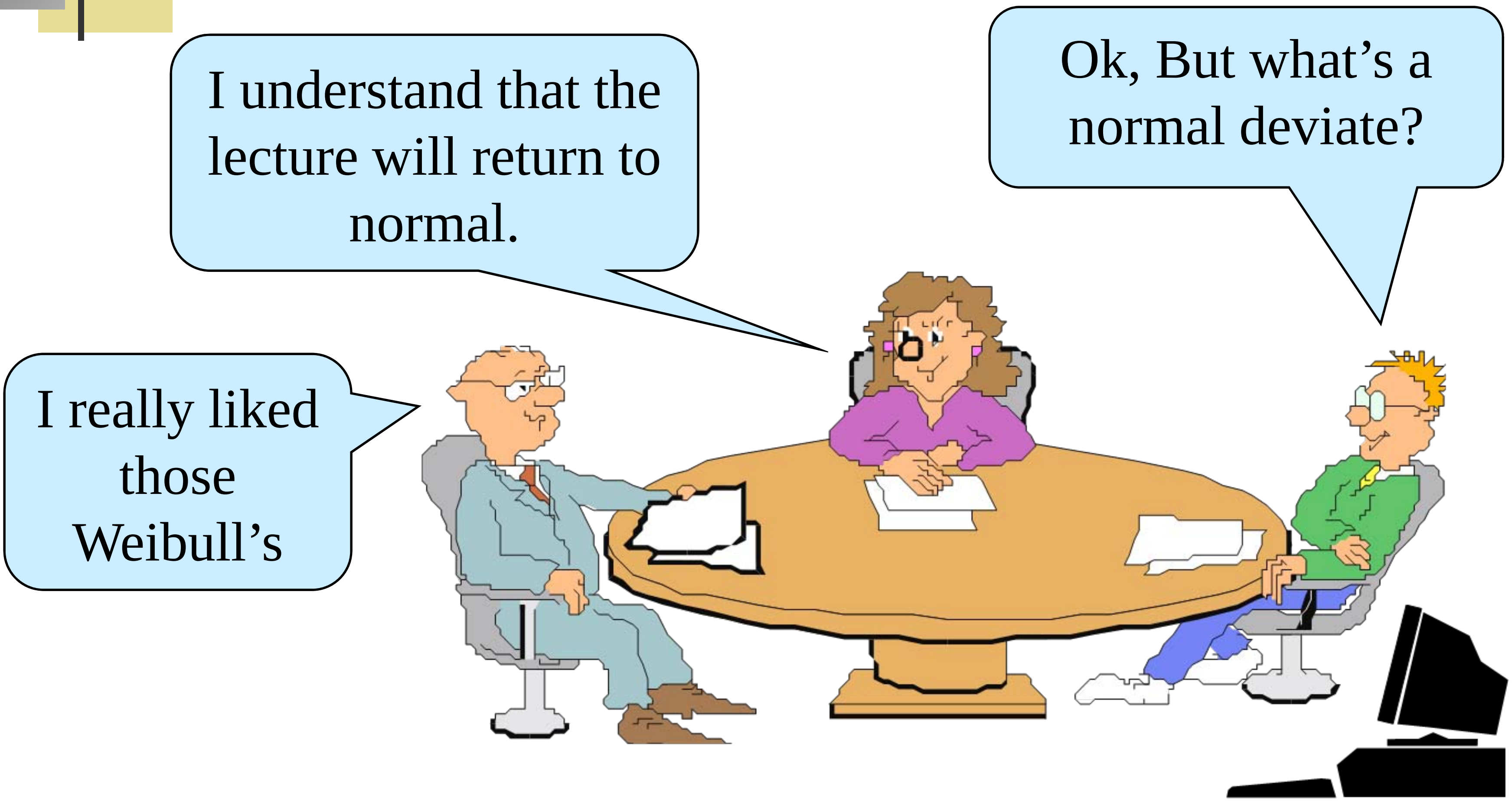
Chapter 4 Part II
Time-Dependent Failure Models

