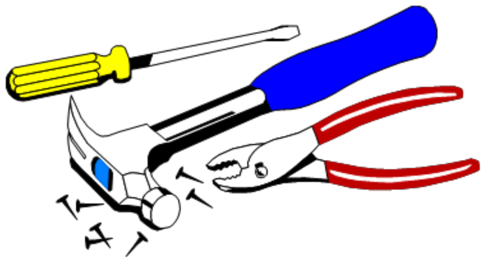
Chapter 9 Part I Maintainability

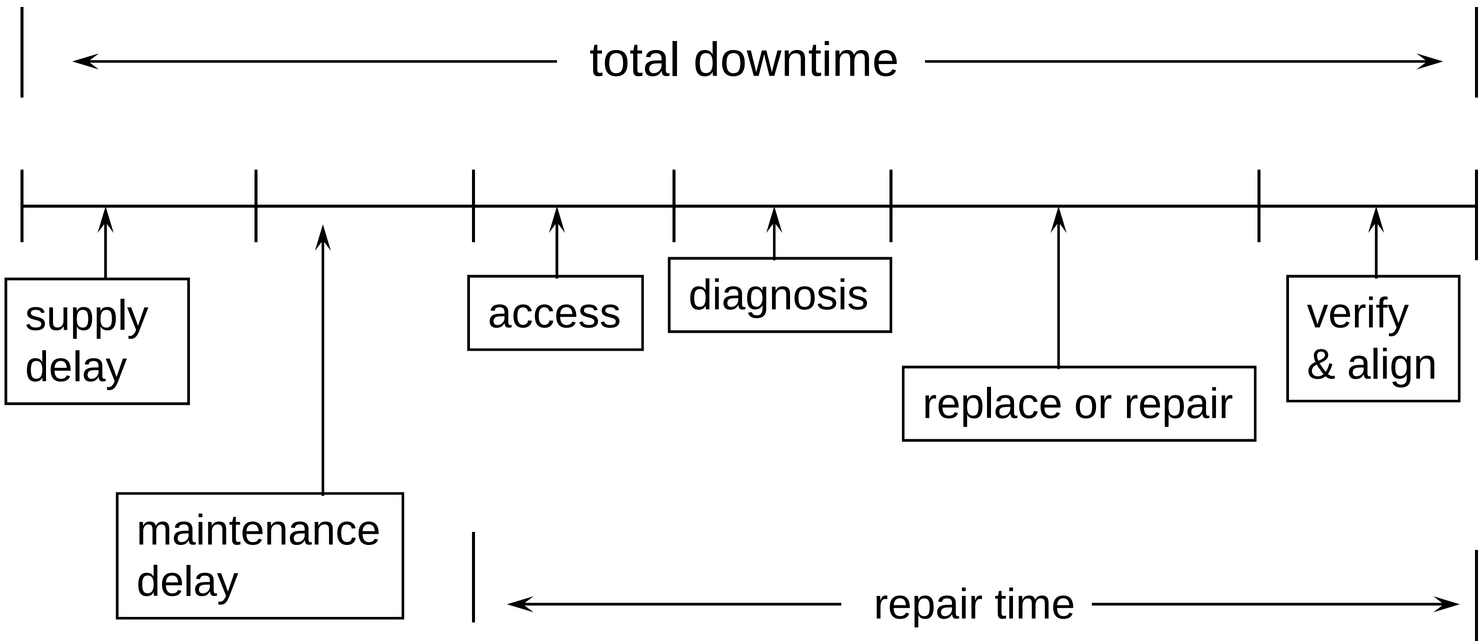


9.1 Analysis of Downtime

9.2 The Repair-Time Distribution

9.3 Stochastic Point Processes

9.1 Analysis of Downtime



9.2 The Repair-Time Distribution

Let T = a continuous random variable, the time to repair a failed unit where h(t) is the probability density function (PDF). Then the CDF is

$$P(T \le t) = H(t) = \int_0^t h(t')dt'$$

$$MTTR = \int_0^\infty t\, h(t)dt = \int_0^\infty (1 - H(t))dt$$

$$\sigma^2 = \int_0^\infty (t - MTTR)^2 h(t)dt$$

Example 9.2

Given h(t) = 0.0833 t   1 <= t <= 5 hr, then

$$H(t) \;=\; \int_{1}^{t} .08333t'\,d't = 0.041665t^2 - 0.041665$$

Pr{T<3} = H(3) = .041665 x 9 - .041665 = .333

$$MTTR \;=\; \int_{1}^{5} .08333\,t^2\;dt \;=\; \left.\frac{.08333\,t^3}{3}\right|_{1}^{5} \;=\; 3.44\ hrs$$

