1 Quotient spaces

Let X be a Banach space and $M \subset X$ a closer linear subspace. Let $\sim$ be the equivalence relation defined through $x \sim y$ if and only if $x - y \in M$ and let X/M be the quotient space with respect to $\sim$.

a) For $[x] \in X/M$ let

$$\|[x]\|_{X/M} := \inf_{y \in [x]} \|y\|_X = \inf_{y \in M} \|x - y\|_X$$

Show that $\|\cdot\|_{X/M}$ is a norm on X/M.

Proof. Indeed we have that $\|\cdot\|_{X/M}$ is well-defined, it is in particular finite because $\|\cdot\|_X$ is a norm on X. Moreover we have

$$\|[x]\|_{X/M} = \inf_{y \in [x]} \|y\| = 0$$

Since $\|\cdot\|$ is a norm on X we must have that $y = 0$ i.e. $0 = y \in [x]$ which is the case if and only if $[x] = [0]$.

Next we have

$$\begin{aligned}
\|[x] + [y]\|_{X/M} &= \|[x + y]\|_{X/M} = \inf_{v \in M} \|x + y - v\| \\
&= \inf_{v \in M} \|x - v + y - v + v\| \\
&\leq \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| + \inf_{v \in M} \|v\| \\
&= \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| \\
&= \|[x]\|_{X/M} + \|[y]\|_{X/M}
\end{aligned}$$

Finally we have for all $\lambda \in \mathbb{K}$

$$\begin{aligned}
\|\lambda[x]\|_{X/M} &= \|[\lambda x]\|_{X/M} = \inf_{y \in M} \|\lambda x - y\| \\
&= \inf_{\lambda y \in M} \|\lambda x - \lambda y\| \\
&= \inf_{u \in M} \|\lambda x - \lambda u\| \qquad (*) \\
&= \inf_{u \in M} |\lambda| \|x - u\| = |\lambda| \|[x]\|_{X/M}
\end{aligned}$$

Where we used in (*) that $\varphi_\lambda : M \to M$ given by $\varphi_\lambda(x) = \lambda x$ is a bijection for all $\lambda \in \mathbb{K}$. Thus we have shown that $\|\cdot\|_{X/M}$ is a norm on X/M. $\square$

b) Show that the projection $\pi : X \to X/M,\ x \mapsto [x]$ is continuous.

Proof. Evidently, the function π is linear because we have by definition that

$$[x+y] = [x] + [y]$$
$$\lambda[x] = [\lambda x]$$

For all $x, y \in X, \lambda \in \mathbb{K}$. Moreover we have

$$\|\pi(x)\|_{X/M} = \|[x]\|_{X/M} = \inf_{y \in M} \|x - y\| \leq \inf_{y \in M} \|x\| + \underbrace{\inf_{y \in M} \|y\|}_{=0} = \|x\|_X$$

That is π is a bounded linear operator and thus continuous. $\square$

c) Show that X/M is complete.

Proof. Recall from Exercise Sheet 1 Exercise 4: Normed spaces and Banach spaces where we have shown

Theorem: Let $(X, \|\cdot\|)$ be a normed space. $(X, \|\cdot\|)$ is a Banach space if and only if, all absolutely convergent series are also convergent, i.e. if

$$\sum_{n=0}^{\infty} \|x_n\| < \infty \implies \sum_{n=0}^{\infty} x_n < \infty$$

Thanks to the above theorem in order to prove that X/M is a Banach space it is enough to show that every series in X/M that converges absolutely also converges in X/M.

To this extent let us take an arbitrary sequence $([x_n])_{n\in\mathbb{N}}$ in X/M such that its series is absolutely convergent, i.e. we have

$$\sum_{n=0}^{\infty} \|[x_n]\|_{X/M} < \infty$$

By definition of the quotient norm (via the inf) we have for all $n \in \mathbb{N}$ the existence of some $y_n \in M$ such that $\|x_n - y_n\| \leq \|[x_n]\|_{X/M} + 1/2^n$. Since we've already established the convergence of the right hand side, the convergence of the left hand side is immediate by domination. But X is a Banach space, thus it follows from the Theorem above that

$$\sum_{n=0}^{\infty} (x_n - y_n) = x < \infty$$

We now claim that the series over the sequence $([x_n])_{n\in\mathbb{N}}$ in X/M converges to $[x] \in X/M$.

