HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

SCHOOL OF INFORMATION COMMUNICATION TECHNOLOGY

CAPSTONE PROJECT REPORT:

Chaos control and schedule of shuttle buses

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1 Problem Introduction

The concept of chaos refers to the presence of randomness and unpredictability within a deterministic framework. This phenomenon is frequently observed in various aspects of daily life, such as crowd dynamics, urban traffic flow, and even seemingly mundane tasks like toothbrushing, where each stroke follows no strict pattern. As such, there has been growing interest in identifying the predictable and potentially controllable elements within chaotic systems, with the aim of enhancing the efficiency of real-world processes.

In this report, we aim to replicate the findings of Takashi Nagatani in his study on shuttle bus scheduling. Nagatani's article, "Chaos Control and Schedule of Shuttle Buses", investigates a transportation system composed of two shuttle stops and two shuttle buses. These buses are allowed to pass one another and alternately load and unload passengers. As the number of waiting passengers at a stop increases, the boarding process becomes more time-consuming, causing the leading bus to slow down. This results in a gradual increase in its headway (i.e., the time interval between successive buses). Conversely, the trailing bus, encountering fewer passengers, may maintain or reduce its headway and eventually overtake the slower bus. This process contributes to an irregular and chaotic pattern in the buses' movement. However, the system includes a mechanism to counteract these delays: a speedup parameter, which allows a bus to accelerate and compensate for lost time. This corrective measure is hypothesized to mitigate the chaotic behavior and restore regularity to the schedule.

To evaluate the presence and extent of chaos in shuttle bus operations, it is essential first to define what is meant by chaos in a mathematical sense. We adopt the definition provided by Strogatz in Nonlinear Dynamics and Chaos, which characterizes chaos as "aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions" (Strogatz, p. 323). A deterministic system is one governed entirely by known rules, without any element of randomness, and sensitive dependence implies that even minor variations in initial conditions can lead to significantly divergent outcomes over time.

The variables, assumptions and objectives are defined as follows:

Variables:

- $B_i(m)$: number of passengers boarding bus i at trip m
- $W_i(m)$: number of passengers waiting at the origin just before bus i arrives at trip m
- γ: time it takes for one passenger to board the bus (s)
- η: time it takes for one passenger to leave the bus (s)
- μ: rate at which new passengers arrive at the origin (s^{-1})
- s_i: degree of speedup for bus i (m/s^2)
- L: length between the origin and the destination (m)
- $V_i(m)$: mean speed of bus i at trip m (m/s)

- $t_i(m)$: arrival time of bus i at the origin and trip m (s)
- $h_i(m)$: headway between bus i and the bus ahead of it (s)
- $\Delta t_i(m)$: tour time of bus i (s)

Assumptions:

- $H_i(m) = t_i(m) - t_{i'}(m')$ where bus i' arrived just before bus i at the origin
- $\Delta t_i(m) = t_i(m) - t_i(m-1)$
- $t_i(m+1) = t_i(m) + (\gamma + \eta)B_i(m) + \frac{2L}{V_i(m)}$
- $B_i(m) = W_i(m)$
- $B_i(m) \geq 0$, $V_i(m) \geq 0$, $t_i(m) \geq 0$, $h_i(m) \geq 0$
- γ, η, s_i, L are positive real numbers

Objectives: Consider how speedup suppresses the chaos.

2 Modelling approach

The modeling approach chosen in this study is a **discrete-time nonlinear map**, which is a specific type of **dynamical system** used to describe how a system evolves at **distinct time steps**. A discrete-time nonlinear map is a mathematical function that relates the state of a system at one time step to its state at the next, typically written as:

$$t_1(m_1 + 1) = f(t_1(m_1), t_2(m_2))$$

where $t_1(m_1), t_2(m_2)$ represents the system's state at current state, and f is a **nonlinear function**. This form is particularly suitable for systems like shuttle bus operations, where events occur in discrete trips (e.g., arriving at a stop, boarding passengers, departing, returning). Since it defines how the system's state changes over time, a discrete map is a special case of a dynamical system, which more generally refers to any system whose state evolves over time according to fixed rules whether in continuous time (using differential equations) or in discrete time (using difference equations or maps). The discrete-time nonlinear map was selected here because it simplifies analysis while capturing the essential feedback loops (such as delays and speedups) that give rise to chaotic patterns in bus schedules.