Exponential repair Times

$$H(t) = \int_0^t \frac{e^{-\frac{t'}{MTTR}}}{MTTR} dt' = 1 - e^{-\frac{t}{MTTR}}$$

Let r = 1/MTTR, the constant rate of repair, then

$$H(t) = \int_0^t r\, e^{-rt'} dt' = 1 - e^{-rt}$$

Exponential repair Times

A component can be repaired at the constant rate of 10 per 8 hour day. What is the probability of a single repair exceeding one hour?

Solution: The MTTR = .1 day = .8 hour.
Therefore $P\{T>1\} = 1 - H(1) = e^{-1/.8} = e^{-1.25} = 0.2865$

Lognormal Repair Times

$$h(t) = \frac{1}{\sqrt{2\pi}\, t\, s}\, e^{-\frac{1}{2}\frac{\left(\ln \frac{t}{t_{med}}\right)^2}{s^2}}, \quad t \geq 0$$

$$P\{T \leq t\} = H(t) = \Phi\left(\frac{1}{s}\ln\frac{t}{t_{med}}\right).$$

$$MTTR = t_{med}\, e^{\frac{1}{2}s^2}$$

Lognormal Repair Times

A requirement exists for an engine fuel pump to be repaired (or replaced) within three hours ninety percent of the time. If the repair distribution is lognormal with s = .45, what MTTR should be achieved to meet this goal?

$$\Phi\left(\frac{1}{.45}\ \ln\frac{3}{t_{med}}\right) = .90 \qquad \frac{1}{.45}\ \ln\frac{3}{t_{med}} = 1.28$$

$$t_{med} = \frac{3}{e^{1.28(.45)}} = 1.686 \text{ hours}$$

$$MTTR = 1.686\, e^{(.45)^2/2} = 1.865 \text{ hours}$$

$$t_{MODE} = \frac{t_{med}}{e^{s^2}} = \frac{1.686}{e^{(.45)^2}} = 1.377 \text{ hours}$$

9.3 Stochastic Point Processes

A stochastic point process is characterized by isolated events occurring at instants distributed randomly over a time continuum. The events in this case are failures with T_1 , T_2 , ... , representing their occurrence times.

T_k = a random variable, the (operating) time to the k^{th} failure where $T_0 = 0$, and $X_k = T_k - T_{k-1}$ = a random variable, the (operating) time* between the $(k-1)^{st}$ and k^{th} failure.

$$T_k = \sum_{i=1}^{k} X_i \qquad E(T_k) = \sum_{i=1}^{k} E[X_i]$$

$E[X_i]$ is the Mean Time Between Failures (MTBF)

*or clock time assuming repair time is negligible

Renewal Process

A renewal process is defined to be one in which the random variables, X_i , are independent and identically distributed (IID).

i.e. repair to "as good as new"

$$E(T_k) = k\,E(X_1) = k\mu_1$$

$$Var(T_k) = kVar(X_1) = k\sigma_1^2$$

From the CLT:

$$\Pr\{T_k \leq t\} \approx \Phi\left(\frac{t - k\mu_1}{\sigma_1\sqrt{k}}\right)$$

Central Limit Theorem

If $Y_n = X_1 + X_2 + \ldots + X_n$ where the X_i are independent random variables with means $E(X_i)$ and variances, $V(X_i)$, then

Y_n has an approximate normal distribution with

$$E(Y_n) = \sum_{i=1}^{n} E(X_i)\,;\; V(Y_n) = \sum_{i=1}^{n} V(X_i)$$

Example 9.5

A cutting tool has a time-to-failure distribution which is normal with a mean of 5 operating hr. and std dev = 1 hr. Nine replacement tools are available with which to complete a 40 hr. production run. Find R(40) given the 9 cutting tools.

$$\Pr\{T_9 \geq 40\} = 1 - \Phi\left(\frac{40-45}{\sqrt{9}}\right) = 1 - \Phi(-1.67) = .95254$$

Number of Failures

Letting $N(t)$ be a discrete random variable, the cumulative number of failures in the interval $(0,t)$, then $N(t) \geq 0$ and a nondecreasing function of time. Further

$$\Pr\{N(t) = 0\} = \Pr\{T_1 > t\}$$

$$\Pr\{N(t) = j\} = \Pr\{ T_j <= t < T_{j+1} \} = \Pr\{ T_{j+1} >= t \} - \Pr\{ T_j > t \}$$

for $j = 1, 2, \ldots.$

Example 9.5 (cont.)

