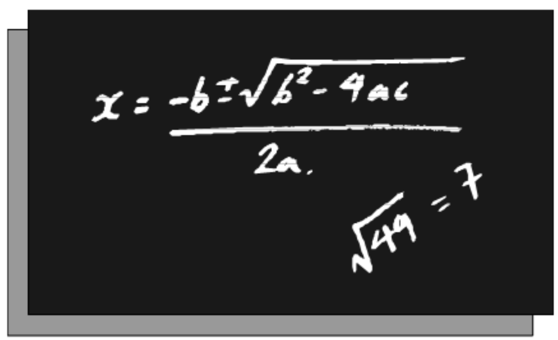
Chapter 2
Basic Reliability Models

The Failure Distribution



The Reliability Function

Let T = a random variable, the time to failure of a component

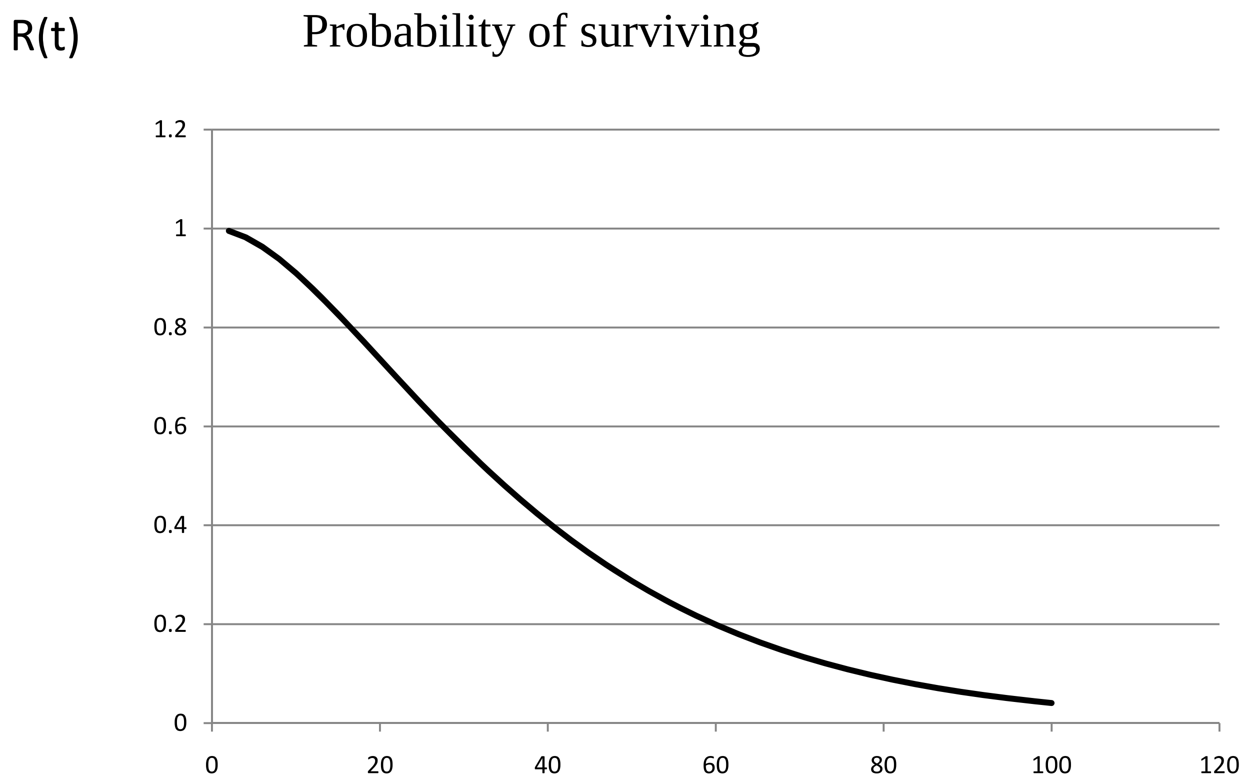
$$R(t) = Pr\{T \geq t\}$$

where $R(t) \geq 0$, $R(0) = 1$, and

$$\lim_{t\to\infty} R(t) = 0$$

Often called the SURVIVAL FUNCTION

Graph of a Reliability Function

