Advanced Dynamics Notes

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Contents

Chapter 1

Survey of Elementary Particles

1.1 Mechanics of a Particle

We begin with a review of common expressions. Recall that

$$\begin{aligned} \boldsymbol{r} &= \text{position vector of a particle from a given origin} \\ \boldsymbol{v} &= \text{particle's vector velocity} \end{aligned}$$

We thus know that

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \quad \text{and} \quad \boldsymbol{p} = m\boldsymbol{v}$$

where

$$\begin{aligned} \boldsymbol{p} &= \text{linear momentum of a particle} \\ m &= \text{the mass of a particle} \\ \boldsymbol{F} &= \text{total force exerted on the particle} \end{aligned}$$

The mechanics or motion of the particle are given by Newton's 2^{nd} Law of Motion:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \dot{\boldsymbol{p}}$$

For an inertial or Galilean reference frame, this can be expressed as

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v})$$

For a particle with constant mass, this simplifies to:

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = m\frac{d^2\boldsymbol{r}}{dt^2}$$

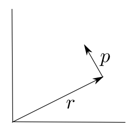
where $\boldsymbol{a}$ is the particle's acceleration. Now, if $\boldsymbol{F} = 0 = \dot{\boldsymbol{p}}$, then $\boldsymbol{p}$ equals a constant, and thus we have Conservation of Linear Momentum of the particle.

We also know that

$$\boldsymbol{L} = \text{angular momentum about a point O}$$

and that

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$



The change of the angular momentum with respect to time is given by

$$\begin{aligned}\frac{d}{dt}\boldsymbol{L} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) &= \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) \\ &= \underbrace{\frac{d\boldsymbol{r}}{dt} \times m\boldsymbol{v}}_{0} + \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) \\ &= \boldsymbol{r} \times \boldsymbol{F} \\ &= \boldsymbol{N}\end{aligned}$$

which is called the moment of force or, more commonly, the torque. Note here that $\boldsymbol{L}$ and $\boldsymbol{N}$ depend on O, the point about which the moments were taken. Now, if $\boldsymbol{N} = \dot{\boldsymbol{L}} = 0$, then $\boldsymbol{L}$ is a constant, and thus we have Conservation of Angular Momentum.

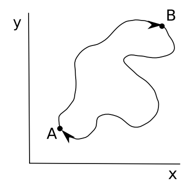
The Work done on a particle by force $\boldsymbol{F}$ is

$$W_{AB} = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{s}$$

where $\boldsymbol{s}$ is the trajectory. Assuming that the mass is constant:

$$\begin{aligned}W_{AB} &= m \int_A^B \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}\, dt \\ &= \frac{m}{2} \int_A^B \frac{d}{dt}(\boldsymbol{v}^2) \\ &= \frac{m}{2}\left(v_B^2 - v_A^2\right) \\ &\equiv T_B - T_A\end{aligned}$$

where $T = \frac{1}{2}mv^2$ is the kinetic energy. Thus we see evidence of the <u>Work-Energy Theorem</u>, which states that the work done is equal to the change in kinetic energy. Now, if W_{AB} depends only on the endpoints A and B (i.e. is path independent), then the force responsible is <u>conservative</u>.



Thus we see that, in a conservative system, the work around a closed path is

$$\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0$$

Remark 1: Dissipative forces such as friction are *not* conservative since

$$\boldsymbol{F} \cdot d\boldsymbol{s} > 0$$

for work done *by* the particle.

For a conservative force (W_{AB} path independent), we can write

$$\boldsymbol{F} = -\nabla V(\boldsymbol{r})$$

where V is the potential or potential energy. Thus we have

$$\boldsymbol{F} \cdot d\boldsymbol{s} = -dV \quad \Rightarrow \quad \boldsymbol{F}_s = -\frac{dV}{d\boldsymbol{s}}$$

Remark 2: Note that

$$\boldsymbol{F} = -\nabla \left(V(\boldsymbol{r}) + \text{constant} \right) = -\nabla V(\boldsymbol{r})$$

and thus the zero of $V(\boldsymbol{r})$ is arbitrary.

Remark 3: For a conservative system, $W_{AB} = V_A - V_B$. Thus we have that

$$V_A - V_B = T_B - T_A \quad \Rightarrow \quad V_A + T_A = V_B + T_B$$

which is our expression for the Conservation of Total Energy!

1.2 Mechanics of a System of Particles

The goal here is to generalize Newton's 2^{nd} law to a system of particles. Starting with the equation of motion for the i^{th} particle:

$$\sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)} = \dot{\boldsymbol{p}}_i$$

where

$$\begin{aligned}\boldsymbol{F}_{ji} &= \text{the internal force on particle } i \text{ by particle } j \text{ and} \\ \boldsymbol{F}_i^{(e)} &= \text{an external force}\end{aligned}$$

Now, assuming that $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$ (Newton's 3^{rd} Law, or the Weak Law of Action and Reaction), then

$$\sum_i \boldsymbol{F}_i = \frac{d^2}{dt^2}\sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} + \underbrace{\sum_{i \neq j} \boldsymbol{F}_{ji}}_{0}$$

We define $\boldsymbol{R}$ as:

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \boldsymbol{r}_i}{M}$$

where $\boldsymbol{R}$ is the center of mass and M is the total mass. Thus we have that

$$M\frac{d^2}{dt^2}\boldsymbol{R} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)}$$

Consequentally, purely internal forces (forces on particles from other particles) vanish by Newtons 3^{rd} law. The total linear momentum can thus be expressed as

$$\boldsymbol{P} = \sum_i m_i \frac{d\boldsymbol{r}_i}{dt} = M\frac{d\boldsymbol{R}}{dt}$$

Thus total linear momentum is conserved when the total external forces are equal to zero.

The total angular momentum is written as

$$\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$$

Hence,

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} = \dot{\boldsymbol{L}} &= \sum_i \frac{d}{dt}(\boldsymbol{r}_i \times \boldsymbol{p}_i) \\ &= \sum_i \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} + \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ji}\end{aligned}$$

but

$$\begin{aligned}\sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} &= \tfrac{1}{2}\sum_{i \neq j}\left[\boldsymbol{r}_i \times \boldsymbol{F}_{ji} + \boldsymbol{r}_j \times \boldsymbol{F}_{ij}\right] \\ &= \tfrac{1}{2}\sum_{i \neq j}(\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ji}\end{aligned}$$

If we define

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$$

then we can write

$$\sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \tfrac{1}{2} \sum_{i \neq j} \boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji}$$

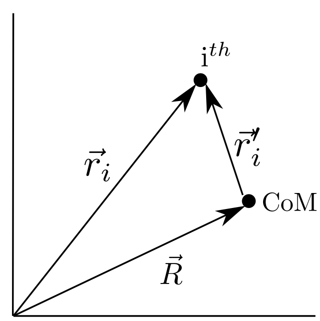
Note that if $\boldsymbol{F}_{ji}$ is parallel to $\boldsymbol{r}_{ij}$, then $\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji} = 0$, which is called the Strong Law of Action and Reaction. If this is true, then

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} + \underbrace{\tfrac{1}{2} \sum_{i \neq j} \boldsymbol{r}_{ij} \times \boldsymbol{F}ji}_{0} = \boldsymbol{N}^{(e)}$$

Thus, if the applied external torque, $\boldsymbol{N}^{(e)}$, equals 0, then $\boldsymbol{L}$ is constant in time and we have Conservation of Angular Momentum.

