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Lecture 22

In this lecture we introduce the matrix completion problem, definitions, and sketch of optimality and uniqueness using weak duality.

1 Matrix Completion Introduction

1.1 Notation and problem statement

The goal of matrix completion is to try to estimate and fill out the *missing* entries of a partially observed matrix, in other words, reconstruct the entire matrix. This problem occurs in many real world scenarios, such as the Netflix problem where one constructs a recommender system to try to predict user ratings from partially observed user ratings.

Throughout the following notes we will use the following: $\cdot$ for inner product, bold capitalized letters for matrices, bold lower cased letters for vectors, and lower case letters for scalars.

We are given a matrix $\mathbf{M} \in \mathbb{R}^{(n,m)}$ that looks like the following where [ ] indicates an observed entry (i, j):

$$\mathbf{M} = \begin{bmatrix} [\,] & & & & \\ & & [\,] & & \\ & [\,] & & & \\ & & & & \\ & & & & [\,] \end{bmatrix}$$

We denote Ω of size *poly* $\log(n)$ as the set of all (i, j) pairs that are observed in $\mathbf{M}$.

1.2 Assumptions

In order to make analysis tractable, we have the following assumptions:

1. $\mathbf{M}$ has a low rank r. This can be interpreted as the $\mathbf{M}$ being described by a few factors leading to a similar assumption (2).

2. $\mathbf{M}$ exists in a compact representation.

3. The observable entries of $\mathbf{M}$ occur at random i.i.d. Bernoulli.
4. $\mathbf{M} \in \mathbb{R}^{(n,m)}$ can be factored into two matrices $\mathbf{U} \in \mathbb{R}^{(n,r)}$ and $\mathbf{V}^T \in \mathbb{R}^{(r,m)}$ for $\mathbf{M} = \mathbf{U} \cdot \mathbf{V}^T$.

Even with these assumptions, analysis is still intractable so we define a notion of coherence.

2 Coherence

In words, coherence can be thought of as *how big is the projection of the standard basis on U*. Mathematically, coherence is defined to be the subspace $\mathbf{U}$ of $\mathbb{R}^n$ of dimension r:

$$\frac{n}{r} \cdot \max_{i=1\ldots n} ||P_u \mathbf{e}_i||_2^2$$

The reason for normalization is that if $\mathbf{U}$ is a random subspace of dimension r, we would expect the coherence to be $\sim O(1)$. This is described below:

In the following we have $\mathbf{U} = span\{\mathbf{u}_1, ..., \mathbf{u}_r\}$ where $\mathbf{u}_i$ are random orthogonal vectors and i is the corresponding standard basis vector with $\mathbf{u}_j$:

$$||P_u \mathbf{e}_i||_2^2 = \sum_{j=1}^{r} \langle \mathbf{u}_j, \mathbf{e}_i \rangle^2$$

Inside the sum we have

$$\langle \mathbf{u}_j, \mathbf{e}_i \rangle^2 \text{ in expectation is } (\frac{1}{\sqrt{n}})^2 = \frac{1}{n}$$

$$\sum_{j=1}^{r} \langle \mathbf{u}_j, \mathbf{e}_i \rangle^2 = \frac{r}{n}$$

3 Main Theorem

Given the following matrices and conditions:

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

$$\mathbf{U} \in \mathbb{R}^{(n_1,r)} \quad (1) \qquad \mathbf{V} \in \mathbb{R}^{(n_2,r)} \quad (3)$$

$$\mathbf{M} \in \mathbb{R}^{(n,m)} \quad (2) \qquad \mathbf{\Sigma} \in \mathbb{R}^{(r,n_2)} \quad (4)$$

1. Columns of $\mathbf{U}$ and columns of $\mathbf{V}$ have coherence $\leq c$
2. $||\mathbf{U}\mathbf{V}^T||_\infty \leq c' \sqrt{\frac{r}{n_1 n_2}}$, in other words, the LHS has a small infinity norm

then, w.h.p. after seeing $O(c + c'^2) * r * (n_1 + n_2) \log^2(n_1 n_2)$ random entries of $\mathbf{M}$, we can exactly reconstruct $\mathbf{M}$.

Note that we must examine *at least* $r(n_1 + n_2)$ entries. We also have the entries of $(\mathbf{U}\mathbf{V}^T)_{ij}$ to be:

$$(\mathbf{U}\mathbf{V}^T)_{ij} = \sum_{k=1}^{r} \mathbf{U}_k^{(i)} * \mathbf{V}_k^{(i)}$$

Giving a variance of $\sim r * \frac{1}{n_1 n_2} = \frac{r}{n_1 n_2}$.