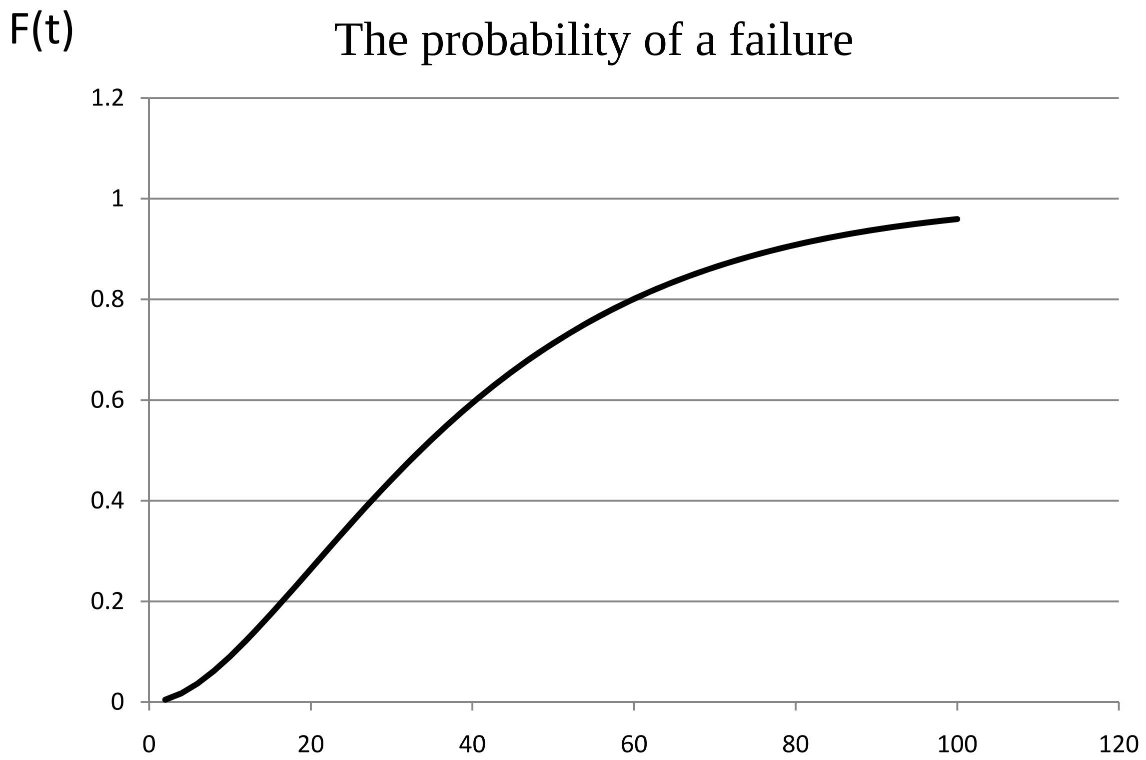


The Cumulative Distribution Function (CDF)

$$F(t) = 1 - R(t) = Pr\{T < t\}$$

$$\text{where } F(0) = 0 \text{ and } \lim_{t \to \infty} F(t) = 1$$

Graph of a CDF



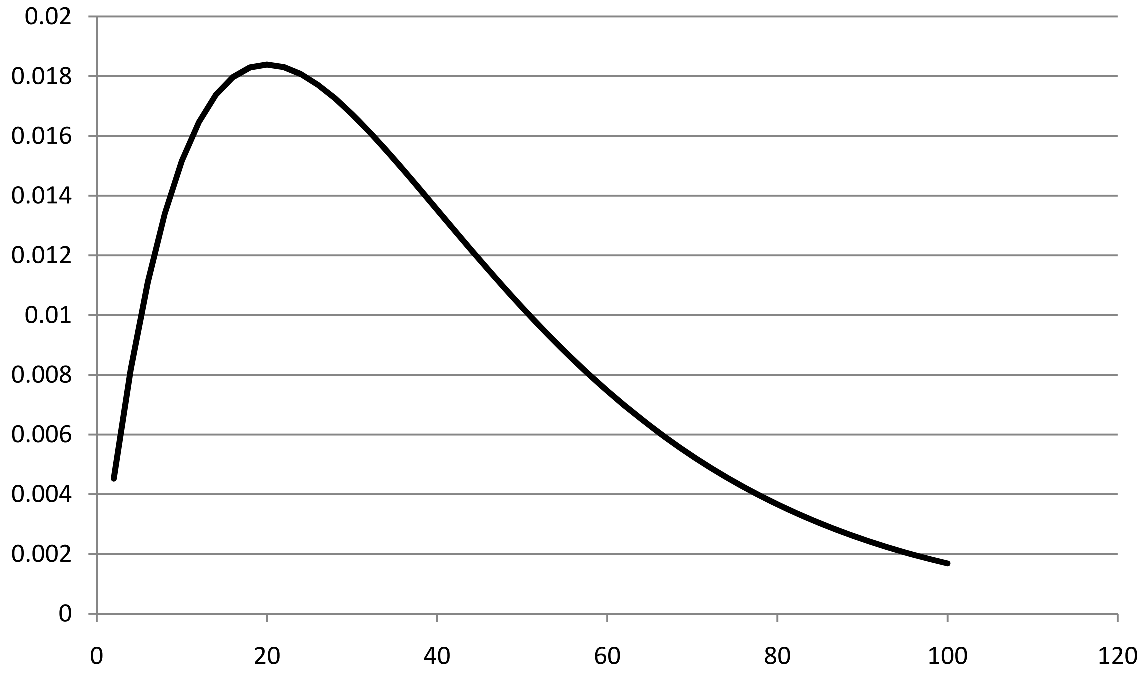
The Density Function (PDF)

$$f(t) = \frac{d\,F(t)}{dt} = -\frac{d\,R(t)}{dt}$$

$$f(t) \geq 0 \quad and \quad \int_0^\infty f(t)dt = 1$$

Graph of a Density Function (PDF)

f(t)

