1 Construction of a Brownian motion

1.1 Goal

We want to construct a one-dimensional Brownian Motion, that is, we want to find a probability space and a collection of random variables $(B_t)_{t\geq 0}$ (t is indexed over $\mathbb{R}_+$) such that:

1. $B_0 = 0$ almost surely.
2. B has independent and stationary increments.
   (a) Stationary increments: For all $t, h \geq 0$, we have $B(t+h)-B(t) \sim B(h)$.
   (b) Independent increments: For all positive integers k and times $0 \leq t_1 < t_2 < \cdots < t_k$ in $\mathbb{R}_+$, the k increments $B(t_1) - B(0), B(t_2) - B(t_1), \ldots, B(t_k) - B(t_{k-1})$ are independent random variables.
3. $B_t \sim \mathcal{N}(0,t)$ for all $t \in \mathbb{R}_+$.
4. There exists a set of probability 1, such that on this set $t \mapsto B_t$ is a continuous function. *(Hardest part in the proof)*

1.2 Preliminaries

Surprisingly, we only require a bare minimum of prerequisites in order to perform the construction of a Brownian motion. Namely:

1. There exists a probability space on which we can define a countable family of independent centered normal variables, i.e. a collection $N_j \sim \mathcal{N}(0,1)$ for all $j \in \mathbb{N}$.
2. If $N \sim \mathcal{N}(0,1)$, then $aN \sim \mathcal{N}(0,a^2)$ for all constants a.
3. $X \sim \mathcal{N}(0,\sigma_X^2)$, $Y \sim \mathcal{N}(0,\sigma_Y^2)$ and if X and Y are independent, then $X+Y \sim \mathcal{N}(0,\sigma_X^2+\sigma_Y^2)$.
4. If $N \sim \mathcal{N}(0,\sigma^2)$ and $N' \sim \mathcal{N}(0,\sigma^2)$ and N, N' are independent, then the two random variables
$$\frac{N+N'}{2} \text{ and } \frac{N-N'}{2}$$
are independent with common law $\mathcal{N}(0,\sigma^2/2)$, notice that their sum adds up to N. This can be seen as a *decomposition law.*

1.3 Construction

1.3.1 Notation and warm-up

For every $n \in \mathbb{N}$ we set $\mathcal{I}_n := \{[j2^{-n}, (j+1)2^{-n}] : j \in \mathbb{N}\}$ and we define $\mathcal{I} := \cup_{n\in\mathbb{N}}\mathcal{I}_n$ (everything nice and countable so far). An element of $\mathcal{I}$ is therefore an interval $I_{j,n} = [j2^{-n}, (j+1)2^{-n}]$ for some n, j. For $I \in \mathcal{I}$ we set $r(I)$ and $l(I)$ to be closed right half, respectively the closed left half of I.

For clarity purposes, here are two examples: Let $I_{j,n} = [j2^{-n}, (j+1)2^{-n}]$ as above, then

$$\begin{aligned} l(I_{j,n}) &= I_{2j,n+1} = [j2^{-n}, (2j+1)2^{-n-1}] \\ r(I_{j,n}) &= I_{2j+1,n+1} = [(2j+1)2^{-n-1}, (j+1)2^{-n}] \end{aligned}$$

For the rest of the construction we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that contains a countable collection of independent centered Gaussian random variables, with

$$\left.\begin{aligned} N_j &\sim \mathcal{N}(0,1) \text{ for all } j \in \mathbb{N}. \\ N_{j,n} &\sim \mathcal{N}(0,2^{-n}) \text{ for all } j,n \in \mathbb{N}. \end{aligned}\right\} \text{Independent}$$

1.3.2 Construction on integer times

We first construct the values of the Brownian Motion at **integer times**. This is easy, we simply define the values of the process A (that we want to be a Brownian motion) as follows: $A(0) = 0$, $A(1) = N_1$,$A(2) = A(1) + N_2 = N_1 + N_2$ and $A(j) = A(j-1) + N_j = N_1 + \cdots + N_j$. In particular we have $A(j) \sim \mathcal{N}(0, j)$, for every increment $A(j) - A(j-1) = N_j \sim \mathcal{N}(0,1) \sim A(1)$ and naturally all increments are independent because the N_j's are, as desired.

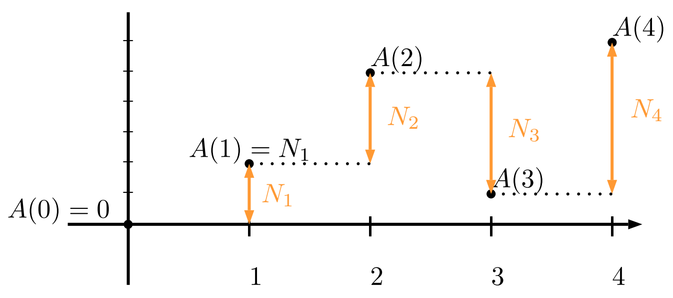

Figure 1: Construction on integer values with increments N_i displayed.

1.3.3 Construction on half integers times

Next we construct the values of the Brownian motion at **half integers times**. We introduce the notation Δ_I, which will denote what will be the increment of the Brownian Motion on the interval I.

