Chapter 4 Part I
Time-Dependent Failure Models

Weibull's wobble but they don't fall down!

-old English saying

The Weibull Distribution
Minimum Extreme Value Distribution

C. Ebeling, *Intro to Reliability & Maintainability Engineering,* 2nd *ed.* Waveland Press, *Inc.*

Time Dependent Failure Mode

$$\lambda(t) = a\,t^{b}\;, with\, a > 0$$

rewrite as:

$$\lambda(t) \;=\; \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1} \;;\; \theta > 0, \beta > 0 \;;\; t \geq 0$$

where

B is the shape parameter and θ is the characteristic life

Reliability Function

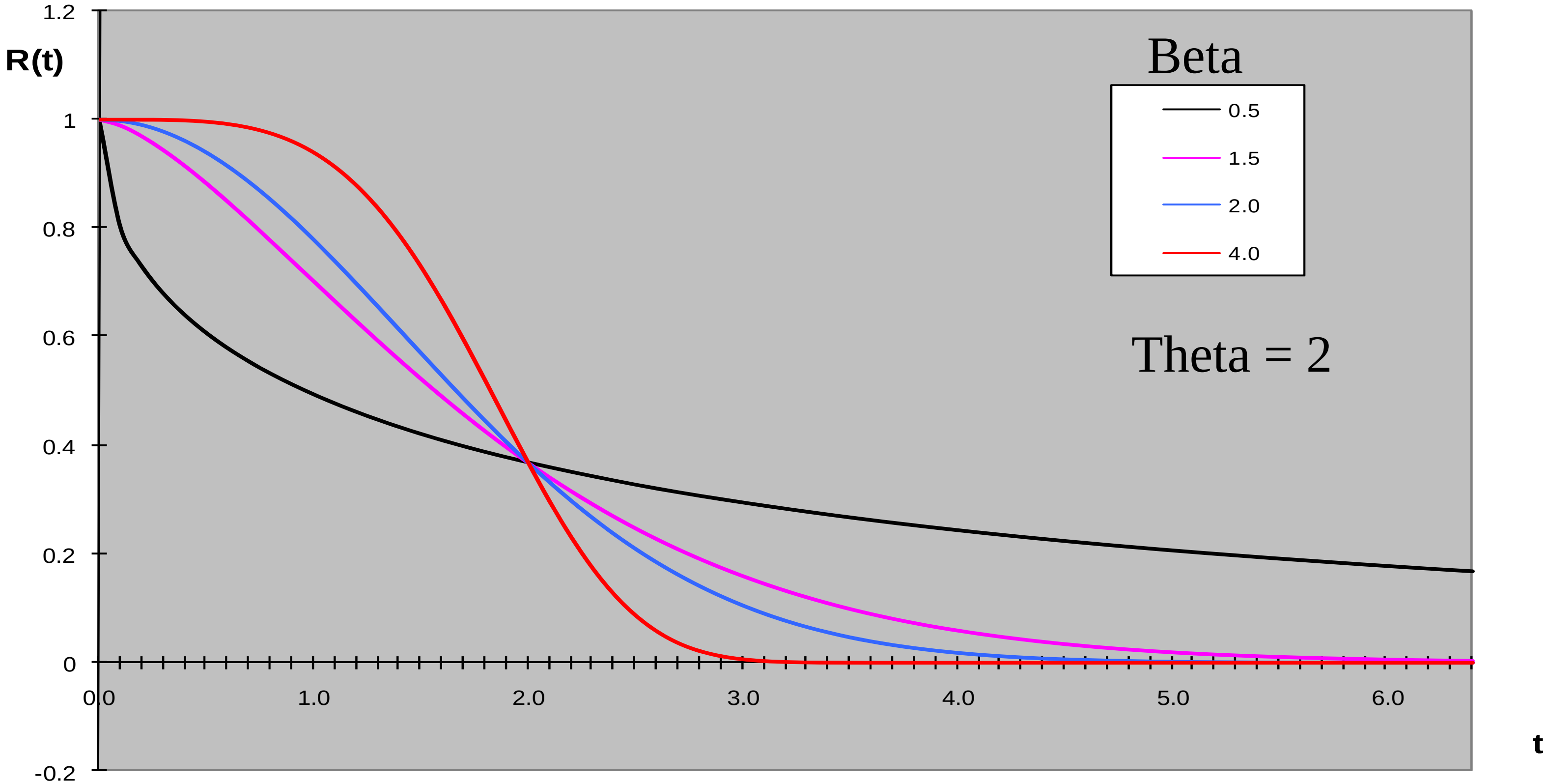
$$R(t) = e^{-\int_0^t \frac{B}{\theta}\left(\frac{t'}{\theta}\right)^{B-1} dt'}$$

