CSC477: Comparative Analysis of Unscented and Extended Kalman Filters in Nonlinear System

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Abstract: This project proposes a comprehensive evaluation of the Unscented Kalman Filter (UKF) and the Extended Kalman Filter (EKF) in the context of estimating the states of nonlinear systems. Given the inherent challenges posed by nonlinear dynamics in various engineering and data assimilation applications, there is an urgent need to deploy estimation filters that provide accurate and efficient performance. By applying UKF and EKF to some nonlinear functions, we intend to compare their performance thoroughly, focusing on accuracy and computational speed.

Keywords: Unscented Kalman Filter, UKF, Extended Kalman Filter, EKF, Nonlinear systems, Lorenz system, Van der Pol oscillator, Duffing equation

1 Introduction

The Kalman Filter [1], a foundational tool in control theory and estimation, originally designed for linear systems, paved the way for its extended and unscented counterparts. While the Extended Kalman filter (EKF) is the nonlinear version of the Kalman filter. In 1997, Julier and Uhlman proposed a new extension of the Kalman filter to nonlinear systems - Unscented Kalman Filter (UKF) [2].

In this study, we delve into the intricate realm of nonlinear system estimation, particularly focusing on its crucial significance within the domain of robotics. Our endeavour revolves around a meticulous examination of the UKF and the EKF, two prominent methodologies for state estimation in the face of nonlinear dynamics.

To embark on this exploration, we are set to construct UKF from scratch and conduct a comprehensive evaluation of UKF and EKF for a series of well-known nonlinear functions commonly found in robotics. This research will include nonlinear functions such as Lorentz systems [3] that model chaotic weather patterns; Van der Pol oscillators [4], which represent non-conservative oscillations

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of systems with nonlinear damping; and the Duffing equation [5], which describes the More complex nonlinear restoring forces for damped oscillator motion. These features are emblematic of the complex nonlinear behaviours observed in robotic navigation and control systems, including trajectory planning and autonomous vehicle behaviour in uncertain environments. To comprehensively illustrate our findings, we will include an analysis and graphs to show the results, providing a visual and quantitative comparison of the performance of UKF and EKF across these nonlinear scenarios. This analysis is designed to highlight each filter's operational strengths and potential limitations within the realm of robotic applications.

2 Related Work

The theoretical foundations and practical applications of the EKF and the UKF are well documented in the literature, particularly in the fields of Bayesian filtering and robotics.

Bayesian Filtering and Smoothing [6], Through meticulous exposition, Särkka elucidates the theoretical underpinnings of Bayesian filtering techniques, offering invaluable insights into their mathematical formulations and algorithmic implementations.

Another seminal contribution that has significantly shaped the discourse on probabilistic robotics is the seminal text *Probabilistic Robotics* [7]. This seminal work transcends traditional boundaries, providing a holistic framework for integrating probabilistic reasoning into robotic perception, planning, and control. By marrying probabilistic models with sensor data and action sequences, the authors illuminate the path toward autonomous systems endowed with robust decision-making capabilities.

In the context of our comparative analysis of the UKF and EKF, the *Unscented Filtering and Nonlinear Estimation* assumes paramount significance [8]. In this work, Julier and Uhlmann present a groundbreaking methodology that revolutionizes nonlinear state estimation. By leveraging a judicious selection of sigma points, the Unscented Kalman Filter emerges as a robust alternative to the traditional Extended Kalman Filter, offering improved accuracy and stability in the face of nonlinearities.

The recent research by Biswas et al. sheds light on a quantified approach for predicting the suitability of employing the UKF in nonlinear applications [9]. Their study introduces a mathematical framework for assessing the performance improvement of UKF over the EKF based on a quantitative measure of non-linearity. By leveraging Non-linearity Indices, their approach offers a systematic means of estimating the potential performance enhancement of UKF relative to EKF, without the need for extensive design and tuning efforts.

3 Method

3.1 Implementation

To ensure a thorough understanding of their inner workings, we will meticulously implement both the UKF and the EKF from scratch. This approach allows us to tailor the filters to our specific test cases and facilitates insight into their underlying algorithms. Our implementation will be carried out in Python, leveraging the computational efficiency of libraries such as NumPy for mathematical operations. By optimizing for computational speed, we aim to ensure that our filters are well-suited for real-time applications.

