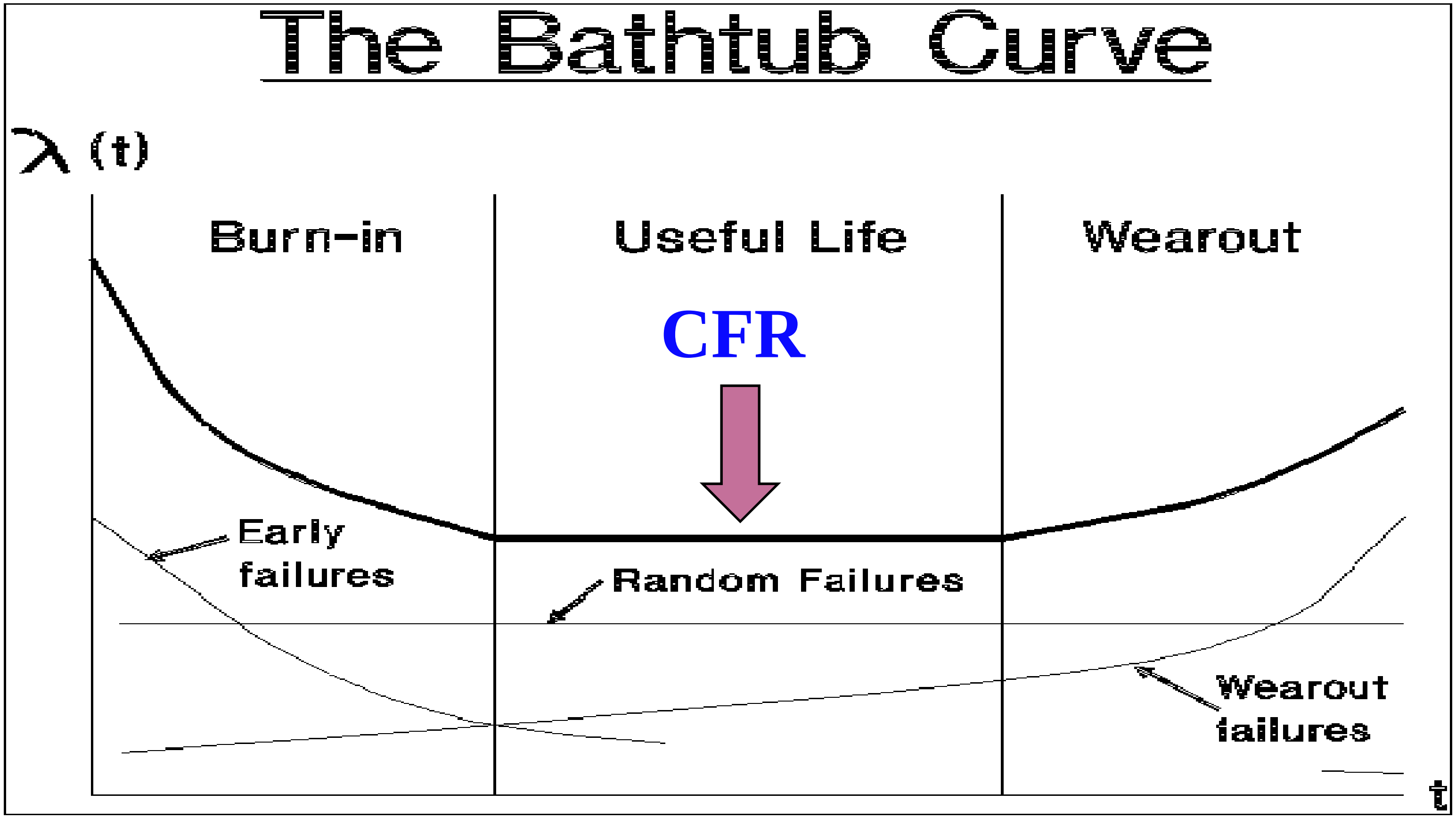
# Chapter 3
# The Constant Failure Rate Model

## Exponential Probability Distribution

Hurry, come here!
Chapter 3 is starting now.
You don't want to miss
any of this.

