# 1 Stochastic integrals

## 1.1 Warm-up

We are now ready to define *stochastic integrals*. Before doing this rigorously, it is useful to informally explain what this means. On the one hand, we are given a continuous martingale $(M_t)_{t\geq 0}$ (or a local martingale) in some filtered probability space. For the time being, we can first think of this continuous martingale to be a Brownian motion $(B_t)_{t\geq 0}$. Then, suppose that $(H_t)_{t\geq 0}$ is a continuous real-valued process that is adapted to the same filtration. We will define a new continuous (local) martingale $(I_t)_{t\geq 0}$, denoted by

$$I_t = \int_0^t H_s dB_s, \tag{Notation}$$

which is called a stochastic integral. It can be understood as follows: Locally, at time $t$, it follows the increments of $B$ but it *"decided"* to multiply those increments by the local factor $H_t$.

In particular, we see that a good approximation of the value of $I_t$ is obainted by considering a nested sequence of subdivisions $\Delta_n$ of $[0,t]$ with $|\Delta_n| \to 0$ and to consider the sum

$$I_n := \sum_{j=0}^{m_n-1} H_{t_j^n}(B_{t_{j+1}^n} - B_{t_j^n}).$$

Indeed, we shall see that $I_n$ does converge in probability to our stochastic integral $I_t$ as $n \to \infty$.

**Remark 1.1.** As we have already discussed, an example of stochastic integral is provided by the process $B_t^2 - t$ that can be viewed as the stochastic integral $\int_0^t 2B_s dB_s$ and we have seen that indeed for a given $t$, this quantity $B_t^2 - t$ is the limit (in $L^2$) of

$$I_n = \sum_{j=0}^{m_n-1} 2B_{t_j^n}(B_{t_{j+1}^n} - B_{t_j^n}).$$

It is very important to notice that in this discrete approximation of $I_n$, one chooses the magnification factor $H_{t_j^n} = B_{t_j^n}$ *just before* the increment $(B_{t_{j+1}^n} - B_{t_j^n})$ occurs (i.e. we choose $B_{t_j^n}$ instead of $B_{t_{j+1}^n}$). So even if the notation $H_s dB_s$ in a stochastic integral describes things at infinitesimal levels, it still means at that infinitesimal level, one *chooses* $H_s$ (infinitesimally) before the increments $dB_s$ occurs. As opposed to the Lebesgue integral, where "orientation" of the time-axis will play an important role in stochastic integrals.

## 1.2 The $L^2$ theory of stochastic integrals with respect to Brownian motion

Let us now first describe how to properly define stochastic integrals with respect to a one-dimensional Brownian motion $B := (B_t)_{t\geq 0}$.

### 1.2.1 Integral of elementary processes, and strategy of construction

**Definition 1.1.** *The process $(K_t)_{t\geq 0}$ is said to be an elementary process if it is of the type*

$$K_t = \sum_{j=0}^{m-1} Y_{a_j} 1_{t\in(a_j,a_{j+1}]}, \tag{*}$$

*where $m \in \mathbb{N}$, $0 \leq a_0 < \cdots < a_m$ are deterministic times and each $Y_{a_j}$ is an $\mathcal{F}_{a_j}$-measurable random variable in $L^2$. We denote by $\mathcal{E}_B$ the set of all elementary processes such that $K_t$ is in $L^2$ (that is exactly the fact if the $Y_{a_j}$ are in $L^2$).*

**Remark 1.2.** $\mathcal{E}_B$ is a vector space.

**Definition 1.2** (Stochastic integral for $\mathcal{E}_B$). *For each elementary process $K_t$ as in the form (*) in $\mathcal{E}_B$, we define the stochastic integral to be the continuous process $I(K) = (I(K)_t)_{t\geq 0}$, such that for all $t \geq 0$*

$$I_t^K := \sum_{j=0}^{m-1} Y_{a_j}(B_{t\wedge a_{j+1}} - B_{t\wedge a_j}).$$

*We also denote this by*

$$I_t^K = \int_0^t K_s dB_s.$$

Let us give a couple of remarks that follow immediately by these definitions:

