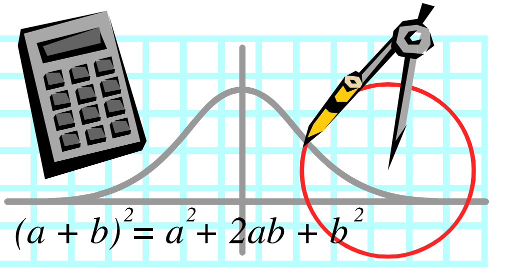
Chapter 15 – Part I
Identifying Failure & Repair Distributions

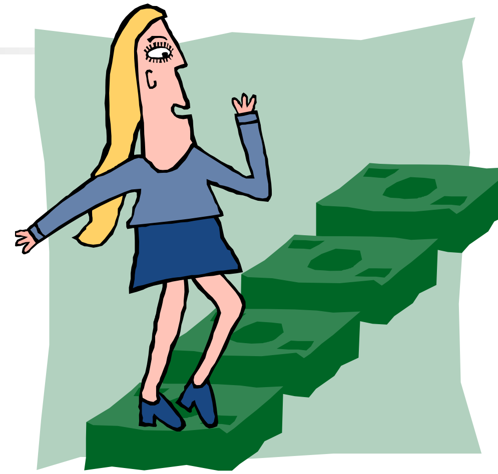
Identifying Candidate Distributions
Probability Plots & Least-squares curve-fitting



Fitting Theoretical Distributions

- Empirical models do not provide information beyond the range of the sample data.
- A sample is only a small (random) subset of the population of failure times, and it is the distribution the sample came from and not the sample itself which we want to establish.
- Often the failure process is a result of some physical phenomena which can be associated with a particular distribution.
- Small sample sizes provide very little information concerning the failure process. However, if the sample is consistent with a theoretical distribution, then much "stronger" results are possible based upon the properties of the theoretical distribution.
- Use can be made of the theoretical reliability model in performing more complex analysis of the failure process.

Identifying Candidate Distributions



3 - STEP PROCESS:

1. Identify candidate distributions
   - Construct a histogram
   - Compute descriptive stats
   - Analyze empirical failure rate
   - Use prior knowledge
   - Use properties of theoretical distribution
   - construct a probability plot
2. Estimate parameters (Maximum Likelihood)
3. Perform a goodness-of-fit tests (Chapter 16)

Histograms
Sturges' rule for grouping data

$$k = \lfloor 1 + 3.3 \log_{10} n \rfloor$$

where k = number of classes,
n = sample size.
$\lfloor x \rfloor$ = integer part of x

For example,

n	k
50	7
500	10
5000	13

Example 15.1

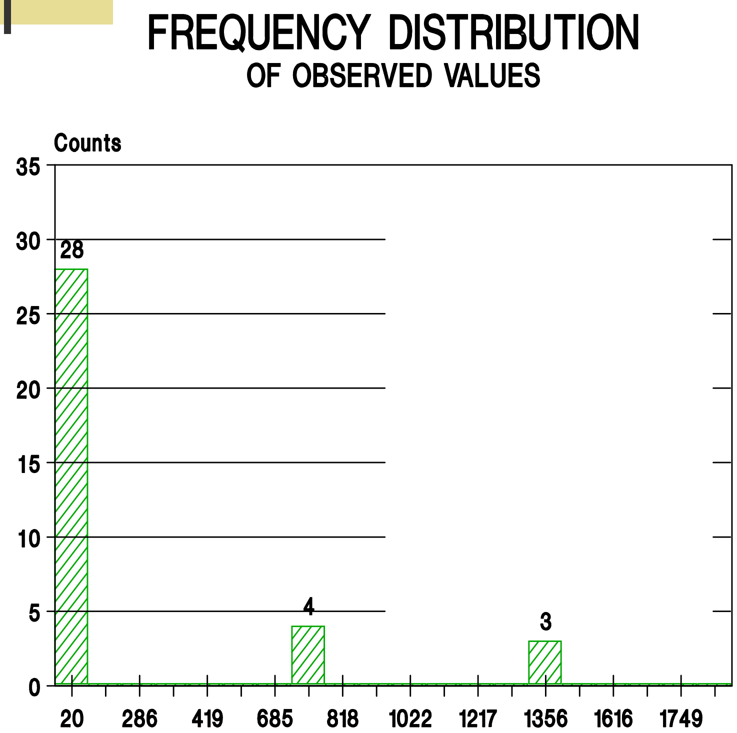
Given the following 35 failure times:

1476	300	98	221	157
1825	499	552	1563	36
246	442	20	796	31
47	438	400	279	247
210	284	553	767	1297
214	428	597	2025	185
467	401	210	289	1024

