

Functional Analysis - The irreducible Minimum

1 Structures

1.1 Topological spaces

We will deal with various structures, in a sense, they build a certain hierarchy. Topological spaces as a framework, then metric spaces, normed spaces, Banach spaces and conclusively Hilbert Spaces.

Definition 1.1. *A topological space is a pair (X, τ) , consisting of a set X and a family $\tau \subset 2^X$ that we call a topology on X , such that*

1. $\emptyset, X \in \tau$.
2. *Stable under countable intersection of open sets.*
3. *Stable under arbitrary union of open sets.*

Remark 1.1. Thanks to the laws of De Morgan we easily conclude from the definition that arbitrary intersections and finite unions of closed sets are again closed.

Definition 1.2. *Let (X, τ) be a topological space and $A \subset X$ a subset. The closure \bar{A} of A is defined by*

$$\bar{A} = \bigcap \{B \subset X : B \text{ is closed and } A \subset B\}$$

by definition, it is the smallest closed set that contains A . The set A is called dense in X if $\bar{A} = X$. The space X is called separable, if it contains a countable dense set. Moreover, the interior of A is defined through

$$A^\circ = \bigcup \{B \subset A : B \text{ is open}\}$$

in other words, A° is the largest open set contained in A . Finally, the boundary of A is defined as $\partial A = \bar{A} \setminus A^\circ$.

Remark 1.2. Evidently, the closure and interior of a set depend on the choice of the topology.

Definition 1.3. *Let (X, τ) be a topological space, $x \in X$. A set $U \subset X$ is called an open neighbourhood of x if $U \in \tau$ (i.e. U is open) and $x \in U$.*

Definition 1.4. Let (X, τ) be a topological space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . We say that x_n converges to $x \in X$ as $n \rightarrow \infty$, written $x_n \rightarrow x$, if

For every open neighbourhood U of x , there exists $n_0 \in \mathbb{N} : x_n \in U \ \forall n \geq n_0$.

Definition 1.5. Let $(X, \tau), (Y, \mathcal{S})$ be two topological spaces. A function $f : X \rightarrow Y$ is called continuous if

$$f^{-1}(V) \in \tau, \ \forall V \in \mathcal{S}.$$

I.e., if the pre-image of every open set in Y is an open set in X .

Remark 1.3. In metric spaces, the notion of convergence completely characterizes the topology. This is however not necessarily true in topological spaces.

Definition 1.6. A topological space (X, τ) is called Hausdorff if

$$\forall x, y \in X, x \neq y \implies \exists U_x, U_y \in \tau \text{ with } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset$$

Definition 1.7. A topological space (X, τ) is called compact if it is Hausdorff and if for every

$$\begin{aligned} & (U_\lambda)_{\lambda \in \Lambda} \text{ family in } \tau \text{ with } \bigcup_{\lambda \in \Lambda} U_\lambda = X \\ \implies & \exists n \in \mathbb{N} \text{ and } \lambda_1, \dots, \lambda_n \in \Lambda : \bigcup_{j=1}^n U_{\lambda_j} = X \end{aligned}$$

i.e. if for every open covering of X there exists a finite sub-covering.

Theorem 1.1. Let (X, τ) be a compact space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Then there exists at least one accumulation point of x_n on X .

Theorem 1.2 (Lemma von Urysohn). Let (X, τ) be a compact space and $A, B \subset X$ disjoint, non-empty, closed subsets of X . Then there exists a continuous function $g : X \rightarrow [0, 1]$ with $g(A) = 0$ and $g(B) = 1$.

Remark 1.4. The Lemma of Urysohn is important because it helps us avoid some of the pathological situations (like only constant functions being continuous). In particular, the theorem implies that on compact spaces the topology is large enough for us to do some meaningful analysis on.

Definition 1.8. Let K be a compact space. We define

$$C_{\mathbb{K}}(K) := \{f : K \rightarrow \mathbb{K} \text{ continuous}\}$$

As a consequence of Urysohn's Lemma we can show that $C_{\mathbb{K}}(K)$ separates the points of K .

Corollary 1.1. Let K be a compact space. Then $C_{\mathbb{K}}(K)$ separates the points of K . In other words, for every $x, y \in K$ with $x \neq y$, there exists $f \in C_{\mathbb{K}}(K)$ such that $f(x) \neq f(y)$.

Proof. Since K is Hausdorff, for every $x \neq y$ we can find two open neighbourhoods U_x, U_y of x and y respectively with $U_x \cap U_y = \emptyset$. We can further find closed sets A, B with $A \subset U_x, B \subset U_y$ with $x \in A$ and $y \in B$. In particular we have $A \cap B = \emptyset$. Urysohn's Lemma then gives that there exists $f \in C_{\mathbb{K}}(K)$ with $f(x) = 0$ and $f(y) = 1$. \square

1.2 Metric Spaces

Definition 1.9. A metric space is a pair (X, d) with X being an arbitrary set and a map $d : X \times X \rightarrow [0, \infty)$ called a metric on X , with the following properties

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. (Δ -inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark 1.5.

1. Every metric space (X, d) is a topological space (X, τ_d) with topology τ_d induced by the metric. The topology τ_d is defined by the condition that $A \in \tau_d$ if and only if for all $x \in A$, there exists $\epsilon > 0$ with $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\} \subset A$.
2. In contrast with general topological spaces, on metric spaces the notion of convergence characterizes the topology. That is we have the convenient characterizations
 - (a) A set $A \subset X$ is closed if and only if for every sequence x_n in A with $x_n \rightarrow x$ in X , we have that $x \in A$.
 - (b) $\overline{A} = \{x \in X : \exists (x_n)_{n \in \mathbb{N}} \text{ sequence in } A \text{ with } x_n \rightarrow x\}$.
 - (c) $A \subset X$ is dense if and only if $\forall x \in X, \exists (x_n)_{n \in \mathbb{N}} \text{ sequence in } A \text{ with } x_n \rightarrow x$.
 - (d) The function $f : X \rightarrow Y$ between the two metric spaces $(X, d_1), (Y, d_2)$ is continuous at the point $x \in X$ if and only if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \rightarrow x$ we have $f(x_n) \rightarrow f(x)$.

Definition 1.10. Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called a *Cauchy sequence* (or is said to have the *Cauchy property*) if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. Every convergent sequence is indeed Cauchy. The metric space (X, d) is called *complete*, if every Cauchy sequence is convergent.

1.3 Normed spaces

Definition 1.11. A *normed space* is a pair $(X, \|\cdot\|)$, consisting of a vector space V over a field \mathbb{K} (\mathbb{R} or \mathbb{C}) and a map $\|\cdot\| : V \rightarrow [0, \infty)$ called a *norm* on V with the following properties

1. $\|x\| = 0$ if and only if $x = 0$.
2. (Homogeneity) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
3. (Δ -inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Remark 1.6. Every norm induces the metric $d(x, y) = \|x - y\|$, hence every normed space is also a metric space and therefore also a topological space.

Definition 1.12. A normed space $(X, \|\cdot\|)$ is called *complete*, if X , equipped with the induced metric $d(x, y) = \|x - y\|$, is a complete metric space. A complete normed space is called a *Banach space*.

Completeness is very important for analysis. It is not a coincidence that we always do analysis on \mathbb{R} instead of \mathbb{Q} . For this reason, it is useful to have a general recipe to complete normed spaces.

Definition 1.13. Let $(X, \|\cdot\|)$ be a normed space. A *completion* of $(X, \|\cdot\|)$ is a 3-tuple $(Y, \|\cdot\|_Y, \phi)$ consisting of a Banach space $(Y, \|\cdot\|_Y)$ and an isometric linear map $\phi : X \rightarrow Y$, with $\phi(X) = Y$.