3 Formulate the problem

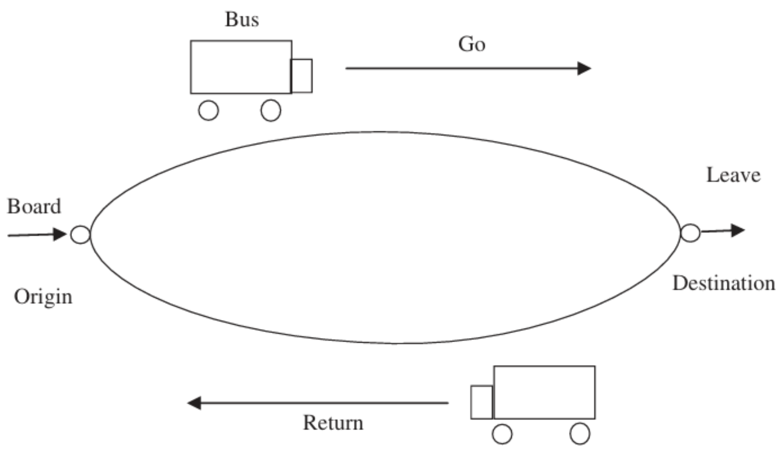

Figure 1: Schematic illustration of the shuttle bus system

The arrival time $t_i(m+1)$ of bus i at the origin and trip $m+1$ is given by:

$$t_i(m+1) = t_i(m) + (\gamma + \eta)B_i(m) + \frac{2L}{V_i(m)}, \quad \text{for } i = 1, 2 \tag{1}$$

If the maximum capacity of bus i is sufficiently large, the number $B_i(m)$ of boarding passengers equals the number $W_i(m)$ of passengers waiting at the origin just before bus i arrives at the origin and trip m. It is expressed by:

$$W_i(m) = B_i(m) = \mu\left(t_i(m) - t_{i'}(m')\right) \tag{2}$$

where bus i' is the one that arrived at the origin just before bus i at trip m, and $t_{i'}(m')$ is the arrival time of bus i' at the origin.

This gives an equation for the motion of the bus:

$$t_i(m+1) = t_i(m) + \mu(\Gamma + \eta)\left(t_i(m) - t_{i'}(m')\right) + \frac{2L}{V_0 + s_i\mu(\gamma+\eta)\left(t_i(m) - t_{i'}(m')\right)} \tag{3}$$

We define the dimensionless arrival time by dividing both sides by $\frac{2L}{V_0}$:

$$T_i(m+1) = T_i(m) + \Gamma\left(T_i(m) - T_{i'}(m')\right) + \frac{1}{1 + S_i\left(t_i(m) - t_{i'}(m')\right)} \tag{4}$$

where:

$$T_i(m) \equiv \frac{t_i(m)V_0}{2L}, \quad \Gamma \equiv \mu(\gamma+\eta), \quad S_i \equiv \frac{s_i\mu(\Gamma+\eta)2L}{V_0^2}$$

Thus, the dynamics of the buses is described by the simplified nonlinear map. The map should be iterated simultaneously for all M buses.

To simulate the dynamical properties of bus motion, we consider:

- Headway of bus 1, which captures inter bus interaction through time spacing:

$$H_1(m) = T_1(m) - T_{i'}(m'), \quad \text{where } i' \text{ is the bus that arrived just before bus 1} \tag{5}$$

- Tour time of bus i, which captures the temporal evolution of an individual bus's performance:

$$\Delta T_i(m) = T_i(m) - T_i(m-1), \quad \text{for } i = 1, 2 \tag{6}$$

4 Solving algorithm

The implementation is divided into two primary sections: (1) the data structure used to store bus arrival information and (2) the visualization module for plotting the simulation results. The core computational logic computes the successive arrival times of each bus at the origin, based on the recurrence relation given in Equation (4) of Nagatani's paper. Since the original article does not explicitly specify the initial conditions, and only provides graphical output for the 901^{st} to $1000th$ trips, we adopt the initial values $T_1^0 = 1$ and $T_2^0 = 2.5$. This choice allows bus 1 to begin at a simple, integer time while bus 2 starts later with a non-integer value. It is important to note that due to the system's sensitivity to initial conditions where different initial values may lead to significantly different simulation outcomes.

The simulation is encapsulated in the function `simulate_bus_system`, which returns a list called `arrival_time`. Each element in this list represents a recorded bus trip and is structured as:

- `time`: The arrival time of the bus at the origin
- `bus_number`: Identifier for the bus (either 1 or 2)
- `trip_number`: The sequential trip count for the corresponding bus
- `used_flag`: A boolean value indicating whether this trip has been used in the calculation of subsequent trips

During the simulation, the next estimated arrival for each bus is calculated and temporarily stored in the auxiliary structures `temp_time` and `temp_trip`. The simulation compares the next potential arrivals of both buses and appends the earlier one to `arrival_time`. This process is iterated until both buses complete more than the specified number of trips (e.g., 1000).