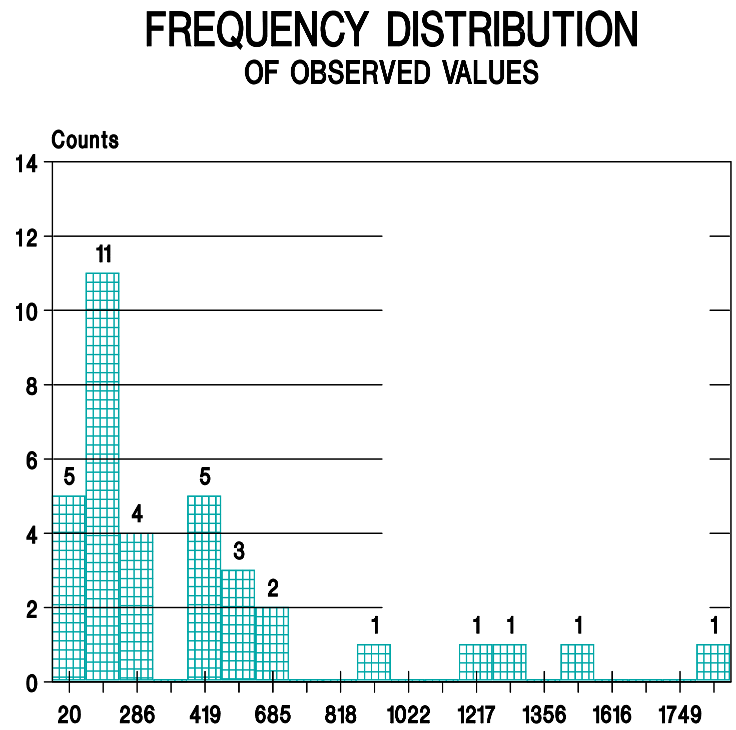
From Sturges' rule:

$$k = 1 + 3.3 \log 10\ 35 = 1 + 3.3\ (1.544) = 6.0954 \approx 6$$

Example 15.1

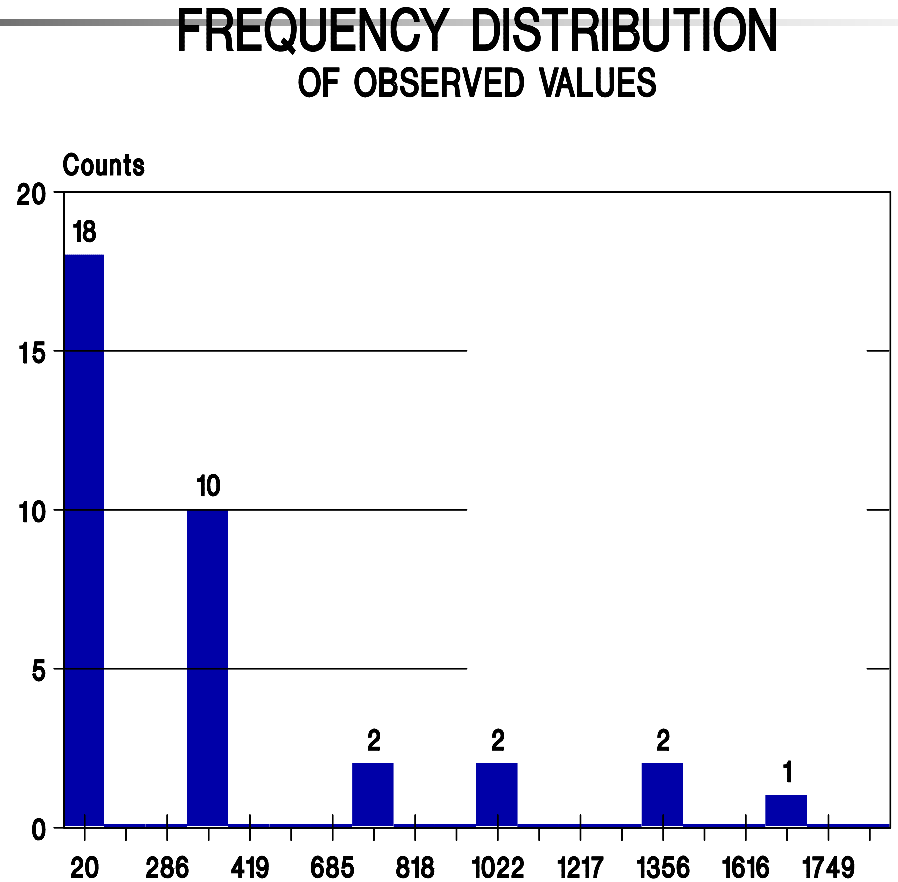


too few classes



too many classes

Example 15.1



JUST RIGHT!

Descriptive Statistics

Rank ordered data:

20	31	36	47	98
157	182	185	210	210
214	221	246	247	279
284	289	**300**	400	401
428	438	442	467	499
552	553	597	767	796
1024	1297	1476	1563	2025

