1 The weak and strong Markov properties for Brownian Motion and first consequences

1.1 The weak Markov property

In what follows, B will denote a real-valued Brownian motion in dimension 1, but the statements will also hold in higher dimensions.

Definition 1.1. *For every $t \geq 0$ we denote by $\mathcal{F}_t$ the σ-algebra generated by the random variables $\{B_r : 0 \leq r \leq t\}$. The Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$ is then just the collection of these σ-fields (in particular we have $\mathcal{F}_t \subset \mathcal{F}_{t+h}$ for all $t, h \geq 0$). Moreover, for each finite time $T \geq 0$, we define the process $B^{(T)} = (B_t^{(T)})_{t\geq 0}$ by*

$$B_t^{(T)} := B_{T+t} - B_T.$$

The definition of Brownian motion (stationary independent increments) ensures immediately that:

Lemma 1.1 (Weak Markov property)**.** *When T is a fixed, deterministic time, then $B^{(T)}$ is a Brownian motion that is independent of $\mathcal{F}_T$.*

1.1.1 Blumenthal's $0-1$ Law and consequences

We also define for each $t \geq 0$, the σ-Field

$$\mathcal{F}_{t+} := \bigcap_{h>0} \mathcal{F}_{t+h},$$

which seems to contain some additional infinitesimal look into the future.

Remark 1.1. We can also think about the use of the above defined σ-field if we consider for instance a local maximum on $[0, t]$, for example a world record, and we wonder if our world record lasts at least for an infinitesimal time into the future $[0, t + h]$.

Proposition 1.1 (Blumenthal's $0-1$ law)**.** *For the Brownian filtration, the σ-field $\mathcal{F}_{0+}$ is trivial in the sense that for all events $A \in \mathcal{F}_{0+}$ we have either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.*

Proof. Let us take $A \in \mathcal{F}_{0+}$, we then have for all $h > 0$ also that $A \in \mathcal{F}_h$. Let us prove that A is necessarily independent of $(B_{t_1}, \dots, B_{t_p})$ for all fixed times $0 < t_1 < \cdots < t_p$. Recall that for this, it is enough to prove that for any bounded continuous function $f : \mathbb{R}^p \to \mathbb{R}$ we have

$$\mathbb{E}(1_A f(B_{t_1}, \dots, B_{t_p}) = \mathbb{P}(A)\mathbb{E}(f(B_{t_1}, \dots, B_{t_p})).$$

Let $h > 0$. By dominated convergence and the continuity of B we easily establish that

$$\begin{aligned}\mathbb{E}(1_A f(B_{t_1}, \dots, B_{t_p})) &= \lim_{h\to 0} \mathbb{E}(1_A f(B_{t_1+h} - B_h, \dots, B_{t_p+h} - B_h)) \\ = \lim_{h\to 0} \mathbb{E}(\overbrace{1_A}^{\in \mathcal{F}_h} f(\underbrace{B^{(h)}_{t_1}, \dots, B^{(h)}_{t_p}}_{\text{indep. of } \mathcal{F}_h})) &\overset{\text{M.P.}}{=} \lim_{h\to 0} \mathbb{P}(A)\mathbb{E}(f(B^{(h)}_{t_1}, \dots, B^{(h)}_{t_p})) \\ &= \mathbb{P}(A)\mathbb{E}(f(B_{t_1}, \dots, B_{t_p}))\end{aligned}$$

By a monotone class argument this establishes that A is independent of $\mathcal{F}_\infty$, but then since $A \in \mathcal{F}_h \subset \mathcal{F}_\infty$ we conclude that A is independent of itself, in particular we have

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2 \implies \mathbb{P}(A) = 0 \text{ or } 1.$$

□

Here is a nice and easy consequence of Blumenthal's $0-1$ law.

Proposition 1.2. *Almost surely,*

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = \infty \textit{ and } \liminf_{t\to 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

Remark 1.2. Note that this implies in particular that for all $\epsilon > 0$, there exists infinitely many times $t \in (0, \epsilon)$ at which $B_t = 0$. Moreover, the Proposition implies that almost surely, Brownian motion is not Hölder continuous with exponent $1/2$ (nor with any exponent greater than $1/2$) and thus Brownian motion is, as expected, nowhere differentiable.

In order to see the second remark, let $0 \le s < t$ be arbitrary, we then have almost surely

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = \infty = \limsup_{t\to s} \frac{B_{t-s}}{\sqrt{t-s}} = \limsup_{t\to s} \frac{B_t - B_s}{\sqrt{t-s}}$$

in particular we cannot have $|B_t - B_s| \le C\sqrt{t-s}$ for some $C \in \mathbb{R}_{>0}$.