Theorem 1.3. Every normed space $(X, \|\cdot\|)$ has a completion, which is unique, up to linear isometric isomorphisms.

Proof. The proof is constructive, we give a sketch. Let \mathcal{C}_X denote the set of all Cauchy sequence on X . We can easily give this space the structure of a vector space over \mathbb{K} . Next we define the linear subspace $\mathcal{N}_X \subset \mathcal{C}_X$ consisting of all null-sequence on X , i.e.

$$\mathcal{N}_X := \{x = (x_n)_{n \in \mathbb{N}} \in \mathcal{C}_X : x_n \rightarrow 0\}$$

Moreover we define $Y := \mathcal{C}_X / \mathcal{N}_X$ as the quotient space of \mathcal{C}_X w.r.t. the equivalence relation defined by $x \sim y : \iff x - y \in \mathcal{N}_X$. In other words, in Y , we identify Cauchy sequences whose difference converges to zero. Y is also a vector space over \mathbb{K} .

Next we want to introduce a norm on Y . To this end, we define the function $p : \mathcal{C}_X \rightarrow [0, \infty)$ through

$$p(x) = \lim_{n \rightarrow \infty} \|x_n\|$$

Thanks to the reverse triangle inequality we've got

$$\| \|x_k\| - \|x_l\| \| \leq \|x_k - x_l\| \rightarrow 0 \text{ as } k, l \rightarrow \infty$$

which shows that $(\|x_n\|)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in \mathcal{C}_X . Thus the limit above is well-defined and finite. We now set

$$\|[x]\|_Y := p(x) = \lim_{n \rightarrow \infty} \|x_n\|$$

We then can verify that $\|\cdot\|_Y$ is indeed a norm on Y .

Next we define the map $\phi : X \rightarrow Y$ by $\phi(z) = [(z, z, \dots)]$, i.e. $\phi(z)$ denotes the equivalence class of all sequences on X that converge to z in the limit. This map is clearly linear and since

$$\|\phi(z)\|_Y = \|z\|_X$$

it defines an isometry. We now claim that $(Y, \|\cdot\|_Y, \phi)$ is a completion of $(X, \|\cdot\|_X)$. In order to show that we can show that $(Y, \|\cdot\|_Y)$ is always complete and that $\phi(X)$ is dense in Y . That is we want to show that for all $[x] \in Y$, we can find $\tilde{x} \in X$ with $\|\phi(\tilde{x}) - [x]\|_Y < \epsilon$. We start with the density and then use this to show that the space is complete.

Finally we can show that the uniqueness of the completion is up to isometric isomorphisms. \square

Remark 1.7. The completion of \mathbb{Q} is \mathbb{R} . For $\Omega \subset \mathbb{R}^n$, the completion of the space $C(\bar{\Omega})$ of continuous functions on $\bar{\Omega}$ is $L^p(\Omega)$.

1.4 Hilbert Spaces

Definition 1.14. Let H be a vector space over the field \mathbb{K} . A scalar product (or an inner product) on H is a map $(\cdot, \cdot) : H \times H \rightarrow \mathbb{K}$ with the properties

1. $(z, x + \lambda y) = (z, x) + \lambda(z, y)$ for all $x, y, z \in H, \lambda \in \mathbb{K}$.
2. $(x, y) = \overline{(y, x)}$.
3. $(x, x) > 0$ for all $x \neq 0$.

A pair $(H, (\cdot, \cdot))$ consisting of a vector space H over \mathbb{K} and a scalar product (\cdot, \cdot) is called a pre-Hilbert space.

Remark 1.8. We defend the scalar product to be linear in its second argument and anti-linear in its first argument.

Lemma 1.1 (Cauchy Schwarz Inequality). *Let $(H, (\cdot, \cdot))$ be a pre-Hilbert space. Then*

$$|(x, y)|^2 \leq (x, x)(y, y).$$

Remark 1.9. The Cauchy-Schwarz inequality allows us to use the scalar product to define a norm on every pre-Hilbert space.

Corollary 1.2. *Let $(H, (\cdot, \cdot))$ be a pre-Hilbert space. Then*

$$\|x\| := \sqrt{(x, x)}$$

defines a norm on H .

Remark 1.10. The triangle inequality follows from the Cauchy-Schwarz inequality. The remaining properties follow from the properties of scalar products.

Definition 1.15. *A pre-Hilbert space is called a Hilbert space if H , equipped with the norm $\|x\| = \sqrt{(x, x)}$ induced by the scalar product, is a Banach space (i.e. if H is complete).*

Remark 1.11.

1. Every pre-Hilbert space can be completed into a Hilbert space.
2. Every Hilbert space is a metric space and therefore a topological space. However, clearly, not every Banach space is a Hilbert space, simply because not all norms on a vector space can be induced by a scalar product.

Theorem 1.4. *Let $(H, (\cdot, \cdot))$ be a Hilbert space, $K \subset H$ a closed convex set in H and $x_0 \in H$. Then there exists a unique $y \in K$ such that*

$$\|x_0 - y\| = \text{dist}(x_0, K) := \inf_{x \in K} \|x_0 - x\|$$

As an application of this theorem, we show that every Hilbert space H can be decomposed in the direct sum of an arbitrary closed subspace and of its orthogonal complement.

Theorem 1.5. *Let $(H, (\cdot, \cdot))$ be a Hilbert space and $M \subset H$ a closed linear subspace. Then the orthogonal complement M^\perp of M , defined through*

$$M^\perp := \{x \in H : (x, m) = 0, \text{ for all } m \in M\}$$

is also a linear closed subspace of H and $H = M \oplus M^\perp$, meaning that $H = M + M^\perp$ and $M \cap M^\perp = \{0\}$.

Proof. Clearly M^\perp is linear. In order to see that it is closed take x_n to be a sequence in M^\perp such that $x_n \rightarrow x$ in H . Then we have

$$(x, m) = \lim_{n \rightarrow \infty} (x_n, m) = 0$$

because thanks to the Cauchy-Schwarz inequality we have

$$|(x - x_n, m)| \leq \|x - x_n\| \|m\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

in particular $|(x - x_n, m)| = |(x, m) - (x_n, m)| \rightarrow 0$ and it follows that $x \in M^\perp$, i.e. M^\perp is closed.

Moreover, the fact that $M \cap M^\perp = \{0\}$ follows, because $(x, x) = 0$ implies that $x = 0$. Thus it only remains to show that $M + M^\perp = H$. To this end, we fix $x \in H$. Since $M \subset H$ is a closed linear subspace (and therefore in particular convex) we can apply the previous theorem to find $z \in M$ such that $\text{dist}(x, M) = \|x - z\|$.

We now claim that $x - z \in M^\perp$, which gives that $x = z + (x - z)$ is the desired decomposition of H into M and M^\perp . Let's assume for contradiction that $(x - z) \notin M^\perp$. Then there exists $\alpha \in M$ with $(x - z, \alpha) > 0$. For $t \in [-1, 1]$ let $z_t = z + t\alpha$. Then we have $z_t \in M$ for all t and

$$\|x - z_t\|^2 = \|x - z\|^2 + t^2 \|\alpha\|^2 - 2t(x - z, \alpha) < \|x - z\|^2 = \text{dist}(x, M)$$

for $t > 0$ small enough. But this contradicts the definition of $\text{dist}(x, M)$. \square

Definition 1.16. An orthonormal system in $(H, (\cdot, \cdot))$ is a family $(x_\alpha)_{\alpha \in A} \subset H$ for an arbitrary index-set A with $(x_\alpha, x_\beta) = \delta_{\alpha, \beta}$. In the case when $A = \mathbb{N}$, we also call the orthonormal system an orthonormal sequence.

