1 Weak convergence without norm convergence

Let $f \in C_c^\infty(\mathbb{R})$ and $1 \leq p < \infty$. We define the following sequences $g_n(x) := f(x-n), h_n(x) := n^{-1/p} f(x/n)$ and $k_n(x) := f(x)e^{inx}$. for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Show that for $1 < p < \infty, g_n, h_n$ and k_n converge weakly in $L^p(\mathbb{R})$ to zero, but they do not converge with respect to the norm. Investigate what happens for $p = 1$ as well.

Proof. Recall the characterization of weak convergence:

Lemma 1.1 (Characterization of weak convergence)**.** *Let X be a normed space, $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then we have $x_n \rightharpoonup x$ in X if and only if $f(x_n) \to f(x)$ for all $f \in X^*$.*

From the Lemma we immediately recover the fact, that strong convergence (i.e. convergence with respect to the norm $\|x_n - x\|_X \to 0$ as $n \to \infty$) implies weak convergence. Moreover we recall that $X^* = \mathcal{L}(X, \mathbb{K})$ are continuous linear functionals from X to $\mathbb{K}$.

In our case $X = L^p(\mathbb{R})$ (for some $p \in (1, \infty)$) is indeed a normed space, moreover we have $C_c(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $1 \leq p < \infty$ (in fact it is a dense subset), thus our sequences are also well defined in $L^p(\mathbb{R})$. It is our goal to show that although the sequences do not convergence with respect to the norm on $L^p(\mathbb{R})$ they do nevertheless converge weakly (towards 0).

Furthermore we must recall another important result which discusses the duality of the L^p spaces:

Theorem 1.1. *Let $1 \leq p < \infty$, $1 < q \leq \infty$ with $1/p + 1/q = 1$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $p = 1$, we assume additionally the measure space $(\Omega, \mathcal{A}, \mu)$ to be σ-finite. Then the map*

$$\phi : \begin{cases} L^q(\Omega, \mathcal{A}, \mu) & \longrightarrow (L^p(\Omega, \mathcal{A}, \mu))^* \\ f & \longmapsto \phi_f : \begin{cases} L^p & \longrightarrow \mathbb{K} \\ g & \longmapsto \phi_f(g) := \int_\Omega \overline{f(x)} g(x) d\mu(x) \end{cases} \end{cases}$$

is an (anti-linear) isometric isomorphism, in other words $L^p(\Omega, \mathcal{A}, \mu)^* \cong L^q(\Omega, \mathcal{A}, \mu)$

The Theorem tells us how we can identify the elements of $(L^p)^*$. More explicitly, every continuous linear functional from $L^p(\Omega, \mathcal{A}, \mu)$ to $\mathbb{K}$ for $1 \leq p < \infty$ can be written as

$$L(f) = \int_\Omega g(x) f(x) d\mu(x)$$

for a $g \in L^q(\Omega, \mathcal{A}, \mu)$ where p, q are conjugate.

Let us now fix $p \in (1, \infty)$ and q conjugate to p. Then we know that all $\phi \in (L^p)^*$ can be written in the form above for some $g \in L^q$. We thus obtain:

$$\phi(g_n) = \int_\mathbb{R} g(x) g_n(x) dx = \int_\mathbb{R} g(x) f(x-n) dx$$

We have that $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x-n) = f(-\infty) = 0$ because f is compactly supported. Since we know that ϕ is continuous we obtain that

$$\lim_{n\to\infty} \phi(g_n) = \phi(\lim_{n\to\infty} g_n) = \int_\mathbb{R} g(x) \lim_{n\to\infty} f(x-n) dx = 0 = \phi(0)$$

This shows that $g_n \rightharpoonup 0$ by the Lemma.

Similarly we have

$$\phi(h_n) = \int_\mathbb{R} g(x) h_n(x) dx = \int_\mathbb{R} g(x) n^{-1/p} f(x/n) dx$$
$$\implies \lim_{n\to\infty} \phi(h_n) = \phi(\lim_{n\to\infty} h_n) = \int_\mathbb{R} g(x) \lim_{n\to\infty} \frac{1}{n^p} f(x/n) dx = 0 = \phi(0)$$

Which entails that $h_n \rightharpoonup 0$.

Finally, we obtain in the same fashion

$$\phi(k_n) = \int_\mathbb{R} g(x) f(x) e^{inx} dx$$

HERE THERE IS NOTHING WE CAN DO, maybe Theorem misstated? Conjugation elsewhere? According to Alt it's correct.

We now turn out attention towards the convergence in the L^p-norm. Let us first consider the case of $p = 1$, we then have

$$\begin{aligned}\|g_n - f\| = \int_{\mathbb{R}} |g_n(x) - f(x)|dx = \int_{\mathbb{R}} |f(x-n) - f(x)|dx \\ = \int_{\mathbb{R}} |f(x) - f(x)|dx = 0\end{aligned}$$

That is $g_n \to f$ in $L^1(\mathbb{R})$ as $n \to \infty$.