We look at each interval $I_{j,0} = [j, j+1]$ separately, we already know the values of the Brownian Motion on such intervals, we introduce the more convenient notation $\Delta_{j,0} := \Delta_{I_{j,0}}$ and we already know that

$$\Delta_{j,0} = A(j+1) - A(j) = N_j \sim \mathcal{N}(0,1).$$

For $N_{j,0} \sim \mathcal{N}(0, 2^{-0}) = \mathcal{N}(0,1)$ we know that

$$\frac{\Delta_{j,0} + N_{j,0}}{2} \text{ and } \frac{\Delta_{j,0} - N_{j,0}}{2}$$

are two independent RV with common law $\mathcal{N}(0, 2^{-1})$ whose sum is $\Delta_{j,0}$.

We now define:

$$A\left(j + \frac{1}{2}\right) := A(j) + \left(\frac{\Delta_{j,0} + N_{j,0}}{2}\right) \sim \mathcal{N}\left(0, j + \frac{1}{2}\right). \tag{*}$$

This definition guarantees now that for the left respectively right half of $[j, j+1]$ we get:

$$\begin{aligned}
\Delta_{[j,j+1/2]} &= A(j+1/2) - A(j) = \frac{\Delta_{j,0} + N_{j,0}}{2} \sim \mathcal{N}(0, 2^{-1}), \\
\Delta_{[j+1/2,j+1]} &= A(j+1) - A(j+1/2) \\
&= A(j+1) - A(j) - \frac{\Delta_{j,0} + N_{j,0}}{2} \\
&= \Delta_{j,0} - \frac{\Delta_{j,0} + N_{j,0}}{2} = \frac{\Delta_{j,0} - N_{j,0}}{2} \sim \mathcal{N}(0, 2^{-1}).
\end{aligned}$$

So we have again $A(j+1/2) \sim \mathcal{N}(0, j+1/2)$, all increments satisfy moreover that $A(j+1/2) - A(j) \sim \mathcal{N}(0, 1/2) \sim A(1/2)$ are therefore stationary and by the above calculations of the decomposition we also have that the increments are independent.

Remark: Notice that the equation (*) above can be rewritten as

$$A\left(j + \frac{1}{2}\right) = \frac{A(j) + A(j+1)}{2} + \frac{N_{j,0}}{2}$$

this equation gives rise to a more geometric interpretation of our construction, see figure 2.

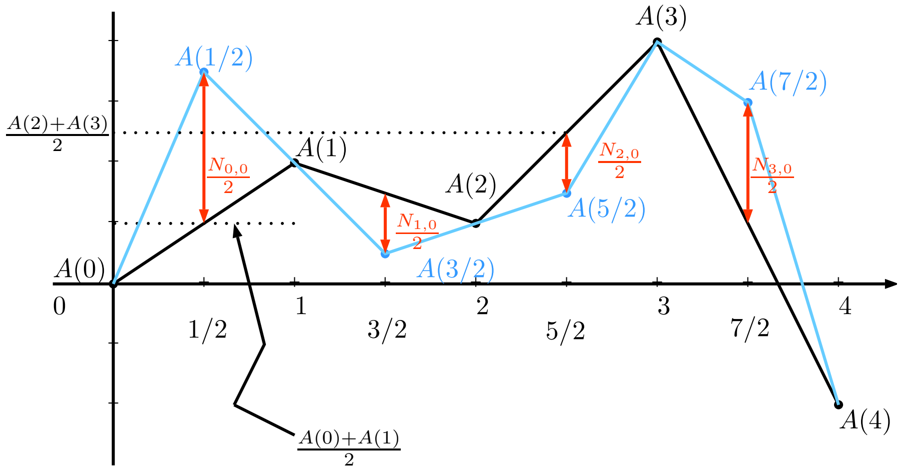

Figure 2: Geometric interpretation of our construction on the half integers

Notice that the blue and the black lines are merely for illustrative purposes. Moreover it's important to remark that $\frac{1}{2}N_{i,0} \sim \mathcal{N}(0, 1/4)$ for all $i \in \mathbb{N}$ and also $((A(i) + A(i+1))/2 \sim \mathcal{N}(0, i + 1/4)$.

1.4 Inductive construction on dyadics

We iterate over $n \in \mathbb{N}$. Let us assume that we know (for a fixed $n \in \mathbb{N}$) the values of $A(j2^{-n})$ for all $j \in \mathbb{N}$. In particular, we know the increments $\Delta_{j,n}$ for all $j \in \mathbb{N}$ and all $(\Delta_{j,n})_{j\geq 0}$ are i.i.d. with common law $\mathcal{N}(0, 2^{-n})$.

We then take again $(N_{j,n})_{j\geq 0}$ i.i.d. $\mathcal{N}(0, 2^{-n})$ and independent of $(\Delta_{j,n})_{j\geq 0}$. We then define for all intervals $I = I_{j,n} = [j2^{-n}, (j+1)2^{-n}]$ the increments $\Delta_{2j,n+1}$, $\Delta_{2j+1,n+1}$ on $l(I)$ respectively on $r(I)$ by:

$$\Delta_{2j,n+1} := \frac{\Delta_{j,n} + N_{j,n}}{2}, \ \Delta_{2j+1,n+1} := \frac{\Delta_{j,n} - N_{j,n}}{2}.$$

The previous arguments (same as on half integers) shows that this time $(\Delta_{j,n+1})_{j\geq 0}$ are i.i.d. $\mathcal{N}(0, 2^{-(n+1)})$. Also, just as in the previous step we define

$$A(\text{middle}(I_{j,n})) := A(j2^{-n}) + \Delta_{2j,n+1} \sim \mathcal{N}(0, (j + 1/2)2^{-n}).$$

With this construction we have now a collection of random variables $(A(q))_{q\in\mathcal{D}}$ where $\mathcal{D} = \{j2^{-n}, j \geq 0, n \geq 0\}$ denotes the set of the dyadics. We have the properties $A(0) = 0$ almost surely, $A(q) \sim \mathcal{N}(0, q)$ on $\mathcal{D}$ and it has stationary independent increments on $\mathcal{D}$.

