Building the Bloomberg Interest Rate Curve – Definitions and Methodology

July 2012

Abstract

The goal of this document is to describe the process of building interest rate curves in Bloomberg terminal. We give short definitions of terms used, e.g. of different rates (simple rate, continuously compounded forward rate, zero rate, etc.). We briefly describe the instruments used in building the curve (cash rates, interest rate (IR) futures and IR swaps). Special attention is given to discussion of a functional form of the curve (a.k.a. 4 interpolation methods) and algorithms of building the curve under these different interpolations (curve *stripping methods*). The discussion is limited to single-currency IR curves, no FX curves or basis curves are discussed here.

Interest Rate Curve – Definition

The Interest Rate (IR) curve is an object which allows one to calculate a discount factor for every date within the curve range in the future thus providing us with the risk-free present value of a unit of currency (say, $1) at that date. It is used to calculate present values of a known set of payments (cash flows). While in some situations one can construct an IR curve which takes into account an additional discount due to risk of default of the counterparty, this document leaves the discussion of default or credit risk out. For the sake of simplicity we will use an assumption that the IR curves describe risk-free present values.

Another use for IR curves is to calculate projected forward rates between two dates (d_1,d_2) in the future. An example of such use is a construction of payments of a 'floating leg' of an IR swap which pays quarterly an amount of interest equal to 3-month LIBOR rate on a given notional amount. While the actual payments which will be made in the future are not known until we reach that point in time when the value of LIBOR is quoted, the present value (PV) of this stream of payments is correct if we use current projections of forward rates from the presently known curve.

Definition of Rates

In this section we provide definitions of different types of interest rates such as simple spot rate, continuously compounded forward rate, etc., necessary for discussion of instruments used to build an IR curve and used to describe the functional shape of the curve.

The *simple spot rate* r_s is defined by the following equation:

$$D(d_0,d)=\frac{1}{1+r_s(t)\cdot t} \tag{1}$$

here d_0 is the current date;

d is the date in the future;

$D(d_0,d)$ is the discount factor from the date d to present date d_0,

$t=t(d_0,d)$ is the time interval between two dates (d_0,d) in years.

The only formula in here which is not defined yet is the $t=t(d_0,d)$, e.g. the method of conversion of a pair of dates (d_0,d) into a time interval t in years. There is more than one way of doing this, and some of such algorithms called *Day Count Conventions* are described in Appendix 1.

A *simple forward rate* r_{sf} between two dates (d1,d2) in the future is defined as:

$$\frac{D(d_0,d_2)}{D(d_0,d_1)}=\frac{1}{1+r_{sf}(t)\cdot t} \tag{2}$$

where $t = t(d_1,d_2)$.

A *continuously compounded forward rate* $r_{cf}(t)$ between two dates (d1,d2) in the future is defined by the equation:

$$D(d_1,d_2)=\exp\left[-\int_{T_1}^{T_2} r_{cf}(t)\cdot dt\right] \tag{3}$$

here times T_1 and T_2 are equal to time intervals between date d_0 now and dates d_1 and d_2 in future correspondingly:

$T_1 = t(d_0,d_1)$ and $T_2 = t(d_0,d_2)$.

The *continuously compounded zero rate* r_{cz} is defined by equation:

$$D(d_0,d)=\exp(-r_{cz}(t)\cdot t) \tag{4}$$

here time t is time interval between date d_0 now and date d in the future.

Building the IR Curve

The process of building of the IR curve a.k.a. *curve stripping* is a process of creating a curve object which would correctly price a set of N given instruments, e.g. produces correct discount factors and forward rates used in these instruments.

Let us consider the choice of instruments. Currently in Bloomberg curve building the instruments belong to 3 groups:

1) cash or deposit rates;

2) IR futures or Forward Rate Agreements (FRAs);

3) IR swaps.

User can choose the set of instruments to use in curve construction by typing **SWDF** on Bloomberg terminal, choosing the right country/currency, selecting the source 8 (custom curve), saving his choice (**1<go>**) and clicking on the curve of interest. A screen like this should appear:

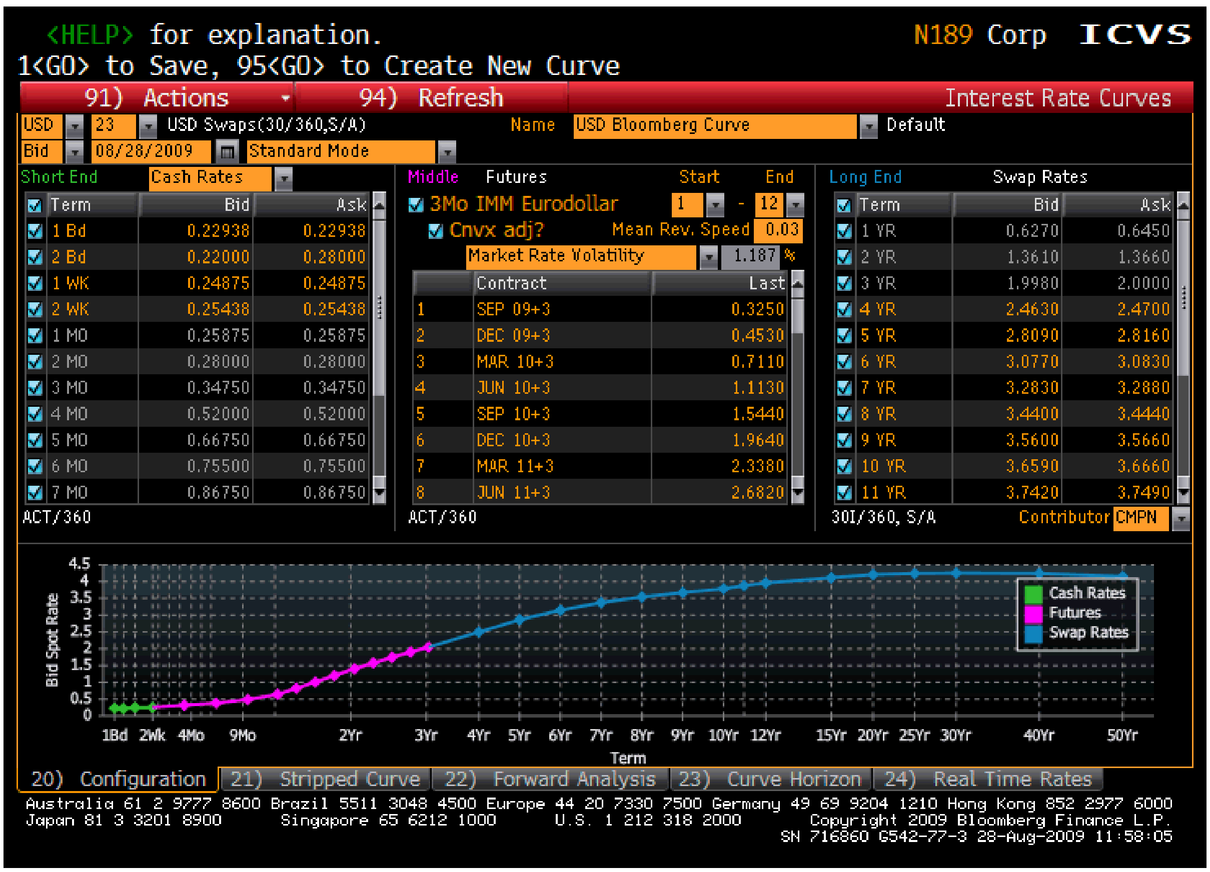


Fig. 1a. Example of Bloomberg screen to choose a set of instruments for building USD IR curve 23.

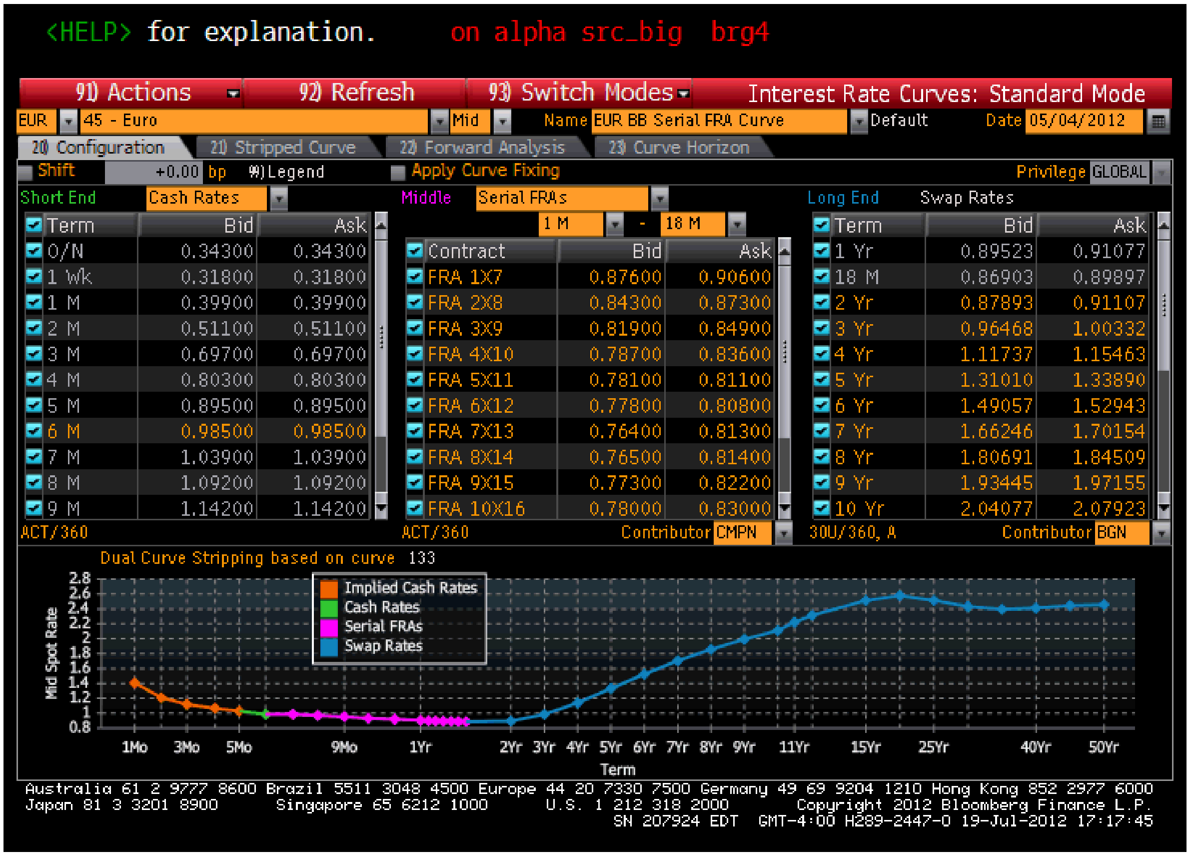


Fig. 1b. Example of Bloomberg screen showing instruments in EUR serial FRA IR curve 45.

