3

The yield curve

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Introduction

Fundamental to any trading and risk management activity is the ability to value future cash flows of an asset. In modern finance the accepted approach to valuation is the discounted cash flows (DCF) methodology. If $C(t)$ is a cash flow occurring t years from today, according to the DCF model, the value of this cash flow today is

$$V_0 = C(t)Z(t)$$

where $Z(t)$ is the present value (PV) factor or discount factor. Therefore, to value any asset the necessary information is the cash flows, their payment dates, and the corresponding discount factors to PV these cash flows. The cash flows and their payment dates can be directly obtained from the contract specification but the discount factor requires the knowledge of the *yield curve*. In this chapter we will discuss the methodology for building the yield curve for the bond market and swap market from prices and rates quoted in the market.

The yield curve plays a central role in the pricing, trading and risk management activities of all financial products ranging from cash instruments to exotic structured derivative products. It is a result of the consensus economic views of the market participants. Since the yield curve reflects information about the microeconomic and macroeconomic variables such as liquidity, anticipated inflation rates, market risk premia and expectations on the overall state of the economy, it provides valuable information to all market participants. The rationale for many trades in the financial market are motivated by a trader's attempt to monetize their views about the future evolution of the yield curve when they differ from that of the market. In the interest rate derivative market, yield curve is important for calibrating interest rate models such as Black, Derman and Toy (1990), Hull and White (1990) and Heath, Jarrow and Morton (1992) and Brace, Gatarek and Musiela (1997) to market prices. The yield curve is built using liquid market instrument with reliable prices. Therefore, we can identify hedges by shocking the yield curve and evaluating the sensitivity of the position to changes in the yield curve. The time series data of yield curve can be fed into volatility estimation models such as GARCH to compute Value-at-Risk (VaR).

Strictly speaking, the yield curve describes the term structure of interest rates in any market, i.e. the relationship between the market yield and maturity of instruments with similar credit risk. The market yield curve can be described by a number of alternative but equivalent ways: discount curve, par-coupon curve, zero-coupon or spot curve and forward rate curve. Therefore, given the information on any one, any of the other curves can be derived with no additional information.

The discount curve reflects the discount factor applicable at different dates in the future and represents the information about the market in the most primitive fashion. This is the most primitive way to represent the yield curve and is primarily used for valuation of cash flows. An example of discount curve for the German bond market based on the closing prices on 29 October 1998 is shown in Figure 3.1. The par, spot, and forward curves that can be derived from the discount curve is useful for developing yield curve trading ideas.

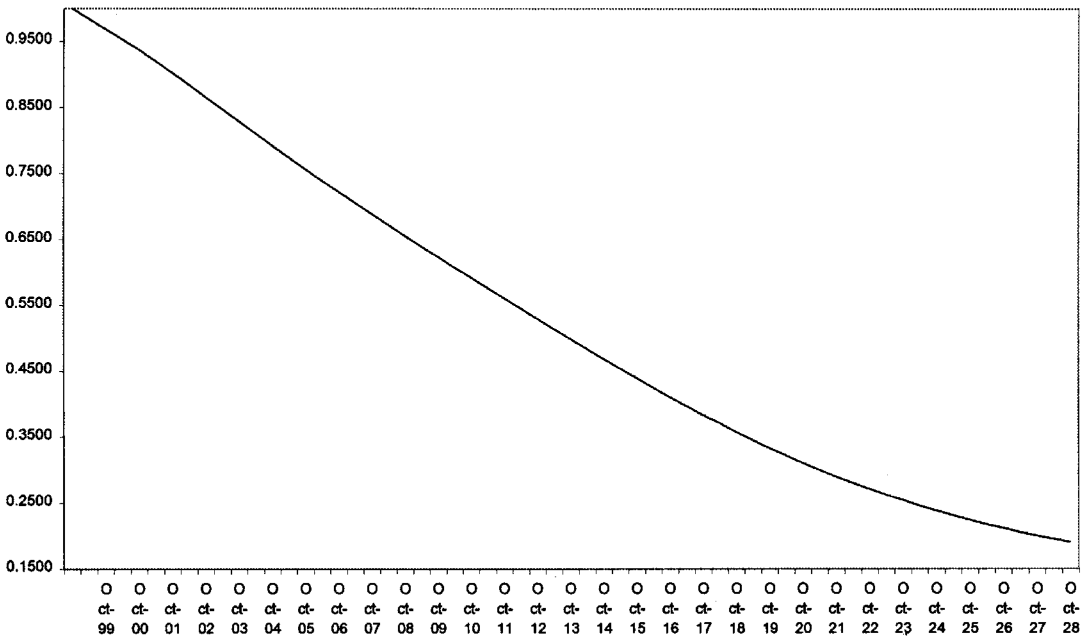


Figure 3.1 DEM government discount curve.

The par-coupon curve reflects the relationship between the yield on a bond issued at par and maturity of the bond. The zero curve or the spot curve, on the other hand, indicates the yield of a zero coupon bond for different maturity. Finally, we can also construct the forward par curve or the forward rate curve. Both these curves show the future evolution of the interest rates as seen from today's market yield curve. The forward rate curve shows the anticipated market interest rate for a specific tenor at different points in the future while the forward curve presents the evolution of the entire par curve at a future date. Figure 3.2 shows the par, spot, forward curves German government market on 29 October 1998. For example the data point (20y, 5.04) in the 6m forward par curve tell us that the 20-year par yield 6m from the spot is 5.04%. The data point (20y, 7.13) on the 6m forward rate curve indicates that the 6-month yield 20 years from the spot is 7.13%. For comparison, the 6-month yield and the 20-year par yield on spot date is 3.25% and 4.95% respectively.

Since discount factor curve forms the fundamental building block for pricing and trading in both the cash and derivative markets we will begin by focusing on the methodology for constructing the discount curve from market data. Armed with the

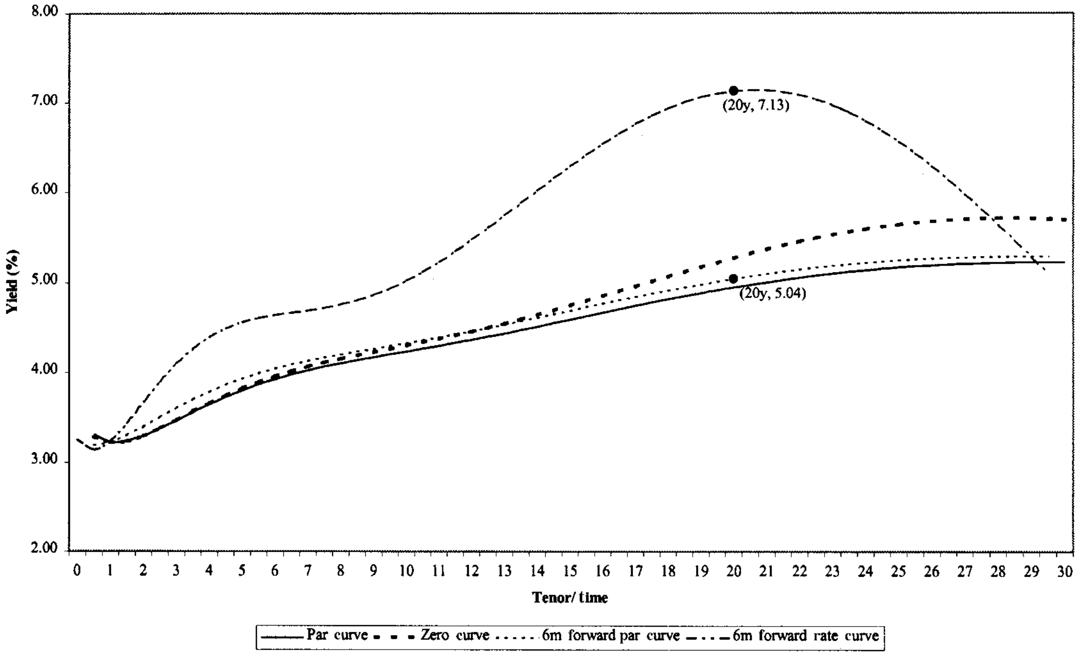


Figure 3.2 DEM par-, zero-, and forward yield curves.

knowledge of discount curve we will then devote our attention to developing other representation of market yield curve. The process for building the yield curve can be summarized in Figure 3.3.

Bootstrapping swap curve

Market participant also refers to the swap curve as the LIBOR curve. The swap market yield curve is built by splicing together the rates from market instruments that represent the most liquid instruments or dominant instruments in their tenors. At the very short end, the yield curve uses the cash deposit rates, where available the International Money Market (IMM) futures contracts are used for intermediate tenors and finally par swap rates are used for longer tenors. A methodology for building the yield curve from these market rates, referred to as *bootstrapping or zero coupon stripping*, that is widely used in the industry is discussed in this section.

The LIBOR curve can be built using the following combinations of market rates:

- Cash deposit + futures + swaps
- Cash deposit + swaps

The reason for the popularity of the bootstrapping approach is its ability to produce a no-arbitrage yield curve, meaning that the discount factor obtained from bootstrapping can recover market rates that has been used in their construction. The downside to this approach, as will be seen later, is the fact that the forward rate curve obtained from this process is not a smooth curve. While there exists methodologies to obtain smooth forward curves with the help of various fitting

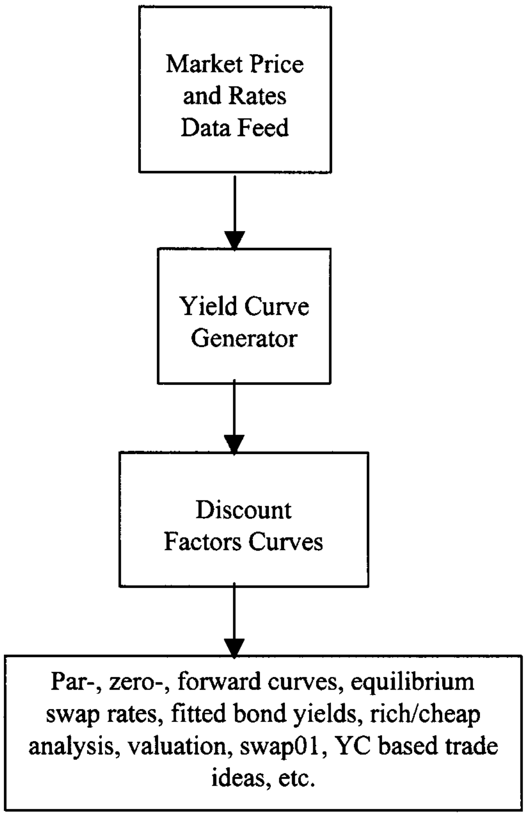


Figure 3.3 Yield curve modeling process.

algorithms they are not always preferred as they may violate the no-arbitrage constraint or have unacceptable behavior in risk calculations.

Notations

In describing the bootstrapping methodology we will adopt the following notations for convenience:

t_0	Spot date
$S(T)$:	Par swap rate quote for tenor T at spot date
$Z(T)$:	Zero coupon bond price or discount factor maturing on date T at spot date
$\alpha(t_1, t_2)$:	Accrual factor between date t_1 and t_2 in accordance to day count convention of the market (ACT/360, 30/360, 30E/360, ACT/ACT)
$F(T_1, T_2)$:	Forward rate between date T_1 and T_2 as seen from the yield curve at spot date
$P(T_1, T_2)$:	IMM futures contract price deliverable on date T_1 at spot date
$f(T_1, T_2)$:	Futures rate, calculated as 100-$P(T_1, T_2)$ at spot date
$d(T)$:	Money market cash deposit rates for maturity T at spot date

Extracting discount factors from deposit rates

The first part of the yield curve is built using the cash deposit rates quoted in the market. The interest on the deposit rate accrue on a simple interest rate basis and as such is the simplest instrument to use in generating discount curve. It is calculated using the following fundamental relationship in finance:

$$\text{Present value} = \text{future value} \times \text{discount factor}$$

The present value is the value of the deposit today and the future value is the amount that will be paid out at the maturity of the deposit. Using our notations we can rewrite this equation as

$$1 = (1 + d(T)\alpha(t_0, T) \times Z(T)$$

or equivalently,

$$Z(T) = \frac{1}{(1 + d(T)\alpha(t_0, T)} \tag{3.1}$$

The accrual factor, $\alpha(t_0, T)$, is calculated according to money market day count basis for the currency. In most currencies it is Actual/360 or Actual/365. For example, consider the deposit rate data for Germany in Table 3.1.

