}e^{\frac{(t_i-\mu)}{\alpha}}-(n-r)e^{\frac{\left(t_s^{+}-\mu\right)}{\alpha}}$$

$$\text{where } t_s=\begin{cases}1 & \text{for complete data}\\ t_* & \text{for Type I data}\\ t_r & \text{for Type II data}\end{cases}$$

Minimum Extreme Value Distribution

$$\frac{\partial \ln L(\alpha,\mu)}{\partial \alpha} = \frac{\partial \ln L(\alpha,\mu)}{\partial \mu} = 0$$

$$-\hat{\alpha} - \frac{1}{r}\sum_{i=1}^{r} t_i + \frac{\sum_{i=1}^{r} t_i e^{t_i/\hat{\alpha}} + (n-r)t_s^+ e^{t_s^+/\hat{\alpha}}}{\sum_{i=1}^{r} e^{t_i/\hat{\alpha}} + (n-r)e^{t_s^+/\hat{\alpha}}} = 0$$

$$\hat{\mu} = \hat{\alpha}\ln\left[\frac{1}{r}\sum_{i=1}^{r} e^{t_i/\hat{\alpha}} + \left(\frac{n-r}{r}\right)e^{t_s^+/\hat{\alpha}}\right]$$

Minimum Extreme Value Distribution

Complete data

Method of Moments:

$$m_1 = \sum_{i=1}^{n} t_i / n; \; m_2 = \sum_{i=1}^{n} t_i^2 / n$$

$$m_1 = \mu - \gamma\alpha; \;\; m_2 = \frac{\alpha^2 \pi^2}{6} + \left(\mu - \gamma\alpha\right)^2$$

solving:

$$\tilde{\alpha} = \sqrt{\frac{6\left(m_2 - m_1^2\right)}{\pi^2}}; \;\; \tilde{\mu} = m_1 + \gamma\tilde{\alpha}; \;\;\; \gamma \approx .577215665$$

Gamma Distribution

Complete data, likelihood function:

$$L(t_i, i=1,\ldots,n \mid \gamma, \alpha) = \prod_{i=1}^{n} \frac{t_i^{\gamma-1} e^{-t_i/\alpha}}{\alpha^{\gamma}\Gamma(\gamma)}$$

$$\ln L = (\gamma - 1)\sum_{i=1}^{n} \ln t_i - \frac{1}{\alpha}\sum_{i=1}^{n} t_i - n\gamma \ln \alpha - n \ln \Gamma(\gamma)$$

$$\frac{\partial \ln L(\alpha,\gamma)}{\partial \alpha} = 0 \text{ and solving for } \alpha : \hat{\alpha} = \frac{1}{n\gamma}\sum_{i=1}^{n} t_i$$

Substituting for α: $$\ln L(\gamma) = (\gamma - 1)\sum_{i=1}^{n} \ln t_i - n\gamma - n\gamma \ln\left(\frac{1}{n\gamma}\sum_{i=1}^{n} t_i\right) - n \ln \Gamma(\gamma)$$

Maximize directly

Gamma Distribution

Complete data
Method of Moments:

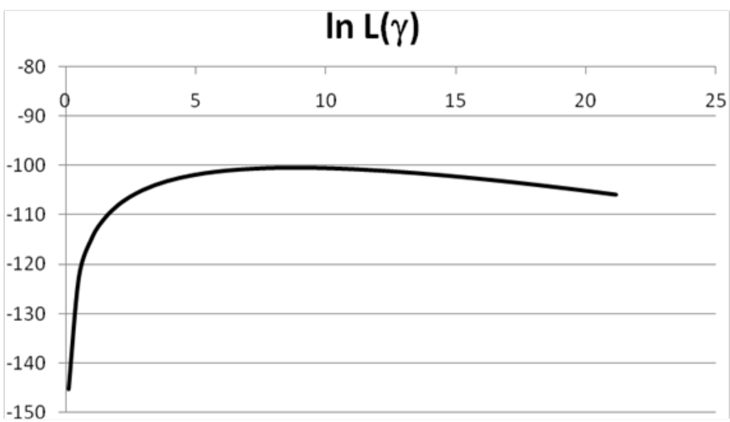
$$m_1 = \alpha\gamma; \;\; m_2 = \gamma\alpha^2 + \left(\gamma\alpha\right)^2$$

$$\tilde{\gamma} = \frac{m_1^2}{m_2 - m_1^2}; \;\; \tilde{\alpha} = \frac{m_2 - m_1^2}{m_1}$$