The first group usually covers the short-term part of the curve; the securities present in the market are available up to a year (say, for USD or EUR), but they are most often used for curve construction up to about 3 months.

The second group picks up where the first group ends (the method of connecting the cash and future rates is discussed in Appendix 3). The futures time interval overlaps with swaps, so the decision of where to switch from futures to swap rates is up to user, and as shown in the Fig.1a above it is around 4 years for a default Bloomberg USD curve and 2 years for EUR curve. For USD curve one can use IR futures, but for other curves, for instance, EUR curve 45, user has a choice between IR futures and FRAs. Both contiguous and serial futures and FRAs are supported. Fig. 1b shows the Bloomberg-defined 6MO EUR curve configured with serial FRAs.

The third group – swap rates – covers time interval from where user decided to end the use of futures and up to 50 yrs for USD or EUR.

While Bloomberg terminal provides default Bloomberg curves, user can customize the curve by choosing a different set of instruments using screens 1a-b, and save the result as a new curve under a user-defined name and set it as a default curve for a given currency.

Now let us consider more closely each of these instruments and what kind of constraint on IR curve or what kind of equation it provides.

The cash rates are very straight-forward: we are given a simple spot rate r_s as defined by formula (1) for a given maturity time t. The correspondent equation to solve is:

$$r_s(T_n, curve) - r_s^{quoted}(T_n) = 0 \tag{5a}$$

here the first term is the simple rate r_s calculated using the curve being built for the time T_n equal to the maturity of the n-th instrument.

The FRAs provide us with the simple forward rates r_{sf} directly. The correspondent equation to solve is:

$$r_{sf}(T_{n-1}, T_n, curve) - r_{sf}^{quoted}(T_{n-1}, T_n) = 0 \tag{5b}$$

here the first term is the forward rate r_{sf} calculated using the curve being built for the time interval between T_{n-1} and T_n equal to the time period of the n-th future or FRA.

The futures rates can be converted into IR forward rates r_{sf}. To do this conversion, one can use a convexity adjustment as discussed in Appendix 2. The adjustment is based on Hull-White IR model where interest rates undergo a random walk with volatility σ and mean-reversion a. These parameters of the model can be modified by user using screen shown in Fig. 1a.

The last group of securities used to construct the curve is IR swaps. The Interest Rate Swap is an instrument which exchanges a stream of fixed rate payments on some notional M vs. a stream of floating payments on the same notional. In the fixed leg the payments are calculated using a fixed rate r_{sw} defined at the inception of the swap:

$$c_i = M \cdot r_{sw} \cdot t_i$$

here c_i is the i-th coupon payment;

r_{sw} is the fixed rate, a.k.a. the *swap rate*;

M is notional amount (e.g. $10M);

t_i is the time interval between inception and first coupon or i-th coupon and previous coupon.

At the last date together with the last coupon a notional is paid (at least formally as a part of each leg's cash flows).

The payments in the floating leg are calculated in almost the same way, except the rate in each payment is a LIBOR rate at the time of the beginning of the correspondent time interval; so the rate changes (or *floats*) with the changes in quoted LIBOR rate – that's where the name *floating rate* comes from. Note that if we use the same IR curve both for projecting the floating rates and discounting the future payments, the present value of a floating leg which includes the final payment of notional is equal to the notional amount.

The swap at inception has two important parameters: time to maturity (e.g. 1y, 5y, or 50y) and the swap rate r_{sw}. There are also many other parameters, such as frequency of payments, etc. which we will consider here as standard and therefore fixed. At the inception of an IR swap the swap rate r_{sw} is chosen in such a way as to make the present value of the whole swap (with one leg paying and another one receiving payments) equal to zero. The swap which satisfies this condition is said to be *at par*. We have par swap rates quoted in the market as a function of maturity and usually available for maturities starting from 1y and up to 50y. The correspondent equation to solve is:

$$PV(swap(T_n), curve) = 0 \qquad (5c)$$

here T_n is the maturity of the n-th swap.

Now, we need to build the IR curve such that it will price all selected securities correctly, e.g. satisfies a system of equations 5a-c.

To assure that we have a unique solution, the curve should have N degrees of freedom, where N is the number of securities, e.g. equations 5a-c. This is achieved by constructing the curve consisting of N pieces or time intervals: the first time interval starts at time T=0 (now) and intervals end at times T_n (here n=1 .. N; the time T_n correspond to the maturity of n-th security e.g. date of last payment) and having N independent parameters. Below we will consider two methods of building a curve:

1. a *bootstrap* method, where each of curve's pieces has one independent degree of freedom varying which does not affect previous time intervals; in this case the curve is built by adjusting one piece at a time while moving from shorter maturities to longer ones;
2. a *global* method where all or at least some degrees of freedom of the curve affect it's shape everywhere, and therefore one needs to solve a general system of N non-linear equations with N variables.

Currently Bloomberg terminal allows user to choose one of the 4 functional forms for the IR curve (a.k.a. *interpolation methods*): on Bloomberg terminal typing **SWDF**, choosing **33) More**, then **24) User Preferences**, one can see the following screen:

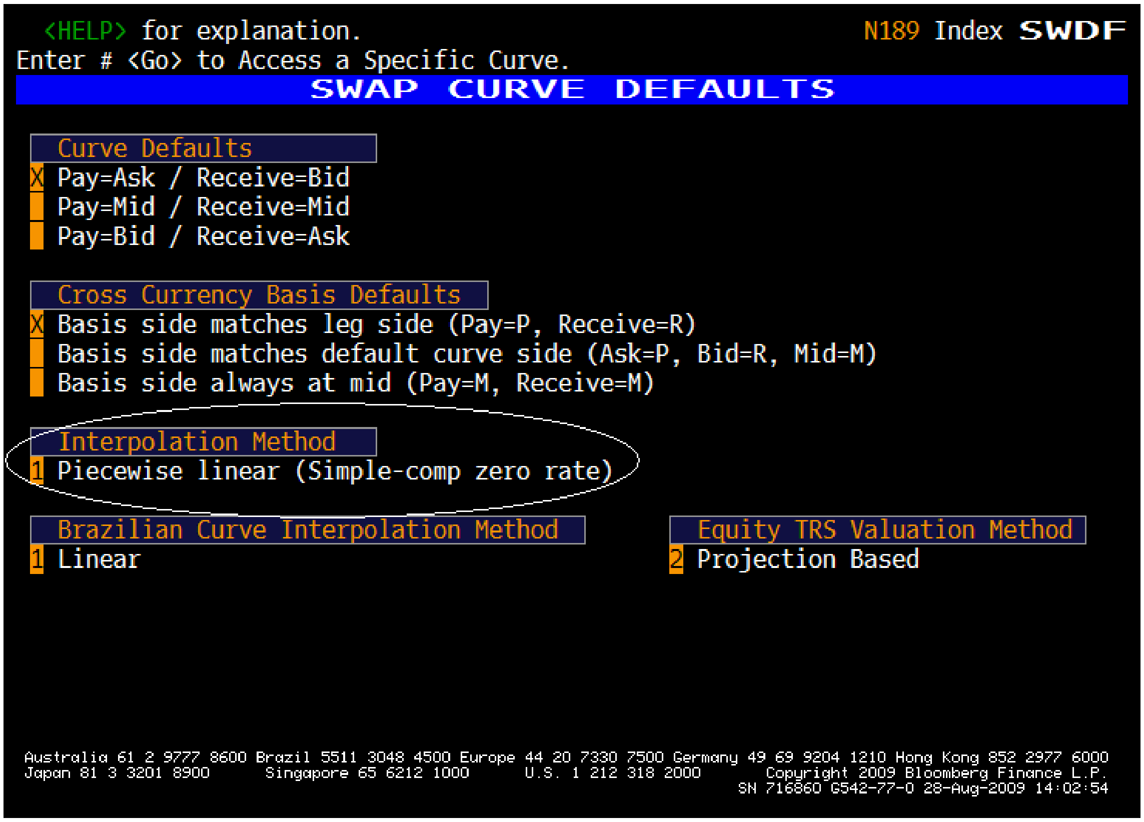


Fig. 2: Screen allowing user to choose curve interpolation method (under SWDF command). The part of the screen which deals with the choice of the functional form of the IR curve (Interpolation Method) is circled.

On that screen one can choose 1 through 4 under "Interpolation Method":

1. Piecewise linear (Simple-compounded zero rate);
2. Smooth forward/Piecewise quadratic (Continuously-compounded forward rate);
3. Step-function forward (Continuously-compounded forward rate);
4. Piecewise linear (Continuously-compounded zero rate).

The interpolation method 1 -- piecewise linear simple-compounded zero rate -- means that the simple rate r_s defined by the formula (1) is a piecewise linear continuous function. An example of the shape of the spot rate and forward rates as functions of time when using this interpolation method is shown on Fig. 3a. For dates outside the range defined by the rates with the shortest and longest maturities on the curve, this function is extrapolated as follows: constant forward in the short end and constant (non-annualized) zero in the long end. An example of long-end extrapolation effect is illustrated in Fig. 3a, where the 50Y swap rate has been removed from the curve in order to show the extrapolated forward and spot rates from 40Y to 50Y.