Pseudocode of `simulate_bus_system` function

Algorithm 1: simulate_bus_system

```
Input: T1_initial, T2_initial, Γ, S_1, S_2, num_trips
Output: arrival_time
Initialize arrival_time with two entries:
    [0, 2, −1, False]                                  // Dummy initial bus
    [T1_initial, 1, 0, False]                          // First trip of bus 1
Initialize:
    temp_time ← [None, T2_initial]
    temp_trip ← [None, [T2_initial, 2, 0, False]]
    count_trip ← [0, 0]                                // Trips per bus
    i ← 1                                              // Index to process
while either bus has not completed num_trips + 1 do
    if i ≥ length of arrival_time then
        break
    end if
    current_row ← arrival_time[i]
    if current_row is dummy or already used then
        i ← i + 1
        continue
    end if
    bus_id ← current_row[1]
    for j ← i − 1 to 0 step −1 do
        if arrival_time[j] not used then
            pre_row ← arrival_time[j]
            break
        end if
    end for
    δ ← current_row[0] − pre_row[0]
    if δ < 0 then
        print warning and continue
    end if
    Set S ← S_1 if bus_id = 1, else S_2
    next_time ← current_row[0] + Γ · δ + 1/(1 + S · δ)
    Mark pre_row as used
    Save next_time into temp_time[bus_id - 1], and corresponding temp_trip
    if temp_time[0] ≠ None and temp_time[1] ≠ None then
        if temp_time[0] < temp_time[1] then
            Append temp_trip[0] to arrival_time
            Increment count_trip[0]
            Reset temp_time[0], temp_trip[0] ← None
        end if
        else
            Append temp_trip[1] to arrival_time
            Increment count_trip[1]
            Reset temp_time[1], temp_trip[1] ← None
        end if
    end if
    i ← i + 1
end while
return arrival_time
```

5 Simulation Result

To simulate the dynamical behaviour of shuttle buses by the use of iteration of map (4), Nagatani calculates headways and tour time with varying trips for the typical case of two buses in 4 cases: (a) $S_1 = S_2 = 0$, (b) $S_1 = S_2 = 0.2$, (c) $S_1 = 0.3$, $S_2 = 0.2$ and (d) $S_1 = 0.5$, $S_2 = 0.2$. We use `matplotlib` library to visualize and the outcome is fairly consistent with the original article.