EXAMPLE 15.20

Twenty units believed to have a gamma distribution were placed on an accelerated life test with failures in days occurring at the times shown: 152 152 115 109 137 88 94 77 160 165 125 40 128 123 136 101 62 153 83 69. Using Excel Solver,

$$\underset{\gamma}{Max} \ln L(\gamma) = 93.469281(\gamma - 1) - 20\gamma - 20\gamma \ln\left(\frac{2269}{20\gamma}\right) - 20 \ln \Gamma(\gamma)$$



$$\hat{\gamma} = 8.7992 \qquad \hat{\alpha} = 12.893$$

The corresponding method of moments estimators are

$$\tilde{\gamma} = 10.5773 \ and \ \tilde{\alpha} = 10.7258$$

Parameter Estimation for Interval Data

n_j , is the number of failures that occur within the interval (a_{j-1}, a_j) where j = 1, ,…,k. Any right censored units are counted in the interval (a_k, ∞). The likelihood function can be stated as

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{k+1} \left[R(a_{j-1} \mid \boldsymbol{\theta}) - R(a_j \mid \boldsymbol{\theta}) \right]^{n_j}$$

EXAMPLE 15.21 (Weibull)

Monthly failures of fifty units in operation were recorded over a six month period with the following results:

Month	Upper bound in days	Number of Failures
1	30.0	10
2	60.0	11
3	90.0	7
4	120.0	4
5	150.0	3
6	180.0	2

$$L(\theta,\beta)=\prod_{j=1}^{k+1}\left[e^{-\left(\frac{a_{j-1}}{\theta}\right)^{\beta}}-e^{-\left(\frac{a_j}{\theta}\right)^{\beta}}\right]^{n_j} \longrightarrow \qquad \ln L(\theta,\beta)=\sum_{j=1}^{k+1} n_j \ln\left[e^{-\left(\frac{a_{j-1}}{\theta}\right)^{\beta}}-e^{-\left(\frac{a_j}{\theta}\right)^{\beta}}\right] \longrightarrow$$

EXAMPLE 15.21 (Weibull)

Using Excel Solver, maximize

$$\ln L(\theta,\beta)=\ln L(\theta,\beta)=10\ln\left[1-e^{-\left(\frac{30}{\theta}\right)^{\beta}}\right]+11\ln\left[e^{-\left(\frac{30}{\theta}\right)^{\beta}}-e^{-\left(\frac{60}{\theta}\right)^{\beta}}\right]+7\ln\left[e^{-\left(\frac{60}{\theta}\right)^{\beta}}-e^{-\left(\frac{90}{\theta}\right)^{\beta}}\right]$$

$$+4\ln\left[e^{-\left(\frac{90}{\theta}\right)^{\beta}}-e^{-\left(\frac{120}{\theta}\right)^{\beta}}\right]+3\ln\left[e^{-\left(\frac{120}{\theta}\right)^{\beta}}-e^{-\left(\frac{150}{\theta}\right)^{\beta}}\right]+2\ln\left[e^{-\left(\frac{150}{\theta}\right)^{\beta}}-e^{-\left(\frac{180}{\theta}\right)^{\beta}}\right]+13\ln\left[e^{-\left(\frac{180}{\theta}\right)^{\beta}}\right]$$

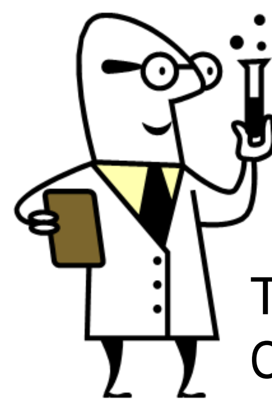
$$\hat{\beta}=.9486 \text{ and } \hat{\theta}=125.612$$

Next Time

Chapter 16

Goodness-of-Fit Testing

The Chi-Square test



Testing for
Chi-squares