# Geometry over Fields

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    - $\rightsquigarrow$ Analytic geometry.

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    - Considerations of limit and continuity developed, which made $\mathbb{R}$ into the standard model for analytic geometry, calculus.
    - Formalization of abstract structures in mathematics led to the concept of a field.
    - Standard model for field is still $\mathbb{R}$, but one could also consider a geometry over any abstract field.

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  - Diverge from purely axiomatic track.
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    - In this framework the theory is built on a logical platform given by the algebraic definition of a field and its operations.

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  - Possibly let the two frameworks (axiomatic and analytic) *converge*.

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*(Out of season April fools joke)*

# The Real Cartesian Plane

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**Definition**

*We call a **point** an ordered pair $P = (a, b)$ of real numbers, and the set of all such ordered pairs is the Cartesian plane. We call the set of points $(a, 0)$ the $x$-axis, and the set of points $(0, b)$ the $y$-axis. The intersection of the two axis $(0, 0)$ is called the **origin**.*

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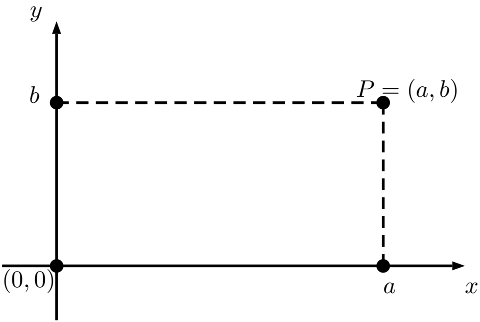

### Definition

*A* ***line*** *in the Cartesian plane is the subset (of the Cartesian plane) defined by a linear equation of the form* $ax + by + c = 0$*, with* $a, b$ *not both zero. We will write these lines in the canonical form* $y = mx + q$*, we call* $m$ *the* ***slope*** *of the line and* $q$ *its* $y$*-intercept. We call* $x = a$ *a vertical line and agree that it has slope* $\infty$*.*

**Definition**

*Two lines $l_1, l_2$ are called **parallel**, denoted by $l_1 \| l_2$, if they are equal or if they have no points in common, i.e. $l_1 \cap l_2 = \emptyset$.*

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• If the lines are distinct, parallel, then the slope must be equal for else they would meet at a unique intersection point.
• Converse arguments are the same.
• All of the above can be verified by looking at the equations for lines. □

**Corollary**

*If $l_1, l_2, l_3$ are three distinct lines, and $l_1 \| l_2$ and $l_2 \| l_3$, then $l_1 \| l_3$.*

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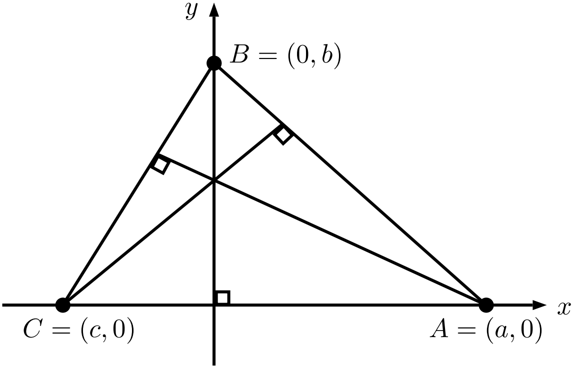
## Remark

*In Euclid's Elements, this result appears as (I.30) and is proved there using the parallel postulate plus earlier results from Book I, in particular it is non-trivial. Here, in the Cartesian plane, we have a trivial proof just by looking at the equations of the lines.*

**Proposition**

*In the real Cartesian plane, the three altitudes of any triangle all meet at a single point.*



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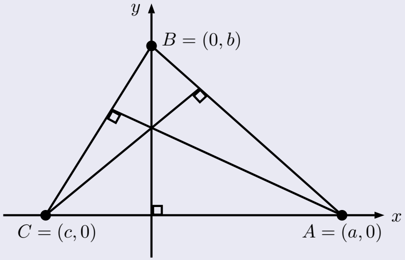


• W.l.o.g. we can arrange the triangle as depicted above. In particular the $y$-Axis will be one of the altitudes.

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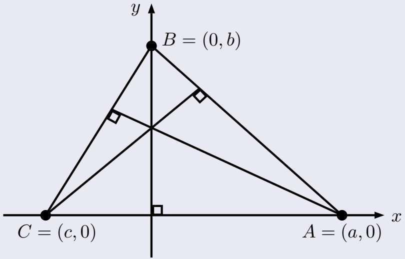


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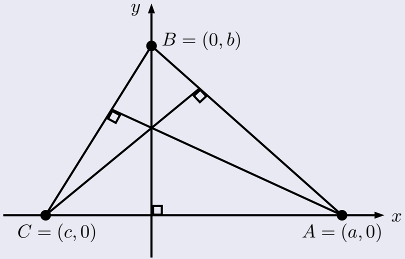


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• Idea: Find equations of two remaining altitudes and verify they meet the $y$-axis at same point.
• To this end, use that two lines with perpendicular slope satisfy $m_1 m_2 = -1$.

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Let us reflect for a moment on the significance of this proof. If we compare this particular proof with the one presented in Section 5 (Some Newer Results), we notice that there is quite a difference in its *complexity*.

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Answer might be subtle:

- Modern maths has abandoned the naive position that any framework is universally more potent than another point of view.
  - Want to know what can be proved within each logical framework, within each separate mathematical theory.
- Since result holds in this framework, we expect it to be true in the framework of axiomatic Euclidean geometry. But proof gives no insight about how to find a proof in the abstract axiomatic geometry, or if proof even exists.

**Theorem (Descartes)**

*Suppose we are given points* $P_1 = (a_1, b_1), \ldots, P_n = (a_n, b_n)$ *in the real Cartesian plane and also assume that we are given the points* $(0,0)$ *and* $(1,0)$ *(in order to construct a unit). Then it is possible to construct a point* $Q = (\alpha, \beta)$ *with ruler and compass if and only if* $\alpha$ *and* $\beta$ *can be obtained from* $a_1, \ldots, a_n, b_1, \ldots, b_n$ *by field operations* $+, -, \cdot, \div$ *and the solution of a finite number of successive linear and quadratic equations, involving the square roots of positive real numbers.*

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### Remark

*Descartes discovered that the ruler and compass constructions of Euclid's geometry correspond to the solution of linear and quadratic equations in algebra.*

As Descartes said after the discovery of his theorem:

*One can construct all the problems of ordinary geometry without doing anything more than what little is contained in the four figures which I am about to explain; which is something I do not believe that ancients had noticed: for otherwise they would not have taken the trouble to write so many fat books, where already the order of their propositions makes it clear that they did not have the true method for finding them all, but merely collected those which they happened to come across.*

**Proof of Theorem.**

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• A circle with center $(a, b)$ and radius $r$ has equation $(x - a)^2 + (y - b)^2 = r^2$. This is a quadratic equation whose coefficients depend on $a, b$ and $r$.

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• Summary: to find the coordinates point $Q = (\alpha, \beta)$ obtained by a ruler and compass from the initial data $P_1, \ldots, P_n$, we must solve a finite number of linear and quadratic eqn. whose coefficients depend on the coordinates $(a_i, b_i)$ and on quantities constructed the steps above.

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• We will now discuss how we can obtain these operations from a ruler and compass construction.

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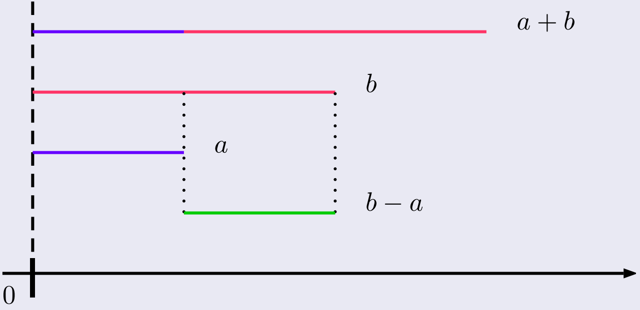


- For the sum and difference of two line segments, simply lay them out on the same line, end to end for the sum, or overlapping for the difference.

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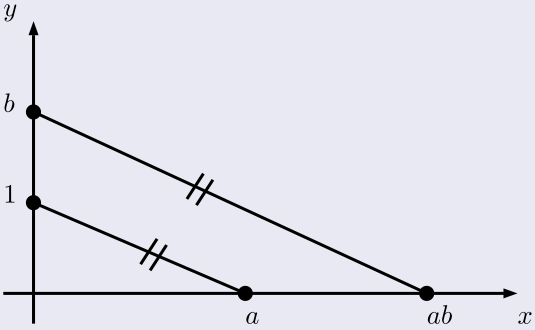
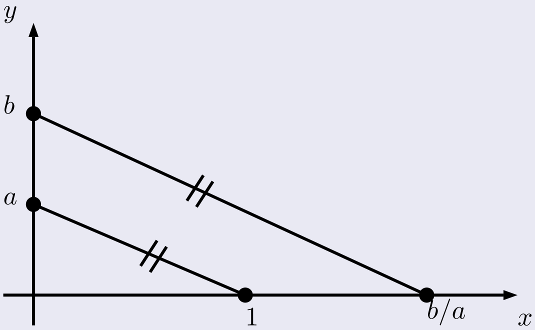



• For the product, lay the segment $a$ on the $x$-axis, and the segments 1, $b$ on the $y$-axis. Draw the line from 1 to $a$, which will have equation $y = -a^{-1}x + 1$. The parallel line that passes through $b$ is given by $y = -a^{-1}x + b$ and has $x$-intercept $ab$.

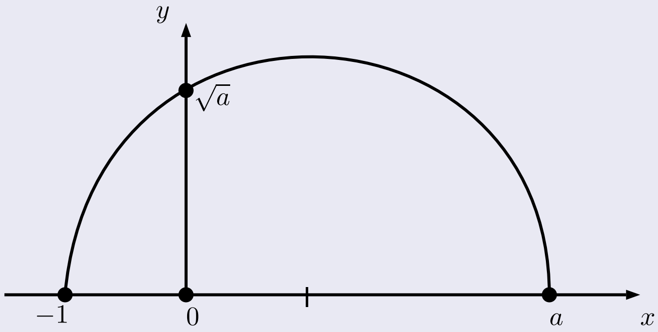
**Proof of Theorem.**




• For the quotient, put 1 on the $x$-axis, and $a, b$ on the $y$-axis. A similar construction as for the product gives the point $b/a$ on the $x$-axis.

# The Real Cartesian Plane

## Proof of Theorem.




• To construct the square root of a segment $a$, lay out $a$ on the positive $x$-axis, and $-1$ on the negative $x$-axis. Bisect the segment from $-1$ to $a$, and draw the semicircle having that segment as diameter. The circle has equation $\left(x - \frac{a-1}{2}\right)^2 + y^2 = \left(\frac{a+1}{2}\right)^2$ and $y$-intercept $\sqrt{a}$.

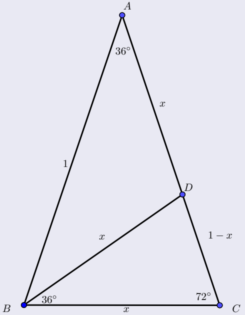
**Proof of Theorem.**

To summarize, we have shown how all the field operations $+, -, \cdot, \div$ can be recovered by the mere use of a ruler and a compass where we use lengths that correspond to the coefficients of equations and a unit length of 1. Moreover we have shown how we can extract the square root of a positive number by using a ruler and a compass. □

**Proposition**

*In a circle of radius* $1$, *the length of the side of a regular decagon is* $\frac{1}{2}(\sqrt{5}-1)$.

## Proof of Proposition.




- The triangles $ABD$ and $BCD$ are isosceles and $BCD \sim ABC$.

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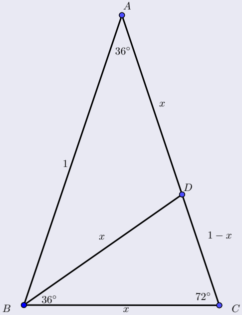


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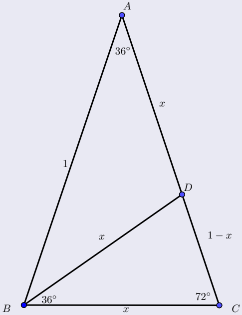



- The triangles $ABD$ and $BCD$ are isosceles and $BCD \sim ABC$.
- $BD = x$, $AD = x$ and $CD = 1 - x$.
- Ratio of corresponding sides of similar triangles yields

$$\frac{1-x}{x} = \frac{x}{1} \implies x^2 + x - 1 = 0 \implies x = \frac{1}{2}(\sqrt{5} - 1).$$

□

**Remark**

*With a little bit of more work, this result allows us to give an analytic proof of the construction of the regular pentagon (seen in a previous talk).*

Idea behind analytic proof for regular pentagon:

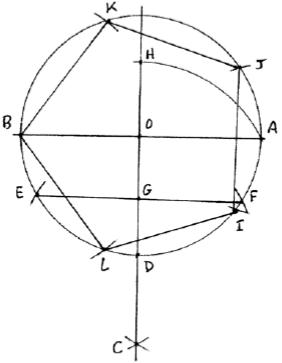
# The Real Cartesian Plane

Idea behind analytic proof for regular pentagon:

1. Draw any line through $O$. Get $A$, $B$.
2. Circle $AB$.
3. Circle $BA$, get $C$.
4. $OC$, get $D$.
5. Circle $DO$. Get $E$, $F$.
6. $EF$, get $G$.
7. Circle $GA$, get $H$.
8. Circle center $A$, radius $OH$, get $I$, $J$.
9. Circle center $B$, radius $IJ$, get $K$, $L$.

10–14. Draw $BK$, $KJ$, $JI$, $IL$, $LB$.

Then $BKJIL$ is the required pentagon.



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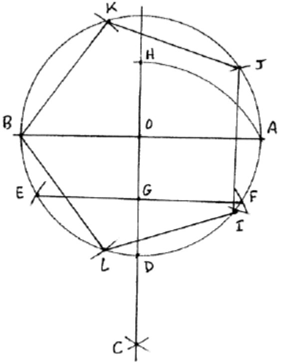
5. Circle $DO$. Get $E$, $F$.
6. $EF$, get $G$.
7. Circle $GA$, get $H$.
8. Circle center $A$, radius $OH$, get $I$, $J$.
9. Circle center $B$, radius $IJ$, get $K$, $L$.

10–14. Draw $BK$, $KJ$, $JI$, $IL$, $LB$.

Then $BKJIL$ is the required pentagon.

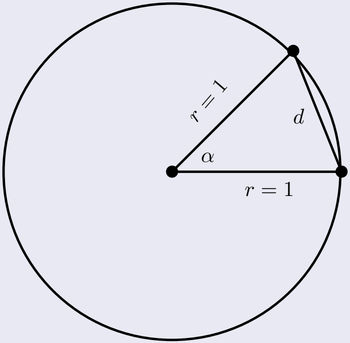



• $0A = 1$, then $0G = 1/2$ and (Pythagoras) $GA = \sqrt{5}/2$, hence $0H = \frac{1}{2}(\sqrt{5} - 1)$. Thus $A, I, J$ are vertices of regular decagon, so $IJ$ is a side of a regular pentagon.

## Proposition

*The length of the chord $d$ of a circle of radius 1 subtending an angle $\alpha$ at the center of the circle is given by*

$$d = \sqrt{2 - 2\cos\alpha}.$$



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**Proof.**

The law of cosines gives $d^2 = 1^2 + 1^2 - 2\cos\alpha$.
$(c^2 = a^2 + b^2 - 2ab\cos\gamma.)$ □

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- Hence $\cos 72^\circ = \frac{1}{4}(\sqrt{5}-1)$ and $d = \frac{1}{2}\sqrt{10-2\sqrt{5}}$. $\square$

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- Proofs are rather short and elegant, ideas are clear.
- The relationship between the algebraic and axiomatic approach to geometry is not (yet) clear.
- Motivated by the algebraic approach of the real field $\mathbb{R}$, it might be fruitful to study geometries over more general fields.

Thank you