The core algorithms for both the Extended Kalman Filter (EKF) and the Unscented Kalman Filter (UKF) were encapsulated within class structures, enhancing modularity and scalability. Special attention was given to the numerical stability of the algorithms, particularly in the handling of matrix inversions and the computation of the Kalman Gain, critical components for accurate state estimation.

Pseudocode for EKF and UKF Initialization and Update: Below, we provide pseudocode that details the initialization and update processes for both EKF and UKF, illustrating the computational steps involved and emphasizing the systematic approach used in these filters.

```
Algorithm 1: Initialize EKF/UKF
Input: Initial state estimate x0, initial covariance P0,
       process noise Q, measurement noise R
Output: Initialized filter object with parameters set
1: function InitializeFilter(x0, P0, Q, R)
2:     Create filter object
3:     Set initial state and covariance
4:     Define noise characteristics
5:     return filter
```

```
Algorithm 2: Update EKF/UKF
Input: Measurement z
Output: Updated state and covariance
1: function UpdateFilter(filter, z)
2:     Predict state and covariance using system dynamics
3:     Compute Kalman Gain
4:     Update state and covariance with measurement
5:     return updated state, covariance
```

This detailed approach to implementation is essential for realizing the theoretical advantages of UKF and EKF in practical scenarios. Each step has been crafted to ensure robustness and reliability, necessary for real-time applications where precise state estimation is paramount.

3.2 Selection of Nonlinear Systems

Our study will encompass a diverse selection of well-established nonlinear functions and systems, chosen for their relevance to real-world robotics applications. Among these systems are the iconic Lorenz system, renowned for its chaotic behaviour modelling; the Van der Pol oscillator, representing non-conservative oscillations prevalent in various mechanical systems; and the Duffing equation, which encapsulates the complex nonlinear restoring forces encountered in damped oscillator motion. These systems serve as emblematic examples of the nonlinear challenges ubiquitous in robotic navigation and control, providing a comprehensive testing ground for our comparative analysis.

3.3 Evaluation Criteria

Our comparison will be guided by two fundamental criteria: accuracy and computational speed. To assess accuracy, we will compare the estimated states produced by each filter against the ground truth states of the systems. Root mean square error (RMSE) will serve as a quantitative measure of accuracy, providing insights into the fidelity of each filter's estimations. In parallel, computational speed will be evaluated by measuring the time taken by each filter to perform state estimations. This assessment will offer valuable insights into the practical efficiency of the filters, particularly in real-time robotic applications where rapid decision-making is paramount.

3.4 Analysis and Graphical Representation

Upon conducting simulations across our selected nonlinear systems, we will embark on a comprehensive analysis of the obtained results. Through meticulous examination, we aim to elucidate trends, strengths, and weaknesses in the performance of the UKF and EKF. Graphical representations, including error plots and computational time charts, will be employed to visualize and contextualize the observed differences between the filters. This combined visual and quantitative analysis will serve as the cornerstone of our conclusions regarding the comparative advantages of the UKF and EKF in nonlinear system estimation within the realm of robotics.

4 Results and Evaluation

Our comparative evaluation of the UKF and the EKF across various nonlinear systems has yielded insightful findings regarding their performance in state estimation and revealed distinct performance characteristics and limitations. In our evaluation, we employed a diverse set of nonlinear functions, including iconic models such as the Lorenz system, Van der Pol oscillator, and Duffing equation. These systems were specifically selected for their relevance to real-world robotics applications and their ability to exhibit complex nonlinear behaviours. Below, we present a summary of the key results obtained from our simulations.

4.1 Van der Pol Oscillator

In our assessment of the Van der Pol oscillator, we observed notable differences in the accuracy between the EKF and the UKF. While both filters provided reasonably accurate estimations of the system states, the UKF exhibited superior accuracy compared to the EKF. Specifically, the total absolute difference, a measure of estimation error, was significantly lower for the UKF (306.54) compared to the EKF (500.67). This suggests that the UKF better captured the nonlinear dynamics inherent in the Van der Pol oscillator, resulting in more precise state estimations.

However, it is essential to note that this improvement in accuracy came at the cost of computational efficiency. Despite its superior accuracy, the UKF demonstrated a longer average runtime per step (6.75e-05 seconds) compared to the EKF (1.96e-05 seconds). This discrepancy underscores the trade-off between accuracy and computational efficiency, where the UKF's enhanced accuracy necessitates increased computational complexity.