- $t \mapsto I_t^K$ is continuous and on each interval $[a_j, a_{j+1}]$ its increments are $Y_{a_j}$ times the increments of $B$.
- $(I_t^K)_{t\geq 0}$ is indeed a continuous martingale in $L^2$ (difference of $L^2$ martingales). Moreover it is constant for all $t \geq a_m$, and that if we denote the constant value $I_{a_m}^K$ by $I_\infty^K$, then (using that BM has independent increments, and $B_{a_{j+1}} - B_{a_j}$ is independent of $\mathcal{F}_{a_j} \ni Y_{a_j}$)
$$\begin{aligned}\mathbb{E}((I_\infty^K)^2) &= \mathbb{E}\left[\left(\sum_{j=0}^{m-1} Y_{a_j}(B_{a_j+1} - B_{a_j})\right)^2\right] \\ &= \sum_{j=0}^{m-1} \mathbb{E}(Y_{a_j}^2)(a_{j+1} - a_j) = \sum_{k=0}^{m-1} \mathbb{E}(Y_{a_j}^2) \int_0^\infty 1_{t\in(a_j,a_{j+1}]} dt \\ &= \mathbb{E}\left(\int_0^\infty \sum_{j=0}^{m-1} Y_{a_j}^2 1_{t\in(a_j,a_{j+1}]} dt\right) = \mathbb{E}\left(\int_0^\infty K_t^2 dt\right).\end{aligned}$$
Which shows that
$$\mathbb{E}\left[\left(\int_0^\infty K_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^\infty K_s^2 ds\right].$$
So, the martingale $I^K$ is bounded in $L^2$ if and only if $K \in \mathcal{E}_B$.
- The quadratic variation of $I^K$ is given by the process
$$t \mapsto \int_0^t K_s^2 ds$$
- Similarly, as in the item described above, simply because the difference between two elementary processes $K$ and $K'$ is again an elementary process, and because $I^K$ is clearly a linear function in $K$, we see that for all $K, K' \in \mathcal{E}_B$,
$$\mathbb{E}\left[\left(\int_0^\infty (K_s - K_s')dB_s\right)^2\right] = \mathbb{E}[(I(K - K')_\infty)^2] = \mathbb{E}\left[\int_0^\infty (K_s - K_s')^2 ds\right]$$
Hence, one can measure the distance between $I(K)$ and $I(K')$ in terms of some distance of $K$ and $K'$. This identity implies also that the crossvariation between $I(K)$ and $I(K')$ is given by
$$\int_0^t K_s K_s' ds.$$

### 1.2.2 Plan of the construction

Let us briefly outline the strategy of the construction of the stochastic integral $I_t^K = \int_0^t K_s dB_s$ for more general processes $K$:

1. We have already constructed $\int_0^t K_s dB_s$ when $K$ is an elementary process $K \in \mathcal{E}_B$.
2. We will consider a larger class of processes $(K_t)_{t\geq 0}$. Heuristically speaking will it be a class of processes that can be *"nicely"* approximated by elementary processes. Typically $K$ will be in this class if there exists a sequence $K^n$ of elementary processes such that
$$\mathbb{E}\left(\int_0^\infty (K_s - K_s^n)^2 ds\right) \to 0, \text{ as } n \to \infty.$$
3. When $K$ is in that class, we are going to show that the sequence of martingales $(I_t^{K_n})_{t\geq 0}$ will converge to a limiting process $(I_t)_{t\geq 0}$ that we will define to be $(I_t^K)_{t\geq 0}$ and call it the stochastic integral of $K$ with respect to $B$.
4. In the end, we will see that $I^K$ is a continuous martingale, that its quadratic variation is given by $\int_0^t K_s^2 ds$ and if $K$ is a continuous process in that class and $\Delta_n$ is a nested sequence of subdivisions of $[0,t]$ with mesh-size going to zero, then we have
$$I_t^K = \int_0^t K_s dB_s = \overset{\mathbb{P}}{\lim_{n\to\infty}} \sum_{i=0}^{m_n-1} K_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}).$$

