1 Quotient spaces

Let X be a Banach space and $M \subset X$ a closer linear subspace. Let $\sim$ be the equivalence relation defined through $x \sim y$ if and only if $x - y \in M$ and let X/M be the quotient space with respect to $\sim$.

a) For $[x] \in X/M$ let

$$\|[x]\|_{X/M} := \inf_{y \in [x]} \|y\|_X = \inf_{y \in M} \|x - y\|_X$$

Show that $\|\cdot\|_{X/M}$ is a norm on X/M.

Proof. Indeed we have that $\|\cdot\|_{X/M}$ is well-defined, to see this let $[x], [y] \in X/M$ be such that $[x] = [y]$, i.e. $x \sim y$ which means that we find $m_0 \in M$ such that $x - y = m_0$, thus

$$\begin{aligned}
\|[x]\|_{X/M} = \inf_{m \in M} \|x - m\|_X &= \inf_{m \in M} \|y - (m - m_0)\| \\
&= \inf_{\tilde{m} \in M} \|y - \tilde{m}\| = \|[y]\|_{X/M}
\end{aligned}$$

since we know that M is a subspace. Also since $\|\cdot\|$ is a norm on X we know that the for all $[x] \in X/M$ we have that $\|[x]\|_{X/M} < \infty$. Moreover

$$\|[x]\|_{X/M} = \inf_{y \in M} \|x - y\| = 0$$

Thus we can construct a sequence y_n in M such that $\|x - y_n\| \to 0$ as $n \to \infty$. But since $M \subset X$ is closed we have that $y_n \to x \in M$, henceforth $[x] = 0$.

Next we have

$$\begin{aligned}
\|[x] + [y]\|_{X/M} = \|[x + y]\|_{X/M} &= \inf_{v \in M} \|x + y - v\| \\
&= \inf_{v \in M} \|x - v + y - v + v\| \\
&\leq \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| + \inf_{v \in M} \|v\| \\
&= \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| \\
&= \|[x]\|_{X/M} + \|[y]\|_{X/M}
\end{aligned}$$

Finally we have for all $\lambda \in \mathbb{K}$

$$\begin{aligned}\|\lambda[x]\|_{X/M} = \|[\lambda x]\|_{X/M} &= \inf_{y\in M} \|\lambda x - y\| \\ &= \inf_{\lambda y \in M} \|\lambda x - \lambda y\| \\ &= \inf_{u\in M} \|\lambda x - \lambda u\| \qquad (*) \\ &= \inf_{u\in M} |\lambda|\|x-u\| = |\lambda|\|[x]\|_{X/M}\end{aligned}$$

Where we used in (*) that $\varphi_\lambda : M \to M$ given by $\varphi_\lambda(x) = \lambda x$ is a bijection for all $\lambda \in \mathbb{K}$. Thus we have shown that $\|\cdot\|_{X/M}$ is a norm on X/M. □

b) Show that the projection $\pi : X \to X/M,\ x \mapsto [x]$ is continuous.

Proof. Evidently, the function π is linear because we have by definition that

$$\begin{aligned}[x+y] &= [x] + [y] \\ \lambda[x] &= [\lambda x]\end{aligned}$$

For all $x, y \in X, \lambda \in \mathbb{K}$. Moreover we have

$$\|\pi(x)\|_{X/M} = \|[x]\|_{X/M} = \inf_{y\in M} \|x-y\| \leq \inf_{y\in M}\|x\| + \underbrace{\inf_{y\in M}\|y\|}_{=0} = \|x\|_X$$

That is π is a bounded linear operator and thus continuous. □

c) Show that X/M is complete.

Proof. Recall from Exercise Sheet 1 Exercise 4: Normed spaces and Banach spaces where we have shown

Theorem: Let $(X, \|\cdot\|)$ be a normed space. $(X, \|\cdot\|)$ is a Banach space if and only if, all absolutely convergent series are also convergent, i.e. if

$$\sum_{n=0}^{\infty} \|x_n\| < \infty \implies \sum_{n=0}^{\infty} x_n < \infty$$

Thanks to the above theorem in order to prove that X/M is a Banach space it is enough to show that every series in X/M that converges absolutely also converges in X/M.