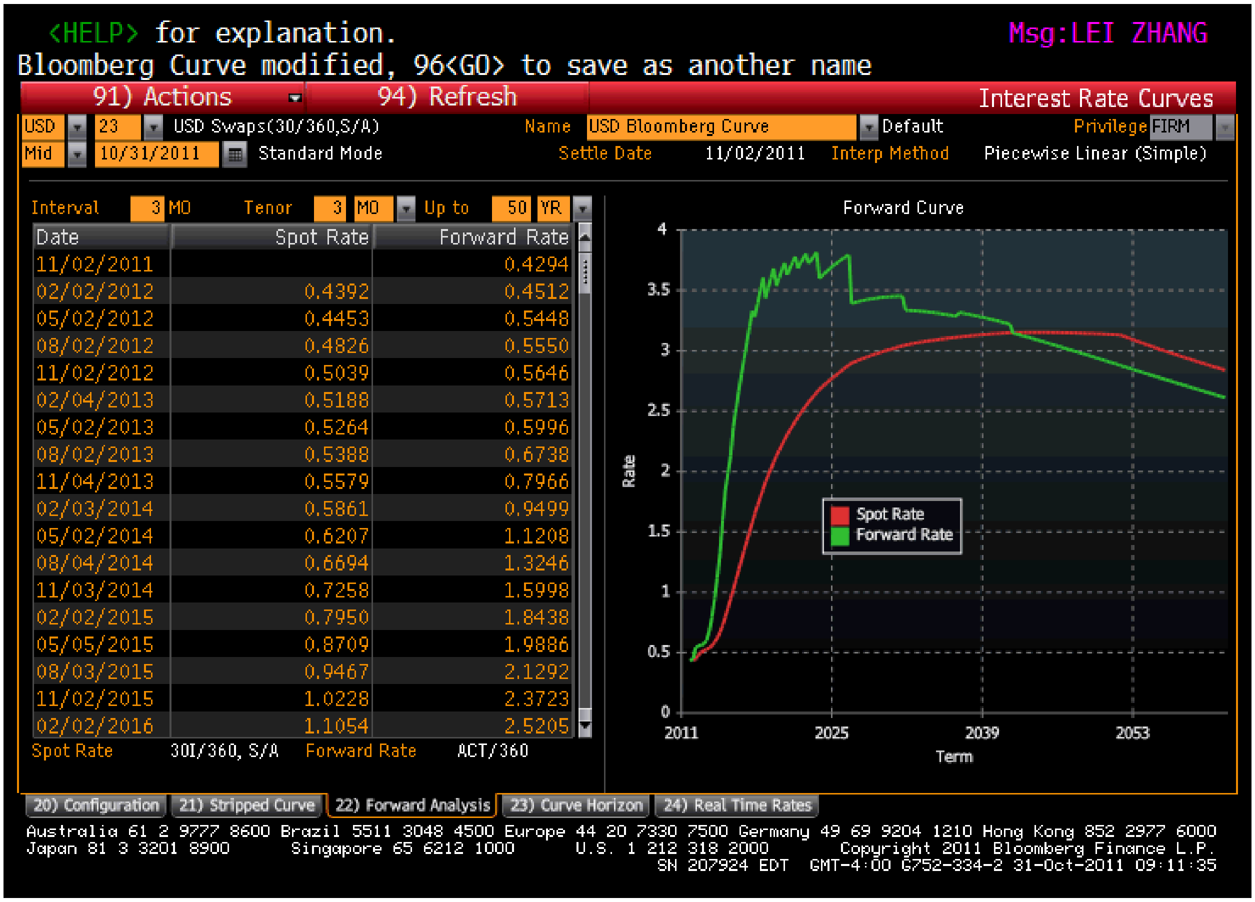


Fig. 3a: Spot (red) and forward (green) rate graphs for USD curve, interpolation method 1.

The interpolation method 2 – 'smooth forward' e.g. piecewise quadratic continuously-compounded forward rate -- means that the forward rate r_{cf} defined by formula (3) is piecewise-quadratic. The neighboring pieces of the forward curve are connected in such way that the first derivative of the forward rate is continuous, which is reflected in name 'smooth'. The building of the curve requires the global method as defined above. For details of the functional form of the curve see Appendix 4. The process of solving the system of N non-linear equations is described in Appendix 5. An example of the shape of the spot and forward rates as functions of time when using this interpolation method is shown on Fig. 3b. Constant forward extrapolation is used in both the short end and the long end, and the long-end extrapolation effect from 40Y to 50Y is illustrated in Fig. 3b with the 50Y swap rate removed from the curve configuration.

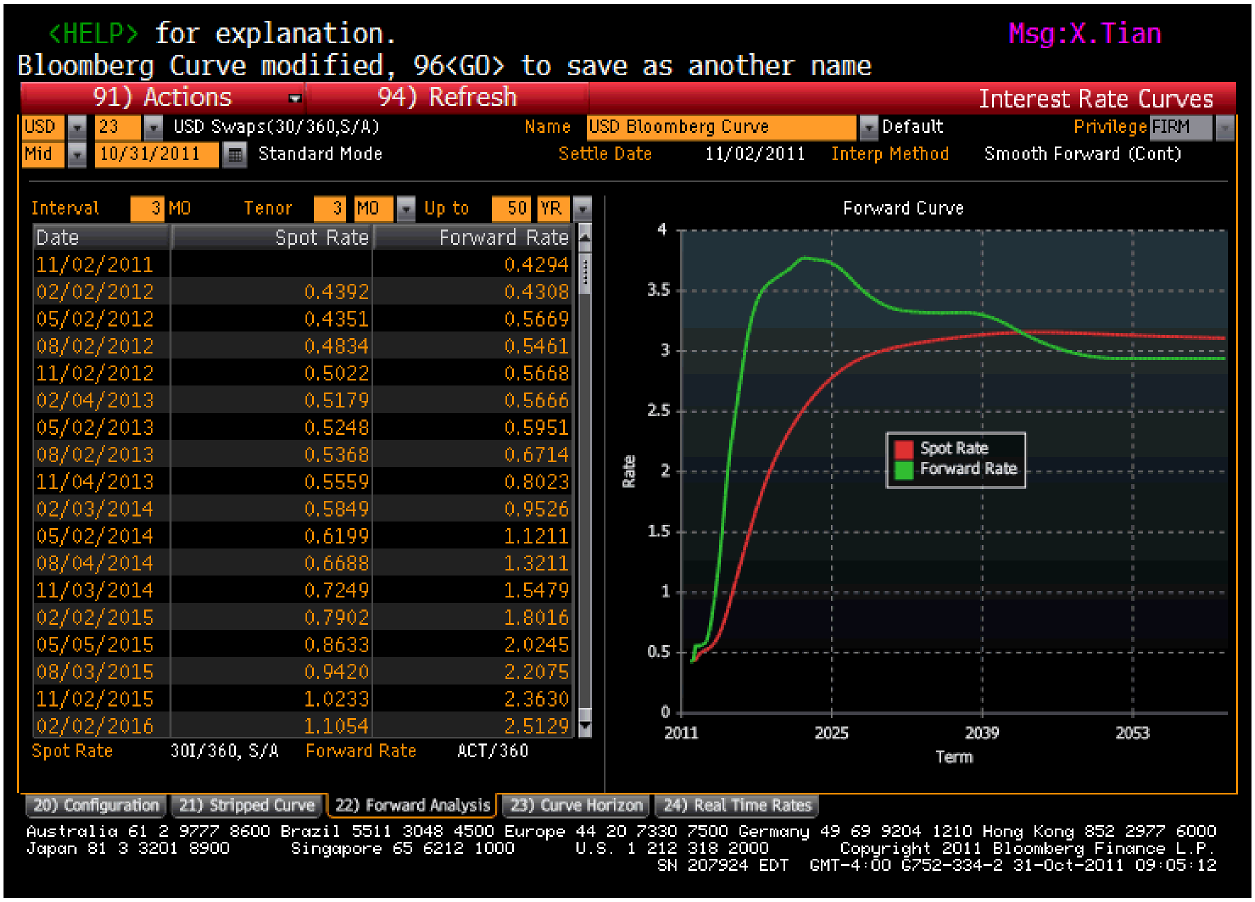


Fig. 3b: Spot (red) and forward (green) rate graphs for USD curve, interpolation method 2.

The interpolation method 3 -- step-function forward continuously-compounded forward rate – means that the forward rate r_{cf} defined by the formula (3) is piecewise constant. An example of the shape of the spot and forward rates as functions of time when using this interpolation method is shown on Fig. 3c. Constant forward extrapolation is used in both the short end and the long end, and the long-end extrapolation effect from 40Y to 50Y is illustrated in Fig. 3c with the 50Y swap rate removed from the curve configuration.