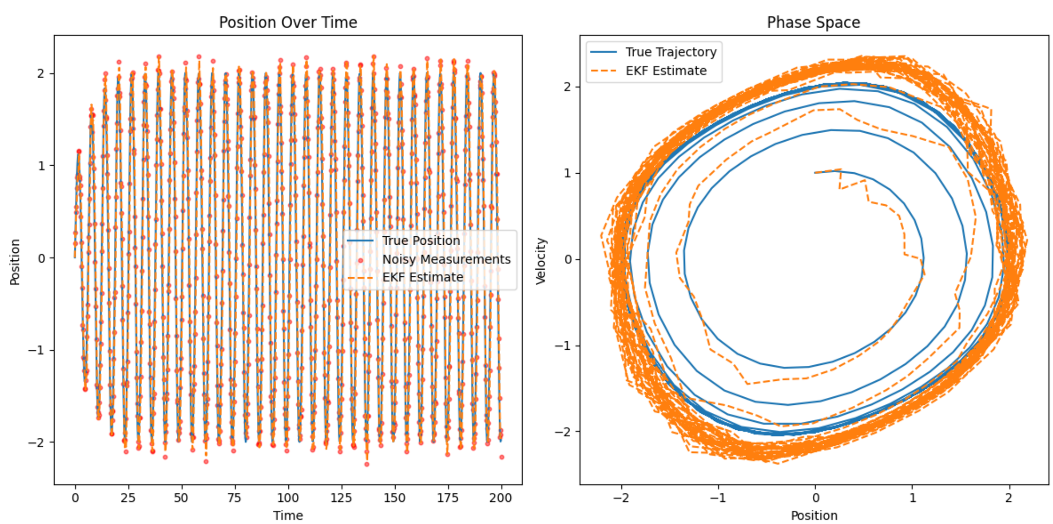

Figure 1: EKF Prediction of Van der Pol Oscillator

Overall, while the EKF showcased commendable accuracy in estimating the states of the Van der Pol oscillator, the UKF emerged as the preferred choice for applications prioritizing accuracy over computational speed. However, the selection between these filters should consider the specific requirements and constraints of the target application, balancing the trade-offs between accuracy and computational efficiency.