Lemma 1.2. Let H be a Hilbert space, $(x_n)_{n \in \mathbb{N}}$ an orthonormal system (orthonormal sequence) and $(\alpha_n)_{n \in \mathbb{N}}$ a sequence in \mathbb{K} . Then we have

1. $\sum_{k=1}^{\infty} \alpha_k x_k$ converges if and only if $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.
2. $\|\sum_{k=1}^n \alpha_k x_k\|^2 = \sum_{k=1}^n |\alpha_k|^2$.
3. If $\sum_{k=1}^{\infty} \alpha_k x_k$ converges, then the limit is independent of the order of the terms.

Lemma 1.3. Let $(H, (\cdot, \cdot))$ be a Hilbert space, A an arbitrary set and $(x_\alpha)_{\alpha \in A}$ an orthonormal system in H . Then $\sum_{\alpha \in A} (x_\alpha, x) x_\alpha$ converges for every $x \in H$. Moreover, the linear map $\phi : H \rightarrow H$ defined through $\phi(x) = \sum_{\alpha \in A} (x_\alpha, x) x_\alpha$ is the continuous projection onto

$$M := \overline{\text{span}\{x_\alpha : \alpha \in A\}} \text{ along its orthogonal complement } M^\perp$$

In particular for $x \in M$, we find that $x = \sum_{\alpha \in A} (x_\alpha, x) x_\alpha$.

Remark 1.12. In particular, if $M = \text{span}\{x_\alpha : \alpha \in A\}$ is dense in H , i.e. if $\overline{M} = H$, then gives us the previous Lemma a representation for every vector $x \in H$. In this case, we say that $(x_\alpha)_{\alpha \in A}$ is a Hilbert space basis.

Definition 1.17. Let H be a Hilbert space. A Hilbert space basis is an orthonormal system $(x_\alpha)_{\alpha \in A}$ with

$$\overline{\text{span}\{x_\alpha : \alpha \in A\}} = H$$

Theorem 1.6 (Characterizations of Hilbert space bases). *Let H be a Hilbert space, and $(x_\alpha)_{\alpha \in A}$ an orthonormal system. Then the following statements are equivalent:*

1. $(x_\alpha)_{\alpha \in A}$ is a Hilbert space basis.
2. $x = \sum_{\alpha \in A} (x_\alpha, x) x_\alpha$, for all $x \in H$.
3. $\|x\|^2 = \sum_{\alpha} |(x_\alpha, x)|^2$ for all $x \in H$.
4. $(x_\alpha, x) = 0$ for all $\alpha \in A$ implies that $x = 0$.
5. $(x_\alpha)_{\alpha \in A}$ is a maximal orthonormal system in the sense of inclusions.

Remark 1.13. Using the maximality property, i.e. point 5 in the theorem above, it follows easily from the Lemma of Zorn that every pre-Hilbert space has a Hilbert space basis. In particular, it follows that every separable Hilbert space admits a countable orthonormal basis. The observation that separable Hilbert spaces have countable orthonormal bases can be used to identify separable Hilbert spaces with the sequence space $\ell^2(\mathbb{K})$.

Theorem 1.7. *Let H be an infinite dimensional separable Hilbert space over \mathbb{K} . Then there exists a linear Isomorphism $\phi : H \rightarrow \ell^2(\mathbb{K})$ with*

$$(\phi(x), \phi(y))_{\ell^2} = (x, y)_H$$

for all $x, y \in H$. In particular the isomorphism is isometric.

2 Function Spaces

2.1 Continuous Functions on Compact Spaces

Lemma 2.1. *Let $f \in C_{\mathbb{K}}(K)$ where K is compact. Then f is bounded and its supremum and infimum are attained.*

This allows us to define the sup/max norm of f on the space $C_{\mathbb{K}}(K)$.

Definition 2.1. *For $f \in C_{\mathbb{K}}(K)$ let*

$$\|f\| := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|$$

It is then simple to check that $\|\cdot\|$ defines a norm on $C_{\mathbb{K}}(K)$ (well-defined because of the above Lemma). Hence, the pair $(C_{\mathbb{K}}(K), \|\cdot\|)$ is a normed space.

Theorem 2.1. *$(C_{\mathbb{K}}(K), \|\cdot\|)$ is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $C_{\mathbb{K}}(K)$. Then, trivially we have for any $x \in K$ that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

It follows that $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy Sequence in \mathbb{K} , but since \mathbb{K} is complete we conclude that $f_n(x)$ converges. Let $f(x) := \lim f_n(x)$ denote its limit. We easily note that

$$|f(x)| \leq \limsup_{n \rightarrow \infty} |f_n(x)| \leq \limsup_{n \rightarrow \infty} \|f_n\|.$$

Moreover, since for every $x \in K$, we have

$$|f(x) - f_n(x)| = \lim_{m \rightarrow \infty} |f_m(x) - f_n(x)| \leq \limsup_{m \rightarrow \infty} \|f_m - f_n\|$$

we conclude that

$$\sup_{x \in K} |f(x) - f_n(x)| \leq \limsup_{m \rightarrow \infty} \|f_m - f_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

since this convergences in uniform over all $x \in K$, we conclude that f is indeed continuous, thus our Cauchy sequence f_n converges to $f \in C_{\mathbb{K}}(K)$ and our space is complete. \square

Definition 2.2. $\mathbb{A} \subset C_{\mathbb{K}}(K)$ is called a subalgebra if \mathbb{A} is a linear subspace of $C_{\mathbb{K}}(K)$ and if, for every $f, g \in \mathbb{A}$, we also have $f \cdot g \in \mathbb{A}$. We say that the subalgebra \mathbb{A} separates the points of K , if

$$\forall x, y \in K \text{ with } x \neq y, \exists f \in \mathbb{A} : f(x) \neq f(y).$$

Remark 2.1. We already know that if $\mathbb{A} = C_{\mathbb{K}}(K)$ then indeed $C_{\mathbb{K}}(K)$ is a subalgebra separating the points of K .

Theorem 2.2 (Stone-Weierstrass Theorem, $\mathbb{K} = \mathbb{R}$). *Let \mathbb{A} be a subalgebra of $C_{\mathbb{K}}(K)$ separating the points of K . Then we have either $\overline{\mathbb{A}} = C_{\mathbb{K}}(K)$ or there exists a unique point $x_0 \in K$ such that $\overline{\mathbb{A}} = \{f \in C_{\mathbb{K}}(K) : f(x_0) = 0\}$.*

Remark 2.2. The Stone-Weierstrass Theorem allows us for example to approximate continuous functions on compact subsets of \mathbb{R}^n through series of polynomials, see next example.

Example 2.1. Let $K \subset \mathbb{R}^n$ be compact and \mathbb{A} be the set of all polynomials in the variables x_1, \dots, x_n , i.e.

$$\mathbb{A} = \left\{ p(x) = \sum_{\alpha: |\alpha| \leq n} b_{\alpha} x^{\alpha}, \quad m \in \mathbb{N}, \quad b_{\alpha} \in \mathbb{R} \right\}$$

It is then easy to check that \mathbb{A} is a subalgebra of $C_{\mathbb{K}}(K)$, separating the points of K , moreover that $\{f \in C_{\mathbb{K}}(K) : f(x_0) = 0\}$ cannot hold true, because evidently we have $1 \in \mathbb{A}$ and if we were in the second case, then there would exist $x_0 \in K$ such that $0 = 1(x_0) = 1$ which is a contradiction. Thus by Stone-Weierstrass we conclude that $\overline{\mathbb{A}} = C_{\mathbb{K}}(K)$.

