1 Generalization of Hahn-Banach

Let X be a vector space over $\mathbb{R}$ and $Y \subset X$ a linear subspace. Let $p : X \to \mathbb{R}$ be a sublinear functional and $f : Y \to \mathbb{R}$ linear with $f \leq p$ on Y.

Consider now $G \subset \mathcal{L}(X) = \mathcal{L}(X, X)$ a subset of bounded linear operators with the properties that $\mathrm{id}_X \in G$ and for all $A, B \in G, AB \in G$ and moreover $AB = BA$. Assume that for all $A \in G$ we have $p(Ax) \leq p(x)$ for all $x \in X$, $Ay \in Y$ and $f(Ay) = f(y)$ for all $y \in Y$.

Claim: There exists $F : X \to \mathbb{R}$ linear with $F_{|Y} = f, F \leq p$ on X and $F(Ax) = F(x)$ for all $x \in X$ and $A \in G$.

Proof. Given the hint to consider $q(x) := \inf_{A_1,\dots,A_n} \frac{1}{n} p(A_1 x + \cdots + A_n x)$ for $x \in X$. Here the infimum is taken over finitely many $A_i \in G$. Then it is clear that q is sublinear because all the $A_i : X \to X$ are linear and p itself is sublinear. More formally this can be shown as described in the Hints to this Exercise Sheet, but it really breaks down to using these two assumptions on p, thus I will be very sketchy about the *proof.*

It is trivial to verify that for $\lambda \geq 0$ we have $q(\lambda x) = \lambda q(x)$ on X. Let now $x, y \in X$ be arbitrary by the definition of the infimum we can find for $\epsilon > 0$, $A_1, \dots, A_n \in G$ and $B_1, \dots, B_m \in G$ such that

$$q(x) \leq \frac{1}{n} p(A_1 x + \cdots + A_n x) \leq q(x) + \epsilon/2$$
$$q(y) \leq \frac{1}{m} p(B_1 y + \cdots + A_m y) \leq q(y) + \epsilon/2$$

We easily find the two lower bounds

$$\begin{aligned} \frac{1}{n} p(A_1 x + \cdots + A_n x) &= \frac{1}{mn} \sum_{k=1}^{m} p(A_1 x + \cdots + A_n x) \\ &\geq \frac{1}{mn} \sum_{k=1}^{m} pp(B_k A_1 x + \cdots + B_k A_n x) \end{aligned}$$

and similarly

$$\frac{1}{m} p(B_1 y + \cdots + B_m y) \geq \frac{1}{mn} \sum_{k=1}^{n} p(A_k B_1 y + \cdots + A_k B_m y)$$

Bringing both of this estimates together, using that p is sublinear, the linearty of the A_i and the commutativity of G eventually leads to

$$q(x) + q(y) + \epsilon \geq \frac{1}{mn} p \left(\sum_{k=1}^{m} \sum_{l=1}^{n} B_k A_l (x + y) \right) \geq q(x + y)$$

Since $\epsilon > 0$ was arbitrary we conclude the sublinearty of q.

Moreover for $y \in Y$ and $A_1, \ldots, A_n \in G$ where $n \in \mathbb{N}$ we have

$$f(A_1 y + \cdots + A_n y) = f(A_1 y) + \cdots + f(A_n y) = n f(y)$$

But also $f(A_1 y + \cdots + A_n y) \leq p(A_1 y + \cdots + A_n y)$ by assumption and thus we have $f(y) \leq \frac{1}{n} p(A_1 y + \cdots + A_n y)$ for all $A_1, \ldots, A_n \in G$. That is f is a lower bound of the expression $\frac{1}{n} p(A_1(\cdot) + \cdots + A_n(\cdot))$ on Y, by the definition of the infimum as the greatest lower bound we conclude that $f(y) \leq q(y)$ on Y.

By **Hahn-Banach Theorem** there exists $F : X \to \mathbb{R}$ linear with $F_{|Y} = f$ and $F(x) \leq q(x)$ for all $x \in X$. However since we also have that $p(A_1 x + \cdots + A_n x) \leq p(A_1 x) + \cdots + p(A_n x) \leq n p(x)$ it also follows that $q \leq p$ on X which takes care of the first part of the claim.

Let now $A_1, \ldots, A_n \in G$ be arbitrary, then we have that

$$\begin{aligned} q(Ax - x) &\leq \frac{1}{n} p(A_1(Ax - x) + \cdots + A_n(Ax - x)) \\ &= \frac{1}{n} p(A_1 A x + \cdots + A_n A x - (A_1 x + \cdots + A_n x)) \\ &= \frac{1}{n} p(A A_1 x + \cdots + A A_n x - (A_1 x + \cdots + A_n x)) \end{aligned}$$

If we now apply this to the special case where $A_i = A^{i-1} \in G$ for $i = 1, \ldots, n$ where $A^0 = id_X \in G$ we obtain from the emerging telescoping sum that for all $n \in \mathbb{N}$ we have

$$q(Ax - x) \leq \frac{1}{n} p(A^n x - x) \leq \frac{1}{n} (p(A^n x) + p(-x)) \leq \frac{1}{n} (p(x) + p(-x))$$

Passing this statement to the limit as $n \to \infty$ we obtain that $q(Ax - x) \leq 0$.