### 1.2.3 The space of continuous martingales that are bounded in $L^2$

In order to implement this strategy, let us first describe what notion of convergence of sequences of continuous martingales we will use: We consider the space $\mathcal{M}^2$ of continuous martingales that are started from 0 and bounded in $L^2$ (i.e. $\sup_{t\geq 0} \mathbb{E}(M_t^2) < \infty$). We then endow this vector space with the scalar product $(M,N) := \mathbb{E}(M_\infty N_\infty)$ for all $M, N \in \mathcal{M}^2$.

**Remark 1.3.** For $M \in \mathcal{M}^2$, we always have $M \xrightarrow{t\to\infty} M_\infty$ almost surely and in $L^2$ (because $M$ is bounded in $L^2$). moreover, all of $(M_t)_{t\geq 0}$ can be recovered from $M_\infty$ by
$$M_t = \mathbb{E}(M_\infty \mid \mathcal{F}_t).$$
Thus if $(M,M) = 0$, then $M_\infty = 0$ almost surely, and therefore $\forall t \geq 0$, $M_t = \mathbb{E}(M_\infty \mid \mathcal{F}_t) = 0$ almost surely, which implies that almost surely $\forall t \geq 0$, $M_t = 0$. Hence $(M,N)$ is indeed a scalar product.

**Lemma 1.1.** *The space $\mathcal{M}^2$ is a Hilbert space.*

*Proof.* We only need to prove that $\mathcal{M}^2$ is complete. Suppose that $(M^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^2$, i.e. we have

$$\mathbb{E}((M^n_\infty - M^l_\infty)^2) \to 0, \text{ as } n, l \to \infty. \tag{*}$$

(*) above shows that the sequence $(M^n_\infty)_{n\in\mathbb{N}}$ is Cauchy in $L^2$ but we know that $L^2$ is complete, thus there exists $M_\infty \in L^2$ such that $M^n_\infty \to M_\infty$ in $L^2$ as $n \to \infty$. By properties of conditional expectations this gives

$$M^n_t = \mathbb{E}(M^n_\infty \mid \mathcal{F}_t) \xrightarrow[n\to\infty]{L^2} \mathbb{E}(M_\infty \mid \mathcal{F}_t) =: M_t$$

We have to show that $M := (M_t)_{t\geq 0}$ is in $\mathcal{M}^2$. Certainly $M$ is a martingale (because its written as a closed martingale), that is bounded in $L^2$, because

$$M_t^2 \leq \mathbb{E}(M^2_\infty \mid \mathcal{F}_t) \implies \mathbb{E}(M_t^2) \leq \mathbb{E}(M^2_\infty) < \infty.$$

We thus need to check if $(M_t)_{t\geq 0}$ is continuous. By Doob's $L^2$ inequality we get

$$\mathbb{E}[(\sup_{t\geq 0}(M^n_t - M^l_t))^2] \leq 4\mathbb{E}((M^n_\infty - M^l_\infty)^2) \xrightarrow{n,l\to\infty} 0.$$

Hence there exists a deterministic sequence $n_k \to \infty$ such that

$$\mathbb{E}[(\sup_{t\geq 0}(M^{n_k}_t - M^{n_{k+1}}_t))^2] \leq \frac{1}{8^k},$$

and consequently by Markov's inequality

$$\mathbb{P}[(\sup_{t\geq 0}(M^{n_k}_t - M^{n_{k+1}}_t))^2 \geq 4^{-k}] \leq \frac{1}{2^k}.$$