Theorem 2.3 (Stone-Weierstrass, $\mathbb{K} = \mathbb{C}$). *Let \mathbb{A} be a subalgebra of $C_{\mathbb{C}}(K)$, separating the points of K and such that for all $f \in \mathbb{B}$ we have $\bar{f} \in \mathbb{A}$. Then we either have $\overline{\mathbb{A}} = C_{\mathbb{C}}(K)$ or*

$$\exists! x_0 \in K \text{ with } \overline{\mathbb{A}} = \{f \in C_{\mathbb{C}}(K) : f(x_0) = 0\}.$$

2.2 Lebesgue Spaces

This should already be known, we will only repeat the most relevant results without much explanation in this section. We work on some measure space (Ω, Σ, μ) and consider the space

$$\tilde{L}^p(\Omega, d\mu) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ measurable}, \int_{\Omega} |f|^p d\mu < \infty\}$$

Theorem 2.4 (Hölder's inequality). *Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let (Ω, Σ, μ) be a measure space and $f \in L^p(\Omega), g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\left| \int_{\Omega} fg d\mu \right| \leq \int_{\Omega} |f| |g| d\mu \leq \|f\|_p \cdot \|g\|_q$$

Theorem 2.5 (Fischer-Riesz). *Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. Then $L^p(\Omega, \Sigma, \mu)$ is complete.*

Remark 2.3. For $p = 2$ the Lebesgue space L^2 is even a Hilbert space with scalar product on $L^2(\Omega, \Sigma, \mu)$ given by

$$(f, g)_{L^2} = \int \overline{f(x)}g(x)d\mu(x)$$

an exercise from class entails that this is on the only Lebesgue Space that is a Hilbert Space, i.e. only for the case $p = 2$.

Definition 2.3. *Let $(X, \|\cdot\|)$ be a normed space over \mathbb{K} . We define the dual space X^* of X as the space of all continuous linear functionals on X , i.e.*

$$X^* := \{f : X \rightarrow \mathbb{K} \mid f \text{ is linear and continuous}\}$$

We will discuss this space in much more detail later on, for now we note that X^ is always a Banach space w.r.t. the norm*

$$\|f\|_{X^*} := \sup_{x \in X: \|x\|=1} |f(x)|$$

Theorem 2.6 (Duality of L^p spaces). *Let $1 \leq p < \infty, 1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $p = 1$, we assume additionally that the measure space $(\Omega, \mathcal{A}, \mu)$ is σ -finite. Then the map*

$$\phi : \begin{cases} L^q(\Omega, \mathcal{A}, \mu) & \longrightarrow (L^p(\Omega, \mathcal{A}, \mu))^* \\ f & \longmapsto \phi_f \end{cases}$$

with

$$\phi_f(g) = \int_{\Omega} \overline{f(x)}g(x)d\mu(x)$$

for all $g \in L^p(\Omega, \mathcal{A}, \mu)$ is an anti-linear isometric isomorphism. In other words, we have

$$L^p(\Omega, \mathcal{A}, \mu)^* \cong L^q(\Omega, \mathcal{A}, \mu)$$

Remark 2.4. The theorem determines the dual space of $L^p(\Omega, \mathcal{A}, \mu)$, it shows that every continuous linear functional L on $L^p(\Omega, \mathcal{A}, \mu)$ for $1 \leq p < \infty$ has the form

$$L(f) = \int_{\Omega} \overline{g(x)}f(x)d\mu(x)$$

for a $g \in L^q(\Omega, \mathcal{A}, \mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Our last goal in this section is to show that smooth functions are dense in L^p , for all $1 \leq p < \infty$. For simplicity we work on \mathbb{R}^n , then the next theorem shows how to approximate L^p -functions with sequences of C^∞ -functions.

Theorem 2.7. *Let $j \in L^1(\mathbb{R}^n)$ with $\int j d\lambda_n(x) = 1$. For $\epsilon > 0$ we let $j_\epsilon(x) = \epsilon^{-1}j(x/\epsilon)$ so that $\int j_\epsilon d\lambda_n = 1$ for all $\epsilon > 0$. Let now $f \in L^p(\mathbb{R}^n)$ for a $1 \leq p < \infty$ and set $f_\epsilon = f * j_\epsilon$. Then $f_\epsilon \in L^p(\mathbb{R}^n)$ with $\|f_\epsilon\|_p \leq \|j\|_1 \|f\|_p$ and $f_\epsilon \rightarrow f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \rightarrow 0$. If $j \in C_c^\infty(\mathbb{R}^n)$, then we have $f_\epsilon \in C^\infty(\mathbb{R}^n)$ and $D^\alpha f_\epsilon = (D^\alpha j_\epsilon) * f$ for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.*

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ open, $K \subset \Omega$ compact. Then there exists $J_K \in C_c^\infty(\Omega)$ with $0 \leq J_K(x) \leq 1$ for all $x \in \Omega$, $J_K(x) = 1$ for all $x \in K$. Therefore, there exists a sequence $(g_j)_{j \in \mathbb{N}}$ in $C_c^\infty(\Omega)$ with $0 \leq g_j(x) \leq 1$ for all $j \in \mathbb{N}$ and $\lim g_j(x) = 1$ for all $x \in \Omega$. Hence if $(f_j)_{j \in \mathbb{N}}$ is a sequence in $C^\infty(\Omega)$ with $f_j \rightarrow f$ in $L^p(\Omega)$ for $1 \leq p < \infty$, then $g_j f_j \in C_c^\infty(\Omega)$ and $g_j f_j \rightarrow f$ in $L^p(\Omega)$. The Theorem above implies therefore that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \leq p < \infty$.*

As an application of the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ we have the following theorem.

Theorem 2.8. *For every measurable $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \infty$, the Banach space $L^p(\Omega)$ is separable.*

2.3 Sobolev Spaces

Functions in L^p -spaces are characterized by their integrability properties. In analysis, however, it is often important to consider derivatives. The notion of derivative does not combine very well with L^p spaces, functions in L^p spaces are typically not differentiable. For this reason, we are going to introduce a weaker notion of derivative, Sobolev spaces consists then of L^p -functions, whose weak derivative is again an L^p -function.

We work with $\Omega \subset \mathbb{R}^n$ open and let $\Sigma \subset 2^\Omega$ be the Borel σ -algebra over Ω , moreover let dx denotes Lebesgue mass. We define the normed space

$$X = \{f \in C^\infty(\Omega) : \|f\|_X < \infty\}, \text{ with } \|f\|_X = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}$$

where the sum above runs over all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$. It is easy to check that $(X, \|\cdot\|_X)$ is indeed a normed space but that it is not complete. We define \tilde{X} as the completion of this normed space and we want to find a good characterization of \tilde{X} .

To this end, let $[(f_j)_{j \in \mathbb{N}}] \in \tilde{X}$, i.e. let $(f_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in

X . Then by the definition of the norm on X , $(D^\alpha f_j)_{j \in \mathbb{N}}$ defines a Cauchy sequence on $L^p(\Omega)$ because we have for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$

$$\|D^\alpha f_j - D^\alpha f_l\|_{L^p} \leq \|D^\alpha f_j - D^\alpha f_l\|_X \rightarrow 0 \text{ as } j, l \rightarrow \infty.$$

Since $L^p(\Omega)$ is a complete space, there exists $f^{(\alpha)} \in L^p(\Omega)$ such that $D^\alpha f_j \rightarrow f^{(\alpha)}$ in L^p as $j \rightarrow \infty$. For any $\xi \in C_c^\infty(\Omega)$ we obtain by the above convergence, integration by parts and the fact that ξ is compactly supported that

$$\begin{aligned} \int_{\Omega} D^\alpha \xi f^{(0)} dx &= \lim_{j \rightarrow \infty} \int_{\Omega} D^\alpha \xi f_j dx \\ &= (-1)^{|\alpha|} \lim_{j \rightarrow \infty} \int_{\Omega} \xi D^\alpha f_j dx = (-1)^{|\alpha|} \int_{\Omega} \xi f^{(\alpha)} dx \end{aligned}$$

This identity gives a relationship between $f^{(\alpha)}$ and $f^{(0)} = f$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$.