Remark 1: The Strong Law of Action and Reaction requires the internal forces to be central.

We can also describe the particle position with respect to the center of mass:



So we have

$$\begin{aligned} \boldsymbol{r}_i &= \boldsymbol{R} + \boldsymbol{r}_i' \\ \boldsymbol{v}_i &= \boldsymbol{V} + \boldsymbol{v}_i' \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{V} &= \frac{d\boldsymbol{R}}{dt} = \text{velocity of center of mass} \\ \boldsymbol{v}_i' &= \frac{d\boldsymbol{r}_i'}{dt} = \text{velocity about the center of mass} \end{aligned}$$

Then

$$\begin{aligned}
\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i &= \sum_i \left(\boldsymbol{R} + \boldsymbol{r}_i'\right) \times \left(m_i \boldsymbol{V} + m_i \boldsymbol{v}_i'\right) \\
&= \sum_i \boldsymbol{R} \times m_i \boldsymbol{V} + \sum_i \boldsymbol{R} \times \underbrace{m_i \boldsymbol{v}_i'}_{0} + \sum_i \underbrace{\boldsymbol{r}_i' \times m_i \boldsymbol{V}}_{0} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i' \\
&= \boldsymbol{R} \times M\boldsymbol{V} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i'
\end{aligned}$$

So the total angular momentum is:

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{R} \times M\boldsymbol{V} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i' \\
&= \text{angular momentum of the CM} + \text{angular momentum about the CM}
\end{aligned}$$

Now,

$$\begin{aligned}
W_{AB} = \sum_i \int_A^B \boldsymbol{F}_i \cdot d\boldsymbol{s}_i &= \sum_i \int_A^B \boldsymbol{F}_i^{(e)} \cdot d\boldsymbol{s}_i + \sum_{i \neq j} \int_A^B \boldsymbol{F}_{ji} \cdot \boldsymbol{s}_i \\
&= \sum_i \int_A^B m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{v}_i \, dt \\
&= \sum_i \int_A^B d\left(\tfrac{1}{2} m_i v_i^2\right) \\
&= T_B - T_A
\end{aligned}$$

with total kinetic energy $T = \frac{1}{2}\sum_i m_i v_i^2$. We can also write

$$\begin{aligned}
T = \tfrac{1}{2}\sum_i m_i v_i^2 &= \tfrac{1}{2}\sum_i m_i \left(\boldsymbol{V} + \boldsymbol{v}_i'\right) \cdot \left(\boldsymbol{V} + \boldsymbol{v}_i'\right) \\
&= \tfrac{1}{2} M V^2 + \sum_i \underbrace{m_i \boldsymbol{v}_i'}_{0} \cdot \boldsymbol{V} + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2 \\
&= \tfrac{1}{2} M V^2 + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2 \\
&= \text{KE of CM} + \text{KE about CM}
\end{aligned}$$

We now go back to

$$W_{AB} = \sum_i \int_A^B \boldsymbol{F}_i^{(e)} \cdot d\boldsymbol{s}_i + \sum_{i \neq j} \int_A^B \boldsymbol{F}_{ji} \cdot \boldsymbol{s}_i$$

For conservative *external* forces, we have that:

$$\sum_i \int_A^B \boldsymbol{F}_i^{(e)} \cdot d\boldsymbol{s}_i = -\sum_i \int_A^B \nabla_i V_i \cdot d\boldsymbol{s}_i = -\sum_i V_i \Big|_A^B$$

For conservative *internal* forces:

$$\boldsymbol{F}_{ji} = -\nabla_j V_{ji}$$

where $V_{ji} = V_{ji}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$ to satisfy the Strong Law of Action and Reaction

$$\begin{aligned} &= \nabla_i V_{ij} \\ &= -\boldsymbol{F}_{ij} \end{aligned}$$

where there is no implied sum over repeated indices above. Therefore

$$\begin{aligned} W_{AB} &= -\sum_i V_i\Big|_A^B + \tfrac{1}{2}\sum_{i\neq j}\int_A^B (\boldsymbol{F}_{ji}\cdot d\boldsymbol{s}_i + \boldsymbol{F}_{ij}\cdot d\boldsymbol{s}_j) \\ &= -\sum_i V_i\Big|_A^B + \tfrac{1}{2}\sum_{i\neq j}\int_A^B \boldsymbol{F}_{ji}\cdot (d\boldsymbol{s}_i - d\boldsymbol{s}_j) \\ &= -\sum_i V_i\Big|_A^B + \tfrac{1}{2}\sum_{i\neq j}\int_A^B \boldsymbol{F}_{ji}\cdot d\boldsymbol{s}_{ij} \\ &= -\sum_i V_i\Big|_A^B - \tfrac{1}{2}\sum_{i\neq j}\int_A^B \nabla_{ji} V_{ji}\cdot d\boldsymbol{s}_{ij} \\ &= -\sum_i V_i\Big|_A^B - \tfrac{1}{2}\sum_{i\neq j} V_{ji}\Big|_A^B \end{aligned}$$

Thus the total potential energy is

$$V = \sum_i V_i - \tfrac{1}{2}\sum_{i\neq j} V_{ij}$$

Remark 2: $T + V$ is conserved for T, V the total kinetic and potential energy.

Remark 3: For a rigid body, internal potential energy is constant, and thus internal forces due no work.

1.3 Constraints

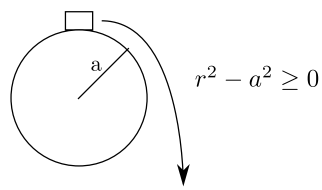
Constraints limit the motion of a system: ie, beads on a string, ball on a circular track, or gas molecules in a container.

Holonomic Constraints: A Holonomic constraint is one where the constraint can be expressed as an equation connecting the space and time coordinates of a particle with the form:

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N, t) = 0$$

Examples include rigid bodies (which have equations of the form $(\boldsymbol{r}_i - \boldsymbol{r}_j)^2 - C_{ij}^2 = 0$) and particles constrained to move on a curve or surface.

Non-Holonomic Constraints: A Non-Holonomic Constraint is one where the constraint can not be expressed as above. Examples include gas molecules in a box or a particle placed on a sphere surface (but *not* stuck to it)



Rheonomous Constraints: A Rheonomous constraint is one which depends explicitly on time

Scleronomous Constraints: A Scleronomous constraint is one which has no explicit time dependence.