Indeed, let us consider for $N \in \mathbb{N}$

$$\left\| [x] - \sum_{n=0}^{N} [x_n] \right\|_{X/M} = \left\| [x - \sum_{n=0}^{N} x_n] \right\|_{X/M} \leq \left\| x - \sum_{n=0}^{N} x_n \right\|_{X} \to 0$$

as $N \to \infty$. Where the last inequality just follows by the very definition of the quotient norm. $\square$

d) Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$.
We define the kernel of $T, \ker T := \{x \in X : Tx = 0\}$ and the range of T, $\mathrm{ran}(T) := \{Tx : x \in X\} \subset Y$. Let $\iota : \mathrm{ran}T \to Y, x \mapsto x$ be the inclusion map. Show that then there exists a bijective operator $\hat{T} \in \mathcal{L}(X/\ker T, \mathrm{ran}T)$ with $T = \iota \hat{T} \pi$ and $\|\hat{T}\| = \|T\|$.

Proof. We have indeed that $\ker T \subset X$ is a closed linear subspace. Now we define

$$\hat{T} : X/\ker T \to \mathrm{ran}T, \ [x] \mapsto T(x)$$

We notice that that $\hat{T}$ is well-defined. Indeed, let $[x], [y] \in X/\ker T$ be such that $[x] = [y]$, this is the case if and only if $x \sim y$ i.e. $x - y \in \ker T$, or equivalently $x \in y + \ker T$. This entails that we can write $x = y + t$ for some $t \in \ker T$. Thus we get

$$\hat{T}([x]) = T(x) = T(y + t) = T(y) + \underbrace{T(t)}_{=0} = T(y) = T([y])$$

Which shows that $\hat{T}$ is well-defined. Moreover, $\hat{T}$ is by definition (defined through T) surjective onto the range of T. Furthermore $\hat{T}$ is injective, let $[x], [y] \in X/\ker T$ be such that

$$\begin{aligned} \hat{T}([x]) = \hat{T}([y]) \iff T(x) = T(y) &\implies T(x - y) = 0 \\ &\iff x - y \in \ker T \\ &\iff x \sim y \\ &\iff [x] = [y] \end{aligned}$$

The linearity of $\hat{T}$ follows immediatly by the linearty of π and the linearty of T. Finally, we have that π is continuous, and by assumption T is continuous too.

We have seen that $\|[x]\|_{X/\ker T} \leq \|x\|_X$ and by the continuity of T we obtain for some $C > 0$ that

$$\|\hat{T}([x])\|_Y = \|T(x)\|_Y \leq C\|x\|_X$$

let us now set $\tilde{C} := \max(1, C)$. This yields that

$$\|\hat{T}([x])\|_Y \leq C\|x\|_X \leq \tilde{C}\|x\|_X = \tilde{C}\|x - 0\|_X$$

Since $0 \in \ker T$ we conclude that

$$\|\hat{T}([x])\|_Y \leq \inf_{t \in \ker T} \|x - t\| = \|x\|_{X/\ker T}$$

showing that indeed $\hat{T} \in \mathcal{L}(X/\ker T, \mathrm{ran} T)$.

Further, by the very definition of $\hat{T}$ we have that $T = \iota\hat{T}\pi$.

Lastly, we observe that since $0 \in \ker T$ we have that if $\|x\|_X \leq 1$ for an arbitrary $x \in X$, then $\|x - 0\|_X = \|x\|_X \leq 1$ and thus by definition of the infimum as the greatest lower bound we must also have $\inf_{y \in \ker T} \|x - y\| \leq 1$.

Conversely, if $\inf_{y \in \ker T} \|x - y\| \leq 1$. Then again by definition of the greatest upper bound we must also have that $\|x - 0\| = \|x\| \leq 1$. This shows that

$$\begin{aligned}
\|\hat{T}\| &= \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|\hat{T}([x])\|_Y = \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|T(x)\|_Y \\
&= \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y = \|T\|
\end{aligned}$$

□