Motivated by this remark, we define the Sobolev space of order $m \in \mathbb{N}$ with exponent $1 \leq p \leq \infty$ as

$H^{m,p}(\Omega) = \{f \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^n \text{ with } 1 \leq |\alpha| \leq m, \exists f^{(\alpha)} \in L^p(\Omega) \text{ such that}$

$$\int_{\Omega} D^\alpha \xi f dx = (-1)^{|\alpha|} \int_{\Omega} \xi f^{(\alpha)} dx \text{ for all } \xi \in C_c^\infty(\Omega)\}$$

We equip $H^{m,p}(\Omega)$ with the norm

$$\|f\|_{H^{m,p}} := \sum_{|\alpha| \leq m} \|f^{(\alpha)}\|_{L^p}.$$

It is then easy to check that $(H^{m,p}(\Omega), \|\cdot\|_{H^{m,p}})$ is a normed space. For $f \in H^{m,p}(\Omega)$ we call the functions $f^{(\alpha)}$, $1 \leq |\alpha| \leq m$, the weak derivatives of f . We often use the notation $f^{(\alpha)} = \partial^\alpha f$. The following Lemma shows that the weak derivatives of a function are unique.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^p(\Omega, dx)$, $1 \leq p \leq \infty$, with*

$$\int_{\Omega} \xi f dx = 0, \text{ for all } \xi \in C_c^\infty(\Omega)$$

then we have $f = 0$.

Remark 2.5. If $f \in L^p(\Omega) \cap C^m(\Omega)$, then the classical derivatives $D^\alpha f$ coincide with the weak derivatives, this can be shown by using integration by parts. The advantage of the weak derivatives is of course the fact that they exist for a much larger class of functions.

Next we show that the completeness of the L^p spaces implies also the completeness of the Sobolev spaces.

Theorem 2.9. For all $m \in \mathbb{N}$, $1 \leq p \leq \infty$, $H^{m,p}(\Omega)$ is a Banach space.

Proof. Let $(f_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in $H^{m,p}(\Omega)$, then clearly f_j is also a Cauchy sequence in $L^p(\Omega)$, hence there exists $f \in L^p(\Omega)$ with $f_j \rightarrow f$ in $L^p(\Omega)$. Moreover, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$, we also have that $\partial^\alpha f$ is a Cauchy sequence in $L^p(\Omega)$. Hence there exists $g_\alpha \in L^p(\Omega)$ with $\partial^\alpha f_j \rightarrow g_\alpha$ in $L^p(\Omega)$. We claim that in fact $f \in H^{m,p}(\Omega)$ and that $\partial^\alpha f = g_\alpha$. In fact, we observe that for arbitray $\xi \in C_c^\infty(\Omega)$ we have

$$\int_{\Omega} D^\alpha \xi f dx = \lim_{j \rightarrow \infty} \int_{\Omega} D^\alpha \xi f_j dx = (-1)^{|\alpha|} \lim_{j \rightarrow \infty} \int_{\Omega} \xi \partial^\alpha f_j dx = (-1)^{|\alpha|} \int_{\Omega} \xi g_\alpha dx$$

□

Theorem 2.10 (Approximation Theorem). Let $f \in H^{m,p}(\Omega)$, $1 \leq p < \infty$, $m \in \mathbb{N}$. Then there exists a sequence in $H^{m,p}(\Omega) \cap C^\infty(\Omega)$ with $\|f_j - f\|_{H^{m,p}}$ as $j \rightarrow \infty$. Moreover, if $\Omega = \mathbb{R}^n$, then we can choose $f_k \in C_c^\infty(\mathbb{R}^n)$ with $\|f_k - f\|_{H^{m,p}} \rightarrow 0$ as $k \rightarrow \infty$.

Remark 2.6. The theorem shows that, for all $1 \leq p < \infty$, $H^{m,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm of $H^{m,p}$.

3 Compactness

3.1 Compact Sets in Metric Spaces

Definition 3.1. Let (X, d) be a metric space. A subset $A \subset X$ is called pre-compact if, for all $\epsilon > 0$, A has a finite covering with ϵ -balls, i.e. if there are $x_1, \dots, x_n \in X$ with $\cup_{i=1}^n B_\epsilon(x_i) \supset A$.

Remark 3.1. We have the following

1. Subsets of pre-compact sets are pre-compact.
2. $A \subset X$ pre-compact implies that A is bounded.
3. $A \subset X$ pre-compact implies that \overline{A} is closed and pre-compact.

Theorem 3.1. Let (X, d) be a metric space. For $A \subset X$, the following statements are equivalent:

1. A is compact.
2. A is sequentially compact, i.e. every sequence in A has a convergent subsequence.
3. (A, d) is complete and A is pre-compact.

Remark 3.2.

1. We see that what's missing for a pre-compact set to be compact is completeness.
2. $A \subset X$ compact implies that A is closed (since A complete implies that A is closed)
3. If X is complete, then $A \subset X$ is pre-compact if and only if \overline{A} is compact (since closed subsets of complete spaces are complete).

On finite dimensional vector spaces (for example normed spaces with finite dimensional V vector space) all norms are equivalent.

Lemma 3.1. *Let X be a finite dimensional vector space over \mathbb{K} . Then all norms on X are equivalent. In other words, if $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X , then there exists $C > 0$ such that*

$$\frac{1}{C}\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2, \text{ for all } x \in X.$$

Corollary 3.1. *Every finite dimensional subspace of a normed space is complete and therefore closed.*

Theorem 3.2 (Heine-Borel). *Let $(X, \|\cdot\|)$ be a finite dimensional vector space. Then $A \subset X$ is compact if and only if A is bounded and closed.*

Proof. " \implies " Readily follows from the Theorem above, because A is compact if and only if (A, d) is complete and A is pre-compact, which implies that A is bounded and closed.

" \impliedby " Assume that $A \subset X$ is bounded and closed. Since $X \subset X$ is a finite dimensional subspace, X is complete and therefore closed. Thus it is enough to show that $A \subset X$ is pre-compact and we know that this is the case if and only if $\overline{A} = A$ is complete.

Let $\{e_1, \dots, e_n\}$ be a basis of X . Then we define the norm

$$\|x\|_\infty = \max_{j=1, \dots, n} |x_j|, \text{ for } x = \sum_{k=1}^n x_k e_k \in X$$

on X . Since all norms are equivalent, it is enough to prove the claim w.r.t. to this norm $\|\cdot\|_\infty$. To this end, it is enough to show that for $R > 0, \epsilon > 0$ there exists $n(R, \epsilon) \in \mathbb{N}$ and $x^{(1)}, \dots, x^{(n(R, \epsilon))} \in X$ such that

$$B_R(0) \subset \bigcup_{j=1}^{n(R, \epsilon)} B_\epsilon(x^{(j)})$$

□

Proposition 3.1. *Let X be a normed space. Then $\overline{B_1(0)}$ is compact if and only if $\dim X < \infty$*

3.2 Compact Subsets of $C(K)$

For finite dimensional normed spaces we found a simple characterization of compactness, namely if $\dim X < \infty$ then $A \subset X$ is compact if and only if A is closed and bounded. We try to find a similar simple characterization of compact subsets for the case of infinite dimensional spaces. We denote $C(K) = C_{\mathbb{K}}(K)$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Definition 3.2. $S \subset C(K)$ is called *equicontinuous at $x \in K$* if

$$\forall \epsilon > 0, \exists U_x \text{ open neighbourhood of } x \text{ in } K : |f(x) - f(y)| \leq \epsilon, \forall y \in U_x, \forall f \in S.$$

$S \subset C(K)$ is called *equicontinuous*, if S is equicontinuous at x , for all $x \in K$.