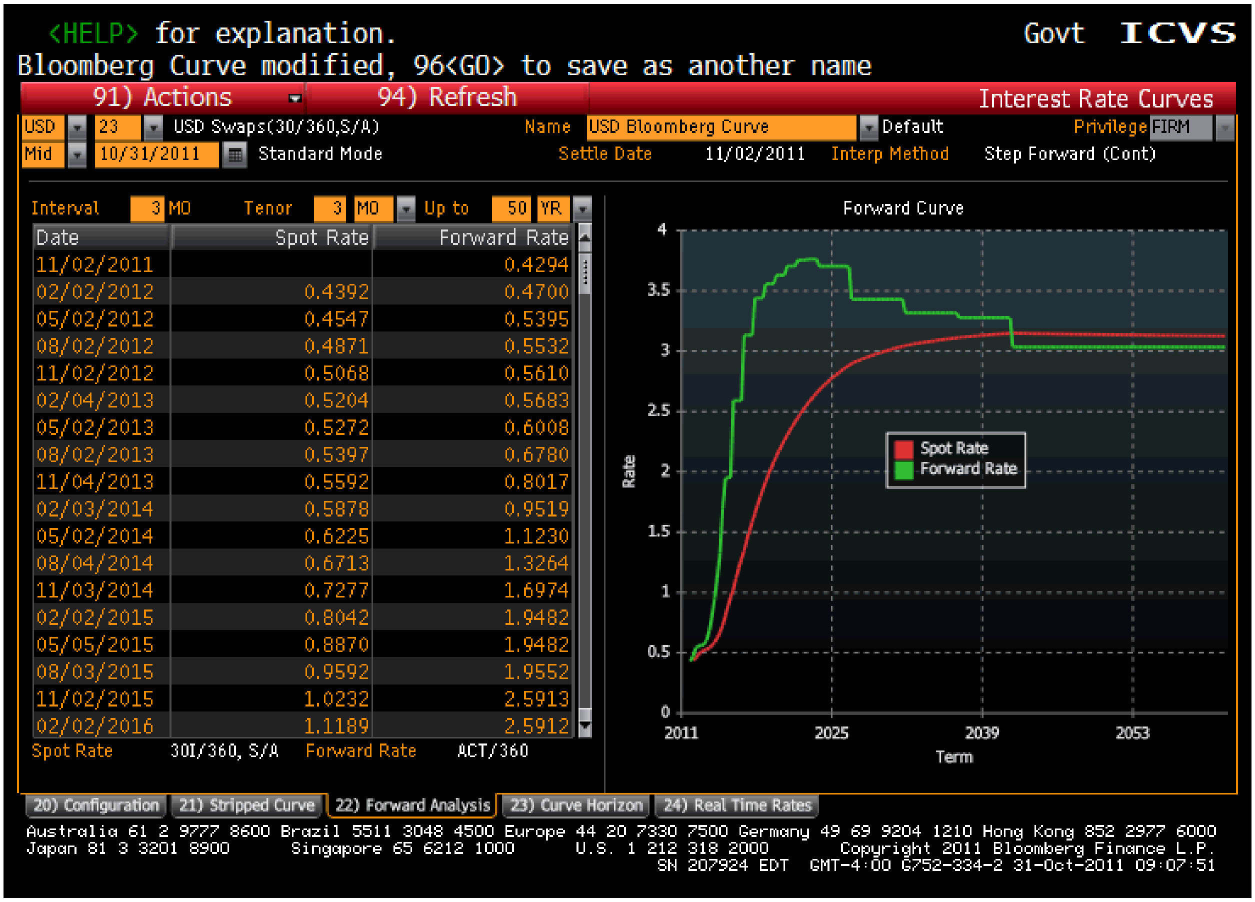


Fig. 3c: Spot (red) and forward (green) rate graphs for USD curve, interpolation method 3.

The interpolation method 4 -- piecewise linear continuously-compounded zero rate – means that the zero rate r_{cz} as defined by the formula (4) is piecewise linear continuous function. An example of the shape of the spot and forward rates as functions of time when using this interpolation method is shown on Fig. 3d. Similar to interpolation method 1, constant forward in the short end and constant zero in the long end are used for rate extrapolation. The long-end extrapolation effect from 40Y to 50Y is illustrated in Fig. 3d. Note that extrapolation forwards are constant but have a different value than the forward at 40Y maturity.

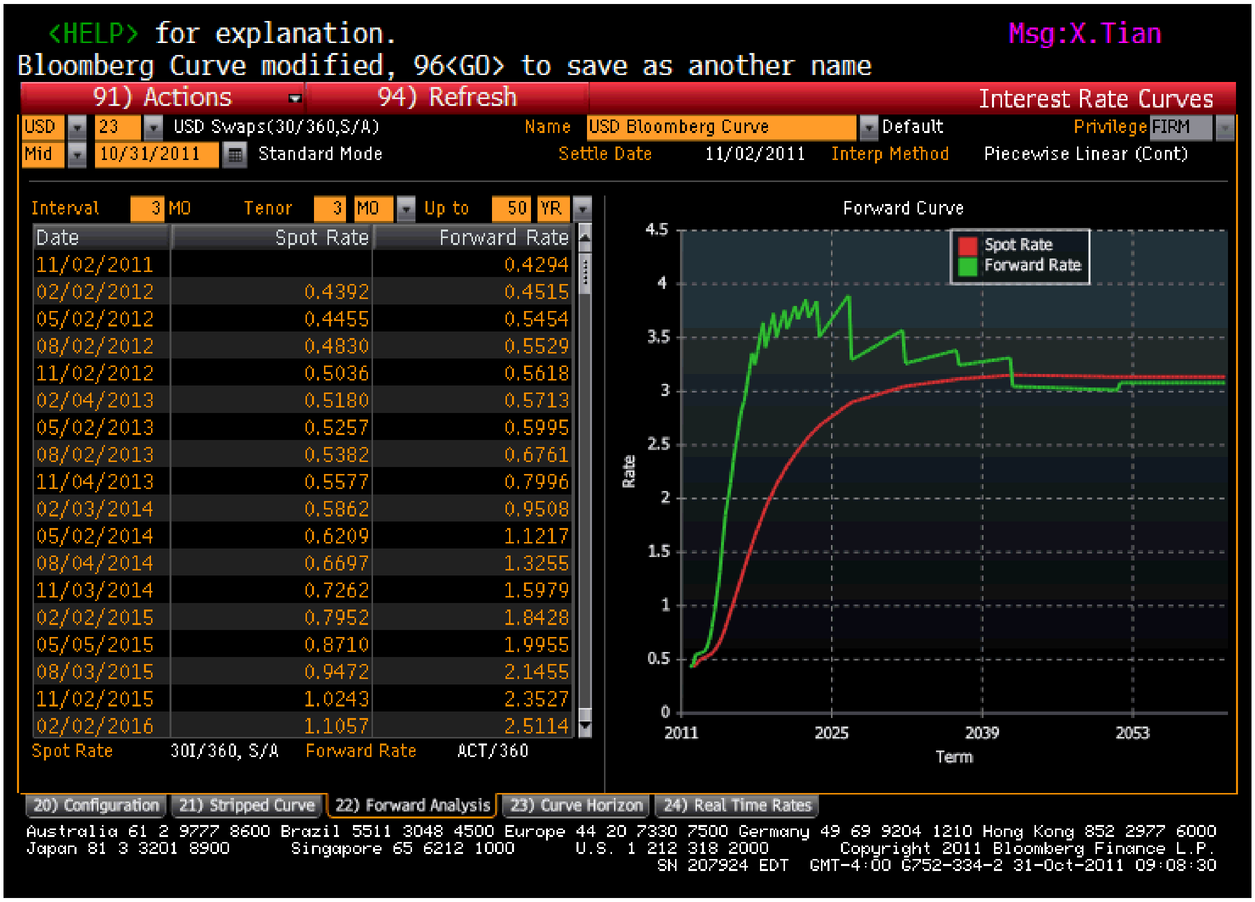


Fig. 3d: Spot (red) and forward (green) rate graphs for USD curve, interpolation method 4.

The three functional forms – 1, 3, and 4 – have one internal parameter per piece, allowing to modify the behavior of one piece at a time while not changing the shape of the curve in previous pieces. This makes it possible to build the curves of these types using the bootstrap method, as mentioned above. It means that we build the curve from left to right, one piece at a time by solving one equation at a time – we thus reduce the problem of solving a system of N equations with N variables to N consecutive solutions of one equation with one variable. In programming terms it means that we use 1-dimensional root finder N times.

In case of a smooth curve (interpolation method number 2) the situation is different. The curve is defined in such a way that change in each internal parameter or degree of freedom affects the shape of the whole curve, and therefore the solution of a non-linear system of N equations with N variables in a general way. The functional form of this curve in current Bloomberg software is described in Appendix 4. The algorithm of this solution is discussed in Appendix 5.

Special Considerations for Curves with Serial FRAs

Incorporation of serial FRAs in curve building by itself does not require any special treatment from the stripping algorithm, in the sense that the stripper will succeed in producing a curve that matches all input instruments. However, converting FRAs to corresponding discount factors relies implicitly on stub rates (i.e. zero rates at FRAs' start dates) that are often interpolated from the curve. While the stubs do not affect the FRAs themselves, they may create artifacts on the forwards on the dates that these FRAs mature.

Let us use the example of 6M EUR serial FRA curve in Fig.1.b to illustrate this susceptibility to artifacts. When that curve is stripped without any special consideration, the associated forward curve exhibits a slight dip between 6/10/13 and 11/8/13 as shown in Fig. 4.a. To explain why such artifacts may happen, let us examine how the forward rates are calculated. For instance, the 6M forward on 6/10/13 (i.e. FRA 13x19) is calculated from the 13M and 19M zero rates, which are in turn functions of FRAs 1x7 and 7x13 and the 1M zero for the former, and 6M cash, FRAs 6x12, 12x18 and 2Y swap rates for the latter. Note that all these rates are market quotes except the 1M zero which is a stub for FRA 1x7. One seemingly sensible approach is to use the 1M zero extrapolated from the curve as the stub, which is done currently. Unfortunately this exposes the resulting FRA 13x19 to assumptions that extrapolation may impose, as any increase in the stub rate reduces this forward rate by almost the same amount. The observed dip is precisely an unfortunate result of the stubs being oblivious of the desired characteristics of the forward curve.

One way to prevent the stubs from polluting the forward curve is to set them explicitly so that the dependent forwards will behave in a desirable fashion. Specifically, we make the forwards that are dependent on stubs to take linearly interpolated values from adjacent forwards that are determined solely based on market quotes. In the case of EUR serial FRA curve in Fig 1.b, the forwards between 6/10/13 and 11/8/13 will have rates that are linearly interpolated from FRA 12x18 and 6M forward on 5/8/13 (equivalent to FRA 18x24). By setting the stubs to the 1M to 5M zero rates that are implied from the above linearly interpolated forwards (Fig. 1.b), the current stripper is able to reproduce the desired forwards as shown in Fig. 4.b. Currently this forward adjustment is applied to all curves that include serial FRAs.