Table 3.1 DEM cash deposit rates data

Tenor	Bid	Accrual basis
O/N	3.35	
T/N	3.38	
T/N	3.38	
1M	3.45	
2M	3.56	Actual/360
3M	3.55	
6M	3.53	
9M	3.44	
12M	3.47	

The discount factor for 1W is

$$\frac{1}{\left(1 + 3.38\% \dfrac{7}{360}\right)} = 0.9993$$

Similarly, using expression (3.1) we can obtain the discount factor for all other dates as well. The results are shown in Table 3.2. These calculations should be performed after adjusting the maturity of cash rates for weekends and holidays where necessary.

In the above calculation the spot date is trade date plus two business days as per the convention for the DEM market and the discount factor for the spot date is defined to be 1. If instead of the spot date we define the discount factor for the trade date to be 1.0 then the above discount factor needs to be rebased using the overnight rate and tomorrow next rate. First, we can calculate the overnight discount factor as:

$$\frac{1}{\left(1 + 3.35\% \dfrac{7}{360}\right)} = 0.9999$$

Table 3.2 DEM cash deposit discount factor curve at spot date

Tenor	Maturity	Accrued days	Discount factor
Trade date	22-Oct-98		
Spot	26-Oct-98	0	1.0000
1W	02-Nov-98	7	0.9993
1M	26-Nov-98	31	0.9970
2M	28-Dec-98	63	0.9938
3M	26-Jan-99	92	0.9910
6M	26-Apr-99	182	0.9824
9M	26-Jul-99	273	0.9745
12M	26-Oct-99	365	0.9660

Next, we use the tomorrow next rate to calculate the discount factor for the spot date. The tomorrow next rate is a forward rate between trade day plus one business day to trade date plus two business day. Therefore, the discount factor for the spot date is:

$$0.9999 \times \frac{1}{\left(1 + 3.38\% \dfrac{3}{360}\right)} = 0.9996$$

The discount factors to trade date can be obtained by multiplying all the discount factors that has been previously calculated to spot date by 0.9996. This is shown in Table 3.3.

Table 3.3 DEM cash deposit discount curve at trade date

Tenor	Maturity	Accrued days	Discount factor
Trade Date	22-Oct-98	0	1.00000000
O/N	23-Oct-98	1	0.99990695
Spot	26-Oct-98	4	0.99962539
1W	02-Nov-98	11	0.99896885
1M	26-Nov-98	35	0.99666447
2M	28-Dec-98	67	0.99343628
3M	26-Jan-99	96	0.99063810
6M	26-Apr-99	186	0.98209875
9M	26-Jul-99	277	0.97421146
12M	26-Oct-99	369	0.96565188

Extracting discount factors from futures contracts

Next we consider the method for extracting the discount factor from the futures contract. The prices for IMM futures contract reflect the effective interest rate for lending or borrowing 3-month LIBOR for a specific time period in the future. The contracts are quoted on a price basis and are available for the months March, June, September and December. The settlement dates for the contracts vary from exchange to exchange. Typically these contracts settle on the third Wednesday of the month and their prices reflect the effective future interest rate for a 3-month period from the settlement date.

The relationship between the discount factor and the futures rate is given by the expression below.

$$t_0 \qquad\qquad T_1 \qquad\qquad\qquad\qquad T_2$$

$$Z(T_2) = Z(T_1)\,\frac{1}{[1 + f(T_1, T_2)\alpha(T_1, T_2)} \tag{3.2}$$

The futures rate is derived from the price of the futures contract as follows:

$$f(T_1, T_2) = \frac{100 - P(T_1, T_2)}{100}$$

Thus, with the knowledge of discount factor for date T_1 and the interest rate futures contract that spans time period (T_1, T_2) we can obtain the discount factor for date T_2.

If the next futures contract span (T_2, T_3) then we can reapply expression (3.2) and use for $Z(T_2)$ the discount factor calculated from the previous contract. In general,

$$Z(T_i) = Z(T_{i-1})\,\frac{1}{(1 + f(T_{i-1}, T_i)\alpha(T_{i-1}, T_i)}$$

An issue that arises during implementation is that any two adjacent futures contract may not always adjoin perfectly. This results in gaps along the settlement dates of the futures contract making the direct application of expression (3.2) difficult. Fortunately, this problem can be overcome. A methodology for dealing with gaps in the futures contract is discussed later.

Building on the example earlier, consider the data in Table 3.4 for 3-month Euromark futures contract in LIFFE. The settlement date is the third Wednesday of the contract expiration month. We assume that the end date for the 3-month forward period is the settlement date of the next contract, i.e. ignore existence of any gaps.

Table 3.4 DEM futures price data

Contract	Price	Implied rate (A/360 basis)	Settle date	End date	Accrued days
DEC98	96.5100	3.4900%	16-Dec-98	17-Mar-99	91
MAR99	96.7150	3.2850%	17-Mar-99	16-Jun-99	91
JUN99	96.7500	3.2500%	16-Jun-99	15-Sep-99	91
SEP99	96.7450	3.2550%	15-Sep-99	15-Dec-99	91
DEC99	96.6200	3.3800%	15-Dec-99	15-Mar-00	91
MAR00	96.6600	3.3400%	15-Mar-00	21-Jun-00	91
JUN00	96.5600	3.4400%	21-Jun-00	20-Sep-00	91
SEP00	96.4400	3.5600%	20-Sep-00	20-Dec-00	91
DEC00	96.2350	3.7650%	20-Dec-00	21-Mar-01	91
MAR01	96.1700	3.8300%	21-Mar-01	20-Jun-01	91
JUN01	96.0800	3.9200%	20-Jun-01	19-Sep-01	91
SEP01	95.9750	4.0250%	19-Sep-01	19-Dec-01	91
DEC01	95.8350	4.1650%	19-Dec-01	20-Mar-02	91
MAR02	95.7750	4.2250%	20-Mar-02	19-Jun-02	91
JUN02	95.6950	4.3050%	19-Jun-02	18-Sep-02	91
SEP02	95.6100	4.3900%	18-Sep-02	18-Dec-02	91

The price DEC98 futures contract reflects the interest rate for the 91-day period from 16 December 1998 to 17 March 1999. This can be used to determine the discount factor for 17 March 1999 using expression (3.2). However, to apply expression (3.2) we need the discount factor for 16 December 1998. While there are several approaches to identify the missing discount factor we demonstrate this example by using linear interpolation of the 1-month (26 November 1998) and 2-month (28 December 1998) cash rate. This approach gives us a cash rate of 3.5188% and discount factor for 0.99504 with respect to the spot date. The discount factor for 17 March 1998 is

$$0.9950\left(\frac{1}{1+3.49\%\dfrac{91}{360}}\right)=0.9863$$

In the absence of any gaps in the futures contract the above discount factor together with the MAR99 contract can be used determine the discount factor for 16 June 1999 and so on until the last contract. The results from these computations are shown in Table 3.5.

Table 3.5 DEM discount curve from futures prices

Date	Discount factor	Method
26-Oct-98	1.00000	Spot
16-Dec-98	0.99504	Interpolated cash rate
17-Mar-99	0.98634	DEC98
16-Jun-99	0.97822	MAR99
15-Sep-99	0.97024	JUN99
15-Dec-99	0.96233	SEP99
15-Mar-00	0.95417	DEC99
21-Jun-00	0.94558	MAR00
20-Sep-00	0.93743	JUN00
20-Dec-00	0.92907	SEP00
21-Mar-01	0.92031	DEC00
20-Jun-01	0.91148	MAR01
19-Sep-01	0.90254	JUN01
19-Dec-01	0.89345	SEP01
20-Mar-02	0.88414	DEC01
19-Jun-02	0.87480	MAR02
18-Sep-02	0.86538	JUN02
18-Dec-02	0.85588	SEP02

Extracting discount factor from swap rates

As we go further away from the spot date we either run out of the futures contract or, as is more often the case, the futures contract become unsuitable due to lack of liquidity. Therefore to generate the yield curve we need to use the next most liquid instrument, i.e. the swap rate.

Consider a par swap rate $S(t_N)$ maturing on t_N with cash flow dates $\{t_1, t_2, \ldots t_N\}$.

The cash flow dates may have an annual, semi-annual or quarterly frequency. The relationship between the par swap rate and the discount factor is summarized in the following expression:

$$1 - Z(t_N) = S(t_N) \sum_{i=1}^{N} \alpha(t_{i-1}, t_i) Z(t_i) \tag{3.3}$$

The left side of the expression represents the PV of the floating payments and the right side the PV of the fixed rate swap payments. Since a par swap rate by definition has zero net present value, the PV of the fixed and floating cash flows must to be equal. This expression can be rearranged to calculate the discount factor associated with the last swap coupon payment:

$$Z(t_N) = \frac{1 - S(t_N) \sum_{i=1}^{N-1} \alpha(t_{i-1}, t_i) Z(t_i)}{1 + \alpha(t_{N-1}, t_N) S(t_N)} \tag{3.4}$$

To apply the above expression we need to know the swap rate and discount factor associated with all but the last payment date. If a swap rate is not available then it has to be interpolated. Similarly, if the discount factors on the swap payment dates are not available then they also have to be interpolated.

Let us continue with our example of the DEM LIBOR curve. The par swap rates are given in Table 3.6. In this example all swap rates are quoted in the same frequency and day count basis. However, note that this need not be the case; for example, the frequency of the 1–3y swap rate in Australia dollar is quarterly while the rest are semi-annual. First we combine our cash curve and futures curve as shown in Table 3.7. Notice that all cash discount factors beyond 16 December 1998 have been dropped. This is because we opted to build our yield curve using the first futures contract. Even though the cash discount factors are available beyond 16 December 1998 the futures takes precedence over the cash rates.

Since we have already generated discount factor until 18 December 2002 the first relevant swap rate is the 5y rate. Before applying expression (3.3) to bootstrap the

Table 3.6 DEM swap rate data

Tenor	Swap Rate	Maturity	Frequency/basis
2Y	3.4600%	26-Oct-00	Annual, 30E/360
3Y	3.6000%	26-Oct-01	Annual, 30E/360
4Y	3.7600%	28-Oct-02	Annual, 30E/360
5Y	3.9100%	27-Oct-03	Annual, 30E/360
6Y	4.0500%	26-Oct-04	Annual, 30E/360
7Y	4.1800%	26-Oct-05	Annual, 30E/360
8Y	4.2900%	26-Oct-06	Annual, 30E/360
9Y	4.4100%	26-Oct-07	Annual, 30E/360
10Y	4.4900%	27-Oct-08	Annual, 30E/360
12Y	4.6750%	26-Oct-10	Annual, 30E/360
15Y	4.8600%	28-Oct-13	Annual, 30E/360
20Y	5.0750%	26-Oct-18	Annual, 30E/360
30Y	5.2900%	26-Oct-28	Annual, 30E/360

Table 3.7 DEM cash plus futures discount factor curve

Date	Discount factor	Source
26-Oct-98	1.00000	Spot
27-Oct-98	0.99991	Cash
02-Nov-98	0.99934	Cash
26-Nov-98	0.99704	Cash
16-Dec-98	0.99504	Interpolated cash
17-Mar-99	0.98634	Futures
16-Jun-99	0.97822	Futures
15-Sep-99	0.97024	Futures
15-Dec-99	0.96233	Futures
15-Mar-00	0.95417	Futures
21-Jun-00	0.94558	Futures
20-Sep-00	0.93743	Futures
20-Dec-00	0.92907	Futures
21-Mar-01	0.92031	Futures
20-Jun-01	0.91148	Futures
19-Sep-01	0.90254	Futures
19-Dec-01	0.89345	Futures
20-Mar-02	0.88414	Futures
19-Jun-02	0.87480	Futures
18-Sep-02	0.86538	Futures
18-Dec-02	0.85588	Futures

discount factor for 27 October 2003 we have to interpolate the discount factor for all the prior payment dates from the 'cash plus futures' curve we have so far. This is shown in Table 3.8 using exponential interpolation for discount factors.