Remark 3.3. The open neighbourhood of x U_x may solely depend on ϵ and the point x , it mustn't depend on $f \in S$.

Theorem 3.3 (Arzelá-Ascoli). *A subset $S \subset C(K)$ is pre-compact if and only if S is bounded and equicontinuous.*

Remark 3.4. We know that $S \subset C(K)$ is pre-compact if and only if \overline{S} is compact. Hence from Arzelá-Ascoli we obtain that $S \subset C(K)$ is compact if and only if S is closed, bounded and equicontinuous. In particular, in comparison with Heine-Borel Theorem we require to satisfy 1 additional conditional, namely the condition of equicontinuity.

Proof. " \implies " Let $\epsilon > 0$. Since S is pre-compact, it has a finite covering with ϵ -balls. In other words, there exists $f_1, \dots, f_n \in S$ with $S \subset \bigcup_{j=1}^n B_{\epsilon}(f_j)$. Hence, for all $f \in S$, there exists $j \in \{1, \dots, n\}$ such that $f \in B_{\epsilon}(f_j)$ and therefore for all $f \in S$

$$\|f\| \leq \|f_j\| + \epsilon$$

This proves that

$$\sup_{f \in S} \|f\| \leq \max_{j=1, \dots, n} \|f_j\| + \epsilon$$

and also that S is bounded. Now, fix $x \in K$. Since the f_i are continuous, we have for $i = 1, \dots, n$ there exists an open neighbourhood U_i of x in K such that

$$|f_i(x) - f_i(y)| \leq \epsilon, \forall y \in U_i$$

Let now $U := \bigcap_{i=1}^n U_i$. Then U is an open neighbourhood of $x \in K$ for all $f \in S$ and for all $y \in U$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq 2\|f - f_i\| + |f_i(x) - f_i(y)| \leq 3\epsilon \end{aligned}$$

for an appropriate choice of $i \in \{1, \dots, n\}$. This shows the equicontinuity of S .

For " \Leftarrow ", since S is equicontinuous we can for $\epsilon > 0$ and $x \in K$ always find U_x open neighbourhood of x such that

$$|f(x) - f(y)| \leq \epsilon \quad \forall f \in S, \forall y \in U_x.$$

Then $\{U_x\}_{x \in K}$ is an open covering of K . I.e. there exists $x_1, \dots, x_n \in K$ with $K = \bigcup_{j=1}^n U_{x_j}$. Since moreover S is bounded, we have that

$$R = \{(f(x_1), \dots, f(x_n)) : f \in S\}$$

is a bounded subset of \mathbb{C}^n , equipped with the max-norm. Therefore, R is pre-compact in \mathbb{C}^n , i.e. there exists $f_1, \dots, f_m \in S$, with

$$R \subset \bigcup_{i=1}^m B_\epsilon((f_i(x_1), \dots, f_i(x_n)))$$

we can then show that

$$S \subset \bigcup_{i=1}^m B_{3\epsilon}(f_i).$$

which shows that S is pre-compact. □

3.3 Compact Subsets of L^p -Spaces

Theorem 3.4 (Riesz's Theorem). *Let $1 \leq p < \infty$, $A \subset L^p(\mathbb{R}^n)$. Then A is pre-compact if and only if the following conditions are satisfied:*

1. A is bounded
2. $\sup_{f \in A} \|f(\cdot + h) - f\|_p \rightarrow 0$ as $h \rightarrow 0$
3. $\sup_{f \in A} \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \rightarrow 0$ as $R \rightarrow \infty$.

Remark 3.5. The theorem holds true for subsets of $L^p(\mathbb{R}^n)$. There are also extensions to subsets of $L^p(\Omega)$, for measurable $\Omega \subset \mathbb{R}^n$.

4 Linear Operators & Functionals on Normed Spaces

4.1 Continuous Operators

Definition 4.1. Let X, Y be two normed spaces. A (linear) operator $T : X \rightarrow Y$ is a linear map. A continuous operator $T : X \rightarrow Y$ is a continuous linear map. An operator $T : X \rightarrow Y$ is called bounded, if there exists a constant $C > 0$ with

$$\|Tx\|_Y \leq C\|x\|_X, \text{ for all } x \in X$$

Proposition 4.1 (Characterization of continuous operators). Let X, Y be normed spaces and $T : X \rightarrow Y$ a linear operator. The following statements are equivalent:

1. T is continuous.
2. T is bounded.
3. T is continuous at $x = 0$.

Definition 4.2. Let X, Y be normed spaces and \mathbb{K} be \mathbb{R} or \mathbb{C} . Then we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X to Y . $\mathcal{L}(X, Y)$ has the structure of a vector space over \mathbb{K} when we consider the canonical operators for additions and multiplication with scalars. Moreover, for $T \in \mathcal{L}(X, Y)$ we define

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$$

in particular $(\mathcal{L}(X, Y), \|\cdot\|)$ is a normed space. We also denote $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Proposition 4.2. Let X be a normed space and Y be a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space.

Proof. Let $(T_l)_{l \in \mathbb{N}}$ be a Cauchy sequence on $\mathcal{L}(X, Y)$ and $x \in X$. Then we always have the inequality (!)

$$\|T_l x - T_k x\| \leq \|T_l - T_k\| \|x\| \rightarrow 0 \text{ as } l, k \rightarrow \infty.$$

Thus $(T_l x)_{l \in \mathbb{N}}$ is a Cauchy sequence in Y . Since Y is by assumption complete, there exists

$$Tx := \lim_{l \rightarrow \infty} T_l x$$

The map $T : X \rightarrow Y$ given as above is clearly linear. Moreover, $T \in \mathcal{L}(X, Y)$ since

$$\|Tx\| = \lim_{l \rightarrow \infty} \|T_l x\| \leq \limsup_{l \rightarrow \infty} \|T_l\| \|x\| \leq C\|x\|,$$

since $(T_l)_{l \in \mathbb{N}}$ is a Cauchy sequence and therefore bounded. We finally get

$$\|Tx - T_l x\| = \lim_{k \rightarrow \infty} \|T_k x - T_l x\| \leq \limsup_{k \rightarrow \infty} \|T_k - T_l\| \|x\| \rightarrow 0 \text{ as } l \rightarrow \infty$$

because $(T_k)_{k \in \mathbb{N}}$ is a Cauchy sequence, this completes the proof. \square

Definition 4.3. Let X be a normed space over the field \mathbb{K} . We define the dual space X^* of X through $X^* := \mathcal{L}(X, \mathbb{K})$. That is elements of X^* are continuous linear functionals (i.e. continuous linear operators on X with values on \mathbb{K}). Since \mathbb{K} is always complete, it follows that X^* is always a Banach space, independently of the completeness of X .

4.2 The Hahn-Banach Theorem and its Applications

Let us recall again the statement of the Lemma of Zorn. A partial order on a set P is a relation \preceq on P with the following properties:

1. $a \preceq a$ (Reflexivity)
2. $a \preceq b$ and $b \preceq a$ implies that $a = b$ (Antisymmetry)
3. If $a \preceq b$ and $b \preceq c$ then $a \preceq c$ (Transitivity)

A subset $M \subset P$ of a partially ordered set P is called totally ordered (or a chain) if for all $a, b \in M$ with $a \neq b$ we either have $a \preceq b$ or $b \preceq a$. Moreover, an element $b \in P$ is called an upper bound for $M \subset P$ if $a \preceq b$ for all $a \in M$. Naturally, we call $b \in P$ a maximal element in P , if there is no other upper bound $a \in P$ with $a \neq b$ and $b \preceq a$ for said upper bound.

Lemma 4.1 (Lemma of Zorn). *Let P be a partially ordered set, so that every totally ordered subset of P has an upper bound. Then P contains at least one maximal element.*

Definition 4.4. Let X be a vector space over \mathbb{R} . A map $p : X \rightarrow \mathbb{R}$ is called a sublinear functional if

1. $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda \geq 0$.
2. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Lemma 4.2. *Let X be a vector space over \mathbb{R} . $M \subset X$ a linear subspace, $p : X \rightarrow \mathbb{R}$ a sublinear functional, $f : M \rightarrow \mathbb{R}$ a linear functional, $x_0 \in X \setminus M$. Assume $f(x) \leq p(x)$ for all $x \in M$. Then there exists $F : \widetilde{M} := M + \mathbb{R}x_0 \rightarrow \mathbb{R}$ linear, with $F(x) \leq p(x)$ for all $x \in \widetilde{M}$ and $F|_M = f$.*

Theorem 4.1 (Hahn-Banach, vector spaces over \mathbb{R}). *Let X be a vector space over \mathbb{R} , $M \subset X$ a linear subspace and $f : M \rightarrow \mathbb{R}$ linear. Let $p : X \rightarrow \mathbb{R}$ be a sublinear functional with $f(x) \leq p(x)$ for all $x \in M$. Then there exists $F : X \rightarrow \mathbb{R}$ linear with $F(x) \leq p(x)$ for all $x \in X$ and $F|_M = f$.*

Proof. We consider the set

$$\mathcal{F} := \{(\widetilde{M}, \tilde{f}) : \widetilde{M} \subset X \text{ linear with } \widetilde{M} \supset M, \tilde{f} \text{ a linear function on } \widetilde{M}, \\ \text{with } \tilde{f}|_M = f \text{ and } \tilde{f}(x) \leq p(x) \text{ for all } x \in \widetilde{M}\}$$

We have $\mathcal{F} \neq \emptyset$ since by assumption $(M, f) \in \mathcal{F}$. On \mathcal{F} we define a partial order by setting $(\widetilde{M}, \tilde{f}) \preceq (\widetilde{N}, \tilde{g})$ if $\widetilde{M} \subset \widetilde{N}$ and $\tilde{g}|_{\widetilde{M}} = \tilde{f}$. We claim that every totally ordered subset of \mathcal{F} has an upper bound.

To this end let $\mathcal{G} = \{(M_i, g_i) : i \in I\}$ be a totally ordered subset of \mathcal{F} . Since \mathcal{G} is totally ordered we can define the linear space

$$\widetilde{M} = \bigcup_{i \in I} M_i$$

and $\tilde{g} : \widetilde{M} \rightarrow \mathbb{R}$, so that for $x \in M_i$, $\tilde{g}(x) = g_i(x)$. We claim that \tilde{g} is linear. Indeed, since \mathcal{G} is totally ordered we have for any given $x, y \in \widetilde{M}$ and $\lambda \in \mathbb{R}$ there exists $i \in I$ (since \mathcal{G} is totally ordered) such that $x, y, \lambda y \in M_i$, hence we have

$$\tilde{g}(x + \lambda y) = \tilde{g}(x) + \lambda \tilde{g}(y)$$

since the g_i are linear. Moreover we clearly have $\tilde{g}|_M = f$ and $\tilde{g}(x) \leq p(x)$ for all $x \in \widetilde{M}$.

Thus $(\widetilde{M}, \tilde{g}) \in \mathcal{F}$ is an upper bound for \mathcal{G} . The Lemma of Zorn implies that there exists a maximal element $(N, F) \in \mathcal{F}$. We want to establish that $N = X$. Assume that $N \neq X$ then $X \setminus N \neq \emptyset$ and we can choose $x_0 \in X \setminus N$, if we apply now the previous Lemma to said x_0 then there exists G with $(N + \mathbb{R}x_0, G) \in \mathcal{F}$, $G|_N = F$. But we clearly have $(N, F) \preceq (N + \mathbb{R}x_0, G)$ and $(N, F) \neq (N + \mathbb{R}x_0, G)$. This is a contradiction to the maximality of (N, F) and thus $N = X$. \square

We want to extend the theorem to vector spaces over \mathbb{C} , we thus need to introduce the notion of a seminorm.

Definition 4.5. *Let X be a vector space over \mathbb{K} and $q : X \rightarrow \mathbb{R}$ a map with the properties $q(\alpha x) = |\alpha|q(x)$ for all $\alpha \in \mathbb{K}, x \in X$ and with $q(x + y) \leq q(x) + q(y)$ for all $x, y \in X$. Then we call q a seminorm.*

Theorem 4.2 (Hahn-Banach, general version, for real and complex vector spaces). *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $q : X \rightarrow \mathbb{R}$ a seminorm, $M \subset X$ a linear subspace and $f : M \rightarrow \mathbb{K}$ a linear functional with $|f(x)| \leq q(x)$ for all $x \in M$. Then there exists a linear functional $F : X \rightarrow \mathbb{K}$ with $F|_M = f$ and $|F(x)| \leq q(x)$ for all $x \in X$.*

Proof. If $\mathbb{K} = \mathbb{R}$ we can apply Hahn-Banach Theorem (real version) to find $F : X \rightarrow \mathbb{R}$ linear with $F|_M = f$ and with $F(x) \leq q(x)$ for all $x \in X$. Then we have

$$-F(x) = F(-x) \leq q(-x) = q(x) \implies F(x) \geq -q(x), \text{ thus } |F(x)| \leq q(x).$$

So, let us now consider the case when $\mathbb{K} = \mathbb{C}$. We can consider X and M as vector spaces over \mathbb{R} . Notice that $\operatorname{Re} f : M \rightarrow \mathbb{R}$ is \mathbb{R} -linear with $|\operatorname{Re} f(x)| \leq |f(x)| \leq q(x)$. We apply Hahn-Banach for vector spaces over \mathbb{R} and find $G : X \rightarrow \mathbb{R}$, \mathbb{R} -linear with $G|_M = \operatorname{Re} f$ and with $|G(x)| \leq q(x)$ for all $x \in X$.

Let us now define $F(x) = G(x) - iG(ix)$. Then F is clearly \mathbb{R} linear, however since we have

$$\begin{aligned} F(ix) &= G(ix) - iG(i^2x) = G(ix) - iG((-1)x) = G(ix) + iG(x) \\ &= iG(x) + G(ix) = i(G(x) - iG(ix)) = iF(x) \end{aligned}$$

we can also conclude that F is \mathbb{C} -linear. Moreover we have for $x \in M$

$$\begin{aligned} \operatorname{Re} F(x) &= G(x) = \operatorname{Re} f(x) \\ \operatorname{Im} F(x) &= -G(ix) = -\operatorname{Re} f(ix) = -\operatorname{Re} if(x) = \operatorname{Im} f(x) \end{aligned}$$

This shows that $F|_M = f$. We still need to verify that $|F(x)| \leq q(x)$ for all $x \in X$. Let $x \in X$ be arbitrary and we write $F(x) = re^{i\theta}$. Then $e^{-i\theta}F(x)$ is real and therefore

$$|F(x)| = |e^{-i\theta}F(x)| = |F(e^{-i\theta}x)| \stackrel{\text{real}}{=} |G(e^{-i\theta}x)| \leq q(e^{-i\theta}x) = q(x).$$

□

The following corollaries are immediate consequences of the Hahn-Banach theorem. Of course, Hahn-Banach's Theorem is used to construct linear functionals.

Corollary 4.1. *Let $(X, \|\cdot\|)$ be a normed space over \mathbb{K} , $M \subset X$, a linear subspace and $f \in M^*$. Then there exists $F \in X^*$ with $F|_M = f$ and $\|F\|_{X^*} = \|f\|_{M^*}$.*

Proof. We define $q(x) = \|f\|_{M^*}\|x\|$. Then q is clearly a seminorm and we have $|f(x)| \leq q(x)$ for all $x \in M$. By Hahn-Banach, there exists a linear functional $F : X \rightarrow \mathbb{K}$ with $F|_M = f$ and $|F(x)| \leq q(x) = \|f\|_{M^*}\|x\|$ for all $x \in X$. This readily implies that $F \in X^*$. Moreover it shows that $\|F\|_{X^*} \leq \|f\|_{M^*}$. On the other hand we have

$$\|f\|_{M^*} = \sup_{x \in M, \|x\| \leq 1} |f(x)| = \sup_{x \in M, \|x\| \leq 1} |F(x)| \leq \sup_{x \in X, \|x\| \leq 1} |F(x)| = \|F\|_{X^*}$$

Hence we established $\|F\|_{X^*} = \|f\|_{M^*}$. \square

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a normed space over \mathbb{K} , $y \in X \setminus \{0\}$. Then there exists $f \in X^*$ with $\|f\| = 1$ and $f(y) = \|y\|$.*

Proof. We define $g : \mathbb{K} \cdot y \rightarrow \mathbb{K}$ through $g(ty) = t\|y\|$. Then clearly $g \in M^*$ with $\|g\|_{M^*} = 1$ and $g(y) = \|y\|$. From the above Corollary, there exists $f \in X^*$ with $f|_M = g$ and hence in particular $f(y) = \|y\|$ and moreover $\|f\|_{X^*} = \|g\|_{M^*} = 1$. \square

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a normed space over \mathbb{K} , $Z \subset X$ a linear subspace and $y \in X \setminus Z$. Let $d = \text{dist}(y, Z) = \inf_{z \in Z} \|z - y\| > 0$. Then there exists $F \in X^*$ with $\|F\| = 1$, $F|_Z = 0$ and $F(y) = d$.*

Proof. Let $M = Z + \mathbb{K}y$ and define $f : M \rightarrow \mathbb{K}$ through $f(z + \alpha y) = \alpha d$, for all $\alpha \in \mathbb{K}$. Then f is clearly linear. We claim that $\|f\| = 1$. In fact, since for $\alpha \in \mathbb{K}$ and $z \in Z$ we have that $-z/\alpha \in Z$ we can conclude that

$$|f(z + \alpha y)| = |\alpha|d \leq |\alpha| \cdot \|y - (-z/\alpha)\| = \|\alpha y + z\|$$

In other words, we have $\|f\|_{M^*} \leq 1$. On the other hand, let $(z_n) \in Z$ be a sequence with $\|y - z_n\| \rightarrow d$. Then

$$d = f(y - z_n) \leq \|f\|_{M^*}\|y - z_n\|$$

As $n \rightarrow \infty$ we find that $\|f\|_{M^*} \geq 1$. We summarize, $f \in M^*$ with $f|_Z = 0$ and $\|f\|_{M^*} = 1$. From Corollary 4.1. there exists $F \in X^*$ with $\|F\|_{X^*} = \|f\|_{M^*} = 1$ and $F|_M = f$ which gives in particular that $F(y) = f(y) = d$ and $F|_Z = 0$. \square

Remark 4.1. Yet another consequence of Hahn-Banach Theorem is that if X is a normed space, then X^* separates the points of X , i.e. for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $f \in X^*$ with $f(x_1) \neq f(x_2)$. In fact, given $x_1, x_2 \in X$ with $x_1 \neq x_2$ we set $y = x_1 - x_2 \neq 0$. Then from Corollary 4.2. there exists $f \in X^*$ with $f(y) = \|y\| \neq 0$ which implies that $f(x_1) \neq f(x_2)$.

Definition 4.6. Let X, Y be two normed spaces and $T : X \rightarrow Y$ a continuous linear operator. The adjoint operator to T , denoted by T^* , is a linear map $T^* : Y^* \rightarrow X^*$, defined by $T^*(f) = f \circ T$, for all $f \in Y^*$.

Remark 4.2. Notice that if $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(X, Z)$, then we have $(ST)^* = T^*S^*$.

Proposition 4.3. Let $T \in \mathcal{L}(X, Y)$. Then $T^* \in \mathcal{L}(Y^*, X^*)$ with $\|T^*\| = \|T\|$.

4.3 Reflexive normed spaces

Let X be a normed space. Apart from the dual space X^* also the bidual space X^{**} plays an important role. We construct the canonical inclusion of X in X^{**} as follows: For $x \in X$ we define the linear function $\tilde{x} : X^* \rightarrow \mathbb{K}$ through

$$\tilde{f}(x) := f(x)$$

Then indeed \tilde{x} is continuous since $f \in X^*$ is continuous, hence we have $\tilde{x} \in X^{**}$. We define $J_X : X \rightarrow X^{**}$ as $J_X(x) = \tilde{x}$ and call it the canonical inclusion of X in X^{**} .

Theorem 4.3. Let X be a normed vector space over \mathbb{K} . Then the canonical inclusion $J_X : X \rightarrow X^{**}$ is a linear isometry.

Definition 4.7. A normed space $(X, \|\cdot\|)$ is called reflexive, if the map J_X is surjective (and therefore an isometric isomorphism, in particular $X \cong X^{**}$).

Remark 4.3. The space $X^{**} = \mathcal{L}(X^*, \mathbb{K})$ is always a Banach space, hence X reflexive implies that X is also a Banach space, i.e. X is complete.

Theorem 4.4. Let X be a reflexive Banach space and $M \subset X$ a closed linear subspace. Then M is reflexive as well.

Theorem 4.5. Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.

4.4 Hilbert Space Methods

Hilbert spaces can be identified with their dual spaces, this is the subject of Riesz's Representation Theorem.

Theorem 4.6 (Riesz's Representation Theorem). Let $(H, (\cdot, \cdot))$ be an Hilbert space. Let $v \in H$ be arbitrary. The map $R_H^v : H \rightarrow H^*$ defined through $R_H^v(u) = (u, v) = \langle u, v \rangle_{H^*}$ is an anti-linear isometric isomorphism.

Remark 4.4. The theorem states that every element of H^* can be written uniquely in this form. We also use the notation $(R_H(u))(v) = (u, v)$.

Corollary 4.4. *Every Hilbert space is reflexive.*

Proof. We claim that $J_H = R_{H^*} \circ R_H$. Then J_H is surjective, as the composition of two surjective maps. Let $u \in H$ and $f \in H^*$ be arbitrary. Then

$$\begin{aligned} ((R_{H^*} \circ R_H)(u))(f) &= (R_{H^*}(R_H(u)))(f) = \langle R_H(u), f \rangle_{H^*} \\ &= \langle R_H^{-1}f, u \rangle_H = (R_H(R_H^{-1}f))(u) = f(u) = (J_H u)(f) \end{aligned}$$

□