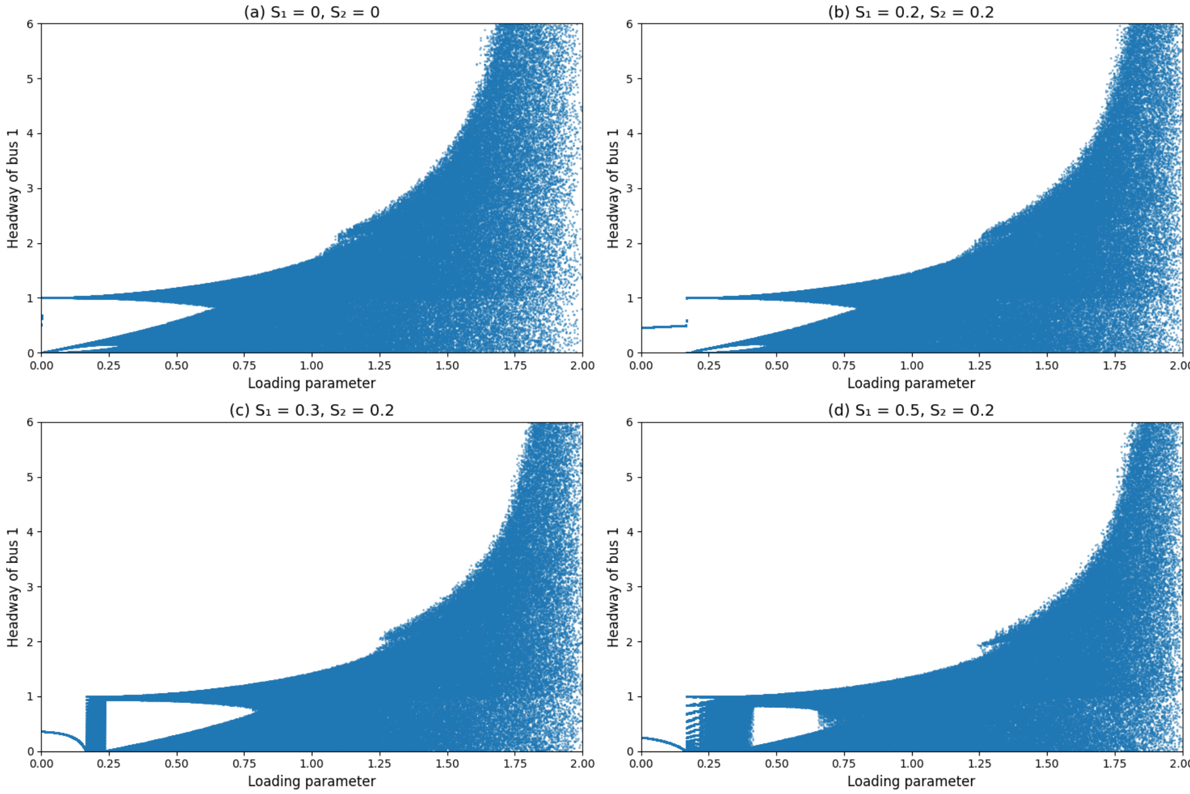

Figure 2: Plots of the time headway $H_1(m)$ of bus 1 against loading parameter Γ from sufficiently large trip $m = 900$–1000. Diagrams (a)–(d) are obtained, respectively, for (a) $S_1 = S_2 = 0$, (b) $S_1 = S_2 = 0.2$, (c) $S_1 = 0.3$, $S_2 = 0.2$ and (d) $S_1 = 0.5$, $S_2 = 0.2$.

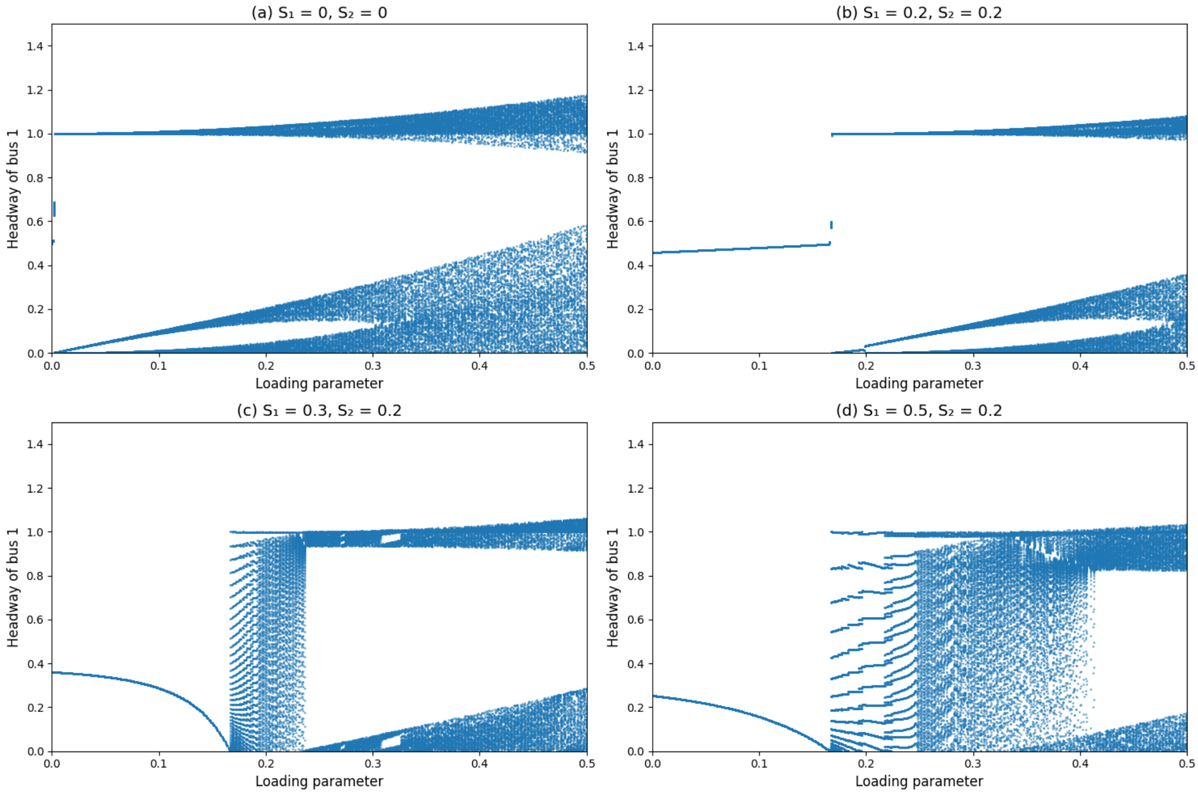

Figure 3: Enlargements of Fig. 2(a)–(d) for $0 < \Gamma < 0.5$.