Moreover we have

$$\begin{aligned}\|h_n - f\|_1 &= \int_{\mathbb{R}} |h_n(x) - f(x)|dx = \int_{\mathbb{R}} |n^{-1}f(x/n) - f(x)|dx \\ &= \int_{\mathbb{R}} |f(x) - f(x)|dx = 0\end{aligned}$$

thus, $h_n \to f$ in $L^1(\mathbb{R})$ as $n \to \infty$.

Let us now consider the case for $p \in (1, \infty)$. We want to show that the sequences g_n, h_n and k_n do not converge with respect to the L^p-norm. The most elegant approach would be to show that these sequences are not Cauchy, in particular they cannot converge (recall that convergent sequences are always Cauchy).

To this goal let $m, n \in \mathbb{N}$ be such that $m > n$.

$$\|g_m - g_n\|_p^p = \int_{\mathbb{R}} |g_m(x) - g_n(x)|^p dx = \int_{\mathbb{R}} |f(x-m) - f(x-n)|^p dx$$

$\square$

2 Criteria for weak and norm convergence

a) Let $1 < p < \infty$, $\mathbb{K}$ be $\mathbb{C}$ or $\mathbb{R}$, and let $(x^{(n)})_{n\in\mathbb{N}}$ be a sequence in $\ell^p(\mathbb{K})$. Let $x \in \ell^p(\mathbb{K})$. Show that $(x^{(n)})_{n\in\mathbb{N}}$ converges weakly to x if and only if $(x^{(n)})_{n\in\mathbb{N}}$ is bounded and $x_i^{(n)} \to x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$

Proof. Let us assume that $(x^{(n)})_{n\in\mathbb{N}}$ converges weakly to x. Then we know already that the sequence must be bounded by **Proposition 6.2.2.**

Moreover we know that weak converges holds if and only if $f(x^{(n)}) \to f(x)$ as $n \to \infty$ for all $f \in \ell^p(\mathbb{K})^*$. In particular it must hold true for the projections $p_i : \ell^p(\mathbb{K}) \to \mathbb{K}$ defined by $x = (x^{(n)})_{n\in\mathbb{N}} \mapsto x_i^{(n)}$, for which we already know that they are linear and continuous.

Thus it follows easily that $x_i^{(n)} \to x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$.

Conversely suppose that $(x^{(n)})_{n\in\mathbb{N}}$ is a bounded sequence and the coordinates $x_i^{(n)}$ converges to x_i as $n \to \infty$ for all $i \in \mathbb{N}$.

Recall from the previous Exercise Sheet 8 Exercise 5, where we have shown the following criteria for weak convergence:

Sheet 8 Exercise 5: Let X be a normed space and $Y \subset X^*$ a dense subset. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then x_n converges weakly to $x \in X$ if and only if $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $f(x_n) \to f(x)$ as $n \to \infty$ for all $f \in Y^*$.

We will apply this exercise to the set $S := \{f_i \in (\ell^p)^* : i \in \mathbb{N}\}$ where $p \in (1, \infty)$ and $f_i : \ell^p \to \mathbb{K}$ are the canonical projections $t \mapsto t_i$. Indeed we have that $S \subset (\ell^p)^*$, moreover we consider its span $Y := \operatorname{span}(S) \subset (\ell^p)^*$ and we claim that Y is dense in $(\ell^p)^*$.

Since $1 < p < \infty$ we have that $(\ell^p)^* \cong \ell^q$ for some q that satisfies the relation $p^{-1} + q^{-1} = 1$. It is therefore enough to show that S is dense in said L^q. To this extend let $t \in \ell^q$ be arbitrary, by definition we can find for $\epsilon > 0$ always an $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} |t_j|^q < \epsilon^q$$

Let us now choose $y \in Y$ such that $y_j = t_j$ for all $j = 1, \dots, N$ and $y_j = 0$ otherwise, then trivially we have $\|t - y\|_q < \epsilon$ which shows the density. $\square$

b) Let H be a Hilbert space, let $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H. Show that if x_n converges weakly to x and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then x_n converges to x in norm.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the Hilbert space H that converges weakly to some x in H. We know that this is the case if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for all $f \in H^*$.

Since this convergence must be satisfied for all $f \in H^*$ it must in particular be the case for

$$f_x := \begin{cases} H & \longrightarrow \mathbb{K} \\ y & \longmapsto \langle x, y \rangle \end{cases}$$

Where we fixed $x \in H$ for the weakly convergent sequence $x_n \rightharpoonup x$. Indeed f_x is by definition linear (the inner product is linear in its second argument and anti-linear in its first argument). Moreover f_x is a bounded linear operator from H to $\mathbb{K}$ by the Cauchy-Schwarz inequality.