We will show using weak duality that there exists a unique and optimal solution.

4 Algorithm

Given Ω as a set of i, j observed pairs with size $poly \log(n)$ and $\mathbf{M}_{i,j} \in \Omega \ \forall i, j$ we have the following algorithms where $\mathbf{X}$ is the recovered matrix we are trying to find:

$$\min \ \text{rank}(\mathbf{X}) \quad s.t.$$

$$\mathbf{X}_{ij} = \mathbf{M}_{ij} \quad \forall i, j \in \Omega$$

or written as a SDP solved with stochastic gradient descent (next lecture) since this is non-convex

$$\min \sum_{i,j \in \Omega} (\mathbf{M}_{ij} - \mathbf{X}_{ij})^2 \quad s.t.$$

$$\text{rank}(\mathbf{X}) = r$$

5 Analysis

For analysis, we will take the convex relaxation of the rank of the first linear program. This is the nuclear norm of the matrix. Then we will use weak duality to argue that the solution we get is the unique optimum.

5.1 Nuclear norm of a matrix

The nuclear norm is defined to be the sum of the singular values of a matrix:

$$||\mathbf{M}||_* = \sum_i \sigma_i$$

and $\text{rank}(\mathbf{M})$ = number of non-zero singular values. We also have the Frobeus norm, which is the max singular value.

5.2 SDP with the nuclear norm

With the nuclear norm, we can rewrite the LP as the following:

$$\min ||\mathbf{X}||_* \quad s.t.$$

$$\mathbf{X}_{ij} = \mathbf{M}_{ij} \quad \forall i, j \in \Omega$$

Since $\mathbf{M}$ is a square symmetric matrix (recall $\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^T$) where $\boldsymbol{\Sigma}$ are the non-zero eigenvalues, we have $||\mathbf{M}||_* = \text{trace}(\mathbf{M})$.

Now we can rewrite this as a semi definite program. Shown below is the primal and then the dual.

$$\min \ \text{trace}(\mathbf{X}) \quad s.t.$$

$$\mathbf{X}_{ij} = \mathbf{M}_{ij} \quad \forall i, j \in \Omega$$

$$\mathbf{X} \geq 0$$

Dual, where $\mathbf{I}$ is the identity and $\mathbf{Y}$ are the dual variables:

$$\max \sum_{i,j} Y_{i,j} \quad s.t.$$

$$\mathbf{Y} \preceq \mathbf{I}$$

$$\mathbf{Y}_{i,j} = 0 : i, j \notin \Omega$$

5.3 Weak duality

Suppose $\mathbf{X}$ is feasible for the primal and $\mathbf{Y}$ is feasible for the dual, then we have

$$\text{trace}(\mathbf{X}) = \mathbf{I} \cdot \mathbf{X} \geq \mathbf{Y} \cdot \mathbf{X} = \sum_{i,j \in \Omega} \mathbf{Y}_{i,j}\mathbf{X}_{i,j} = \sum_{i,j \in \Omega} \mathbf{M}_{i,j}\mathbf{Y}_{i,j}$$

In the previous, if the inequality were an equality then this would mean the primal solution is equal the dual solution, showing that we have an optimal solution.

We want to prove, under the assumptions outlined in the main theorem in section 3 that we have

$$\mathbf{Y} : \mathbf{Y} \preceq \mathbf{I}, \quad \mathbf{Y}_{i,j} = 0 \quad \forall i, j \notin \Omega \quad s.t.$$

$$\mathbf{I} \cdot \mathbf{M} = \mathbf{Y} \cdot \mathbf{M} \qquad \text{for optimality}$$

$$\forall \mathbf{X} \neq \mathbf{M}, \quad \mathbf{X}_{i,j} = \mathbf{M}_{i,j} \quad \forall i, j \in \Omega \quad s.t.$$

$$\mathbf{X} \text{ is feasible }, \mathbf{I} \cdot \mathbf{X} > \mathbf{Y} \cdot \mathbf{X} \qquad \text{for uniqueness}$$

5.4 Decomposition of all matrices $\in \mathbb{R}^{(n,n)}$

Consider a decomposition of $\mathbb{R}^{n,n}$ into 2 orthogonal subspaces denoted as $\mathbf{T}$ and $\mathbf{T}^{\perp}$ of size $\mathbb{R}^{(n^2,n^2)}$. We have the following:

$$\mathbf{A} = P_{\mathbf{T}}(\mathbf{A}) + P_{\mathbf{T}^{\perp}}(\mathbf{A})$$

where using the two decomposition operations, we have

$$P_{\mathbf{T}^{\perp}}(\mathbf{A}) = (\mathbf{I} - \mathbf{P_U})\mathbf{A}(\mathbf{I} - \mathbf{P_U})$$

$$P_{\mathbf{T}}(\mathbf{A}) = \mathbf{A} - P_{\mathbf{T}^{\perp}}(\mathbf{A}) = \mathbf{P_U}\mathbf{A} + \mathbf{A}\mathbf{P_U} - \mathbf{P_U}\mathbf{A}\mathbf{P_U}$$