$$= e^{-\left(\frac{t}{\theta}\right)^{B}}$$

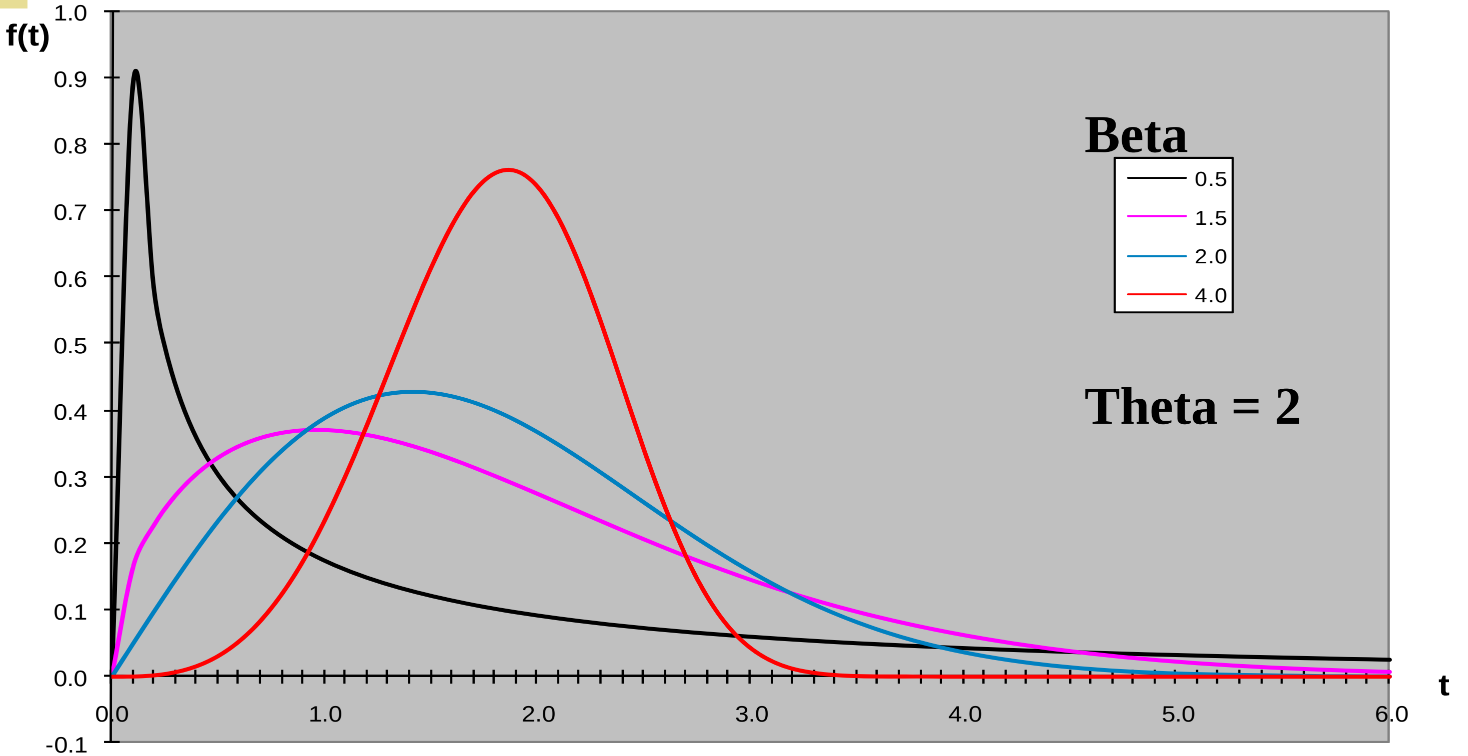
note that:

$$R(\theta) = e^{-\left(\frac{\theta}{\theta}\right)^{\beta}} = e^{-1} = .368$$

Graph of the Reliability Function

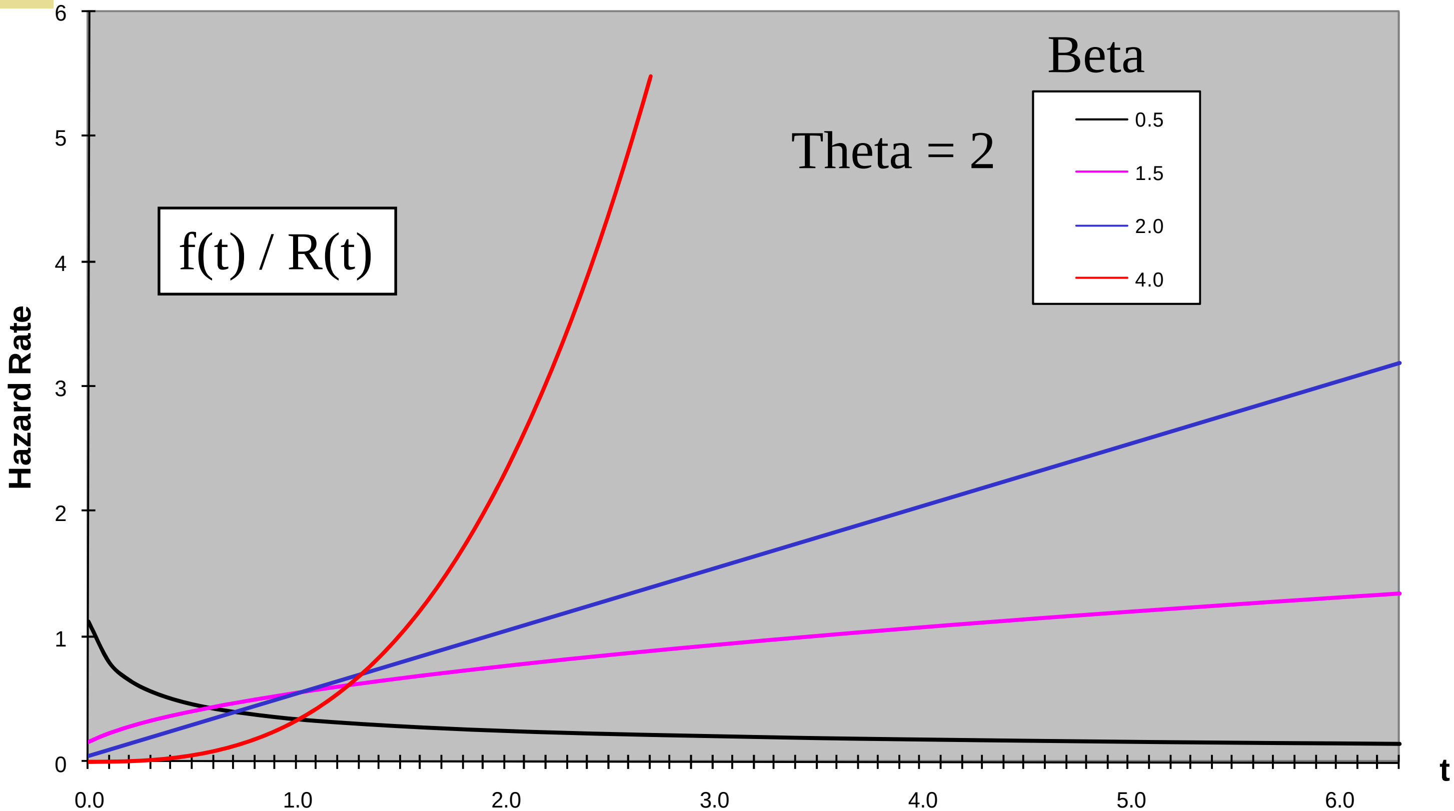


The Probability Density Function (PDF)



$$f(t) = -\frac{dR(t)}{dt} = \frac{B}{\theta}\left(\frac{t}{\theta}\right)^{B-1} e^{-\left(\frac{t}{\theta}\right)^{B}}$$

