Exercise Sheet 7
MAT602 – Functional Analysis

Hand in on Friday November 10 in class (or by 3pm in the mailbox of your assistant)

Exercise 1: Generalization of Hahn-Banach (10 points)

Let X be a vector space over $\mathbb{R}$ and $Y \subseteq X$ a linear subspace. Let $p : X \to \mathbb{R}$ be a sublinear functional and $f : Y \to \mathbb{R}$ linear with $f \leq p$ on Y.

Let G be a commutative monoid of bounded linear operators on X, i.e. $G \subseteq \mathcal{L}(X)$ with $\mathrm{id}_X \in G$ and such that for all $A, B \in G$, $AB \in G$ and $AB = BA$. Assume that for all $A \in G$, $p(Ax) \leq p(x)$ for all $x \in X$, $Ay \in Y$ and $f(Ay) = f(y)$ for all $y \in Y$.

Prove that there exists $F : X \to \mathbb{R}$ linear with $F|_Y = f$, $F \leq p$ on X and $F(Ax) = F(x)$ for all $x \in X, A \in G$.

Hint: Set $q(x) := \inf_{A_1,\ldots,A_n} \frac{1}{n} p(A_1 x + \ldots + A_n x)$, where the infimum here is taken over all finite subsets of G (i.e. $n \in \mathbb{N}$ varies as well). Show that q is sublinear and $f(y) \leq q(y)$ for all $y \in Y$, and then use Hahn-Banach.

Exercise 2: Reflexivity (5 points)

Let X and Y be Banach spaces with an isometric linear map $f : X \to Y^*$ such that $f^* : Y^{**} \to X^*$ is also isometric. Moreover, let X be reflexive. Show that there exists isometric isomorphims $Y \simeq X^*$ and $X \simeq Y^*$.

Hint: Hahn-Banach.

Exercise 3: Hellinger-Toeplitz theorem (5 points)

Let H be a Hilbert space and $A : H \to H$ linear and symmetric, i.e.

$$\langle y, Ax \rangle = \langle Ay, x \rangle \quad \forall x, y \in H.$$

Show that then A is bounded.

Exercise 4: The bi-harmonic equation (10 points)

Let $U \subseteq \mathbb{R}^n$ be open and bounded. For U with smooth boundary ∂U, the bi-harmonic equation is given by

$$\begin{aligned} -\Delta^2 u &= f \quad \text{on } U \\ u &= 0 \quad \text{on } \partial U \\ \nu \cdot \nabla u &= 0 \quad \text{on } \partial U, \end{aligned}$$

for $u \in C^4(U) \cap C^1(\overline{U})$, where ν denotes the unit normal to ∂U.

Find a "weak formulation" of this boundary value problem (in a suitable subspace of $H^{2,2}$) and show existence and uniqueness of a solution when $f \in L^2(U)$.

Exercise 5: Bilinear functionals (5 + 5 points)

Let X be a normed vector space over $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). A *bilinear functional* on X is a map $B : X \times X \to \mathbb{K}$ such that for all $x, y \in X$ the maps $B(x, \cdot) : X \to \mathbb{K}$ and $B(\cdot, y) : X \to \mathbb{K}$ are linear functionals on X.

a. Let X be a Banach space and B a bilinear functional on X which is continuous in each variable separately, i.e. for every fixed $x, y \in X$, the maps $B(x, \cdot)$ and $B(\cdot, y)$ are continuous. Show that there exists a constant $C > 0$ such that $|B(x, y)| \leq C\|x\|\|y\|$ for all $x, y \in X$. Conclude that B is continuous with respect to the norm $\|(x, y)\| := \|x\| + \|y\|$ on $X \times X$.

b. Let $\mathcal{P}$ be the vector space of real polynomials in one variable, equipped with the norm $\|p\| = \int_0^1 |p(t)|dt$ for $p \in \mathcal{P}$. Let

$$B(p, q) = \int_0^1 p(t)q(t)dt.$$

Show that B is a (real valued) bilinear functional on $\mathcal{P}$ which is continuous in each variable separately, but that B is not continuous on $\mathcal{P} \times \mathcal{P}$.