Proof. We define for each $\epsilon > 0$, the random variable

$$V_\epsilon := \sup_{s\in(0,\epsilon]} \frac{B_s}{\sqrt{s}}.$$

Thanks to the continuity of Brownian motion we can write $V_\epsilon = \sup\{B_s/\sqrt{s} : s \in (0,\epsilon] \cap \mathbb{Q}\}$, and we notice that V_ϵ is $\mathcal{F}_\epsilon$ measurable. Moreover, obviously $t \mapsto V_t$ is increasing and thus we can define

$$V_{0+} := \lim_{\epsilon\to 0} V_\epsilon = \inf_{\epsilon>0} V_\epsilon.$$

So we see that the random variable V_{0+} is $\mathcal{F}_\epsilon$-measurable for all $\epsilon > 0$ and therefore $\mathcal{F}_{0+}$ measurable. Hence, by Blumenthal's $0-1$ law, we have that it is constant either 0 or 1.

Recall that the scaling property of a Brownian motion tells us that for all $a > 0$ we have $\frac{1}{a}B_{a^2t}$ is also a Brownian motion, in particular $\frac{1}{a}B_{a^2t} \sim B_t$. We note by applying this scaling property, that the law of V_ϵ does in fact not depend on ϵ. Indeed, let us fix $M \geq 1$ integer, then we have almost surely

$$\begin{aligned}\left\{\sup_{s\in(0,\epsilon]} \frac{B_s}{\sqrt{s}} \geq M\right\} &\overset{t:=s/\epsilon}{=} \left\{\sup_{t\in(0,1]} \frac{B_{\epsilon t}}{\sqrt{\epsilon t}} \geq M\right\} = \left\{\sup_{t\in(0,1]} \frac{\frac{1}{\sqrt{\epsilon}}B_{\epsilon t}}{\sqrt{t}} \geq M\right\} \\ &= \left\{\sup_{t\in(0,1]} \frac{B_t}{\sqrt{t}} \geq M\right\}\end{aligned}$$

Where in the last step the scaling property with $a = \sqrt{\epsilon}$ was used. So we have shown that $\mathbb{P}(V_\epsilon \geq M) = \mathbb{P}(V_1 \geq M)$, thus we finally get,

$$\begin{aligned}\mathbb{P}(V_{0+} \geq M) = \lim_{\epsilon\to 0} \mathbb{P}(V_\epsilon \geq M) = \mathbb{P}(V_1 \geq M) &= \mathbb{P}\left(\sup_{t\in(0,1]} \frac{B_t}{\sqrt{t}} \geq M\right) \\ &\geq \mathbb{P}(\sup_{t\in(0,1]} B_t \geq M) \geq \mathbb{P}(B_1 \geq M) > 0.\end{aligned}$$

So by Blumenthal's $0-1$ law we get that for all $M \in \mathbb{N}_{\geq 1}$, $\mathbb{P}(V_{0+} \geq M) = 1$. Thus we can exchange almost surely with for all M integer to get

$$\mathbb{P}(\forall M \geq 1, V_{0+} \geq M) = 1,$$

i.e. $V_{0+} = \infty$ almost surely. We can apply this also to the Brownian motion $-B$, to get the second statement. $\square$

1.2 Stopping times and the strong Markov property

The goal of this section is to extend the weak Markov property to the case where T is replaced by special random times.

Definition 1.2. *A random variable $T \in \mathbb{R}_+ \cup \{\infty\}$ is a stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$ if for every $t \geq 0$, the event $\{T \leq t\}$ is in $\mathcal{F}_t$. We also define the σ-field of the past before T as*

$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

It intuitively corresponds to all the information (about everything) that has happened up to random time T.

Remark 1.3. A good intuitive way to think about the definition of stopping times is that when T has occurred, then one actually knows it. Here is a neat example to have in mind of a stopping time: You're driving on the highway, when a red car passes you, you take the next exit. Here is an example of a non-stopping time: One year before the next big earthquake.