The Hazard Rate Function



$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

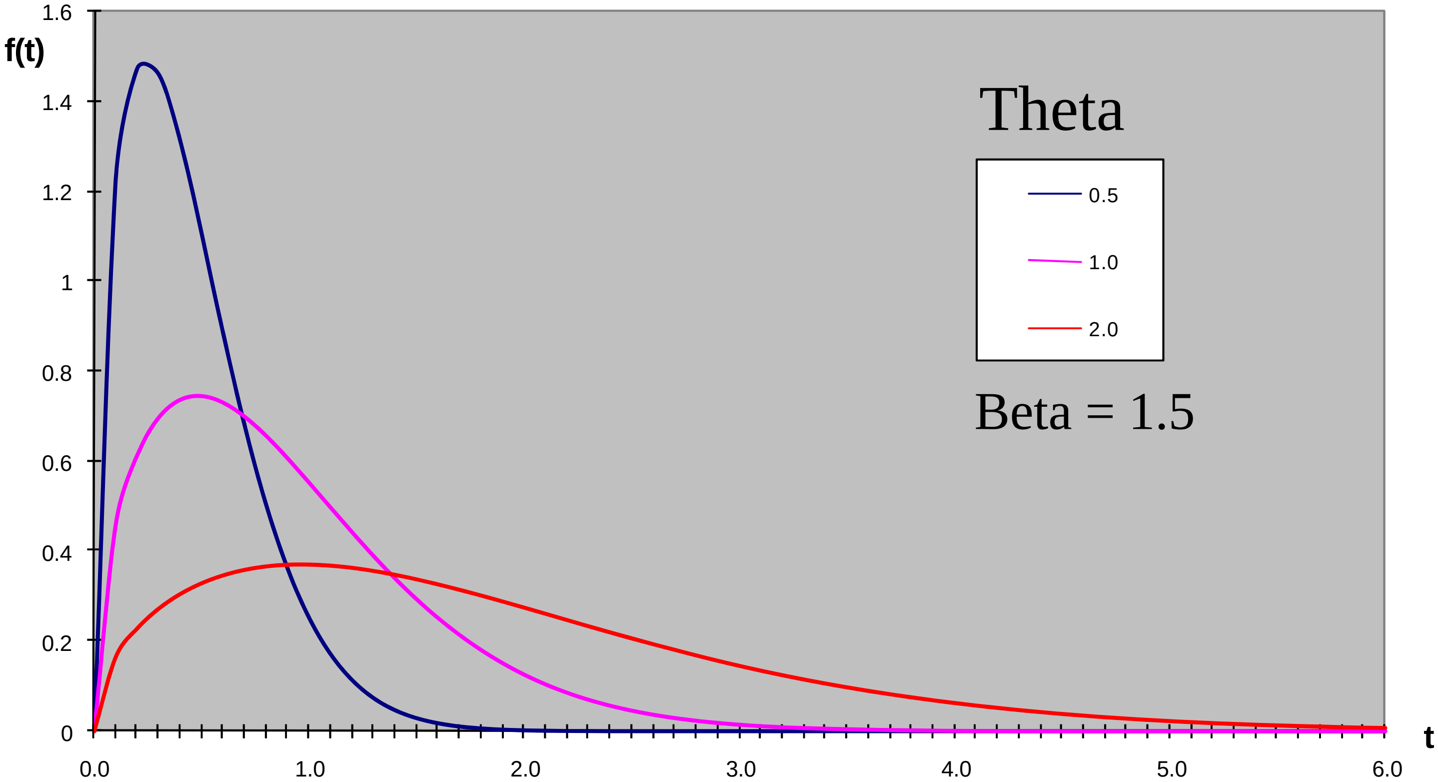
Weibull Shape Parameter

Value	Property
$0 < \beta < 1$	Decreasing Failure Rate (DFR)
$\beta = 1$	Exponential Distribution (CFR)
$1 < \beta < 2$	IFR-concave
$\beta = 2$	Rayleigh Distribution (LFR)
$\beta > 2$	IFR - Convex
$3 <= \beta <= 4$	IFR - Approaches Normal Distribution - Symmetrical

$\beta = 3.43927$ Most closely approximates the normal

$\beta = 3.43938$ Mean = median

The Characteristic Life



The Mean Time to Failure (MTTF)

$$MTTF = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

$$\Gamma(x) = \text{the gamma function} = \int_0^\infty y^{x-1} e^{-y}\, dy$$

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

$$e.g.\ \Gamma(4.23) = (3.23)\Gamma(3.23) = (3.23)(2.23)\Gamma(2.23)$$
$$= (3.23)(2.23)(1.12023) = 8.0689$$

$$\lim_{\beta\to\infty} MTTF = \lim_{\beta\to\infty} \theta\Gamma(1 + \frac{1}{\beta}) = \theta\Gamma(1) = \theta$$

Gamma Function - selected values

x	Gamma(x)
1.53	.88757
1.54	.88818
1.55	.88887
1.56	.88964
1.57	.89049
1.58	.89142
1.59	.89243
1.6	.89352
1.61	.89468
1.62	.89592
1.63	.89724
1.64	.89864
1.67	.9033
1.68	.905
1.69	.90678

x	Gamma(x)
2.23	**1.12023**
2.24	1.12657
2.25	1.133
2.26	1.13954
2.27	1.14618
2.28	1.15292
2.29	1.15976
2.3	1.16671
2.31	1.17377
2.32	1.18093
2.33	1.18819
2.34	1.19557
2.35	1.20305

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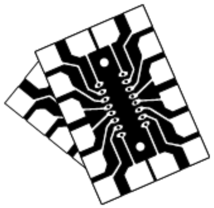
Example Problems

Let T = a random variable, the time to failure of a circuit card T has a Weibull distribution with beta equal to .5 and theta equal to 500 (thousands of hours).