Remark 1: Constraints are equivalent to saying that there are forces that can not be specified explicitly; only by their effect on the systems motion.

Now we will consider Holonomic constraints and introduce generalized coordinates. A system of N particles has $d \cdot N$ degrees of freedom (independent coordinates) in d spacial dimensions. If the holonomic constraints are expressed in K equations, then we have $d \cdot N - K$ degrees of freedom expressed in terms of the generalized coordinates $q_1, q_2, q_3, \ldots, q_{dN-K}$.

Transformation Equations relate the original variables $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$ in terms to q_i via

$$\boldsymbol{r}_j = \boldsymbol{r}_j \left(q_1, q_2, q_3, \ldots, q_{dN-K}, t\right)$$

Examples would include a particle constrained to move on a sphere of fixed radius, a pendalum with a sliding attach-point, or a double pendalum.

1.4 D'Alembert's Principle and Lagrange's Equations

Let

$$\underbrace{\boldsymbol{F}_i}_{\substack{\text{total force}\\ \text{on i}^{th}\\ \text{particle}}} = \underbrace{\boldsymbol{F}_i^{(a)}}_{\substack{\text{applied}\\ \text{forces}}} + \underbrace{\boldsymbol{f}_i}_{\substack{\text{force of}\\ \text{constraint}}}$$

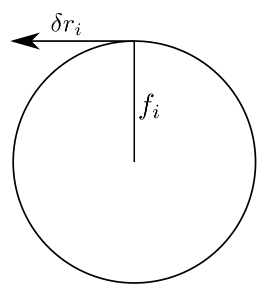
By Newton's 2^{nd} law:

$$\boldsymbol{F}_i = \dot{\boldsymbol{p}}_i \Rightarrow \boldsymbol{F}_i - \dot{\boldsymbol{p}}_i = 0$$

which gives

$$\sum_i (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i) \cdot \underbrace{\delta \boldsymbol{r}_i}_{\substack{\text{infinitesimal}\\ \text{displacement}}} = 0 = \sum_i \left(\boldsymbol{F}_i^{(a)} - \dot{\boldsymbol{p}}_i\right) \cdot \delta \boldsymbol{r}_i + \sum_i (\boldsymbol{f}_i - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i$$

Consider the case where $\sum_i \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_i = 0$. This is akin to



i.e. the forces of constraint do no work. This leaves us with:

$$\sum_i \left(\boldsymbol{F}_i^{(a)} - \dot{\boldsymbol{p}}_i\right) \cdot \delta \boldsymbol{r}_i = 0 \quad \Rightarrow \text{D'Alembert's Principle} \tag{†}$$

Note that the above equation contains no constraints, so we'll drop the "(a)" designation unambiguously in the future.

For $\boldsymbol{r}_i = \boldsymbol{r}_i\,(q_1, q_2, q_3, \ldots, q_n, t)$, we will write (†) in terms of the generalized coordinates q_i:

$$\delta \boldsymbol{r}_i = \sum_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j$$

$$\Rightarrow \quad \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{i,j} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \equiv \sum_j Q_j \, \delta q_j$$

where

$$Q_j = \sum_i \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \text{ the generalized force}$$

Working on the other half of the equation, we have:

$$\begin{aligned}\Rightarrow \sum_i \dot{\boldsymbol{p}}_i \cdot \delta \boldsymbol{r}_i &= \sum_{i,j} m_i \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \\ &= \sum_{i,j} \left[\frac{d}{dt} \left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) - m_i \boldsymbol{v}_i \cdot \frac{d}{dt} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \right] \delta q_j\end{aligned}$$

Using the fact that

$$\dot{\boldsymbol{v}}_i = \frac{d\boldsymbol{r}_i}{dt} = \frac{\partial}{\partial t}\boldsymbol{r}_i + \sum_k \dot{q}_k \frac{\partial \boldsymbol{r}_i}{\partial q_k} \qquad \text{by chain rule}$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

Thus:

$$\begin{aligned}\sum_i \dot{\boldsymbol{p}}_i \cdot \delta \boldsymbol{r}_i &= \sum_{i,j} \left[\frac{d}{dt} \left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} \right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j} \right] \delta q_j \\ &= \sum_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \tfrac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \tfrac{1}{2} m_i v_i^2 \right) \right\} \delta q_j\end{aligned}$$

And D'Alembert's principle becomes:

$$\sum_i (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = \sum_j \left\{ Q_j - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} \right\} \delta q_j = 0$$

with $T = \sum_i \frac{1}{2} m_i v_i^2$. Now if the system is Holonomic, then it is possible to find q_j such that δq_j is independent of δq_k for all $j \neq k$. Thus, the individual coefficients must vanish:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

If $\boldsymbol{F}_i = -\nabla U$ (conservative force), then

$$Q_j = \sum_i \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_i \nabla_i U \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\frac{dU}{dq_j} = -\frac{dU}{dq_j} + \underbrace{\frac{d}{dt}\left(\frac{dU}{d\dot{q}_j} \right)}_{0}$$

where the last term is 0 since we are taking U to be velocity independent. Thus we have

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T - U) - \frac{\partial}{\partial q_j}(T - U) = 0$$

Define the Lagrangian to be $\mathcal{L} = T - U$. Thus Lagrange's equations become:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Note that the Lagrangian is *not* unique! (See HW)

1.5 Velocity Dependent Potentials and Dissipation Functions

If $U = U(q_j, \dot{q}_j)$, then

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad \text{and} \qquad \mathcal{L} = T - U$$

An example would be a charged particle in an EM field (See HW). If frictional forces are present, then we'll have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tilde{Q}_j = \text{ forces } \textit{not} \text{ arising from a potential}$$

Example: Consider Rayleigh's dissipation function:

$$\mathcal{F} = \tfrac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$$

And

$$F_{fx} = -\frac{\partial \mathcal{F}}{\partial v_x}; \qquad \boldsymbol{F}_f = -\nabla_v \mathcal{F}$$

where F_{fx} is the x component of the frictional force and the sum over i is over all particles. Thus we have that $\tilde{Q}_j$ (the generalized force due to friction) is

$$\begin{aligned}\tilde{Q}_j &= \sum_i \boldsymbol{F}_{f_i} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \\ &= -\sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}\end{aligned}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

1.6 Simple Applications of Lagrangian Formalism

Example 1: Consider a single free particle in space:

$$\begin{aligned}&T = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \ldots + \dot{x}_d^2\right) \quad \text{in d-dimensional space} \\ &U = 0 \\ &Q_i = \sum_j \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i}\end{aligned}$$

where, in cartesian coordinates, $\frac{\partial x_j}{\partial q_i} = \delta_{ij}$

$$\mathcal{L} = T$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

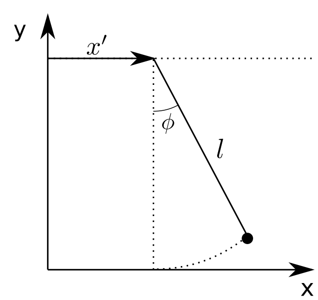
$$\Rightarrow \quad m\ddot{x}_i = F_{x_i} = 0$$

Example 2: Consider a particle constrained on a sphere. Using polar coordinates, we have

$$\begin{aligned} x_1 &= R\sin\theta\cos\phi \\ x_2 &= R\sin\theta\sin\phi \\ x_3 &= R\cos\theta \end{aligned}$$

Calculate $\dot{x}_1, \dot{x}_2, \dot{x}_3$ and substitute into T it get the kinetic energy for the system whose natural coordinates are spherical.