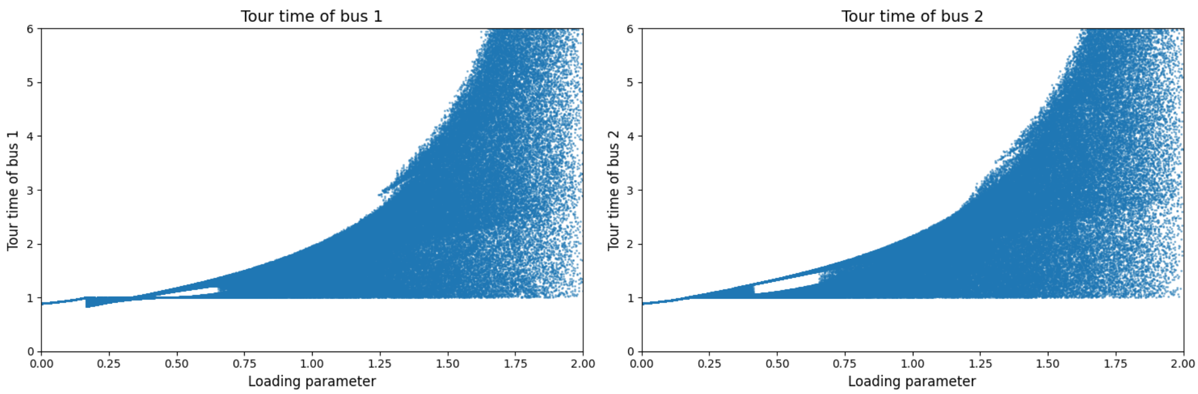

Figure 4: Plots of tour times of buses 1 and 2 against loading parameter Γ from sufficiently large trip $m = 900$–1000 for the case of $S_1 = 0.5$ and $S_2 = 0.2$.

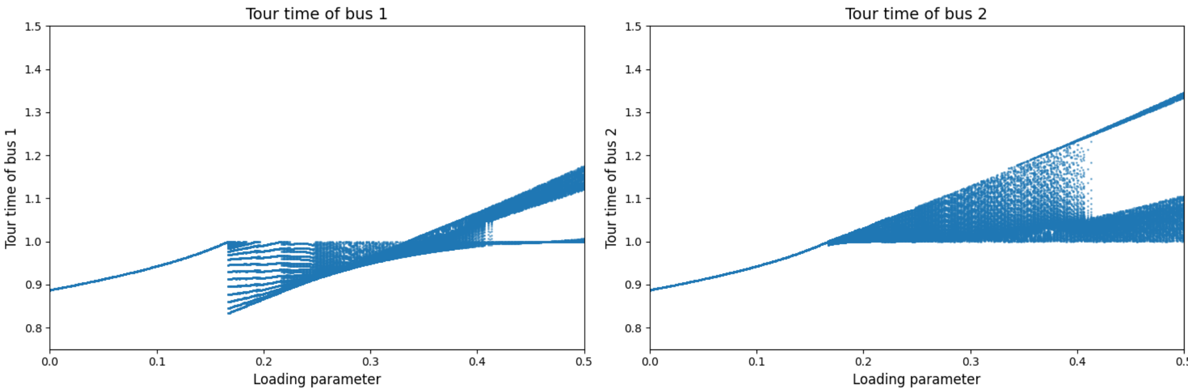

Figure 5: Enlargements of Fig. 4(a) and (b) for $0 < \Gamma < 0.5$.