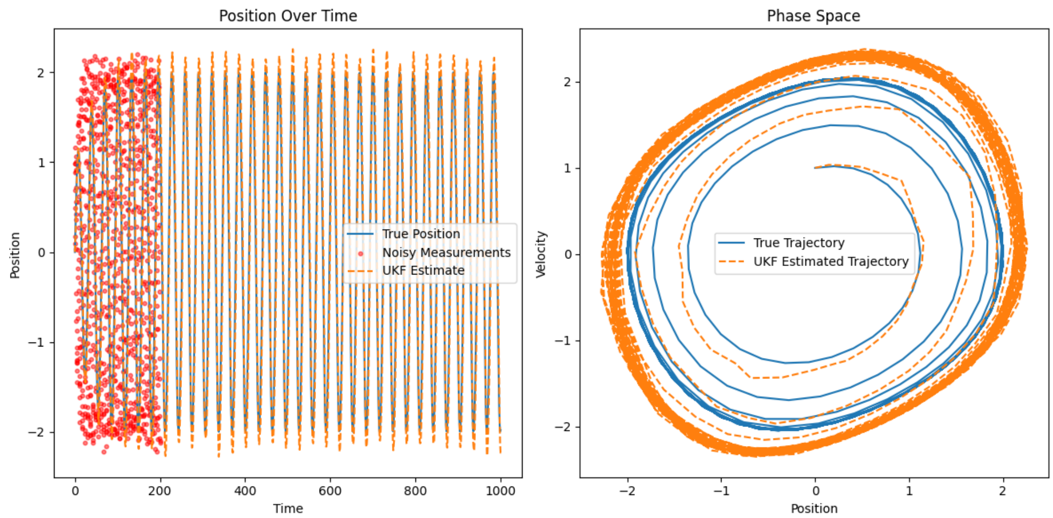

Figure 2: UKF Prediction of Van der Pol Oscillator

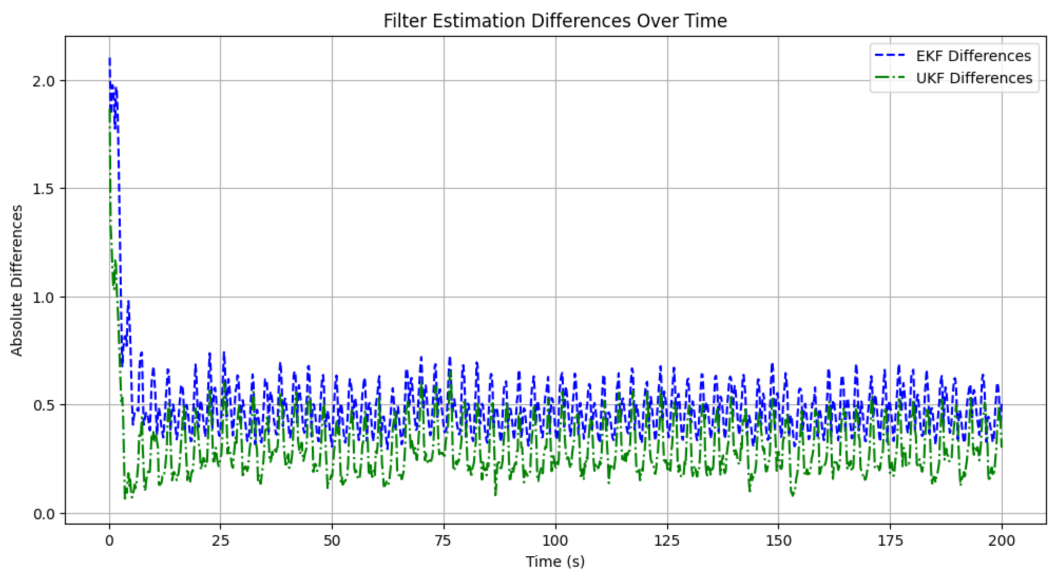

Figure 3: UKF and EKF Prediction Difference Over Time of Van der Pol Ocillator

4.2 Lorentz Systems

Our examination of the Lorentz system has similar results as the Van der Pol oscillator. It underscores significant disparities in the performance between the EKF and the UKF, particularly concerning accuracy and computational efficiency.

Regarding accuracy, the UKF demonstrated markedly superior performance compared to the EKF. The total squared difference, indicative of estimation error, was substantially lower for the UKF (2022.12) than for the EKF (13550.38). This implies that the UKF provided more accurate estimations of the system states, effectively capturing the nonlinear dynamics inherent in the Lorentz system.

However, this improvement in accuracy came at the expense of computational efficiency. The UKF exhibited a significantly longer average runtime per step (0.1415 seconds) compared to the EKF (0.0274 seconds). This disparity highlights the trade-off between accuracy and computational efficiency, where the UKF's enhanced accuracy necessitated increased computational complexity.