Since the above is summable, we get by the Borel-Cantelli lemma that almost surely, for all but finitely many $k$'s, we have

$$\sup_{t\geq 0}|M^{n_{k+1}}_t - M^{n_k}_t| \leq 2^{-k} = \sqrt{4^{-k}}.$$

This implies that $(M^{n_k}_t)_{t\geq 0}$ converges uniformly on $\mathbb{R}_+$ to some continuous function that we call $t \mapsto \tilde{M}_t$. But, since we already know that $M^n_t \to M_t$ in $L^2$, we must have for all $t \geq 0$, $M_t = \tilde{M}_t$ almost surely. Hence, we can conclude that $(M_t)_{t\geq 0}$ is a continuous martingale that is bounded in $L^2$, and we see that $M^n$ converges to $M$ in $\mathcal{M}^2$ as $n \to \infty$ because:

$$(M - M^n, M - M^n) = \mathbb{E}((M_\infty - M^n_\infty)^2) \to 0$$

by the convergence of $M^n_\infty$ to $M_\infty$ in $L^2$. $\square$

### 1.2.4 Progressively measurable processes

Let us now look more closely at the class of processes that can be nicely approximated by processes in $\mathcal{E}_B$. Note that we want to look at processes $(K_t)_{t\geq 0}$ such that the integral $\int_0^t K_s^2 ds$ makes sense, so that some measurability with respect to time is required to be sure that this integral is a well-defined random variable.

**Definition 1.3.** *The process $(K_t)_{t\geq 0}$ is said to be progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if there exists a measurable set $E$ of probability $1$, such that for all $t \geq 0$, the map $(s,\omega) \mapsto K_s(\omega)$ defined on $[0,t]\times\Omega$ is measurable with respect to the product $\sigma$-field $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$.*

**Remark 1.4.** In essence this (cryptic) definition means that at each time $t$, one can look at once at the whole function $s \mapsto K_s$ defined on $[0,t]$ as a random measurable function on $[0,t]$ and this function is $\mathcal{F}_t$ measurable.

It is easy to see that most adapted processes with some regularity will be progressively measurable as the next Lemma suggests (see exercise sheets):

**Lemma 1.2.** *An adapted process that has right-continuous paths with probability $1$ is progressively measurable. An adapted process that has left-continuous paths with probability $1$ is progressively measurable.*

*Proof.* Exercise sheets. □

**Definition 1.4.** *The set of progressively measurable processes will be denoted by $\mathcal{P}$. We can then define the subset $\mathcal{P}_B$ of $\mathcal{P}$ consisting of progressively measurable processes such that*

$$\mathbb{E}\left(\int_0^\infty K_s^2 ds\right) < \infty.$$

*This set is naturally endowed with the scalar product*

$$(K,K')_B := \mathbb{E}\left(\int_0^\infty K_s K'_s ds\right).$$

**Remark 1.5.** In Exercise 10.1 we will derive that $\mathcal{P}_B$ is the $L^2$ space on $\Omega\times\mathbb{R}_+$ equipped with the progressive $\sigma$-algebra (10.1.a) and product measure $\mathbb{P}\otimes d\lambda$ (where $\lambda$ denotes the Lebesgue measure), so that in particular $\mathcal{P}_B$ is a Hilbert space.

We recall from Functional Analysis the following important Proposition:

**Proposition 1.1.** *A subspace $M$ of a Hilbert space $H$ is dense in $H$ if and only if $M^{\perp} = \{0\}$ (i.e. trivial orthogonal complement).*

**Lemma 1.3.** *The subspace $\mathcal{E}_B$ (of $\mathcal{P}_B$) is dense in the Hilbert space $\mathcal{P}_B$. In other words, for any progressively measurable process in $\mathcal{P}_B$, one can find a sequence of elementary processes $K^n$ in $\mathcal{E}_B$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^{\infty} (K_s - K_s^n)^2 ds\right) = 0.$$

*Proof.* It suffices to show that a process $K$ in $\mathcal{P}_B$ that is orthogonal to $\mathcal{E}_B$ is necessarily equal to 0, i.e. for all $\tilde{K} \in \mathcal{E}_B$ we have

$$(K, \tilde{K})_B = 0. \tag{*}$$

Let us first note that by Cauchy-Schwarz inequality, for each $t \geq 0$, the random variable $X_t := \int_0^t K_s ds$ is in $L^1$, thus we can choose $t \mapsto X_t$ to be continuous almost surely, moreover it then has finite variation because $X$ is the difference of the non-decreasing integral of $K1_{K>0}$ and of $-K1_{K<0}$.