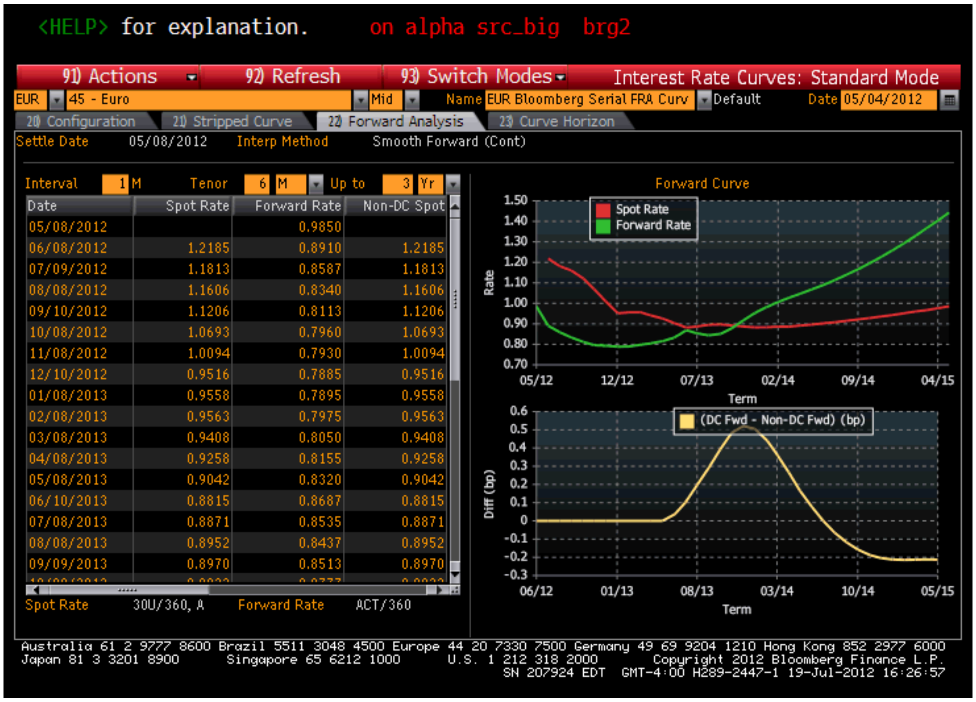


Fig. 4.a Forward curve showing artifacts in forwards on 7/8/2013 to 11/8/2013

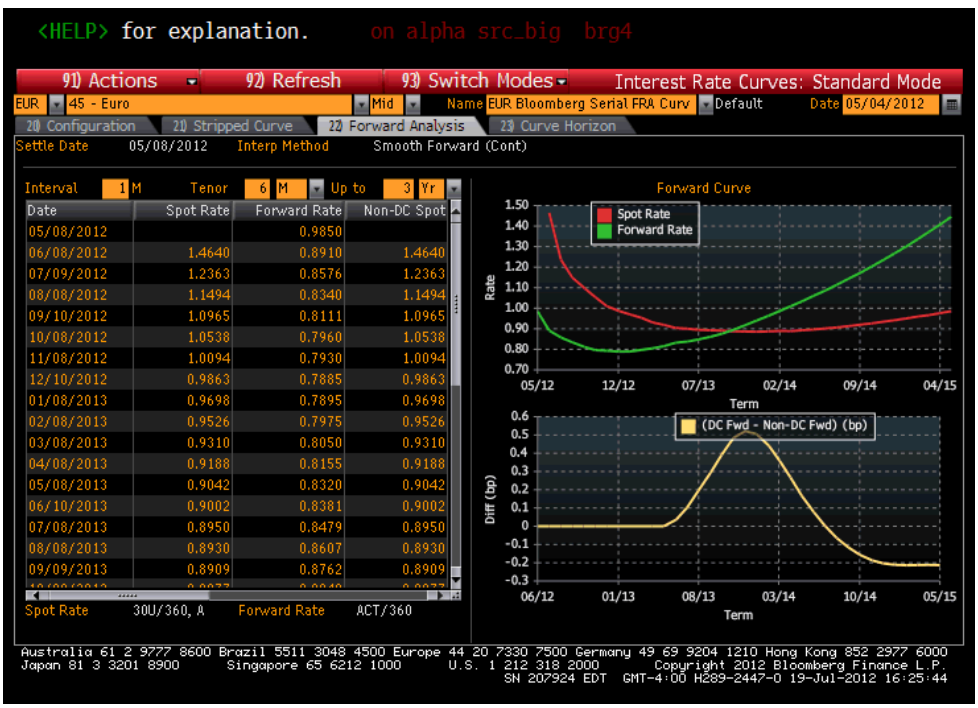


Fig. 4.b Forward curve with artifact correction

Building the OIS-Discounted IR Curves

OIS discounting means discounting the expected cash flows of a derivative using a nearly risk free curve such as an overnight index swap (OIS) curve. OIS-discounted IR curves are built using a dual-curve (DC) stripping technique. To enable OIS-discounting, run Bloomberg function {SWDF DFLT<go>} and make appropriate choice of drop-down menu “Enable OIS Discounting / Dual Curve Stripping” as shown below. Note that this setting is global and applies to all currencies where an OIS curve is available.

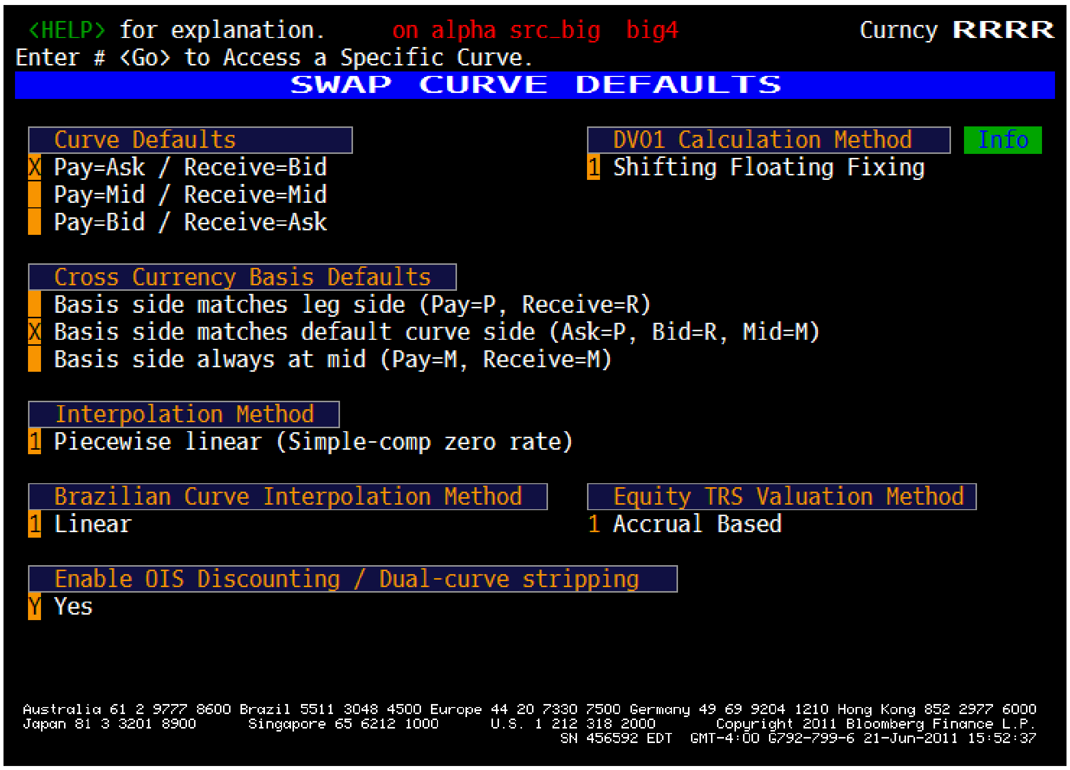


Fig. 5 Enabling/disabling of OIS-discounting using SWDF.

With OIS-discounting, swap rates are calculated using a different formula as its single-curve counterpart because the PV of the floating leg is no longer at par. Let L_i be the fixed rate to be exchanged at time T_i for the LIBOR rate $\mathrm{L}(T_{i-1}, T_i)$ so that the swap has zero value at time 0, and $\mathrm{D}(T_0, T_i)$ be the OIS discount factor from T_2 *to* T_0. Then the swap rate with maturity T_N , $r_{sw}(T_N)$, is calculated per [5] as

$$r_{sw}(T_N) = \frac{\sum_{i=1}^{N} L_i \cdot t_i \cdot D(T_0, T_i)}{\sum_{i=1}^{N} t_i \cdot D(T_0, T_i)} \tag{5}$$

where $T_0 = 0$ and $t_i = T_i - T_{i-1}$.

Given a DC-striped IR curve and associated interpolation method, L_i can be calculated as the forward rate from T_{i-1} to T_i. Therefore DC-stripping simply needs to change how it calibrates to swap rates by using Eq. (5). Fig.6 shows an example of DC-stripped USD IR curve S23 and the forward rate changes when DC-striping is enabled. Note that there are no changes in near-end forward rates because the calibration to cash and FRA/futures rates remains the same in DC-striping.

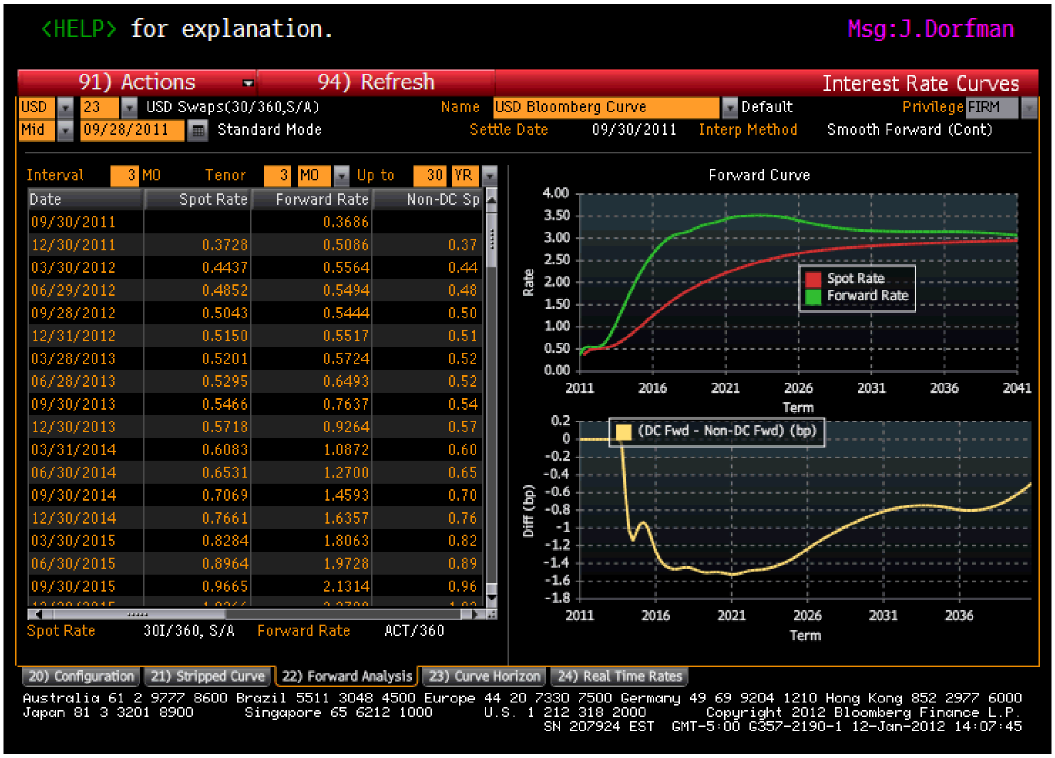


Fig.6 Example of DC-striped S23 and comparison of forwards to single-stripped curve

While effects on the forward rates from using dual-curve stripping are convoluted, the formula below can be used to provide a helpful estimate. Let $\Delta_r(T) = r_{sw}(T) - r_{OIS}(T)$ be the spread in bp between the swap rate and the OIS rate for maturity T, $d_{sw}(T)$ be the annual rate of change in bp of the swap rate curve for maturity T (a measure of curve steepness). Then the forward rate change in bp, $\Delta_f(T)$, can be approximated as:

$$\Delta_f(T) \approx -\frac{0.5 \times \Delta_r(T) \times d_{sw}(T) \times T^2}{10000}$$

For example, if the spread between Libor and OIS at 10 years is 30bp and the swap rate curve is changing by about 14bp a year at 10 year point, then $\Delta_f(10) = -0.5 \times 30 \times 14 \times 100/10000$ = -2.1bp (the observed actual difference is -1.97bp).