1.5 Continuous extension to $\mathbb{R}_+$

We will see that almost surely the function $\mathcal{D} \ni q \mapsto A(q)$ can be extended into a continuous function $t \mapsto \tilde{A}(t)$ on $\mathbb{R}_+$ and we will show that indeed $t \mapsto \tilde{A}(t)$ is a Brownian motion.

We define for all $n \in \mathbb{N}$, the function $f_n(t)$ where $t \geq 0$ to be the linear interpolation of $A(0), A(1 \cdot 2^{-n}), A(2 \cdot 2^{-n}), \dots, A(j2^{-n}), \dots$ (i.e. linear on each $I_{j,n}$).

Our goal is to show that f_n converges uniformly to some continuous function f on any compact interval $[0, K]$, where K is an integer.
In order to achieve this goal, we notice that the function $f_{n+1} - f_n$ is

- a continuous linear function on each interval $I_{j,n+1}$.
- equal to 0 at each $j/2^n$ for all j.
- equal to $N_{j,n}/2$ at each middle point of $I_{j,n}$.

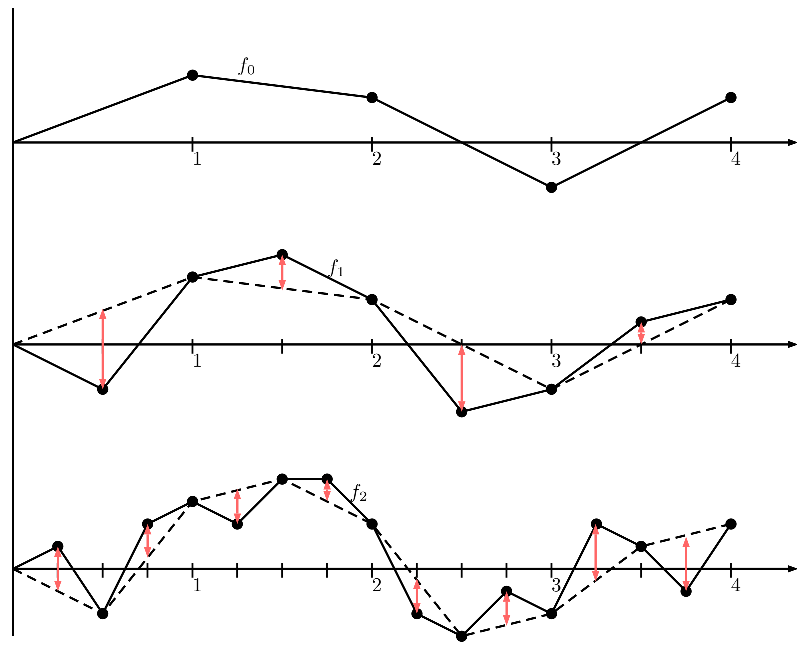

Figure 3: Illustration of the iterative construction: First on the integers, then on the half-integers and so on. The increment of f_n on the interval $I_{j,n} = [j2^{-n}, (j+1)2^{-n}]$ is $\Delta_{j,n}$. The difference $f_{n+1} - f_n$ at the middle point of each $I_{j,n}$ is $N_{j,n}/2$ (red), and at each end point of $I_{j,n}$ is zero.

With figure 3 in mind we obtain for all integers K and $x \geq 1$,

$$\begin{aligned}\mathbb{P}(\max_{[0,K]} |f_{n+1} - f_n| \geq x2^{-n/2}) &\overset{1)}{=} \mathbb{P}(\exists j \leq K2^n - 1 : |N_{j,n}| \geq 2x2^{-n/2}) \\ &\overset{2)}{\leq} \sum_{j \leq K2^n - 1} \mathbb{P}(|N_{j,n}| \geq 2x2^{-n/2}) \overset{3)}{=} K2^n \mathbb{P}(|N_1| \geq 2x) \overset{4)}{\leq} K2^n e^{-2x^2}.\end{aligned}$$

Where we used:

1. If the maximum is attained, then it's attained at the mid point of the intervals (see figure 3) where its value is $N_{j,n}/2$ (it cannot be at the start/end points because there $f_{n+1} - f_n \equiv 0$). Moreover we translate the interval $[0, K]$ for $I_{j,n} = [j2^{-n}, (j + 1)2^{-n}]$ i.e. j must run from 0 to $K2^n - 1$.

2. Union bound.

3. We have $N_{j,n}$ i.i.d. $\mathcal{N}(0, 2^{-n})$ which implies that $2^{n/2}N_{j,n} \sim \mathcal{N}(0, 1)$ and still i.i.d. so we just take $N_1 \sim \mathcal{N}(0, 1)$ (independent of j).

4. Very crude upper bound for a standard Gaussian, easy to show.

If we now choose $x = n \in \mathbb{N}_{\geq 1}$ in our estimate above, we notice that the upper bound is summable, i.e. $\sum_{n \geq 1} \mathbb{P}(\max_{[0,K]} |f_{n+1} - f_n| \geq n2^{-n/2}) < \infty$. By the Borel-Cantelli Lemma, we can conclude that almust surely, there exists a (random) $n_0 = n_0(\omega)$ such that for all $n \geq n_0$ we have

$$|f_{n+1} - f_n| \leq n2^{-n/2} \text{ on } [0, K].$$

Since $\sum n2^{-n/2} < \infty$, it is very easy to conclude from the above that $(f_n)_{n\geq 1}$ is uniformly Cauchy on $[0, K]$ (since the sum over all increments $|f_{n+1} - f_n|$ converges, we must have for large enough p, n that $|f_p - f_n|$ converges to 0).