This implies that $F(Ax - x) \leq 0$ because $F \leq q$ on X. Thus we've got the inequality

$$F(Ax) \leq F(x)$$

But since both F and $A \in G$ are linear we obtain that

$$-F(Ax) = F(-Ax) = F(A(-x)) \leq F(-x) = -F(x)$$

Multiplying by (-1) we get that $F(Ax) \geq F(x)$ and thus $F(Ax) = F(x)$. $\square$

2 Reflexivity

Let X and Y be Banach spaces with an isometric linear map $f : X \to Y^*$ such that $f^* : Y^{**} \to X^*$ is also isometric. Moreover, let X be reflexive. Show that there exists isometric isomorphisms $Y \cong X^*$ and $X \cong Y^*$.

Proof. First consider the diagram below, seriously, it's pretty sweet.

$$\begin{tikzcd}
Y^* \arrow[d, hook, "J_{Y^*}"'] & X \arrow[l, hook', "f"'] \arrow[rr, "g"] \arrow[d, hook, two heads, "J_X"'] & & Y \arrow[d, hook, "J_Y"] & \\
Y^{***} & X^{**} \arrow[l, "f^{**}"'] \arrow[rr, "g^{**}"] & & Y^{**} \arrow[r, hook, "f^*"] & X^*
\end{tikzcd}$$

In the previous exercise sheet I have already shown that the above diagram commutes, i.e. we have $g^{**} \circ J_X = J_Y \circ g$ under the assumption that g is a linear map. *(A similar proof can be found in the proof of Thm 4.3.3.)*

Now we recall that J_X, J_Y are linear isometries and quite generally, the composition of (linear) isometries is again an (linear) isometry. Moreover, isometries are always continuous and injective. Finally, if we have a bijective isometry, then quite trivially the inverse of said isometry is again an isometry.

Recall from **Thereom 4.3.4.** That for a Banach Space X we have X is reflexive if and only if X^* is reflexive. We will need that later.

Claim 1: Let $f : X \to Y$ be an linear, continuous and injective map, then $f^* : Y^* \to X^*$ is surjective.

Proof of Claim 1: Let $x^* \in X^*$ be arbitrary. Since f is linear $\mathrm{Im}(f) \subset Y$ is a linear subspace. We define

$$\Psi : \begin{cases} \mathrm{Im}(f) & \longrightarrow \mathbb{K} \\ f(x) & \longmapsto \Psi(f(x)) = x^*(x) \end{cases}$$

We notice that Ψ is well defined because f is assumed to be injective, in particular f has a 1 to 1 correspondence to its image. Clearly Ψ is linear, because both f and x^* are linear. Further we have

$$\|\Psi(f(x))\| \leq \|x^*\|\|x\| \text{ for all } x \in X$$

Hence we have that Ψ is continuous and thus $\Psi \in \mathrm{Im}(f)^*$. By **Corollary 4.2.7. to Hahn Banach** we have that Ψ extends continuously and linearly on Y i.e. $\Psi \in Y^*$. But then by definition of the adjoint we have that

$$f^*(\Psi) = \Psi \circ f = x^*$$

which entails that f^* is surjective as claimed. $\square$

Applying this result now to our $f : X \to Y^*$ which is a linear isometry we obtain that $f^* : Y^{**} \to X^*$ is surjective. Moreover, by assumption f^* is an linear isometry we now have that f^* is in fact an continuous isomorphism between Y^{**} and X^*.

Since by assumption f^* is an linear isometry (in particular injective) we also get that $f^{**} : X^{**} \to Y^{***}$ is surjective.
But now we have that $J_{Y^*} \circ f = f^{**} \circ J_X : X \to Y^{***}$ is surjective as the composition of surjective maps *(recall that X is reflexive by assumption)*, thus J_{Y^*} has to be surjective too. This entails that Y^* is reflexive and this is the case if and only if Y is reflexive, i.e. J_Y is an isometric isomorphisms.

This now shows that $f^* \circ J_Y : Y \to X^*$ is an isometric isomorphism.

In order to show that $X \cong Y^*$ we start with another claim.

Claim 2: Let $f : X \to Y$ be an isometric isomorphism. Then $f^* : Y^* \to X^*$ is a linear isometry. In Particular by the first **Claim 1** it follows that f^* is an isometric isomorphism.

