1 Dual Space of L^1

If Ω is a σ-finite space, we have $(L^1(\Omega))^* = L^\infty(\Omega)$. In this exercise we show that this is not always the case, if Ω is not σ-finite.

Let $X = \mathbb{R}$ and $\mathcal{A} = \{B \subset X : B \text{ is countable or } B^c \text{ is countable}\}$. Let μ be the counting measure, i.e. for $B \in \mathcal{A}$ let $\mu(B)$ be the number of elements of B, possibly ∞.

a) Show that $\mathcal{A}$ is a σ-Algebra and μ is a measure, but the measure space $(X, \mathcal{A}, \mu)$ is not σ-finite.

Proof. We will show the different statements

Claim 1: $\mathcal{A}$ is a σ-Algebra.

Proof of Claim 1: Evidently we have $X \in \mathcal{A}$ because $X^c = \emptyset$ is countable. Moreover, by the very definition of $\mathcal{A}$ we have that if $A \in \mathcal{A}$ then always $A^c \in \mathcal{A}$.

Finally consider a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ in $\mathcal{A}$. Then either there is at least one $A_i \in \mathcal{A}$ cocountable for some $i \in \mathbb{N}$ or all the A_n's are countable. In the first case we have that

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subset A_i^c$$

Which is countable as the subset of the countable set A_i^c. In the second case we simply remark that the countable union of countable sets is again countable. This entails that $\mathcal{A}$ is a σ-Algebra. □

Claim 2: $\mu : \mathcal{A} \to [0, \infty]$ is a measure.

Proof of Claim 2: The positivity of the measure is clear because we work with the cardinality of the sets. Further it is clear that the empty set has cardinality 0 and thus $\mu(\emptyset) = 0$. We only need to verify the countable additivity property of μ.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$.

Then by definition of the counting measure we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \left|\bigcup_{n\in\mathbb{N}} A_n\right| = \sum_{n\in\mathbb{N}} |A_n| = \sum_{n\in\mathbb{N}} \mu(A_n)$$

Because our sets are mutually disjoints. We also notice that this expression makes sense in any case, i.e. the RHS is infinite/finite so is the LHS. $\square$

Claim 3: The measure space $(X, \mathcal{A}, \mu)$ is not σ-finite.

Proof of Claim 3: We say that $(X, \mathcal{A}, \mu)$ is σ-finite if we can find a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with μ-finite measure such that $X = \bigcup_{n\in\mathbb{N}} A_n$.

Assume for contradiction that $(X = \mathbb{R}, \mathcal{A}, \mu)$ is σ-finite. Then we have

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} A_n \text{ with } \mu(A_i) = |A_i| < \infty \text{ for all } i \in \mathbb{N}$$

But if all the A_n have finite counting measure, they must in particular be countable because all finite sets are countable. This however would entail that $\mathbb{R}$ is also countable as the countable union of countable sets. This is clearly a contradiction. $\square$

b) Show that there exists a $\Psi \in (L^1(X,\mu))^*$ that can not be represented as

$$\Psi(f) = \int fgd\mu \text{ for all } f \in L^1(X,\mu)$$

with some $g \in L^\infty(X,\mu)$

Proof. Let $E \subset X = \mathbb{R}$ be a set that is not in our sigma algebra $\mathcal{A}$. Let $f \in L^1(X,\mu)$ be arbitrary and let us define

$$\Psi(f) := \sum_{x \in E} f(x)$$

This is well-defined, because we assume $f \in L^1(X,\mu)$. Clearly we also have that Ψ is a linear function $\Psi : L^1(X,\mu) \to \mathbb{K}$. Moreover since $f \in L^1(X,\mu)$ we have that

$$|\Psi(f)| = \left|\sum_{x \in E} f(x)\right| \leq \sum_{x \in E} |f(x)| \leq \|f\|_{L^1}$$

