CS-102: Discrete Structures Tutorial #3

Summary	
argument	a sequence of statements
argument form	a sequence of compound propositions involving propositional variables
premise	a statement, in an argument, or argument form, other than the final one
conclusion	the final statement in an argument or argument form
valid argument form	a sequence of compound propositions involving propositional variables where the truth of all the premises implies the truth of the conclusion
valid argument	an argument with a valid argument form
rule of inference	a valid argument form that can be used in the demonstration that arguments are valid
fallacy	an invalid argument form often used incorrectly as a rule of inference (or sometimes, more generally, an incorrect argument)
circular reasoning or begging the question	reasoning where one or more steps are based on the truth of the statement being proved
theorem	a mathematical assertion that can be shown to be true
conjecture	a mathematical assertion proposed to be true, but that has not been proved
proof	a demonstration that a theorem is true
axiom	a statement that is assumed to be true and that can be used as a basis for proving theorems
lemma	a theorem used to prove other theorems
corollary	a proposition that can be proved as a consequence of a theorem that has just been proved
vacuous proof	a proof that $p \rightarrow q$ is true based on the fact that p is false
trivial proof	a proof that $p \rightarrow q$ is true based on the fact that q is true
direct proof	a proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true
proof by contraposition	a proof that $p \rightarrow q$ is true that proceeds by showing that p must be false when q is false
proof by contradiction	a proof that p is true based on the truth of the conditional statement $\sim p \rightarrow q$, where q is a contradiction
exhaustive proof	a proof that establishes a result by checking a list of all possible cases
proof by cases	a proof broken into separate cases, where these cases cover all possibilities
without loss of generality	an assumption in a proof that makes it possible to prove a theorem by reducing the number of cases to consider in the proof
counterexample	an element x such that $P(x)$ is false
constructive existence proof	a proof that an element with a specified property exists that explicitly finds such an element
Non-constructive existence proof	a proof that an element with a specified property exists that does not explicitly find such an element
Fallacy of affirming the conclusion	an incorrect reasoning in proving p → q by starting with assuming q and proving p
Fallacy of denying the hypothesis	an incorrect reasoning in proving p → q by starting with assuming ~p and proving ~q
Fallacy of begging the question or circular reasoning	an incorrect reasoning when one or more steps of a proof is based on the statement being proved.
the principle of mathematical induction	the statement $\forall n\, P(n)$ is true if $P(1)$ is true and $\forall k[P(k) \rightarrow P(k+1)]$ is true. $k \in Z^+$.
• **basis step**	Establish that $P(1)$ is true.
• **inductive step**	on the assumption that $P(k)$ is true for an arbitrary k, $P(k) \rightarrow P(k+1)$ for all positive integers k; $\therefore\ \forall n P(n)$
strong induction	the statement $\forall n P(n)$ is true if $P(1)$ is true and $\forall k[(P(1) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)]$ is true
recursive definition of a function	a definition of a function that specifies an initial set of values and a rule for obtaining values of this function at integers from its values at smaller integers

recursive definition of a set	a definition of a set that specifies an initial set of elements in the set and a rule for obtaining other elements from those in the set
structural induction	a technique for proving results about recursively defined sets
recursive algorithm	an algorithm that proceeds by reducing a problem to the same problem with smaller input
linear homogeneous recurrence relation with constant coefficients	a recurrence relation that expresses the terms of a sequence, except initial terms, as a linear combination of previous terms
characteristic roots of a linear homogeneous recurrence relation with constant coefficients	the roots of the polynomial associated with a linear homogeneous recurrence relation with constant coefficients
linear non-homogeneous recurrence relation with constant coefficients	a recurrence relation that expresses the terms of a sequence, except for initial terms, as a linear combination of previous terms plus a function that is not identically zero that depends only on the index
divide-and-conquer algorithm	an algorithm that solves a problem recursively by splitting it into a fixed number of smaller non-overlapping sub-problems of the same type

Rules of Inference for Propositional Logic

Rule	***Tautology***	***Name***
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	*Modus ponens* *Or Mode that affirms*
$\sim q$ $p \to q$ $\therefore \sim p$	$(\sim q \wedge (p \to q)) \to \sim p$	*Modus tollens*
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	*Hypothetical syllogism*
$p \vee q$ $\sim p$ $\therefore q$	$[(p \vee q) \wedge \sim p] \to q$	*Disjunctive syllogism*
p $\therefore p \vee q$	$p \to (p \vee q)$	*Addition*
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	*Simplification*
p q $\therefore p \wedge r$	$[(p) \wedge (q)] \to (p \wedge q)$	*Conjunction*
$p \vee q$ $\sim p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\sim p \vee r)] \to (q \vee r)$	*Resolution*