Since $(f_n)_{n\geq 1}$ is uniformly Cauchy on $[0, K]$ we know that it must converge uniformly on the interval $[0, K]$ as $n \to \infty$ to some function f, and that this limit f is almost surely a continuous function on $[0, K]$.

We remark that this is true for each integer K, so we can interchange "almost surely" and "for all K" (recall that $\forall k, \mathbb{P}(A_k)$ is 'weaker' than $\mathbb{P}(\forall k, A_k) = \mathbb{P}(\cap_k A_k)$) to conclude that: *Almost surely, the function f_n converges uniformly on any compact subset of $\mathbb{R}_+$ to a limiting function f, and this limiting function f is continuous on $\mathbb{R}_+$.* We used that $\forall k \geq 0$ (integer i.e. countable) $\mathbb{P}(A_k^c) = 0$ then $\mathbb{P}(\cup_{k\in\mathbb{N}} A_k^c) \leq \sum_{k\in\mathbb{N}} \mathbb{P}(A_k^c) = 0$ i.e. $\mathbb{P}(\cap_{k\in\mathbb{N}} A_k) = 1$.

1.5.1 The continuous extension is a Brownian motion

We will now prove that the law of the stochastic process $(f(t))_{t\geq 0}$ that has been constructed in the previous section is indeed that of a Brownian motion. We do already know that it is continuous and that $f(0) = 0$ almost surely. It remains to check that for any $0 = t_0 < t_1 < \cdots < t_k$, the random variables

$$f(t_1) - f(t_0), f(t_2) - f(t_1), \ldots, f(t_k) - f(t_{k-1})$$

are independent, and that the law of $f(t+h) - f(t)$ is a centered Gaussian variable with variance h.

To establish this, we first choose for each time t_l where $l = 1, \ldots, k$, a sequence $t_l(n)$ of dyadic times that converges to t_l. (For instance, we take $t_l(n)$ to be the smallest multiple of 2^{-n} that is larger than t_l). Because of the continuity of f, we know that almost surely, $f(t_l(n)) \to f(t_l)$ as $n \to \infty$. This readily implies that the random variables

$$f(t_1) - f(t_0), f(t_2) - f(t_1), \ldots, f(t_k) - f(t_{k-1})$$

are independent as the almost sure limits of the independent random variables

$$f(t_1(n)), f(t_2(n)) - f(t_1(n)), \ldots, f(t_k(n)) - f(t_{k-1}(n)).$$

Moreover we know that $f(t_l(n)) - f(t_{l-1}(n))$ is a centered Gaussian with variance $t_l(n) - t_{l-1}(n)$. Furthermore, $t_l(n) - t_{l-1}(n)$ converges to $t_l - t_{l-1}$ as $n \to \infty$. Therefore we get for all $\lambda \in \mathbb{R}$, that

$$\begin{aligned} \mathbb{E}(\exp(i\lambda(f(t_l) - f(t_{l-1}))) &= \lim_{n\to\infty} \mathbb{E}(\exp(i\lambda(f(t_l(n)) - f(t_{l-1}(n)))) \\ &= \lim_{n\to\infty} \exp(-\lambda^2/(2(t_l(n) - t_{l-1}(n)))) = \exp(-\lambda^2/(2(t_l - t_{l-1}))). \end{aligned}$$

This establishes that $f(t_l) - f(t_{l-1})$ is indeed a centered Gaussian random variable with variance $t_l - t_{l-1}$. This concludes the construction and shows that the process f is indeed a Brownian motion.

Remark 1.1. We see that this last part of the proof is in essence nothing more than using that the set of dyadic times $\mathcal{D}$ is dense in $\mathbb{R}_+$, i.e. we can approximate times $t \in \mathbb{R}_+$ by times $\tilde{t}$ in $\mathcal{D}$.

1.6 Kolmogorov's continuity criterion

Since it matches the current presentation quite well, we will also briefly discuss Kolmogorov's continuity criterion. The underlying question one should have in mind is *if continuity can be read off from the law of a stochastic process or not.* Certainly the answer is no, because it is quite easy to construct two stochastic processes which have the same law, one being continuous (always equal to 0) and the other having jumps.

However, we do add quite a bit more subtlety to this question if we consider *continuous modifications* of a stochastic process. Assume we are given the law of a stochastic process $(X_t)_{t\in I}$. We can then check whether it is possible to find some probability space, and some random process $(Y_t)_{t\in I}$ with this given law, such that there exists a measurable set with probability 1, such that for all ω in this set, $t \mapsto Y_t$ is continuous on I. It is possible to show that for all $t \in I$ we have $X_t = Y_t$ almost surely (i.e. $\forall t \in I,\ \mathbb{P}(X_t = Y_t) = 1$), we then say that the process Y is a continuous modification of X.

Theorem 1.1 (Kolmogorov's continuity criterion). *If $(X_t)_{t\geq 0}$ is a stochastic process such that for all $T > 0$, there exists $\epsilon > 0, \alpha > 0, C > 0$ such that for all $t, s \in [0, T]$ we have*

$$\mathbb{E}(|X_t - X_s|^\alpha) \leq C|t-s|^{1+\epsilon}, \tag{ς}$$

then X admits a continuous modification.

Remark 1.2. Before we write down the proof, it is important to mention that we follow very similar arguments as in the construction of a Brownian motion to establish the proof.