$$\hat{t}_{med} = 300$$

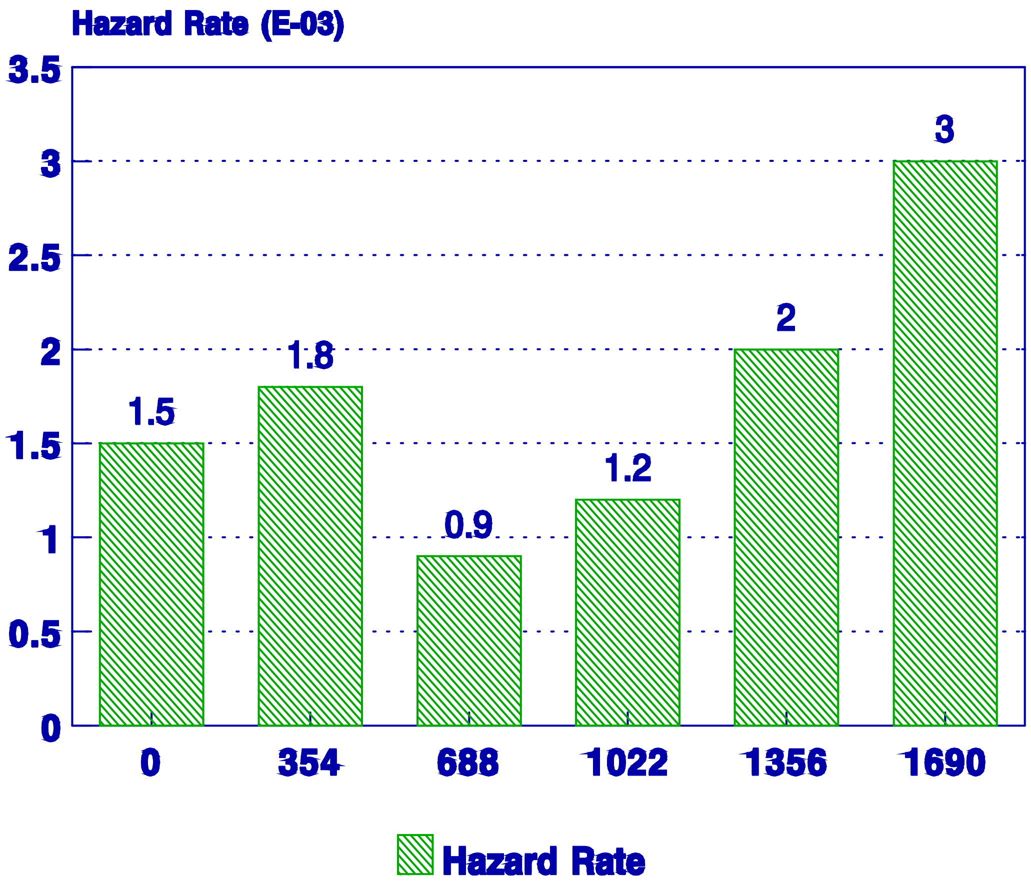
$$\widehat{MTTF} = 485.2$$

supports exponential

s = 469.8

$s^2 = (20^2 + 31^2 + \ldots + 2025^2 - 35 \times 485.2^2)/34 = 220{,}712.3$

Empirical Hazard Rate Curve



Probability Plots

$$\left[t_i, \hat{F}(t_i) \right], \; i = 1,2,\ldots,n$$

plotting position

plots as a straight line on appropriate graph paper

y = a + b x

Exponential Plots

$F(t) = 1 - e^{-\lambda t}$ or $1 - F(t) = e^{-\lambda t}$

$\ln [1 - F(t)] = - \lambda t$ or

$- \ln [1 - F(t)] = \ln [1/(1-F(t))] = \lambda t$

On special graph paper, given failure times

$t_1, t_2, \ldots , t_n$, plot the points $\left[t_i \; , \hat{F}(t_i) \right]$

vertical scale: $\hat{F}(t) \;\rightarrow\; \text{--} \longrightarrow \;\ln\left[\dfrac{1}{1-\hat{F}(t)}\right]$

Exponential Plots - least-squares

parameter estimation: Since F(MTTF) = 1 - e^{-1} = .632
Find the value of t which corresponds to F(t) = 0.632.

alternatively use least-square to fit a line passing through the origin:

$$\hat{\lambda} = b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$\hat{MTTF} = 1/\hat{\lambda} = 1/b$$

where $y_i = \ln\{1/[1-F(t)]\}$ and $x_i = t_i$.