Table 3.8 DEM 5y swap payment date discount factor from exponential interpolation

Cash flow dates	Accrual factor	Discount factor	Method
26-Oct-99	1.0000	0.96665	Exponential interpolation from cash + futures curve
26-Oct-00	1.0000	0.93412	
26-Oct-01	1.0000	0.89885	
28-Oct-02	1.0056	0.86122	
27-Oct-03	0.9972	?	

Therefore, the discount factor for 27 October 2003 is

$$\frac{1-3.91\%(1.00\times 0.9665+1.00\times 0.93412+1.00\times 0.89885+1.0056\times 0.86122)}{1+3.91\%\times 0.9972}$$

$$=0.82452$$

This procedure can be continued to derive all the discount factors. Each successive swap rate helps us identify the discount factor associated with the swap's terminal date using all discount factors we know up to that point. When a swap rate is not available, for example the 11y rate, it has to be interpolated from the other available swap rates. The results are shown in Table 3.9 below, where we apply linear interpolation method for unknown swap rates.

In many markets it may be that the most actively quoted swap rates are the annual tenor swaps. However, if these are semi-annual quotes then we may have more discount factors to bootstrap than available swap rates. For example, suppose that we have the six-month discount factor, the 1y semi-annual swap rate and 2y semi-annual swap rate. To bootstrap the 2y discount factor we need the 18 month discount factor which is unknown:

Spot1	6m	1y	18y	2y
Z_0	Z_{6m}	Z_{1y}	Z_{18M}	Z_{2Y}

A possible approach to proceed in building the yield curve is to first interpolate (possibly linear interpolation) the 18-month swap rate from the 1y and 2y swap rate. Next, use the interpolated 18-month swap rate to bootstrap the corresponding discount factor and continue onwards to bootstrap the 2y discount factor. Another alternative is to solve numerically for both the discount factors simultaneously. Let G be an interpolation function for discount factors that takes as inputs the dates and adjacent discount factors to return the discount factor for the interpolation date, that is,

$$Z_{18\mathrm{m}} = G(T_{18\mathrm{m}}, Z_{1\mathrm{y}}, Z_{2\mathrm{y}})$$

The 2y equilibrium swap is calculated as

$$S_{2\mathrm{y}} = \frac{1 - Z_{2\mathrm{y}}}{[\alpha_{\mathrm{spot},6\mathrm{m}} Z_{6\mathrm{m}} + \alpha_{6\mathrm{m},1\mathrm{y}} Z_{1\mathrm{y}} + \alpha_{1\mathrm{y},18\mathrm{y}} Z_{18\mathrm{m}} + \alpha_{18\mathrm{m},2\mathrm{y}} Z_{2\mathrm{y}}]}$$

where α's are the accrual factor according to the day count basis. Substituting the relationship for the 18m discount factor we get

$$S_{2\mathrm{y}} = \frac{1 - Z_{2\mathrm{y}}}{[\alpha_{\mathrm{spot},6\mathrm{m}} Z_{6\mathrm{m}} + \alpha_{6\mathrm{m},1\mathrm{y}} Z_{1\mathrm{y}} + \alpha_{1\mathrm{y},18\mathrm{y}} G(T_{18\mathrm{m}}, Z_{1\mathrm{y}}, Z_{2\mathrm{y}}) + \alpha_{18\mathrm{m},2\mathrm{y}} Z_{2\mathrm{y}}]}$$

The above expression for the 2y swap rate does not require the 18m discount factor as input. We can then use a numerical algorithm such as Newton–Raphson to determine the discount factor for 2y, $Z_{2\mathrm{y}}$, that will ensure that the equilibrium 2y swap rate equals the market quote for the 2y swap rate.

Finally, putting it all together we have the LIBOR discount factor curve for DEM in Table 3.10. These discount factors can be used to generate the par swap curve, forward rate curve, forward swap rate curve or discount factors for pricing various structured products.

The forward rate between any two dates T_1 and T_2 as seen from the yield curve on spot date t_0 is

$$F(T_1, T_2) = \left(\frac{Z(T_1)}{Z(T_2)} - 1 \right) \frac{1}{\alpha(T_1, T_2)}$$

Table 3.9 DEM discount factors from swap rates

Tenor	Maturity	Swap rate	Accrual factor	Discount factor
5y	27-Oct-03	3.9100%	0.9972	0.82452
6y	26-Oct-04	4.0500%	0.9972	0.78648
7y	26-Oct-05	4.1800%	1.0000	0.74834
8y	26-Oct-06	4.2900%	1.0000	0.71121
9y	26-Oct-07	4.4100%	1.0000	0.67343
10y	27-Oct-08	4.4900%	1.0028	0.63875
11y	26-Oct-09	4.5824%	0.9972	0.60373
12y	26-Oct-10	4.6750%	1.0000	0.56911
13y	26-Oct-11	4.7366%	1.0000	0.53796
14y	26-Oct-12	4.7981%	1.0000	0.50760
15y	28-Oct-13	4.8600%	1.0056	0.47789
16y	27-Oct-14	4.9029%	0.9972	0.45122
17y	26-Oct-15	4.9459%	0.9972	0.42543
18y	26-Oct-16	4.9889%	1.0000	0.40045
19y	26-Oct-17	5.0320%	1.0000	0.37634
20y	26-Oct-18	5.0750%	1.0000	0.35309
21y	28-Oct-19	5.0966%	1.0056	0.33325
22y	26-Oct-20	5.1180%	0.9944	0.31445
23y	26-Oct-21	5.1395%	1.0000	0.29634
24y	26-Oct-22	5.1610%	1.0000	0.27900
25y	26-Oct-23	5.1825%	1.0000	0.26240
26y	28-Oct-24	5.2041%	1.0056	0.24642
27y	27-Oct-25	5.2256%	0.9972	0.23127
28y	26-Oct-26	5.2470%	0.9972	0.21677
29y	26-Oct-27	5.2685%	1.0000	0.20287
30y	26-Oct-28	5.2900%	1.0000	0.18959

This can be calculated from the discount factor curve after applying suitable interpolation method to identify discount factors not already available. For example, using exponential interpolation we find that the discount factor for 26 April 1999 is 0.98271 and that for 26 October 1999 is 0.96665. The forward rate between 26 April 1999 and 26 October 1999 is

$$\left(\frac{0.98271}{0.96665}-1\right)\frac{1}{\left(\dfrac{183}{360}\right)}=3.27\%$$

The 6m forward rate curve for DEM is shown in Table 3.11. Forward rate curves are important for pricing and trading a range of products such as swaps, FRAs, Caps and Floors and a variety of structured notes.

Similarly, the equilibrium par swap rate and forward swap rates can be calculated from the discount from

$$S(t_s, t_{s+N})=\frac{Z(t_s)-Z(t_{s+N})}{\sum_{i=1}^{N}\alpha(t_{s+i-1}, t_{s+i})Z(t_{s+i})}$$

Table 3.10 DEM swap or LIBOR bootstrapped discount factor curve

Cash		Futures		Swaps	
Date	Discount factor	Date	Discount factor	Date	Discount factor
26-Oct-98	1.00000	17-Mar-99	0.98634	27-Oct-03	0.82452
27-Oct-98	0.99991	16-Jun-99	0.97822	26-Oct-04	0.78648
02-Nov-98	0.99934	15-Sep-99	0.97024	26-Oct-05	0.74834
26-Nov-98	0.99704	15-Dec-99	0.96233	26-Oct-06	0.71121
16-Dec-98	0.99504	15-Mar-00	0.95417	26-Oct-07	0.67343
		21-Jun-00	0.94558	27-Oct-08	0.63875
		20-Sep-00	0.93743	26-Oct-09	0.60373
		20-Dec-00	0.92907	26-Oct-10	0.56911
		21-Mar-01	0.92031	26-Oct-11	0.53796
		20-Jun-01	0.91148	26-Oct-12	0.50760
		19-Sep-01	0.90254	28-Oct-13	0.47789
		19-Dec-01	0.89345	27-Oct-14	0.45122
		20-Mar-02	0.88414	26-Oct-15	0.42543
		19-Jun-02	0.87480	26-Oct-16	0.40045
		18-Sep-02	0.86538	26-Oct-17	0.37634
		18-Dec-02	0.85588	26-Oct-18	0.35309
				28-Oct-19	0.33325
				26-Oct-20	0.31445
				26-Oct-21	0.29634
				26-Oct-22	0.27900
				26-Oct-23	0.26240
				28-Oct-24	0.24642
				27-Oct-25	0.23127
				26-Oct-26	0.21677
				26-Oct-27	0.20287
				26-Oct-28	0.18959

$S(t_s, t_{s+N})$ is the equilibrium swap rate starting at time t_s and ending at time t_{s+N}. If we substitute zero for s in the above expression we get the equilibrium par swap rate. Table 3.12 shows the par swap rates and forward swap rates from our discount curve. For comparison we also provide the market swap rates. Notice that the market swap rate and the equilibrium swap rate computed from the bootstrapped does not match until the 5y swaps. This is due to the fact that we have used the futures contract to build our curve until 18 December 2002. The fact that the equilibrium swap rates are consistently higher than the market swap rates during the first 4 years is not surprising since we have used the futures contract without convexity adjustments (see below).

Curve stitching

Cash rates and futures contracts

In building the yield curve we need to switch from the use of cash deposit rates at the near end of the curve to the use of futures rates further along the curve. The choice of the splice date when cash deposit rate is dropped and futures rate is picked up is driven by the trader's preference, which in turn depends on the instruments

Table 3.11 DEM 6 month forward rates from discount factor curve

Date	Discount factor	Accrual factor	6m forward rate
26-Oct-98	1.00000		
26-Apr-99	0.98271	0.5083	3.27%
26-Oct-99	0.96665	0.5083	3.35%
26-Apr-00	0.95048	0.5083	3.44%
26-Oct-00	0.93412	0.5056	3.73%
26-Apr-01	0.91683	0.5083	3.93%
26-Oct-01	0.89885	0.5056	4.16%
26-Apr-02	0.88035	0.5083	4.32%
26-Oct-02	0.86142	0.5056	4.32%
26-Apr-03	0.84300	0.5083	4.38%
26-Oct-03	0.82463	0.5083	4.64%
26-Apr-04	0.80562	0.5083	4.79%
26-Oct-04	0.78648	0.5056	4.89%
26-Apr-05	0.76749	0.5083	5.03%
26-Oct-05	0.74834	0.5056	5.02%
26-Apr-06	0.72981	0.5083	5.14%
26-Oct-06	0.71121	0.5056	5.39%
26-Apr-07	0.69235	0.5083	5.52%
26-Oct-07	0.67343	0.5083	5.21%
26-Apr-08	0.65605	0.5083	5.30%
26-Oct-08	0.63884	0.5056	5.60%
26-Apr-09	0.62125		

Table 3.12 DEM equilibrium swap rates from discount factor curve

Tenor	Market swap rate	Equilibrium swap rate	6m forward start swap rate
2y	3.4600%	3.4658%	3.5282%
3y	3.6000%	3.6128%	3.7252%
4y	3.7600%	3.7861%	3.8917%
5y	3.9100%	3.9100%	4.0281%
6y	4.0500%	4.0500%	4.1678%
7y	4.1800%	4.1800%	4.2911%
8y	4.2900%	4.2900%	4.4088%
9y	4.4100%	4.4100%	4.5110%
10y	4.4900%	4.4900%	4.5970%

that will be used to hedge the position. However, for the methodology to work it is necessary that the settlement date of the first futures contract (referred to as a 'stub') lie before the maturity date of the last cash deposit rate. If both cash deposit rate and futures rates are available during any time period then the futures price takes precedence over cash deposit rates.