Proof. Let us assume that $Z_q = X_q$ for all dyadics $q \in \mathcal{D}$. Let us try to bound $|Z_{(j+1)2^{-n}} - Z_{j2^{-n}}|$ for all $j \leq 2^n T - 1$.

$$\mathbb{P}(\exists j \leq T2^n - 1 : |Z_{(j+1)2^{-n}} - Z_{j2^{-n}}| \geq x_n) \leq \sum_{j\leq T2^n-1} \mathbb{P}(|X_{(j+1)2^{-n}} - X_{j2^{-n}}|^\alpha \geq x_n^\alpha)$$

$$\overset{\text{Markov}}{\leq} \sum_{j\leq T2^n-1} \frac{1}{x_n^\alpha}\mathbb{E}(|X_{(j+1)2^{-n}} - X_{j2^{-n}}|^\alpha) \overset{\varsigma}{\leq} T2^n C\frac{(2^{-n})^{1+\epsilon}}{x_n^\alpha} = TC\frac{2^{-n\epsilon}}{x_n^\alpha} = TC2^{-n\epsilon/2}$$

Where in the last step we specialized on $x_n = 2^{-n\epsilon/(2\alpha)}$. Since $\sum 2^{-n\epsilon/2} < \infty$, we have by the Borel-Cantelli Lemma that almost surely, there exists $n_0(\omega)$ such that for all $n \geq n_0(\omega), j \leq T2^n - 1$ we have

$$|X_{(j+1)2^{-n}} - X_{j2^{-n}}| < 2^{-n\epsilon/(2\alpha)}$$

We now define f_n to be the linear interpolation of the values of X on the dyadics $\mathcal{D}$ (multiples of 2^{-n}).

$\max(|f_{n+1} - f_n|)$ on $[0,T]$ is attained at one of the middle points of $I_{j,n}$ and is therefore bounded by the maximum over all $j \leq 2^{(n+1)}T - 1$, thus almost surely there exists $n_0(\omega)$, for all $n \geq n_0$ such that

$$\max_{[0,T]} |f_{n+1} - f_n| \leq 2^{-\frac{(n+1)\epsilon}{2\alpha}}.$$

Since $\sum 2^{-n\epsilon/(2\alpha)} < \infty$, we get that the series $\sum |f_{n+1} - f_n|$ is uniformly summable and thus $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $[0,T]$. Therefore there exists a continuous function f on $[0,T]$ such that $f_n \to f$ uniformly on $[0,T]$.

We now claim that for all fixed $t > 0$ we have $f(t) = X(t)$ almost surely, that is f is a continuous modification of X. To this extent let $q_n(t)$ be a sequence of dyadics that converges to t. Since f is almost surely continuous at t we have (almost surely)

$$f(t) = \lim_{n\to\infty} f(q_n(t)) = \lim_{n\to\infty} X(q_n(t)). \tag{*}$$

On the other hand, by (ς) we also have for all $\delta > 0$

$$\begin{aligned}\mathbb{P}(|X(t) - X(q_n(t))| > \delta) &\leq \delta^{-\alpha}\mathbb{E}(|X(t) - X(q_n(t))|^\alpha) \\ &\leq C\delta^{-\alpha}|t - q_n(t)|^{1+\epsilon} \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

so we have $X(q_n(t)) \to X(t)$ in Probability as $n \to \infty$. Since the almost sure convergence in (*) above implies convergence is Probability and the limit of convergence in probability is unique we must have that $f(t) = X(t)$ almost surely. $\square$

Lemma 1.1. *Kolmogorov's continuity criterion is satisfied by the law of Brownian motion.*

Proof. We know that $B_t - B_s \sim \mathcal{N}(0, t-s)$ for all $t > 0, s < t$. We first notice that $\mathbb{E}((B_t - B_s)^2) = |t-s|^1$, so we have no chance for this exponent to satisfy Kolmogorov's continuity criterion. However, we can easily see that $\sqrt{t-s}B_1 \sim \mathcal{N}(0, t-s)$ and consequently

$$\mathbb{E}((B_t - B_s)^4) = \mathbb{E}((\sqrt{t-s}B_1)^4) = |t-s|^2 \underbrace{\mathbb{E}(B_1^4)}_{<\infty}$$

So we choose $\epsilon = 1, \alpha = 4, C = \mathbb{E}(B_1)^4$ and conclude by Kolmogorv's continuity criterion. $\square$

Remark 1.3. It is possible to adapt the proof of Kolmogorov's continuity criterion to show that if the conditions are satisfied, then the modification will not only be continuous, but also γ-Hölder continuous for any $\gamma < \epsilon/\alpha$ in the sense that for all $T > 0$, there almost surely exists $C = C(T, \gamma)$ such that for all $0 \leq s < t \leq T$ we have

$$|B_t - B_s| \leq C|t - s|^\gamma.$$

Since for Brownian motion (using the same argument as in the proof of the lemma before), one can take $\alpha = 2k$ and $\epsilon = k - 1$ this shows that Brownian motion is almost surely Hölder continuous of exponent γ for all $\gamma < 1/2$. We will later establish that Brownian motion is not Hölder of exponent $1/2$.

1.7 Brownian motion as a Gaussian process, and consequences

Before we give another approach to constructing a Brownian motion, which will be more from a Functional Analytic point of view, it will be fruitful to introduce Gaussian processes in order to describe Brownian motion.