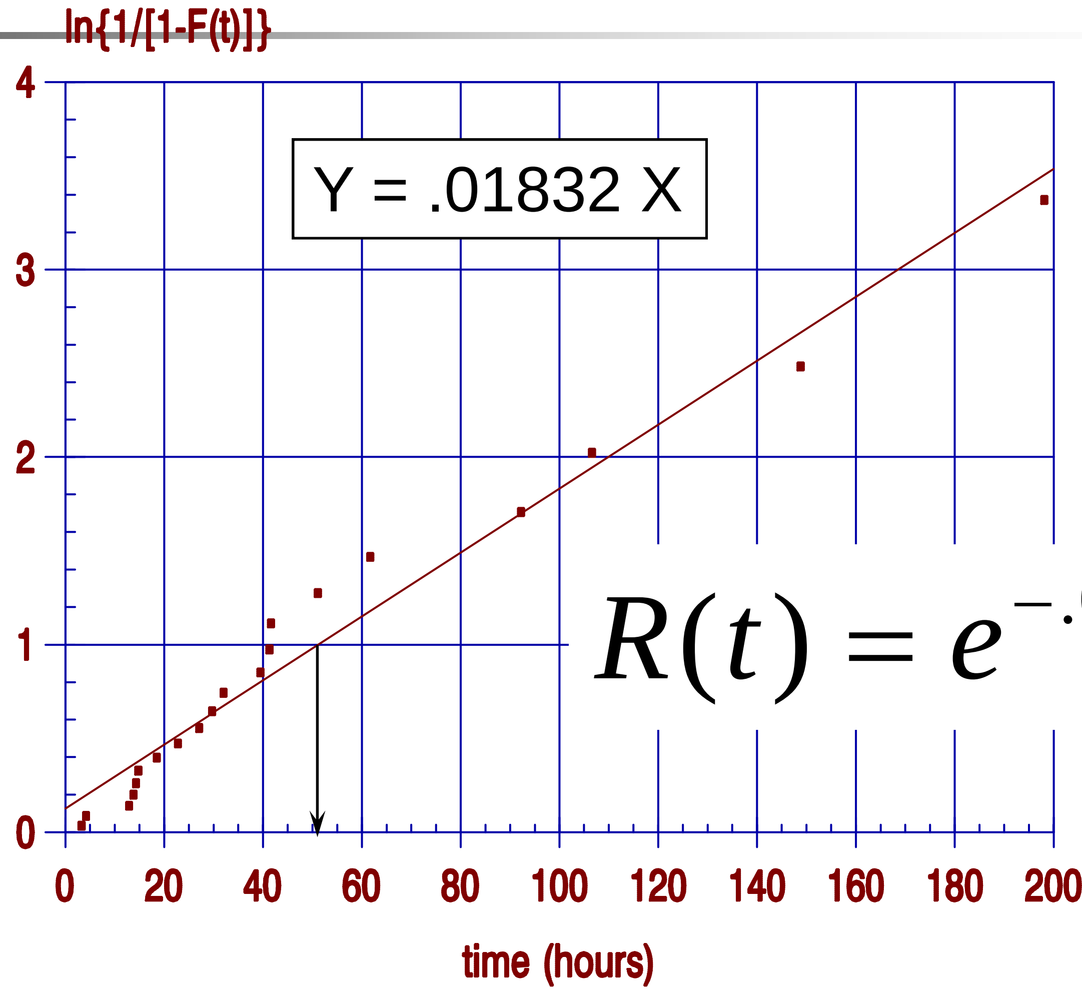
Example 15.4

FAILURE TIME (x_i)	$F(t_i)$	y_i =Ln [1/(1-$F(t_i)$)]
3.3	3.431373E-02	3.491616E-02
4.2	8.333334E-02	8.701131E-02
12.9	.132353	.1419703
13.8	.1813726	.2001262
14.3	.2303922	.2618741
14.8	.2794118	.3276874
18.5	.3284314	.398139
22.8	.377451	.4739329
27.1	.4264706	.5559461
29.7	.4754902	.6452911
32	.5245098	.743409
39.5	.5735295	.8522118
41.3	.622549	.9743145
41.6	.6715686	1.113427
51.1	.7205883	1.275069
61.7	.7696078	1.467972
92.2	.8186275	1.707202
106.6	.8676471	2.022284
148.8	.9166668	2.484908
198.1	.9656863	3.372211

$F(t_i) = (i-0.3)/(n+0.4)$

SLOPE - b = 01832
Estimated MTTF
= 1/b = 54.6
Index of Fit = .979
SAMPLE MTTF = 48.7

Example 15.4 – Exponential Plot



ln{1/(1-.632)} = 0.99967

MTTF = 50+

$$R(t) = e^{-.01832t}$$

Weibull Plots

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}} \qquad \ln\left[\frac{1}{(1-F(t))}\right] = \left(\frac{t}{\theta}\right)^{\beta}$$

$$\ln\,\ln\left[\frac{1}{(1-F(t))}\right] = \beta \ln t - \beta \ln \theta$$

plot: $$\left(\ln t_i, \ln\ln\left[\frac{1}{(1-\hat{F}(t_i))}\right]\right)$$

Weibull Graphs parameter estimation

Find the characteristic life from: $F(\theta) = 0.632$

Find the shape parameter from the slope of the fitted line:

$$\ln \ln\left[\frac{1}{(1 - F(t))}\right] = \beta \ln t - \beta \ln \theta$$

or solve:

$$\beta = \ln \ln\left[\frac{1}{(1 - F(t))}\right] / (\ln t - \ln \theta)$$

find average value over several t_i

Example 15.5

i	Failure Time	(i-.3)/(5+.4)
1	32 hrs	.13
2	51	.31
3	74	.50
4	90	.69
5	120	.87

From F(t) =.632

$\hat{\theta}$ = 85 hr.

for i = 2:

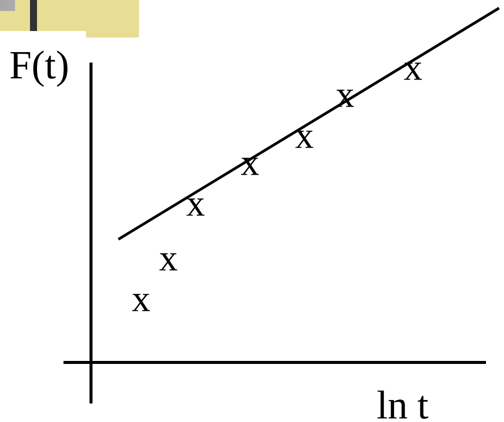
$$\hat{\beta} = \frac{\ln\ln\frac{1}{1-.31}}{\ln 51 - \ln 85} = 1.94$$

for i = 5:

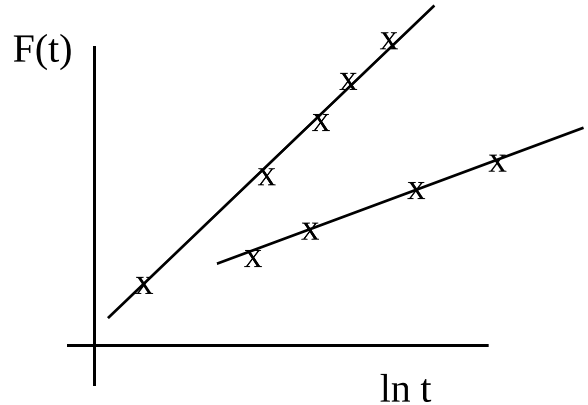
$$\hat{\beta} = \frac{\ln\ln\frac{1}{1-.87}}{\ln 120 - \ln 85} = 2.06$$