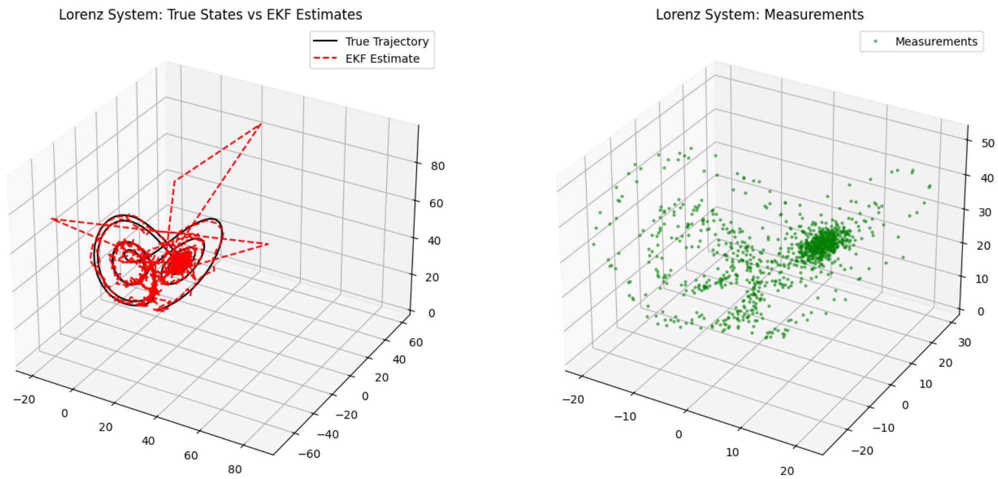

Figure 4: EKF Prediction of Lorentz Systems

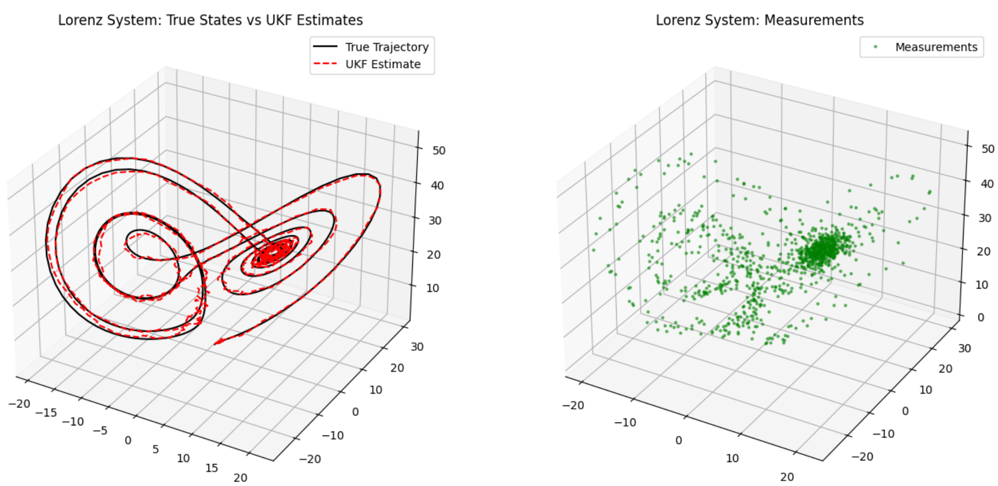

Figure 5: UKF Prediction of Lorentz Systems

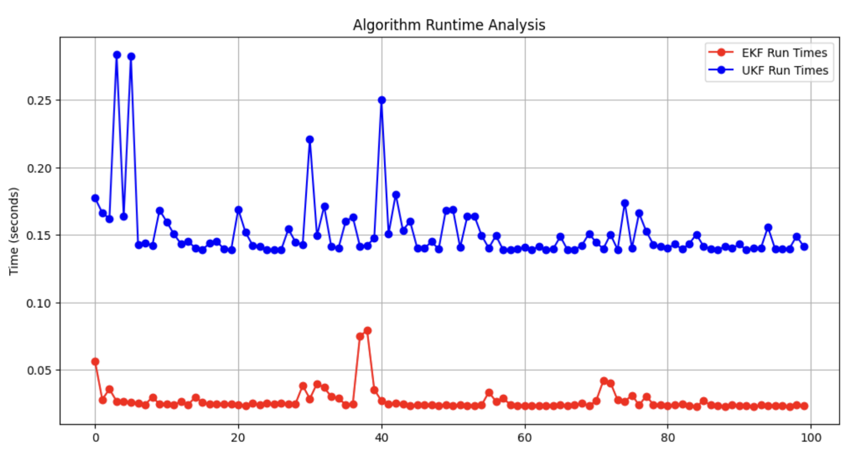

Figure 6: Runtime Analysis of Lorenezk System

In summary, while the EKF demonstrated respectable accuracy in estimating the states of the Lorentz system, the UKF outperformed it significantly in terms of accuracy. However, the choice between these filters should consider the specific requirements and constraints of the application, balancing the trade-offs between accuracy and computational efficiency.

4.3 Duffing Equation

Our investigation of the Duffing equation reveals further insights into the comparative performance of the EKF and the UKF. In this context, we observed contrasting outcomes compared to our assessments of the Van der Pol oscillator and the Lorenz system.

Concerning accuracy, the EKF demonstrated notably superior performance compared to the UKF. The total absolute difference, reflecting estimation error, was substantially lower for the EKF (158.81238527988162) compared to the UKF (258.42205403233413). This suggests that, for the Duffing equation, the EKF provided more accurate estimations of the system states, effectively capturing the nonlinear dynamics inherent in this specific system.