The Normal Distribution

LogNormal Distribution

Gamma Distribution

The Normal Distribution

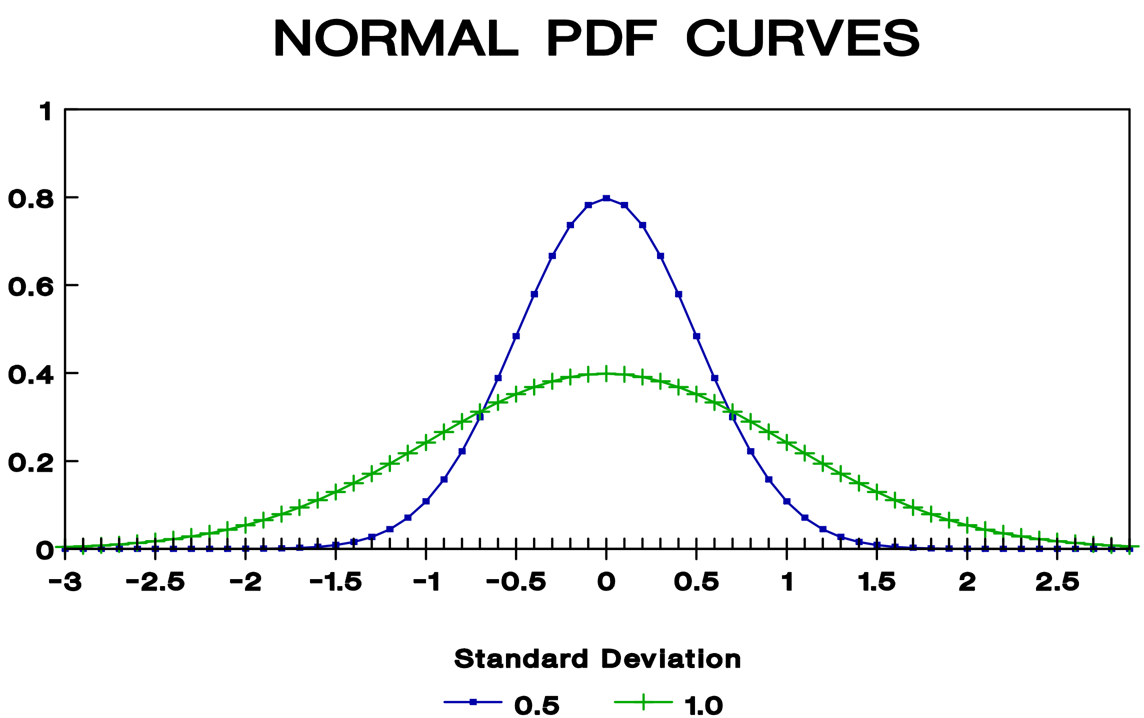


The Normal Probability Density Function

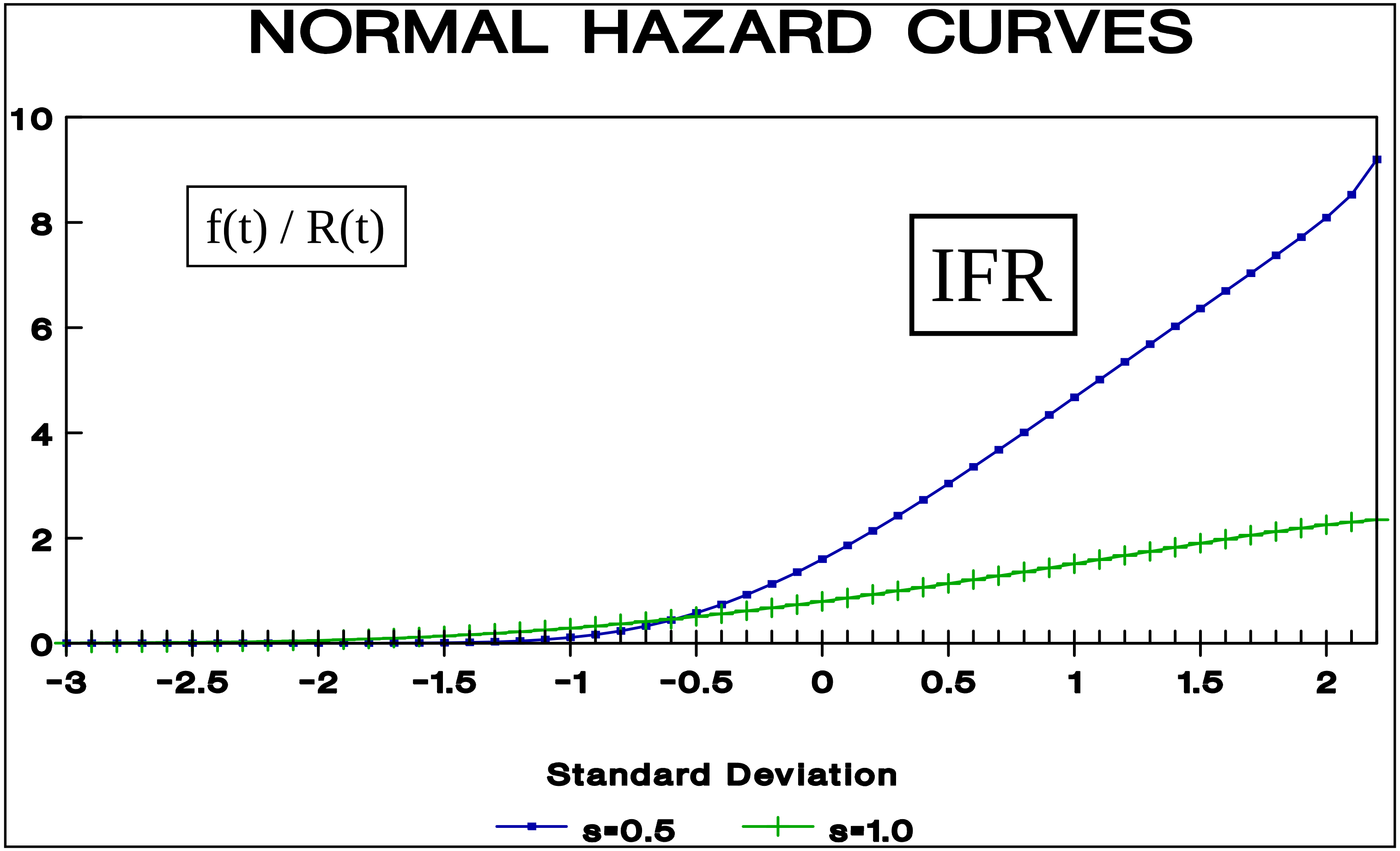
$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}}, -\infty < t < \infty$$