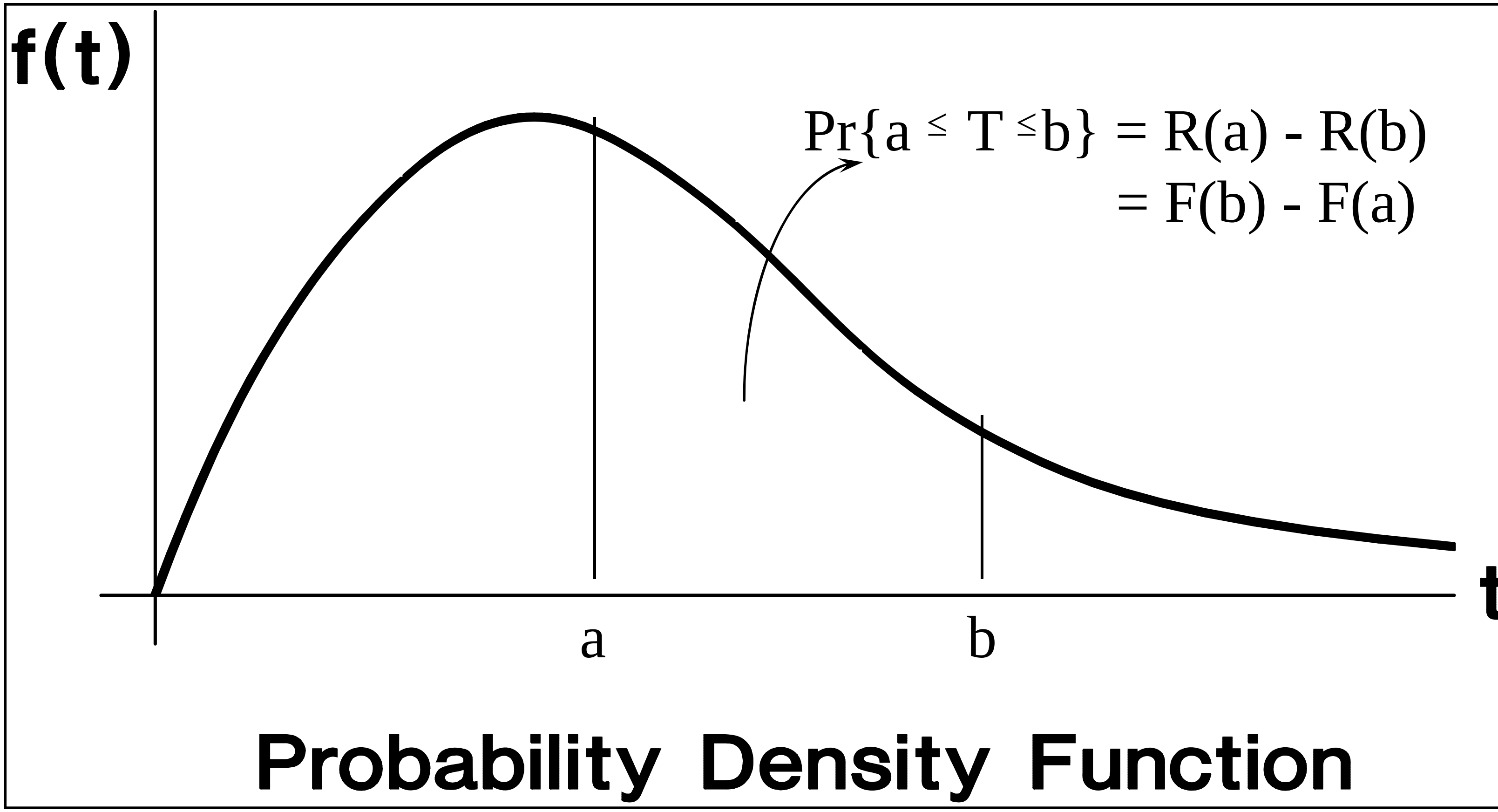


Relationship between PDF and CDF

$$F(t) = \int_0^t f(t')\,dt'$$

$$R(t) = \int_t^\infty f(t')\,dt'$$

Finding Failure Probabilities



Example

The passive components of a distribution system for natural gas has the following reliability function:

$$R(t) = 1 - \frac{t^2}{100}; \quad 0 \le t \le 10 \text{ yrs}$$

Find:
a. R(3 yrs)
b. The CDF, F(t)
c. Pr{1<T<3}
d. The density function, f(t)

Example - solution

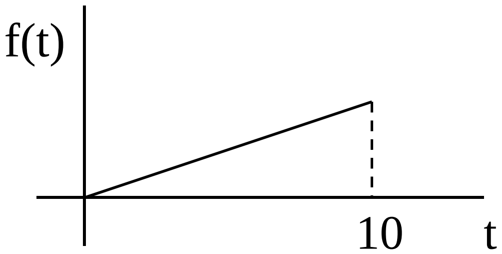
$$a. \quad R(3) = 1 - \frac{3^2}{100} = .91$$

$$\boxed{R(t) = 1 - \frac{t^2}{100}; \quad 0 \le t \le 10 \ yrs}$$

b. $F(t) = 1 - [1 - t^2 / 100] = t^2 / 100$

c. $\Pr\{1<T<3\} = F(3) - F(1) = .09 - .01 = .08$

$$d. \quad f(t) = \frac{dF(t)}{dt} = \frac{2t}{100} = \frac{t}{50}; \quad 0 \le t \le 10$$



Mean Time to Failure (MTTF)

$$MTTF = \int_0^\infty t\ f(t)\,dt = \int_0^\infty R(t)\,dt$$

Note alternate notation: MTTF = E[T]

Derivation of MTTF (Appendix 2A)

$$MTTF = \int_0^\infty t f(t)\, dt = \int_0^\infty -t \frac{dR(t)}{dt} dt$$

Integration by parts:

$$MTTF = -tR(t)\Big|_0^\infty + \int_0^\infty R(t) dt = \int_0^\infty R(t)\, dt$$

Since

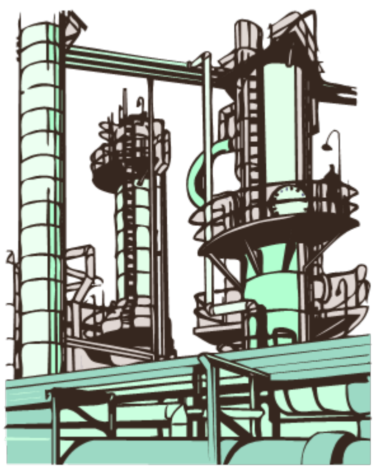
$$-tR(t)\Big|_0^\infty = \lim_{t\to\infty} t\, R(t) - 0R(0) = \lim_{t\to\infty} t \exp\left[-\int_0^t \lambda(t')dt'\right] = 0$$

Example - MTTF

For the distribution system, find the MTTF.

$$MTTF = \int_0^{\infty} t\ f(t)dt = \int_0^{10} t\,\frac{t}{50}\,dt$$

$$= \frac{t^3}{150}\bigg|_0^{10} = \frac{10^3}{150} = \frac{100}{15} = 6\frac{2}{3}\ yr$$



Example - MTTF revisited

For the distribution system, find the MTTF.

$$MTTF = \int_0^{\infty} R(t)dt = \int_0^{10} (1 - \frac{t^2}{100})dt$$

$$= t - \frac{t^3}{300}\Bigg|_0^{10} = 10 - \frac{1000}{300} = 6\frac{2}{3}\ yr$$