Distribution of the number of failures in the first 12 hours:

$$\Pr\{N(12)=0\}=\Pr\{T_1>12\}=1-\Phi\left(\frac{12-5)}{1}\right)=1-\Phi(7)=0$$

$$\Pr\{N(12)=1\}=\Pr\{T_2\geq 12\}-\Pr\{T_1>12\}$$

$$=1-\Phi\left(\frac{12-10}{\sqrt{2}}\right)-0=1-\Phi(1.414)=.07927$$

$$\Pr\{N(12)=2\}=\Pr\{T_3\geq 12\}-\Pr\{T_2>12\}$$

$$=1-\Phi\left(\frac{12-15}{\sqrt{3}}\right)-\left[1-\Phi\left(\frac{12-10}{\sqrt{2}}\right)\right]=-\Phi(-1.732)+\Phi(1.414)=.87891$$

Example 9.5 (cont.)

Distribution of the number of failures in the first 12 hours:

$\Pr\{N(12) = 3\} = .04179$

$\Pr\{N(12) = 4\} = .00003$

$\Pr\{N(12) >= 5\} = 0$

$E\{N(12)\} = 0 + 1(.07927) + 2\,(.87891) + 3\,(.04179) + 4\,(.00003) = 1.96.$

Gamma Failure Distribution

If X_i has a gamma distribution with shape parameter γ and scale parameter α, then from the additive property of the gamma distribution, T_k will also have a gamma distribution with shape parameter $k\gamma$ and scale parameter α. Therefore

$$\Pr\{T_k \le t\} = \frac{1}{\Gamma(k\gamma)} \int_0^{t/\alpha} y^{k\gamma-1} e^{-y}\, dy \;\; = \frac{I\left(\dfrac{t}{\alpha}, k\gamma\right)}{\Gamma(k\gamma)}$$

EXAMPLE 9.6

A 40-gal hot water heater having a gamma failure distribution with $\gamma = 1.5$ and $\alpha = 3$ yr is replaced upon failure. For t = 10 yr,

$$\Pr\left\{N(10)=0\right\} = \Pr\left\{T_1 > 10\right\} = R(10) = 1 - \frac{I\left(\frac{10}{3}, 1.5\right)}{\Gamma(1.5)} = .0833$$

$$\Pr\left\{N(10)=1\right\} = \Pr\left\{T_2 > 10\right\} - \Pr\left\{T_1 > 10\right\} = 1 - \frac{I\left(\frac{10}{3}, 3\right)}{\Gamma(3)} - .0833 = .3528 - .0833 = .2695$$

$$\Pr\left\{N(10)=2\right\} = \Pr\left\{T_3 > 10\right\} - \Pr\left\{T_2 > 10\right\} = 1 - \frac{I\left(\frac{10}{3}, 4.5\right)}{\Gamma(4.5)} - .3528 = .3190$$

$$\Pr\left\{N(10)=3\right\} = .207, \text{ etc. with } E\left[N(10)\right] \approx 2.056$$

$E[T_2]$ = 9 yr. and $E[T_3]$ = 13.5 yr.

Homogeneous Poisson Process (HPP)

If X_i is exponential with parameter λ; then T_k is gamma with parameters k and λ ; and

$$\Pr\{N(t)=j\}=e^{-\lambda t}\left[\sum_{i=0}^{j}\frac{(\lambda t)^i}{i!}-\sum_{i=0}^{j-1}\frac{(\lambda t)^i}{i!}\right]=\frac{(\lambda t)^j e^{-\lambda t}}{j!}$$

That is N(t) has a Poisson distribution with E{ N(t) } = λt

Renewal Function

$$m(t) = E[N(t)] = \sum_{j=1}^{\infty} j \Pr\{N(t) = j\}$$

Elementary Renewal Theorem:

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

μ = MTBF

Corollary:

$$\lim_{t \to \infty} m(t+T) - m(t) = \frac{T}{MTBF}, \quad T > 0$$

Example 9.6

A motor has a (first) time to failure distribution which is Weibull with theta = 2400 hrs and a beta of 1.8 . Determine its reliability at 500 hours if the motor has failed at 200 hours and is restored to as "good as new" condition. How many failures are expected in the first 10,000 hours of use?

$$R(300) = e^{-\left(\frac{300}{2400}\right)^{1.8}} = .9766$$

Since the MTTF = 2400 $\Gamma(1 + 1/1.8)$ = 2135 hours, then the expected number of failures, m(t),

is approximated by 10,000 / 2135 = 4.68.