$$MTTF = \mu$$

$$Std\ Dev = \sigma$$

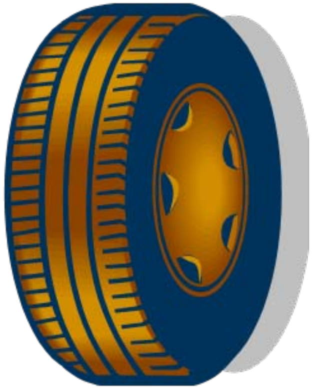


Normal Hazard Rate Function



Normal Distribution - Applications

- Tool failures
- Brake lining wear
- Tire tread wear



It’s the additive effect of temperature variation, material wear, friction, and other random stresses over time, isn’t it?.

Finding Normal Cumulative Probabilities

If T is normally distributed, then let $z = \frac{T-\mu}{\sigma}$

Then z has a normal distribution with a mean of 0 and a standard deviation of 1. The PDF for z is given by

Its cumulative distribution is then given by $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$\Pr\{Z \le z\} = \Phi(z) = \int_{-\infty}^{z} \phi(z')\, dz'$$

z is the standardized normal deviate

Normal Probability Tables

Z	Φ(Z)	1-Φ(Z)
-0.55000	0.29116	0.70884
-0.54000	0.29460	0.70540
-0.53000	0.29806	0.70194
-0.52000	0.30153	0.69847
-0.51000	0.30503	0.69497
-0.50000	0.30854	0.69146
-0.49000	0.31207	0.68793
-0.48000	0.31561	0.68439
-0.47000	0.31918	0.68082
-0.46000	0.32276	0.67724
-0.45000	0.32636	0.67364
-0.44000	0.32997	0.67003

Pr{Z < -.5} = .30854

Pr{Z > - .46 = .67724

Normal Reliability Function

$$R(t) = \int_t^{\infty} \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{(t-\mu)^2}{2\sigma^2}}\ dt'$$

$$R(t) = \Pr\{T \geq t\} = \Pr\left\{\frac{T-\mu}{\sigma} \geq \frac{t-\mu}{\sigma}\right\}$$

$$= \Pr\left\{z \geq \frac{t-\mu}{\sigma}\right\} = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

Example Problem - Normal

The time to failure of a fan belt is normally distributed with a MTTF = 220 (in hundreds of vehicle miles) and a standard deviation of 40 (in hundreds of vehicle miles).

$R(100) = 1 - \Phi[(100-220)/40] = 1 - \Phi(-3) = .99865$

$R(200) = 1 - \Phi[(200-220)/40] = 1 - \Phi(-.5) = .69146$

$R(300) = 1 - \Phi[(300-220)/40] = 1 - \Phi(2) = .02275$

$R(100|200) = R(300) / R(200) = .02275 / .69146 = .0329$

note: both the median and mode = MTTF = 22,000 miles

Normal Example problem - design life

A new fan belt is developed from a higher grade of material. It has a time to failure distribution which is normal with a mean of 35,000 vehicle miles and a standard deviation of 7,000 vehicle miles. Find its designed life if a .97 reliability is desired.

