LOGM 634-ASAM - Homework #1 (Solution)

20 September 2017

From the Ebeling text - Exercise 2.2

A component has the following linear hazard rate, where t is in years:

$$\lambda(t) = 0.4t \qquad t \geq 0$$

a. Find $R(t)$ and determine the probability of a component failing within the first month of its operation.

   To find $R(t)$, utilize the following relationship

$$R(t) = \exp\left[-\int_0^t \lambda(t)\; dt\right]$$

   Which can be used to convert between the hazard and reliability functions. Using this relationship we can find that $R(t)$ is expressed as

$$R(t) = \exp\left[-\int_0^t 0.4t\; dt\right] = \exp\left[-0.2t^2\right]$$

   The reliability of this componet after one month can be found by substituting $t = 1/12$ into the reliability function above

$$\begin{aligned} R(t = 1/12 \text{ yr}) &= \exp\left[-0.2 \times \frac{1}{12^2}\right] \\ &= \exp\left[-0.2 \times \frac{1}{144}\right] \\ &= \exp\left[-0.2 \times \frac{1}{12^2}\right] \\ &= \exp\left[-0.001388889\right] = 0.9986121 \end{aligned}$$

The failure probability is then $1 - 0.9986 =$ **0.0014**.

b. What is the design life if a reliability of 0.95 is desired?

The design life $t_{0.95}$ is the value of t at which $R(t) = 0.95$. This value can be found by setting the expression for $R(t)$ that we just determined above equla to 0.95 and algebraically solving for t, as shown below.

$$R(t_{0.95}) = 0.95$$

$$\exp\left[-0.2 t_{0.95}^2\right] = 0.95$$

$$t_{0.95} = \sqrt{\frac{-\ln[0.95]}{0.2}} = \mathbf{0.5064252} \text{ years}$$

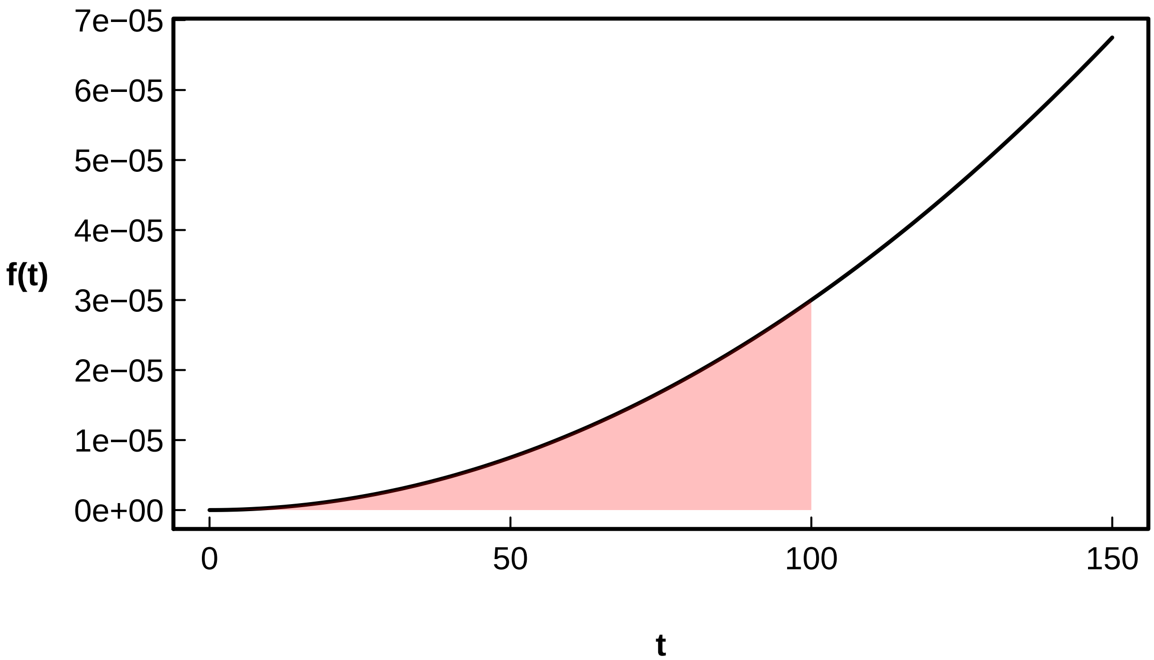
From the Ebeling text - Exercise 2.4

The failure distribution is defined by

$$f(t) = \frac{3t^2}{10^9} \qquad 0 \leq t \leq 1000 \text{ hr}$$

a. What is the probability of failure within a $100 - \text{hr}$ warranty period?