Proposition 1.3 (Strong Markov property)**.** *Let T be a stopping time (with respect to the Brownian filtration) such that $T < \infty$ almost surely. Then the process $B^{(T)} := (B_t^{(T)} = B_{T+t} - B_T)_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.*

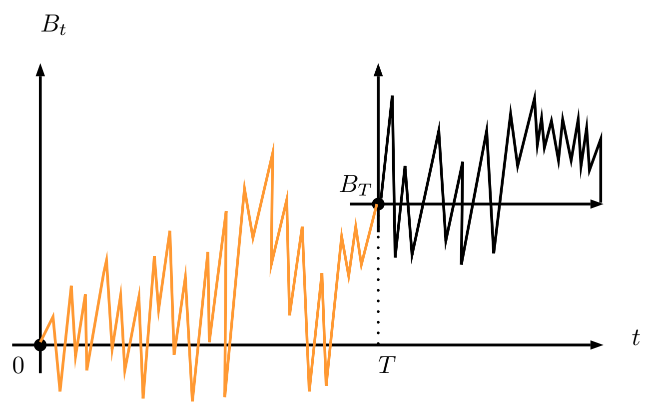

Figure 1: Qualitative picture of the strong Markov property. If we "reboot" the Brownian Motion at random stopping time T (notice the shift of axis, $B^{(T)}$ starts at 0), then what we observe is again a Brownian motion that is independent of its past $\mathcal{F}_T$ (orange).

Proof. Let $A \in \mathcal{F}_T$, we want to prove that for all $t_1 < \cdots < t_p$ the event A is independent of $(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})$ and that $(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})$ has the same law as $(B_{t_1}, \ldots, B_{t_p})$. In order to prove these two statements, it is enough to check that for all $f : \mathbb{R}^p \to \mathbb{R}$ continuous and bounded, we have

$$\mathbb{E}(1_A f(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})) = \mathbb{P}(A)\mathbb{E}(f(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})),$$

and

$$\mathbb{E}(f(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})) = \mathbb{E}(f(B_{t_1}, \ldots, B_{t_p}))$$

We work with the discrete approximation of the stopping T. I.e. we define T_n to be the smallest multiple of 2^{-n} such that $T \leq T_n$ (T_n = smallest $j2^{-n}$ with $j2^{-n} \geq T$). It is then an easy exercise to see that T_n are also a stopping times w.r.t. the Brownian filtration and that $\{T_n = j2^{-n}\} \in \mathcal{F}_{j2^{-n}}$.

We then define for each given n, $A_j = A \cap \{T_n = j2^{-n}\}$. Then, each A_j is in $\mathcal{F}_{j2^{-n}}$ and $A = \cup_j A_j$ as a disjoint union of the A_j's. So we get:

$$\begin{aligned}
\mathbb{E}(1_A f(B^{(T_n)}_{t_1}, \ldots, B^{(T_n)}_{t_p})) &= \sum_{j=0}^{\infty} \mathbb{E}(1_{A_j} f(B^{(T_n)}_{t_1}, \ldots, B^{(T_n)}_{t_p})) \\
= \sum_{j=0}^{\infty} \mathbb{E}(1_{A_j} f(B^{(j2^{-n})}_{t_1}, \ldots, B^{(j2^{-n})}_{t_p})) &\overset{\text{W.M.P.}}{=} \sum_{j=0}^{\infty} \mathbb{P}(A_j)\mathbb{E}(f(B^{(j2^{-n})}_{t_1}, \ldots, B^{(j2^{-n})}_{t_p})) \\
&= \sum_{j=0}^{\infty} \mathbb{P}(A_j)\mathbb{E}(f(B_{t_1}, \ldots, B_{t_p})) = \mathbb{P}(A)\mathbb{E}(f(B_{t_1}, \ldots, B_{t_p}))
\end{aligned}$$

But by dominated convergence and the continuity of Brownian motion we also have $\mathbb{E}(1_A f(B^{(T_n)}_{t_1}, \ldots, B^{(T_n)}_{t_p})) \to \mathbb{E}(1_A f(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p}))$ as $n \to \infty$ almost surely. So finally we get

$$\mathbb{E}(1_A f(B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_p})) = \mathbb{P}(A)\mathbb{E}(f(B_{t_1}, \ldots, B_{t_p})).$$

□

Remark 1.4. An equivalent way of phrasing the strong Markov property will be that (under the same conditions, T being a finite stopping time) for all $f : \mathbb{R}^{\times \mathbb{R}_+} \to \mathbb{R}$ bounded and continuous (measurable would be enough) we have for all $x \in \mathbb{R}$,

$$\mathbb{E}_x[f((B_{T+t})_{t\geq 0}) \mid \mathcal{F}_T] = \mathbb{E}_{B_T}[f((B_t)_{t\geq 0})]$$

i.e. conditionally on $\mathcal{F}_T$ the process $(B_{T+t})_{t\geq 0}$ is again a Brownian motion, it is independent of $\mathcal{F}_T$ and has the same law as B started from B_T.