From the orthogonality condition (*) we get that for all $a < b$ and for all $Y_a \in L^2(\mathcal{F}_a)$, the process $\tilde{K}_t := (Y_a 1_{t\in(a,b]})_{t\geq 0}$ is in $\mathcal{E}_B$ and thus orthogonal to $K \in \mathcal{P}_B$, which by (*) translates to

$$\begin{aligned}(K, \tilde{K})_B = \mathbb{E}(Y_a \int_a^b K_s ds) = \mathbb{E}(Y_a(X_b - X_a)) &\overset{(*)}{=} 0 \\ \implies \mathbb{E}(X_b \mid \mathcal{F}_a) &= X_a\end{aligned}$$

So $(X_t)_{t\geq 0}$ is a continuous martingale started from 0 and its of bounded variation, we know that this implies that $t \mapsto X_t$ is identically 0. Hence, $K$ itself is equal to 0. $\square$

### 1.2.5 Definition of the stochastic integral

With the previous Lemma at hand we are now ready to define the stochastic integral $I(K)$ of a progressively measurable process $K \in \mathcal{P}_B$ with respect to $B$, where $B$ is a Brownian motion.

- We first consider any sequence $K^n$ of elementary processes in $\mathcal{E}_B$ that converges to $K$ in $\mathcal{P}_B$ (this is guaranteed to work by the previous Lemma, i.e. $\mathcal{E}_B$ is dense in $\mathcal{P}_B$).
- The sequence $(K^n)_{n\in\mathbb{N}}$ is then Cauchy with respect to the norm of $\mathcal{P}_B$ (norm induced by scalar product), because every convergent sequence is Cauchy, i.e. we have

$$\mathbb{E}\left(\int_0^\infty (K_s^n - K_s^l)^2 ds\right) \xrightarrow{n,l\to\infty} 0.$$

  This implies that the sequence $I(K^n)$ is Cauchy in $\mathcal{M}^2$ because we know that the norm of $I(K^n) - I(K^l)$ in $\mathcal{M}^2$ is equal to the distance between $K^n$ and $K^l$ in $\mathcal{P}_B$. Recall that we have shown this isometry:

$$\mathbb{E}((I_\infty^{K_n} - I_\infty^{K_l})^2) = \mathbb{E}((I_\infty^{K_n - K_l})^2) = \mathbb{E}\left(\int_0^\infty (K_s^n - K_s^l)^2 ds\right) \xrightarrow{n,l\to\infty} 0.$$

  - Notice that since $(K^n)_{n\in\mathbb{N}}$ is a sequence of elementary processes in $\mathcal{E}_B$ we know already how to define the stochastic integral for such processes $(I_t^{K_n} = \int_0^t K_s^n dB_s)_{t\geq 0}$ and $I^{K_n}$ is a continuous martingale.
- The space $\mathcal{M}^2$ is complete (it is in fact a Hilbert space), so there exists a continuous martingale $I(K)$ in $\mathcal{M}^2$ such that $I(K^n) \to I(K)$ in this space.
- We note that the continuous martingale $I(K)$ does not depend on our choice of sequence $(K^n)_{n\in\mathbb{N}}$ we did begin with in our first step to approximate $K$ in $\mathcal{P}_B$. Indeed, if $(\tilde{K}^n)_{n\in\mathbb{N}}$ is another such sequence, then $I(K^n - \tilde{K}^n)$ does converge to 0 in $\mathcal{M}^2$ by using again the aforementioned isometry.