Example 3: Here we will consider a pendalum with a sliding pivot:



So we have that

$$\begin{aligned} x &= x' + l\sin\phi \\ y &= l - l\cos\phi \end{aligned}$$

Thus we have:

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \\ &= \tfrac{1}{2}m\left(\left(\dot{x}'+l\cos\phi\dot{\phi}\right)^2+\left(l\sin\phi\dot{\phi}\right)^2\right) \\ &= \tfrac{1}{2}m\left(\dot{x'}^2+2l\cos\phi\dot{x}'\dot{\phi}+l^2\cos^2\phi\dot{\phi}^2+l^2\sin^2\phi\dot{\phi}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{x'}^2+2l\cos\phi\dot{x}'\dot{\phi}+l^2\dot{\phi}^2\right) \end{aligned}$$

$$U = mgy = mg\left(l-l\cos\phi\right)$$

$$\mathcal{L} = \tfrac{1}{2}m\left(\dot{x'}^2+2l\cos\phi\dot{x}'\dot{\phi}+l^2\dot{\phi}^2\right)-mgl\left(1-\cos\phi\right)$$

Remark 1: Generalized coordinates are not necessarily orthogonal!

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}-\frac{\partial\mathcal{L}}{\partial q_i}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}'}-\frac{\partial\mathcal{L}}{\partial x'} &= \frac{d}{dt}\left(m\dot{x}'+ml\cos\phi\dot{\phi}\right)-0=0 \\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}-\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{d}{dt}\left(ml\cos\phi\dot{x}'+ml^2\dot{\phi}\right)+ml\sin\phi\dot{x}'\dot{\phi}+mgl\sin\phi=0 \end{aligned}$$

To find the equilibrium points:

$$U = mgl-mgl\cos\phi$$

$$\begin{aligned} \frac{\partial U}{\partial\phi} &= mgl\sin\phi=0 \qquad \Rightarrow \qquad \phi=0,\pi \\ \frac{\partial^2U}{\partial\phi^2} &= mgl\cos\phi|_{\phi=0,\pi}=\begin{cases} mgl>0, & \phi=0 \\ -mgl<0, & \phi=\pi \end{cases} \end{aligned}$$

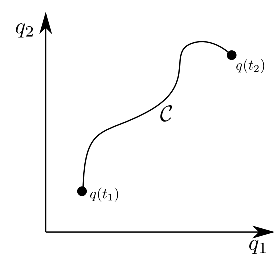
So we have a stable equilibrium at $\phi=0$ and an unstable equilibrium at $\phi=\pi$ (as we'd expect).

Chapter 2

Variational Principles and Lagrange's Equations

2.1 Hamilton's Principle and Calculus of Variations

Consider a system with n generalized coordinates: $q_1, q_2, \ldots, q_n$. These coordinates make up an n-dimensional configuration space with each q corresponding to an axis in the space.



1. t is a parameter of curve $\mathcal{C}$ with $t \in [t_-, t_+] =$ interval

2. $q_- = q(t_-), \quad q_+ = q(t_+)$

3. $\frac{d}{dt} q(t) \equiv \dot{q}(t)$ is the tangent vector

4. If $L(q(t), \dot{q}(t), t)$ is a function of the q's and their tangents, then we define a number that characterizes the path:

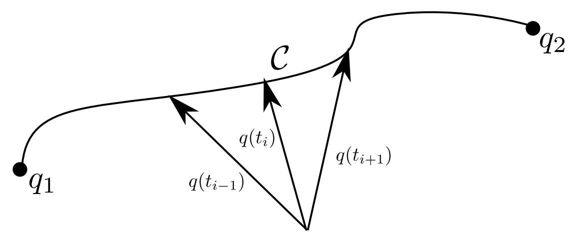
$$S(\mathcal{C}) = \sum_{t_-}^{t_+} L(q(t), \dot{q}(t), t)\, dt$$

   Note that this is more general than $t =$ time and $L =$ Lagrangian. The above is true of *any* function of $q, \dot{q}$ and t.

5. S is called a functional (a functional is an animal that eats a function and spits out a number).

6. If $\mathcal{C} \to \mathcal{C} + \delta\mathcal{C}$, then $S \to S + \delta S$ continuously

Example 1: Let $n = 2$ and we'll work in 2-d Euclidean space. We have that $q = q(x_1, x_2)$ and $\mathcal{C}$ is a curve with parameter t.



We can find the length of $\mathcal{C}$ by:

$$\begin{aligned} l(\mathcal{C}) &= \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\left(q(t_{i+1}) - q(t_i)\right)^2} \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\left(\frac{q(t_{i+1}) - q(t_i)}{t_{i+1} - t_i}\right)^2} (t_{i+1} - t_i) \\ &= \int_{t_1}^{t_2} \sqrt{(\dot{q}(t))^2}\, dt \end{aligned}$$

$$L(q, \dot{q}, t) = L(\dot{q}) = \sqrt{\dot{q}^2}$$
$$\Rightarrow S = l$$

Example 2: Now we'll look at a point moving on curve $\mathcal{C}$ with velocity $v(q)$ that takes time T to go from q_1 to q_2.

$$\begin{aligned} T &= \lim_{N\to\infty} \sum_{i=1}^{n} \frac{1}{v(q(t_i))} \sqrt{\left(q(t_{i+1}) - q(t_i)\right)^2} \\ &= \int_{t_1}^{t_2} \frac{1}{v(q(t))} \sqrt{(\dot{q})^2} \end{aligned}$$

$$\Rightarrow L(q, \dot{q}, t) = L(q, \dot{q}) = \frac{\sqrt{\dot{q}^2}}{v(q)} \qquad \text{and} \qquad S = T$$

Example 3: Again, we have a moving particle with time$=t$ and position$=q$. We can write the kinetic and potential energy as:

$$T = \frac{\dot{q}^2}{2m}, \qquad U(q)$$

Thus:

$$L(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}) = T(\dot{q}) - U(q)$$

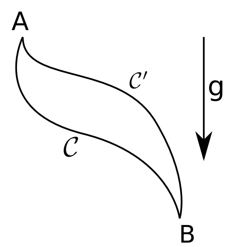
$$S = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t)\, dt$$

So S is the Action. Hamilton's principle says that S has a stationary value for the actual path of motion:

$$\Rightarrow \delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt = 0$$

2.2 The 3 Classic Problems of the Calculus of Variations

The Brachistochrone Problem: Brachistochrone means "short time", so these type of problems are attempting to minimize the time. A massive particle moves from A to B under the force



of gravity along path $\mathcal{C}$. Which $\mathcal{C}$ gives the shortest travel time?

