THE NEW SWAP MATH

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ABSTRACT

In this short document we describe the valuation formula for swaps in a multi-curve context and hint at the dual-curve bootstrapping of LIBOR projections from market interest-rate data.

ASSUMPTION ON THE DISCOUNT CURVE

We introduce the following assumption on the (assumed single) discount curve, for a given currency:

The discount curve is the OIS zero-coupon curve, which is stripped from market OIS swap rates and defined for every possible maturity, T:

$$T \rightarrow P_D(0, T) = P^{OIS}(0, T),$$

Where $P_D(t, T)$ denotes the discount factor (zero-coupon bond) at time t for maturity T, which is assumed to coincide with the corresponding OIS-based zero-coupon bond for maturity T. The $_D$ stands for "discount curve."

The rationale behind this assumption is that in the interbank derivatives market, a collateral agreement (CSA) is often negotiated between two counterparties. The CSA is set to mitigate the credit risk of both parties, allowing them to establish bilateral mark-to-market collateral arrangements. We assume here that the collateral, typically a bond or cash, is revalued daily at a rate equal (or close) to the overnight rate, which can thus justify the use of OIS rates for discounting.

In the following, as in Kijima, et al. (2009), the pricing measures we will consider are those associated with the discount curve. This is also consistent with the results of Fujii, et al. (2009) and Piterbarg (2010), since we assume CSA agreements where the collateral rate to be paid equals the (assumed risk-free) overnight rate.

DEFINITION OF FRA RATE AND ITS PROPERTIES

Definition 1

Classically, an FRA is defined according to the following[1]:

[1] This definition of the FRA rate slightly differs from that implied by the actual market contract. This abuse of terminology is justified because this "theoretical" FRA rate and the market coincide in a single-curve setting. In our multi-curve case they are different, but their difference can be shown to be negligible under typical market conditions.

Consider times t, $T1$ and T_2, $t \leq T_1 < T_2$. The time-t FRA rate ***FRA***$(t;T_1,T_2)$ is defined as the fixed rate to be exchanged at time T_2 for the Libor rate $L(T_1,T_2)$ so that the swap has zero value at time t.

Denoting by QTD the T-forward measure whose associated numeraire is the zero-coupon bond $P_D(t, T)$, by (risk-adjusted) no-arbitrage pricing, we immediately have

$$\text{(1)} \quad \mathbf{FRA}(t;T_1,T_2) = E_D^{T_2}\left[L(T_1,T_2)|\mathcal{F}_t\right],$$

Where E^T_D denotes expectation under Q^T_D and F_t denotes the "information" available in the market at time t.

In the classic single-curve valuation, i.e., when the Libor curve corresponding to tenor $T_2 - T_1$ coincides with the discount curve, the FRA rate ***FRA***$(t;T_1,T_2)$ coincides with the forward rate:

$$\text{(2)} \quad F_D(t;T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{P_D(t,T_1)}{P_D(t,T_2)} - 1\right]$$

In fact, the Libor rate $L(T_1,T_2)$ can be defined by the classic relation:

$$\text{(3)} \quad L(T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{P_D(T_1,T_2)} - 1\right] = F_D(T_1;T_1,T_2),$$

So that we can write:

$$\mathbf{FRA}(t;T_1,T_2) = E_D^{T_2}\left[F_D(T_1;T_1,T_2)|\mathcal{F}_t\right]$$

Since $F_D(t;T_1,T_2)$ is a martingale under $Q^{T}{}_{2D}$, we can then conclude that:

$$\mathbf{FRA}(t;T_1,T_2) = F_D(t;T_1,T_2)$$

However, in our dual-curve setting, (3) no longer holds, since the simply compounded rates defined by the discount curve are different, in general, from the corresponding Libor fixings.

Our FRA rate is the natural generalization of a forward rate to the dual-curve case. In particular, we notice that at its reset time T_1 the FRA rate ***FRA***$(T_1;T_1,T_2)$ coincides with the Libor rate $L(T_1,T_2)$. Moreover, the FRA rate is a martingale under the corresponding pricing measure. These properties will prove to be very convenient when pricing swaps and options on Libor rates.