**Definition 1.5.** *We call this process $I(K)$ the stochastic integral of $K$ with respect to $B$ and we will denote it by*

$$I_t^K := \int_0^t K_s dB_s.$$

**Remark 1.6.** A more compact and equivalent way to summarize this construction is to say that $K \mapsto I(K)$ is an isometry from the set $\mathcal{E}_B$ into its image in $\mathcal{M}^2$, i.e. $\|I(K)\|_{\mathcal{M}^2} = \|K\|_B$. The extension of this map to $\mathcal{P}_B = \overline{\mathcal{E}}_B$ is the map $K \mapsto I(K)$.

## 1.3 Basic properties of stochastic integrals

We now list some of the basic properties of the stochastic integral with respect to Brownian motion. The proofs of these facts often proceeds as follows: First one checks directly that it holds in the case where the process $K$ is an elementary process, and one then extends the result to the case $K \in \mathcal{P}_B$ by continuity.

- **An isometry:** Let $K \in \mathcal{P}_B$ and $K^n$ be a sequence of elementary processes that approximates $K$. We know, since $K^n$ is elementary that we have
$$\mathbb{E}[(I(K^n)_\infty)^2] \overset{\text{def}}{=} \mathbb{E}\left[\left(\int_0^\infty K_s^n dB_s\right)^2\right] = \mathbb{E}\left(\int_0^\infty (K_s^n)^2 ds\right). \qquad (*)$$
But we also know that $I(K^n)_\infty \to I(K)_\infty$ in $L^2$ (because $I(K^n) \to I(K)$ in $\mathcal{M}^2$). But on the other hand, by the very definition of convergence of $K^n$ to $K$ in $\mathcal{P}_B$, we know that
$$\int_0^\infty (K_s^n)^2 ds \xrightarrow[n\to\infty]{L^1} \int_0^\infty K_s^2 ds.$$
Thus by taking the limit as $n \to \infty$ in (*) above we obtain
$$\mathbb{E}\left[\left(\int_0^\infty K_s dB_s\right)^2\right] = \mathbb{E}\left(\int_0^\infty K_s^2 ds\right).$$

- **The quadratic variation of $I(K)$:** The quadratic variation of the continuous martingale $I(K)$ is the process $t \mapsto \int_0^t K_s^2 ds$. Indeed, one checks it first directly in the case for elementary processes. Then, using the same approximation methods as above we see that
$$(I(K^n)_\infty)^2 - \int_0^\infty (K_s^n)^2 ds \xrightarrow[n\to\infty]{L^1} (I(K)_\infty)^2 - \int_0^\infty K_s^2 ds$$
The conditional expectations with respect to $\mathcal{F}_t$ do therefore converge as well. But the conditional expectation of the LHS (LHS we know is a martingale) is equal to $(I(K^n)_t)^2 - \int_0^t (K_s^n)^2 ds$. With the help of Doob's $L^2$-inequality
$$\begin{aligned}\sup_{t\geq 0} \mathbb{E}\left[(I(K^n)_t - I(K)_t)^2\right] &\leq \mathbb{E}\left[\sup_{t\geq 0} (I(K^n)_t - I(K)_t)^2\right] \\ &\leq 4\mathbb{E}\left[(I(K^n)_\infty - I(K)_\infty)^2\right] \xrightarrow{n\to\infty} 0,\end{aligned}$$
which shows that $(I(K^n)_t)^2 \to (I(K)_t)^2$ in $L^1$ as $n \to \infty$. Similarly one shows that $\int_0^t (K_s^n)^2 ds \to \int_0^t K_s^2 ds$ in $L^1$. Hence we have shown that
$$(I(K^n)_t)^2 - \int_0^t (K_s^n)^2 ds \xrightarrow[n\to\infty]{L^1} (I(K)_t)^2 - \int_0^t K_s^2 ds =: M_t$$
which shows that $M_t$ is indeed a continuous martingale.