Median Time to Failure and Mode

$$R(t_{med}) = .5 = P\{T \geq t_{med}\}$$

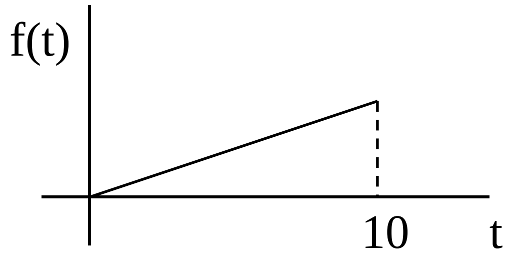
$$f(t_{mode}) = \max_{0 \leq t < \infty} f(t)$$

Example - median and mode

$R(t) = 1 - t^2 / 100 = .5$

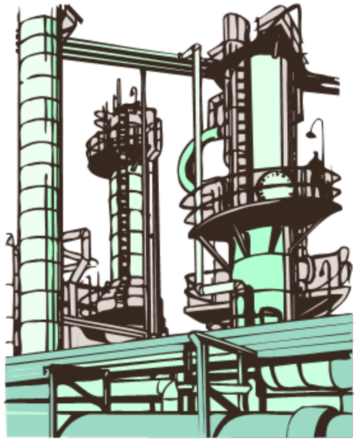
$t^2 / 100 = .5$

$t^2 = 50$ or $\quad t_{med} = 7.07$ yrs

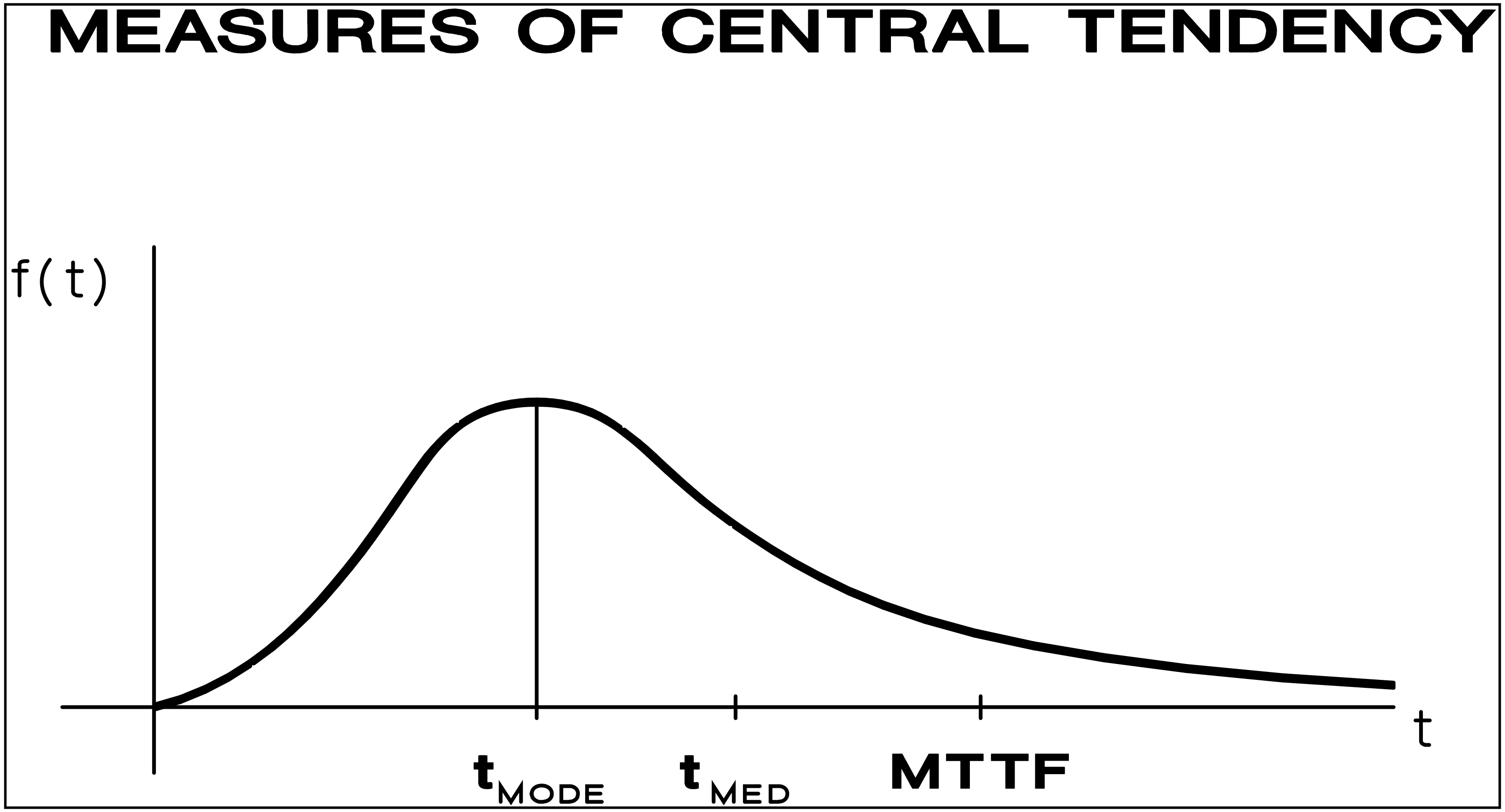


by inspection

$t_{mode} = 10$ yrs



Comparison of MTTF, Median, & Mode



Design Life

Find t_R such that $R(t_R) = R$

For example:

Find that time, $t_{.99}$ such that $R(t_{.99}) = .99$

Then $t_{.99}$ is the 99 percent design life.