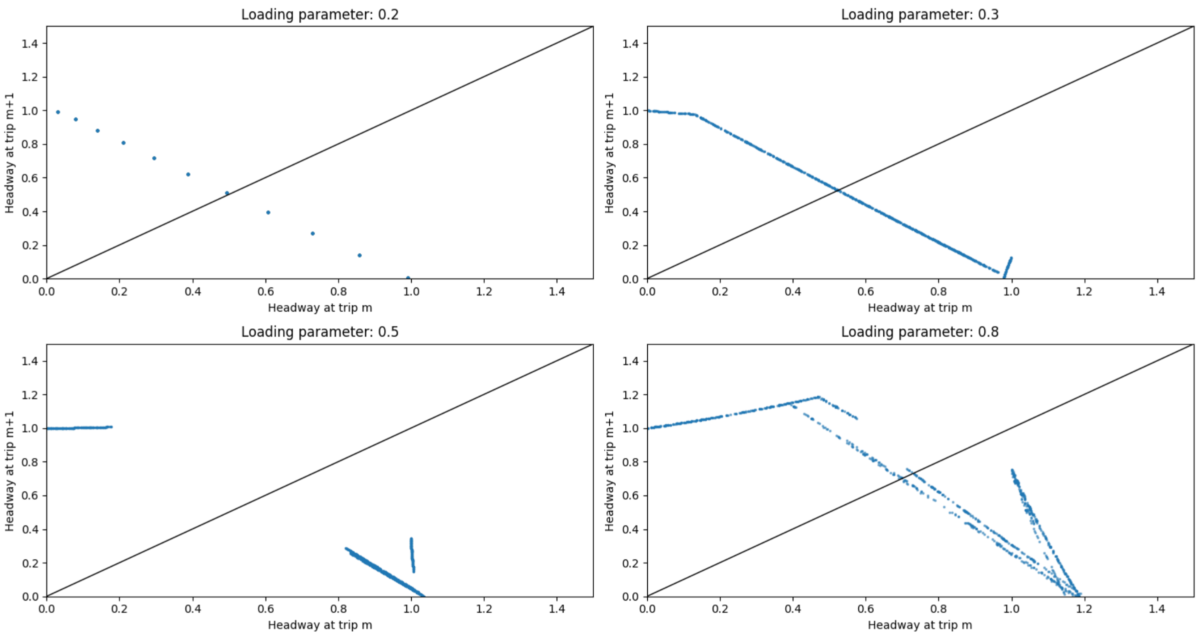

Figure 6: Plots of $H_1(m+1)$ against $H_1(m)$ from $m = 1000$–2000 for speedup parameters $S_1 = 0.2$ and $S_2 = 0.2$.

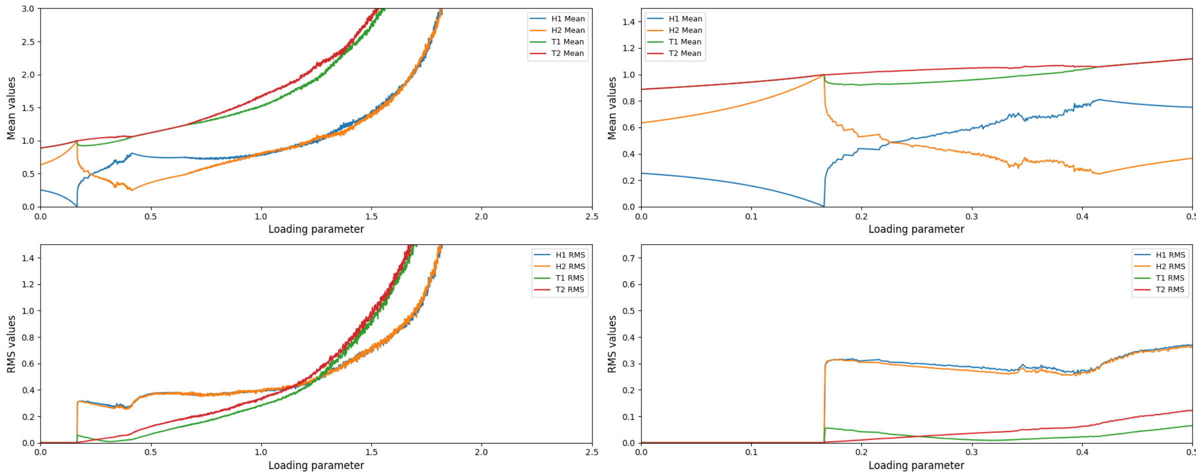

Figure 7: Plots of mean and rms headway $H_{1\text{mean}}, H_{2\text{mean}}, H_{1\text{rms}}, H_{2\text{rms}}$ and tour times $\Delta T_{1\text{mean}}, \Delta T_{2\text{mean}}, \Delta T_{1\text{rms}}, \Delta T_{2\text{rms}}$ against loading parameter Γ for speedup parameters $S_1 = 0.5$ and $S_2 = 0.2$.

