Seminar Euclidean Geometry FS19

Geometry over Fields

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1 Introduction

Beginning with the familiar example of the real Cartesian plane, we show how to construct a geometry satisfying Hilbert's axioms over an abstract field. We have seen that the geometry developed in Euclid's *Elements* does not make use of numbers to measure lengths or angles or areas. it is purely geometric in that it deals with points lines, circles, triangles, and the relationships among these.

In the centuries after Euclid, geometers began using numbers more and more. A big step was taken by René Descartes (1596-1650), who showed in his book *La Géométrie* how to construct the product, quotient, and square root of line segments, having once fixed a unit line segment. We shall see this result in this exposition as well. Descartes' use of algebra in geometry led to the idea of representing points in the plane by pairs of numbers, and thus to the modern discipline called *analytic geometry*.

Meanwhile, the concept of numbers were expanded from rational numbers to include irrational numbers and then transcendental numbers as they were discovered. By the end of the nineteenth century, considerations of limits and continuity made the real numbers $\mathbb{R}$ into the standard to be used in analytic geometry, calculus, and topology. Also, at the end of the nineteenth century, the formalization of abstract structures in mathematics led to the concept of a field, so that by analogy with the standard model over $\mathbb{R}$, one could also consider a geometry over any abstract field.

So far we've followed the axiomatic approach of Euclid and Hilbert, starting with geometrical postulates and proving results in a logical sequence from them. In this talk, we will diverge from this track and rather follow the algebraic methods of Descartes, that is the geometry over a field. In this setup the theory is built on a logical platform given by the algebraic definition of a field and its operations. We will start by studying the real Cartesian plane.

2 The Real Cartesian Plane

In this section we will make it clear what we mean by the *real Cartesian plane*, which is the plane geometry over the real numbers. Our proofs will be informal in the sense that we will use well-known results from high-school geometry and analytic geometry.

For the sake of exposition, we accept the field of real numbers $\mathbb{R}$ as given.

Definition 1. *We call a **point** an ordered pair $P = (a, b)$ of real numbers, and the set of all such ordered pairs is the Cartesian plane. We call the set of points $(a, 0)$ the x-axis, and the set of points $(0, b)$ the y-axis. The intersection of the two axis $(0, 0)$ is called the **origin**.*

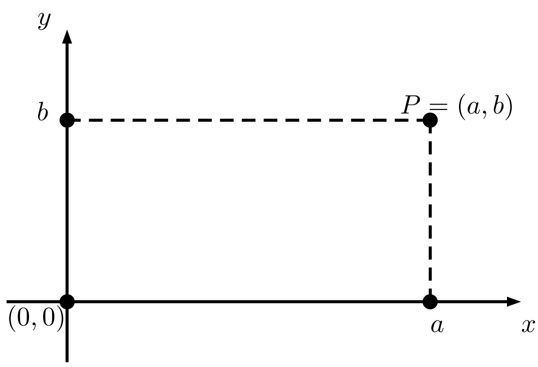

Figure 1: The point $P = (a, b)$ with respect to the x- and y-axis and the origin $(0, 0)$.

Definition 2. *A **line** in the Cartesian plane is the subset (of the Cartesian plane) defined by a linear equation of the form $ax + by + c = 0$, with a, b not both zero. We will write these lines in the canonical form $y = mx + q$, we call m the **slope** of the line and q its y-intercept. We call $x = a$ a vertical line and agree that it has slope ∞.*

Definition 3. *Two lines l_1, l_2 are called **parallel**, denoted by $l_1 \| l_2$, if they are equal or if they have no points in common, i.e. $l_1 \cap l_2 = \emptyset$.*

Lemma 1. *Two lines are parallel if and only if they have the same slope.*

Proof. If we assume the lines are equal, then of course they have the same slope. If the lines are distinct, but parallel, then they must have the same slope, else they would meet at an unique point, which would be a contradiction since distinct parallel lines have no points in common. Conversely, if two lines have the same slope, then they can either be equal or they have no points in common. $\square$

Corollary 1. *If l_1, l_2, l_3 are three distinct lines, and $l_1 \| l_2$ and $l_2 \| l_3$, then $l_1 \| l_3$.*

Proof. Indeed, all three must have the same slope as a consequence of the previous Lemma. □

Remark 1. *In Euclid's Elements, this result appears as (I.30) and is proved there using the parallel postulate plus earlier results from Book I, in particular it is non-trivial. Here, in the Cartesian plane, we have a trivial proof just by looking at the equations of the lines.*

Let us give another, less trivial, example of how useful the analytic method can be for proving geometric results. We will show that the three altitudes of a triangle meet at a point. (Compare this with the geometric proofs given earlier in Section 5, Proposition 5.6, skipped in the seminar).

Proposition 1. *In the real Cartesian plane, the three altitudes of any triangle all meet at a single point.*

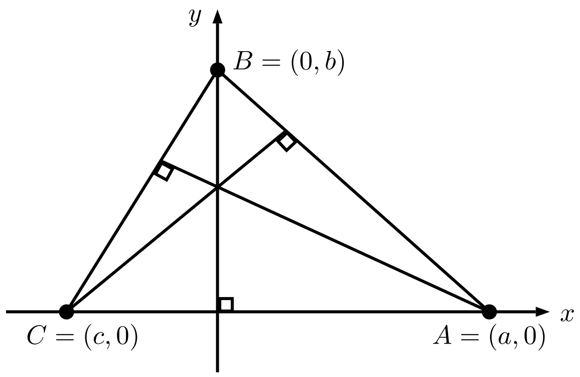

Figure 2: The situation as described in the Proposition. Recall that an *altitude* of a triangle is the line through one vertex that is perpendicular to the opposite side. The 3 lines intersect at the same point.

Proof. Without loss of generality we can move the triangle, so that one edge lies along the x-axis, and the opposite vertex lies on the y-axis (see Figure 2). With this, the y-axis is by construction one of the altitudes of the triangle and we can call the vertices $A = (a, 0)$, $B = (0, b)$ and $C = (c, 0)$.

Our strategy is to find the equations of the other two altitudes, see where they meet the y-axis (which is by construction one altitude), and verify that they meet it at the same point.

The line AB has slope $-b/a$ (using $m = \Delta y/\Delta x$), hence the altitude through C which is perpendicular to this line will have slope a/b, where we use the fact that if two perpendicular lines have slopes m_1 and m_2, then $m_1 m_2 = -1$. Thus the equation of the altitude through the point C, using the point-slope formula, is given by

$$y = \frac{a}{b}(x - c).$$

To intersect this with the y-axis, we set $x = 0$ and obtain $y = -ac/b$.

Now consider the line BC. It has slope $-b/c$, so the altitude through A will have slope c/b. Its equation becomes

$$y = \frac{c}{b}(x - a).$$

Setting $x = 0$, we obtain again $y = -ac/b$. Since this is the same point as the previous calculation yielded, we find that the three altitudes meet at the same point as claimed. $\square$

Let us reflect for a moment on the significance of this proof. If we compare this particular proof with the one presented in Section 5, we notice that there is quite a difference in its *complexity*. A serious question that might arise is, how do we respond to someone who says, with a simple analytic proof like the one presented above, why bother with geometric proofs from axioms?

The answer is subtle. Modern mathematics has abandoned the naive position that any framework is (even a posteriori) universally more potent than another point of view. Instead it asks, what can be proved within each logical framework, within each separate mathematical theory. The proof presented above shows that the result is true within the logical framework of the real Cartesian plane, using algebra of the real numbers as a logical base.

Having found the result to be true in this framework, we certainly expect it to be true in the framework of axiomatic Euclidean geometry. However, this proof gives no insight at all about how to find a proof in the abstract axiomatic geometry. In other words, if an analytic proof shows that a result is true in the geometry of the real Cartesian plane, that does not imply a proof, or even guarantee the existence of a proof, in the abstract axiomatic geometry.

Next, we turn to one of the great insights provided by the algebraic perspective, namely Descartes' discovery that the ruler and compass constructions of Euclid's geometry correspond to the solution of quadratic equations in algebra. To be more precise, let us regard a construction problem as giving certain points in the plane, and requiring the construction of certain other points.

Theorem 1 (Descartes). *Suppose we are given points $P_1 = (a_1, b_1), \ldots, P_n = (a_n, b_n)$ in the real Cartesian plane and also assume that we are given the points $(0, 0)$ and $(1, 0)$ (in order to construct a unit). Then it is possible to construct a point $Q = (\alpha, \beta)$ with ruler and compass if and only if α and β can be obtained from $a_1, \ldots, a_n, b_1, \ldots, b_n$ by field operations $+, -, \cdot, \div$ and the solution of a finite number of successive linear and quadratic equations, involving the square roots of positive real numbers.*

This theorem is a striking example of the insight into geometrical questions given by the algebraic point of view. As Descartes (1637) says:

One can construct all the problems of ordinary geometry without doing anything more than what little is contained in the four figures which I am about to *explain; which is something I do not believe that ancients had noticed: for otherwise they would not have taken the trouble to write so many fat books, where already the order of their propositions makes it clear that they did not have the true method for finding them all, but merely collected those which they happened to come across.*

Proof of Theorem. "$\Longleftarrow$" A ruler and compass construction consists of drawing lines through given points, constructing circles with given center and radius, and finding intersections of lines and circles.

• Given two points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, the line passing through them has equation

$$y - b_1 = \frac{b_2 - b_1}{a_2 - a_1}(x - a_1).$$

Its coefficients are obtained by field operations from the initial data a_1, a_2, b_1, b_2.

• A circle with center (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

This is a quadratic equation whose coefficients depend on a, b and r.

• To find the intersection of two lines, we solve two linear equations, which can be done using only field operations.

• To intersect a line with a circle, we solve the equations simultaneously, which requires solving a quadratic equation in x. Assuming that the line meets the circle, we will need to take square roots of positive numbers only.

• To intersect two circles, we first subtract the two equations, which eliminates the x^2 and y^2 terms. Then we must solve a quadratic equation with a linear equation, leading to another quadratic equation in x.

To summarize, to find the coordinates of a point $Q = (\alpha, \beta)$ obtained by a ruler and compass construction from the initial data $P_1, \ldots, P_n$, we must solve a finite number of linear and quadratic equations whose coefficients depend on the coordinates (a_i, b_i) and on quantities constructed in earlier steps.

"$\Longrightarrow$" Conversely, the roots of any linear or quadratic equation can be constructed by ruler and compass, given lengths corresponding to the coefficients of the equations, and given a standard unit length of 1. Indeed, thanks to the quadratic equation such equations can be solved by a finite number of applications of field operations $+, -, \cdot, \div$ and extractions of of square roots of positive numbers, and each of these five operations can be accomplished using ruler and compass.

We will now discuss how we can obtain the operations from a ruler and compass construction.

• For the sum and difference of two line segments, simply lay them out on the same line, end to end for the sum, or overlapping for the difference.

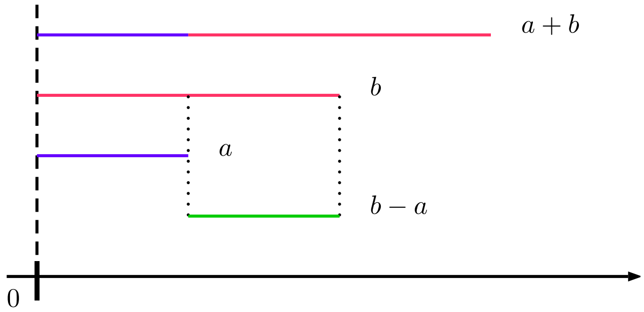

Figure 3: We put two line segments a, b on the same line end by end, to obtain the addition $a+b$ and overlapping to get the difference $b-a$.

• For the product, lay the segment a on the x-axis, and the segments 1, b on the y-axis. Draw the line from 1 to a, which will have equation

$$y = -\left(\frac{1}{a}\right)x + 1.$$

The parallel line through b has equation

$$y = -\left(\frac{1}{a}\right)x + b.$$

This intersects the x-axis in the point $(ab, 0)$. Hence we have constructed the segment ab out of the segments $1, a, b$.

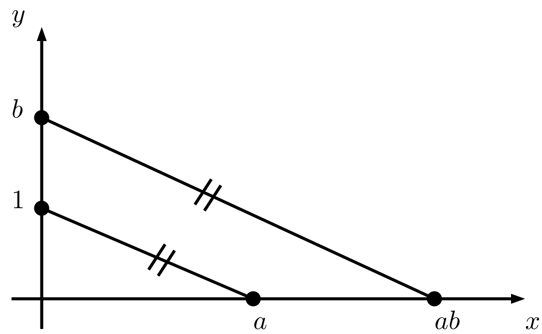

Figure 4: The product ab, can be obtained in the above described way from the segments $1, a, b$.

• For the quotient, put 1 on the x-axis, and a, b on the y-axis. A similar construction as for the product gives the point b/a on the x-axis.

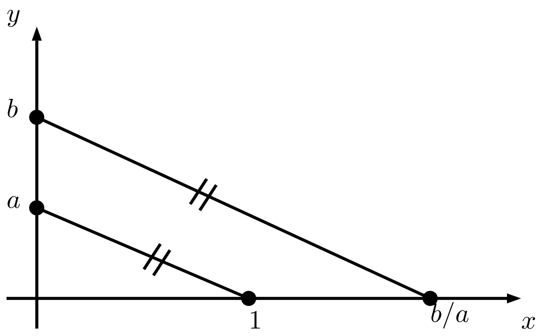

Figure 5: A similar construction as considered for the product allows us to construct the quotient.

• To construct the square root of a segment a, lay out a on the positive x-axis, and -1 on the negative x-axis. Bisect the segment from -1 to a, and draw the semicircle having that segment as diameter. The circle will satisfy the equation

$$\left(x - \frac{a-1}{2}\right)^2 + y^2 = \left(\frac{a+1}{2}\right)^2$$

Setting $x = 0$ to get the y-axis intercept and recalling that we work with the upper semicircle then yields

$$y^2 + \left(\frac{a}{2}\right)^2 - \frac{a}{2} + \frac{1}{4} = \left(\frac{a}{2}\right)^2 + \frac{a}{2} + \frac{1}{4} \implies y = \sqrt{a}.$$

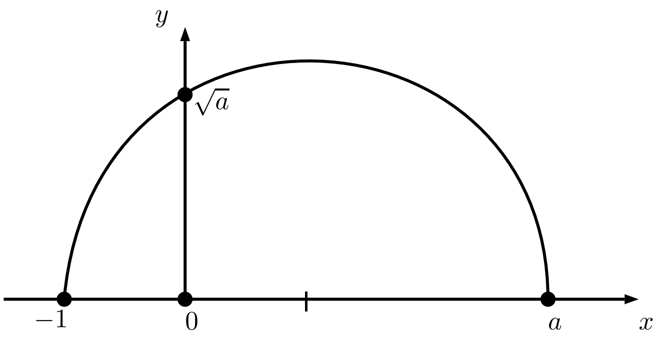

Figure 6: The above constructions yields the square root of a positive value a.

To summarize, we have shown how all the field operations $+, -, \cdot, \div$ can be recovered by the mere use of a ruler and a compass where we use lengths that correspond to the coefficients of equations and a unit length of 1. Moreover we have shown how we can extract the square root of a positive number by using a ruler and a compass. $\square$

As a practical application of this result, we will find expressions using nested square roots for some lengths that are constructible with ruler and compass, such as the sides of regular polygons inscribed in a circle.

Proposition 2. *In a circle of radius* 1*, the length of the side of a regular decagon is* $\frac{1}{2}(\sqrt{5}-1)$*.*

Proof. Let us consider the triangle ABC formed by the two radii of length 1 and one side of the decagon x. Then $AB = AC = 1$ and $BC = x$ is the side of the decagon. The angle at A is $2\pi/10$ or $36°$, so the angles at B and C are $72°$ each. Let BD bisect the angle at B. Then the two halves are both $36°$ angles. From this it follows that ABD is an isosceles triangle, and BCD is an isosceles triangle, similar to the original triangle ABC.

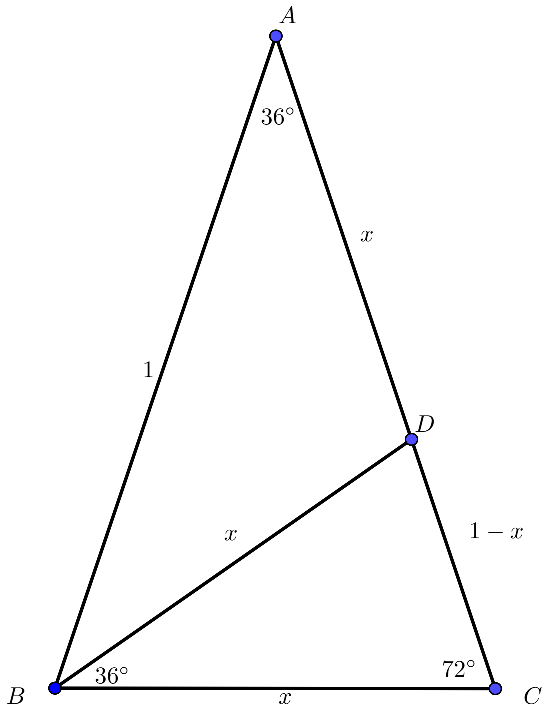

Figure 7: Visual aid for the proof.

Therefore, $BD = x$ and $AD = x$ and $CD = 1 - x$. Writing the ratios of corresponding sides of the similar triangles BCD and ABC gives

$$\frac{1-x}{x} = \frac{x}{1}.$$

Hence $x^2 + x - 1 =$, and solving this quadratic equation with the quadratic formula yields $x = \frac{1}{2}(\sqrt{5}-1)$, as required. □

Proposition 3. *The length of the chord d of a circle of radius 1 subtending an angle α at the center of the circle is given by*

$$d = \sqrt{2 - 2\cos\alpha}.$$

Proof. Recall that the law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$, i.e. $d^2 = 1^2 + 1^2 - 2\cos\alpha$, from which the result follows. $\square$

Remark 2. *We did not use the above Proposition in order to proof Proposition 2, because we do not know how to express $\cos 36°$ in terms of a ruler and a compass construction.*

Proposition 4. *In a circle of radius 1, the side of the regular pentagon is $\frac{1}{2}\sqrt{10 - 2\sqrt{5}}$.*

Proof. Applying the law of cosines to the triangle ABC as seen in the proof of Proposition 2 yields

$$1^2 = 1^2 + x^2 - 2x\cos 72°.$$

But we have seen that $x = \frac{1}{2}(\sqrt{5} - 1)$, from which it follows that $\cos 72° = \frac{1}{4}(\sqrt{5} - 1)$. Since a side of the regular pentagon subtends an angle of 72° at the center of the circle, from Proposition 3, we have that the side of the pentagon is given by

$$d = \sqrt{2 - 2\cos 72°} = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}.$$

$\square$