PRICING OF INTEREST RATE SWAPS

Let us consider a set of times $T_a,...,T_b$ compatible with a given tenor[2] and an IRS where the floating leg pays at each time Tk the LIBOR rate $L(Tk-1,Tk)$ set at the previous time $Tk-1$, where $k = a +1,...,b$, and the fixed leg pays the fixed rate K at times $TcS +1,...,T dS$.

Under our assumptions on the discount curve, swap valuation is straightforward.[3] Applying Definition 1 and setting:

[2] For instance, if the tenor is three months the times T_k must be three-month spaced.

[3] Details of the derivation can be found in Chibane and Sheldon (2009), Henrard (2009), Kijima, et al. (2009) and Mercurio (2009).

$$L_k(t) := \mathbf{FRA}(t; T_{k-1}, T_k) = E_D^{T_k}[L(T_{k-1}, T_k)|\mathcal{F}_t]$$

The IRS time-t value, to the fixed-rate payer, is given by:

$$\mathbf{IRS}(t, K; T_a, \ldots, T_b, T_{c+1}^S, \ldots, T_d^S) = \sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)$$

Where T_k and T^S_j denote, respectively, the floating-leg year fraction for the interval $(T_{k-1},T_k]$, and the fixed-leg year fraction for the interval $(T^S_{j-1},T^S_j]$.

The corresponding forward swap rate, which is the fixed rate K that makes the IRS value equal to zero at time t, is then defined by:

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} \quad (4)$$

In the particular case of a spot-starting swap, with payment times for the floating and fixed legs given by $T_1,\ldots,T_b$ and $T_1^S,\ldots,T_d^S$, respectively, with $T_b = T_d^S$, the swap rate becomes:

$$S_{0,b,0,d}(0) = \frac{\sum_{k=1}^{b} \tau_k P_D(0, T_k) L_k(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)} \quad (5)$$

Where $L_1(0)$ is the constant first floating payment (known at time 0).

As already noticed by Kijima, et al. (2009), neither leg of a spot-starting swap needs to be worth par (when a fictitious exchange of notionals is introduced at maturity). However, this is not a problem, since the only requirement for quoted spot-starting swaps is that their initial NPV must be equal to zero.

A comparison between the two swap-rate formulas in the single- and dual-curve setups is provided in Table 1. In the single-curve case, the uniquely defined zero-coupon curve coincides with the discount curve.

Swap rate	Formulas
OLD	$\frac{\sum_{k=a+1}^{b} \tau_k P_D(t,T_k) F_D(t;T_{k-1},T_k)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)} = \frac{P_D(t,T_a) - P_D(t,T_b)}{\sum_{j=1}^{d} \tau_j^S P_D(t,T_j^S)}$
NEW	$\frac{\sum_{k=a+1}^{b} \tau_k P_D(t,T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)}$

Table 1: Comparison between "old" and "new" formulas for forward swap rates.

STRIPPING THE LIBOR PROJECTIONS

As traditionally done in any bootstrapping algorithm, equation (5) can be used to infer the expected (risk-free) rates L_k implied by the market quotes of spot-starting swaps, which by definition have zero value. Given that, by the above assumption, the discount curve has already been bootstrapped from market OIS rates, the discount factors $P_D(0,T)$, $T \in \{T_1,\ldots,T_b,T_1^S,\ldots,T_d^S\}$ entering formula (5) are all known. The FRA rates $L_k(0)$ can thus be iteratively derived by matching the market quotes of rates based on the same Libor tenor as the one under consideration.[4]

The bootstrapped L_k can then be used, in conjunction with any interpolation tool, to price off-the-market swaps based on the same underlying tenor. As already noticed by Boenkost and Schmidt (2005) and by Kijima, et al. (2009), these other swaps will have different values, in general, than those obtained by stripping discount factors through a classic (single-curve) bootstrapping method applied to swap rates:

$$S_{0,d}(0) = \frac{1 - P_D(0,T_d^S)}{\sum_{j=1}^{d} \tau_j^S P_D(0,T_j^S)}$$

So that the choice of discount factors $P_D(0,T_j^S)$ heavily affects the IRS value of off-the-market fixed rates K.

[4] Details on a similar curve construction methodology can be found in Chibane and Sheldon (2009), Henrard (2009) and Fujii, et al. (2009). The analysis in Fujii, et al. (2009) is more thorough, since they consider a general collateral rate in a multi-currency environment.

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