Definition 1.1. *A random vector $(X_1, \ldots, X_n) \in \mathbb{R}^n$ is a centered Gaussian vector if any linear combination of the X_i's is a centered Gaussian random variable.*

We can make some useful remarks (that should be reminders):

- If $N_1, \ldots, N_k$ are i.i.d. centered Gaussian random variables, then any vector whose entries are (fixed) linear combinations of the $N_1, \ldots, N_k$ is a centered Gaussian vector. In other words, if there exists Q a $n \times n$ Matrix such that $(X_1, \ldots, X_n) = (N_1, \ldots, N_d) \cdot Q$, then $(X_1, \ldots, X_n)$ is a Gaussian vector.
- A nice useful property is that the law of a centered Gaussian vector X is completely characterized by its covariance matrix $\sum_X = (\mathbb{E}(X_i X_j))_{1 \leq i,j \leq n}$.
- If $(X_1, \ldots, X_n)$ is a centered Gaussian vector and if for some $n_0 < n$, one has $\mathbb{E}(X_i, X_j) = 0$ for all $1 \leq i \leq n_o < j \leq n$, then the vectors $(X_1, \ldots, X_{n_0})$ and $(X_{n_0+1}, \ldots, X_n)$ are independent.

Definition 1.2. *A stochastic process $(X_t)_{t \in I}$ is said to be a centered Gaussian process for all $n \in \mathbb{N}$ and for all $t_1, \ldots, t_k \in I$ the finite dimensional vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector. The covariance function of a stochastic process $X = (X_t)_{t \in I}$ is the function $\sum_X$ defined on $I \times I$ by $\sum_X(s,t) = \mathbb{E}(X_s X_t)$.*

Remark 1.4. We know that the finite-dimensional distributions of X are completely characterized by $\sum_X$, so that the law of the whole process is also characterized by the covariance function $\sum_X$. *(Recall that the law of a stochastic process is determined by its finite-dimensional distributions).*

The important statement in this subsection is the following:

Proposition 1.1. *A Brownian Motion $(B_t)_{t \geq 0}$ is a (centered) Gaussian process with covariance function given by*

$$\Sigma_B(s,t) = \mathbb{E}(B_t B_s) = \min(s,t).$$

Proof. Let us choose $t_1 < t_2 < \cdots < t_k$ for $k \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. We want to show that $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ is a centered Gaussian variable. This is easy because we can write this as a linear combination of the independent centered Gaussian variables

$$B_{t_1}, (B_{t_2} - B_{t_1}), \ldots, (B_{t_k} - B_{t_k - 1})$$

using telescopic sums for example. Let us now assume that $s < t$, then

$$\begin{aligned}\mathbb{E}(B_t B_s) &= \mathbb{E}((B_s + B_t - B_s)B_s) = \mathbb{E}(B_s^2) + \mathbb{E}((B_t - B_s)B_s) \\ &= s + \mathbb{E}((B_t - B_s)(B_s - B_0)) = s + \mathbb{E}(B_t - B_s)\mathbb{E}(B_s) = s\end{aligned}$$

where we used that for $s < t$, $B_t - B_s$ is independent of $B_s - B_0 = B_s$. □

Corollary 1.1 (Characterization). *$(B_t)_{t\geq 0}$ is a Brownian motion if and only if $t \mapsto B_t$ is continuous on an event of probability 1* ***and*** *$(B_t)_{t\geq 0}$ is a Gaussian process with covariance function $\mathbb{E}(B_t B_s) = \min(s, t)$.*

1.7.1 Some invariance properties of Brownian motion

The following facts are immediate consequences of the previous description of a Brownian motion as a centered Gaussian process with covariance function $\mathbb{E}(B_s B_t) = s \wedge t$.

Proposition 1.2. *Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion, then:*

- *(Scaling invariance). For every $a > 0$, the process $(a^{-1}B_{a^2 t})_{t\geq 0}$ is a Brownian motion.*
- *(Inversion invariance). The process $(tB_{1/t})_{t>0}$ is distributed like $(B_t)_{t>0}$.*
- *(Invariance under time reversal) The process $(B_{1-t} - B_1)_{t\in[0,1]}$ is distributed like $(B_t)_{t\in[0,1]}$.*

Proof. We already know that all three processes involved are continuous (resp. on $[0, \infty), (0, \infty), [0, 1]$). Since B is a centered Gaussian process, it follows that the same is true for all three processes. Hence it only remains to check the covariance conditions: For the first one

$$\mathbb{E}(B_{a^2 t} B_{a^2(t+h)}/a^2) = \frac{1}{a^2}\min(a^2 t, a^2(t+h)) = \frac{1}{a^2}a^2 t = t.$$

For the second one:

$$\mathbb{E}(tB_{1/t}(t+h)B_{1/(t+h)}) = t(t+h)/(t+h) = t$$

For the last one: Let $s, t \in [0,1]$ such that $s < t$, then

$$\begin{aligned}\mathbb{E}((B_{1-t} - B_1)(B_{1-s} - B_1)) &= \mathbb{E}(B_{1-t}B_{1-s}) - \mathbb{E}(B_1 B_{1-t}) - \mathbb{E}(B_1 B_{1-s}) + \mathbb{E}(B_1^2) \\ &= (1-t) - (1-t) - (1-s) + 1 = s\end{aligned}$$

which concludes the proof. □

1.7.2 The Brownian bridge

The Gaussian processes framework can be useful to describe processes that are derived from a Brownian motion via some linear operations.

Definition 1.3. *Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion, let us define the process $\beta = (\beta_t)_{t\in[0,1]}$ via*

$$\beta_t = B_t - tB_1.$$

The process β is called a (standard) Brownian bridge.