Minimal repair

Define Intensity function:

let ρ(t) = (unconditional) failure rate at time t

$$\rho(t) \approx \frac{N(t+\Delta t) - N(t)}{\Delta t}$$

$$\rho(t)\Delta t \approx E\left[N(t)\right]$$

$$E[N(t)] = \int_0^t \rho(t')dt'$$

$$\rho(t) = \frac{d\,E\{N(t)\}}{dt}$$

Minimal repair

$$E[N(t)] = \int_0^t \rho(t')\,dt'$$

$$MTBF(t) = \frac{1}{\rho(t)}$$

$$MTBF(t_1, t_2) = \frac{t_2 - t_1}{m(t_1, t_2)}$$

where

$$m(t_1 t_2) = E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \rho(t)\,dt$$

Example 9.8

A manufacturing machine has an intensity function given by $\rho(t) = \exp(-6.5+0.0002t)$ with t measured in operating hours.

After 1 year of operation (3000 hours), $\rho(3000) = .0027394$ and the instantaneous MTBF = $1/\rho(3000) = 1/.0027394 = 365$ hours.

The expected number of failures over the second year is

$$m(3000{,}6000) = \int_{3000}^{6000} e^{-6.5+.0002t}\,dt = 11.26$$

Nonhomogeneous Poisson Process (NHPP)

N(0) = 0

$$\Pr\{N(t_2) - N(t_1) = j\} = \frac{m(t_1, t_2)^j e^{-m(t_1, t_2)}}{j!}$$

$$R(t) = \Pr\{N(t) = 0\} = e^{-m(0,t)}$$

$$R(t|T) = \Pr\{N(T+t) - N(T) = 0\} = e^{-m(T, T+t)}$$

Power Law Process

$$\rho(t) = a\,b\,t^{b-1} \qquad , a,b > 0$$

A six year old regional transit bus experiences minimal repair upon failure. It was found to have an intensity function given by $\rho(t) = .0464\, t^{2.1}$ with t measured in years.

1. Then MTBF (instantaneous) = 1 / [(.0464) $(6)^{2.1}$] = .5 years

2. The expected number of failures during the 7th year is:

$$m(6,7) = \int_6^7 .0464 t^{2.1} dt = 2.37$$

Power Law Process (continued)

3. Pr{exactly one failure in the 7th year} = Pr{ N(7) - N(6) = 1} = $2.37\, e^{-.2.37}$ = .222

4. The conditional reliability for the 7th year given it has survived through the 6th year:

R(1|6) = Pr{N(7) - N(6) = 0} = $e^{-2.37}$ = .093

5. The unconditional reliability:

$$R(t) = \exp[-\int_0^t .0464\, y^{2.1} dy] = e^{-.0149677\, t^{3.1}}$$

Weibull with β = 3.1 and θ = 3.878

Other Intensity Functions

Log-linear:

$$\rho(t) = e^{a+bt}, \; -\infty < a, b < \infty, \; t > 0$$

$$E\left[N(t)\right] = \frac{e^{a}}{b}\left(e^{bt} - 1\right)$$

Bounded:

$$\rho(t) = a\left(1 - e^{-bt}\right), \; a, b > 0, t > 0$$

$$E\left[N(t)\right] = \int_{0}^{t} a\left(1 - e^{-bt'}\right) dt' = a\left[t + \frac{e^{-bt}}{b} - \frac{1}{b}\right]$$

Example

A system under minimal repair has the intensity function

$$\rho(t) = .002\left(1 - e^{-.001t}\right)$$

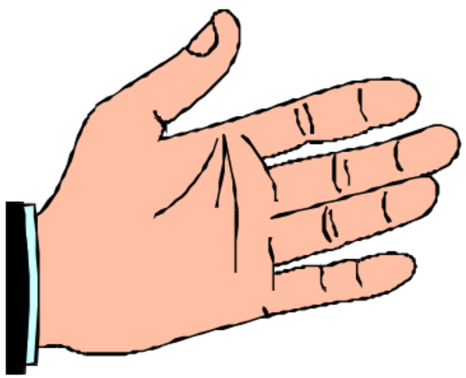
where t is measured in days. The system has been operating for 2 years (730 days). Its instantaneous MTBF is