# Recall the Bathtub Curve



# The Exponential Distribution

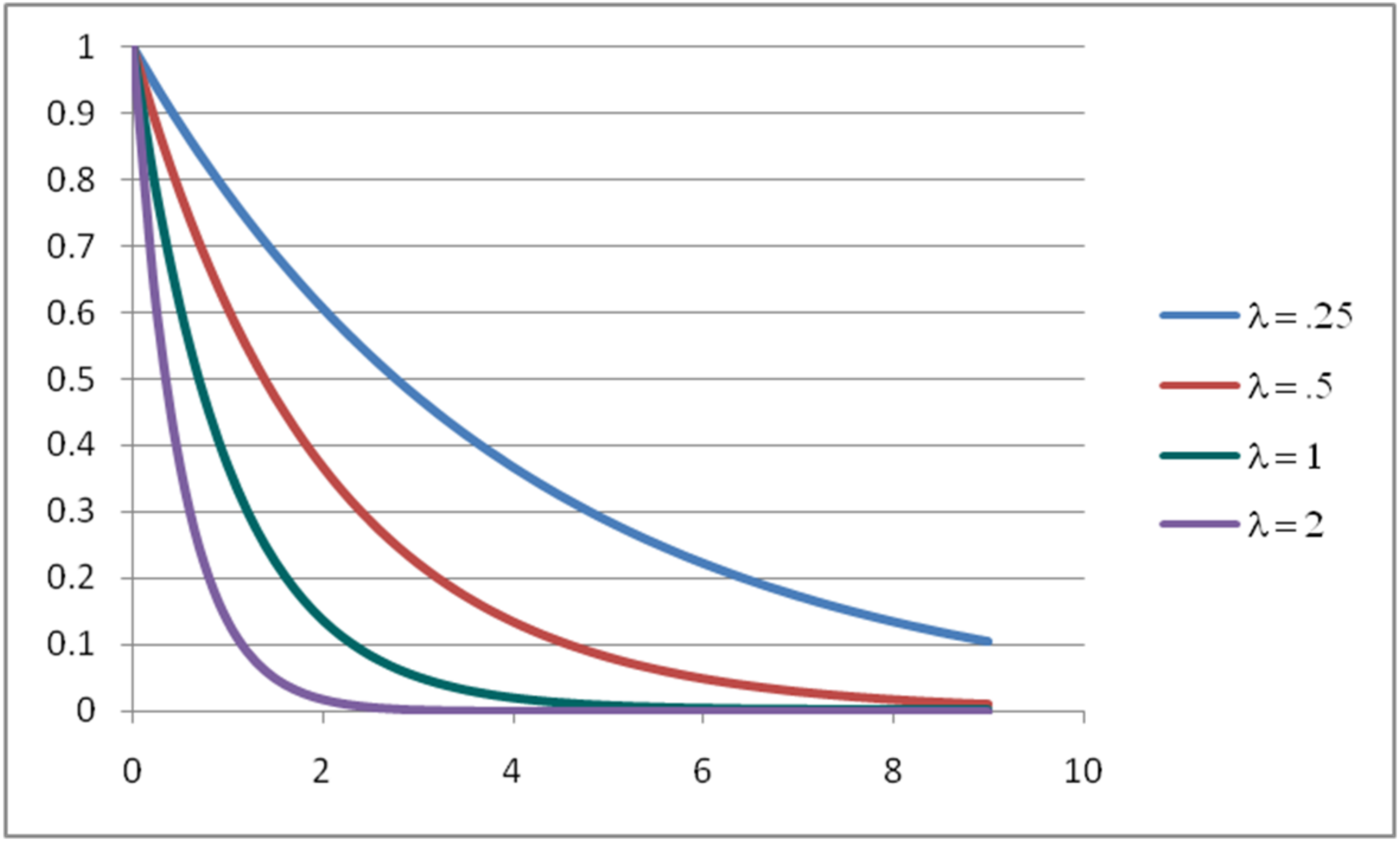
Let

$$\lambda(t) = \lambda \;, \qquad t \geq 0$$

Then

$$R(t) = e^{-\int_0^t \lambda dt'} = e^{-\lambda t} \;, \; t \geq 0$$

# The Reliability Function

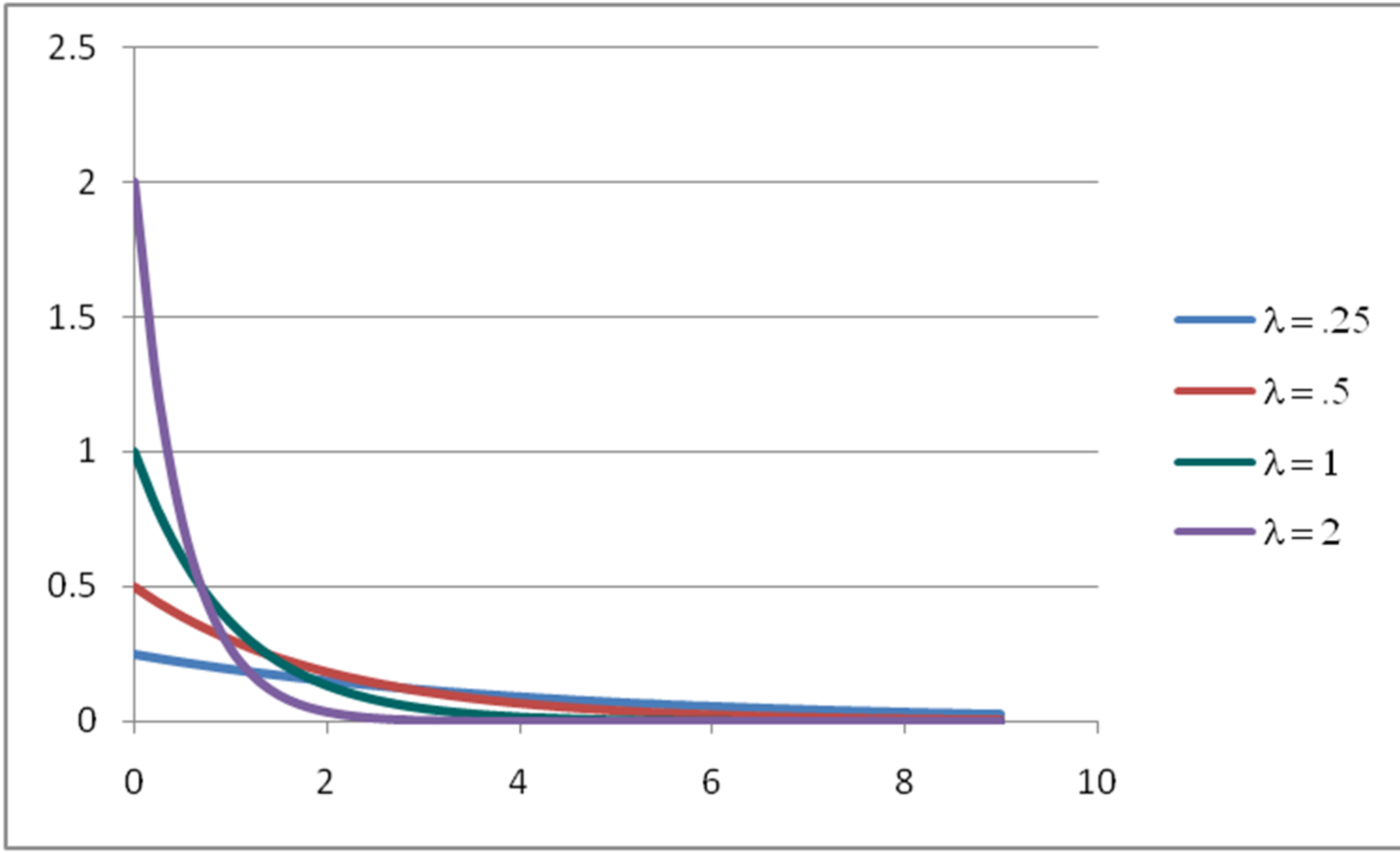


# The CDF and PDF

$$F(t) = 1 - e^{-\lambda t}$$

$$f(t) = \frac{d\,F(t)}{dt} = \lambda e^{-\lambda t}$$

# Probability Density Function (PDF)



# The MTTF

$$MTTF = \int_0^\infty e^{-\lambda t} dt = \frac{e^{-\lambda t}}{-\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