Once the futures contracts have been identified all discount factors until the last futures contract is calculated using the bootstrapping procedure. To bootstrap the curve using the futures contract, the discount factor corresponding to the first futures delivery date (or the 'stub') is necessary information. Clearly, the delivery date of the first futures contract selected may not exactly match the maturity date of one of the cash deposit rates used in the construction. Therefore, an important issue in the curve building is the method for identifying discount factor corresponding to the first futures contract or the stub rate.

The discount factor for all subsequent dates on the yield curve will be affected by the stub rate. Hence the choice of the method for interpolating the stub rate can have a significant impact on the final yield curve. There are many alternative approaches to tackle this problem, all of which involves either interpolation method or fitting algorithm. We will consider a few based on the example depicted in Figure 3.4.

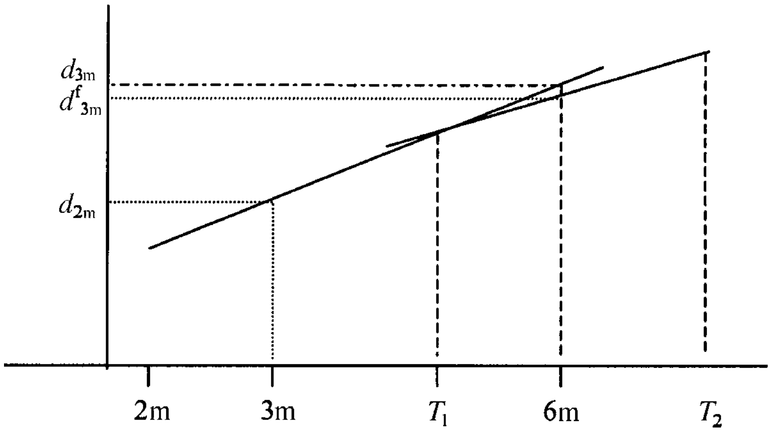


Figure 3.4 Cash-futures stitching.

The first futures contract settles on date T_1 and spans from T_1 to T_2. One alternative is to interpolate the discount factor for date T_1 with the 3m and 6m discount factor calculated from expression (3.1). The second alternative is to directly interpolate the 3m and 6m cash deposit rates to obtain the stub rate and use this rate to compute the discount factor using expression (3.1). The impact of applying different interpolation method on the stub is presented in Table 3.13.

Table 3.13 DEM stub rate from different interpolation methods

Data	Interpolation method	Discount factor	Cash rate (stub rate)
Spot date: 26-Oct-98	Linear cash rate	0.99504	3.5188%
1m (26-Nov-99): 3.45%	Exponential DF	0.99504	3.5187%
2m (28-Dec-98): 3.56%	Geometric DF	0.99502	3.5341%
Basis: Act/360	Linear DF	0.99502	3.5332%
Stub: 16-Dec-98			

The exponential interpolation of discount factor and linear interpolation of the cash rate provide similar results. This is not surprising since exponential interpolation of discount factor differs from the linear interpolation of rate in that it performs a linear interpolation on the equivalent continuously compounded yield.

To see the impact of the stitching method on the forward around the stub date we have shown the 1m forward rate surrounding the stub date in Figure 3.5 using different interpolation methods. Also, notice that the forward rate obtained from the bootstrapping approach is not smooth.

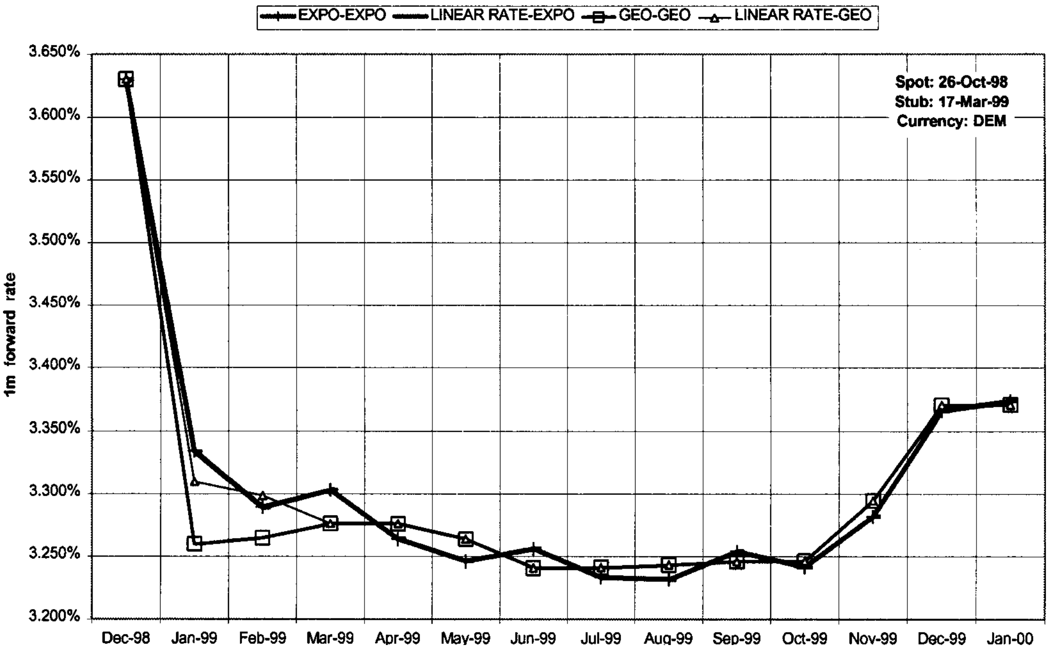


Figure 3.5 Six-month forward rate from different interpolation.

Going back to Figure 3.4, once the 3m and 6m rate has been used for interpolating the discount factor for T_1 all cash rates beyond the 3m is dropped. The yield curve point after the 3m is T_2 and all subsequent yield curve point follow the futures contract (i.e. 3 months apart). Since the 6m has been dropped, the 6m cash rate interpolated from the constructed yield curve may not match the 6m cash rate that was initially used.

This immediately raises two issues. First, what must be the procedure for discounting any cash flow that occurs between T_1 and 6m? Second, what is the implication of applying different methods of interpolation of the discount curve on the interpolated value of the 6m cash rate versus the market cash rate. Further, is it possible to ensure that the interpolated 6m cash rate match the market quoted 6m cash rate?

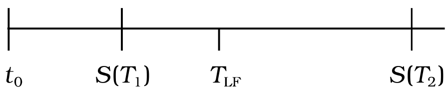
If recovering the correct cash rate from the yield curve points is an important criterion then the methodologies discussed earlier would not be appropriate. A possible way to handle this issue is to change the method for obtaining the stub rate by directly solving for it. We can solve for a stub rate such that 6m rate interpolated from the stub discount factor for T_1 and discount factor for T_2 match the market.

In some markets such as the USD the short-term swap dealers and active cash dealers openly quote the stub rate. If so then it is always preferable to use the market-quoted stub rate and avoid any interpolation.

Futures strip and swap rates

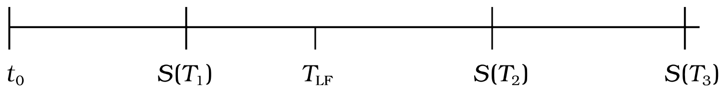
To extend the curve beyond the last futures contract we need the swap rates. The required swap rate may be available as input data or may need to be interpolated.

Consider the following illustration where the last futures contract ends on T_{LF}. If $S(T_2)$, the swap rate that follows the futures contract, is available as input then we can apply expression (3.3) to derive the discount factor for date T_2. This is a straightforward case that is likely to occur in currencies such as DEM with annual payment frequency:



The discount factor corresponding to all payment dates except the last will need to be interpolated.

In the next scenario depicted below suppose that swap rate $S(T_2)$ is not available from the market:



We have two choices. Since we have market swap rate, $S(T_1)$ and $S(T_3)$, we could use these rates to interpolate $S(T_2)$. Alternatively, since we have discount factor until date T_{LF} we can use them to calculate a equilibrium swap rate, $S'(T_1)$, for tenor T_1. The equilibrium swap rate $S'(T_1)$ and market swap rate $S(T_3)$ can be used to interpolate the missing swap rate $S(T_2)$.

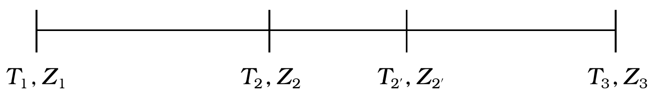
In some circumstances we may have to interpolate swap rates with a different basis and frequency from the market-quoted rates. In these cases we recommend that the market swap rates be adjusted to the same basis and frequency as the rate we are attempting to interpolate. For example, to get a 2.5y semi-annual, 30E/360 swap rate from the 2y and 3y annual, 30E/360 swap rate, the annual rates can be converted to an equivalent semi-annual rate as follows:

$$S_{\text{semi-annual}} = 2 \times [(1 + S_{\text{annual}})^{1/2} - 1]$$

Handling futures gaps and overlaps

In the construction of the discount factor using expression (3.2), we are implicitly assuming that all futures contract are contiguous with no gaps or overlaps. However, from time to time due to holidays it is possible that the contracts do not line up exactly.

Consider the illustration below:



The first futures contract span from T_1 to T_2 while the next futures contract span from $T_{2'}$ to T_3 resulting in a gap. An approach to resolve this issue is as follows. Define $G(T, Z_a, Z_b)$ to be a function representing the interpolation method (e.g. exponential) for discount factor. We can apply this to interpolate Z_2 as follows:

$$Z_{2'} = G(T_{2'}, Z_2, Z_3) \tag{3.4}$$

From expression (3.2) we also know that

$$Z_3 = Z_{2'} \frac{1}{[1 + f(T_{2'}, T_3)\alpha(T_{2'}, T_3)]} \equiv \Psi(f, Z_{2'}) \tag{3.5}$$

Combining expressions (3.4) and (3.5) we have

$$Z_3 = \Psi(f, G(T_{2'}, Z_2, Z_3)) \tag{3.6}$$

All variables in expression (3.6) are known except Z_3. This procedure can be applied to find the discount factor Z_3 without the knowledge of $Z_{2'}$ caused by gaps in the futures contract.

For example, if we adopt exponential interpolation

$$Z_{2'} = G(T_{2'}, Z_2, Z_3) = Z_2^{\frac{(T_{2'} - t_0)}{(T_2 - t_0)}\lambda} Z^{\frac{(T_{2'} - t_0)}{(T_3 - t_0)}(1-\lambda)}$$

and

$$\lambda = \frac{(T_3 - T_2)}{(T_3 - T_2)}$$

Therefore,

$$Z_3 = Z_2^{\frac{(T_{2'} - t_0)}{(T_2 - t_0)}\lambda} Z^{\frac{(T_{2'} - t_0)}{(T_3 - t_0)}(1-\lambda)} \frac{1}{[1 + f(T_{2'}, T_3)\alpha(T_{2'}, T_3)]}$$

This can be solved analytically for Z_3. Specifically,

$$Z_3 = \text{Exp}\left\{ \frac{\ln\left(Z_2^{\frac{(T_{2'} - t_0)}{(T_2 - t_0)}\lambda} \dfrac{1}{[1 + f(T_{2'}, T_3)\alpha(T_{2'}, T_3)]} \right)}{\left(1 - \dfrac{(T_{2'} - t_0)}{(T_3 - t_0)}(1-\lambda) \right)} \right\} \tag{3.7}$$

As an example consider an extreme case where one of the futures price is entirely missing. Suppose that we know the discount factor for 16 June 1999 and the price of SEP99 futures contract. The JUN99 contract is missing. Normally we would have used the JUN99 contract to derive the discount factor for 15 September 1999 and then use the SEP99 contract to obtain the discount factor for 15 December 1999 (Table 3.14).