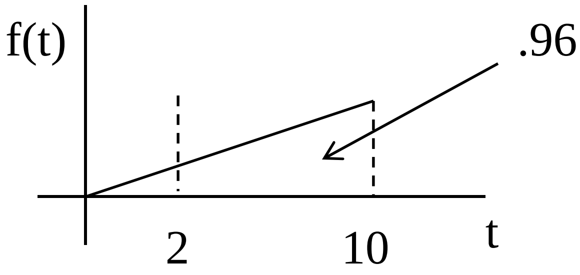
One percent will fail before time $t_{.99}$

Example – design life

$R(t) = 1 - t^2 / 100 = .96$

$t^2 /100 = .04$

$t^2 = 4$ or $t_{.96} = 2$ yrs



Variance & Standard Deviation

definitional form:

$$\sigma^2 = \int_0^\infty (t - MTTF)^2 f(t)dt$$

computational form:

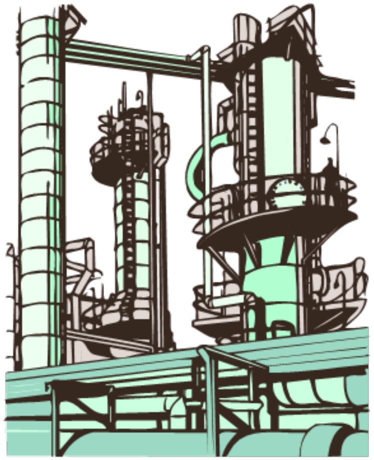
$$\sigma^2 = \int_0^\infty t^2 f(t)dt - (MTTF)^2$$

Example - standard deviation

$$\sigma^2 = \int_0^{10} \frac{t^3}{50} dt - \left(6\frac{2}{3}\right)^2$$

$$= \left.\frac{t^4}{200}\right|_0^{10} - \left(6\frac{2}{3}\right)^2 = \frac{10,000}{200} - 44.444 = 5.55$$

$$or \quad \sigma = \sqrt{5.55} = 2.36 \ yr.$$



Hazard Rate Function

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

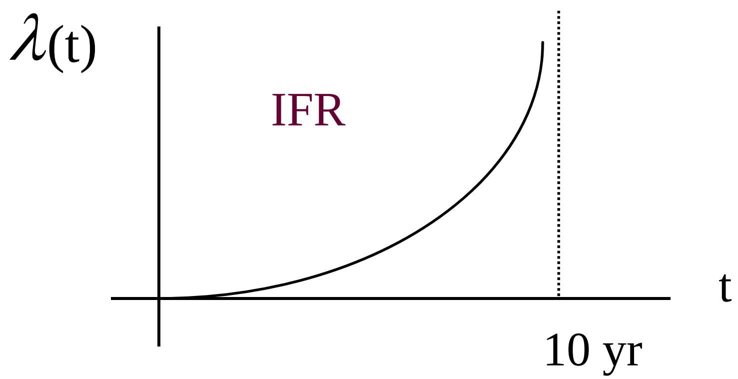
$$Pr\{t\le T\le t+\Delta t\}=R(t)-R(t+\Delta t)$$

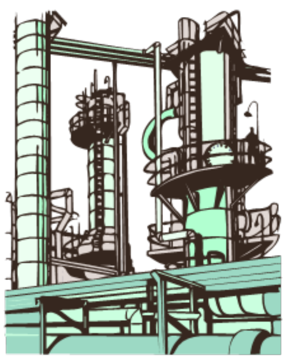
$$Pr\{t\le T\le t+\Delta t\,|\,T\ge t\}=\frac{R(t)-R(t+\Delta t)}{R(t)}$$

$$\lambda(t)=\lim_{\Delta t\to 0}\frac{-[R(t+\Delta t)-R(t)]}{\Delta t}\;\frac{1}{R(t)}=\frac{-dR(t)}{dt}\frac{1}{R(t)}=\frac{f(t)}{R(t)}$$

Example - hazard rate function

$$\lambda(t) = \frac{t/50}{1 - t^2/100} = \frac{t/50}{\frac{100 - t^2}{100}} = \frac{2t}{100 - t^2}$$





Derivation of R(t) from the Hazard Rate Function

$$\lambda(t) = -\frac{d\,R(t)}{dt}\frac{1}{R(t)}$$

$$\lambda(t)dt = -\frac{d\,R(t)}{R(t)} \longrightarrow \int_0^t \lambda(t')dt' = -\int_1^{R(t)} \frac{d\,R(t')}{R(t')}$$

$$-\int_0^t \lambda(t')dt' = \ln R(t)$$

$$R(t) = \exp\left[-\int_0^t \lambda(t')dt'\right]$$

Hazard Rate Function & R(t)