Rules of Inference for Quantified Statements

Rule	***Name***
$\forall x P(x)$ $\therefore P(c)$	*Universal instantiation*
$P(c)$ *for an* ***arbitrary*** c $\therefore \forall x P(x)$	*Universal generalization*
$\exists x P(x)$ $\therefore P(c)$ *for some element* c	*Existential instantiation*
$P(c)$ *for some element* c $\therefore \exists x P(x)$	*Existential generalization*

Logical Equivalences involving Universal and Existential quantifiers

$\sim (\forall x P(x)) \equiv \exists x \sim P(x)$	$\sim (\exists x P(x)) \equiv \forall x (\sim P(x))$
$\exists x (P(x) \to Q(x)) \equiv \forall x P(x) \to \exists x Q(x)$	$\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$
$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$	
$((\forall x P(x)) \vee (\forall x Q(x))) \to \forall x (P(x) \vee Q(x))$ is a tautology.	
$\exists x (P(x) \wedge Q(x)) \to \exists x P(x) \wedge \exists x Q(x)$ is a tautology.	

Examples of Tautologies

$(p \wedge q) \to p$	$(p \wedge q) \to q$
$p \to (p \vee q)$	$q \to (p \vee q)$
$\sim p \to (p \to q)$	$\sim (p \to q) \to p$
$(p \wedge (p \to q)) \to q$	$(\sim p \wedge (p \vee q)) \to q$
$(\sim q \wedge (p \to q)) \to \sim p$	$((p \to q) \wedge (q \to r)) \to (p \to r)$

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "$\forall\, n \geq b, P(n)$" for a given integer b.
2. Explicitly indicate the "Basic Step" and "Inductive Steps".
3. In the "Basis Step" establish that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
4. In the "Inductive Step", state and clearly identify, the inductive hypothesis, in the form "assume $that\ P(k)$ is true for an arbitrary integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Ensure that the proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, by stating "$this\ completes\ the\ inductive\ step$."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Rules of Inference

1. What rule of inference is used in each of these arguments?

   a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

   b) It is hotter than 45°C today or the pollution is dangerous. It is less than 45°C outside today. Therefore, the pollution is dangerous.

   c) Vijay is an excellent swimmer. If Vijay is an excellent swimmer, then he can work as a lifeguard. Therefore, Vijay can work as a lifeguard.

   d) Shiva will work at a grocery store this summer. Therefore, this summer Shiva will work at a grocery store or he will spend his time at the Goa Beach.

   e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the content of the course CO-205. Therefore, if I work all night on this homework, then I will understand the content of the course CO-205.

2. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? State the rules of inference used to obtain each conclusion from the premises.

   a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

   b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

   c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

   d) "Every student has an Internet account." "Henry does not have an Internet account." "Mamta has an Internet account."

   e) "All foods that are healthy to eat do not taste good." "Bitter-gourd is healthy to eat." "You only eat what tastes good." "You do not eat bitter-gourd." "Cheeseburgers are not healthy to eat."

   f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

3. Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, q and $\sim s$ and conclusion r is valid.

3.1. Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, and $\sim s$ and conclusion $q \rightarrow r$ is valid

4. Confirm the validity of each of these arguments (state the rules of inference are used for each step).

   a) "Linda, a student in this class, owns a red car. Everyone who owns a red car has got at least one speeding ticket. Therefore, someone in this class has got a speeding ticket."

   b) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

5. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

   a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$; then, $n > 1$.

   b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$; then, $n \leq 3$.

   c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$; then, $n^2 \leq 4$.

6. Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy."

6.1. Based on the following premises about the trip of four friends to a hill station: -

Premise A: Amit paid for the trip and Bobby paid for the hotel reservations or Charlie paid for everything.

Premise B: If Charlie paid for everything, then David is going on the trip.

Can it be concluded that – "Amit paid for the trip or David is going on the trip"? Justify your answer.

6.2. Is the following argument valid? Justify your answer.

Premise A: "If the birthday gift is a box of chocolates, then the child will be happy".

Premise B: "The child is happy"

Therefore, it can be concluded that "the birthday gift is a box of chocolates".