or $$MTTF = \int_0^\infty \lambda t e^{-\lambda t} dt = \frac{1}{\lambda}$$

note that $$R(MTTF) = e^{\frac{-MTTF}{MTTF}} = e^{-1} = .368$$

# The Standard Deviation

$$\sigma^2 = \int_0^\infty \left( t - \frac{1}{\lambda} \right)^2 \lambda e^{-\lambda t} \, dt = \frac{1}{\lambda^2}$$

$$and \quad \sigma = \frac{1}{\lambda} = MTTF$$

# The Median Time to Failure

$$R(t) = e^{-\lambda t} = .5$$

$$t_{med} = -\frac{1}{\lambda} \ln .5 = \frac{.69315}{\lambda}$$

$$= .69315 \text{ MTTF}$$

# The Design Life

For a given reliability R, let

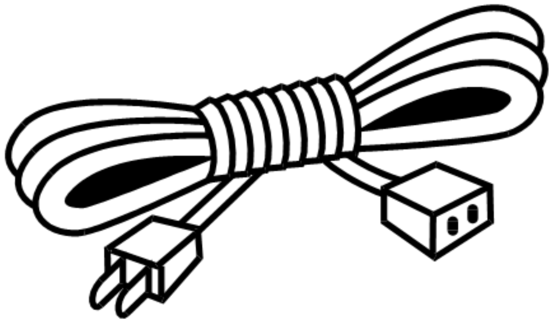
$$R(t_R) = e^{-\lambda t_R} = R$$

$$\text{then} \quad t_R = -\frac{1}{\lambda} \ln R$$

# Example #1 - CFR model

An electrical transmission line has been tested and found to have a CFR of .001 defects per foot. Find:

a. R(100)
b. R(1000)
c. MTTF
d. Standard Deviation
e. $t_{med}$
f. 90 percent design life



# Example #1 - solution

$$R(100) = e^{-.001(100)} = .9048$$

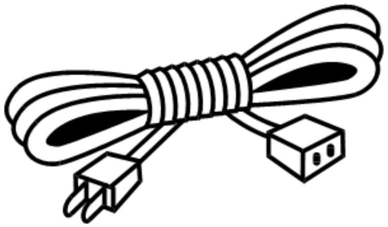
$$R(1000) = e^{-.001(1000)} = .368$$

$$MTTF = 1/.001 = 1000 \ \ ft.$$

$$\sigma = MTTF = 1000 \ \ ft.$$

$$t_{med} = .69315\, MTTF = 693.15 \ \ ft.$$

$$t_{.90} = -MTTF \ln(.90) = 105.4 \ \ ft.$$



# Memoryless Property

$$R(t \mid T_0) = \frac{R(t + T_0)}{R(T_0)} = \frac{e^{-\lambda(t+T_0)}}{e^{-\lambda T_0}}$$

$$= \frac{e^{-\lambda t} \cdot e^{-\lambda T_0}}{e^{-\lambda T_0}} = e^{-\lambda t} = R(t)$$

# Failure Modes

$R_i(t)$ is the reliability function for the ith failure mode, then, assuming <u>independence</u> among the failure modes, the system reliability, R(t) is found from

$$R(t) = \prod_{i=1}^{n} R_i(t)$$

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

# More on Failure Modes

Let $R_i(t) = e^{-\int_0^t \lambda_i (t')d't}$

Then

$$R(t) = \prod_{i=1}^{n} e^{-\int_0^t \lambda_i (t')dt'} = \exp\left[-\int_0^t \sum_{i=1}^{n} \lambda_i (t')dt'\right]$$

$$= e^{-\int_0^t \lambda (t')d't}$$ where $\lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$

# Failure Modes and CFR

If a system consists of n independent, serially related components each with CFR, then

$$\lambda(t) = \lambda = \sum_{i=1}^{n} \lambda_i$$

and

$$R(t) = e^{-\int_0^t \lambda \, d't} = e^{-\lambda t}$$

# Example - Failure Modes

An engine tune-up kit consists of 3 parts each having CFRs (in failures per mile) of .000034, .000017, and .0000086.