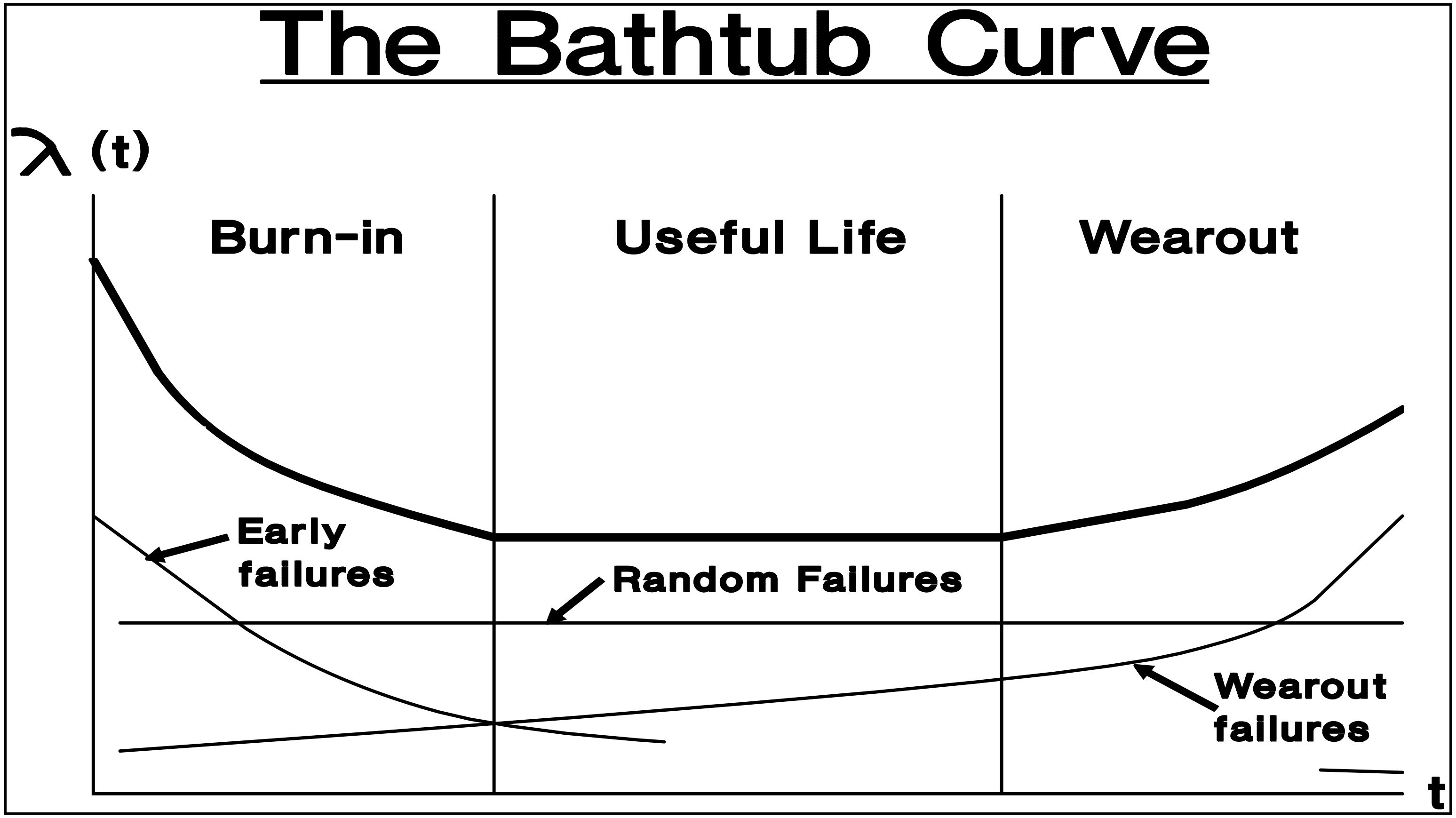
$$R(t) = e^{-\int_0^t \lambda(t')dt'}$$

Example:

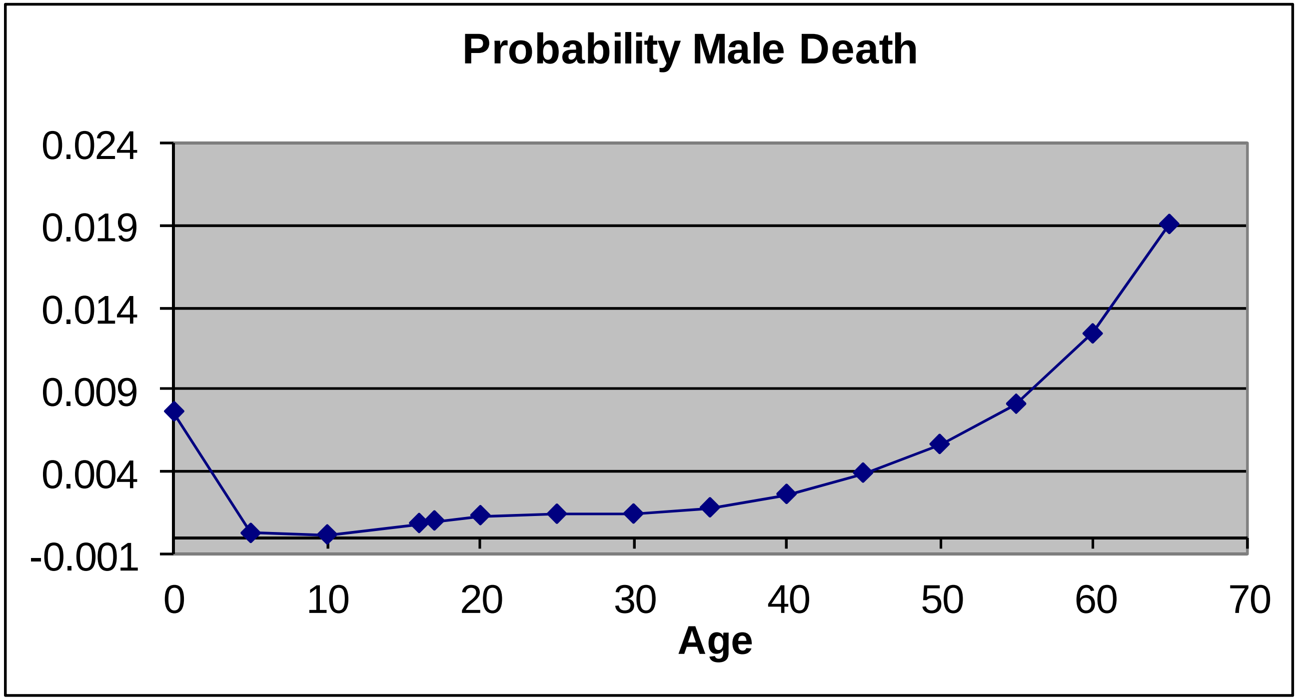
$$\lambda(t) = .02t\,,\ then$$

$$R(t) = e^{-\int_0^t .02t'dt'} = e^{-.01t^2}$$

The Bathtub Curve



Human Mortality Curve



age	Male	Female
0	0.007644	0.006275
5	0.000202	0.000152
10	0.00011	0.000113
16	0.00081	0.000375
17	0.000964	0.000423
20	0.00129	0.000456
25	0.001379	0.000499
30	0.001389	0.000628
35	0.00177	0.000953
40	0.002589	0.001514
45	0.003891	0.002264
50	0.005643	0.003227
55	0.008106	0.004884
60	0.012405	0.007732
65	0.019102	0.012199
70	0.029824	0.019312
75	0.046499	0.030582
80	0.073269	0.050396
85	0.120186	0.086443
90	0.192615	0.147616