Find R(50,000) and the MTTF.



$$R(50) = e^{-\left(\frac{50}{500}\right)^{.5}} = .729$$

$$MTTF = 500\Gamma(1+2) = 500(2) = 1000$$

Let T = a random variable, the time to failure of a fuse.

T has a Weibull distribution with beta equal to 1.5 and theta equal to 500 (thousands of hours).

Find R(50,000) and the MTTF.

$$R(50) = e^{-\left(\frac{50}{500}\right)^{1.5}} = .969$$

$$MTTF = 500\Gamma\left(1 + \frac{2}{3}\right) = 500(.903) = 451$$

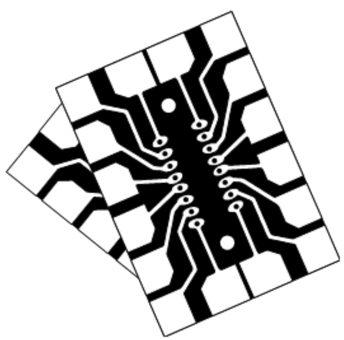
The Variance and Standard Deviation

$$\sigma^2 = \theta^2 \left\{ \Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2 \right\}$$

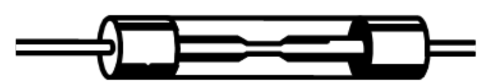
Note: $\lim_{\beta\to\infty} \sigma^2 = 0$

Example Problem - standard deviation

$$\sigma^2 = 500^2[\Gamma(5) - 2^2]$$
$$= 500^2[24 - 4] = 5{,}000{,}000$$
$$\sigma = 2236 \ \ (\mathit{thousands\ of\ hr.})$$



$$\sigma^2 = 500^2[\Gamma(1 + 4/3) - .903^2]$$
$$= 500^2[1.18819 - .81595]$$
$$= 93059.8$$
$$\sigma = 305 \ (\mathit{thousands\ of\ \ hr.})$$



Design Life and Median

set $$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\beta}} = R$$

then $$t_R = \theta(-\ln R)^{\frac{1}{\beta}}$$

and $$t_{.50} = t_{med} = \theta(-\ln .5)^{\frac{1}{\beta}}$$

The Mode

$$f(t_{mode}) = \underset{t \geq 0}{MAX} \; f(t)$$

$$t_{mode} = \begin{cases} \theta \left(1 - \dfrac{1}{\beta} \right)^{\frac{1}{\beta}} & \text{for } \beta > 1 \\ 0 & \text{for } \beta \leq 1 \end{cases}$$

Example - design life, median, & mode

beta = .5

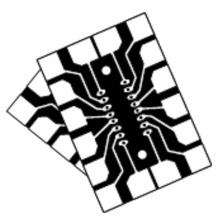
$t_{.9} = 500\ (-\ln.90)^{2}$

$= 5.55\ (1000\ \text{hr.})$

$t_{med} = 500\ (.69315)^{2}$

$= 240\ (1000\ \text{hr.})$

$t_{mode} = 0$



beta = 1.5 = 3/2

$t_{.9} = 500\ (-\ln.90)^{2/3}$

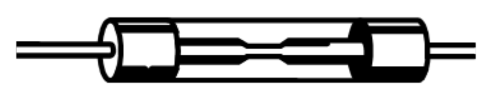
$= 111.5\ (1000\ \text{hr.})$

$t_{med} = 500\ (.69315)^{2/3}$

$= 391.6\ (1000\ \text{hr.})$

$t_{mode} = 500\ [1-2/3)]^{2/3}$

$= 240\ (1000\ \text{hr.})$



Conditional Reliability

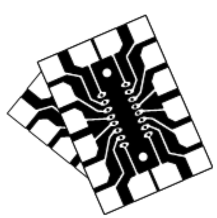
$$R(t|T_0) = \frac{e^{-\left(\frac{t+T_0}{\theta}\right)^{\beta}}}{e^{-\left(\frac{T_0}{\theta}\right)^{\beta}}} = e^{-\left(\frac{t+T_0}{\theta}\right)^{\beta}+\left(\frac{T_0}{\theta}\right)^{\beta}}$$