Thus Ψ is bounded and therefore continuous, so indeed we have $\Psi \in (L^1(X,\mu))^*$. Let us now suppose for contradiction that there is some $g \in L^\infty(X,\mu)$ such that

$$\Psi(f) = \int fgd\mu = \sum_{x \in X} f(x)g(x) \stackrel{!}{=} \sum_{x \in E} f(x)$$

This inequality can only hold if $g = \mathcal{X}_E$, but clearly $g \notin L^\infty(X,\mu)$ because by assumption we have that E is not a measurable set. $\square$

2 Integral equation

3 Theorem of Baire

A subset A of a metric space M is called **nowhere dense** if $\overset{\circ}{\overline{A}} = (\overline{A})^\circ = \emptyset$. The subset A is called **meagre**, if there exists a sequence of nowhere dense sets A_i, such that $A = \bigcup_{i=1}^\infty A_i$.

a) Show that A is nowhere dense if $M \setminus \overline{A}$ is dense in M.

Proof. I don't think that there is anything to show, we have:

$$\begin{aligned} A \text{ is nowhere dense} &\iff \left(\overline{A}\right)^\circ = \emptyset \\ &\iff M \setminus \left(\overline{A}\right)^\circ = M \text{ (by passing to the complement)} \\ &\iff \overline{M \setminus \overline{A}} = M \text{ (see claim below)} \\ &\iff \overline{M \setminus \overline{A}} \text{ is dense in } M \end{aligned}$$

Claim: Let (X, d) be a metric space and $A \subset X$, then we have $\overline{A^c} = (A^\circ)^c$.

Proof of Claim: We have by definition

$$\begin{aligned} \overline{A} &= \bigcap \{F \supset A \mid F \text{ is closed in } X\} \\ A^\circ &= \bigcup \{G \subset A \mid G \text{ is open in } X\} \end{aligned}$$

Use De Morgan Laws and we're done. □

□

b) Show that there exists a meagre set $A \subset \mathbb{R}$, such that $\mathbb{R} \setminus A$ has Lebesgue measure zero.

Proof. We first recall the central results from Class:

Lemma 5.1.1. Let (X, τ) be a topological space. The following statements are equivalent:

1. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of closed sets in X. If the interior of each A_i is empty , then also the interior of $\bigcup_{i=1}^{\infty} A_i$ is empty.

2. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of open sets in X. If each B_i is dense in X, then also $\bigcap_{i=1}^{\infty} B_i$ is dense in X.

Definition: A topological space (X, τ) is called a Baire space, if condition 1) or 2) above (and thus both) are satisfied.

The big result that Baire provided tells us now that in fact complete metric spaces are Baire spaces, i.e. build a category.

Theorem 5.1.3. (Baire's Category Theorem): Every complete metric space is a Baire space.

Since $\mathbb{R}$ is complete, it is in particular a Baire space, thus we have the equivalence of 1) and 2) above.

Let now $k \in \mathbb{N}$ and $q_n \in \mathbb{Q}$ for all $\mathbb{N}$. We then set

$$U_k := \bigcup_{n\in\mathbb{N}} \left(q_n - 2^{-(n+k)}, q_n + 2^{-(n+k)}\right) = \bigcup_{n\in\mathbb{N}} I_{n,k}$$

All the $I_{n,k}$ are open have Lebesgue measure (length) $2^{1-(n+k)}$. Using the subadditivity of the Lebesgue measure and the geometric sum we get that

$$\lambda(U_k) \leq 2^{1-k} \sum_{n\in\mathbb{N}} 2^{-n} = 2^{2-k} < \infty \text{ for all } k \in \mathbb{N}$$

Now we set

$$N := \bigcap_{k\in\mathbb{N}} U_k$$

Since we have $U_{k+1} \subset U_k$ and $\lambda(U_k) < \infty$ we get that

$$\lambda(N) = \lambda\left(\bigcap_{k\in\mathbb{K}} U_k\right) = \lim_{k\to\infty} \lambda(U_k) \leq \lim_{k\to\infty} 2^{2-k} = 0$$

which entails that N is a nullset.