Proof of Claim 2: We have by definition

$$\|f^*(y^*)\|_{X^*} = \|y^* \circ f\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |y^*(f(x))|$$

Since $f : X \to Y$ is an isomorphism we can find for all $y \in Y$ with $\|y\| \leq 1$ an $x \in X$ such that $f(x) = y$, but then we have because f is also isometric that

$$\|y\|_Y = \|f(x)\|_Y = \|x\|_X \leq 1$$

Similarly if $x \in X$ such that $\|x\| \leq 1$ then also $\|f(x) = y\| = \|x\| \leq 1$. Thus we obtain that

$$\|f^*(y^*)\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |y^*(f(x))| = \sup_{\substack{y \in Y \\ \|y\| \leq 1}} y^*(y)| = \|y^*\|_{Y^*}$$

That is f^* is an isometry $\square$

Now if we bring this all together, we have that $J_X : X \to X^{**}$ and $J_{Y^*} : Y^* \to Y^{**}$ are isometric isomorphisms, moreover, by our efforts above we have that $f^{**} : X^{**} \to Y^{***}$ is an isometric isomorphism.

Remark: Alternatively we could also have said that because $f^{**} \circ J_X = J_{Y^*} \circ f$, it follows immediately that f^{**} is an isometry, because $f^{**} = J_{Y^*} \circ f \circ (J_X)^{-1}$ is the composition of linear isometries.

We conclude that $(J_{Y^*})^{-1} \circ f^{**} \circ J_X : X \to Y^*$ is an isometric isomorphism, that is $X \cong Y^*$ which concludes the proof. $\square$

3 Hellinger-Toeplitz Theorem

Let H be a Hilber space and $A : H \to H$ linear and symmetric, i.e.

$$\langle y, Ax\rangle = \langle Ay, x\rangle \text{ for all } x, y \in H.$$

Show that then A is bounded.

Proof. We first recall **Theorem 5.1.4.**

Theorem 5.1.4. (Banach-Steinhaus): Let X be a Banach space, Y a normed space and $\mathcal{F} \subset \mathcal{L}(X, Y)$. Assume that for all $x \in X$ there exists $c_x \geq 0$ such that

$$\sup_{T \in \mathcal{F}} \|Tx\| \leq c_x$$

Then there exists $c \geq 0$ with

$$\sup_{T \in \mathcal{F}} \|T\| \leq c$$

We take $X = H$ (recall that every Hilbert Space is also a Banach Space) and $Y = \mathbb{K}$. For $y \in H$ we define

$$f_y : H \to \mathbb{K}, \ f_y(x) := \langle Ay, x\rangle$$

Then we have that f_y is linear because by our definition of the scalar product we have that $\langle \cdot, \cdot \rangle$ is linear in its second argument. Moreover the function f_y is continuous, because it's defined as an inner product. Thus we have that $f_y \in \mathcal{L}(H, \mathbb{K})$.

Let us now define

$$\mathcal{F} := \{f_y \in \mathcal{L}(X, \mathbb{K}) \mid \|y\| = 1\} \subset \mathcal{L}(X, \mathbb{K})$$

Let now $f_y \in \mathcal{F}$ be arbitrary. Then for $x \in H$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\|f_y(x)\| = |\langle Ay, x\rangle| = |\langle y, Ax\rangle| \leq \|y\|\|Ax\| = \|Ax\| =: c_x \\ \implies \sup_{f_y \in \mathcal{F}} \|f_y(x)\| \leq c_x\end{aligned}$$

By **Banach-Steinhaus** there exists $c \geq 0$ such that $\sup_{f_y \in \mathcal{F}} \|f_y\| \leq c$

We now have established the existence of a constant $c \geq 0$ such that

$$\sup_{f_y \in \mathcal{F}} \sup_{\|x\|=1} \|f_y(x)\| \leq c$$

Now we obtain that

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \|x\| \langle A(x/\|x\|), Ax \rangle = \|x\| f_{x/\|x\|}(Ax)$$

We have for $y = x/\|x\|$ that $\|y\| = 1$ and we use the bound that we established through the Banach-Steinhaus theorem, i.e.

$$f_{x/\|x\|}(Ax) \leq c\|Ax\|, \text{ where } c = \sup_{f_y \in \mathcal{F}} \|f_y\|$$

Thus we have

$$\|Ax\|^2 \leq c\|x\|\|Ax\| \implies \|Ax\| \leq c\|x\|$$

That is A is bounded. $\square$

4 Exercise 5: Bilinear functionals

Let X be a normed vector space over $\mathbb{K}$. A bilinear functional on X is a map $B : X \times X \to \mathbb{K}$ such that for all $x, y \in X$ the maps $B(x, \cdot) : X \to \mathbb{K}$ and $B(\cdot, y) : X \to \mathbb{K}$ are linear functionals on X.

a) Let X be a Banach space and B a bilinear functional on X which is continuous in each variable separately, i.e. for every fixed $x, y \in X$, the maps $B(x, \cdot)$ and $B(\cdot, y)$ are continuous. Show that there exists a constant $C > 0$ such that $|B(x, y)| \le C\|x\|\|y\|$ for all $x, y \in X$. Conclude that B is continuous with respect to the norm $\|(x, y)\| := \|x\| + \|y\|$ on $X \times X$.