Table 3.14

Contract	Price	Implied rate (A/360 basis)	Settle date	End date	Discount factor
Spot				26-Oct-98 (t_0)	1.0000
				16-Jun-99 (T_2)	0.9782
JUN99	Missing	Missing	16-Jun-99	15-Sep-99 ($T_{2'}$)	N/A
SEP99	96.7450	3.2550%	15-Sep-99	15-Dec-99 (T_3)	?

In this example we can apply expression (3.7) to obtain the discount factor for 15 December 1999:

$$Z_{15\text{-Dec-99}} = \exp\left\{\frac{\ln\left(Z_2^{\frac{0.8877}{0.6384}\times 0.5}\dfrac{1}{[1+3.255\%\times 91/360]}\right)}{\left(1-\left(\dfrac{0.8877}{1.1370}\right)(1-0.5)\right)}\right\}$$

$$= 0.96218$$

Earlier when we had the price for the JUN99 contract the discount factor for 15 December1999 was found to be 0.96233.

Solution to overlaps are easier. If the futures contracts overlap (i.e. $T_{2'} < T_2$) then the interpolation method can be applied to identify the discount factor $Z_{2'}$ corresponding to the start of the next futures contract.

Futures convexity adjustment

Both futures and FRA are contracts written on the same underlying rate. At the expiration of the futures contract, the futures rate and the forward rate will both be equal to the then prevailing spot LIBOR rate. However, these two instruments differ fundamentally in the way they are settled. The futures contracts are settled daily whereas the FRAs are settled at maturity. As a result of this difference in the settlement procedure the daily changes in the value of a position in futures contract and that of a position in FRA to anticipated changes in the future LIBOR rate are not similar. The futures contract react linearly to changes in the future LIBOR rate while the FRA reacts non-linearly. This convexity effect creates an asymmetry in the gains/losses between being long or short in FRA and hedging them with futures contracts. To be more precise, there is an advantage to being consistently short FRA and hedging them with short futures contracts. This is recognized by the market and reflected in the market price of the futures contract. The convexity effect implies that the forward rate obtained from the futures price will be high. Since the futures rate and forward rate converge as we approach the maturity date, the futures rate must drift downwards. Hence while building the LIBOR yield curve it is important that the forward rates implied from the futures price be adjusted (downwards) by the drift.

In most markets the drift adjustments tend to be fairly small for futures contracts that expire within one year from the spot date, but can get progressively larger beyond a year. Consider an N futures contract for periods (t_i, t_{i+1}), $i = 1, 2, \ldots N$ and $(t_{i+1} - t_i) = \Delta t$. A simple method to calculate the drift adjustments for the kth futures contract is given below:[1]

$$\mu_k = \sum_{i=1}^{k} f(t_i, t_{i+1})\rho_{fZ}\sigma_{f(t_i, t_{i+1})}\sigma_{Z(t_{i+1})}\Delta t$$

where $f(t_i, t_{i+1})$ is the futures rate for the period t_i to t_{i+1}, $\sigma_{Z(t_{i+1})}$ is the volatility of the zero coupon bond maturing on t_{i+1}, $\sigma_{f(t_i, t_{i+1})}$ is the volatility of the forward rate for the corresponding period and ρ_{fZ} is the correlation between the relevant forward rate and zero coupon bond price. The kth forward rate can be calculated from the futures contract as follows.

$$F(t_k, t_{k+1}) = f(t_k, t_{k+1}) + \mu_k$$

Since the correlation between the forward rate and the zero coupon price is expected to be negative the convexity adjustment would result in the futures rate being adjusted downwards.

We demonstrate the calculations for the convexity adjustment in the table below.

Table 3.15

Contract	Date	Futures rate (1)	Futures rate volatility (2)	Zero coupon volatility (3)	Correlation n (4)	Δt (5)	Drift (bp) = (1) × (2) × (3) × (4) × (5) (6)	Cumulative drift or convexity bias (bp) (7)	Convexity adjusted futures rate (8) = (1) + (7)
SPOT	26-Oct-98								
DEC98	16-Dec-98	3.49%	5%	0.25%	−0.99458	0.25	−0.0108	−0.01	3.49%
MAR99	17-Mar-99	3.29%	12%	0.75%	−0.98782	0.25	−0.0728	−0.08	3.28%
JUN99	16-Jun-99	3.25%	14%	1.25%	−0.97605	0.25	−0.1383	−0.22	3.25%
SEP99	15-Sep-99	3.26%	18%	1.50%	−0.96468	0.25	−0.2112	−0.43	3.25%
DEC99	15-Dec-99	3.38%	20%	2.00%	−0.95370	0.25	−0.3212	−0.75	3.37%
MAR99	15-Mar-00	3.34%	15%	2.25%	−0.94228	0.27	−0.2850	−1.04	3.33%
JUN00	21-Jun-00								

Typically the convexity bias is less that 1 basis point for contracts settling within a year from the spot. Between 1 year and 2 years the bias may range from 1 basis point to 4 basis point. For contracts settling beyond 2 years the bias may be as high as 20 basis point – an adjustment that can no longer be ignored.

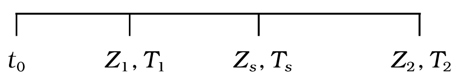
Interpolation

The choice of interpolation algorithm plays a significant role in the process of building the yield curve for a number of reasons: First, since the rates for some of the tenors is not available due to lack of liquidity (for example, the 13-year swap rate) these missing rates need to be determined using some form of interpolation algorithm. Second, for the purposes of pricing and trading various instruments one needs the discount factor for any cash flow dates in the future. However, the bootstrapping methodology, by construction, produces the discount factor for specific maturity dates based on the tenor of the interest rates used in the construction process. Therefore, the discount factor for other dates in the future may need to be identified by adopting some interpolation algorithm. Finally, as discussed earlier, the discount factor corresponding to the stub will most of the time require application of interpolation algorithm. Similarly, the futures and swap rates may also need to be joined together with the help of interpolation algorithm.

The choice of the interpolation algorithm is driven by the requirement to balance the need to control artificial risk spillage (an important issue for hedging purposes) against the smoothness of the forward curve (an important issue in the pricing of exotic interest rate derivatives).

Discount factor interpolation

Consider the example described below:



(a) Linear interpolation

The linear interpolation of discount factor for date T_s is obtained by fitting a straight line between the two adjacent dates T_1 and T_2. According to linear interpolation, the discount factor for date T_s is:

$$Z_s = Z_1 + \frac{Z_2 - Z_1}{T_2 - T_1}(T_s - T_1)$$

or

$$Z_s = \left(\frac{T_2 - T_s}{T_2 - T_1}\right) Z_1 + \left(\frac{T_s - T_1}{T_2 - T_1}\right) Z_2$$

Linear interpolation of the discount factor is almost never due to the non-linear shape of the discount curve, but the error from applying it is likely to be low in the short end where there are more points in the curve.

(b) Geometric (log-linear) Interpolation

The geometric or log linear interpolation of discount factor for date T_s is obtained by applying a natural logarithm transformation to the discount factor function and then performing a linear interpolation on the transformed function. To recover the interpolated discount factor we take the exponent of the interpolated value as shown below:

$$\ln(Z_s) = \left(\frac{T_2 - T_s}{T_2 - T_1}\right) \ln(Z_1) + \left(\frac{T_s - T_1}{T_2 - T_1}\right) \ln(Z_2)$$

or

$$Z_s = \exp\left(\left(\frac{T_2 - T_s}{T_2 - T_1}\right) \ln(Z_1) + \left(\frac{T_s - T_1}{T_2 - T_1}\right) \ln(Z_2)\right)$$

or

$$Z_s = Z_1^{\left(\frac{T_2 - T_s}{T_2 - T_1}\right)} Z_2^{\left(\frac{T_s - T_1}{T_2 - T_1}\right)}$$

(c) Exponential interpolation

The continuously compounded yield can be calculated from the discount factor as follows:

$$y_1 = -\frac{1}{(T_1 - t_0)} \ln(Z_1) \qquad \text{and} \quad y_2 = -\frac{1}{(T_2 - t_0)} \ln(Z_2)$$

To calculate the exponential interpolated discount factor we first perform a linear interpolation of the continuously compounded yields as follows:

$$y_s = y_1 \frac{(T_2 - T_s)}{(T_2 - T_1)} + y_2 \frac{(T_s - T_1)}{(T_2 - T_1)}$$

Next we can substitute for yield y_1 and y_2 to obtain:

$$y_s = -\frac{1}{(T_1 - t_0)} \ln(Z_1)\lambda - \frac{1}{(T_2 - t_0)} \ln(Z_2)(1-\lambda)$$

where

$$\lambda = \frac{(T_2 - T_s)}{(T_2 - T_1)}$$

The exponentially interpolated value for date T_s is

$$Z_s = \exp(-y_s(T_s - t_0))$$

or

$$Z_s = Z_1^{\frac{(T_s - t_0)}{(T_1 - t_0)}\lambda} Z_2^{\frac{(T_s - t_0)}{(T_2 - t_0)}(1-\lambda)}$$

Interpolation example

Consider the problem of finding the discount factor for 26 February 1999 using the data in Table 3.16.

Table 3.16 Discount factor interpolation data

Date	Days to spot	Discount factor
26-Oct-98	0	1.00000
26-Jan-99	92	0.99101
26-Feb-99	123	?
26-Apr-99	182	0.98247

Linear interpolation:

$$Z_{26\text{-Feb-99}} = \left(\frac{182-123}{182-92}\right)0.99101 + \left(\frac{123-92}{182-92}\right)0.98247$$

$$= 0.98806$$

Geometric interpolation:

$$Z_{26\text{-Feb-99}} = 0.99101^{\left(\frac{182-123}{182-92}\right)} 0.98247^{\left(\frac{123-92}{182-92}\right)}$$

$$= 0.98805$$

Exponential interpolation:

$$\lambda = \frac{(182-123)}{(182-92)}$$

$$= 0.6555$$

$$Z_s = 0.99101^{\frac{(123-0)}{(92-0)}0.6555} 0.98247^{\frac{(123-0)}{(182-0)}(1-0.6555)}$$

$$= 0.988039$$

(d) Cubic interpolation

Let $\{t = t_0, t_1, t_2, \ldots t_n = T\}$ be a vector of yield curve point dates and $Z = \{Z_0, Z_1, Z_2, \ldots Z_n\}$ be the corresponding discount factors obtained from the bootstrapping process. Define

$$Z_i = a_i + b_i t_i + c_i t_i^2 + d_i t_i^3$$

to be a cubic function defined over the interval $[t_i, t_{i+1}]$. A cubic spline function is a number of cubic functions joined together smoothly at a number of knot points. If the yield curve points $\{t, t_1, t_2, \ldots T\}$ are defined to be knot points, then coefficients of the cubic spline function defined over the interval $[t, T]$ can be obtained by imposing the following constraints:

$$\begin{cases} Z_i = a_i + b_i t_i + c_i t_i^2 + d_i t_i^3 & i = 0 \text{ to } n-1;\ n \text{ constraints} \\ Z_{i+1} = a_i + b_i t_{i+1} + c_i t_{i+1}^2 + d_i t_{i+1}^3 & i = 0 \text{ to } n-1;\ n \text{ constraints} \\ b_i + 2c_i t_i + 3d_i t_i^2 = b_{i+1} + 2c_{i+1} t_i + 3d_{i+1} t_i^2 & i = 0 \text{ to } n-2;\ n-2 \text{ constraints} \\ 2c_i + 6d_i t_i = 2c_{i+1} + 6d_{i+1} t_i & i = 0 \text{ to } n-2;\ n-2 \text{ constraints} \end{cases}$$

The first sets of n constraints imply that the spline function fit the knot points exactly. The second sets of n constraints require that the spline function join perfectly at the knot point. The third and the fourth sets of constraints ensure that the first and second derivatives match at the knot point. We have a $4n$ coefficient to estimate and $4n$-2 equation so far. The two additional constraints are specified in the form of end point constraints. In the case of natural cubic spline these are that the second derivative equals zero at the two end points, i.e.

$$2c_i + 6d_i t_i = 0 \qquad i = 0 \text{ and } n$$

The spline function has the advantage of providing a very smooth curve.