Proposition 1.3. *The Browindian bridge $(\beta_t = B_t - tB_1, t \in [0,1])$ is a Gaussian process, with covariance function given by (when $0 \leq t \leq s \leq 1$) $\sum_\beta = t(1-s)$ and it is independent of B_1.*

Proof. β is clearly a centered Gaussian process, because B is a centered Gaussian process, thus $((\beta_t)_{t\leq 1}, B_1)$ is a Gaussian process and we have for all $t \leq 1$

$$\mathbb{E}(\beta_t B_1) = \mathbb{E}(B_t B_1 - tB_1^2) = \mathbb{E}(B_t B_1) - t\mathbb{E}(B_1^2) = t - t \cdot 1 = 0,$$

which readily implies that β is independent of B_1. Moreover we have for all $0 \leq t \leq s \leq 1$,

$$\begin{aligned}\mathbb{E}(\beta_t\beta_s) &= \mathbb{E}((B_t - tB_1)(B_s - sB_1)) = \mathbb{E}(B_tB_s - tB_1B_s - sB_1B_t + tsB_1^2) \\ &= t \wedge s - ts - st + ts = t - ts = t(1-s).\end{aligned}$$

which concludes the proof. □

1.8 L^2 considerations of constructing a Brownian Motion

1.8.1 Preliminaries

In this subsection we summarize the tools we require.

We work in the space $L^2([0,1])$, more precisely we consider the space of L^2 functions on $[0,1]$. We know that L^2 is the only space amongst the L^p spaces that is a Hilbert space, i.e. that has an inner product (geometry) given by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Moreover we will use Parseval's identity:

Theorem 1.2 (Parseval's Identity)**.** *Let V be a pre-Hilbert space and $S \subset V$ be an orthonormalsystem, i.e. all elements of S are orthogonal with respect to one another and have Norm (induced by the inner product on V) of 1. Then S is a complete orthonormalbasis of V if for all $v \in V$ parseval's identity holds:*

$$\|v\|^2 = \langle v, v\rangle = \sum_{s\in S} |\langle v, s\rangle|^2.$$

More generally, Parseval's identity holds for all $x, y \in V$

$$\langle x, y\rangle = \sum_{s\in S}\langle x, s\rangle\langle y, s\rangle.$$

Last let us recall that if we have a sequence X_n of centered Gaussian random variables such that X_n converges in probability to some finite random variable X, then X is also a centered Gaussian random variable. Indeed, convergence in probability implies the convergence in law, so that for all $\lambda \in \mathbb{R}$ we have

$$\mathbb{E}(\exp(i\lambda X)) = \lim_{n\to\infty} \exp\left(-\frac{\lambda^2}{2\sigma_{X_n}^2}\right).$$

For the right hand side to convergence to some non-zero limit (which has to be the case when λ is small), it is necessary that $\sigma_{X_n}^2$ converges to some finite limit σ^2, and it follows that the law of X is indeed $\mathcal{N}(0, \sigma^2)$.

1.8.2 Construction

Let us take an ONB of $L^2([0,1])$: $(\varphi_n)_{n\in\mathbb{N}}$. Let us define for all $t>0$ and for all $n\in\mathbb{N}$

$$\Psi_n(t) := \int_0^t \varphi_n(s)ds = \langle 1_{[0,t]}, \varphi_n\rangle.$$

Let us now consider a sequence of i.i.d. centered standard Gaussians $(N_n)_{n\in\mathbb{N}}$ (i.e. $N_n \sim \mathcal{N}(0,1)$ for all $n\in\mathbb{N}$), and let us define the functions

$$S_m(t) = \sum_{n=0}^{m} N_n \Psi_n(t).$$

We notice that the above defines a sum of independent Gaussian random variables with mean 0, in particular it defines a Gaussian process. Moreover we have

$$\begin{aligned}\mathbb{E}(S_m(t)^2) = \sum_{n=0}^{m} \Psi_n(t)^2 \mathbb{E}(N_n^2) = \sum_{n=0}^{m} \Psi_n(t)^2 = \sum_{n=0}^{m} \langle 1_{[0,t]}, \varphi_n\rangle^2 \\ \leq \sum_{n=0}^{\infty} \langle 1_{[0,t]}, \varphi_n\rangle^2 \overset{P.I.}{=} \langle 1_{[0,t]}, 1_{[0,t]}\rangle = t.\end{aligned}$$

Where in the above P.I. stands for Parseval's Identity. Since $S_m(t)$ defines a sum of independent random variables where the sum of variances converges (as $m\to\infty$) by the above, we know (easy to show) that for each t, the series $S_m(t)$ converges almost surely and in L^2 to a random variable $S(t)$ *(alternatively, S_m defines a martingale bounded in L^2 and thus must converge a.s. and in L^2).* Since each $S_m(\cdot)$ is a centered Gaussian process, the same is true (by our preliminaries) for the limiting process $(S(t))_{t\in[0,1]}$.

Furthermore using the convergence in L^2 of $S_m(t)$ to $S(t)$, we get that

$$\begin{aligned}\mathbb{E}(S(t)S(s)) &= \lim_{m\to\infty} \mathbb{E}(S_m(t)S_m(s)) = \lim_{m\to\infty} \sum_{n=0}^{m} \Psi_n(t)\Psi_n(s) \\ &= \sum_{n=0}^{\infty} \Psi_n(t)\Psi_n(s) = \sum_{n=0}^{\infty} \langle 1_{[0,t]}, \varphi_n\rangle \langle 1_{[0,s]}, \varphi_n\rangle \overset{P.I.}{=} \langle 1_{[0,t]}, 1_{[0,s]}\rangle \\ &= \int_0^1 1_{[0,t]}(u) 1_{[0,s]}(u) du = s\wedge t.\end{aligned}$$

So, we see that the law of the process $(S(t))_{t\geq 0}$ is that of a Brownian motion.