To this extent let us take an arbitrary sequence $([x_n])_{n\in\mathbb{N}}$ in X/M such that its series is absolutely convergent, i.e. we have

$$\sum_{n=0}^{\infty} \|[x_n]\|_{X/M} < \infty$$

By definition of the quotient norm (via the inf) we have for all $n \in \mathbb{N}$ the existence of some $y_n \in M$ such that $\|x_n - y_n\| \leq \|[x_n]\|_{X/M} + 1/2^n$. Since we've already established the convergence of the right hand side, the convergence of the left hand side is immediate by domination. But X is a Banach space, thus it follows from the Theorem above that

$$\sum_{n=0}^{\infty} (x_n - y_n) = x < \infty$$

We now claim that the series over the sequence $([x_n])_{n\in\mathbb{N}}$ in X/M converges to $[x] \in X/M$.

Indeed, let us consider for $N \in \mathbb{N}$

$$\left\| [x] - \sum_{n=0}^{N} [x_n] \right\|_{X/M} = \left\| [x - \sum_{n=0}^{N} x_n] \right\|_{X/M} \leq \left\| x - \sum_{n=0}^{N} x_n \right\|_{X} \to 0$$

as $N \to \infty$. Where the last inequality just follows by the very definition of the quotient norm. $\square$

d) Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$.
We define the kernel of $T, \ker T := \{x \in X : Tx = 0\}$ and the range of T, $\operatorname{ran}(T) := \{Tx : x \in X\} \subset Y$. Let $\iota : \operatorname{ran}T \to Y, x \mapsto x$ be the inclusion map. Show that then there exists a bijective operator $\hat{T} \in \mathcal{L}(X/\ker T, \operatorname{ran}T)$ with $T = \iota \hat{T} \pi$ and $\|\hat{T}\| = \|T\|$.

Proof. We have indeed that $\ker T \subset X$ is a closed linear subspace. Now we define

$$\hat{T} : X/\ker T \to \operatorname{ran}T, \ [x] \mapsto T(x)$$

We notice that that $\hat{T}$ is well-defined. Indeed, let $[x], [y] \in X/\ker T$ be such that $[x] = [y]$, this is the case if and only if $x \sim y$ i.e. $x - y \in \ker T$, or equivalently $x \in y + \ker T$. This entails that we can write $x = y + t$ for some $t \in \ker T$. Thus we get

$$\hat{T}([x]) = T(x) = T(y + t) = T(y) + \underbrace{T(t)}_{=0} = T(y) = T([y])$$

Which shows that $\hat{T}$ is well-defined. Moreover, $\hat{T}$ is by definition (defined through T) surjective onto the range of T. Furthermore $\hat{T}$ is injective, let $[x], [y] \in X/\ker T$ be such that

$$\begin{aligned} \hat{T}([x]) = \hat{T}([y]) \iff T(x) = T(y) &\implies T(x - y) = 0 \\ &\iff x - y \in \ker T \\ &\iff x \sim y \\ &\iff [x] = [y] \end{aligned}$$

The linearity of $\hat{T}$ follows immediatly by the linearty of π and the linearty of T. Finally, we have that π is continuous, and by assumption T is continuous too.

We have seen that $\|[x]\|_{X/\ker T} \leq \|x\|_X$ and by the continuity of T we obtain for some $C > 0$ that

$$\|\hat{T}([x])\|_Y = \|T(x)\|_Y \leq C\|x\|_X$$

let us now set $\tilde{C} := \max(1, C)$. This yields that

$$\|\hat{T}([x])\|_Y \leq C\|x\|_X \leq \tilde{C}\|x\|_X = \tilde{C}\|x - 0\|_X$$

Since $0 \in \ker T$ we conclude that

$$\|\hat{T}([x])\|_Y \leq \inf_{t \in \ker T} \|x - t\| = \|x\|_{X/\ker T}$$

showing that indeed $\hat{T} \in \mathcal{L}(X/\ker T, \mathrm{ran}T)$.

Further, by the very definition of $\hat{T}$ we have that $T = \iota\hat{T}\pi$.