In Figure 3.6 we present the discount factor and 3-month forward rate derived from exponential and cubic interpolation. Although the discount curves in both interpolation seem similar; comparison of the 3-month forward rate provides a clearer picture of the impact of interpolation technique. The cubic spline produces a smoother forward curve.

Swap rate interpolation

As in the discount curve interpolation, the swap rate for missing tenor can be interpolated using the methods discussed earlier for the discount factor. The exponential or geometric interpolation is not an appropriate choice for swap rate. Of the remaining methods linear interpolation is the most popular.

In Figure 3.7 we compare the swap rate interpolated from linear and cubic splines for GBP. The difference between the rate interpolated by linear and cubic spline ranges from $+0.15$ bp to -0.25 basis points. Compared to the swap rate from linear interpolation, the rate from cubic spline more often higher, particularly between 20y and 30y tenors. Unfortunately, the advantage of the smooth swap rate curve from the cubic spline is overshadowed by the high level of sensitivity exhibited by the

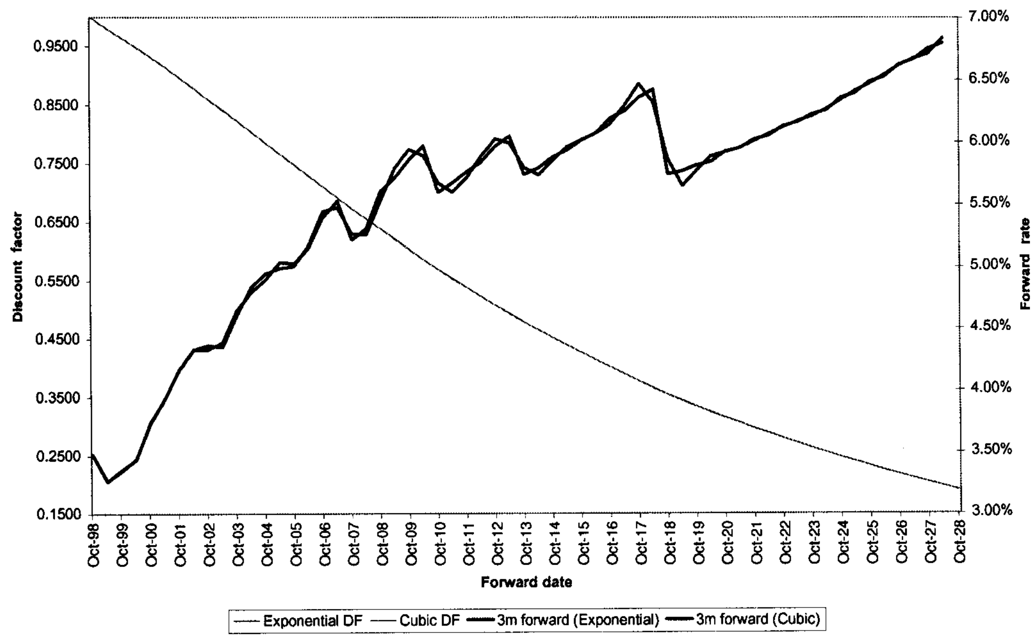


Figure 3.6 Forward rate and discount factor from cubic spline and exponential interpolation.

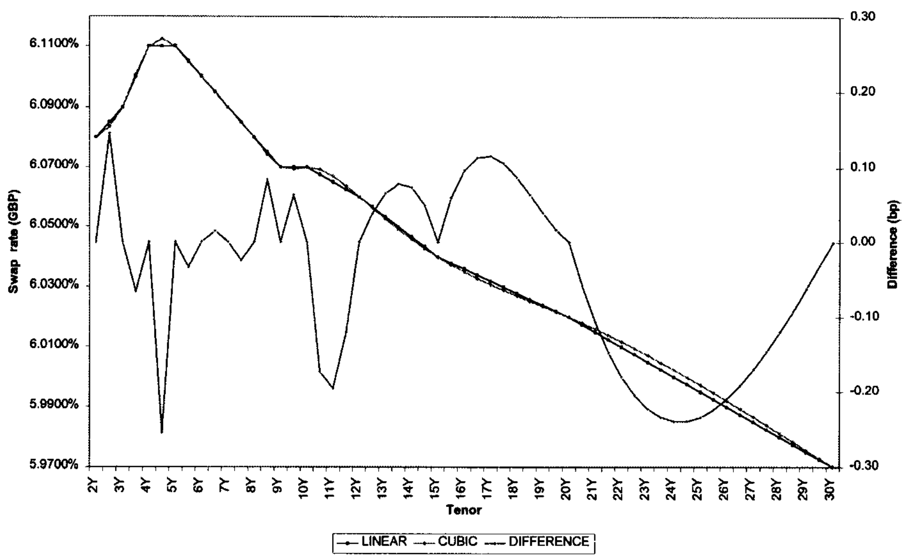


Figure 3.7 Linear and cubic spline swap rate interpolation.

method to knot point data. This can give rise to artificial volatility with significant implications for risk calculations.

Figure 3.8 shows the changes in the interpolated swap rate (DEM) for all tenors corresponding to a 1 basis point change in the one of the swap tenors. A 1 basis point change in one swap rate can change the interpolated swap rates for all tenors irrespective of their maturities! That is, all else being equal, the effects of a small change in one swap rate is not localized. For example, a 1 bp shift in the 2y swap rate results in a −0.07 bp shift in the 3.5y swap rate and a 1 bp shift in the 5y swap rate results in a −0.13 bp change in the 3.5y swap rate. This is an undesirable property of cubic spline interpolation, and therefore not preferred in the market.

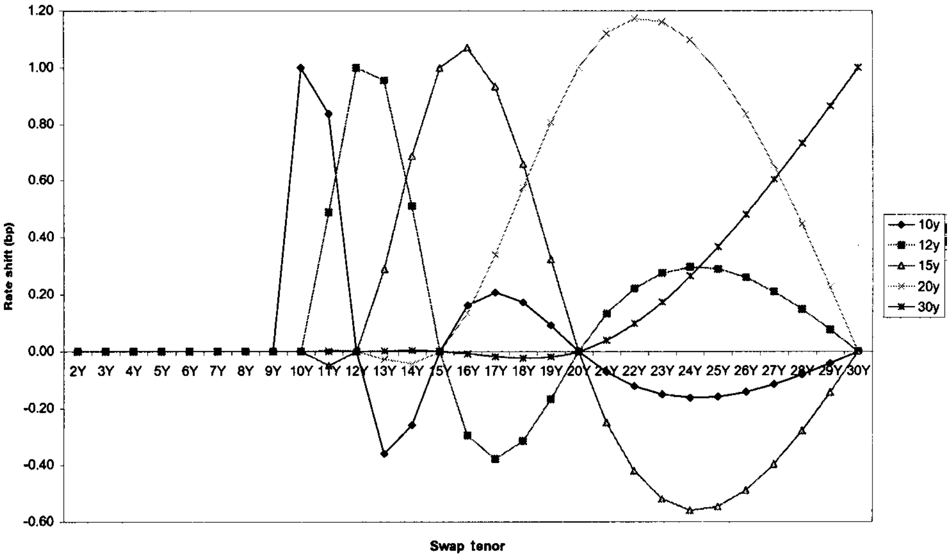


Figure 3.8 Sensitivity of cubic spline interpolation.

Figure 3.9 displays the implications of applying linear interpolation. The interpolation method, while not smooth like the cubic spline, does keep the impact of a small change in any swap rate localized. The effect on swap rates outside the relevant segment is always zero. This property is preferred for hedge calculations.

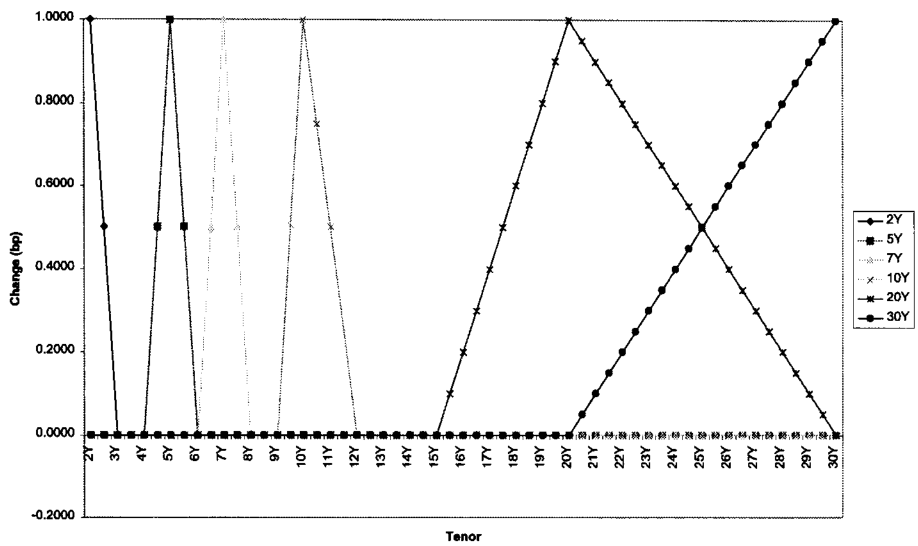


Figure 3.9 Sensitivity of linear interpolation.

Government Bond Curve

The bond market differs from the swap market in that the instruments vary widely in their coupon levels, payment dates and maturity. While in principle it is possible to follow the swap curve logic and bootstrap the discount factor, this approach is not recommended. Often the motivation for yield curve construction is to identify bonds that are trading rich or cheap by comparing them against the yield curve. Alternatively, one may be attempting to develop a time series data of yield curve for use in econometric modeling of interest rates. Market factors such as liquidity effect and coupon effect introduce noise that makes direct application of market yields unsuitable for empirical modeling. In either application the bootstrapping approach that is oriented towards guaranteeing the recovery of market prices will not satisfy our objective. Therefore the yield curve is built by applying statistical techniques to market data on bond price that obtains a smooth curve.

Yield curve models can be distinguished based on those that fit market yield and those that fit prices. Models that fit yields specify a functional form for the yield curve and estimate the coefficient of the functions using market data. The estimation procedure fits the functional form to market data so as to minimize the sum of squared errors between the observed yield and the fitted yield. Such an approach while easy to implement is not theoretically sound. The fundamental deficiency in this approach is that it does not constrain the cash flows occurring on the same date to be discounted at the same rate.

Models that fit prices approach the problem by specifying a functional form for the discount factor and estimate the coefficient using statistical methods. Among the models that fit prices there is also a class of models that treat forward rate as the fundamental variable and derive the implied discount function. This discount function is estimated using the market price data.

In this section we limit our discussion to the later approach that was pioneered by McCulloch (1971). This approach is well accepted, although there is no agreement among the practitioners on the choice of the functional form for discount factor. There is a large volume of financial literature that describes the many ways in which this can be implemented. The discussions have been limited to a few approaches to provide the reader with an intuition into the methodology. A more comprehensive discussion on this topic can be found in papers listed in the References.

Parametric approaches

The dirty price of a bond is simply the present value of its future cash flows. The dirty price of a bond with N coupon payments and no embedded options can be expressed as:

$$P(T_N) + A(T_N) = \sum_{i=1}^{N} c_N Z(T_i) + F_N Z(T_N) \tag{3.8}$$

where $P(T_N)$ is the clean price of a bond on spot date t_0 and maturing on T_N, $A(T_N)$ is its accrued interest, c_N is the coupon payment on date t, $Z(T_i)$ is the discount factor at date t and F_N is the face value or redemption payment of the bond. The process of building the yield curve hinges on identifying the discount factors corresponding to the payment dates. In the swap market we obtained this from bootstrapping the cash, futures and swap rates. In contrast, in the bond market we assume it to be one of the many functions available in our library of mathematical function and then estimate the function to fit the data.