$$\text{average } \hat{\beta} = 2.0$$

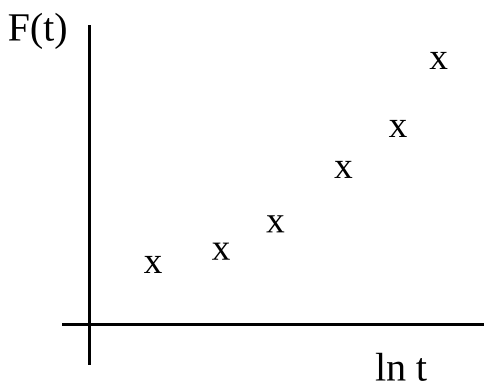
Nonlinear Weibull Plots



(a) Correct for t_0



(b) Competing Failure Modes



(c) Non-Weibull Distribution

Least-Squares Approach

$$\ln \ln\left[\frac{1}{(1 - F(t))}\right] = \beta \ln t - \beta \ln \theta$$

$$x_i = \ln t_i \quad and \quad y_i = \ln \ln\left[\frac{1}{1 - \hat{F}(t_i)}\right]$$

$$b = \frac{\sum_{i=1}^{n} x_i y_i - \bar{x} \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$\hat{B} = b$$

$$a = \bar{y} - b\bar{x}$$

$$\hat{\theta} = e^{-a/b}$$

Example 15.5 (continued)

FAILURE TIME	$F(t_i) = (i-0.3)/(n+0.4)$	$\ln \ln[1/(1-F(t_i))]$
32	.1296296	-1.974459
51	.3148148	-.9726862
74	.5	-.3665129
90	.6851852	.1447674
120	.8703704	.7144555

INTERCEPT - a	-8.951651
SLOPE - b	2.01553
ESTIMATED BETA	2.01553
ESTIMATED THETA	84.88845
Index-of-Fit (R)	.9986

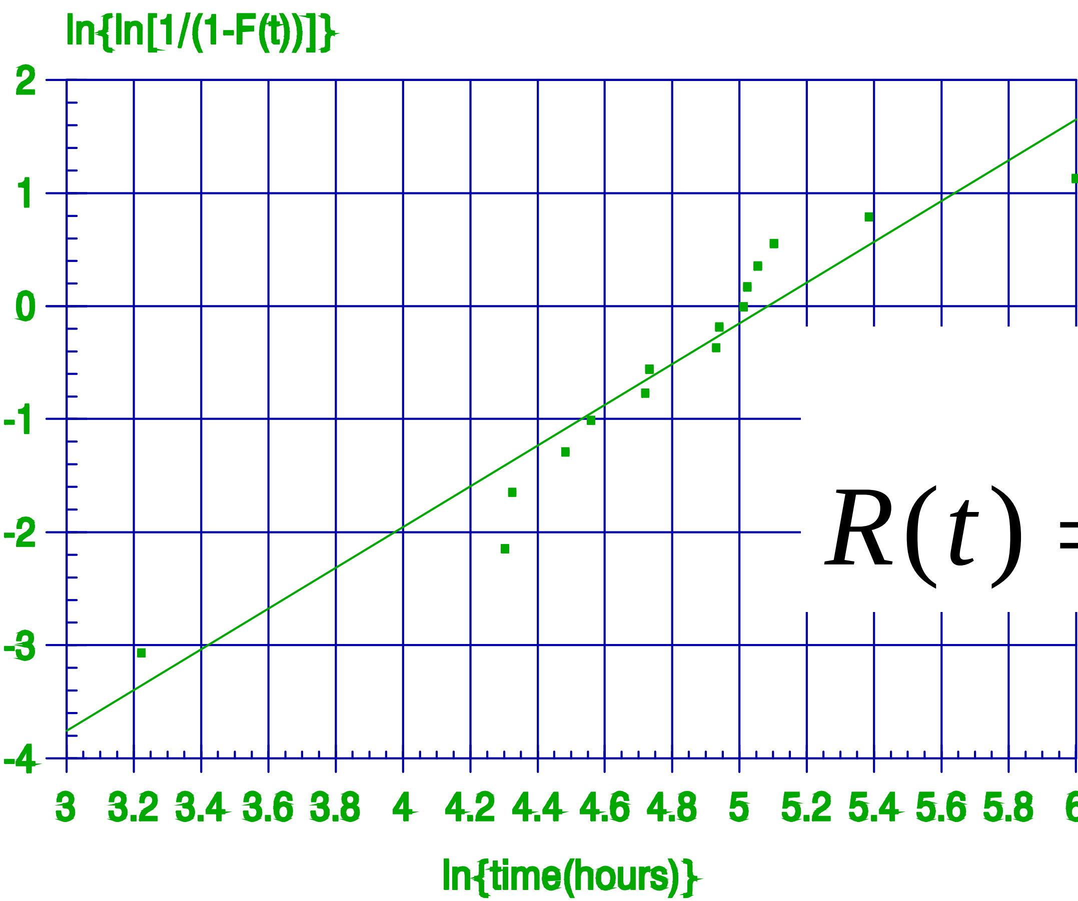
$$R(t) = e^{-\left(\frac{t}{84.88845}\right)^{2.01553}}$$

Example 15.6

i	t_i	$\hat{F}(t_i) = \frac{i-.3}{n+.4}$	y_i = ln ln [1/(1-F(t_i))]
1	25.1	0.0455	-3.067874
2	73.9	0.1104	2.145824
3	75.5	0.1753	-1.646281
4	88.5	0.2403	-1.291789
5	95.5	0.3052	-1.010262
6	112.2	0.3701	-0.7716678
7	113.6	0.4351	-0.5602884
8	138.5	0.5000	-0.3665131
9	139.8	0.5649	-0.1836104
10	150.3	0.6299	-6.117305E-03
11	151.9	0.6948	0.1712648
12	156.8	0.7597	0.3548976
13	164.5	0.8247	0.5545261
14	218.0	0.8896	0.7901556
15	403.1	0.9545	1.128508