$R(t) = 1 - \Phi[(t - 350)/70] = .97$; find t !

From the normal table, $1 - \Phi(-1.88) = .96995$

Therefore; $(t - 350) / 70 = -1.88$

and $t_{.97} = 350 - 1.88\,(70) = 218.4$ or 21,840 vehicle miles

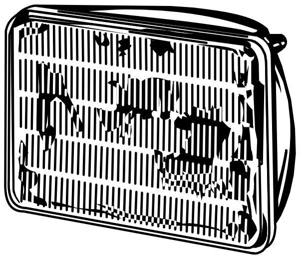
Student Exercise - Normal

The operating hours until failure of a halogen headlamp is normally distributed with a mean of 1200 hr. and a standard deviation of 450 hr.

Find:

a. The 5 year reliability if normal driving results in the use of the headlamp an average of .2 hr. a day.

b. The .90 design life in years.



Student exercise - solution

a. $t = .2$ hr./da. x 365 da./yr. x 5 yr. = 365 hr.

$R(365) = 1 - F[\,(365 - 1200)/450]$

$= 1 - F[\,-1.86] = .96856$

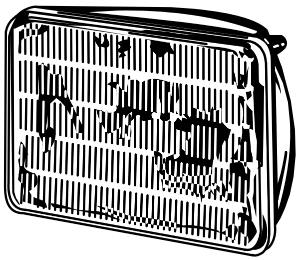
b. $R(t_{.90}) = .90$

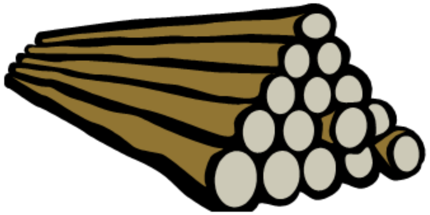
or $1 - F[\,(t_{.90} - 1200)/450] = .90$

$(\,t_{.90} - 1200\,)\,/\,450 = -1.28$

$t_{.90} = 1200 - 1.28\,(450) = 624$ hr.

or $t_{.90} = 624\ /\ (.2 \times 365) = 8.5$ yr.





some normal logs

The Lognormal Failure Process

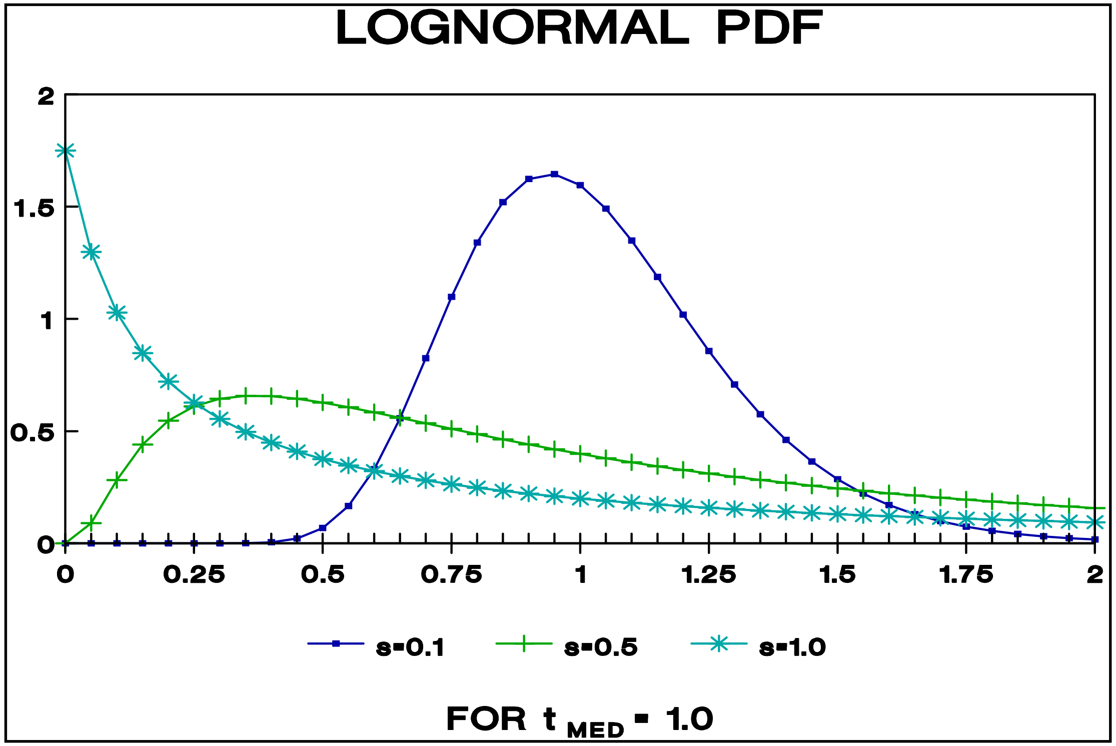
Let T = a random variable, the time to failure. If T has a lognormal distribution, then the logarithm of T has a normal distribution.