   In this question, we are asked to find the probability of failure over the time interval $[0, 100]$. This probability is shown graphically in the figure below.



The probability of failure over an interval can be determined by the CDF $F(t)$. To find an expression for $F(t)$, we integrate the pdf $f(t)$ as shown below.

$$F(t) = \int_0^t f(t)\, dt = \int_0^t \frac{3t^2}{10^9}\, dt = \frac{t^3}{10^9}$$

The probability of failure in the time interval $[0, 100]$ (corresponding to the shaded area in the figure above) is then computed as

$$F(100) - F(0) = \frac{100^3}{10^9} - \frac{0}{10^9} = \mathbf{0.001}.$$

b. Compute the MTTF

The mean (or expected value) of any random variable is defined as

$$\text{MTTF} = E[T] = \int_0^\infty t f(t)\, dt$$

Using the provided expression for $f(t)$ we can find the MTTF as

$$\begin{aligned}
\text{MTTF} &= \int_0^{1000} t f(t)\, dt \\
&= \int_0^{1000} \frac{3t^3}{10^9}\, dt \\
&= \left.\frac{3t^4}{4 \times 10^9}\right|_0^{1000} \\
&= \frac{3 \times 10^{12}}{4 \times 10^9} - \frac{0}{4 \times 10^9} \\
&= \frac{3}{4} \times \frac{10^{12}}{10^9} \\
&= 0.75 \times 1000 = \mathbf{750} \text{ hours}
\end{aligned}$$

---

c. Find the design life for a reliability of 0.99

The design life $t_{0.99}$ is the value of t at which $R(t) = 0.99$. This value can be found as follows:

$$\begin{aligned}
R(t_{0.99}) &= 0.99 \\
1 - \frac{t_{0.99}^3}{10^9} &= 0.99 \\
t_{0.99} &= \left[0.01 \times 10^9\right]^{1/3} = \mathbf{215.4435} \text{ hours}
\end{aligned}$$

From the Ebeling text - Exercise 2.11

A new fuel injection system is experiencing high failure rates. This reliability function has been found to be

$$R(t) = (t+1)^{-3/2} \qquad t \geq 0$$

where t is measured in years. The reliability over its intended life of 2 yr is 0.19, which is unacceptable. Will a burn-in period of 6 months significantly improve upon this reliability? If so, by how much?

This question asks us to compute the conditional reliability $R(2|0.5)$. That is, the reliability of a population of items after two years, considering the items that have already survived a 6-month burn-in test prior to fielding.

This quantity is found as follows:

$$\begin{aligned} R(2|0.5) &= \frac{R(2.5)}{R(0.5)} \\ &= \frac{(2.5+1)^{-3/2}}{(0.5+1)^{-3/2}} \\ &= \frac{0.1527207}{0.5443311} = 0.2805659 \end{aligned}$$

The difference in reliability is then $0.2805 - 0.19 = \mathbf{0.0905}$

From the Ebeling text - Exercise 3.1

A component experiences chance (CFR) failures with an MTTF of 1100 hr. Find the following:

a. The reliability for a 200-hr mission

A constant failure rate (CFR) implies an exponential distribution. The reliability function for the exponential distribution is expressed as

$$R(t) = \exp[-\lambda t]$$

We are told that MTTF = 1100, which implies that $\lambda = 1/\text{MTTF} = 0.00091$. With λ known, we find that

$$R(200) = \exp[-0.00091 \times 200] = \mathbf{0.8336013}$$

---

b. The design life for a 0.90 reliability

The design life $t_{0.90}$ is the value of t at which $R(t) = 0.90$. This value can be found as follows:

$$R(t_{0.90}) = 0.90$$

$$\exp[-0.00091 \cdot t_{0.90}] = 0.90$$

$$t_{0.90} = \left[\frac{-\ln[0.90]}{0.00091}\right] = \mathbf{115.7808} \text{ hours}$$