Example 15.6
Weibull Failure Data

INTERCEPT - a	-9.1649
SLOPE - b	1.8027
Estimated BETA (ß)	1.8027
Estimated THETA (θ)	161.41
Index of Fit	.9545



$$R(t) = e^{-\cdot\left(\frac{t}{161.41}\right)^{1.8027}}$$

Normal Plots

$$F(t) = \Phi\left(\frac{t-\mu}{\sigma}\right) = \Phi(z)$$

$$z_i = \Phi^{-1}\left[F(t_i)\right] = \frac{t_i - \mu}{\sigma} = \frac{t_i}{\sigma} - \frac{\mu}{\sigma}$$

set: $x_i = t_i$ and $y_i = z_i$ and apply the L-S fomulae

$$\hat{\sigma} = \frac{1}{b} \quad and \quad \hat{\mu} = -a\,\hat{\sigma} = -\frac{a}{b}$$

Example

I	$x_i = t_i$	$F(t_i)$	$y_i = z_i$
1	68.0	0.0343	-1.8211
2	69.6	0.0833	-1.3832
3	71.1	0.1324	-1.1151
4	71.4	0.1814	-0.9100
5	74.3	0.2304	-0.7375
6	74.6	0.2794	-0.5846
7	75.5	0.3284	-0.4443
8	77.6	0.3775	-0.3121
9	77.8	0.4265	-0.1853
10	78.0	0.4755	-0.0615
11	78.2	0.5245	0.0615
12	80.2	0.5735	0.1853
13	80.3	0.6225	0.3121
14	81.9	0.6716	0.4443
15	83.0	0.7206	0.5846
16	85.6	0.7696	0.7375
17	87.4	0.8186	0.9100
18	87.7	0.8676	1.1151
19	88.4	0.9167	1.3832
20	98.3	0.9657	1.8211

(i - 0.3) / (20 + 0.4)