Lognormal Density Function

$$f(t) = \frac{1}{\sqrt{2\pi}s\,t} e^{-\frac{1}{2s^2}\left(\ln \frac{t}{t_{MED}}\right)^2} \quad ; t \geq 0$$

t_{med} = median time to failure

s = shape parameter



Lognormal/Normal Relationship

Given T is a lognormal random variable, then

	<u>T</u>	<u>Log T</u>
Distribution	Lognormal	Normal
Mean	$t_{med}\, e^{s^2/2}$	$\ln t_{med}$
Variance	$t_{med}^2 e^{s^2}[e^{s^2} - 1]$	s^2
Mode	$t_{mode} = \dfrac{t_{med}}{e^{s^2}}$	$\ln t_{med}$

Lognormal Failure & Reliability Distribution

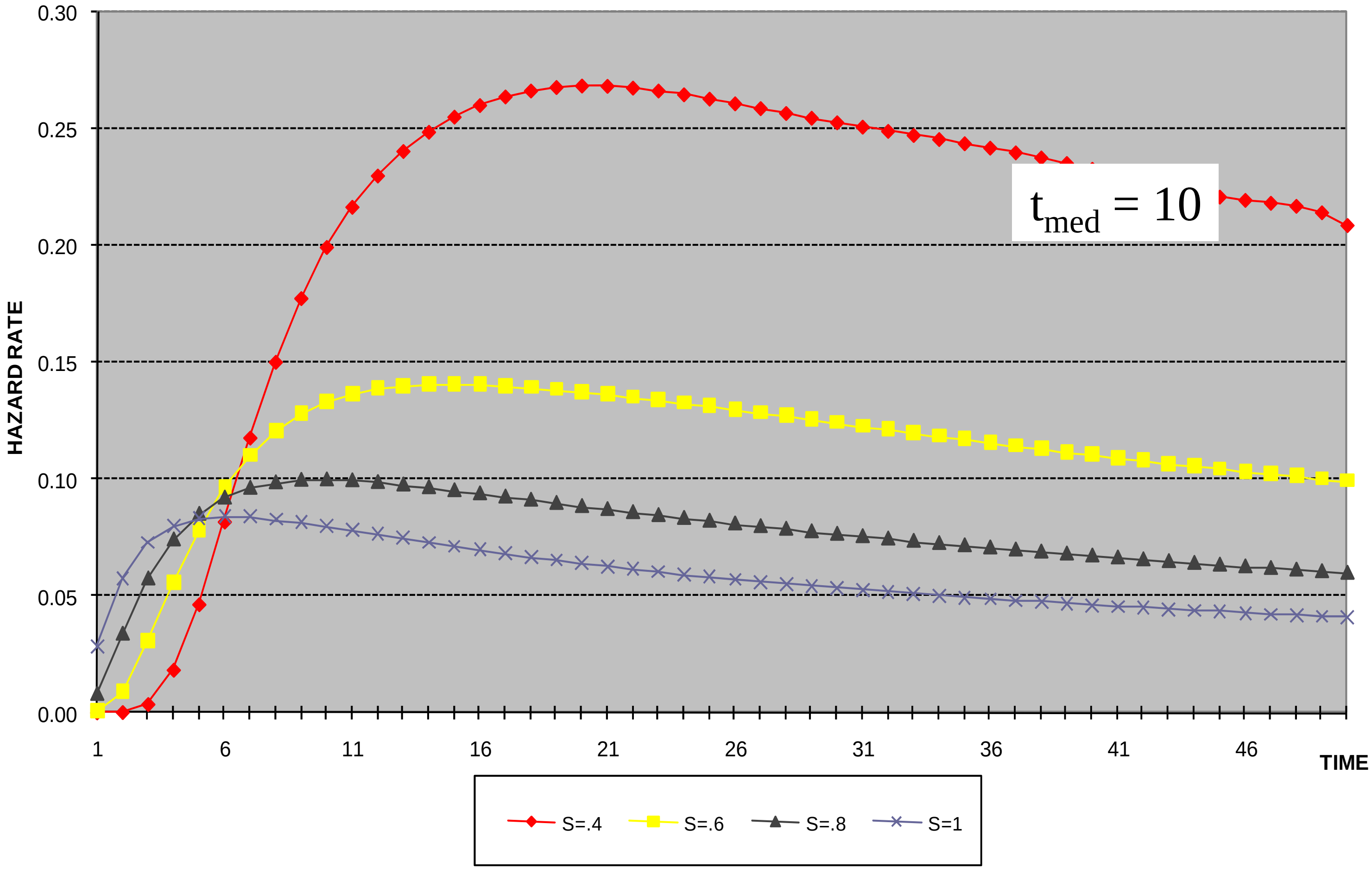
$$F(t) = P\{T \le t\} = P\{\ln T \le \ln t\}$$