Proof. We define

$$\mathcal{F} := \{B(\cdot, y) : \|y\| = 1\}$$

By definition $B(\cdot, y)$ is linear because it's a bilinear functional. Further by assumption it is continuous in it's first argument, thus we have for all $x \in X$ that

$$|B(x, y)| \le c_y\|x\|$$

Hence we indeed have that $\mathcal{F} \subset \mathcal{L}(X, \mathbb{K})$. Further, because B is also linear in its second argument we also have for any $x \in X$ and any $y \in X$ with $\|y\| = 1$ that

$$|B(x, y)| \le c_x\|y\| = c_x$$

Thus we conclude that

$$\sup_{T \in \mathcal{F}} |T(x)| \le c_x$$

Thanks to Banach-Steinhaus we now know that there exists a constant $c \ge 0$ such that

$$\sup_{T \in \mathcal{F}} \|T\| \le c$$

Let now $x, y \in X$ be arbitrary.

$$\begin{aligned} |B(x, y)| &= \|x\|\|y\||B(x/\|x\|, y/\|y\|)| \le \|x\|\|y\| \sup_{\|\xi\|=1} |B(\xi, y/\|y\|)| \\ &\le \sup_{\|\zeta\|=1} \sup_{\|\xi\|=1} |B(\xi, \zeta)| = \|x\|\|y\| \sup_{\|\zeta\|=1} |B(\cdot, \zeta)| \le c\|x\|\|y\| \end{aligned}$$

Which takes care of the first part of the claim.

Next we need to show that with respect to the norm $\|(x,y)\| := \|x\|+\|y\|$ on $X \times X$ the function B is continuous.

To this extent consider the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ and assume that this sequence converges to $(x,y) \in X \times X$. Then by definition of the norm on $X \times X$ we have that

$$0 \xleftarrow{n\to\infty} \|(x_n, y_n) - (x,y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\|$$

Which shows that $x_n \to x$ and $y_n \to Y$ in X. In particular the sequence y_n is bounded as a convergent sequence in X, i.e. there exists $M \geq 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Finally we obtain that

$$\begin{aligned}|B(x_n, y_n) - B(x,y)| &= |B(x_n, y_n) - B(x, y_n) + B(x, y_n) - B(x,y)| \\ &= |B(x_n - x, y_n) + B(x, y_n - y)| \\ &\leq |B(x_n - x, y_n)| + |B(x, y_n - y)| \\ &\leq c\|x_n - x\|\|y_n\| + c\|x\|\|y_n - y\| \\ &\leq cM\|x_n - x\| + \|x\|\|y_n - y\| \to 0 \text{ as } n \to \infty\end{aligned}$$

This shows that that B is continuous. $\square$

b) Let $\mathcal{P}$ be the vector space of real polynomials in one variable, equipped with the norm $\|p\| = \int_0^1 |p(t)|dt$ for $p \in \mathcal{P}$. Let

$$B(p,q) = \int_0^1 p(t)q(t)dt$$

Show that B is a (real valued) bilinear functional on $\mathcal{P}$ which is continuous in each variable separately, but that B is not continuous on $\mathcal{P} \times \mathcal{P}$.

Proof. The Bilinearty of B follows immediately by the fact, that the integral is linear. Fix now $q \in \mathcal{P}$ arbitrary, then we have for all $p \in \mathcal{P}$ that

$$\begin{aligned}|B(p,q)| = \left|\int_0^1 p(t)q(t)dt\right| \leq \int_0^1 |p(t)||q(t)|dt &\leq \sup_{t\in[0,1]} |q(t)| \int_0^1 |p(t)|dt \\ &\leq c_q\|p\|\end{aligned}$$

Since $q \in \mathcal{P}$ is continuous. This shows that $B(\cdot, q)$ is continuous, i.e. B is continuous in its first variable. By the exact same reasoning we establish that B is continuous in its second variable.

We still need to show that B is not continuous on $\mathcal{P} \times \mathcal{P}$. We remark that this is no contradiction to our statement that we've just showed in a). Indeed our vector space $\mathcal{P}$ of real valued polynomials in one variable is not complete, in particular its no Banach space.

There is a hint given on how to approach this problem. But I have a daunting feeling that it's incorrect, so I will leave this problem open. $\square$