While implementing this approach two factors must be kept in mind. First, the discount factor function selected to represent the present value factor at different dates in the future must be robust enough to fit any shape for the yield curve. Second, it must satisfy certain reasonable boundary conditions and characteristics. The discount curve must be positive monotonically non-increasing to avoid negative forward rates. Mathematically, we can state these conditions as

(a) $Z(0) = 1$
(b) $Z(\infty) = 0$
(c) $Z(T_i) > Z(T_{i=1})$

Conditions (a) and (b) are boundary conditions on present value factors based upon fundamental finance principles. Condition (c) ensures that the discount curve is strictly downward sloping and thus that the forward rates are positive.

A mathematically convenient choice is to represent the discount factor for any date in the future as a linear combination of k basis functions:

$$Z(T_i) = 1 + \sum_{j=1}^{k} a_i f_i(T_i)$$

where $f_j(t)$ is the jth basis function and a_j is the corresponding coefficient. The basis function can take a number of forms provided they produce sensible discount function. Substituting equation (3.9) into (3.8) we get

$$P(T_N) + A(T_N) = c_N \sum_{i=1}^{N} \left(1 + \sum_{j=1}^{k} a_j f(T_i)\right) + F_N \left(1 + \sum_{j=1}^{k} a_j f(T_N)\right) \tag{3.10}$$

This can be further simplified as

$$P(T_N) + A(T_N) - Nc_N - F_N = \sum_{j=1}^{k} a_j \left(c_N \sum_{i=1}^{N} f(T_i) = F_N f(T_N)\right) \tag{3.11}$$

Equivalently,

$$y_N = \sum_{j=1}^{k} a_j x_N \tag{3.12}$$

where

$$y_N = P(T_N) + A(T_N) - Nc_N - F_N \tag{3.13}$$

$$x_N = c_N \sum_{i=1}^{N} f(T_i) + F_N f(T_N) \tag{3.14}$$

If we have a sample of N bonds the coefficient of the basis function can be estimated using ordinary least squares regression. The estimated discount function can be used to generate the discount factors for various tenors and the yield curves.

McCulloch (1971) modeled the basis function as $f(T) = T^j$ for $j = 1, 2, \ldots k$. This results in the discount function being approximated as a kth degree polynomial. One of the problems with this approach is that it has uniform resolution power. This is not a problem if the observations are uniformly distributed across the maturities. Otherwise, it fits well wherever there is greatest concentration of observations and poorly elsewhere. Increasing the order of the polynomial while solving one problem can give rise to another problem of unstable parameters.

Another alternative suggested by McCulloch (1971) is to use splines. A polynomial spline is a number of polynomial functions joined together smoothly at a number of knot points. McCulloch (1971, 1975) shows the results from applying quadratic spline and cubic spline functions. The basis functions are represented as a family of quadratic or cubic functions that are constrained to be smooth around the knot points. Schaefer (1981) suggested the use of Bernstein polynomial with the constraint that discount factor at time zero is 1. A major limitation of these methods is that the forwards rates derived from the estimated discount factors have undesirable properties at long maturity.

Vasicek and Fong (1982) model the discount function as an exponential function and describe an approach that produces asymptotically flat forward curve. This approach is in line with equilibrium interest rate models such as Vasicek (1977) and Hull and White (1990) that show that zero coupon bond price or the discount factor to have exponential form.

Rather than modeling the discount curve Nelson and Siegel (1987) directly model the forward rate. They suggest the following functional form for the forward rate:

$$F(t)=\beta_0+\beta_1 \exp\left(-\frac{t}{\alpha_1}\right)+\beta_2\left[\left(\frac{t}{\alpha_1}\right)\exp\left(-\frac{t}{\alpha_1}\right)\right] \tag{3.15}$$

This implies the following discount curve:

$$Z(t)=\exp\left\{-t\left[\beta_0+(\beta_1+\beta_2)\left(1-\exp\left(-\frac{t}{\alpha_1}\right)\right)\frac{\alpha_1}{t}-\beta_2\exp\left(-\frac{t}{\alpha_1}\right)\right]\right\} \tag{3.16}$$

Coleman, Fisher and Ibbotson (1992) also model the forward rates instead of the discount curve. They propose instantaneous forward rate to be a piecewise constant function. Partitioning the future dates into N segments, $\{t_0, t_1, t_2, \ldots t_N\}$, their model define the forward rate in any segment to be

$$F(t)=\lambda_i \qquad t_{i-1}<t\leqslant t_i \tag{3.17}$$

This model implies that the discount factor at date t between t_{k-1} and t_k is

$$Z(t)=\exp\left\{-\left[\lambda_1 t_1+\sum_{i=2}^{k-1}\lambda_i(t_i-t_{i-1})=\lambda_k(t-t_{k-1})\right]\right\} \tag{3.18}$$

The discount curve produced by this model will be continuous but the forward rate curve will not be smooth.

Chambers, Carleton and Waldman (1984) propose an exponential polynomial for the discount curve. The exponential polynomial function for discount factor can be written as

$$Z(t)=\exp\left\{-\sum_{j=1}^{k} a_j m^j\right\} \tag{3.19}$$

They recommend that a polynomial of degree 3 or 4 is sufficient to model the yield curve.

Finally, Wiseman (1994) model the forward curve as an exponential function

$$F(t) = \sum_{j=0}^{k} a_j \mathrm{e}^{-k_j t} \tag{3.20}$$

Exponential model

To see how the statistical approaches can be implemented consider a simplified example where the discount curve is modeled as a linear combination of m basis functions. Each basis function is assumed to be an exponential function. More specifically, define the discount function to be:

$$Z(t) = \sum_{k=1}^{m} a_k (e^{-\beta t})^k \tag{3.21}$$

where α and β are unknown coefficients of the function that need to be estimated. Once these parameters are known we can obtain a theoretical discount factor at any future date. These discount factors can be used to determine the par, spot and forward curves.

Substituting the condition that discount factor must be 1 at time zero we obtain the following constraint on the α coefficients:

$$\sum_{k=1}^{m} \alpha_k = 1 \tag{3.22}$$

We can rearrange this as

$$\alpha_m = 1 - \sum_{k=1}^{m-1} \alpha_k \tag{3.23}$$

Suppose that we have a sample of N bonds. Let $P(T_i)$, $i = 1, 2, \ldots N$, be the market price of the ith bond maturing T_i years from today. If q_i is the time when the next coupon will be paid, according to this model the dirty price of this bond can be expressed as:

$$\begin{aligned} P(T_i) + A(T_i) &= \sum_{t=q_i}^{T_i} c_j Z(t) + 100 Z(T_i) + \varepsilon_i \\ &= \sum_{t=q_i}^{T_i} c_i \left(\sum_{k=1}^{m} a_k e^{-k\beta t} \right) + 100 \left(\sum_{k=1}^{m} \alpha_k e^{-k\beta T_i} \right) + \varepsilon_i \end{aligned} \tag{3.24}$$

One reason for specifying the price in terms of discount factors is that price of a bond in linear in discount factor, while it is non-linear in either forward or spot rates. We can simplify equation (3.24) further and write it as:

$$\begin{aligned} P(T_i) + A(T_i) = &\sum_{k=1}^{m-1} a_k \left(c_i \sum_{t=q_i}^{T_i} e^{-k\beta t} + 100 e^{-k\beta T_i} \right) \\ &+ \left(1 - \sum_{k=1}^{m-1} \alpha_k \right) \left(c_i \sum_{t=q_i}^{T_i} e^{-m\beta t} + 100 e^{-m\beta T_i} \right) + \varepsilon_i \end{aligned} \tag{3.25}$$

Rearranging, we get

$$P(T_i)+A(T_i)-\left(c_i\sum_{t=q_i}^{T_i}e^{-m\beta t}+100e^{-m\beta T_i}\right)$$
$$=\sum_{t=q_i}^{T_i}\alpha_k\left[\left(c_i\sum_{t=q_i}^{T_i}e^{-k\beta t}+100e^{-k\beta T_i}\right)-\left(c_i\sum_{t=q_i}^{T_i}e^{-m\beta t}+100e^{-m\beta T_i}\right)\right]+\varepsilon_i \tag{3.26}$$

or

$$y_i=\sum_{k=1}^{m-1}\alpha_k x_{i,k}+\varepsilon_i \tag{3.27}$$

where

$$z_{i,k}=\left(c_i\sum_{t=q_i}^{T_i}e^{-k\beta t}+100e^{-k\beta T_i}\right)$$
$$y_i=P(T_i)+A(T_i)-z_{i,m} \tag{3.28}$$
$$x_{i,k}=z_{i,k}-z_{i,m}$$

To empirically estimate the discount function we first calculate y_i and $x_{i,k}$ for each of the N bonds in our sample. The coefficient of the discount function must be selected such that they price the N bonds correctly or at least with minimum error. If we can set the β to be some sensible value then the α's can be estimated using the ordinary least squares regression.

$$\begin{pmatrix} y_1\\ y_2\\ \cdot\\ y_N\end{pmatrix}=\begin{bmatrix} x_{1,1} & x_{1,2} & \cdot & x_{1,m}\\ x_{2,1} & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ x_{N,1} & \cdot & \cdot & x_{N,m}\end{bmatrix}\begin{pmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \cdot\\ \hat{\alpha}_m\end{pmatrix}+\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \cdot\\ \varepsilon_N\end{pmatrix} \tag{3.29}$$

The $\hat{\alpha}$'s estimated from the ordinary least squares provide the best fit for the data by minimizing the sum of the square of the errors, $\Sigma_{i=1}^{N}\varepsilon_i^2$. The estimated values of $\hat{\alpha}$ and β can be substituted into (3.21) to determine the bond market yield curve.

The model is sensitive to the number of basis functions therefore it should be carefully selected so as not to over-fit the data. Also, most of the models discussed are very sensitive to the data. Therefore, it is important to implement screening procedures to identify bonds and exclude any bonds that are outliers. Typically one tends to exclude bonds with unreliable prices or bonds that due to liquidity, coupon or tax reasons is expected to have be unusually rich or cheap. A better fit can also be achieved by iterative least squares. The model can be extend in several ways to obtain a better fit to the market data such as imposing constraint to fit certain data point exactly or assuming that the model is homoscedastic in yields and applying generalized least squares.

Exponential model implementation

We now present the results from implementing the exponential model. The price data for a sample of German bond issues maturing less than 10 years and settling on 28 October 1998 is reported in Table 3.17.