Given that the cash and FRA/future rates produce the same forwards since they are stripped identically with or without DC[1], we need to be mindful of artifacts that are present in forwards with maturities

[1] For Smooth Forward interpolation, this assertion is technically an approximation with negligible errors.

between the last FRA/future and the first swap rate. In order to calibrate to the same swap rate, these few forwards have to be adjusted to compensate for applying difference discounting to the other forwards in Eq. (5). These artifacts become more pronounced as more and more FRA/futures are included in the curve. For instance, Fig. 7 shows forwards of the USD swap curve S23 constructed with 20 futures, as opposed to 12 the Bloomberg default setting (Fig.6). In this case there is a sharp difference between 5th year DC and Non-DC forwards.

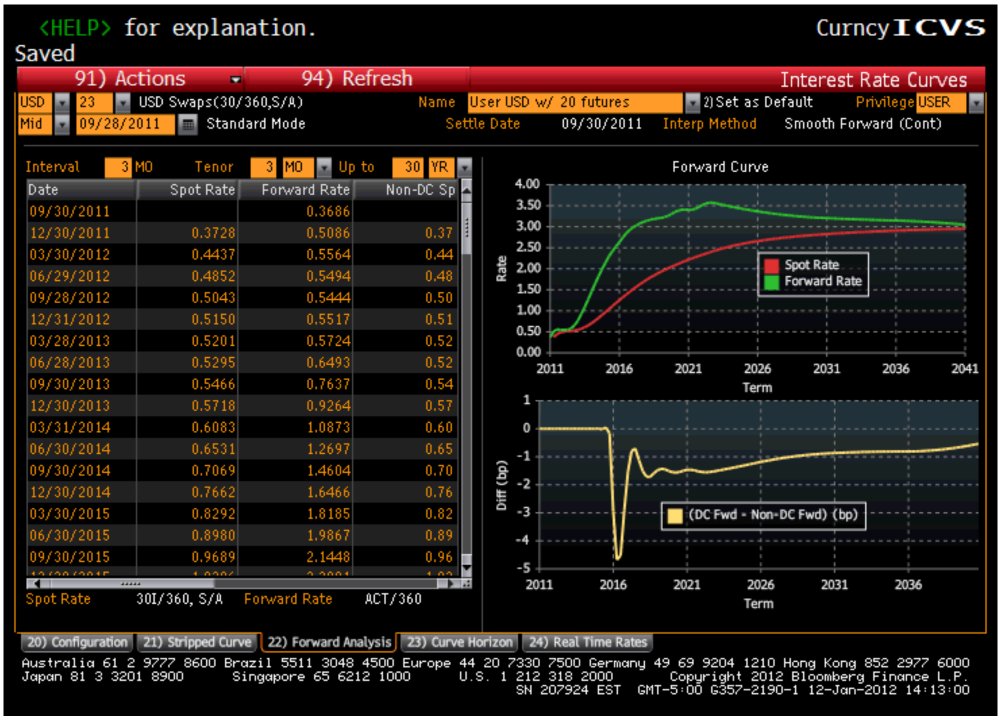


Fig. 7 Example of forwards artifacts

DV01 Calculation and Effects of DV01 and DC Settings

Currently the SWPM function calculates DV01 as follows: shift all curves up and down by 10 bps (all par rates except basis spreads), calculate PVs using the up- and down-shifted curves, and scale the PV difference by 20.

In addition, two DV01 options are offered, via the drop-down menu in {SWDF DFLT<go>} (see Fig.5), to control the behavior of the LIBOR fixing index that is embedded in the swap rates when the par curve is shifted[2]:

1. *(L)ibor Fixing Shifts*: (L)IBOR index fixing shifts with the curve; or
2. *(L)ibor Fixing Constant*: (L)IBOR index fixing remains constant.

[2] Kirill Levin, "DV01 Calculation Method", Bloomberg IDOC #2057712, Sept, 2010.

The current method of DV01 calculation of shifting the par curve can produce counter-intuitive values for some combinations of the DV01 and DC setting. The following table provides an example that summarizes DV01 values under different scenarios for EUR 6M fixed-float swaps entered on 10/3/2011, with those having counter-intuitive values shown in red. Note that DV01 values for 1 year swap are not impacted because the shortest swap maturity on the default curve S45 is 2 years.

Swap Maturity	LIBOR Discounting (Non-DC)				OIS Discounting (DC)			
	Shifting Index		Constant Index		Shifting Index		Constant Index	
	fixed	float	fixed	float	fixed	float	fixed	float
1 yr	1,008.82	-503.83	1,008.82	-503.83	1,011.82	-502.14	1,011.82	-502.14
2 yr	1,961.81	-503.83	2,465.64	-503.83	2,035.87	-556.80	2,035.87	-50.14
3 yr	2,914.05	-503.83	3,417.89	-503.83	3,037.66	-587.31	3,037.66	-80.64
4 yr	3,846.66	-503.83	4,350.49	-503.83	4,020.64	-616.57	4,020.64	-109.90
5 yr	4,758.04	-503.83	5,261.88	-503.83	4,989.53	-650.82	4,989.53	-144.16
10 yr	8,928.94	-503.83	9,432.81	-503.83	9,487.16	-831.71	9,487.16	-325.05
20 yr	15,516.27	-503.83	16,020.19	-503.83	16,721.22	-1,174.81	16,721.22	-668.15
30 yr	20,579.44	-503.83	21,083.42	-503.83	22,404.43	-1,504.61	22,404.43	-997.94

The main cause for the observed counter-intuitive DV01 behavior is the fact that a parallel shift of a par curve does not necessarily result in a parallel shift of the forward curve. As illustrated in Fig. 8, parallel up-shift of the par curve almost always results in humps in forward rate shifts when instruments on the curve transition from cash/futures/FRA rates to swap rates (starting at year 2 in this example). Note that the magnitude of humps is much larger for DV01 option 2 of *Libor Fixing Constant* than for option 1 of *Libor Fixing Shifts*. With *constant fixing*, the first coupon remains constant as the curve shifts, hence the remaining forwards that are bootstrapped from this first swap rate need to have larger shifts to realize the shifted swap rate. The subsequent forwards humps are result of ripple-effects from the first one. Similar humps in forward shift are observed when DC is enabled.

These humps on forward shifts will invariably be reflected in discount factors as well, when the shifted curve is used for discounting. Fig. 8 shows the ratio of discount factors before and after shifting the EUR 6M curve. For reference, discount ratios for the EONIA curve are also shown there. (Only one set of discount ratios is displayed, because EONIA resets daily and the effect of DV01 option is negligible). Note that the large hump in the forward rate observed in Fig.8 is now manifested here as a small decrease in discount factor after year 2. Effects of down-shifting the par curve are near mirror images of up-shifting.

With the help of Figs 8 and 9, we can explain why DV01 may behave in a counter-intuitive manner.

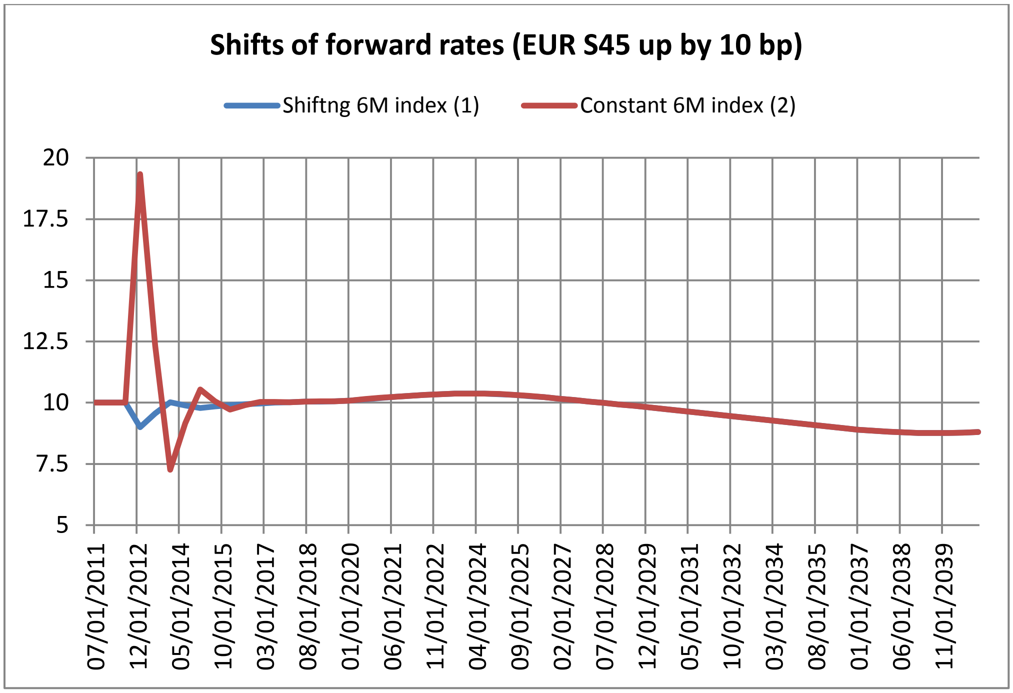


Fig. 8 Forward shifts as a result of par curve shifting (EUR 6M S45: mid/non-DC/curve date = 7/25/11)

