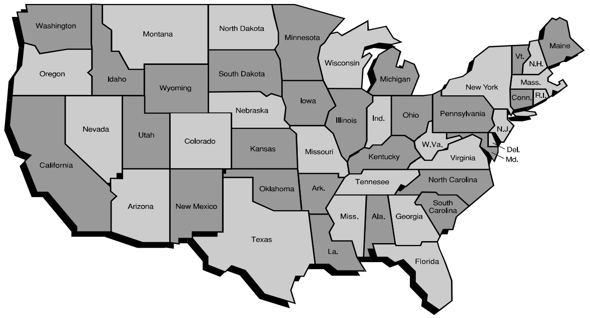


Chapter 6
STATE DEPENDENT SYSTEMS

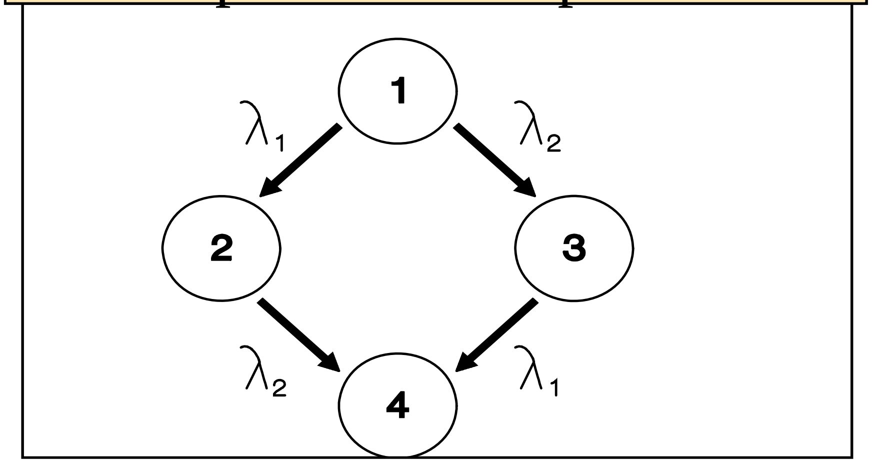
Markov Analysis

Load Sharing Systems

Standby Systems

Markov Analysis

Independent Components

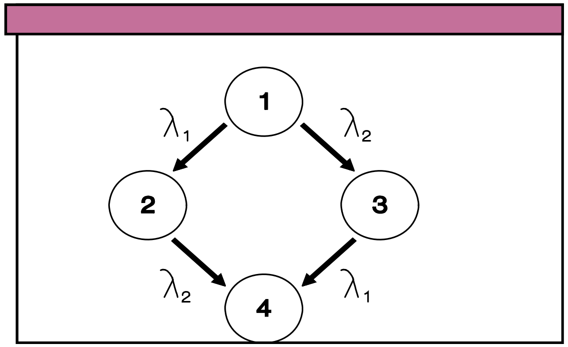


STATE	COMPONENT 1	COMPONENT 2
1	operating	operating
2	failed	operating
3	operating	failed
4	failed	failed

Independent Components

Let Pi(t) = probability of being in state i at time t

$$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$$



For two components in series:

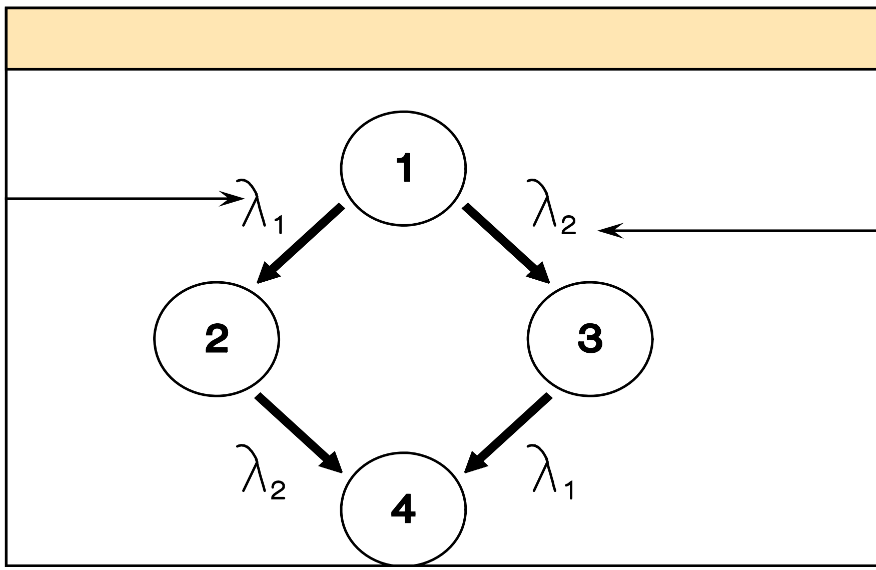
$$R_s(t) = P_1(t)$$

For two components in parallel:

$$R_p(t) = P_1(t) + P_2(t) + P_3(t)$$

State Equation

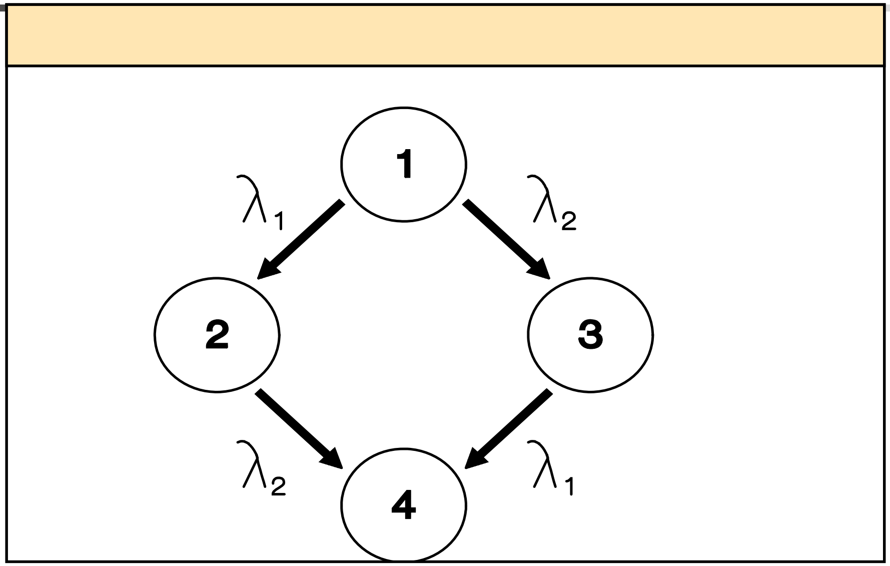
rate out of state 1
into state 2



rate out of state 1
into state 3

Prob of being in state 1 at time t+Δ t is equal to the prob of being in state 1 at time t and not transitioning to states 2 or 3 in time Δ t.

More State Equation



$P_1(t)$ = prob of being in state 1 at time t

$$P_1(t+\Delta t) = P_1(t) - \lambda_1 \Delta t\, P_1(t) - \lambda_2 \Delta t\, P_1(t)$$

prob of being in state 1 at time t + Δ t

prob of being in state 1 at time t and transitioning to state i in time Δ t

Even More State Equations

State 2:

$$P_2(t+\Delta t) \;=\; P_2(t) + \lambda_1 \Delta t\, P_1(t) - \lambda_2 \Delta t\, P_2(t)$$

State 3:

$$P_3(t+\Delta t) \;=\; P_3(t) + \lambda_2 \Delta t\, P_1(t) - \lambda_1 \Delta t\, P_3(t)$$

State 4:

$$P_4(t+\Delta t) \;=\; P_4(t) + \lambda_2 \Delta t\, P_2(t) + \lambda_1 \Delta t\, P_3(t)$$

Rewriting the Equation for State 1:

$$P_1(t+\Delta t) = P_1(t) - \lambda_1 \Delta t\, P_1(t) - \lambda_2 \Delta t\, P_1(t)$$

$$\frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = -(\lambda_1 + \lambda_2)\, P_1(t)$$

then

$$\lim_{\Delta t \to 0} \frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = \frac{d\, P_1(t)}{dt} = -(\lambda_1 + \lambda_2)\, P_1(t)$$