Example - conditional reliability

Beta = .5; R(50) = .7289

R(50|50) = R(100)/R(50)
$=\exp[-(100/500)^{.5}]$ / .7289

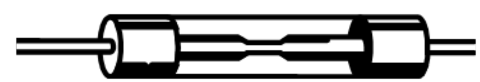
= .6394 / .7289 = .8772



Beta = 1.5; R(50) = .969

R(50|50) = R(100)/R(50)
$=\exp[-(100/500)^{1.5}]$ / .969

= .9144 / .969 = .9437



Failure Modes

$$\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta_i} \left(\frac{t}{\theta_i} \right)^{\beta-1} = \beta\, t^{\beta-1} \left[\sum_{i=1}^{n} \left(\frac{1}{\theta_i} \right)^{\beta} \right]$$

which is a Weibull hazard rate function with a shape parameter of β and a characteristic life of $\left[\sum_{i=1}^{n} \left(\frac{1}{\theta_i} \right)^{\beta} \right]^{-\frac{1}{\beta}}$

Equate $\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \frac{\beta t^{\beta-1}}{\theta^{\beta}} = \beta\, t^{\beta-1}\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^{\beta}\right]$

and solve for θ

Identical Weibull Components

n components have identical hazard rate functions:

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1} \quad ,$$

then

$$\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1} = \frac{n\beta}{\theta^{\beta}}(t\,)^{\beta\text{-}1}$$

and $R(t) = e^{-n\left(\frac{t}{\theta}\right)^{\beta}}$

Three Parameter Weibull

Let t_0 be the minimum life such that $T>t_0$, then

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t - t_0}{\theta}\right)^{\beta-1}, \; t \geq t_0$$

$$R(t) = e^{-\left(\frac{t-t_0}{\theta}\right)^{\beta}}, \; t \geq t_0$$

$$MTTF = t_0 + \theta\Gamma\left(1 + \frac{1}{\beta}\right) \qquad t_{med} = t_0 + \theta(.69315)^{\frac{1}{\beta}}$$

Redundancy - Two Identical Weibull Components

$$R_s(t) = 1 - \left[1 - R(t)\right]^2$$

$$= 1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^2 = 2e^{-\left(\frac{t}{\theta}\right)^{\beta}} - e^{-2\left(\frac{t}{\theta}\right)^{\beta}}$$

MTTF - Redundant Weibull Components

$$MTTF = \int_0^\infty R_p(t)dt = \int_0^\infty \left[2e^{-\left(\frac{t}{\theta}\right)^\beta} - e^{-2\left(\frac{t}{\theta}\right)^\beta}\right]dt$$

$$= 2\int_0^\infty e^{-\left(\frac{t}{\theta}\right)^\beta} dt - \int_0^\infty e^{-2\left(\frac{t}{\theta}\right)^\beta} dt$$

$$= 2\theta\Gamma\left(1+\frac{1}{\beta}\right) - \frac{\theta}{2^{1/\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$$

$$= \theta\Gamma\left(1+\frac{1}{\beta}\right)\left[2 - 2^{-1/\beta}\right]$$

Hazard Rate Function
Redundant Weibull Components

$$\lambda_s(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \frac{\left[2-2e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]}{\left[2-e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]}$$

Derivation of Hazard Rate Function 2 component redundant system

$$\frac{dR(t)}{dt} = 2e^{-\left(\frac{t}{\theta}\right)^{\beta}}\left[-\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1}\right] - e^{-2\left(\frac{t}{\theta}\right)^{\beta}}\left[-\frac{2\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1}\right]$$

$$= \frac{-\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}\left[2 - 2e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]$$

$$\lambda_p(t) = -\frac{dR(t)}{dt}\bullet\frac{1}{R(t)} = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta\text{-}1}\frac{\left[2-2e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]}{\left[2-e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]}$$

Minimum Extreme Value Distribution Appendix 4C

Let T_i = failure time of i^{th} component, then $R_i(t_i) = e^{-\left(\frac{t_i - t_0}{\theta}\right)^{\beta}}$

with $T = \min\{t_1, t_2, \ldots, t_n\}$

$$F(t) = P\{T \le t\} = P\{t_1 < t \text{ or } t_2 < t, \ldots \text{or} ., t_n < t\}$$

$$= 1 - P\{t_1 > t,\ t_2 > t,\ \ldots\ t_n > t\}$$

$$= 1 - P\{t_1 > t\} P\{t_2 > t\} \ldots P\{t_n > t\} = 1 - \left[e^{-\left(\frac{t - t_0}{\theta}\right)^{\beta}}\right]^n$$

$$R(t) = 1 - F(t) = e^{-n\left(\frac{t - t_0}{\theta}\right)^{\beta}} = e^{-\left(\frac{t - t_0}{\theta / n^{1/\beta}}\right)^{\beta}}$$