Remark 1.5. Here is a heuristic interpretation of what we just derived. The "derivative" of a Brownian motion on $[0,1]$ (recall that a Brownian motion is nowhere differentiable) is given by

$$\sum_{n\geq 0} N_n \varphi_n, \tag{$\mho$}$$

because we have seen that

$$S(t) := \sum_{n\geq 0} N_n \Psi_n(t) = \sum_{n\geq 0} N_n \int_0^t \varphi_n(s)ds$$

has the law of a Brownian Motion. In particular we can read from ($\mho$) that the "derivative" of a BM satisfies that for each orthonormal basis of L^2, the coordinates are just given by i.i.d. standard Gaussians. We stress that this does not make sense, because the sum given in ($\mho$) does not converge, but we guess that there is something there.

This "derivative" of Brownian motion, given at ($\mho$), is sometimes called **white noise**.

Let us now consider two concrete examples of orthonormal basis of $L^2([0,1])$.

First we want to consider the **Haar basis**. For each dyadic interval $I_{j,n} \subset [0,1]$, we define $\varphi_0 = 1$ and we index the others by the dyadic intervals $I_{j,n} = [j2^{-n}, (j+1)2^n]_{n\geq 0}$ for $j \leq 2^n - 1$ as $\varphi_{j,n} = 2^{n/2}$ on the left-half of $I_{j,n}$ and as $\varphi_{j,n} = -2^{n/2}$ on the right-half of $I_{j,n}$.

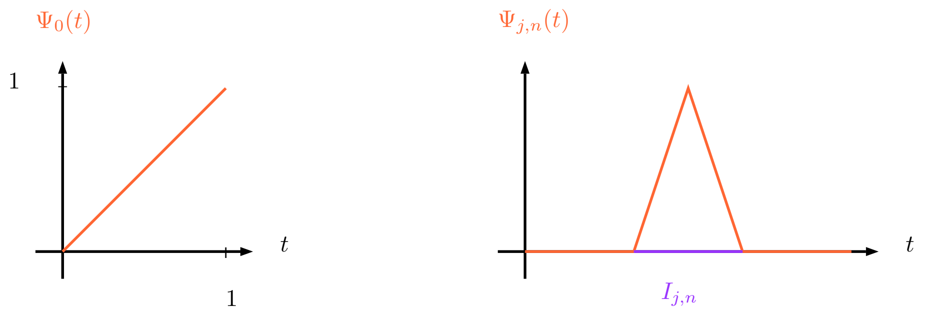

Figure 4: Depiction of Ψ_0 and $\Psi_{j,n}$ for the Haar basis.

We then realize that $S(t) = \sum_j N_{j,n}\Psi_{j,t}(t) = f_{n+1}(t) - f_n(t)$ as we have defined them in the previous construction. So this construction of S leads exactly to our previous construction of Brownian Motion in the first chapter.

1.8.3 The Fourier series of $(B_t)_{t\in[0,1]}$ and the Fourier Basis

In order to motivate the second example of ONB, namely the Fourier basis, we first start with a few considerations. Let us consider a Brownian motion $(B_t)_{t\in[0,1]}$ defined on the time-interval $[0,1]$. We define the Brownian bridge $\beta_t := B_t - tB_1$ (recall that is also a centered Gaussian process) and we know that β is independent of B_1). Since β is almost surely a continuous function with $\beta_0 = \beta_1$, we can almost surely decompose it into a Fourier series (in the sense of L^2 functions). More precisely, we have for all $t \in [0,1]$, the sequence of functions

$$S_m(t) := \sum_{n=1}^{m} b_n \sin(\pi n t), \text{ where } b_n = \int_0^1 \beta_t \sin(\pi n t) dt.$$

We then have that this sequence S_m converges in $L^2([0,1])$ to the function $t \mapsto \beta_t$ as $m \to \infty$. So, if we set $\tilde{b_0} = B_1$ and $\tilde{b}_n = \pi n b_n/\sqrt{2}$, then we obtain

$$B_t = \tilde{b_0} t + \lim_{m\to\infty} \sum_{n=1}^{m} \frac{\tilde{b_n}\sqrt{2}}{\pi n} \sin(\pi n t), \tag{$\clubsuit$}$$

where the limit is in the sense of L^2 limits of functions.

We now use the Fourier basis. The ONB of $L^2([0,1])$ is in this case given by $\varphi_0(t) = 1$ and for all $n \geq 1$ we set $\varphi_n(t) = \sqrt{2}\cos(\pi t n)$. Consequently,

$$\Psi_0(t) = t, \ \Psi_n(t) = \frac{\sqrt{2}}{\pi n} \sin(\pi n t) \text{ for all } n \geq 1.$$

We then know that $S(t) :=$ "$\sum N_n \Psi_n(t)$" has the law of a Brownian motion on $[0,1]$ and

$$S(t) = \sum_{n=1}^{\infty} \frac{N_n\sqrt{2}}{\pi n} \sin(\pi n t) + N_0 t, \tag{$\spadesuit$}$$

since the Fourier coefficients are unique, by comparison of ($\clubsuit$) with ($\spadesuit$) we can identify the Fourier coefficients to be independent centered standard Gaussian random variables.