Moreover we have that all the U_k are open as the union of open sets and they are also dense in $\mathbb{R}$, because we have chosen the dense sequence $q_n \in \mathbb{Q}$. More precisely, we have that U_k is an open covering of $\mathbb{Q}$, in particular we have:

$$\mathbb{Q} \subset U_k \subset \mathbb{R} \implies \mathbb{R} = \overline{\mathbb{Q}} \subset \overline{U_k} \subset \mathbb{R} \implies \overline{U_k} = \mathbb{R}$$

By Baire's theorem we thus get that also N as the intersection of said U_k is dense. Now we obtain that

$$M := N^c = \bigcup_{k \in \mathbb{N}} \overline{U_k^c}$$

We claim that $M \subset \mathbb{R}$ is meagre. Indeed, from **a)** we know that U_k^c is nowhere dense if and only if $\overline{U_k}$ is dense in $\mathbb{R}$, but we already have that established. $\square$

4 Induced Topologies

a) Let X be a set and $(\tau_\alpha)_{\alpha\in A}$ be a family of topologies on X, where A is an arbitrary index set. Show that the intersection $\bigcap_{\alpha\in A}\tau_\alpha$ is again a topology on X.

Proof. If $A=\emptyset$ then $\bigcup_{\alpha\in A}\tau_\alpha=\mathcal{P}(X)$ and we obtain the discrete topology on X. Assume now that $A\neq\emptyset$. We have $\emptyset, X\in\bigcap_{\alpha\in A}\tau_\alpha$, indeed we have $\tau, X\in\tau_\alpha$ for every $\alpha\in A$ because they are all topologies on X.

Let now I be an arbitrary index set and suppose that for all $i\in I$ we have $A_i\in\bigcap_{\alpha\in A}\tau_\alpha$. It follows that $A_i\in\tau_\alpha$ for all $\alpha\in A$. Since all the τ_α's are topologies we obtain that

$$\bigcup_{i\in I}A_i\in\tau_\alpha \text{ for all } \alpha\in A \implies \bigcup_{i\in I}A_i\in\bigcap_{\alpha\in A}\tau_\alpha.$$

Let now $n\in\mathbb{N}$ and let $A_1,\dots,A_n\in\bigcap_{\alpha\in A}\tau_\alpha$. We then have for $i=1,\dots,n$ that $A_i\in\tau_\alpha$ for all $\alpha\in A$. Since all the τ_α are topologies we have that

$$\bigcap_{i=1}^{n}A_i\in\tau_\alpha \text{ for all } \alpha\in A \implies \bigcap_{i=1}^{n}A_i\in\bigcap_{\alpha\in A}\tau_\alpha$$

consequently $\bigcap_{\alpha\in A}\tau_\alpha$ is a topology, as claimed. □

b) Let X be a set and $\mathcal{F}$ a family of maps $f:X\to Y_f$, where each Y_f is a topological space. We define

$$\mathcal{S}:=\{f^{-1}(V): f\in\mathcal{F}, V\subset Y_f \text{ open}\}$$

and

$$\tau_{\mathcal{F}}:=\bigcap\{\tau:\tau \text{ is a topology on } X \text{ and } \mathcal{S}\subset\tau\}$$

Prove that

$$\tau_{\mathcal{F}}=\{\mathcal{O}:\mathcal{O} \text{ is an arbitrary union of finite intersections of sets } B\in\mathcal{S}\}.$$

Proof. First we remark that by **a)** $\tau_{\mathcal{F}}$ is indeed a topology on X. We will show both inclusions separately. For the $\supset$ inclusion, we consider any $\mathcal{O}$ that can be written as the arbitrary union of finite intersections of sets $B\in\mathcal{S}$. By definition $\tau_{\mathcal{F}}$ is the smallest topology on X such that the maps $f:X\to Y_f$ are continuous, thus by definition of $\tau_{\mathcal{F}}$ and the axioms of a topology (stable under arbitrary union and finite intersection) the inclusion $\supset$ is given.