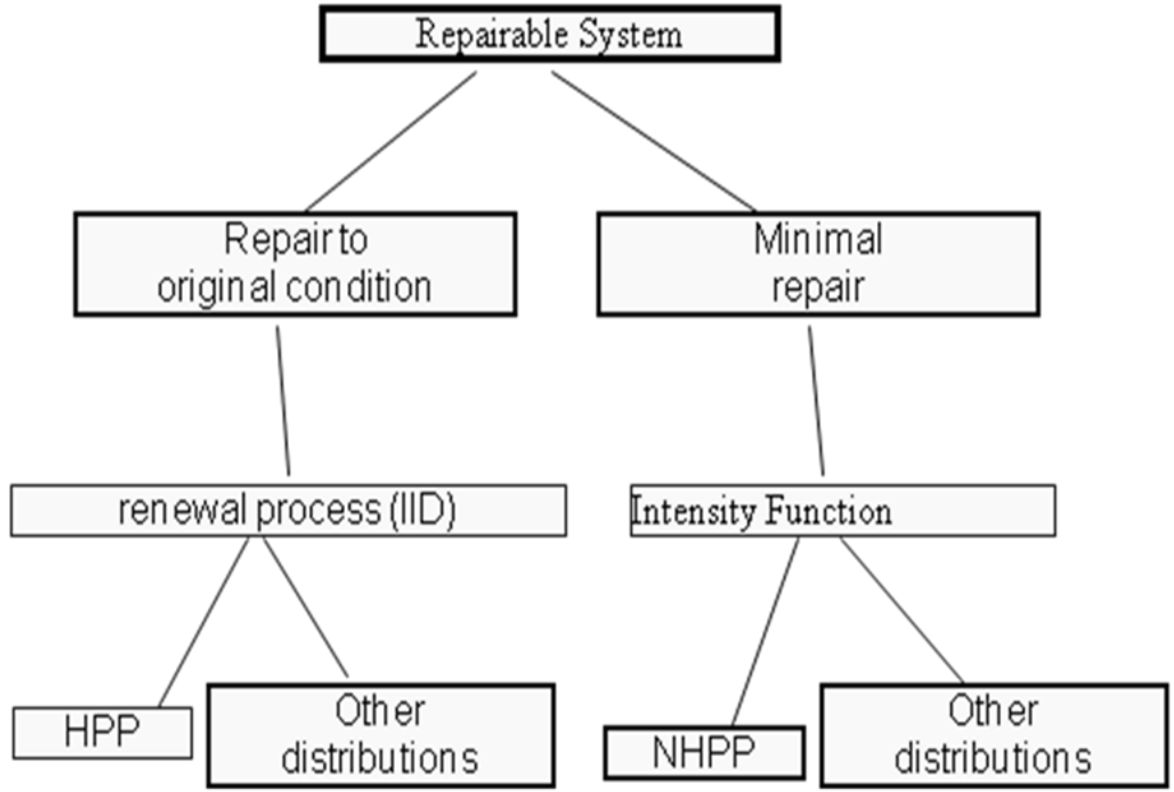
$$1/\rho(730) = \left[.002\left(1 - e^{-.001(730)}\right)\right]^{-1} = 965 \text{ days}$$

and its reliability over the next year is $\exp\left[-\int_{730}^{1095} .002\left(1 - e^{-.001t}\right) dt\right] = e^{-.4353} = .647$

By the tenth year, $m(10) = \int_{0}^{3650} .002\left(1 - e^{-.001t}\right) dt = 5.35$

Stochastic Point Process





Renewal vs. Minimal repair

Repair to "as good as new"	Minimal Repair
MTBF = MTTF	MTBF = $(t_2 - t_1) / m(t_1,t_2)$
$E[N(t_2) - N(t_1)] = m(t_1,t_2) = \dfrac{t_2 - t_1}{MTBF}$	$E[N(t_2) - N(t_1)] = m(t_1,t_2) = \int_{t_1}^{t_2} \rho(t)dt$
HPP (CFR): $\Pr\{N(t) = j\} = \dfrac{(\lambda t)^j e^{-\lambda t}}{j!}$ $R(t) = e^{-\lambda t}$	NHPP: $\Pr\{N(t) = j\} = \dfrac{m(t_1,t_2)^j e^{-m(t_1,t_2)}}{j!}$ $R(t) = e^{-m(0,t)}$
$m(t) = \dfrac{t}{MTBF}$	Power law process: $\rho(t) = abt^{b-1}$; $m(t) = \int_0^t abt'^{b-1} dt' = at^b$