Henceforth f_x is a continuous linear function from H to $\mathbb{K}$, i.e.$f_x \in H^*$. Consider now:

$$\begin{aligned} \|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle &= \|x_n\|^2 + \|x\|^2 - \langle x, x_n \rangle - \langle x_n, x \rangle \\ &= \|x_n\|^2 + \|x\|^2 - \langle x, x_n \rangle - \overline{\langle x, x_n \rangle} \\ &= \|x_n\|^2 + \|x\|^2 - f_x(x_n) - \overline{f_x(x_n)} \end{aligned}$$

Since $x_n \rightharpoonup x$ we have by our efforts above that $\langle x, x_n \rangle = f_x(x_n) \to f(x) = \langle x, x \rangle = \|x\|^2$ and by assumption we have $\|x_n\| \to \|x\|$ as $n \to \infty$. Thus taking the limit in our derived expression above we conclude that

$$\|x_n - x\|^2 \to 0 \text{ as } n \to \infty$$

That is x_n converges to x in norm. $\square$

3 Norm for weak-* convergence

Let $(X, \|\cdot\|)$ be a separable normed vector space. We will also denote by $\|\cdot\|$ the norm on X^*. Let $\sigma = \underline{(x_n)_{n\in\mathbb{N}}}$ be a sequence in the unit ball $S_X := \{x \in X : \|x\| = 1\}$ such that $\overline{\text{span}(\sigma)} = X$. For $x^* \in X^*$ we define

$$\|x^*\|_\sigma := \sum_{k=1}^{\infty} 2^{-k}|x^*(x_n)|$$

a) Show that $\|\cdot\|_\sigma$ is a norm on X^* and that it satisfies the inequality $\|x^*\|_\sigma \leq \|x^*\|$.

Proof. By the fact that $\sigma = (x_n)_{n\in\mathbb{N}}$ is a sequence in the unit ball and the geometric series we obtain that

$$\|x^*\|_\sigma = \sum_{k=1}^{\infty} 2^{-k}|x^*(x_k)| \leq \sum_{k=1}^{\infty} 2^{-k}\|x^*\|\|x_k\| = \|x^*\| \sum_{k=1}^{\infty} 2^{-k} = \|x^*\|$$

Which shows that $\|\cdot\|_\sigma$ is well defined and already gives the claimed inequality. Next we establish that $\|\cdot\|_\sigma$ is indeed a norm.

Evidently we have $\|\lambda x^*\|_\sigma = |\lambda|\|x^*\|_\sigma$ for all $\lambda \in \mathbb{K}$ and $x^* \in X^*$. Moreover the triangle inequality follows by the triangle inequality of the norm $|\cdot|$ on $\mathbb{K}$.

Finally, we claim that $\|x^*\| = 0 \iff x^* = 0$. Indeed the necessary condition is as always trivial, for the sufficient condition we consider

$$\|x^*\|_\sigma = \sum_{k=1}^{\infty} 2^{-k}|x^*(x_k)| = 0 \implies |x^*(x_k)| = 0 \text{ for all } k \in \mathbb{N}$$

Since $(x_k)_{k\in\mathbb{N}}$ is an arbitrary sequence in the unit ball we obtain that

$$\sup_{\sigma \in S_X} |x^*(x_k)| = \|x^*\| = 0 \implies x^* = 0$$

Because $\|\cdot\|_{X^*}$ is a norm on X^*. $\square$

4 Sequential compactness

For $\epsilon > 0$ and $f \in L^\infty((0,1))$ let

$$I_\epsilon(f) := \epsilon^{-1} \int_0^\epsilon f(x)dx$$

a) Show that $I_\epsilon \in L^\infty((0,1))^*$ with $\|I_\epsilon\| = 1$ for every $\epsilon > 0$.

Proof. By the linearity of the integral it is clear that I_ϵ is a linear functional from $L^\infty(0,1)$ to $\mathbb{K}$. Assume now that $\epsilon \in (0,1)$ is fixed, then we have for all $f \in L^\infty(0,1)$ that

$$\begin{aligned} \|I_\epsilon(f)\| = \left|\frac{1}{\epsilon}\int_0^\epsilon f(x)dx\right| \leq \frac{1}{\epsilon}\int_0^\epsilon |f(x)|dx \leq \frac{1}{\epsilon}\int_0^\epsilon \|f\|_{L^\infty(0,1)}dx \\ = \|f\|_{L^\infty(0,1)} \end{aligned}$$

Thus I_ϵ is bounded and therefore as an linear operator also continuous.

Moreover we have

$$\begin{aligned} \|I_\epsilon\| = \sup_{\|f\|_\infty=1} \|I_\epsilon(f)\| = \sup_{\|f\|_\infty=1} \left|\frac{1}{\epsilon}\int_0^\epsilon f(x)dx\right| \leq \sup_{\|f\|_\infty=1} \frac{1}{\epsilon}\int_0^\epsilon |f(x)|dx \\ \leq \sup_{\|f\|_\infty=1} \frac{1}{\epsilon}\int_0^\epsilon \|f\|_\infty dx = 1 \end{aligned}$$

Thus we have $\|I_\epsilon\| \leq 1$. NO CLUE ABOUT THE OTHER INEQUALITY. □