Conversely, to establish the $\subset$ inclusion we remark again that $\tau_{\mathcal{F}}$ is by definition the smallest topology on X such that the maps $f : X \to Y_f$ are continuous. Thus it suffices to prove that

$$\mathcal{T} = \{\mathcal{O} : \mathcal{O} \text{ is an arbitrary union of finite intersections of sets } B \in \mathcal{S}\}.$$

is a topology on X, then the $\subset$ inclusion follows.

Towards that, we first establish the following result:

Lemma: Let X be a set and let $\mathfrak{O} \subset \mathcal{P}(X)$ be a collection of subsets of X, such that

1. $\emptyset$ and X are in $\mathfrak{O}$.
2. $\mathfrak{O}$ is closed under finite intersections.

Then

$$\mathcal{T} := \left\{ \bigcup_{O \in \mathcal{O}} O \mid \mathcal{O} \subset \mathfrak{O} \right\} \text{ is a topology on } X.$$

Proof of Lemma: By definition, $\mathcal{T}$ contains the emptyset as well as X since those were already contained in $\mathfrak{O}$. Moreover, again by definition, $\mathcal{T}$ is closed under arbitrary unions. Thus we only need to show that $\mathcal{T}$ is closed under finite intersections.

Let A_1 and A_2 be two elements of $\mathcal{T}$. Then, by definition of $\mathcal{T}$, there exist $\mathcal{O}_1$ and $\mathcal{O}_2$, subsets of $\mathfrak{O}$, such that

$$A_1 = \bigcup_{O_1 \in \mathcal{O}_1} O_1 \text{ and } A_2 = \bigcup_{O_2 \in \mathcal{O}_2} O_2$$

We then have

$$A_1 \cap A_2 = \bigcup_{O_1 \in \mathcal{O}_1} O_1 \cap \bigcup_{O_2 \in \mathcal{O}_2} O_2 = \bigcup_{\substack{O_1 \in \mathcal{O}_1 \\ O_2 \in \mathcal{O}_2}} O_1 \cap O_2$$

Setting $\mathcal{O} = \{O_1 \cap O_2 \mid O_1 \in \mathcal{O}_1,\ O_2 \in \mathcal{O}_2\}$, then $\mathcal{O}$ is a subset of $\mathfrak{O}$ since by assumption $\mathfrak{O}$ is closed under finite intersections, we thus get

$$A_1 \cap A_2 = \bigcup_{O \in \mathcal{O}} O \in \mathcal{T}$$

By induction over $n \in \mathbb{N}$, using the same construction as above for the induction step, we obtain that $\mathcal{T}$ is closed under finite intersections. $\square$

Corollarly to the above Lemma we can now establish that

$$\mathcal{T} = \{\mathcal{O} : \mathcal{O} \text{ is an arbitrary union of finite intersections of sets } B \in \mathcal{S}\}.$$

is a Topology on X. In the above case we have

$$\mathfrak{O} = \{U_1 \cap \cdots \cap U_n \mid n \in \mathbb{N}, U_i \in \mathcal{S} \text{ for all } i = 1, \ldots, n\} \subset \mathcal{P}(X)$$

Where

$$\mathcal{S} = \{f^{-1}(V) : f \in \mathcal{F}, V \subset Y_f \text{ open}\}$$

By definition, we have that $\mathfrak{O}$ is closed under finite intersections. Further we have $\emptyset \in \mathcal{S}$ because $\emptyset \subset Y_f$ is open and $f^{-1}(\emptyset) = \emptyset$, thus it follows that $\emptyset \in \mathfrak{O}$.