$$= P\left\{\frac{\ln T - \ln t_{MED}}{s} \le \frac{\ln t - \ln t_{MED}}{s}\right\}$$

$$= P\left\{z \le \frac{1}{s}\ln\frac{t}{t_{MED}}\right\} = \Phi\left(\frac{1}{s}\ln\frac{t}{t_{MED}}\right)$$

$$R(t) = 1 - \Phi\left(\frac{1}{s}\ln\frac{t}{t_{MED}}\right)$$

Lognormal Hazard Rate Function



Lognormal Hazard Rate Function

s	1.0	.8	.6	.4
Mode	3.7	5.3	7.0	8.5
MTTF	16.5	13.8	12.0	10.8
Max $\lambda(t)$	7	10	16	20

$$t_{med} = 10$$

Design Life

$$1 - \Phi\left(\frac{1}{s}\ln\frac{t_R}{t_{med}}\right) = R$$

Find $z_{1\text{-}R}$ such that:

$$\Phi(z_{1-R}) = 1 - R$$

$$\Phi\left(\frac{1}{s}\ln\frac{t_R}{t_{med}}\right) = 1 - R$$

$$\frac{1}{s}\ln\frac{t_R}{t_{med}} = z_{1-R}$$

$$t_R = t_{med}\, e^{s z_{1-R}}$$

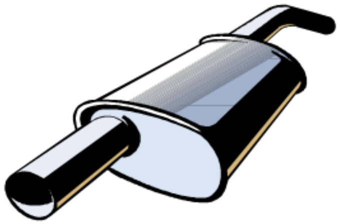
Lognormal Example

The failure distribution of an exhaust system is lognormal with t_{med} = 50,000 vehicle miles and $s = .8$. Therefore:

a. MTTF = $50{,}000\, e^{.64/2} = 68{,}856$ mi.

b. $t_{mode} = 50{,}000 / e^{.64} = 26{,}640$ mi.

c. variance = $50{,}000^2\, e^{.64}\, [e^{.64} - 1]$ and the standard deviation = 65,195 mi.

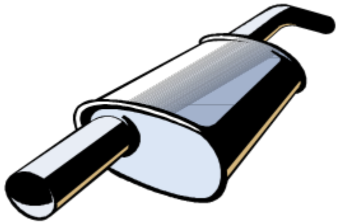


Lognormal Example (continued)

$$R(10{,}000) = 1 - \Phi\left(\frac{1}{.8}\ln\frac{10{,}000}{50{,}000}\right) = 1 - \Phi(-2.01) = .9779$$

$$R(20{,}000) = 1 - \Phi\left(\frac{1}{.8}\ln\frac{20{,}000}{50{,}000}\right) = 1 - \Phi(-1.15) = .8749$$

R(10,000 | 10,000) = R(20,000) / R(10,000) = .8749 / .9770 = .8955



Example (continued)

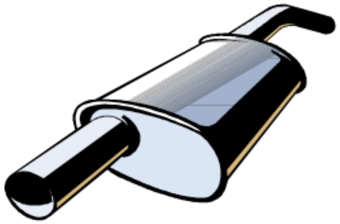
Find the design life corresponding to a 90 percent reliability.

$$R(t_{.9}) = 1 - \Phi\left(\frac{1}{.8}\ln\frac{t_{.9}}{50{,}000}\right) = .90$$

from the normal probability tables:

$$\left(\frac{1}{.8}\ln\frac{t_{.9}}{50{,}000}\right) = -1.285$$

$$or \quad t_{.9} = 50{,}000\ e^{-1.285(.8)} = 17{,}886\ mi.$$



Class Exercise - Lognormal

Reliability testing of the new 1.6 liter automotive engine has resulted in a time to failure distribution which is lognormal with t_{med} = 100,000 mi. and s = .70. Find:

a. R(36,000 mi.)

b. MTTF and Std. Dev.

c. R(100,000|36,000)

d. $t_{.95}$

Class Exercise - solution

a. $R(36{,}000) = 1 - \Phi[\,(1/.7)\ln(36{,}000/100{,}000)]$

$= 1 - \Phi[\,-1.46] = .92786$

b. $MTTF = 100{,}000\ e^{.49/2} = 127{,}762$ mi.