The Geodesics Problem: A ship is traveling from Portland, OR to Hawaii along the surface of a sphere. Which route is the shortest?

The Isoperimetric Problem (Dido's Problem): For a curve $\mathcal{C}$ with given length l, which form gives the maximum area?

2.3 Calculus of Variations, Hamilton's Principle, and Lagrange's Equations

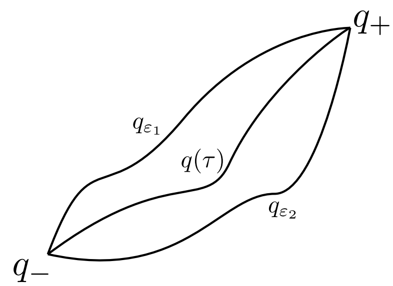
Consider a curve $\mathcal{C}_\varepsilon$ in configuration space.

$$\mathcal{C}_\varepsilon: \quad q_\varepsilon(\tau) = q_\varepsilon\left(q_1(\tau, \varepsilon), \ldots, q_n(\tau, \varepsilon)\right)$$

Some things to note:

1. $q_\varepsilon(\tau)$ is a path for every fixed ε.

2. $q_{\varepsilon=0}(\tau) = q(\tau)$



3.
$$\left.\begin{aligned} q_\varepsilon(\tau_-) &= q(\tau_-) \\ q_\varepsilon(\tau_+) &= q(\tau_+) \end{aligned}\right\} \text{ for all } \varepsilon$$

4. $q_\varepsilon(\tau)$ are continuously differentiable with respect to ε (fixed τ).

Consider a function L and functional S:

$$S_L(\mathcal{C}) = \int_{\tau_-}^{\tau_+} L(q(\tau), \dot{q}(\tau), \tau)\, d\tau$$

Then, under variations of the path $\mathcal{C}$:

1. $S(\mathcal{C})$ is minimal if $S(\mathcal{C}_\varepsilon) > S(\mathcal{C})$ for all ε in the neighborhood of $\mathcal{C}$
2. $S(\mathcal{C})$ is maximal if $S(\mathcal{C}_\varepsilon) < S(\mathcal{C})$ for all ε in the neighborhood of $\mathcal{C}$
3. $S(\mathcal{C})$ is stationary, and $\mathcal{C}$ is an extremal of S if

$$\begin{aligned} \delta S &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[S(\mathcal{C}_\varepsilon) - S(\mathcal{C}) \right] = 0 \quad \forall\, \mathcal{C}_\varepsilon \text{ in nbhd} \\ &= \text{ variation of functional } S \end{aligned}$$

Remark 1: We find the extremals to solve classical problems.

$$\begin{aligned} \delta S &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[S(\mathcal{C}_\varepsilon) - S(\mathcal{C}) \right] \\ &= \lim_{\varepsilon \to 0} \int_{\tau_-}^{\tau_+} \frac{1}{\varepsilon} \left[L(q_\varepsilon, \dot{q}_\varepsilon, \tau) - L(q, \dot{q}, \tau) \right] \, d\tau \end{aligned}$$

We then Taylor expand $S(\mathcal{C}_\varepsilon)$ for small ε. Let $q_\varepsilon(\tau) = q(\tau) + \varepsilon\eta(\tau)$. Then

$$\delta S = \lim_{\varepsilon \to 0} \int_{\tau_-}^{\tau_+} \frac{1}{\varepsilon} \left[L(q, \dot{q}, \tau) + \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_{\varepsilon,i}} \frac{\partial q_{\varepsilon,i}}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}_\varepsilon} \frac{\partial \dot{q}_\varepsilon}{\partial \varepsilon} \right) \varepsilon + \mathcal{O}(\varepsilon^2) - L(q, \dot{q}, \tau) \right] d\tau$$

Note that, since $q_\varepsilon(\tau) = q(\tau) + \varepsilon\eta(\tau)$, we have

$$\frac{\partial q_\varepsilon}{\partial \varepsilon} = \eta(\tau)$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_\varepsilon} = \frac{\partial L}{\partial q}\underbrace{\frac{\partial q}{\partial q_\varepsilon}}_{1} = \frac{\partial L}{\partial q}$$

Thus

$$\begin{aligned}
\delta S &= \lim_{\varepsilon\to 0}\int_{\tau_-}^{\tau_+}\frac{1}{\varepsilon}\left[\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right]d\tau \\
&= \int_{\tau_-}^{\tau_+}\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\eta_i\,d\tau + \sum_{i=1}^{n}\left[\underbrace{\frac{\partial L}{\partial q_i}\eta_i\Big|_{\tau_-}^{\tau_+}}_{0} - \int_{\tau_-}^{\tau_+}\eta_i\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,d\tau\right] \\
&= \int_{\tau_-}^{\tau_+}\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]\eta_i(\tau)\,d\tau \\
&= 0 \text{ since } S \text{ is stationary.}
\end{aligned}$$

Now, applying the fundamental lemma of the calculus of variations:

Lemma: Let $f(\tau)$ be continuous for $\tau \in [\tau_-, \tau_+]$ and

$$\int_{\tau_-}^{\tau_+} f(\tau)\eta(\tau)\,d\tau = 0 \quad \forall\eta$$

which are 2 times differentiable and obey

$$\eta(\tau_+) = \eta(\tau_-) = 0$$

Then

$$f(\tau) = 0$$

since $\eta_i(\tau)$ is arbitrary and vanishes at the endpoints.

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 = \text{ Euler Equations} \qquad (\dagger)$$

Remark 2: When $L = T - U$ in (†), these are the Euler-Lagrange equations.

Remark 3: The functional $S = \int L(q(\tau), \dot{q}(\tau), \tau)\,d\tau$ is stationary only if the functional L obeys the Euler-Lagrange equations.

Remark 4: (†) implies that

$$\frac{\partial L}{\partial q_i} - \sum_k \frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\ddot{q}_k - \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_k - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\Rightarrow \quad \sum_k L_{ik}\ddot{q}_k = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial \tau} - \sum_k \left(\frac{\partial^2 L}{\partial q_k \partial \dot{q}_i}\right)\dot{q}_i \qquad (\ddagger)$$

and

$$L_{ik} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_k} \quad \leftarrow \text{ symmetric n}\times\text{n matrix}$$

So the Euler-Lagrange equations (†) are equivalent to n-coupled ODE's of second order for $q(\tau)$. Therefore, the initial point of $q(\tau_-)$ and initial tangent $\dot{q}(\tau_-)$ completely determine the path.