More on the Bathtub Curve

	Burn-in	Useful Life	Wearout
Characterized by	DFR	CFR	IFR
Caused by	Manufacturing defects Welding flaws, Cracks, Defective parts, Poor quality control, Contamination, Poor workmanship	Environment Random loads Human error "Acts of God" Chance events	Fatigue Corrosion Aging Friction Cyclical loading
Reduced by	Burn-in testing Screening Quality control Acceptance testing	Redundancy Excess strength	Derating Preventive Maint. Parts replacement Technology

Average Failure Rate (AFR)

$$AFR(t_1,t_2)=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\lambda(t')\,dt'=\frac{\ln R(t_1)-\ln R(t_2)}{t_2-t_1}$$

since:

$$R(t)=e^{-\int_0^t \lambda(t')dt'}$$

Note: AFR(0,t) = AFR(t) = -ln R(t) / t

Example - Average Failure Rate

$$AFR(t) = \frac{-\ln\left[1 - \frac{t^2}{100}\right]}{t}$$

AFR(5 yr) = - ln [1 - .25] / 5 = .0575 failures / yr

Conditional Reliability

$$\boxed{R(t|T_0) = P\{\underbrace{T > T_0 + t}_{\text{Event A}} \mid \underbrace{T > T_0}_{\text{Event B}}\}}$$

Event A

Event B

$$\boxed{= \frac{P\{T > T_0 + t\}}{P\{T > T_0\}} = \frac{R(T_0 + t)}{R(T_0)}}$$

Residual MTTF

$$MTTF(T_0) = \int_0^{\infty} R(t \mid T_0)\,dt = \int_0^{\infty} \frac{R(t+T_0)}{R(T_0)}\,dt$$

$$= \frac{1}{R(T_0)} \int_{T_0}^{\infty} R(t')\,dt'$$

where $t' = t + T_0$

Example - conditional reliability

$$R(t) = 1 - \frac{t^2}{100}$$

$$R(t|1) = \frac{R(t+1)}{R(1)} = \frac{1-(t+1)^2/100}{1-(1/100)}$$

$$= \frac{100-(t+1)^2}{100(.99)} = 1.01-(t+1)^2/99$$

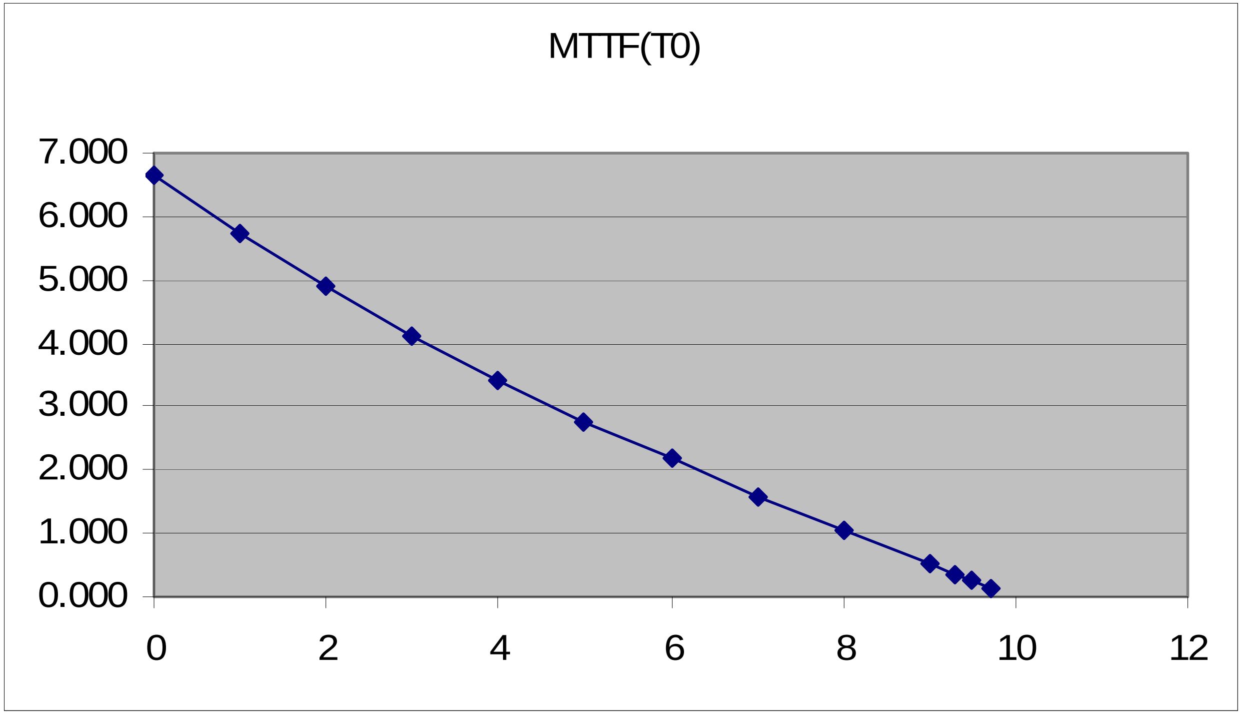
Therefore: R(5|1) = 1.01-36/99=.646
where R(5) = 1 - .25 = .75

$$MTTF(T_0) = \frac{1}{1-T_0^2/100}\int_{T_0}^{10} 1-\frac{t^2}{100}dt = \frac{1}{.99}\left[t-\frac{t^3}{300}\right]_{T_0=1}^{10}$$

$$or\ \ MTTF(1) = \frac{1}{.99}\left[10-\frac{10^3}{300}-1+\frac{1}{300}\right] = \frac{5.67}{.99} = 5.72\,yr.$$

Residual MTTF(T_0)

T0	MTTF(T0)
0	6.667
1	5.727
2	4.889
3	4.128
4	3.429
5	2.778
6	2.167
7	1.588
8	1.037
9	0.509
9.5	0.252
9.8	0.100
9.9	0.050