Normal Tables

INTERCEPT - a = -9.81565

SLOPE - b = 0.123553

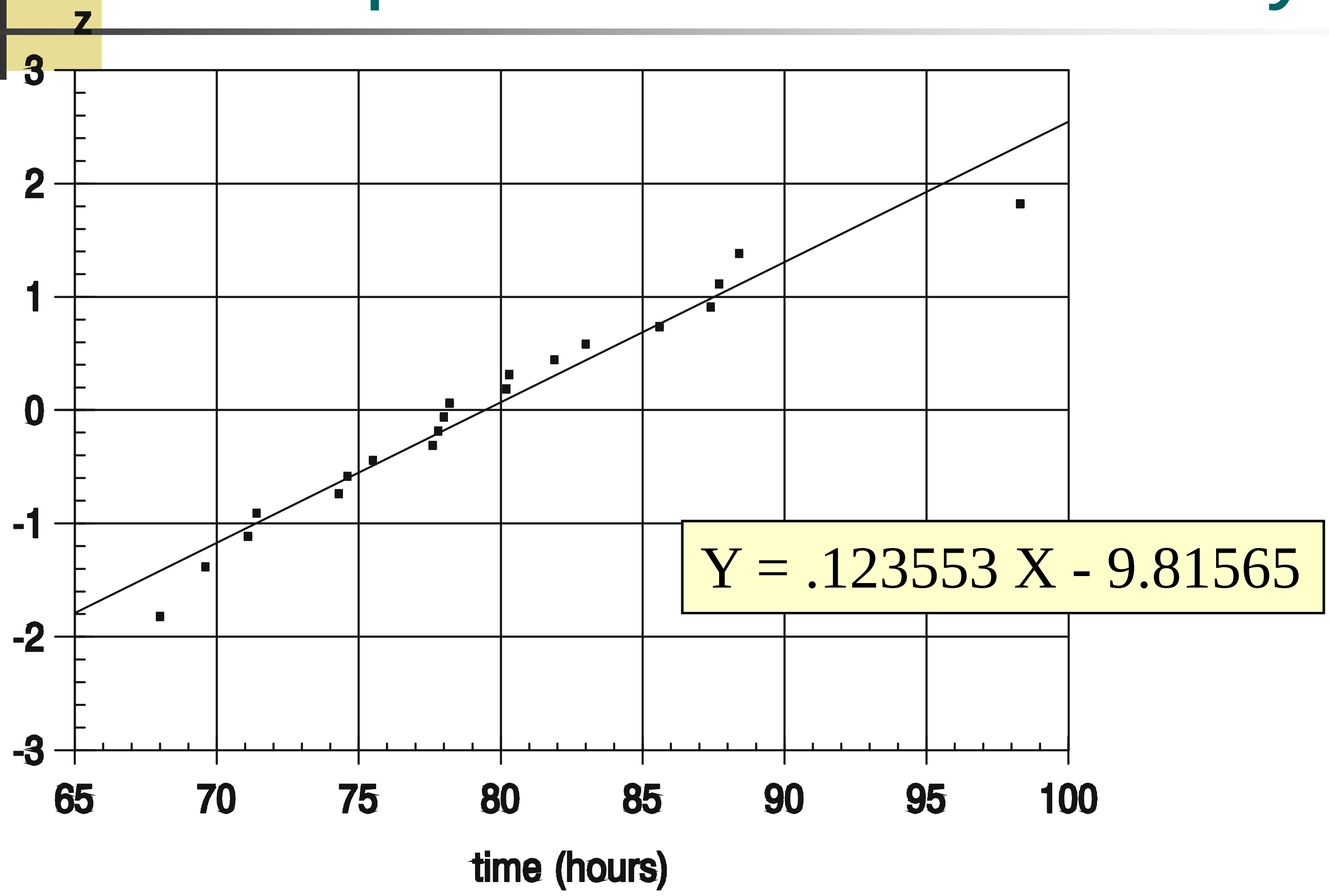
Estimated SIGMA (σ) = 1/b = 8.0937

Estimated MEAN = -a/b = 79.445

Index of Fit = R = 0.979

$$R(t) = 1 - \Phi\left(\frac{t - 79.445}{8.0937}\right)$$

Example – Normal Probability Plot



Lognormal Plots

$$F(t) = \Phi\left(\frac{1}{s}\ln\frac{t}{t_{med}}\right) = \Phi(z)$$

$$z_i = \Phi^{-1}\left[F(t_i)\right] = \frac{1}{s}\ln t - \frac{1}{s}\ln t_{med}$$

set: $x_i = \ln t_i$ and $y_i = z_i$ and apply the L-S fomulae

$$\hat{s} = \frac{1}{b} \quad and \quad \hat{t}_{med} = e^{-a/b}$$

Example

Repair times (in min.) of a mechanical pump are believed to follow a lognormal distribution.

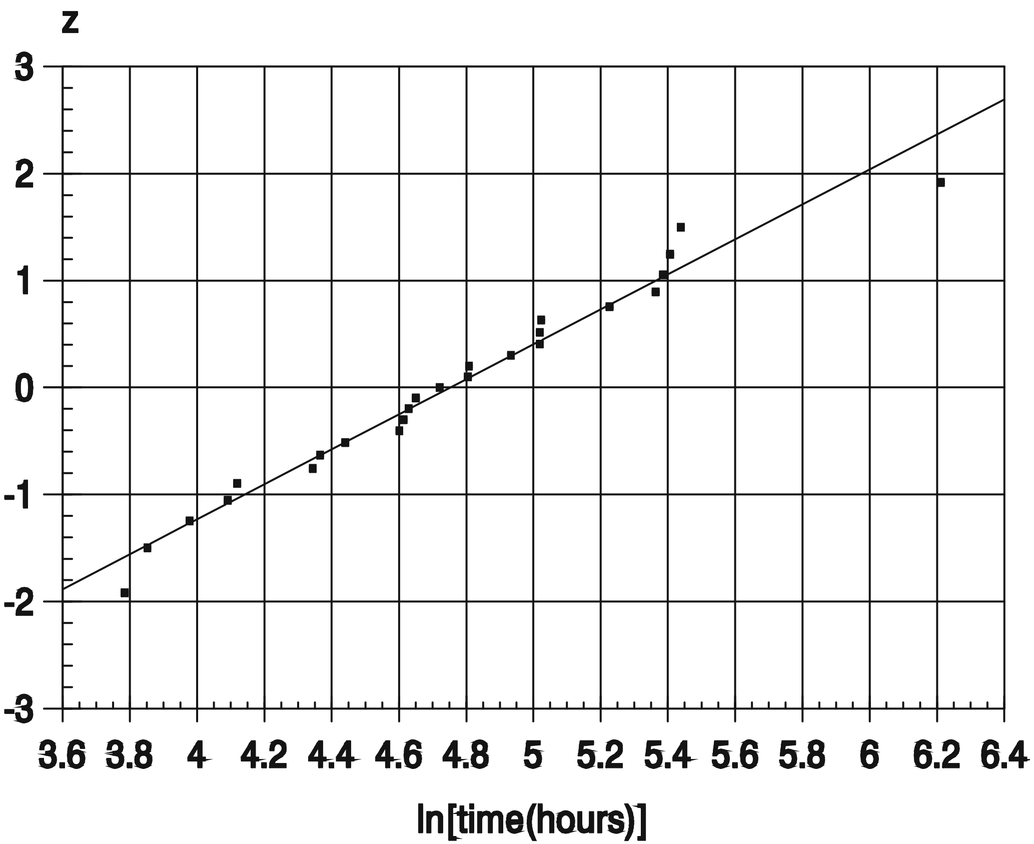
i	t_i	$F(t_i)$	z_i
1	44.0	0.0276	-1.9173
2	47.1	0.0669	-1.4993
3	53.4	0.1063	-1.2465
4	59.8	0.1457	-1.0551
5	61.5	0.1850	-0.8965
6	77.0	0.2244	-0.7574
7	78.7	0.2638	-0.6317
8	84.8	0.3031	-0.5155
9	99.6	0.3425	-0.4057
10	100.8	0.3819	-0.3005
11	102.4	0.4213	-0.1986
12	104.6	0.4606	-0.0989
13	112.3	0.5000	0.0000
14	122.1	0.5394	0.0990
15	122.5	0.5787	0.1986

i	t_i	$F(t_i)$	z_i
			0.3005
17	151.3	0.6575	0.4057
18	151.3	0.6969	0.5155
19	151.9	0.7362	0.6317
20	186.2	0.7756	0.7574
21	213.5	0.8150	0.8965
22	218.2	0.8543	1.0551
23	222.8	0.8937	1.2461
24	230.1	0.9331	1.4993
25	498.4	0.9724	1.9173