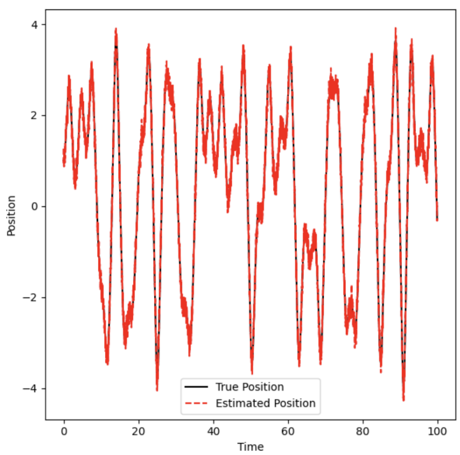

Figure 7: EKF Prediction of Duffing Equation

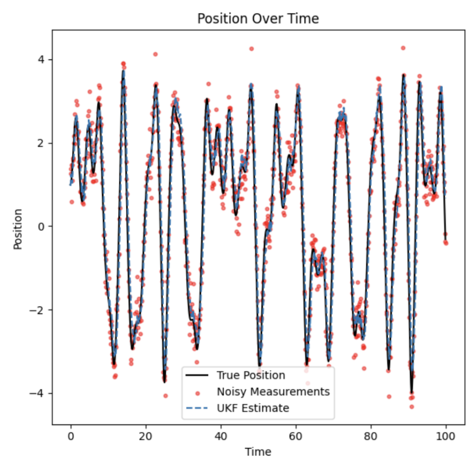

Figure 8: UKF Prediction of Duffing Equation

However, in terms of computational efficiency, the UKF exhibited a longer average runtime per step (8.10e-05 seconds) compared to the EKF (2.26e-05 seconds). This disparity suggests that the UKF's enhanced accuracy came at the expense of increased computational complexity, highlighting the trade-off between accuracy and computational efficiency.

In summary, while the EKF demonstrated superior accuracy in estimating the states of the Duffing equation, the UKF showcased better performance in previous assessments of the Van der Pol oscillator and the Lorenz system. The choice between these filters should be carefully considered in the context of the specific nonlinear system being analyzed, balancing the trade-offs between accuracy and computational efficiency.

5 Limitations

While our study endeavours to provide a comprehensive comparison between the UKF and the EKF in the context of nonlinear system estimation, certain limitations should be acknowledged.

Firstly, our evaluation is constrained by the selection of nonlinear systems chosen for analysis. While we have carefully curated a diverse set of systems representative of common challenges in robotics, the applicability of our findings may vary across different domains and scenarios. Thus, the generalizability of our conclusions to all possible nonlinear systems may be limited.

Additionally, the effectiveness of our methods and conclusions might be contingent upon the accuracy of the models used for simulation. Real-world systems often exhibit nuances and dynamics that are challenging to capture fully in mathematical representations. Thus, the performance of our techniques in practical applications could differ from what is observed in simulated environments.

Lastly, the robustness of our findings to variations in sensor noise, model uncertainty, and other sources of perturbation has not been exhaustively investigated. While we have considered noise in our simulations, the extent to which our methods can tolerate or adapt to uncertainties in real-world conditions remains an open question.

Despite these limitations, we believe that our study contributes valuable insights into the comparative performance of UKF and EKF in nonlinear system estimation, thereby informing future research directions and practical applications in robotics and beyond.

6 Conclusion

In this study, we conducted a comprehensive comparison between the UKF and the EKF in the realm of nonlinear system estimation, with a focus on applications in robotics. Through meticulous implementation, rigorous evaluation, and insightful analysis, we have gleaned valuable insights into the relative strengths and weaknesses of these two prominent estimation filters.

Despite the inherent trade-offs and limitations associated with each filter, our study contributes valuable insights to the field of nonlinear system estimation, informing practitioners and researchers alike in their quest for robust and efficient state estimation techniques.

Moving forward, future research directions may explore hybrid approaches that leverage the strengths of both UKF and EKF, as well as novel methodologies that address the limitations identified in our study. By continually advancing the state-of-the-art in estimation theory and practice, we can pave the way for enhanced performance and reliability in a wide range of robotic applications and beyond.

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Appendix

Each author contributed equal works