Table 3.17 DEM government bond price data

	Issue	Price	Yield	Accrued	Coupon	Maturity (years)
1	TOBL5 12/98	100.21	3.3073	4.3194	5	0.1361
2	BKO3.75 3/99	100.13	3.3376	2.2813	3.75	0.3917
3	DBR7 4/99	101.68	3.3073	3.6556	7	0.4778
4	DBR7 10/99	103.41	3.3916	0.1556	7	0.9778
5	TOBL7 11/99	103.74	3.3907	6.4750	7	1.0750
6	BKO4.25 12/99	100.94	3.3851	3.6715	4.25	1.1361
7	BKO4 3/0	100.81	3.3802	2.4556	4	1.3861
8	DBR8.75 5/0	108.02	3.3840	3.7917	8.75	1.5667
9	BKO4 6/0	100.95	3.3825	1.4667	4	1.6333
10	DBR8.75 7/0	108.81	3.3963	2.3819	8.75	1.7278
11	DBR9 10/0	110.56	3.3881	0.2000	9	1.9778
12	OBL 118	104.05	3.3920	3.6021	5.25	2.3139
13	OBL 121	103.41	3.5530	4.4597	4.75	3.0611
14	OBL 122	102.80	3.5840	3.0750	4.5	3.3167
15	OBL 123	102.94	3.5977	2.0125	4.5	3.5528
16	OBL 124	103.06	3.6218	0.8625	4.5	3.8083
17	THA7.75 10/2	114.81	3.6309	0.5813	7.75	3.9250
18	OBL 125	104.99	3.6480	4.8056	5	4.0389
19	THA7.375 12/2	113.78	3.6846	6.6785	7.375	4.0944
20	OBL 126	103.35	3.6409	3.1250	4.5	4.3056
21	DBR6.75 7/4	114.74	3.8243	1.9313	6.75	5.7139
22	DBR6.5 10/5	114.85	4.0119	0.2528	6.5	6.9611
23	DBR6 1/6	111.75	4.0778	4.8833	6	7.1861
24	DBR6 2/6	111.91	4.0768	4.2000	6	7.3000
25	DBR6 1/7	112.05	4.2250	4.9000	6	8.1833
26	DBR6 7/7	112.43	4.2542	1.9000	6	8.6833
27	DBR5.25 1/8	107.96	4.1860	4.2875	5.25	9.1833
28	DBR4.75 7/8	104.81	4.1345	1.5042	4.75	9.6833

Suppose that we choose to model the discount factor with 5 basis functions and let β equal to the yield of DBR4.75 7/2008. The first step is to calculate the $z_{i,j}$, $i=1$ to 28, $j=1$ to 5. An example of the calculation for DBR6.75 7/4 is described below. This bond pays a coupon of 6.75%, matures in 5.139 years and the next coupon is payable in 0.7139 years from the settle date.

$$z_{21,2} = \left(6.75 \sum_{t=0}^{5} e^{-2\times 0.0413\times(t+0.7139)} + 100e^{-2\times 0.0413\times 5.7139}\right)$$

$$= 93.70$$

Similarly, we can calculate the $z_{i,j}$ for all the bonds in our sample and the results are shown in Table 3.18.

Next we apply equation (3.28) to obtain the data for the ordinary least square regression estimation. This is reported in Table 3.19.

Finally we estimate the α's using ordinary least square regression and use it in equation (3.20) to generate the discount factors and yield curves. The coefficient

Table 3.18 DEM government bond $z_{i,j}$ calculation results

Issue	$z_{i,1}$	$z_{i,2}$	$z_{i,3}$	$z_{i,4}$	$z_{i,5}$
TOBL5 12/98	104.41	103.82	103.24	102.66	102.09
BKO3.75 3/99	102.08	100.44	98.83	97.24	95.68
DBR7 4/99	104.91	102.86	100.84	98.87	96.94
DBR7 10/99	102.76	98.69	94.78	91.02	87.42
TOBL7 11/99	109.33	104.86	100.58	96.49	92.57
BKO4.25 12/99	103.69	99.10	94.73	90.55	86.56
BKO4 3/0	102.14	96.61	91.39	86.45	81.78
DBR8.75 5/0	110.48	103.89	97.70	91.89	86.45
BKO4 6/0	101.11	94.66	88.63	82.98	77.71
DBR8.75 7/0	109.74	102.51	95.77	89.48	83.61
DBR9 10/0	109.09	100.86	93.26	86.25	79.77
OBL 118	105.80	96.75	88.50	80.99	74.16
OBL 121	105.94	94.41	84.21	75.20	67.24
OBL 122	103.90	91.57	80.78	71.34	63.07
OBL 123	102.89	89.80	78.45	68.61	60.07
OBL 124	101.81	87.92	76.00	65.77	56.98
THA7.75 10/2	113.09	97.76	84.63	73.37	63.71
OBL 125	107.64	92.87	80.31	69.61	60.50
THA7.375 12/2	118.30	102.50	89.06	77.61	67.85
OBL 126	104.18	88.77	75.79	64.85	55.64
DBR6.75 7/4	114.51	93.70	77.03	63.63	52.85
DBR6.5 10/5	113.75	89.48	70.87	56.57	45.54
DBR6 1/6	115.70	91.23	72.64	58.47	47.64
DBR6 2/6	115.15	90.37	71.62	57.38	46.53
DBR6 1/7	116.98	89.92	70.05	55.40	44.54
DBR6 7/7	114.58	86.27	65.84	51.01	40.17
DBR5.25 1/8	111.97	83.45	63.28	48.92	38.62
DBR4.75 7/8	105.62	76.72	56.68	42.67	32.80

estimates and the resultant discount factor curve are reported in Tables 3.20 and 3.21, respectively.

The discount factor can be used for valuation, rich-cheap analysis or to generate the zero curve and the forward yield curves.

Model review

The discount factors produced by the yield curve models are used for marking-to-market of position and calculation of end-of-day gains/losses. Others bump the input cash and swap rates to the yield curve model to generate a new set of discount factors and revalue positions. This provides traders with an estimate of their exposure to different tenor and hedge ratios to manage risk of their positions. Models such as those of Heath, Jarrow and Morton (1992) and Brace, Gaterak and Musiela (1995) use the forward rates implied from the yield curve as a starting point to simulate the future evolution of the forward rate curve. Spot rate models such as those of Hull and White (1990), Black, Derman and Toy (1990) and Black and Karasinsky (1990) estimate parameters for the model by fitting it to the yield curve data. The model to

Table 3.19 DEM government bond $x_{i,j}$ regression data

Issue	y_i	$x_{i,1}$	$x_{i,2}$	$x_{i,3}$	$x_{i,4}$
TOBL5 12/98	2.4427	2.3240	1.7381	1.1555	0.5761
BKO3.75 3/99	6.7305	6.4027	4.7630	3.1495	1.5620
DBR7 4/99	8.3987	7.9702	5.9182	3.9064	1.9339
DBR7 10/99	16.1479	15.3430	11.2716	7.3615	3.6064
TOBL7 11/99	17.6444	16.7562	12.2852	8.0077	3.9153
BKO4.25 12/99	18.0511	17.1321	12.5442	8.1659	3.9876
BKO4 3/0	21.4836	20.3622	14.8299	9.6033	4.6653
DBR8.75 5/0	25.3652	24.0300	17.4390	11.2536	5.4485
BKO4 6/0	24.7094	23.3980	16.9496	10.9182	5.2768
DBR8.75 7/0	27.5772	26.1283	18.8960	12.1524	5.8641
DBR9 10/0	30.9865	29.3115	21.0826	13.4866	6.4741
OBL 118	33.4955	31.6457	22.5896	14.3445	6.8368
OBL 121	40.6304	38.7041	27.1683	16.9745	7.9648
OBL 122	42.8011	40.8274	28.4955	17.7065	8.2648
OBL 123	44.8833	42.8227	29.7297	18.3797	8.5376
OBL 124	46.9444	44.8324	30.9431	19.0232	8.7895
THA7.75 10/2	51.6844	49.3862	34.0537	20.9184	9.6586
OBL 125	49.2943	47.1380	32.3718	19.8071	9.1107
THA7.375 12/2	52.6131	50.4555	34.6570	21.2112	9.7599
OBL 126	50.8381	48.5455	33.1304	20.1514	9.2172
DBR6.75 7/4	63.8172	61.6541	40.8501	24.1722	10.7782
DBR6.5 10/5	69.5667	68.2133	43.9415	25.3377	11.0347
DBR6 1/6	68.9955	68.0584	43.5888	25.0013	10.8359
DBR6 2/6	69.5806	68.6234	43.8422	25.0908	10.8533
DBR6 1/7	72.4097	72.4375	45.3753	25.5126	10.8640
DBR6 7/7	74.1636	74.4180	46.1074	25.6739	10.8408
DBR5.25 1/8	73.6259	73.3496	44.8310	24.6579	10.2981
DBR4.75 7/8	73.5156	72.8177	43.9244	23.8769	9.8696

Table 3.20 DEM exponential model coefficient estimations

Coefficient	Estimate
β	4.13%
α_1	16.97
α_2	−77.59
α_3	139.55
α_4	−110.08
α_5	32.15

generate the yield curve model is not as complicated as some of the term structure models. However, any small error made while building the yield curve can have a progressively amplified impact on valuation and hedge ratios unless it has been reviewed carefully. We briefly outline some of the issues that must be kept in mind while validating them.

First, the yield curve model should be arbitrage free. A quick check for this would

Table 3.21 DEM government bond market discount factor curve

Time	Discount factor	Par coupon yield
0.00	1.0000	
1.00	0.9668	3.44%
2.00	0.9353	3.40%
3.00	0.9022	3.49%
4.00	0.8665	3.64%
5.00	0.8288	3.80%
6.00	0.7904	3.96%
7.00	0.7529	4.09%
8.00	0.7180	4.17%
9.00	0.6871	4.20%
10.00	0.6613	4.18%

be to verify if the discount factors generated by the yield curve model can produce the same cash and swap rates as those feed into the model. In addition there may be essentially four possible sources of errors – use of inappropriate market rates data, accrual factor calculations, interpolation algorithms, and curve-stitching.

The rates used to build the curve for the short-term product will not be the same as the rates used for pricing long-term products. The individual desk primarily determines this so any curve builder model should offer flexibility to the user in selecting the source and the nature of rate data. Simple as it may seem, another common source of error is incorrect holiday calendar and market conventions for day count to calculate the accrual factors. Fortunately the computation of accrual factors is easy to verify. Interpolation algorithms expose the yield curve model to numerical instability. As we have mentioned earlier, some interpolation methods such as the cubic spline may be capable of producing a very smooth curve but performs poorly during computation of the hedge ratio. A preferable attribute for the interpolation method is to have a local impact on yield curve to changes in specific input data rather than affecting the entire curve. There are many systems that offer users a menu when it comes to the interpolation method. While it is good to have such flexibility, in the hands of a user with little understanding of the implications this may be risky. It is best to ensure that the model provides sensible alternatives and eliminate choices that may be considered unsuitable after sufficient research.

In the absence of reasonable market data curve stitching can be achieved by interpolating either the stub rate or the discount factor. A linear interpolation can be applied for rates but not if it is a discount factor.

Summary

In this chapter we have discussed the methodology to build the market yield curve for the swap market and the bond market. The market yield curve is one of the most important pieces of information required by traders and risk managers to price, trade, mark-to-market and control risk exposure. The yield curve can be described as discount factor curves, par curves, forward curves or zero curves. Since the discount factors are the most rudimentary information for valuing any stream of cash flows it is the most natural place to start. Unfortunately discount factors are not directly observable in the market, rather they have to be derived from the market-quoted interest rates and prices of liquid financial instruments.

The swap market provides an abundance of par swap rates data for various tenors. This can be applied to extract the discount factors using the bootstrap method. The bootstrap approach produces a discount factor curve that is consistent with the market swap rates satisfying the condition of no-arbitrage condition. When a swap rate or discount factor for a specific date is not available then interpolation methods may need to be applied to determine the value. These methods must be carefully chosen since they can have significant impact on the resultant yield curve, valuations and risk exposure calculations. Linear interpolation for swap rates and exponential interpolation for discount factors is a recommended approach due to their simplicity and favorable performance attributes.

In the bond market due to non-uniform price data on various tenors, coupon effect and liquidity factors a statistical method has to be applied to derive the discount factor curves. The objective is not necessarily to derive discount factors that will price every bond to the market exactly. Instead we estimate the parameters of the model that will minimize the sum of squares of pricing errors for the sample of bonds used. In theory many statistical models can be prescribed to fit the discount curve function. We have reviewed a few and provided details on the implementation of the exponential model. An important criterion for these models is that they satisfy certain basic constraints such as discount factor function equal to one on spot date, converge to zero for extremely long tenors, and be a decreasing function with respect to tenors.

Note

[1] For an intuitive description of the futures convexity adjustment and calculations using this expression see Burghartt and Hoskins (1996). For other technical approaches to convexity adjustment see interest rate models such as Hull and White (1990) and Heath, Jarrow and Morton (1992).

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