These operations allow for any matrix to be decomposed into 2 parts and the following:

1. first set as the set of vectors in kern($\mathbf{M}$)
2. second set as all other vectors
3. $P_{\mathbf{T}}(\mathbf{M}) = \mathbf{M}$

5.5 Restriction

Next we define a restriction denoted as $R_{\Omega}(\mathbf{A}) = \mathbf{A}$ where the entries not in Ω are replaced with 0. Note that this operation is a linear matrix operator.

5.6 Recover Y

In order to recover $\mathbf{Y}$ (Candes, Recht), we define the following:

$$\mathbf{Y} = R_{\Omega} P_{\mathbf{T}} (P_{\mathbf{T}} R_{\Omega} P_{\mathbf{T}})^{-1} \cdot \mathbf{U}\mathbf{U}^T \quad \in \mathbb{R}^{(n,n)}$$

$$\mathbf{M} = \sum_{k=1}^{r} \lambda_k \mathbf{u}_k \mathbf{u}_k^T \qquad \mathbf{U}\mathbf{U}^T = \sum_{k=1}^{r} \mathbf{u}_k \mathbf{u}_k^T$$

Using the previously defined decomposition we have:

$$\mathbf{Y} = P_{\mathbf{T}}(\mathbf{Y}) + P_{\mathbf{T}^{\perp}}(\mathbf{Y})$$

where we have

$$P_{\mathbf{T}}(\mathbf{Y}) = \mathbf{U}\mathbf{U}^T \qquad ||P_{\mathbf{T}^{\perp}}(\mathbf{Y})|| < 1$$

Referring back to the dual (section 5.3) we have:

$$\begin{aligned}
\mathbf{M} \cdot \mathbf{Y} &= P_{\mathbf{T}}(\mathbf{M}) P_{\mathbf{T}}(\mathbf{Y}) + P_{\mathbf{T}^{\perp}}(\mathbf{M}) P_{\mathbf{T}^{\perp}}(\mathbf{Y}) \\
&= \mathbf{M} \cdot P_{\mathbf{T}}(\mathbf{Y}) \\
&= \mathbf{M} \cdot \mathbf{U}\mathbf{U}^{T} \\
&= (\sum_k \lambda_k \mathbf{u}_k \mathbf{u}_k^T) \cdot (\sum_k \mathbf{u}_k \mathbf{u}_k^T) \\
&= \sum_{k,n} \lambda_k (\mathbf{u}_k \mathbf{u}_k^T) \cdot (\mathbf{u}_h \mathbf{u}_h^T) \\
&= \sum_k \lambda_k
\end{aligned} \tag{5}$$

So when $\mathbf{X} = \mathbf{M}$, then the primal and dual have an equality meaning we have the optimal solution. Now we want to show uniqueness.

5.7 Uniqueness

Now given a generic matrix $\mathbf{X} = \sum_{k=1}^{r'} \lambda_k \mathbf{x}_k \mathbf{x}_k^T$ that is positive semi-definite and $\mathbf{X} \neq \mathbf{M}$, we want to prove that $\mathbf{X}$ cannot be optimal.

Here we have two cases

1. $span(\mathbf{x}_1, .., \mathbf{x}_{r'}) \nsubseteq span(\mathbf{u}_1, ..., \mathbf{u}_r)$
2. $span(\mathbf{x}_1, .., \mathbf{x}_{r'}) \subseteq span(\mathbf{u}_1, ..., \mathbf{u}_r)$

For case 2., we omit the proof but w.h.p. 2. is no possible. For case 1 we want to show:

$$\mathbf{X} \cdot \mathbf{I} = \text{trace}(\mathbf{X}) > \mathbf{X} \cdot \mathbf{Y} = \mathbf{M} \cdot \mathbf{Y}$$

We want to show:

$$\text{trace}(\mathbf{X}) = \sum_k \lambda_k$$

We start by rewriting the decomposition:

$$\mathbf{X} \cdot \mathbf{Y} = P_{\mathbf{T}}(\mathbf{X}) \cdot P_{\mathbf{T}}(\mathbf{Y}) + P_{\mathbf{T}^{\perp}}(\mathbf{X}) \cdot P_{\mathbf{T}^{\perp}}(\mathbf{Y})$$