$(i - 0.3) / (25 + 0.4)$

Normal Tables

Example – Lognormal Probability



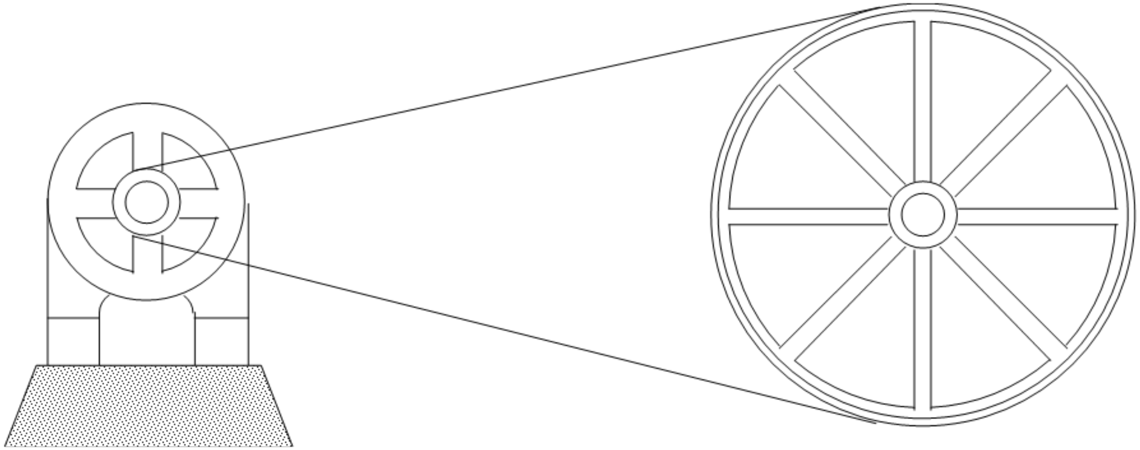
INTERCEPT - a	**= -7.755**
SLOPE - b	**= 1.631**
Estimated s = 1/b	**= 0.613**
Estimated T_{MED} = e^{-sa}	**= 116.0**
Index of Fit	**= 0.986**

$$H(t) = \Phi\left(\frac{1}{.613}\ln\frac{t}{116}\right)$$

Y = 1.631X - 7.755

Multiply Censored Plots

Thirty motors are placed on accelerated test with failures occurring at the following cycle times. A cycle consists of a motor starting up to its maximum number of revolutions per minute then shutting down until it has come to a complete stop. Censored units resulted from motors removed from test to satisfy other demands.



Multiply Censored Plots

Time	Adj Rank (i)	$F(t_i)$	y = ln [1/(1-F)]	ln y
141	1	.023	.0233	-3.735
391	2	.056	.0575	-2.855
399	3	.089	.0930	-2.375
410+				
463	4.04	.123	.1311	-2.031
465	5.07	.157	.1708	-1.767
497	6.11	.191	.2121	-1.550
501+				
559	7.19	.227	.2571	-1.358
563	8.27	.262	.3043	-1.190
579	9.36	.298	.3537	-1.039
580+				
586	10.50	.336	.4086	-.8950
616	11.64	.373	.4666	-.7622
683	12.77	.410	.5282	-.6382
707	13.91	.448	.5939	-.5211
713	15.05	.485	.6641	-.4092
742+				
755+				
764	16.38	.529	.7529	-.2838

$$\boxed{F(t_i) = \frac{i_{t_i} - 0.3}{30 + 0.4}}$$

number at risk

fit exponential

fit Weibull

EXAMPLE 15.10

A newly manufactured diesel engine is experiencing frequent failures as a result of either the ignition system failing (failure mode A) or the fuel injection system failing (failure mode B). Tracking failures under a one-year warranty, the following failure times were recorded in days since beginning operation for the first 40 units sold:

Unit Number	12	3	22	9	31	38	5	8	27	18	23	19	32	13	30	36	24
Failure time	2	8	15	30	35	61	123	123	132	184	186	202	218	232	269	297	333
Failure mode	A	A	A	A	A	B	A	B	B	A	A	A	A	B	B	B	A

EXAMPLE 15.10

For Weibull distribution - Least-squares fit:

Failure Mode A:

$$\hat{\beta} = .4811 \text{ and } \hat{\theta} = 3732.7$$

Failure Mode B:

$$\hat{\beta} = 1.34 \text{ and } \hat{\theta} = 1037.6$$

i	t_i	$F(t_i)$	i	t_i	$F(t_i)$
0	0	0.0000	10	184	0.1737
1	2	0.0244	11	186	0.2003
2	8	0.0488	12	202	0.2270
3	15	0.0732	13	218	0.2537
4	30	0.0976	14	232.0^+	
5	35	0.1220	15	269.0^+	
6	61.0^+		16	297.0^+	
7	123	0.1470	17	333	0.2835
8	123.0^+		18-40	365.0^+	
9	132.0^+				

$$R(t) = e^{-\left(t/3732.7\right)^{.4811}} e^{-\left(t/1037.6\right)^{1.34}}$$

Minimum Extreme Value Distribution

$$F(t) = 1 - \exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right]$$

$$\ln\left[\frac{1}{1-F(t)}\right] = e^{\frac{(t-\mu)}{\alpha}} \text{ or } \ln\left\{\ln\left[\frac{1}{1-F(t)}\right]\right\} = \frac{t}{\alpha} - \frac{\mu}{\alpha}$$

Therefore plot x_i and y_i and fit the line $y = a + bx$.

$$\left(x_i = t_i, y_i = \ln\ln\left[\frac{1}{1-F(t_i)}\right]\right)$$

The least-squares parameter estimates are $\alpha = 1/b$ and $\mu = -a\alpha$.