Find the MTTF, median time to failure, standard deviation, and reliability of the tune-up kit at 10,000 miles.

# Example - solution

$$\lambda_s = .000034 + .000017 + .0000086 = .0000596$$

$$MTTF = \sigma = 1/\lambda_s = 16{,}778.5\ mi.$$

$$t_{med} = .69315\lambda_s = 11{,}630\ mi.$$

$$R(10{,}000) = e^{-.0000596(10{,}000)} = .551$$

# Parts Count Approach

An integrated circuit board consists of the following components each having a CFR.

| Component | a-Failure Rate($10^{-5}$) | b- Quantity | (a) x (b) |
|---|---|---|---|
| Diodes, silicon | .00041 | 10 | .0041 |
| Resistors | .014 | 25 | .3500 |
| Capacitors | .0015 | 12 | .0180 |
| Transformer | .0020 | 2 | .0040 |
| Relays | .0065 | 6 | .0390 |
| Inductive devices | .0004 | 12 | .0048 |
| | | total | $.4199 \times 10^{-5}$ |

Therefore $R_{sys}(t) = e^{-.000004199\,t}$ and MTTF = $1/.4199 \times 10^{5}$

# All Failure Modes are CFR

$$MTTF = \frac{1}{\lambda} = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} \frac{1}{MTTF_i}}$$

$$where \quad MTTF_i = \frac{1}{\lambda_i}$$

If all components have identical failure rates, then:

$$\lambda = n \lambda_1 \quad \text{and} \quad MTTF = \frac{1}{n \lambda_1}$$

# 3.5 Poisson Process

If a component having a constant failure rate λ is immediately repaired or replaced upon failing, the number of failures observed over a time period *t* has a Poisson distribution. The probability of observing *n* failures in time *t* is given by the Poisson probability mass function $p_n(t)$:

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \dots$$

$$E[n] = \sigma^2 = \lambda t$$

# The Poisson and Exponential

$$p_0(t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} = R(t)$$

# The Gamma Distribution

Let $T_i$ = time between failure $i$ – 1 and failure $i$ having an exponential distribution with parameter λ.

$$Y_k = \sum_{i=1}^{k} T_i$$

$Y_k$ = the time of the $k$th failure. The sum of $k$ independent exponential random variables has a gamma distribution with parameters λ and k.

If $k$ is an integer (i.e. the Erlang distribution),

$$f_Y(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)} \text{ for } k, \lambda, t \geq 0$$

# The Gamma Distribution

The CDF is the probability that the $k^{th}$ failure will occur by time $t$.

The mean value for $Y_k$ is $k/\lambda$, and the variance is $k/\lambda^2$.

The mode is $(k-1)/\lambda$.

$$\Pr\left\{Y_k \le t\right\} = F_{Y_k}\left(t\right) = 1 - e^{-\lambda t} \sum_{i=0}^{k-1} \frac{\left(\lambda t\right)^i}{i!}$$

$$P_n\left(t\right) = \Pr\left\{Y_n \le t\right\} - \Pr\left\{Y_{n+1} \le t\right\} = F_{Y_n}\left(t\right) - F_{Y_{n+1}}\left(t\right)$$

$$= \frac{e^{-\lambda t}\left(\lambda t\right)^n}{n!}; \text{ the Poisson}$$

# Related Failure Distributions

| Random Variable | Distribution | Parameter(s) |
| --- | --- | --- |
| T, time to failure | Exponential | $\lambda$ |
| $Y_k$, time of the $k^{th}$ failure | Gamma (Erlang) | $\lambda$, k |
| N, number of failures in time t | Poisson | $\lambda$ |

# EXAMPLE 3.9

A specially designed welding machine has a nonrepairable motor with a constant failure rate of 0.05 failures per year. The company has purchased two spare motors. If the design life of the welding machine is 10 yr, what is the probability that the two spares will be adequate?