$Var = 100{,}000^2\ e^{.49}\ [\ e^{.49} - 1] = 1.032 \times 10^{10}$

Std Dev = 101,594 mi.

c. $R(64{,}000|36{,}000)$

$= R(100{,}000)/R(36{,}000)$

$= 5.\ /\ .92786 = .539$

Class Exercise - solution

d. $R(t_{.95}) = .95$

$1 - \Phi[\,(1/.7)\ \ln(t_{.95}/100{,}000)\,] = .95$

$(1/.7)\ \ln(t_{.95}/100{,}000) = -1.645$

$t_{.95} = 100{,}000\ e^{-1.645 \times .7} = 31{,}616$ mi.

general approach:

$t_R = t_{med}\ e^{z \times s}$

The Gamma Distribution

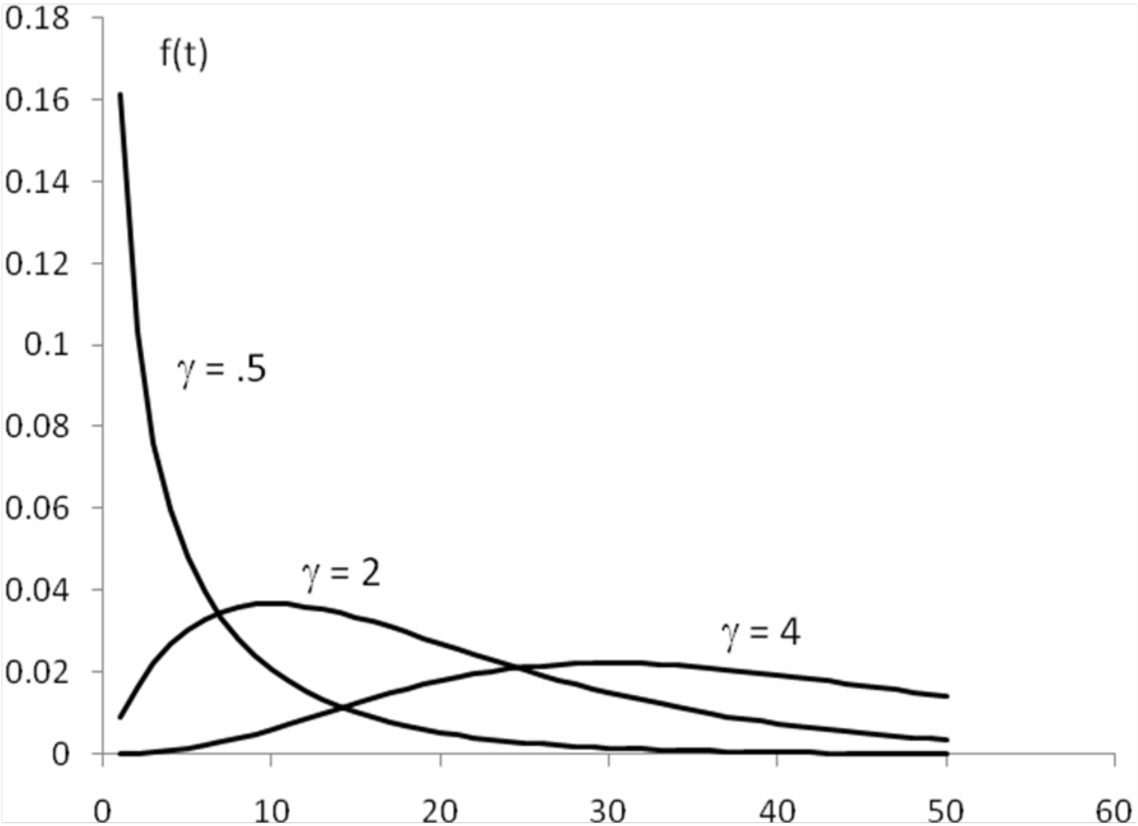
$$f(t) = \frac{t^{\gamma-1} e^{-t/\alpha}}{\alpha^{\gamma}\Gamma(\gamma)} \quad \text{for } \gamma, \alpha > 0 \text{ and } t \geq 0$$

γ - shape parameter

α - scale parameter

When $\gamma = 1$, the resulting distribution is exponential with a mean equal to α.