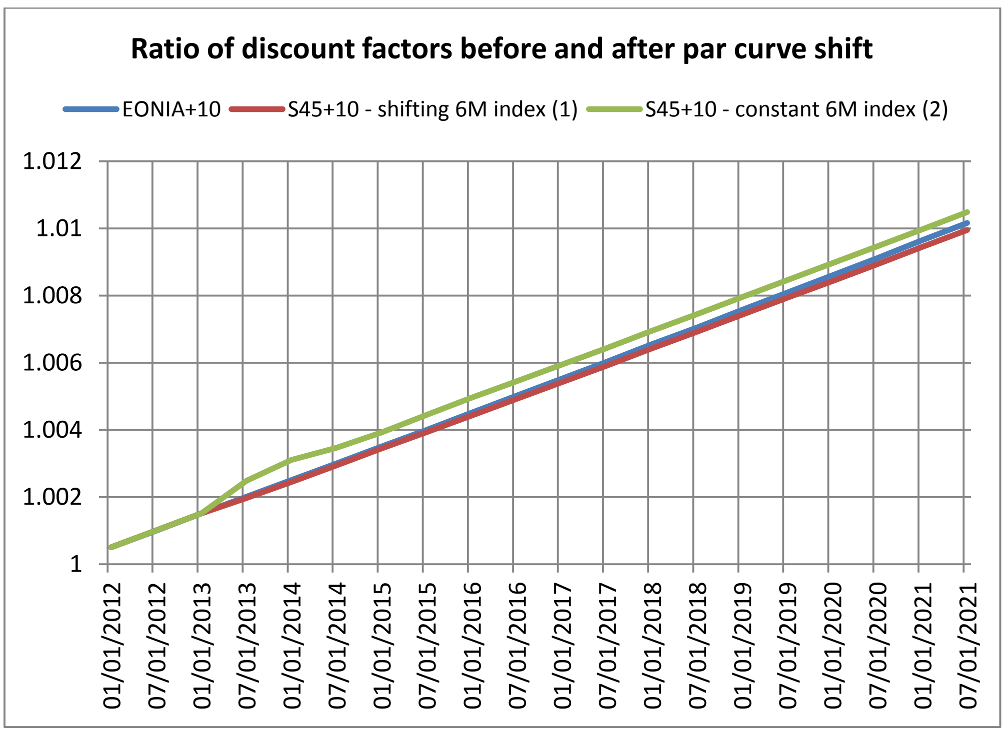


Fig. 9 Ratio of discounts resulting from shifting EUR 6M curve & EONIA curve (mid on 7/25/11)

Without OIS Discounting: A single curve is used in this case for both forward projection and discounting. Since the floating leg PV only depends on the first projected forward rate, its DV01 is hence immune from the aforementioned forward rate humps. But the fixed leg cannot escape the impact of the humps on forward shifts. Due to the bias in discount factors observed in Fig 9, option 2 (constant index) will produce a larger DV01 value than option 1 (shifting index). The difference is approximately equal to the equivalence of the LIBOR fixing index duration. For example, DV01 for 6M EUR swaps with 10M notional will differ by approximately 500 (~6Mo) between the two DV01 options.

With OIS Discounting: With OIS discounting and DC, the opposite effect is observed. Fixed legs no longer suffer from the DV01 step-up when switching from option 1 to option 2, because OIS resets daily and whether the fixing index shifts with the curve or not has virtually no impact on the resulting discount factors. However the float leg PV valuation now requires all cash flows to be explicitly calculated up to the swap maturity and then discounted back to present. Since the humps in forwards can no longer be canceled out by the changes in discounts, they become exposed through the DV01 values in the form of unexpected bumps, especially when DV01 option 2 is selected.

Conclusion

This document describes the process of building the Bloomberg interest rate curves. We have discussed definitions, systems of equations to solve and algorithms used to solve them. The four different functional forms used in Bloomberg terminal for the IR curve are considered. We used an assumption that the IR curve describes risk-free present values. However, in reality some of the instruments used to build the curve have non-zero credit risk, which sometimes affects the shape of the curve thus producing curves with unreasonable dips. Also some of the instruments could have much lower liquidity (or frequency of quotes) than others, thus producing stale quotes. If the curve contains unreasonable oscillations such as on Fig. 3c, which probably do not have a financial meaning, this suggests that the choice of the instruments used to build the curve should be revised. These considerations must be kept in mind by user when choosing the instruments for IR curve construction.

Appendix 1: Short Description of a Few Most Popular Day Count Conventions

According to day count convention called *ACT/360* (where ACT is short for ACTUAL),

$$t(d_1, d_2) = \frac{d_2 - d_1}{360} \tag{1.1}$$

Here the operation $d_2 - d_1$ returns the actual number of calendar days between two dates.

Similarity, the rule *ACT/365* works as follows:

$$t(d_1, d_2) = \frac{d_2 - d_1}{365} \tag{1.2}$$

In the rule *ACT/ACT*, each date counts as 1/366 if it belongs to a leap year and 1/365 otherwise.

The rule 30/360 is the most complicated one and aims to return

$^1/_{12}$ for two dates separated by one month,

¼ for two dates separated by a quarter,

½ for a half-a year and

1 for a year.

We will omit here the description of an actual algorithm for this rule. It is not used in IR curves, it is used when calculating coupon payments of some bonds or fixed legs of IR swaps. The intent of this rule is for a periodically (say, quarterly) paying instrument to have interest rate payments more or less constant despite the fact that actual number of days between the payments varies due to different length of months.

In all formulas for calculating discount factors any rate, no matter simple or compounded, appears only as a product r*t, where t is time in years calculated according to some day count convention. Because the discount factor between two dates is fixed by the market, but day count convention is ours to choose, we can get different times t using different day count conventions, and therefore the correspondent rates will be different. Because of that, when quoting the rate, it is important to note not only which rate it is (and frequency of compounding) but the day count convention too. In this document every time we mention a conversion of a pair of dates into a time interval t in years, it will be assumed that a day count convention is specified.

Appendix 2: Convexity Adjustment or Conversion of a Futures Rate into a Forward Rate

In this appendix we provide a short digest of discussions [1-4].

The forward IR agreement (FRA) is a simple instrument: at time t=0 (e.g. now) it guarantees that one can receive a loan at time t_1 expiring at time t_2 at a rate of r_{sf} for some notional amount M. Once we have reached the time t_1, and the interest rate for a loan from that time to time t_2 is known, one can immediately short a loan for the same amount at a prevailing rate and have a guaranteed risk-free payment of the difference at time t_2. Discounting that difference to time t_1, one can immediately settle the contract for this amount at time t_1. At time t=0 the rate r_{sf} is chosen such that the present value of such contract at inception is 0.

The futures contract is slightly different: instead of being settled once at time t_1, it is settled at the end of each business day between the inception and t_1 in such way as to bring the PV of the contract to 0.

Hull [1] argues that when interest rates are constant, forward and futures rates would be the same. However, under more realistic assumptions that the interest rates undergo some kind of random process, this difference in the method of the settlement causes the rates of forward and futures contracts to differ slightly:

$$forwardRate = futuresRate - convAdjustment \qquad (2.1)$$

where the expressions for *convAdjustment* are discussed below.

Under the Hall-White model the instantaneous forward rate undergoes a random walk with mean reversion described by SDE:

$$dr = (\theta(t) - a \cdot r)dt + \sigma \cdot dW \qquad (2.2)$$

here r is an instantaneous continuously compounded interest rate;

σ is the constant volatility of this rate;

a is the constant mean-reversion coefficient;

$\theta(t)$ is a function of time which is chosen to match the prices of the known instruments, e.g. bonds or IR swaps (a procedure of calculating this function is called *calibration* of the model).

Under this model the convexity adjustment is calculated as [2]:

$$convAdjustment = \frac{B(t_1,t_2)}{t_2-t_1}[B(t_1,t_2)(1-\exp(-2at_1))+2aB(0,t_1)^2]\frac{\sigma^2}{4a} \qquad (2.3)$$

Where

$$B(t,T) = \frac{1-\exp(-a(T-t))}{a} \qquad (2.4)$$

In the limit of mean-reversion coefficient $a \to 0$, the Hall-White model becomes a Ho-Lee model:

$$\text{Dr} = \theta(t)dt + \sigma \cdot dW \qquad (2.5)$$

and the expression (2.3) for the convexity adjustment becomes [4]:

$$convAdjustment = \frac{1}{2}\sigma^2 t_1 t_2 \qquad (2.6)$$

This is the simplest formula for convexity adjustment used to convert futures rates into forward rates. It is convenient for the back-of-the-envelope calculations, and also useful for giving a user an intuitive feeling of the order of magnitude and behavior of the convexity adjustment.

The Bloomberg implementation is close to the one given by expressions (2.3-2.4) with additions to convert the rates compounded according to the specifications of the contract (for instance, if the loan is for t_2-t_1=3 mo, the rate is compounded quarterly) into continuously compounded rate and back.