$\lambda t = 0.05(10) = 0.5.$

$$R_2(10) = \sum_{n=0}^{2} \frac{e^{-0.5} 0.5^n}{n!} = e^{-0.5}\left(1 + 0.5 + \frac{0.25}{2}\right) = 0.9856$$

# EXAMPLE 3.9

Let $Y_3$ be the time of the third failure. $Y_3$ has a gamma distribution with $k$ = 3 and $\lambda$ = 0.05.

The expected, or mean, time to obtain 3 failures is 3/0.05 = 60 yr.

The probability that the third failure will occur within 10 yr is

$$F_{Y_3}(10) = 1 - e^{-0.05\times 10}\left(1 + 0.05\times 10 + \frac{(0.05\times 10)^2}{2!}\right) = 0.0144$$

Note: 0.0144 = 1 – 0.9856 since the probability of two or fewer failures in 10 yr is complementary to the event that the third failure occurs within 10 yr.

# Redundancy 2 identical components with CFR

$$R(t) = 1 - (1 - e^{-\lambda t})^2$$

$$= 1 - (1 - 2e^{-\lambda t} + e^{-2\lambda t})$$

$$= 2e^{-\lambda t} - e^{-2\lambda t}$$

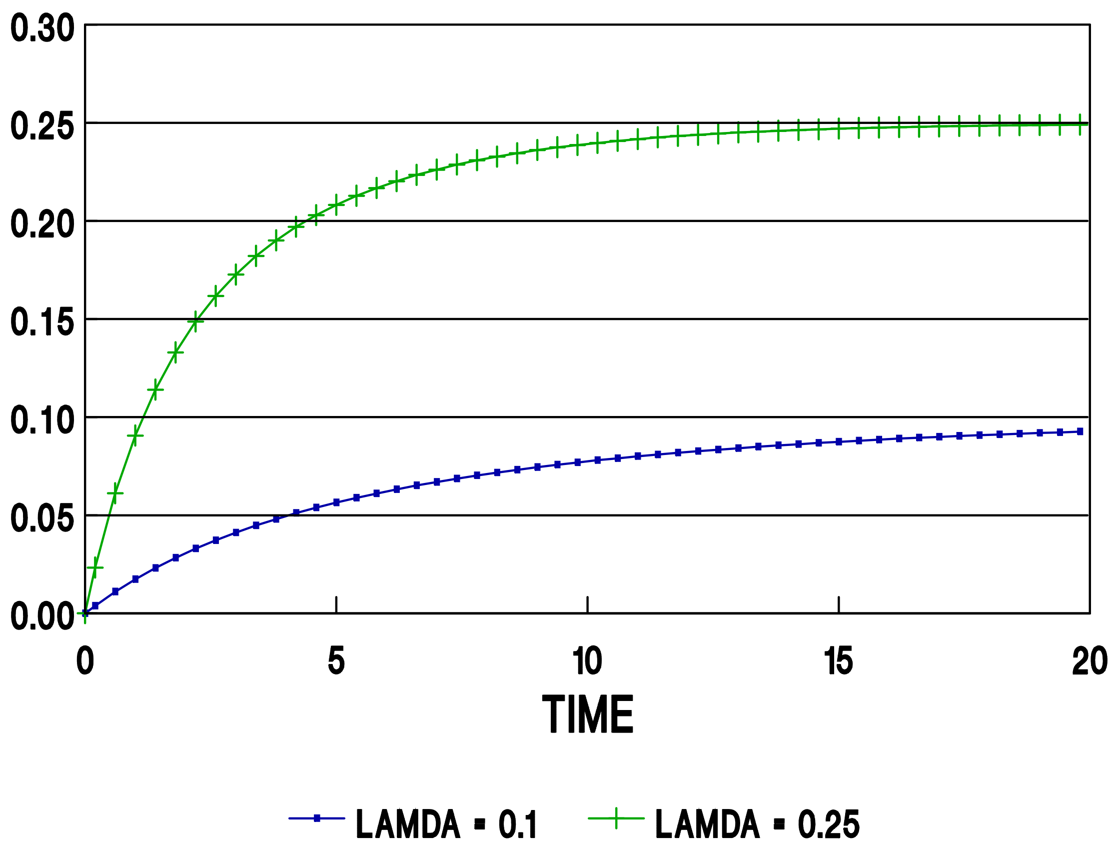
# Redundancy - Hazard Rate Function

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}}{2e^{-\lambda t} - e^{-2\lambda t}}$$

$$= \frac{\lambda(1 - e^{-\lambda t})}{(1 - .5e^{-\lambda t})}$$

Not a CFR process!

# HAZARD RATE FUNCTION PARALLEL COMPONENTS



# Redundancy, CFR & MTTF

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty (2e^{-\lambda t} - e^{-2\lambda t})dt$$

$$= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda} = \frac{1.5}{\lambda}$$

A most useful result: $\int_0^\infty e^{-at}dt = \left.\frac{e^{-at}}{-a}\right|_0^\infty = \frac{1}{a}$

# Two-parameter Exponential

$$f(t) = -\frac{d\,R(t)}{dt} = \lambda e^{-\lambda(t-t_0)} \quad ,0 < t_0 \le t < \infty$$

$$R(t) = e^{-\lambda(t-t_0)} \quad ,t \ge t_0$$