Lastly, we observe that since $0 \in \ker T$ we have that if $\|x\|_X \leq 1$ for an arbitrary $x \in X$, then $\|x - 0\|_X = \|x\|_X \leq 1$ and thus by definition of the infimum as the greatest lower bound we must also have $\inf_{y \in \ker T} \|x - y\| \leq 1$.

Conversely, if $\inf_{y \in \ker T} \|x - y\| \leq 1$. Then again by definition of the greatest upper bound we must also have that $\|x - 0\| = \|x\| \leq 1$. This shows that

$$\begin{aligned} \|\hat{T}\| &= \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|\hat{T}([x])\|_Y = \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|T(x)\|_Y \\ &= \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y = \|T\| \end{aligned}$$

$\square$

2 Reflexivity and weak convergence

a) Let X be a normed space and X^* its dual. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence in X^* and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Let $L_n \overset{*}{\rightharpoonup} L$ in X^* with respect to the weak-$*$ topology and $x_n \to x$ with respect to the norm in X. Show that if X is reflexive, then $L_n(x_n) \to L(x)$ as $n \to \infty$.

Proof. We recall that the space X^{**} is always a Banach space, since by assumption X is reflexive we have $X \cong X^{**}$, i.e. X is a Banach space as well. Thus we know that the reflexivity of X is equivalent to saying that X^* is reflexive.

Claim: Let X be a reflexive Banach space, then weak-$*$ convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ in X^* implies weak convergence.

Proof of Claim: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X^* that converges towards $f \in X^*$ w.r.t. τ_W^*, this is the case if and only if $f_n(x) \to f(x)$ for all $x \in X$ as $n \to \infty$.

We want to show that $f_n \rightharpoonup f$ in X^*, that is by **Lemma 6.2.1.** equivalent to showing that $\lambda(f_n) \to \lambda(f)$ as $n \to \infty$ for all $\lambda \in X^{**}$.

Since X is reflexive, the canonical inclusion $J_X : X \to X^{**}$ is surjective, that is to say that for every $\lambda \in X^{**}$ we find an $x \in X$ such that $J_X(x) = \lambda$. But then by the definition of the canonical inclusion we have

$$\lambda(f_n) = J_X(x)(f_n) = f_n(x) \to f(x) = J_X(x)(f) = \lambda(f)$$

which gives the weak convergence. $\square$

Next we recall that weakly convergent sequences are always bounded. By our efforts above we now have for the sequence $(L_n)_{n\in\mathbb{N}}$ in X^* that $L_n \overset{*}{\rightharpoonup} L$ in X^* implies that $L_n \rightharpoonup L$ in X^* and the sequence $(L_n)_{n\in\mathbb{N}}$ is bounded. Let $M > 0$ be a constant such that $\|L_n\|_{X^*} \le M$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} |L_n(x_n) - L(x)| &= |L_n(x_n) - L_n(x) + L_n(x) - L(x)| \\ &\le |L_n(x_n - x)| + |L_n(x) - L(x)| \\ &\le \|L_n\|_{X^*}\|x_n - x\|_X + |L_n(x) - L(x)| \\ &\le M\|x_n - x\|_X + |L_n(x) - L(x)| \end{aligned}$$

Letting $n \to \infty$ the right hand side converges to zero since $x_n \to x$ with respect to the norm in X and L_n converges in the sense of weak-$*$ converges towards L *(see characterization at begin of proof of claim).* $\square$

b) Let X and Y be normed vector spaces and let $T : X \to Y$ be linear. Furthermore let T be such that

$$x_n \rightharpoonup 0 \text{ in } X \implies Tx_n \rightharpoonup 0 \text{ in } Y$$

Let X be reflexive, show that then T is bounded.

Proof. For the sake of contradiction, assume that T is not bounded, this means that for every $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\|Tx_n\| \geq n^3 \|x_n\|$. We can easily choose x_n to be normed to 1. Next we define $\tilde{x_n} := x_n/n$ and notice that since x_n is normed that $\|\tilde{x_n}\|_X \to 0$ as $n \to \infty$.