6.3. Test the validity of the following argument:

(a) Babies are illogical

(b) Nobody is despised who can manage a crocodile

(c) Illogical persons are despised

Therefore babies cannot manage crocodiles

6.4 Test the validity of the following argument: -

(a) It is not sunny today and it is colder than yesterday.

(b) We will go for swimming only if it is sunny.

(c) If we don't go for swimming, we will go for a trekking trip.

(d) If we go out trekking, then we will come home by sunset.

Therefore, it can be concluded that we will be home by sunset.

Direct / Indirect Proofs

7. Use a direct proof to show that every odd integer is the difference of two squares.

8. Prove or disprove the following:-

(a) the product of a nonzero rational number and an irrational number is irrational.

(b) if x is rational and $x \neq 0$, then $1/x$ is rational.

9. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

10. Prove that if m and n are integers and mn is even, then m is even or n is even.

11. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

12. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

13. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

14. Identify the error(s) in this argument that supposedly shows that if $\exists xP(x) \wedge \exists xQ(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

1.	$\exists xP(x) \vee \exists xQ(x)$	Premise
2.	$\exists xP(x)$	Simplification from (1)
3.	$P(c)$	Existential instantiation from (2)
4.	$\exists xQ(x)$	Simplification from (1)
5.	$Q(c)$	Existential instantiation from (4)
6.	$P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7.	$\exists x(P(x) \wedge Q(x))$	Existential generalization of (6)

15. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct ?

(1) $\sqrt{2x^2 - 1} = x$ is given;

(2) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1);

(3) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2);

(4) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of (3) ;

(5) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.

16. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

17. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

Proof by Cases, Proof by Exhaustion etc

18. Prove: -

(a) $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

(b) the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

(c) that there atleast 100 consecutive positive integers that are not perfect squares. Is your proof constructive or non-constructive?

(d) that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

(e) that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

(f) that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

19. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

20. Prove or disprove that: -

(a) if a and b are rational numbers, then a^b is also rational.

(b) if you have an 8-litre jug of water and two empty jugs with capacities of 5 litres and 3 litres, respectively, then you can measure 4 litres by successively pouring some or all of the water in a jug into another jug.

21. Use forward reasoning (direct proof) to show that if x is a nonzero real number, then $x^2 + \frac{1}{x^2} \geq 2$. {*Hint:* Start with the inequality $\left(x - \frac{1}{x}\right)^2 \geq 0$ which holds for all nonzero real numbers x.}

22. Let T be the transformation that transforms an *even* integer x to $\frac{x}{2}$ and an *odd* integer x to $3x + 1$. A famous conjecture, known as the *3x + 1 conjecture (or Hasse's Algorithm)*, states that for all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1. Verify the $3x + 1$ conjecture for these integers 6 and 21.

Principle of Mathematical Induction

23. Use mathematical induction to prove the summation formulae: -

(a) $P(n)$: $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$ for the positive integer n.

(b) $P(n)$: $1^2+3^2+5^2 \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ for a ***nonnegative*** integer n.

(c) P(n): $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

24. Find a formula for $\frac{1}{1.2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$ by examining the values of this expression for small values of n.

25. Prove that $3^n < n!$ if n is an integer greater than 6.

26. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

27. Use mathematical induction to show that given a set of $n + 1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

Recursion

28. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer

a) $f(0) = 0$, $f(n) = 2f(n - 2)$ for $n \geq 1$

b) $f(0) = 1$, $f(n) = f(n - 1) - 1$ for $n \geq 1$

c) $f(0) = 2$, $f(1) = 3$, $f(n) = f(n - 1) - 1$ for $n \geq 2$

29. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$. b) $a_n = 1 + (-1)^n$.

c) $a_n = n(n + 1)$. d) $a_n = n^2$.

30. Let F be the function such that $F(n)$ is the sum of the first n positive integers. Give a recursive definition of $F(n)$.

31. Solve these recurrence relations:

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

c) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

d) $a_n = -6a_{n-1} - 9a_{n-2}\ for\ n \geq 2, a_0 = 3, a_1 = -3$

e) $a_n + 2a_{n-1} + 4a_{n-2} = 0\ for\ n \geq 3, a_1 = 1, a_2 = 3$

f) $a_n = 2a_{n-1} - a_{n-2} + 2\ for\ n \geq 3, a_1 = 1, a_2 = 5$

g) $a_n = a_{n-1} + \sin\frac{n\pi}{2}\ for\ n \geq 2, a_1 = -1$