$$MTTF = \int_{t_0}^{\infty} \lambda t\, e^{-\lambda(t-t_0)} dt = t_0 + \frac{1}{\lambda}$$

# More 2-parameter Exponential

$$R(t_{med}) = e^{-\lambda(t_{med} - t_o)} = .5$$

$$t_{med} = t_0 + \frac{\ln 0.5}{-\lambda} = t_0 + \frac{0.69315}{\lambda}$$

$$t_R = t_0 + \frac{\ln R}{-\lambda}$$

Note that the variance does not change and the mode occurs at $t_0$.

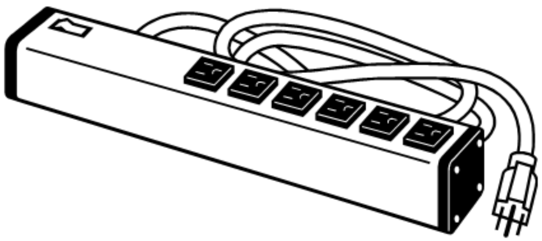
# Example CFR Model

A certain type of surge protector is observed to fail at the constant rate of .0005 failures per day. Find:

a. The MTTF, median, and the standard deviation.

b. Find the reliability during the first year; the second year; the second year given it has survived the first year.

c. The 90% design life.



# Example - solution

a. MTTF = 1/.0005 = 2000 days (5.5 yr.)

Std Dev = 2000 days

$t_{med}$ = 2000 x .69315 = 1386.3 days (3.8 yr.)

b. $R(365) = e^{-365 \times .0005} = .833$

$R(730) = e^{-730 \times .0005} = .694$

$R(365|365) = R(365) = .833$

c. $t_{.90} = -2000 \ln .90 = 210$ days (7 mo.)

# Bonus Round - Return Period

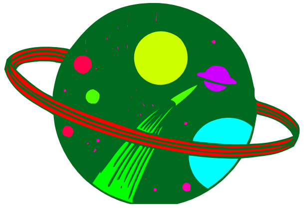
"An asteroid 2/3 of a mile across smashing into the Earth at a high speed which would cause a blast with the power of thousands of hydrogen bombs could happen within the next 300,000 years."

- Dave Morrison (NASA)

Let P = the return period, the mean length of time between a failure event. Then

$$R(t) = e^{-t/P} = e^{-t/300{,}000 \text{ yrs}}$$

and $\quad R(70 \text{ yr.}) = e^{-70/300{,}000} = .9997667$



# The End

Our next assignment will be in Chapter 4. First, however, you will want work the problems identified for chapter 3.