Student Exercise #1

A panel consisting of analog displays has a reliability function given by

$$R(t) = (200-t)/200 \text{ for } 0 < t < 200$$

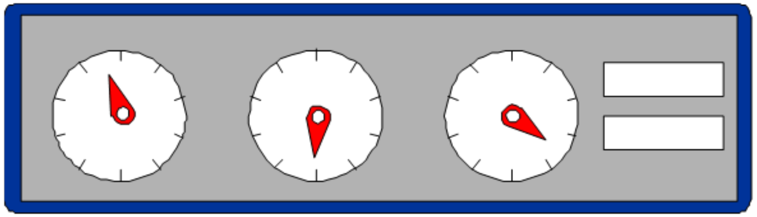
where t is measured in 1000's of hr. Find:

a. R(50,000) and R(12,000)

b. R(50,000 | 12,000)

c. MTTF

d. MTTF(12,000)



Student #1- solution

a. R(50) = (200 - 50)/200 = .75

R(12) = (200 - 12)/200 = .94

b. R(50|12) = R(62) / R(12) = (200 - 62) /200 / .94= .69 / .94 = .734

$$c.\ MTTF = \int_0^{200} 1 - t/200\, dt = t - \frac{t^2}{400}\Big|_0^{200} = 200 - \frac{40000}{400} = 100$$

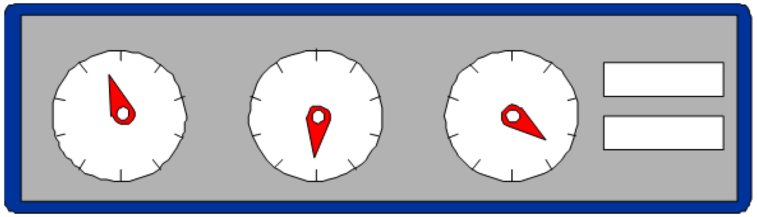
$$d.\ MTTF(12) = \frac{1}{.94}\int_{12}^{200} (1 - t/200)\, dt = \frac{1}{.94}\left[t - \frac{t^2}{400}\right]_{12}^{200} = 94$$

Student Exercise #1 (continued)

e. What is the shape of the density function?

f. Is the hazard rate function increasing or decreasing?

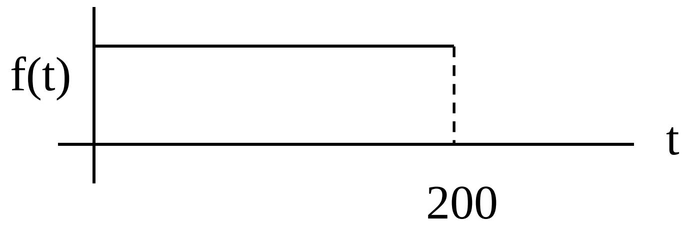
g. Compute the average failure rate over the first 100,000 miles.



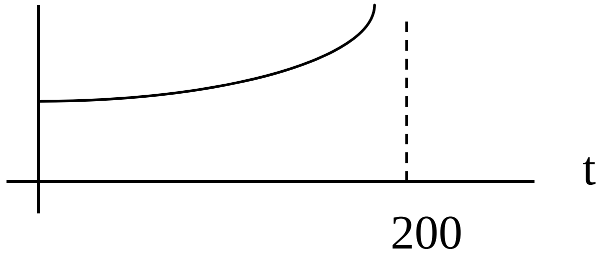
Student Exercise #1-solution

e. f(t) = -dR(t) /dt

= - d {(200-t)/200}/dt =1/200



$$\boxed{f.\ \ \lambda(t) = \frac{1/200}{(200-t)/200} = \frac{1}{200-t}}$$



g. AFR(100) = - ln {(200-100)/200} / 100

= .007 failures/ 1000 mi.

Student Exercise #2

Insulators on a power distribution system have a reliability function with t measured in yr.

$$R(t) = 1 / (1 + .05t) \text{ where } t >= 0$$

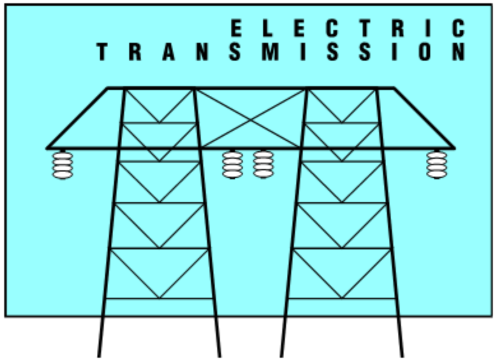
Find:

a. F(1 yr) and R(2)

b. R(2|1)

c. The hazard rate function (optional)

d. AFR(3)



Student Exercise #2 - solution

a. $F(1) = Pr\{T<1\} = 1 - 1/[1+.05(1)] = 1 - .9524 = .0476$

$R(2) = Pr\{T>2\} = 1/[1+.05(2)] = .9091$

b. $R(2|1) = Pr\{T>3 \mid T>1\} = R(3) / R(1)$

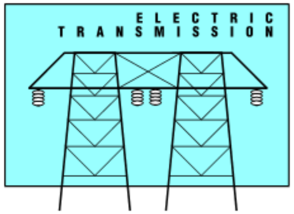
$= [1+.05(3)]^{-1} / .9524 = .913$

c. $f(t) = -d\, [1+.05t]^{-1} /dt = .05[1+.05t]^{-2}$

$\lambda(t) = f(t)/R(t) = .05/[1+.05t]$ which is DFR

d. $AFR(3) = \{-\ln\, [1+.05(3)]^{-1} \} / 3$

$= -\ln .8696 /3 = .0466$ failures per year.



Summary - The Four Functions

f(t), the Probability Density Function (PDF)

F(t), the Cumulative Distribution Function (CDF)

R(t), the Reliability Function

λ(t), the Hazard Rate Function