Remark 5: (‡) has the structure of Newton's Second Law.

Remark 6: Lagrange's equations of motion follow naturally from Hamilton's Principle.

Example: Geodesics in d-dimensional Euclidean space: Recall that the length of curve $\mathcal{C}$ is given by

$$l(\mathcal{C}) = \int_{\tau_i}^{\tau_+} \sqrt{(\dot{q}(\tau))^2}\, d\tau = L(q, \dot{q}, \tau) = \sqrt{\dot{q}^2} = \left[\sum_{j=1}^{d}(q_j(\tau))^2\right]^{1/2}$$

(†) then implies that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{1}{2}\frac{1}{\sqrt{\dot{q}^2}}\cdot 2\dot{q}_i\right) = 0 \quad \forall i$$

$$\Rightarrow \quad \frac{\dot{q}_i}{\sqrt{\dot{q}^2}} = \text{ constant}$$

$$\Rightarrow \quad \dot{q} = a = \text{ constant} \quad \Rightarrow \quad q(\tau) = a\tau + b = \text{ straight lines}$$

So the geodesics must run thru 2 points to specify a and b, implying that we require a 2-d condition.

2.4 Extensions of Hamilton's Principle to Non-Holonomic Systems (Lagrange Multipliers)

$$\delta S = \delta \int_{t_-}^{t_+} L(q, \dot{q}, t)\, dt = 0 = \int_{\tau_-}^{\tau_+} \sum_i \left(\frac{\partial}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) d\tau\, \delta q_i = 0$$

Non-Holonomic constraints imply that the generalized coordinates are *not* independent. This implies that displacements of the path may or may not satisfy the constraints. If displacements satisfy constraints, then the constraints are holonomic. If the displacements do *not* satisfy the constraints, then we want to eliminate the constraints by means of Lagrange Multipliers. Lagrange multipliers work for "semi-holonomic" constraints, which can be put in the form:

$$f_\alpha\left(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n\right) = 0 \tag{$\dagger$}$$

where $\alpha = 1, 2, \ldots, n$.

Remark 1: Semi-holonomic differs from holonomic in that the latter can be expressed in terms of *one* constraint equation (function of generalized coordinates only), whereas the former con be more than one (function of tangents as well).

Remark 2: In terms of path displacements, the semi-holonomic constraints can be expressed:

$$\sum_k a_{ik}\, dq_k + a_{it}\, dt = 0 \qquad \text{where} \quad i = 1, \ldots, m$$

This is more restrictive than (†).

Remark 3: (†) implies that:

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0$$

where

$$\lambda_\alpha = \lambda_\alpha\left(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t\right)$$

are undetermined functions.

Recall Hamilton's Principle:

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0$$

which implies

$$\int_{t_1}^{t_2} \sum_k \left(\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) dt\, \delta q_k = 0$$

And δq_k are no longer independent (if we have non-holonomic constraints). But if the constraints are semi-holonomic, then:

$$\delta \int_{t_1}^{t_2} \left(\mathcal{L} + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt = 0$$

Changing notation slightly, let

$$L = \mathcal{L}_1, \qquad \underbrace{\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = \lambda \mathcal{L}_2}_{\star}$$

where $(\star)$ in the piece that will make the δq_k's independent. Then

$$
\begin{aligned}
&\mathcal{L}_3 = \mathcal{L}_1 + \lambda\mathcal{L}_2 \\
\Rightarrow \quad & \delta \int_{t_1}^{t_2} \left(\mathcal{L}_1 + \lambda\mathcal{L}_2\right) dt = \delta \int_{t_1}^{t_2} \mathcal{L}_3 \, dt = 0 \\
\Rightarrow \quad & \int_{t_1}^{t_2} \sum_k \left(\frac{\partial \mathcal{L}_3}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{q}_i} \right) dt \, \delta q_i = 0
\end{aligned}
$$

And $\mathcal{L}_3$ obeys Lagrange's equations:

$$
\frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_3}{\partial q_i} = 0
$$

Also

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_3}{\partial q_i} &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\mathcal{L}_1 + \lambda\mathcal{L}_2\right) - \frac{\partial}{\partial q_i}\left(\mathcal{L}_1 + \lambda\mathcal{L}_2\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\mathcal{L}_1 - \frac{\partial \mathcal{L}_1}{\partial q_i} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\lambda\mathcal{L}_2\right) - \frac{\partial}{\partial q_i}\left(\lambda\mathcal{L}_2\right) \\
&= 0
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_1}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\lambda\mathcal{L}_2\right) - \frac{d}{dt}\frac{\partial (\lambda\mathcal{L}_2)}{\partial \dot{q}_i} \equiv Q_i \\
\Rightarrow \quad & \frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_1}{\partial q_i} = Q_i = \text{ forces of constraint}
\end{aligned}
$$

with

$$
Q_i = \frac{\partial}{\partial q_i}\left(\lambda\mathcal{L}_2\right) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\lambda\mathcal{L}_2\right)
$$

Remark 4: Lagrange multipliers allow us to map a semi-holonomic system of n-generalized coordinates and m-constraints to a holonomic one with $n+m$ generalized coordinates.

Example:

$$
\mathcal{L} = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(x,y) \equiv \mathcal{L}_1
$$

$$
\text{Constraint:} \quad f(\dot{x},\dot{y},y) = \dot{x}\dot{y} + ky = 0 \equiv \mathcal{L}_2 \qquad \text{where } k = const \tag{1}
$$

$$
\Rightarrow \quad \mathcal{L}_3 = \mathcal{L}_1 + \lambda\mathcal{L}_2 \quad \text{and} \quad \frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_3}{\partial q_i} = 0
$$

$$
\Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{x}} - \frac{\partial \mathcal{L}_3}{\partial x} = \frac{d}{dt}\left(m\dot{x} + \lambda\dot{y}\right) + \frac{\partial U}{\partial x} = 0 \tag{2}
$$

$$
\frac{d}{dt}\frac{\partial \mathcal{L}_3}{\partial \dot{y}} - \frac{\partial \mathcal{L}_3}{\partial y} = \frac{d}{dt}\left(m\dot{y} + \lambda\dot{x}\right) + \frac{\partial U}{\partial y} - k\lambda = 0 \tag{3}
$$

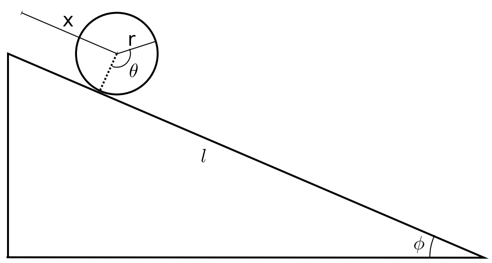
Thus we have 3 unknowns: x, y, λ, and 3 equations: (1), (2), (3).

Remark 5: Physically, the λ's represent forces of constraints.