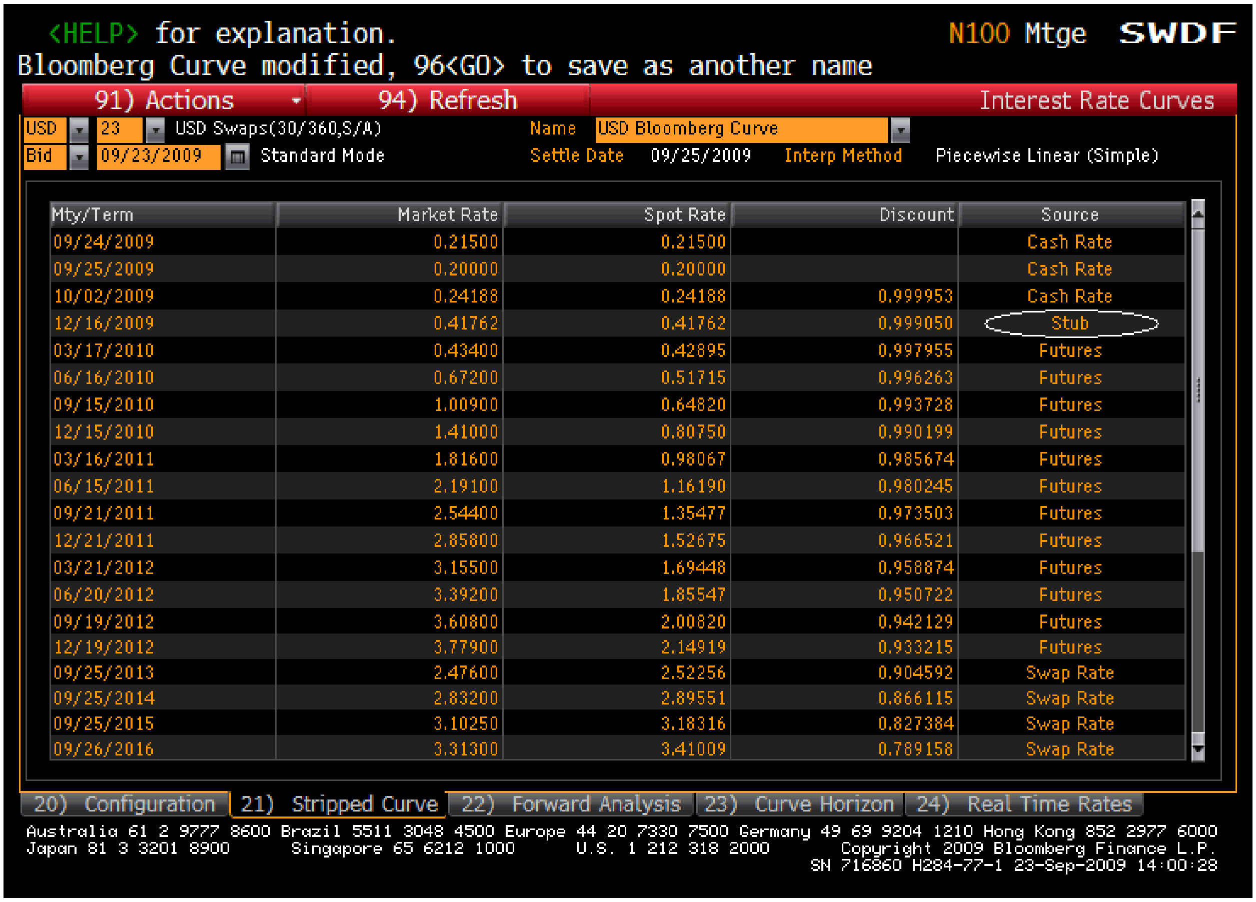
Appendix 3: Stub Rate or connecting the Cash Rates and Futures

There are two possible scenarios:

1. The cash rates end at the date or after the futures (or FRAs) start;
2. Cash rates end before the futures start.

In the first scenario the discount factor at the first futures date is calculated using interpolation of cash rates. This is the case with the default Bloomberg curve for USD.

In case of the second scenario there exists a time gap between the end of the cash rates and start of the futures, and we need to make an estimate of a rate in this gap in order to extend the curve to the point where the futures start. For historical reasons this point is called a *stub*, and one can see whether the scenario 2 has occurred by typing on Bloomberg terminal **SWDF 23<go>,** unselecting enough cash rates and/or futures and hitting **<go>** so that the system can accept the changes, and then clicking on the '**21) Stripped Curve**' tab. If you see the stab point shown on Fig. 10, we have the scenario 2.



<HELP> for explanation. N100 Mtge SWDF
Bloomberg Curve modified, 96<GO> to save as another name
91) Actions 94) Refresh Interest Rate Curves
USD 23 USD Swaps(30/360,S/A) Name USD Bloomberg Curve
Bid 09/23/2009 Standard Mode Settle Date 09/25/2009 Interp Method Piecewise Linear (Simple)

Mty/Term	Market Rate	Spot Rate	Discount	Source
09/24/2009	0.21500	0.21500		Cash Rate
09/25/2009	0.20000	0.20000		Cash Rate
10/02/2009	0.24188	0.24188	0.999953	Cash Rate
12/16/2009	0.41762	0.41762	0.999050	Stub
03/17/2010	0.43400	0.42895	0.997955	Futures
06/16/2010	0.67200	0.51715	0.996263	Futures
09/15/2010	1.00900	0.64820	0.993728	Futures
12/15/2010	1.41000	0.80750	0.990199	Futures
03/16/2011	1.81600	0.98067	0.985674	Futures
06/15/2011	2.19100	1.16190	0.980245	Futures
09/21/2011	2.54400	1.35477	0.973503	Futures
12/21/2011	2.85800	1.52675	0.966521	Futures
03/21/2012	3.15500	1.69448	0.958874	Futures
06/20/2012	3.39200	1.85547	0.950722	Futures
09/19/2012	3.60800	2.00820	0.942129	Futures
12/19/2012	3.77900	2.14919	0.933215	Futures
09/25/2013	2.47600	2.52256	0.904592	Swap Rate
09/25/2014	2.83200	2.89551	0.866115	Swap Rate
09/25/2015	3.10250	3.18316	0.827384	Swap Rate
09/26/2016	3.31300	3.41009	0.789158	Swap Rate

20) Configuration 21) Stripped Curve 22) Forward Analysis 23) Curve Horizon 24) Real Time Rates
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2009 Bloomberg Finance L.P.
SN 716860 H284-77-1 23-Sep-2009 14:00:28

Fig. 10 Stub point on USD curve 23.

In case of the scenario 2 we have further two sub-cases:

a) Among cash rates there is a rate with a tenor which matches the tenor of the futures, (e.g. 3mo);
b) In this case a) is false.

In case a) the software does interpolation between the cash rate with the same tenor as futures and the first futures rate, i.e. the rate in the gap is assumed to be an average of the cash rate and futures rate. Otherwise the first futures rate is extrapolated into the gap between the cash and futures.

Appendix 4: Functional Form of Smooth Curve (Piecewise Quadratic Forward Rate)

The functional form of this curve is the piecewise quadratic continuously compounded forward rate r_{cf}. The first derivative of the rate is continuous over the borders between the pieces. Also the first derivative of the forward rate is zero at the beginning and at the last maturity time:

$$\frac{dr(0)}{dt} = 0 \tag{4.1a}$$

$$\frac{dr(T_N)}{dt} = 0 \tag{4.1b}$$

The discount factor of the smooth curve is defined as:

$$D(t) = \exp(-L(t)) \tag{4.2}$$

where the log discount factor L(t) is:

$$L(t) = \lambda_0 \cdot t + \tfrac{1}{6} \sum_{T_n < t} \lambda_n (t - T_n)^3 - \tfrac{1}{6} P \cdot t^3 \tag{4.3}$$

here T_n is the maturity of the n-th security used to build the curve, e.g. the end time of the curve's n-th piece;

the summation is done for all n=1 .. N-1 such that $T_n < t$, which leads to a curve L(t) being piecewise cubic. At each time $t=T_n$ a new additional cubic polynomial starts to the right from the point $t=T_n$ in a manner which assures that first and second derivatives of L(t) are continuous.

The coefficients λ_n (where n=0 .. N-1) are the independent parameters. The last λ_N is not an independent parameter, but defined as:

$$\lambda_N = -\frac{1}{T_N} \sum_{n=1}^{N-1} \lambda_n T_n$$

The parameter P is defined as:

$$P = \sum_{n=1}^{N} \lambda_n$$

This choice of P assures that for t>T_n the cubic term in L (and quadratic in forward rate) is zero. The choice of λ_N similarly assures that for t>T_n the quadratic term in L (and linear in forward rate) is zero. The curve is 'smooth' e.g. the forward rate has a continuous first derivative. The curve described above satisfies all the boundary conditions, e.g. continuous first derivative of the forward rate between the pieces or zero derivatives at the ends of the curve by construction, and it has N degrees of freedom. However, it is easy to see that change in any parameter λ_n, where n=0 .. N-1, leads to changes in λ_N and P and thus affects the value of the function everywhere. Therefore to build a curve one needs to solve a system of N non-linear equations with N variables in a general way, which is described in Appendix 4.

Appendix 5: Building a Smooth Curve: Solving Non-linear System of N Equations with N Variables

One of the algorithms for solving a system of N non-linear equations with N variables is an N-dimensional Newton-Raphson method.

Let us start with one-dimensional version of this algorithm. Suppose, we need to solve an equation:

$$f(x) = 0$$

In Newton-Raphson method we also assume that for any x we know a first derivative df/dx (this means that we have a fast and exact way of calculating it, and in computer programming terms it means that we have a separate function which returns this first derivative).

One can see that if we have the current value of the function $f(x_i)$, one needs correct x_i in first approximation by

$$dx = -\frac{f(x_i)}{df(x_i)/dx}$$

and therefore the value of x for next iteration is:

$$x_{i+1} = x_i - \frac{f(x_i)}{df(x_i)/dx}$$

One continues these iterations until a condition of the type $|f(x)| < \varepsilon_y$ or $|dx| < \varepsilon_x$ or both is satisfied.

Now let us consider the N-dimensional version of the same: we have an N-dimensional vector function $\vec{f}(\vec{x})$ of N-dimensional vector $\vec{x}$. Given some current approximation to the solution $\vec{x}_i$, the next iteration can be found as:

$$\vec{x}_{i+1} = \vec{x}_i - d\vec{x}_i$$

where $d\vec{x}_i$ is found as a solution of:

$$\vec{f}(\vec{x}_i) = J \cdot d\vec{x}$$

here J is an N x N Jacobian matrix defined as

$$J_{k.m} = \frac{\partial f_k}{\partial x_m}$$

The initial approximation to solution $\vec{x}_0$ is found using one of the interpolation methods which uses the bootstrap algorithm, for instance, interpolation method 1 (linear simple rate). Then we use iterative multi-dimensional Newton-Raphson process described above.

References

[1] John C. Hull, Options, Futures and Other Derivatives, 4th edition, Appendix 3B on p. 85, 6th edition – Appendix on p. 127.

[1] John C. Hull, Options, Futures and Other Derivatives, Fifth edition, chapter 23, page 566

Same in Fourth edition: formula (21.32) on p. 595; absent in 6th edition.

[3] G. Kirikos and D. Novak, Convexity Conundrums, Risk, March 1997, pp 60-61

[4] John C. Hull, Options, Futures and Other Derivatives, 6th edition, formula (6.3) on p. 140.
Same in 4th edition: see section 21.16 on p. 595.

[5] Fabio Mercurio, The New Swap Math, OTC Derivatives and Structured Notes, Bloomberg LP