Minimum Extreme Value Distribution - the Limiting Distribution

Assume the hazard rate function increases exponentially, $\lambda(t) = ae^{bt}$

$$\lambda(t) = \frac{1}{\alpha} e^{\frac{1}{\alpha}(t-\mu)}, \;\; \alpha > 0, \; -\infty < t < \infty$$

$$R(t) = \exp\left[-\frac{1}{\alpha} \int_{-\infty}^{t} e^{\frac{1}{\alpha}(t-\mu)} dt' \right] = \exp\left[-e^{\frac{1}{\alpha}(t'-\mu)} \right]_{-\infty}^{t} = \exp\left[-e^{\frac{(t-\mu)}{\alpha}} \right]$$

α is a scale parameter and μ is a location parameter

Minimum Extreme Value Distribution

$$f(t) = -\frac{d\exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right]}{dt} = \left(\frac{1}{\alpha}\right) e^{\frac{(t-\mu)}{\alpha}} e^{-e^{\frac{(t-\mu)}{\alpha}}}$$

$$MTTF = \mu - \alpha\gamma \text{ where } \gamma = .577215665$$

$$\sigma = \frac{\alpha\pi}{\sqrt{6}}$$

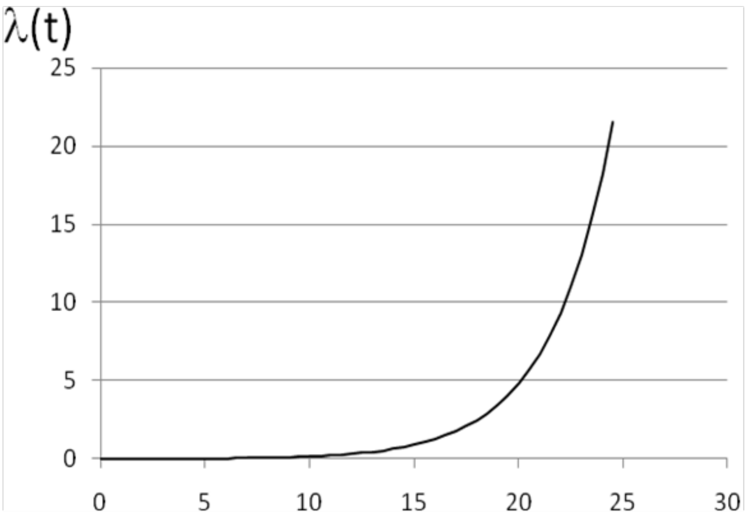
Median time to failure

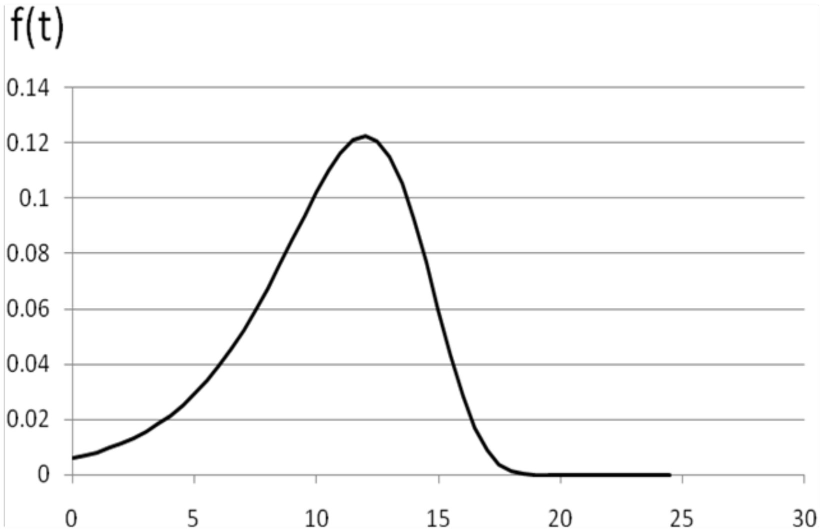
$$\text{setting } R(t) = \exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right] = .5 \text{ and solving for t:}$$

$$\mathrm{t}_{med} = \mu + \alpha \ln\left(-\ln .5\right) = \mu - .366513\alpha$$

The location parameter, μ, is the mode of the distribution

The minimum extreme for $\alpha = 3$ and $\mu = 12$





Relationship with the Weibull

letting t = ln x, α = 1/β and μ = ln θ where x > 0

$$R(t) = \exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right] = \exp\left[-e^{\frac{(\ln x - \ln\theta)}{1/\beta}}\right] = e^{-\left(\frac{x}{\theta}\right)^{\beta}}$$