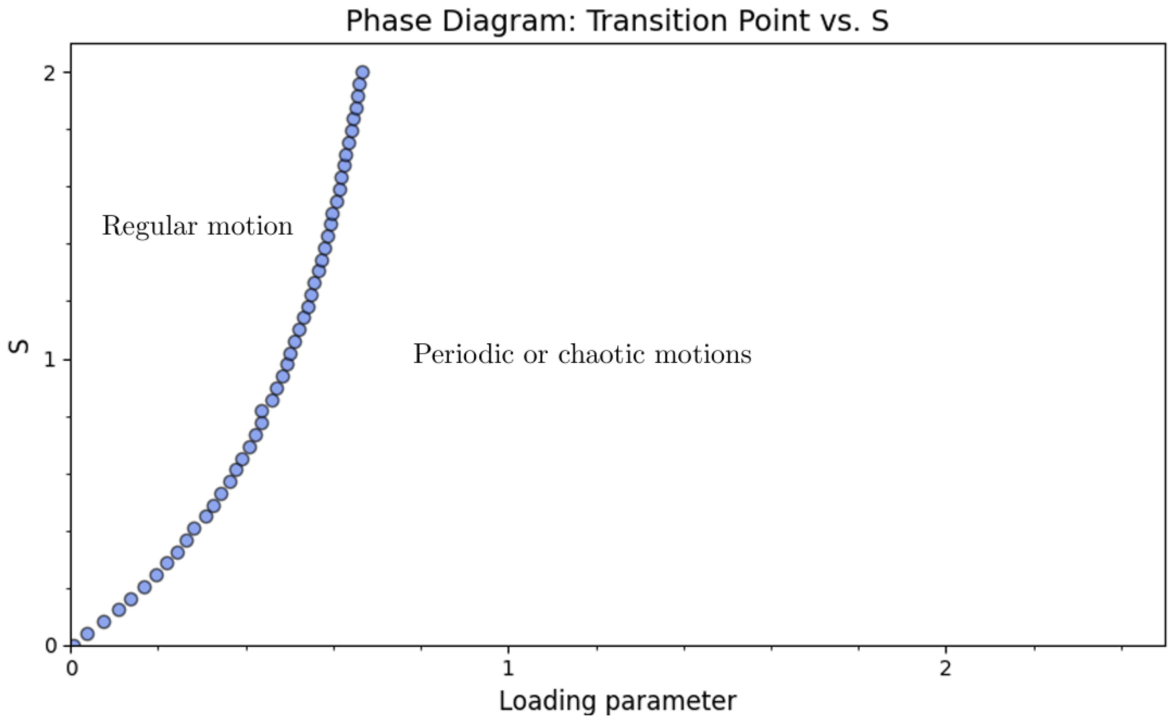

Figure 8: Phase diagram (region map) for the regular and periodic (or chaotic) motions in phase space (Γ, S) of loading and speedup parameter for $S_1 = S_2 = S$. The circles indicate the transition points obtained from simulation.

To determine the transition point, we analyze the second derivative of $T_{2\text{rms}}$. As shown in Figure 7d, the value of $T_{2\text{rms}}$ remains close to zero before the onset of the transition. The transition point is identified as the first moment when the second derivative exhibits a significant increase. In our experiments, we define this point by applying a threshold:

the difference in second derivative values between two adjacent Γ values must exceed 130. This threshold was chosen empirically based on the observed patterns in the data.

6 Discussion & Conclusion

6.1 Disccusion

Sensitivity Analysis

The bus dynamics model incorporates key parameters such as the loading parameter (Γ) and speedup parameters (S_1, S_2), which critically influence the system's behavior. Sensitivity analysis helps us understand how variations in these parameters affect the overall dynamics, especially regarding the onset of periodic and chaotic motions.

Loading Parameter (Γ)

The model shows high sensitivity to changes in the loading parameter. When Γ is below a critical threshold (approximately 0.167), buses maintain regular motion at a constant speed, indicating system stability. However, small increases beyond this threshold lead to significant changes: periodic fluctuations in time headway emerge, and eventually, chaotic behavior develops. This sensitivity suggests that the loading parameter acts as a bifurcation parameter, controlling transitions between different dynamical regimes.

Speedup Parameters (S_1, S_2)

The difference in speedup parameters between buses significantly affects system dynamics. When speedup is balanced or high enough, fluctuations and chaotic behavior are suppressed, indicating that speedup acts as a control mechanism. Conversely, disparities in speedup parameters induce complex, nonlinear behavior, including periodic and chaotic motions.

Robustness of the Model

The model reliably predicts transitions from regular to chaotic bus motion, and speedup helps suppress chaos, demonstrating useful control strategies. However, its robustness may be limited under real-world variability such as random passenger arrivals and traffic. The model assumes unlimited bus capacity and simplifies the calculation of waiting passengers by using the headway between two consecutive buses. This ignores the fact that buses often dwell at the origin to load passengers, which affects the actual headway and boarding time in practice.

Future Research

In Figure 8, the relationship between speedup parameter S ($S_1 = S_2 = S$) and transition point Γ_0 is approximated by a fourth-degree polynomial function. Utilizing the scikit-learn library, the data was fitted to obtain the following functional form:

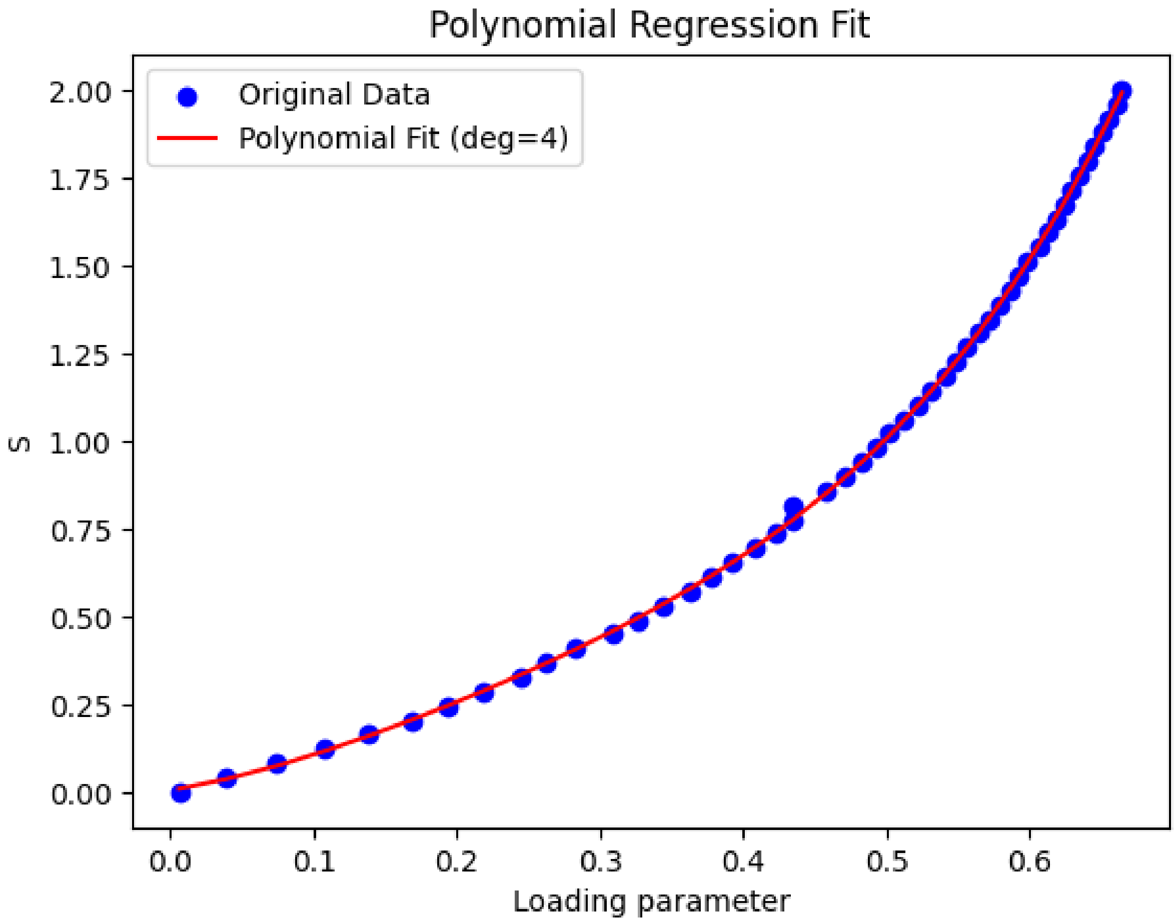

Figure 9: Estimation the relationship between speedup parameter and transition point.

We have the estimation:

$$S \approx 1.0 + 0.1162\Gamma_0 + 0.6534\Gamma_0^2 - 0.9998\Gamma_0^3 + 0.8366\Gamma_0^4$$

$$\text{sensitive}(S, \Gamma_0) = \frac{\partial S}{\partial \Gamma_0}\frac{\Gamma_0}{S} = \frac{0.1162\Gamma_0 + 1.3068\Gamma_0^2 - 2.9994\Gamma_0^3 + 3.3464\Gamma_0^4}{1.0 + 0.1162\Gamma_0 + 0.6534\Gamma_0^2 - 0.9998\Gamma_0^3 + 0.8366\Gamma_0^4}$$

6.2 Conclusion

We re-examined a dynamic model for shuttle bus systems that includes both the delay caused by passenger loading and the compensating effect of speedup. Using nonlinear mapping techniques, we simulated the bus dynamics to analyze how different levels of loading and speedup parameters affect system behavior.

Our simulation results showed that the buses exhibit complex dynamics, including regular trip patterns and deterministic chaos where seemingly unpredictable fluctuations arise from the system's inherent deterministic rules rather than random noise. By varying key parameters, we demonstrated that irregularities in bus arrival times are fundamentally connected to chaotic dynamics.

In particular, we identified that when the loading parameter passes certain critical values, the system experiences multiple dynamic transitions:

- The first transition point marks the change from steady behavior to periodic motion.
- The second point indicates a shift from periodic motion to chaotic dynamics.
- The third point reflects further alterations in the chaotic regime, showing increased complexity.

These transition points are strongly influenced by the speedup parameter, which acts as a control to either trigger or suppress chaotic behavior. Our findings re-emphasize that shuttle bus scheduling stability depends on a careful balance between passenger loading and speedup capabilities, highlighting the importance of parameter tuning to prevent unstable or unpredictable service patterns.

7 References

References

[1] Nagatani, T. (2006). Chaos control and schedule of shuttle buses. *Physica A: Statistical Mechanics and its Applications*, 371(2), 683–691. `https://doi.org/10.1016/j.physa.2006.04.056`

[2] Strogatz, S. H. (2000). *Nonlinear Dynamics and Chaos*. Cambridge, MA: Perseus Books Publishing, LLC.

[3] Avetian, M., Guzman, L., Kumada, R., Soimaru, R., & Zerkoune, S. (2010, April 1). *Chaos Control in Shuttle Bus Schedule*. Math 485. University of Arizona.