Moreover we have $X \in \mathfrak{O}$. Indeed we have $X \in \mathcal{S}$ because $Y_f \subset Y_f$ is open in Y_f and $f^{-1}(Y_f) = X$, i.e. $X \in \mathcal{S}$ and thus $X \in \mathfrak{O}$. By the Lemma that we've established it follows that $\mathcal{T}$ is indeed a topology on X.

Bringing the arguments together one last time for readability. We have shown that $\tau_{\mathcal{F}} \supset \mathcal{T}$. Then we used the Lemma to show that $\mathcal{T}$ is a topology on X. But by definition $\tau_{\mathcal{F}}$ is the smallest topology on X that contains $\mathcal{S}$, i.e. the smallest topology such that the maps $f : X \to Y_f$ are continuous. But we also have clearly have that $\mathcal{S} \in \mathcal{T}$ since we already have $\mathcal{S} \subset \mathfrak{O}$. Thus it follows that $\tau_{\mathcal{F}} \subset \mathcal{T}$. $\square$

5 Criterion for weak convergence

Let X be a normed vector space and $Y \subset X^*$ a dense subset. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that x_n converges weakly to $x \in X$ if and only if $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $f(x_n) \to f(x)$ for all $f \in Y$.

Proof. We recall the Lemma from class which gives a characterization of weak convergence and the proposition that states that weakly convergent sequences are bounded.

Lemma 6.2.1. Let X be a normed space. For a sequence $(x_n)_{n\in\mathbb{N}}$, we have $x_n \rightharpoonup x$ if and only if $f(x_n) \to f(x)$ for all $f \in X^*$.

Proposition 6.2.2. Let X be a normed space, $(x_n)_{n\in\mathbb{N}}$ a sequence on X with $x_n \rightharpoonup x$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded and

$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|$$

Assume now that $x_n \rightharpoonup x$. Then by the above Proposition we already know that $(x_n)_{n\in\mathbb{N}}$ must be bounded, in particular $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Moreover we have that

$$f(x_n) \overset{n\to\infty}{\longrightarrow} f(x) \text{ for all } f \in X^*$$

Since $Y \subset X^*$ and the above convergence is true for all $f \in X^*$, it must also be true for all $f \in Y$.

Conversely assume that $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and that

$$f(x_n) \overset{n\to\infty}{\longrightarrow} f(x) \text{ for all } f \in Y$$

But $Y \subset X^*$ is not only a subset of X^* but also dense in X^*. That means for all $f \in X^*$ we can find a sequence $(f_k)_{k\in\mathbb{N}}$ in Y such that $f_k \to f$ in X^*. Hence let $f \in X^*$ be arbitrary and choose a sequence $(f_k)_{k\in\mathbb{N}}$ as above to approximate it. Thus we have by assumption that

$$f_k(x_n) \overset{n\to\infty}{\longrightarrow} f_k(x) \text{ for all } k \in \mathbb{N}$$

We then obtain by the triangle inequality because by assumption we can bound x_n by some constant $M > 0$ for all $n \in \mathbb{N}$ that

$$\begin{aligned}
|f(x_n) - f(x)| &\leq |f(x_n) - f_k(x_n)| + |f_k(x_n) - f_k(x)| + |f_k(x) - f(x)| \\
&= |(f - f_k)(x_n)| + |f_k(x_n) - f_k(x)| + |(f_k - f)(x)| \\
&\leq \|f - f_k\|_{X^*}\|x_n\| + |f_k(x_n) - f_k(x)| + \|f_k - f\|_{X^*}\|x\| \\
&\leq M\|f - f_k\|_{X^*} + |f_k(x_n) - f_k(x)| + \|f_k - f\|_{X^*}\|x\|
\end{aligned}$$

Choosing $k, n \in \mathbb{N}$ large enough we obtain that $f(x_n) \to f(x)$ as $n \to \infty$ for all $f \in X^*$ and thanks to the Lemma this is equivalent to saying that $x_n \rightharpoonup x$. $\square$