---

c. The median time to failure

The median time to failure is the same as the design life $t_{0.50}$ which is the value of t at which $R(t) = 0.50$. This value can be found as follows:

$$R(t_{0.50}) = 0.50$$

$$\exp[-0.00091 \cdot t_{0.50}] = 0.50$$

$$t_{0.50} = \left[\frac{-\ln[0.50]}{0.00091}\right] = \mathbf{761.7002} \text{ hours}$$

From the Ebeling text - Exercise 3.2

A CFR system with $\lambda = 0.0004$ has been operating for 1000 hr. What is the probability that it will fail in the next 100 hr? The next 1000 hr?

> This exercise is asking us to find the conditional reliabilities $R(100|1000)$ and $R(1000|1000)$. These values can be found as follows.

$$R(100|1000) = \frac{R(1100)}{R(1000)}$$

$$= \frac{\exp[-0.0004 \cdot 1100]}{\exp[-0.0004 \cdot 1000]}$$

$$= \exp[-0.0004 \cdot (1100 - 1000)]$$

$$= \exp[-0.004 \cdot 100]$$

$$= \mathbf{0.67032}$$

$$R(1000|1000) = \frac{R(2000)}{R(1000)}$$

$$= \frac{\exp[-0.0004 \cdot 2000]}{\exp[-0.0004 \cdot 1000]}$$

$$= \exp[-0.0004 \cdot (2000 - 1000)]$$

$$= \exp[-0.004 \cdot 1000]$$

$$= \mathbf{0.0183156}$$

Summary Exercise

A relay circuit has an average failure rate of 4 failures every 3 years. The circuit's failure times follow an exponential distribution.

a. What is the probability that the circuit will survive for one year without failure?

   The relay circuit has a constant failure rate of $\lambda = 4/3$. Therefore, the probability that a circuit will survive the first year is expressed as

$$R(1) = \exp[-4/3 \cdot 1] = \mathbf{0.2635971}$$

---

b. What is the probability that there will be more than two failures in the first year?

   This question pertains to the relationship between the exponential distribution and the Poisson distribution.

   Suppose that we are concerned about the reliability of a circuit in a subsystem. We only have two of these circuits on hand and - due to manufacturing issues - we cannot get any more for 1 year. We're concerned that both of these circuits will fail before we can get more into supply.

   We've been told that the failure times for these circuits follow an exponential distribution with a rate parameter $\lambda = 4/3$. Therefore, over a one-year period we would expect to see $\lambda \cdot t = 4/3 \cdot 1 = 1.333$ failures in a typical year. but what is the probability that this is NOT a typical year and three or more failures occur?

   Mathematically speaking, subsystem failure means that the sum of three exponentially-distributed failure times is less than 1-year. The Poisson distribution expresses the probability that n exponential distributed events occur by some time t. The pdf for the Poisson distribution is expressed as:

$$f(n) = \frac{e^{-\lambda t} \lambda t^n}{n!}$$

   To determine the probability that $n = 3$ events occur in 1 year, we first determine the expected number of events in a year. This value is the rate at which the events occur, which is $\lambda = 4/3$ multipied by the time of interest. Therefore, $\lambda t = 4/3 \times 1 = 4/3$.

   Now that we know this expected number of events, we can find probability that $n = 3$ events occur using the Poisson pdf. This value is

$$f(n = 3) = \frac{e^{-4/3} 4/3^3}{3!} = \mathbf{0.1041371}$$

   Finally, the probability that $n > 2$ is equal to $1 - P(n \leq 2)$ which is expressed as

$$1 - F(n = 2) = 1 - \frac{e^{-4/3} 4/3^0}{0!} + \frac{e^{-4/3} 4/3^1}{1!} + \frac{e^{-4/3} 4/3^2}{2!} = \mathbf{0.1506314}$$

---

c. What is the expected number of failures per year?

   The expected number of failures is computed as λt. For this problem, the expected number of failures during the first year of operation is

$$E[\text{failures}] = \lambda t = \frac{4 \text{ failures}}{3 \text{ years}} \times 1 \text{ years} = \mathbf{1.333} \text{ failures}$$