Remark 6: Forces of constraint do no work in (virtual) displacements (See §2.4 of Goldstein).

Remark 7: Lagrange multipliers can be used for holonomic constraints ($\alpha = 1$) as well.

Example: Hoop Rolling without Slipping Down an Inclined Plane: We are using the



generalized coordinates x, θ. Our constraint in the rolling constraint:

$$\begin{aligned} r\dot{\theta} &= \dot{x} \\ \Rightarrow \quad r\theta &= x \\ \Rightarrow \quad r\theta - x &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 \\ U &= mg(l-x)\sin\phi \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \mathcal{L}_1 &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - mg(l-x)\sin\phi \\ \mathcal{L}_2 &= r\theta - x \\ \mathcal{L}_3 &= \mathcal{L}_1 + \lambda\mathcal{L}_2 \\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - mg(l-x)\sin\phi + \lambda(r\theta - x) \end{aligned}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial\mathcal{L}_3}{\partial\dot{x}} - \frac{\partial\mathcal{L}_3}{\partial x} = m\ddot{x} - mg\sin\phi + \lambda = 0 \tag{1}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}_3}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}_3}{\partial\theta} = mr^2\ddot{\theta} - \lambda r = 0 \tag{2}$$

$$\begin{aligned} \Rightarrow \quad m\ddot{x} &= mg\sin\phi - \lambda \\ &= mg\sin\phi - mr\ddot{\theta} \qquad \text{from (2)} \\ &= mg\sin\phi - m\ddot{x} \qquad \text{by (3)} \end{aligned}$$

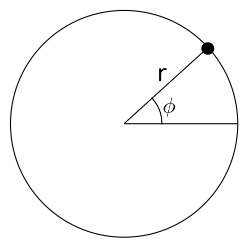
$$\Rightarrow \quad 2m\ddot{x} = mg\sin\phi \qquad \Rightarrow \quad \ddot{x} = \tfrac{1}{2}g\sin\phi$$

So rolling has half the acceleration as the hoop sliding on a frictionless plane!

$$\ddot{\theta} = \frac{g}{2r}\sin\phi$$

$$\lambda = \frac{mg}{2}\sin\phi \rightarrow \text{ “frictional force” of constraint}$$

Example: Bead on a Wire Loop:



The normal way we'd solve this problem would be:

$$T = \tfrac{1}{2}mR^2\dot{\phi}^2, \qquad U = mgy = mgR\sin\phi$$

$$\mathcal{L} = \tfrac{1}{2}mR^2\dot{\phi}^2 - mgR\sin\phi$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = mR^2\ddot{\phi} + mgR\cos\phi = 0$$

$$\Rightarrow \quad \ddot{\phi} = -\frac{g}{R}\cos\phi$$

Now, using Lagrange multipliers:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2), \qquad U = mgy \quad \Rightarrow \quad \mathcal{L}_1 = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

Our constraint is:

$$x^2 + y^2 = R^2 \quad \Rightarrow \quad x^2 + y^2 - R^2 = 0 = \mathcal{L}_2$$

which implies

$$\Rightarrow \quad \mathcal{L}_3 = \mathcal{L}_1 - \lambda\mathcal{L}_2 = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda(x^2 + y^2 - R^2)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}_3}{\partial\dot{x}} - \frac{\partial\mathcal{L}_3}{\partial x} = m\ddot{x} - 2\lambda x = 0$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}_3}{\partial\dot{y}} - \frac{\partial\mathcal{L}_3}{\partial y} = m\ddot{y} + mg - 2\lambda y = 0$$

So our 3 equations are:

$$\begin{aligned} m\ddot{x} &= 2\lambda x \\ m\ddot{y} &= -mg + 2\lambda y \\ R^2 &= x^2 + y^2 \end{aligned}$$

Letting

$$\begin{array}{ll} x = R\cos\phi & y = R\sin\phi \\ \dot{x} = -R\sin\phi\dot{\phi} & \dot{y} = R\cos\phi\dot{\phi} \\ \ddot{x} = -R\cos\phi\dot{\phi}^2 - R\sin\phi\ddot{\phi} & \ddot{y} = -R\sin\phi\dot{\phi}^2 + R\cos\phi\ddot{\phi} \end{array}$$

Taking $m\ddot{x} = 2\lambda x$ and plugging in $\ddot{x}$ implies:

$$m\left(-R\cos\phi\dot{\phi}^2 - R\sin\phi\ddot{\phi}\right) = 2\lambda R\cos\phi$$

$$\Rightarrow \quad \lambda = -\frac{m}{2}\dot{\phi}^2 - \frac{m}{2}\frac{\sin\phi}{\cos\phi}\ddot{\phi}$$

Taking $m\ddot{y} = -mg + 2\lambda y$:

$$\Rightarrow \quad m\left(-R\sin\phi\dot{\phi}^2 + R\cos\phi\ddot{\phi}\right) = -mg + 2\left(-\frac{m}{2}\dot{\phi}^2 - \frac{m}{2}\tan\phi\ddot{\phi}\right)R\sin\phi$$

$$\Rightarrow \quad (mR\cos\phi + mR\tan\phi\,\sin\phi)\,\ddot{\phi} = -mg$$

$$\Rightarrow \ddot{\phi} = -\frac{g}{R}\cos\phi$$

2.5 Conservation Theorems and Symmetry Properties

Consider a function $f(q, \dot{q}, t)$. If

$$\frac{d}{dt}\left(f(q, \dot{q}, t)\right) = 0$$

then

$$f(q, \dot{q}, t) = \mathit{const} \tag{$\dagger$}$$

where (†) is called the

- first integral of the equations of motion
- integrals of motion
- constant of motion

Remark 1: Conservation laws are an example of (†).

Nöther's Theorem: Symmetries in a system are related to conservation laws.

2.5.1 Energy Conservation and Time Homogeneity

Consider the Lagrangian:

$$\mathcal{L} = \mathcal{L}(q(\tau), \dot{q}(\tau), \tau)$$

Time homogeneity (time-translational invariance) means that $\mathcal{L}$ does not depend explicitly on time.

$$\Rightarrow \quad \mathcal{L} = \mathcal{L}(q(\tau), \dot{q}(\tau))$$

Then

$$\frac{d}{dt}\mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \underbrace{\frac{\partial \mathcal{L}}{\partial t}}_{0}$$

Recall Lagrange's equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial q_j}$$

These imply

$$\begin{aligned}
\frac{d}{dt}\mathcal{L} &= \sum_i \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\ddot{q}_j \\
&= \sum_i \frac{d}{dt}\left(\dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) \\
\Rightarrow \quad & \frac{d}{dt}\left[\sum_i \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}\right] = 0 \\
\Rightarrow & \sum_i \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} \equiv H = E
\end{aligned}$$

Then $H = E = const$ is the energy and energy is conserved in a system whose Lagrangian has no explicit time dependence.