Differential Equations or States 2 & 3

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$\frac{d\,P_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1 P_3(t)$$

A fourth differential equation is not needed. Why?

$$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$$

Solution

From the Equation for State 1:

$$\frac{d\,P_1(t)}{P_1(t)} = -(\lambda_1 + \lambda_2)dt$$

Integrating both sides:

$$\ln P_1(t) = -(\lambda_1 + \lambda_2)t$$

or

$$P_1(t) = e^{-(\lambda_1 + \lambda_2)t}$$

Solution

Substituting the solution for $P_1(t)$ into the State 2 Equation:

$$\frac{d\,P_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2 P_2(t)$$

with $e^{\lambda_2 t}$ as an integrating factor,

$$P_2(t)e^{+\lambda_2 t} = +\lambda_1 \int e^{-(\lambda_1+\lambda_2)t}\, e^{+\lambda_2 t}\, dt + C$$

or

$$P_2(t) = -e^{-(\lambda_1+\lambda_2)t} + c\, e^{-\lambda_2 t}$$

with $P_2(0) = 0$, then $c = 1$

$P_3(t)$ is derived in the same manner!

Finally

For a series system:

$$R_s(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

For a parallel system:

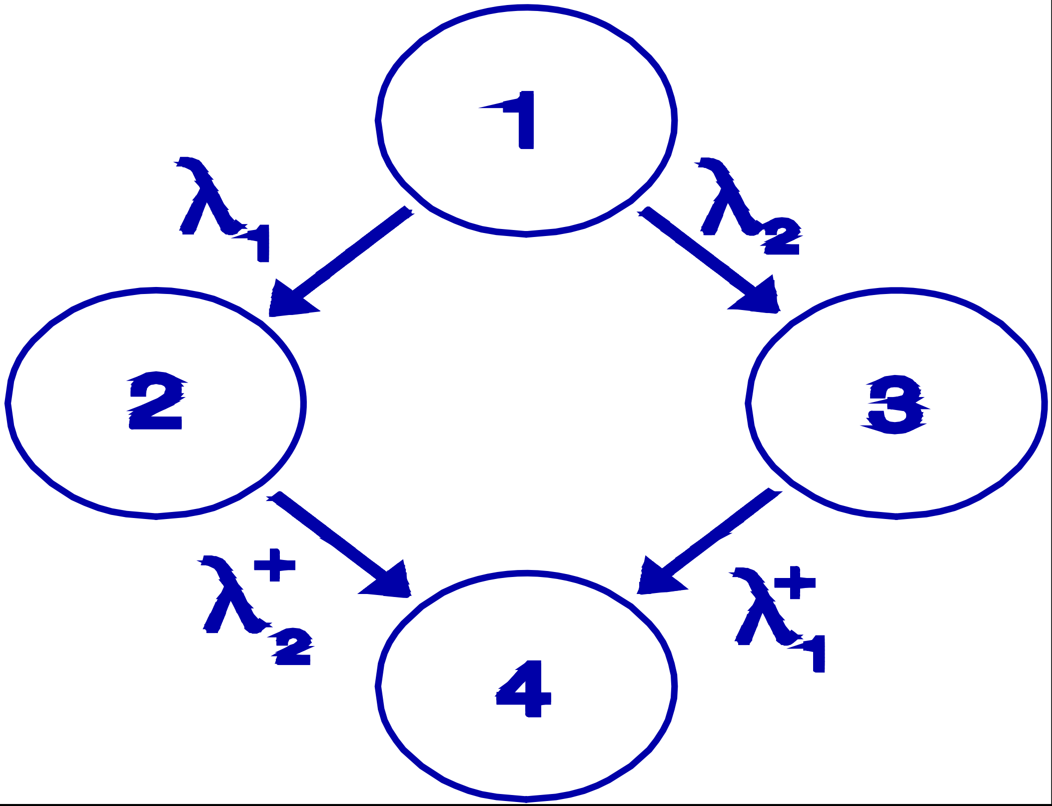
$$R_p(t) = P_1(t) + P_2(t) + P_3(t)$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