Example 4.7

A manufactured part is subjected to a large number of (cyclical) stress points at which a fracture line may initiate. The weakest stress point will show a fracture line the earliest resulting in the part failing. Laboratory tests have determined that the failure time is a minimum extreme value distribution with a location parameter of 5 years and scale parameter of 0.4.

$$MTTF = \mu - \alpha\gamma = 5 - .4(.577215665) = 4.77 \ yr.$$

$$\sigma = \frac{\alpha\pi}{\sqrt{6}} = \frac{.4\pi}{\sqrt{6}} = .513\, yr.$$

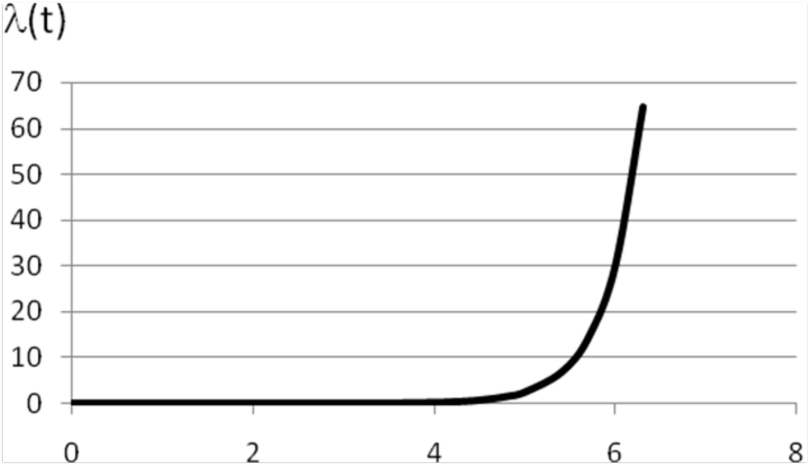
$$t_{med} = \mu - .366513\alpha = 5 - .366513(.4) = 4.85 \ yr.$$

Example continued

$$R(4) = \exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right] = \exp\left[-e^{\frac{4-5}{.4}}\right] = .9212$$

R(4.5 yr) = 0.75 indicating that the part will quickly fail once it reaches its 4th yr as a result of cyclical fatigue.

The 99 percent design life is 3.16 yr.



Student Exercise - Weibull

A certain brand lightning arrester has a Weibull failure distribution with a shape parameter of 2.4 and a characteristic life of 10 years. Find:

a. R(5 yrs)

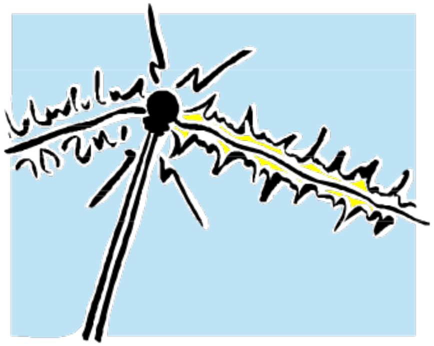
b. MTTF

c. Standard deviation

d. Median and Mode

e. 99% (B1) and 95% B(5) design life

f. R(5|5)



Student Exercise - solution

$$a. \quad R(5) = e^{-\left(\frac{5}{10}\right)^{2.4}} = .8274$$

$$b. \quad MTTF = 10\,\Gamma(1 + 1/2.4) = 10\,\Gamma(1.42)$$

$$10(.88636) = 8.86 \ yrs$$

$$c. \quad \sigma^2 = 10^2\left\{\Gamma(1 + 2/2.4) - .88636^2\right\}$$

$$or \quad \sigma = 3.93 \ yr.$$

Student Exercise - solution (continued)

$$d.\quad t_{med} = 10\left(.69315\right)^{1/2.4} = 8.58\ yr.$$

$$t_{\text{mod}\,e} = 10\left(1 - 1/2.4\right)^{1/2.4} = 8.0\ yr.$$

$$e.\quad t_{.99} = 10\left(-\ln .99\right)^{1/2.4} = 1.5\ yr.$$

$$t_{.95} = 10\left(-\ln .95\right)^{1/2.4} = 2.9\ yr.$$

Student Exercise - solution (continued)

$$f.\quad R(5\,|\,5) = \frac{R(10)}{R(5)} = \frac{e^{-\left(\frac{10}{10}\right)^{2.4}}}{.8274}$$

$$= \frac{.3679}{.8274} = .4446$$