The Density Function Graphed



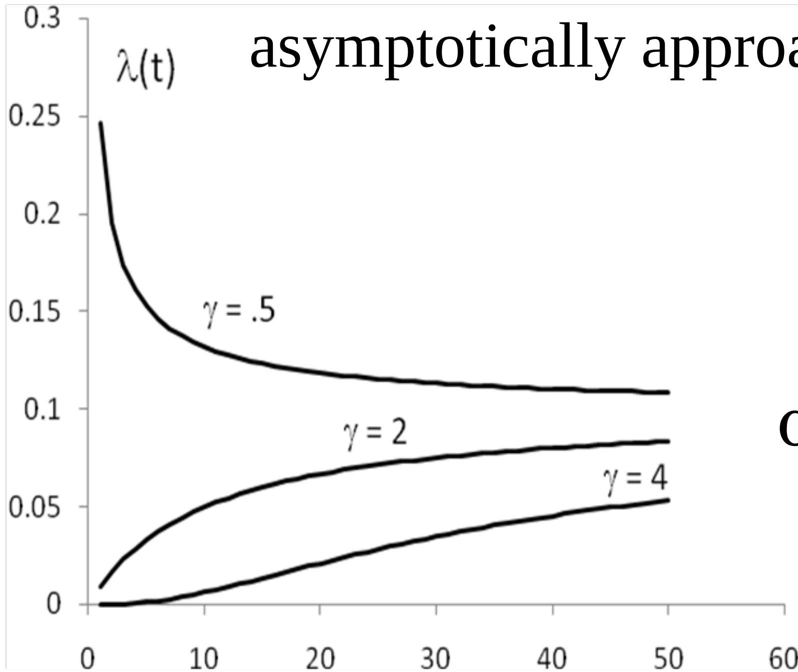
The Reliability Function

$$F(t)=\int_0^t \frac{t'^{\gamma-1}e^{-t'/\alpha}}{\alpha^\gamma\Gamma(\gamma)}dt' = \frac{1}{\Gamma(\gamma)}\int_0^{t/\alpha} y^{\gamma-1}e^{-y}\,dy = \frac{I\left(\frac{t}{\alpha},\gamma\right)}{\Gamma(\gamma)}$$

where $y=t'/\alpha$ and $I\left(\frac{t}{\alpha},\gamma\right)=\int_0^{t/\alpha} y^{\gamma-1}e^{-y}\,dy$

Therefore $R(t)=1-\dfrac{I\left(\frac{t}{\alpha},\gamma\right)}{\Gamma(\gamma)}$

The Hazard Rate Function



asymptotically approaches $1/\alpha$ as $t \to \infty$.

$\alpha = 10$

Shape parameter	Hazard rate function
$0 < \gamma < 1$	DFR
$\gamma = 1$	CFR
$\gamma > 1$	IFR

Distribution Characteristics

$$MTTF = \gamma\alpha$$

$$\sigma^2 = \gamma\alpha^2$$

$$t_{\mathrm{mod}e} = \begin{cases} \alpha(\gamma - 1) \text{ for } \gamma > 1 \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

EXAMPLE 4.11

Failures of a critical machine part due to cyclical vibration has a gamma distribution with a shape parameter of 2.3 and a scale parameter of 2,000 operating hours. Then

MTTF = $\gamma\alpha$ = (2.3) (2,000) = 4600 hr

$$\sigma = \sqrt{\gamma\alpha^2} = \sqrt{(2.3)(2,000)^2} = 3033.15 \text{ hr}$$

$$t_{\text{mod}e} = \alpha(\gamma - 1) = 2,000(2.3 - 1) = 2,600 \text{ hr}$$

Using the Excel Chapter 4 template for the gamma distribution the median is found to be 3,953.25 hr and R(1000) = .9463

Summary

	Reliability	MTTF
Exponential	$R(t) = e^{-\lambda t}$	$1/\lambda$
Weibull	$R(t) = e^{-\left(\frac{t}{\theta}\right)^{B}}$	$\theta\,\Gamma\left(1+\frac{1}{\beta}\right)$
Min Extreme Value	$R(t) = \exp\left[-e^{\frac{(t-\mu)}{\alpha}}\right]$	$\mu - .577215665\alpha$
Normal	$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$	μ
Lognormal	$R(t) = 1 - \Phi\left(\frac{1}{s}\ln\frac{t}{t_{med}}\right)$	$t_{med}\, e^{s^2/2}$
Gamma	$R(t) = 1 - \frac{I\left(\frac{t}{\alpha},\gamma\right)}{\Gamma(\gamma)}$	$\alpha\gamma$

That’s all folks!