- **Approximations of stochastic integrals:** Suppose that $K$ is a **continuous** process in $\mathcal{P}_B$. We then claim that for all $t \geq 0$, if $\Delta_n$ is a nested sequence of subdivisions of $[0,t]$ with $|\Delta_n| \to 0$, then

$$\sum_{i=0}^{m_n-1} K_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}) \xrightarrow[n\to\infty]{L^2} \int_0^t K_s dB_s$$

*Proof.* Let us suppose first that $|K_t| \leq C$ almost surely for all $t \geq 0$. We define $\tilde{K}_s := K_s 1_{s\leq t}$, then we know that $\tilde{K}$ is still in $\mathcal{P}_B$ and we have

$$\int_0^\infty \tilde{K}_s dB_s = \int_0^t K_s dB_s.$$

We also define for all $n \in \mathbb{N}$

$$\tilde{K}_s^n := \sum_{i=0}^{m_n-1} K_{t_i^n} 1_{s\in(t_i^n, t_{i+1}^n]}$$

and we observe that

$$\int_0^\infty \tilde{K}_s^n dB_s = \int_0^t \tilde{K}_s^n dB_s = \sum_{i=0}^{m_n-1} K_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}),$$

so in order to establish the claim it suffices to show that $\tilde{K}^n \to \tilde{K}$ in $\mathcal{P}_B$, since we know that this implies in particular the $L^2$ convergence of the integrals. We notice that by the continuity of $K$ we get that

$$|\tilde{K}_s^n - \tilde{K}_s| \xrightarrow[n\to\infty]{a.s} 0$$

but also

$$|\tilde{K}_s^n - \tilde{K}_s|^2 \leq (2C)^2 1_{s\leq t},$$

thus by dominated convergence we get that

$$\mathbb{E}\left(\int_0^\infty |\tilde{K}_s^n - \tilde{K}_s|^2 ds\right) \overset{n\to\infty}{\to} 0$$

which establishes that $\tilde{K}^n \to \tilde{K}$ in $\mathcal{P}_B$. □

**Remark 1.7.** It is possible to remove the condition of boundedness in the previous proof, however, we will skip that here.

## 1.4 Generalization to stochastic integrals with respect to local martingales

### 1.4.1 The $L^2$ theory of stochastic integrals with respect to a continuous martingale

Let us now consider a continuous martingale $(M_t)_{t\geq 0}$ in our filtered probability space. We want to define the stochastic integral of processes $K$ with respect to $M$, i.e., we want to define the continuous martingale $t \mapsto \int_0^t K_s dM_s$.

In some sense, what we are now going to do is to use the very same idea, modulo the fact that time plays now a different role, because of the time-change that is due to the quadratic variation $A$ of $M$. To start with, we introduce a new set $\mathcal{E}_M$ of elementary processes as follows: These are elementary processes in $\mathcal{E}$, i.e. of the form

$$K_t = \sum_{j=0}^{m-1} K_{a_j} 1_{t\in(a_j,a_{j+1}]},$$

where $0 \leq a_0 < \cdots < a_m$ and each $K_{a_j}$ is an $\mathcal{F}_{a_j}$-measurable random variable such that for each $j$

$$\mathbb{E}((K_{a_j})^2(A_{a_{j+1}} - A_{a_j})) < \infty.$$

Note that when $M$ is a Brownian motion, then this is exactly the class of elementary processes $\mathcal{E}_B$, because if $M = B$, then $A = t$ and $t_{a_{j+1}} - t_{a_j} = a_{j+1} - a_j$.

Next, for each $K \in \mathcal{E}_M$, one can define the process

$$I(K)_t := \int_0^t K_s dM_s := \sum_{j=0}^{m-1} K_{a_j}(M_{t\wedge a_{j+1}} - M_{t\wedge a_j}),$$

the definition of $\mathcal{E}_M$ then guarantees us that the above is an $L^2$ martingale. We then define the set $\mathcal{P}_M$ of progressively measurable processes such that

$$\mathbb{E}\left(\int_0^\infty K_s^2 dA_s\right).$$

This space can be viewed as an $L^2$ space, when endowed with the scalar product

$$(K, K')_M := \mathbb{E}\left(\int_0^\infty K_s K'_s dA_s\right).$$

Now everything is exactly the same as in the construction we've already done. We keep the same space $\mathcal{M}^2$ of continuous martingales started from 0 that are bounded in $L^2$, we get