Moreover we have

$$\|T\tilde{x_n}\| = \frac{1}{n}\|T(x_n)\| \geq \frac{1}{n} \cdot n^3 \|x_n\| = n^2$$

Thus we have shown that we can easily constract a sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges in norm to 0 but for which we have

$$\|Tx_n\| \geq n^2 \text{ for all } n \in \mathbb{N}$$

Let us therefore take such a sequence $(x_n)_{n\in\mathbb{N}}$ as above. We then have that $x_n \to 0$ in the strong (i.e. norm) sense and thus in particular $x_n \rightharpoonup 0$. By our initial assumption this gives that $T(x_n) \rightharpoonup T(0) = 0$ in Y. This means that for all $f \in Y^*$ we have that

$$f(T(x_n)) \to f(0) = 0$$

Furthermore we can view $T(x_n)$ as an element Ψ_n of Y^{**} by using the canonical embedding $J_Y : Y \to Y^{**}$, setting $\Psi_n := J_Y(T(x_n)) \in Y^{**}$. By definition of the canonical embedding we then have

$$\Psi_n(f) := J_Y(T(x_n))(f) = f(T(x_n)) \to 0 \text{ for all } f \in Y^*$$

This yields that we have

$$\sup_{n\in\mathbb{N}} |\Psi_n(f)| \leq c_f$$

Applying the Banach-Steinhaus Theorem to Y^{**} we further get that there exists a constant $c \geq 0$ such that

$$\sup_{n\in\mathbb{N}} \|\Psi_n\|_{Y^{**}} \leq c$$

But we recall that the canonical embedding J_Y is an isometry between Y and Y^{**}. Thus our derived bound yields

$$\sup_{n\in\mathbb{N}} \|\Psi_n\|_{Y^{**}} = \sup_{n\in\mathbb{N}} \|J_Y(T(x_n))\|_{Y^{**}} = \sup_{n\in\mathbb{N}} \|T(x_n)\|_Y \leq c$$

Which is absurd because we have constructed our sequence to satisfy

$$\|T(x_n)\|_Y \geq n^2 \to \infty \text{ as } n \to \infty$$

arriving at a contradiction, we conclude that T must be indeed bounded. $\square$

Remark 2.1. We did not make use of the fact that X is reflexive, maybe it is possible to show the above in a more elegant way when tightening the conditions for X to be reflexive.

3 Eigenvalues of the Laplace operator

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and non-empty. We call $\lambda \in \mathbb{C}$ an eigenvalue of $-\Delta$ with zero boundary values, if there exists $u \in H_0^{1,2}(\Omega)$ such that $\|u\|_{L^2(\Omega)} = 1$ and $-\Delta u = \lambda u$ in a weak sense, i.e. such that

$$\int_\Omega \nabla u \nabla \varphi dx = \lambda \int_\Omega u \varphi dx \text{ for all } \varphi \in H_0^{1,2}(\Omega)$$

Then we call u an eigenfunction to the eigenvalue λ.

a) Let λ_1 and λ_2 be eigenvalues of $-\Delta$ with eigenfunctions u_1 and u_2. Show that if $\lambda_1 \neq \lambda_2$, then u_1 and u_2 are orthogonal in L^2, i.e. $\langle u_1, u_2 \rangle_{L^2(\Omega)} = 0$

Proof. We have

$$\langle u_1, u_2 \rangle_{L^2} = \int_\Omega u_1(x) u_2(x) dx$$

Since u_1, u_2 are both eigenvalues of $-\Delta$ with eigenvalues λ_1 respectively λ_2 we can say that

$$\lambda_1 \int_\Omega u_1(x) u_2(x) dx = \lambda_1 \langle u_1, u_2 \rangle_{L^2}$$

using $\varphi = u_2 \in H_0^{1,2}(\Omega)$ as a test function in the above, similarly we have

$$\lambda_2 \int_\Omega u_1(x) u_2(x) dx = \lambda_2 \langle u_1, u_2 \rangle_{L^2}$$

using $\varphi = u_1 \in H_0^{1,2}(\Omega)$ as a test function in the above definition. We thus obtain that

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle u_1, u_2 \rangle_{L^2} = 0 \implies \langle u_1, u_2 \rangle_{L^2} = 0$$

since by assumption $\lambda_1 \neq \lambda_2$. $\square$