Recall that $\mathbf{X} = \sum_k \lambda_k \mathbf{x}_k \mathbf{x}_k^T$

$$\mathbf{X} \cdot \mathbf{Y} = \sum_k \lambda_k P_{\mathbf{T}}(\mathbf{x}_k \mathbf{x}_k^T) \cdot P_{\mathbf{T}}(\mathbf{Y}) + \sum_k \lambda_k P_{\mathbf{T}^{\perp}}(\mathbf{x}_k \mathbf{x}_k^T) \cdot P_{\mathbf{T}^{\perp}}(\mathbf{Y})$$

Now we examine $P_{\mathbf{T}}(\mathbf{x}_k \mathbf{x}_k^T)$ and $P_{\mathbf{T}^{\perp}}(\mathbf{x}_k \mathbf{x}_k^T)$. Define $\mathbf{x} = \mathbf{u} + \mathbf{z}$ where $\mathbf{u} \in span(\mathbf{u}, .., \mathbf{u}_r)$ and $\mathbf{z} \perp \mathbf{u}$. Plugging this into $\mathbf{x}\mathbf{x}^T$, an instance of k, we get:

$$\mathbf{x}\mathbf{x}^T = \mathbf{u}\mathbf{u}^T + \mathbf{z}\mathbf{z}^T + \mathbf{z}\mathbf{u}^T + \mathbf{u}\mathbf{z}^T$$

Simplifying the decomposition we get

$$P_{\mathbf{T}}(\mathbf{x}\mathbf{x}^T) = \mathbf{u}\mathbf{u}^T + \mathbf{z}\mathbf{u}^T + \mathbf{u}^T\mathbf{z}$$

$$P_{\mathbf{T}^\perp}(\mathbf{x}\mathbf{x}^T) = \mathbf{z}\mathbf{z}^T$$

The latter simplifies to $\mathbf{z}\mathbf{z}^T$ because $\mathbf{u}\mathbf{u}^T$ in the perpendicular decomposition goes to 0, so any term with $\mathbf{u}$ is dropped. Now we look at the second term, $P_{\mathbf{T}}(\mathbf{Y})$ and $P_{\mathbf{T}^\perp}(\mathbf{Y})$.

Recall that $P_{\mathbf{T}}(\mathbf{Y}) = \mathbf{U}\mathbf{U}^T$:

$$\begin{aligned} P_{\mathbf{T}}(\mathbf{x}\mathbf{x}^T) \cdot P_{\mathbf{T}}(\mathbf{Y}) &= (\mathbf{u}\mathbf{u}^T + \mathbf{z}\mathbf{u}^T + \mathbf{u}^{\mathbf{z}}) \cdot \mathbf{U}\mathbf{U}^T \\ &= \mathbf{u}^T\mathbf{U}\mathbf{U}^T\mathbf{u} + \mathbf{z}^T\mathbf{U}\mathbf{U}^T\mathbf{u} + \mathbf{u}^T\mathbf{U}\mathbf{U}^T\mathbf{z} \\ &= ||\mathbf{u}||^2 \end{aligned} \tag{6}$$

For the second term we have:

$$\begin{aligned} P_{\mathbf{T}^\perp}(\mathbf{x}\mathbf{x}^T) \cdot P_{\mathbf{T}^\perp}(\mathbf{Y}) &= \mathbf{z}\mathbf{z}^T \cdot P_{\mathbf{T}^\perp}(\mathbf{Y}) \\ &= \mathbf{z}^T(P_{\mathbf{T}^\perp}(\mathbf{Y})\mathbf{z} \\ &< ||\mathbf{z}||^2 \end{aligned} \tag{7}$$

Finally we put everything back into $\sum_k$ and we define $\mathbf{x}_k = \mathbf{v}_k + \mathbf{z}_k$ where $\mathbf{x}_k$ has at least 1 vector $\in span(\mathbf{u}_1, ..., \mathbf{u}_r)$ and for $\mathbf{z}_k$ at least 1 is not the $\mathbf{0}$ vector.

$$\begin{aligned} \mathbf{X} \cdot \mathbf{Y} &= \sum_k \lambda_k(||\mathbf{v}_k||^2 + \mathbf{z}_k^T(P_{\mathbf{T}^\perp}(\mathbf{Y})\mathbf{z}_k) \\ &< \sum_k \lambda_k(||\mathbf{v}_k||^2 + ||\mathbf{z}_k||^2) \\ &= \sum_k \lambda_k \end{aligned} \tag{8}$$