1.2.1 Reflection principle and consequences

Suppose that T is a stopping time for the Brownian filtration and assume that T is almost surely finite. We now construct a new process $\tilde{B}$ as follows: for all $t \geq 0$

$$\tilde{B}_t := \begin{cases} B_t, & t \leq T \\ B_T - (B_t - B_T), & t \geq T \end{cases}$$

In other words, the increments of $\tilde{B}$ after the stopping time T are the opposite of those of B.

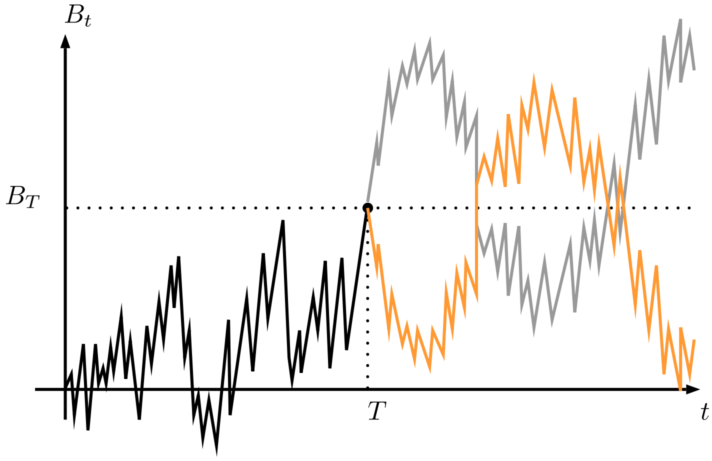

Figure 2: Depiction of the process $\tilde{B}$. We see the Brownian motion B_t (black + grey) and the reflection after stopping time T (depicted in orange) and the process $\tilde{B}$ (black + orange).

Proposition 1.4 (Reflection principle)**.** *The process $\tilde{B}$ is also a Brownian motion.*

Proof. The strong Markov property says that the process $B^{(T)}$ is a Brownian motion and independent of $\mathcal{F}_T$. Hence, the process $-B^{(T)}$ is also a Brownian motion independent of $\mathcal{F}_T$. But we can reconstruct B from the pair $(B_t, t \leq T)$ and $B^{(T)} = (B_{T+t} - B_T)_{t\geq 0}$ in exactly the same way in which $\tilde{B}$ is reconstructed from the pair $(B_t, t \leq T)$ and $-B^{(T)}$, which implies that B and $\tilde{B}$ have the same law. $\square$

Corollary 1.1. *Let B be a Brownian motion in dimension 1. For every $t > 0$, define the running maximum $S_t := \max_{s \le t} B_s$. For every $a \ge 0$ and $h \ge 0$ we have*

$$\mathbb{P}(S_t \ge a, B_t \le a - h) = \mathbb{P}(B_t \ge a + h).$$

Moreover, for each given t, the variable S_t has the same distribution as $|B_t|$.

Proof. Let $T_a = \inf\{t \ge 0 : B_t = a\}$. We know that (best seen in a picture) $\{S_t \ge a\} = \{T_a \le t\}$. We obtain:

$$\begin{aligned} \mathbb{P}(S_t \ge a, B_t \le a - h) &= \mathbb{P}(T_a \le t, B_t - B_{T_a} \le -h) = \mathbb{P}(T_a \le t, B^{(T_a)}_{t-T_a} \le -h) \\ &\overset{1)}{=} \mathbb{P}(T_a \le t, -\tilde{B}^{(T_a)}_{t-T_a} \le -h) = \mathbb{P}(T_a \le t, \tilde{B}^{(T_a)}_{t-T_a} \ge h) = \mathbb{P}(T_a \le t, \tilde{B}_t - \tilde{B}_{T_a} \ge h) \\ &\overset{2)}{=} \mathbb{P}(T_a \le t, \tilde{B}_t - B_{T_a} \ge h) = \mathbb{P}(T_a \le t, \tilde{B}_t \ge a + h) \overset{\text{R.P.}}{=} \mathbb{P}(T_a \le t, B_t \ge a + h) \\ &\overset{3)}{=} \mathbb{P}(B_t \ge a + h). \end{aligned}$$

Where we used:

1. It is easily seen that on $t \ge T_a$ we have $\tilde{B}^{(T_a)}_{t-T_a} = \tilde{B}_t - \tilde{B}_{T_a} \overset{\text{def}}{=} B_{T_a} - (B_t - B_{T_a}) - a = B_{T_a} - B_t$. Geometrically this is obvious when we look at the previous figure.

2. By definition, as already used in 1. above, we have $\tilde{B}_{T_a} = B_{T_a} = a$.

3. We have $\{B_t \ge a + h\} \subset \{T_a \le t\}$, indeed if for some fixed $t \ge 0$ we have $B_t \ge a + h$ then necessarily the first time when B_t meets the height a must occur before time t, i.e. $T_a \le t$.

And R.P. stands for Reflection Principle.

To establish the second claim we choose $h = 0$ to see that

$$\begin{aligned} \mathbb{P}(S_t \ge a) &= \mathbb{P}(S_t \ge a, B_t \le a) + \mathbb{P}(S_t \ge a, B_t \ge a) \\ &\overset{h=0}{=} \mathbb{P}(B_t \ge a) + \mathbb{P}(S_t \ge a, B_t \ge a) = 2\mathbb{P}(B_t \ge a) = \mathbb{P}(|B_t| \ge a). \end{aligned}$$

Where obviously $\{B_t \ge a\} \subset \{S_t \ge a\}$. □

1.2.2 The zero-set of a Brownian motion

We now state and prove another property of a Brownian motion which just further illustrates that a Brownian motion is quite a strange continuous curve. Let us define the zero-set of a Brownian motion as

$$\mathcal{Z} := \{t \geq 0 : B_t = 0\}.$$

Proposition 1.5. *Almost surely, the set $\mathcal{Z}$ is a perfect set (i.e. it is an non-empty closed set with no isolated points).*

Remark 1.5.

1. Recall that a point $t \in Z$ is called isolated by definition if for all $\epsilon > 0$ we have $Z \cap ((t-\epsilon, t+\epsilon) \setminus \{t\}) \neq \emptyset$.
2. It is an elementary exercise to show that a non-empty closed subset of $\mathbb{R}$ with no isolated points has the same cardinality as $\mathbb{R}$.

Proof. The set $\mathcal{Z}$ is closed almost surely thanks to the continuity of B. *(A subset D of a metric space is closed iff it contains all limits of seq. in D).*

For $q \in \mathbb{Q}_+ (= \mathbb{Q} \cap \mathbb{R}_+)$ we define the stopping time $\tau_q = \inf\{t \geq q : B_t = 0\}$. Clearly if we take $T \in \mathcal{Z}$ (i.e. $B_T = 0$) and if we assume that T is isolated from the left (i.e. $\exists \epsilon > 0$ such that $(T-\epsilon, T) \cap \mathcal{Z} = \emptyset$) then $\exists q \in \mathbb{Q}_+$ such that $\tau_q = T$.

On the other hand, we know that for all such q, $B^{(\tau_q)}$ is distributed like a Brownian motion, and we have seen ($\limsup_{t\to 0} B_t/\sqrt{t} = \infty$) that almost surely $B^{(\tau_q)}$ has infinitely many zeros in any interval $(0, \epsilon)$, in particular τ_q is not isolated from the right in $\mathcal{Z}$.

Consequently, we have almost surely for all $q \in \mathbb{Q}_+$, that τ_q is not isolated in $\mathcal{Z}$ from the right. But then almost surely for all $t \in \mathcal{Z}$ which are isolated from the left we have

$$t \in \bigcup_{q \in \mathbb{Q}} \{\tau_q\},$$

so t is not isolated from the right and therefore $\mathcal{Z}$ has no isolated points. $\square$

Remark 1.6. The Lebesgue measure $\lambda(\mathcal{Z})$ of $\mathcal{Z}$ is almost surely equal to zero, indeed, by Fubini's theorem

$$\mathbb{E}(\lambda(\mathcal{Z})) = \mathbb{E}\left(\int_0^\infty 1_{B_t=0} dt\right) = \int_0^\infty \mathbb{P}(B_t = 0) dt = 0$$

which entails that $\lambda(\mathcal{Z}) = 0$ almost surely.

1.3 Analogous results for multidimensional Brownian motion

Let us briefly list which results in the previous section can be immediately generalized to the case where one considers a Brownian motion B in d-dimensional space with $d \geq 2$ instead of $d = 1$.

- The weak Markov property.
- The strong Markov property.
- Blumenthal's $0-1$ law.

The statements and proofs are exactly the same as in the one-dimensional case.

1.3.1 One extension of some 1D results/ideas

We start with an application of Blumenthal's $0-1$ law in higher dimensions. Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d, d \geq 2$, started at the origin 0. Let $\mathcal{C}$ be an open subset of $\mathbb{R}^d \setminus \{0\}$ such for some fixed $r > 0$, it contains a union of balls $\mathcal{B}(x_n, r|x_n|)$, where x_n is some sequence in $\mathbb{R}^d \setminus \{0\}$ with $x_n \to 0$ (we can always assume that $|x_n|$ decreasing with n). That is, $\exists r > 0$ and there exists a sequence $x_n \to 0$ in $\mathbb{R}^d \setminus \{0\}$ such that

$$\bigcup_{n\in\mathbb{N}} \mathcal{B}(x_n, r|x_n|) \subset \mathcal{C}.$$

A good example to have in mind of such a set if when $\mathcal{C}$ is a cone with apex at 0.

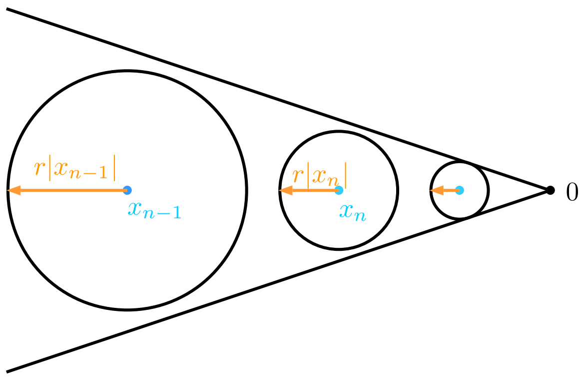

Figure 3: $\mathcal{C}$ is a cone with apex at 0.

Proposition 1.6. *Almost surely, for all $\epsilon > 0$ there exists $t \in (0, \epsilon)$ such that $B_t \in \mathcal{C}$.*

Proof. Let us define for $n_0 \in \mathbb{N}$

$$V_{n_0} := \bigcup_{n \geq n_0} \{B_{|x_n|^2} \in \mathcal{B}(x_n, r|x_n|)\}.$$

Then V_{n_0} is measurable with respect to

$$\mathcal{F}_{\max_{n \geq n_0} |x_n|^2}.$$

We have

$$V_\infty := \bigcap_{n_0 \geq 0} V_{n_0} = \{\exists \text{ infinitely many } n's : B_{|x_n|^2} \in \mathcal{B}(x_n, r|x_n|)\}.$$

Since also $V_{n_0+1} \subset V_{n_0}$ we have for all $\tilde{n}$ that

$$V_\infty = \bigcap_{n_0 > \tilde{n}} V_{n_0},$$

in particular V_∞ is in F_{0+} and therefore we have thanks to Blumenthal's $0-1$ law that $\mathbb{P}(V_\infty) = 0$ or 1. We want to show that this probability is 1.

$$\begin{aligned}
\mathbb{P}(V_\infty) &= \lim_{n_0 \to \infty} \mathbb{P}(V_{n_0}) \geq \lim_{n_0 \to \infty} \mathbb{P}(B_{|x_{n_0}|^2} \in \mathcal{B}(x_{n_0}, r|x_{n_0}|) \\
&= \lim_{n_0 \to \infty} \mathbb{P}\left[\frac{1}{|x_{n_0}|} B_{|x_{n_0}|^2} \in \mathcal{B}\left(\frac{x_{n_0}}{|x_{n_0}|}, r\right)\right] \overset{1)}{=} \lim_{n_0 \to \infty} \mathbb{P}(B_1 \in \mathcal{B}(x_{n_0}/|x_{n_0}|, r)) \\
&\overset{2)}{=} \lim_{n_0 \to \infty} \mathbb{P}(B_1 \in \mathcal{B}(1., r)) = \mathbb{P}(B_1 \in \mathcal{B}(1., r)) > 0.
\end{aligned}$$

Where we used:

1. the scaling invariance at time $t = 1$ $\frac{1}{a} B_{a^2 t} \sim B_t$.
2. The isotropy property of a Brownian motion, it just states that for all linear isometries ϕ we have $\Phi(B)$ is still a BM in $\mathbb{R}^d$ and $1. = (1, 0, 0 \ldots, 0) \in \mathbb{R}^d$.

$\square$