- The mapping $K \mapsto I(K)$ is an isometry from $\mathcal{E}_M$ into its image in $\mathcal{M}^2$.
  - In other words, $\|I(K)\|_{\mathcal{M}^2} = \|K\|_M$ or written out

$$\mathbb{E}\left[\left(\int_0^\infty K_s dM_s\right)^2\right] = \mathbb{E}\left(\int_0^\infty K_s^2 dA_s\right).$$

- The set $\mathcal{E}_M$ is dense in the Hilbert space $\mathcal{P}_M$.
- The mapping $K \mapsto I(K)$ can therefore be extended in a unique way into an isometry from $\mathcal{P}_M$ into the space $\mathcal{M}^2$.

Let us now list the basic properties of these stochastic integrals:

**Proposition 1.2.** *Suppose that $M$ is a continuous martingale and that $K \in \mathcal{P}_M$ and $K' \in \mathcal{P}_M$, then:*

1. *The quadratic variation of the martingale $(\int_0^t K_s dM_s)_{t\geq 0}$ is the process $\int_0^t K_s^2 d\langle M\rangle_s$.*
2. *The cross-variation of the two martingales $(\int_0^t K_s dM_s)_{t\geq 0}$ and $(\int_0^t K'_s dM_s)_{t\geq 0}$ is the process $\int_0^t K_s K'_s d\langle M\rangle_s$.*
3. *If $T$ denotes a stopping time, then if $\tilde{K}_t = K_t 1_{t<T}$, one has for all $t \geq 0$,*

$$\int_0^t \tilde{K}_s dM_s = \int_0^{t\wedge T} K_s dM_s = \int_0^t K_s dM_s^T.$$

4. *If $K$ has almost surely left-continuous paths, and if $(\Delta_n)_{n\geq 0}$ denotes a nested sequence of subdivisions of $[0,t]$ with $|\Delta_n| \to 0$, then*

$$\sum_{i=0}^{m_n-1} K_{t_i^n}(M_{t_{i+1}^n} - M_{t_i^n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t K_s dM_s$$

*Proof.* The proofs are essentially word for word the same as for Brownian motion. □

### 1.4.2 Stochastic integrals with respect to a local martingale

Item (2) in the previous proposition enables us to directly generalize the definition of stochastic integrals to the following setting:

- The process $M$ is a local martingale (and one denotes its quadratic variation by $(A_t)_{t\geq 0}$).
- The process $K$ is a progressively measurable process such that almost surely, for all $t \geq 0$, $\int_0^t K_s^2 dA_s < \infty$.

Indeed, under these conditions, one can define $(\tilde{M}_t = M_t - M_0)_{t\geq 0}$ which will be a local martingale started from 0 and for all $k$, we define the sequence of stopping times

$$T_k := \inf\{t > 0 : t = k \text{ or } \int_0^t K_s^2 dA_s = k \text{ or } |\tilde{M}_t| = k\}.$$

One notes that almost surely, $T_k \nearrow \infty$ as $k \to \infty$, because $\tilde{M}$ is continuous and $\int_0^t K_s^2 dA_s$ is non-decreasing and finite. Thus the stopped process $\tilde{M}^{T_k}$ is then a proper martingale and the process $K$ is in $\mathcal{P}_{\tilde{M}^{T_k}}$.

Then, one can define the stochastic integral of $K$ with respect to the martingale $\tilde{M}^{T_k}$ and check that for each given $t$

$$\int_0^t K_s d\tilde{M}_s^{T_k}$$

is almost surely the same for all integer values of $k$ such that $t \leq T_k$. We define this value to be $\int_0^t K_s dM_s$, and one can then check that this process is a local martingale started from 0, because for each $k$, this process stopped at $T_k$ is a martingale.

Finally, one can check that all the items in the previous stated proposition still remains valid in this most general setting.