Load-Sharing System

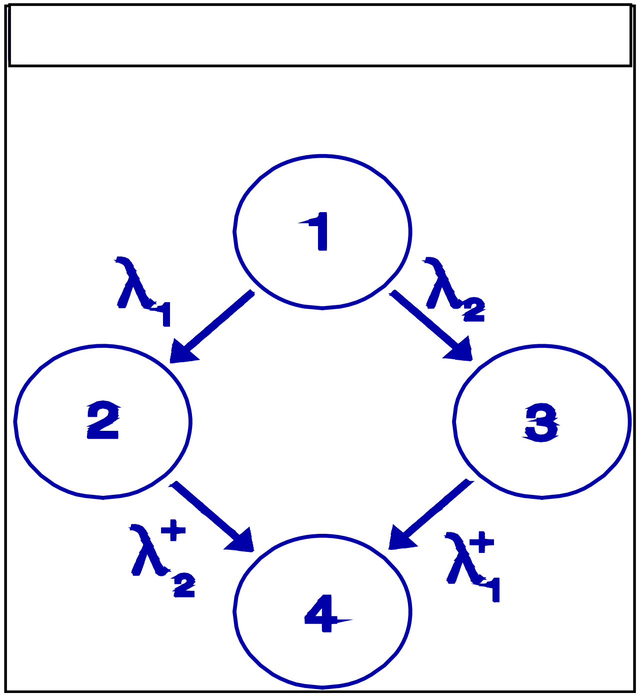
State	Operating
1 -	Both components
2 -	Component 2
3 -	Component 1
4 -	Neither component

Assume Constant Failure Rates



Load-Sharing Systems

$P_i(t)$ = probability of being in state i at time t



$$\frac{d\,P_1(t)}{dt} = -(\lambda_1 + \lambda_2)\,P_1(t)$$

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2^+ P_2(t)$$

$$\frac{d\,P_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1^+ P_3(t)$$

Remember: $P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$!

Load-Sharing Systems

Solution:

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_2^+}\left[e^{-\lambda_2^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1+\lambda_2-\lambda_1^+}\left[e^{-\lambda_1^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$R(t) = P_1(t) + P_2(t) + P_3(t)$$

animated

Load-Sharing Systems

If we let $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^+ = \lambda_2^+ = \lambda^+$, then

$$R(t) = e^{-2\lambda t} + \frac{2\lambda}{2\lambda - \lambda^+}\left[e^{-\lambda^+ t} - e^{-2\lambda t}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{2\lambda} + \frac{2\lambda}{2\lambda - \lambda^+}\left[\frac{1}{\lambda^+} - \frac{1}{2\lambda}\right]$$

Load-Sharing Systems - example

Two generators provide electrical power. If either fails, the other will continue to provide power. However, the increased load results in a higher failure rate of .10 failures per day. When both generators are on-line, the failure rate is .01 failures per day.

$$R(t) = e^{-2(.01)t} + \frac{2(.01)}{2(.01)-.10}\left[e^{-.1t} - e^{-2(.01)t}\right]$$

$$R(10) = e^{-.2t} + \frac{.02}{-.08}\left[e^{-1} - e^{-.2}\right] = .9314$$

$$MTTF = \frac{1}{2(.01)} + \frac{.02}{-.08}\left[\frac{1}{.1} - \frac{1}{.02}\right] = 60\,days$$

animated

Standby Systems

State Operating

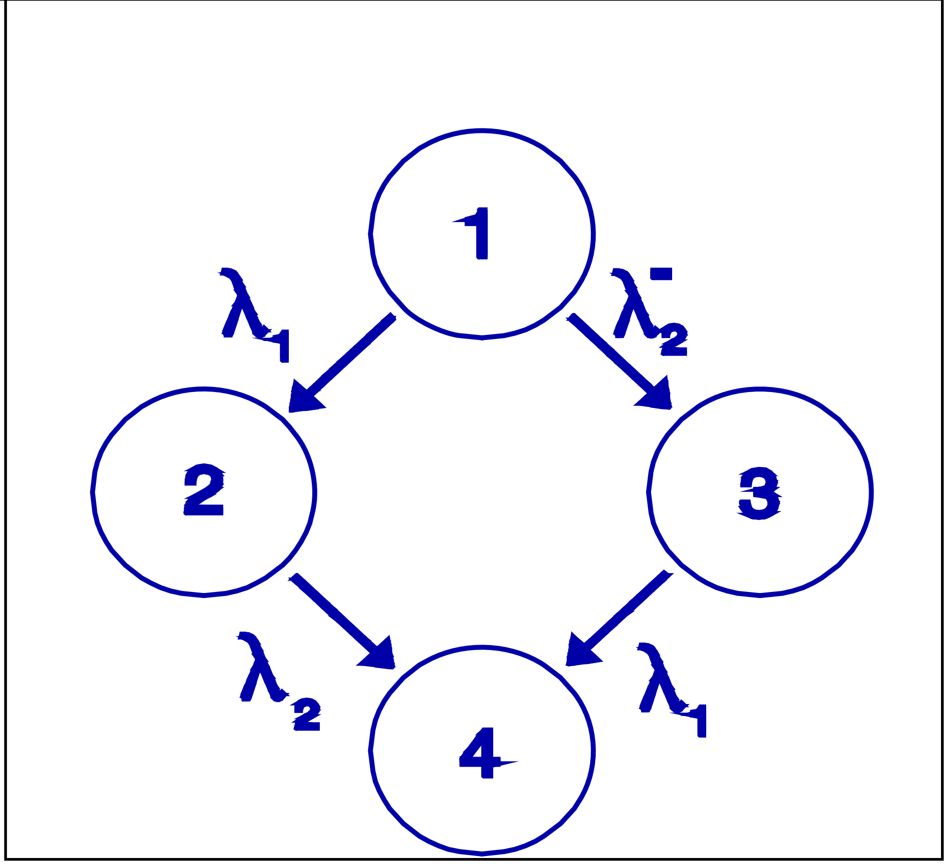
1 - Component 1 with component 2 in standby

2 - Component 2

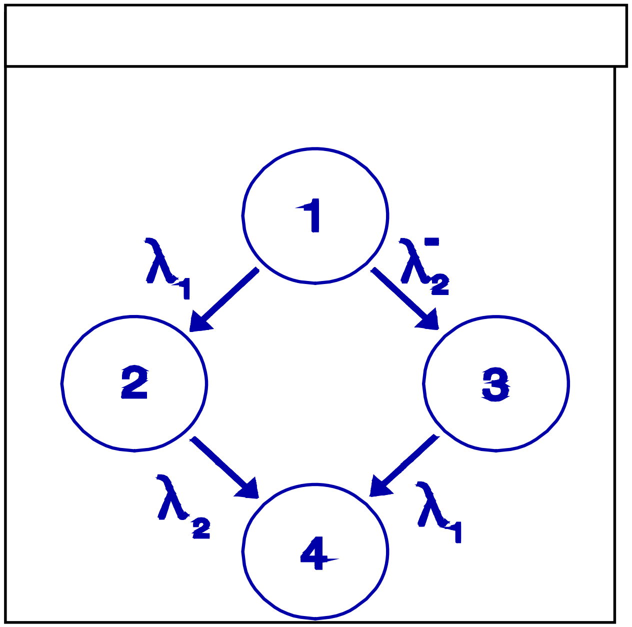
3 - Component 1, component 2 failed in standby

4 - Neither component

Assume Constant Failure Rates



Standby Systems - Model



$$\frac{d\,P_1(t)}{dt} = -(\lambda_1 + \bar{\lambda}_2)\,P_1(t)$$

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$\frac{d\,P_3(t)}{dt} = \bar{\lambda}_2 P_1(t) - \lambda_1 P_3(t)$$

Remember: $P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$!

Standby Systems- Solution

$$P_1(t) = e^{-(\lambda_1+\lambda_2^-)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2^-)t}\right]$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2^-)t}$$

$$R(t) = e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2^-)t}\right]$$

$$MTTF = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2^-}\right] = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_2^-)}$$

Standby Systems - identical units

If we let $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_2^- = \lambda^-$, then

$$R(t) = e^{-\lambda t} + \frac{\lambda}{\lambda^-}\left[e^{-\lambda t} - e^{-(\lambda+\lambda^-)t}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{\lambda} + \frac{\lambda}{\lambda^-}\left[\frac{1}{\lambda} - \frac{1}{(\lambda+\lambda^-)}\right]$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda+\lambda^-}$$

Standby Systems- Example 6.2

An active generator has a .01 failure rate (failures per day). An older standby generator has a .001 failure rate while in standby and a .10 failure rate when on-line. Determine the system reliability for a planned 30 day usage and compute the system MTTF.

$$R(t) = e^{-.01t} + \frac{.01}{.01+.001-.1}\left[e^{-.1t} - e^{-(.011)t}\right]$$

$$R(30) = .741 - .11236\left[0.04978 - 0.7189\right] = .8162$$

$$MTTF = \frac{1}{.01} + \frac{.01}{.1(.01+.001)} = 109.09 \text{ days}$$

Standby Systems- Example 6.3

For a 2 component stand-by system, determine the design life based upon a 95% reliability where both units are identical with λ = .002 failures per hour and λ^- = .0001 failures per hour.

$$.95 = R(t) = e^{-.002t} + \frac{.002}{.0001}\left[e^{-.002t} - e^{-(.0021)t}\right]$$

by trial and error

R(100) = .982
R(200) = .935
R(150) = .961
R(175) = .949
R(173) = .950.

animated

Standby Systems- Identical Units

Assume k identical units with CFR - a primary and k-1 units in standby with no failures in standby.

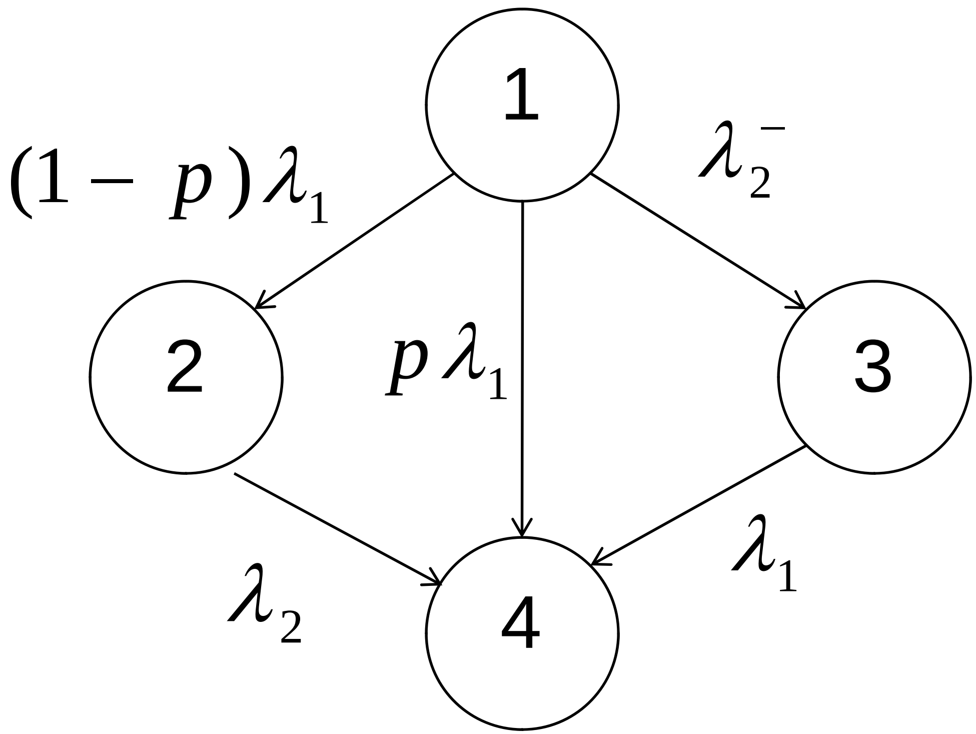
The system fails when the kth failure occurs. The time to kth failure is the sum of k exponential failure times:

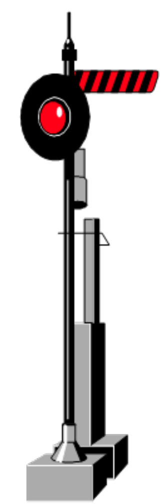
$$T_{failure} = T_1 + T_2 + \ldots + T_k$$

The sum of k independent and identically distributed exponential random variables with parameter λ is gamma with parameters λ and k and MTTF = k/λ.

$$R_k(t) = e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}$$

Standby Systems- Switching Failures





Standby Systems- Switching Failures

$$\frac{d\,P_1(t)}{dt} = -[(1-p)\lambda_1 + p\,\lambda_1 + \lambda_2^-]\,P_1(t) = -(\lambda_1 + \lambda_2^-)\,P_1(t)$$

$$\frac{d\,P_2(t)}{dt} = (1-p)\lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$P_2(t) = \frac{(1-p)\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right]$$

$$R(t) = e^{-\lambda_1 t} + \frac{(1-p)\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right]$$

Only Change

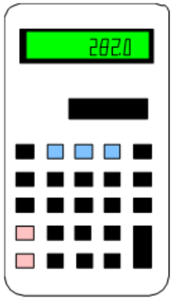
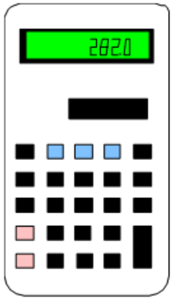
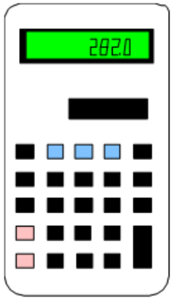
Three-Component System

State	Unit 1	Unit 2	Unit 3
1	on-line	standby	standby
2	failed	on-line	standby
3	failed	failed	on-line
4	failed	failed	failed

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t)$$

$$\frac{d\,P_2(t)}{dt} = \lambda P_1(t) - \lambda P_2(t)$$

$$\frac{d\,P_3(t)}{dt} = \lambda P_2(t) - \lambda P_3(t)$$

animated

Solution to 3 -component system

$$P_1(t) = e^{-\lambda t}$$

$$P_2(t) = \lambda t\, e^{-\lambda t}$$

$$P_3(t) = \frac{\lambda^2 t^2}{2} e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}\left[1 + \lambda t + \frac{\lambda^2 t^2}{2}\right]$$

$$MTTF = \int_0^\infty e^{-\lambda t}\, dt + \int_0^\infty \lambda t\, e^{-\lambda t}\, dt + \int_0^\infty \frac{\lambda^2 t^2}{2} e^{-\lambda t}\, dt = \frac{3}{\lambda}$$

Bonus - Components in Series with Standby Redundancy - together at last

(Problem 6.24)

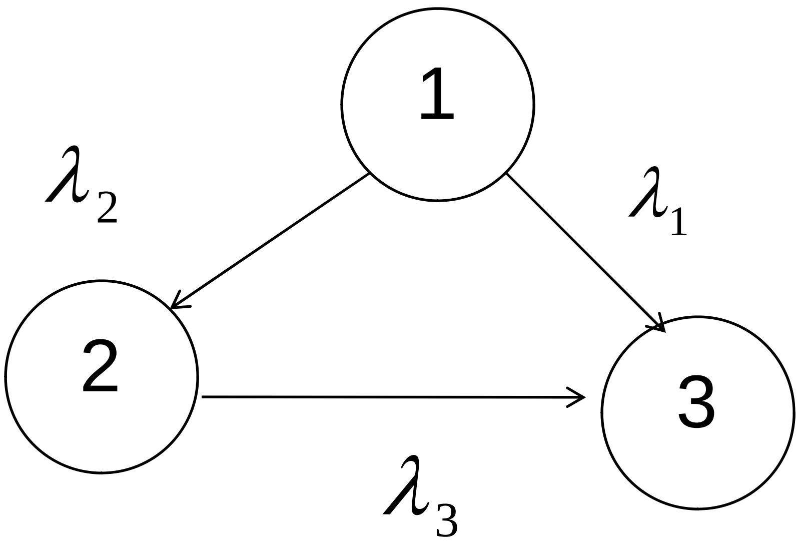
N identical components having failure rate λ with k in series and N-k standby units:

$$R_{sys}(t) = e^{-k\lambda t} \sum_{i=0}^{N-k} \frac{(k\lambda t)}{i!}$$

note: kλt is the expected number of failures during time t of k identical and active units in series. This becomes the mean of a Poisson process. $R_{sys}(t)$ is the equivalent to finding the probability of no more than N-k failures (the number of standby or replacement units) during time t. What is the MTTF?

Degraded Systems

fully operational	state 1
degraded	state 2
failed	state 3



$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_2 P_1(t) - \lambda_3 P_2(t)$$

Degraded Systems Solution

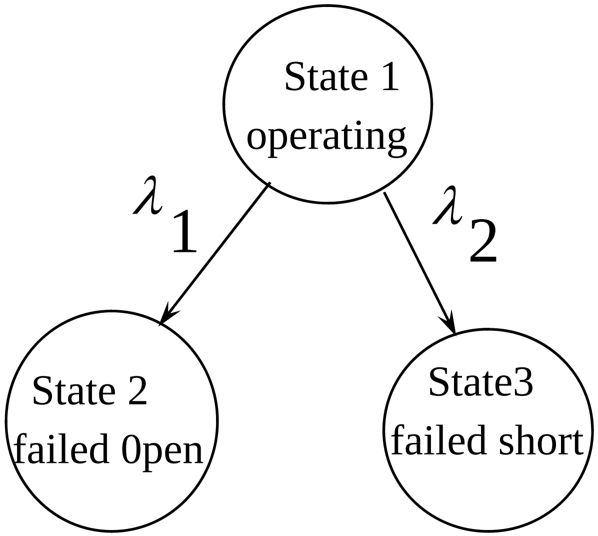
$$P_1(t) = e^{-(\lambda_1 + \lambda_2)t}$$

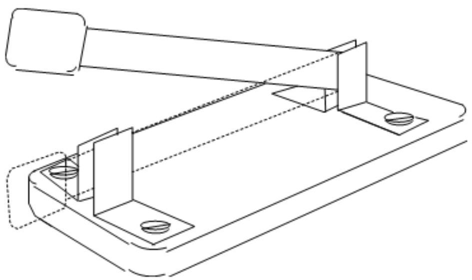
$$P_2(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[e^{-\lambda_3 t} - e^{-(\lambda_1 + \lambda_2)t}\right]$$

$$R(t) = P_1(t) + P_2(t)$$

$$MTTF = \int_0^{\infty}[P_1(t) + P_2(t)]dt = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[\frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2}\right]$$

Three-State Devices





$$\frac{d\ P_1(t)}{dt} = -(\lambda_1 + \lambda_2)\, P_1(t)$$

$$\frac{d\ P_2(t)}{dt} = \lambda_1\, P_1(t)$$

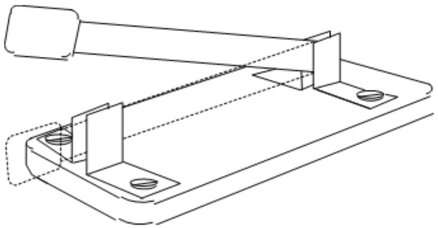
$$\frac{d\ P_3(t)}{dt} = \lambda_2\, P_1(t)$$

Three State Devices Solution

$$R(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$



The Last Slide (for now)

Next Class: Chapter 7

- Physical reliability,
- Covariate Model,
- Interference theory,
- and much, much more…

You will not want to miss the next reliability class. It promises to be the best ever.