Example:

$$\mathcal{L} = T(\dot{q}) - U(q), \qquad T = \tfrac{1}{2}\sum_j m_j \dot{q}_j^2$$

$$\begin{aligned}
\Rightarrow \quad H &= \sum_i \dot{q}_i m_i \dot{q}_i - \tfrac{1}{2}\sum_i m_i \dot{q}_i^2 + U(q) \\
&= \tfrac{1}{2}\sum_i m_i \dot{q}_i^2 + U(q) = T(\dot{q}) + U(q) = \text{ total energy}
\end{aligned}$$

Remark 1: Here we have assumed that $U = U(q)$, so there does not exist dissipative forces (which dissipate energy).

Remark 2: If $\mathcal{L}$ is an explicit function of time:

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

Remark 3: H is sometimes called the energy function, and it's form coincides with the Hamiltonian.

Remark 4: For frictional forces that are derivable from a dissipative function $\mathcal{F}$:

$$\text{From §1.5:} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

Then

$$\begin{aligned}
\frac{d}{dt}\mathcal{L} &= \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \\
&= \sum_i \dot{q}_i \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i}\right) + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \\
&= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \underbrace{\sum_i \dot{q}_i \frac{\partial \mathcal{F}}{\partial \dot{q}_i}}_{=2\mathcal{F} \text{ by §1.5}} + \frac{\partial \mathcal{L}}{\partial t}
\end{aligned}$$

$$\Rightarrow \quad \frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t} - 2\mathcal{F}$$

for $\mathcal{L} = \mathcal{L}(q, \dot{q})$ and $H = E$ then this gives the dissipative rate.

2.5.2 Momentum Conservation and Space Homogeneity

This will correspond to space translational invariance. This implies that the Lagrangian is invariant under $\boldsymbol{r}_i \to \boldsymbol{r}_i + \boldsymbol{\varepsilon}$ (every particle moves by some same displacement). For now, we'll only consider shifts in coordinates, not velocities.

$$\delta \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i}\delta \boldsymbol{r}_i = 0 = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\underbrace{\delta q_i}_{\star}$$

where $(\star)$ is arbitrary, and only for holonomic constraints. This implies that

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = const$$

If $U = U(q)$ only, then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = m\dot{q}_i = p_i$$

$$\Rightarrow \quad p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{ canonical (or conjugate) momentum}$$

p also generalizes to include $U = U(q, \dot{q})$. Now consider $\mathcal{L} = T(\dot{q}) - U(q)$:

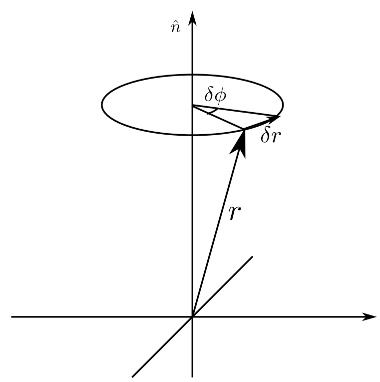
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \dot{p}_i = -\frac{\partial U}{\partial q_i} \equiv Q_i$$

$$\Rightarrow \quad p_i = Q_i = \text{ generalized force}$$

2.5.3 Angular Momentum Conservation and Rotational Invariance

Let $\delta\boldsymbol{\phi}$ be an infinitesimal rotation and

$$\delta\boldsymbol{\phi} = \delta\phi\,\hat{\boldsymbol{n}}$$



$$\delta\mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i}\cdot\delta\boldsymbol{v}_i = 0$$

Note that

$$\delta\boldsymbol{r}_i = \delta\boldsymbol{\phi}\times\boldsymbol{r}_i$$
$$\delta\boldsymbol{v}_i = \delta\boldsymbol{\phi}\times\boldsymbol{v}_i \quad \text{assuming } \delta\dot{\boldsymbol{\phi}} = 0$$

So

$$\delta\mathcal{L} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i}\cdot(\delta\boldsymbol{\phi}\times\boldsymbol{r}_i) + \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i}\cdot(\delta\boldsymbol{\phi}\times\boldsymbol{v}_i)\right]$$

but

$$\boldsymbol{p}_i = \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i} = \frac{d}{dt}\boldsymbol{p}_i$$

$$\Rightarrow \quad \delta\mathcal{L} = \sum_i \left[\frac{d}{dt}\boldsymbol{p}_i \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{r}_i) + \boldsymbol{p}_i \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{v}_i)\right]$$

Now, recall that $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$.

$$\Rightarrow \quad \delta\mathcal{L} = \sum_i \left[\delta\boldsymbol{\phi} \cdot (\boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i) + \delta\boldsymbol{\phi} \cdot (\boldsymbol{v}_i \times \boldsymbol{p}_i)\right]$$

$$= \delta\boldsymbol{\phi} \sum_i \frac{d}{dt}(\boldsymbol{r}_i \times \boldsymbol{p}_i) = 0$$

But $\delta\boldsymbol{\phi}$ is arbitrary, so

$$\frac{d}{dt}\sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i) = 0$$

Now $\boldsymbol{M}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i =$ angular momentum, so we have that

$$\sum_i \boldsymbol{M}_i = \textit{const}$$

Remark 1: The components of angular momentum along any axis (e.g. the z-axis) are given by

$$M_z = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i} \qquad \text{where } \boldsymbol{\phi} = \phi\,\hat{\boldsymbol{z}}$$

Example: We'll take a look at how this works in cylindrical coordinates:

$$M_z = \sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i)_z = \sum_i m_i\,(x_i\dot{y}_i - y_i\dot{x}_i)$$

$$\begin{aligned} x &= r\cos\phi \qquad & \dot{x} &= \dot{r}\cos\phi - r\sin\phi\,\dot{\phi} \\ y &= r\sin\phi \qquad & \dot{y} &= \dot{r}\sin\phi + r\cos\phi\,\dot{\phi} \end{aligned}$$

$$\begin{aligned} \Rightarrow M_z &= \sum_i m_i\left[r_i\cos\phi_i\left(\dot{r}_i\sin\phi_i + r_i\cos\phi_i\dot{\phi}_i\right) + r_i\sin\phi_i\left(\dot{r}_i\cos\phi_i - r_i\sin\phi_i\dot{\phi}_i\right)\right] \\ &= \sum_i m_i r_i^2\dot{\phi}_i \end{aligned}$$

Comparing with

$$\begin{aligned} \sum_i \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i} &= \sum_i \frac{\partial}{\partial\dot{\phi}_i}\left[\tfrac{1}{2}m_i\left(\dot{r}_i^2 + r_i^2\dot{\phi}_i^2 + \dot{z}^2\right) - U(r_i,\phi,z)\right] \\ &= \sum_i m_i r_i^2\dot{\phi}_i \end{aligned}$$

Chapter 3

The Central Force Problem

3.1 2 Body to 1 Body Problem and Reduced Mass

The two body problem: