

1 Quotient spaces

Let X be a Banach space and $M \subset X$ a closed linear subspace. Let \sim be the equivalence relation defined through $x \sim y$ if and only if $x - y \in M$ and let X/M be the quotient space with respect to \sim .

a) For $[x] \in X/M$ let

$$\|[x]\|_{X/M} := \inf_{y \in [x]} \|y\|_X = \inf_{y \in M} \|x - y\|_X$$

Show that $\|\cdot\|_{X/M}$ is a norm on X/M .

Proof. Indeed we have that $\|\cdot\|_{X/M}$ is well-defined, it is in particular finite because $\|\cdot\|_X$ is a norm on X . Moreover we have

$$\|[x]\|_{X/M} = \inf_{y \in M} \|x - y\| = 0$$

Thus we can construct a sequence y_n in M such that $\|x - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. But since $M \subset X$ is closed we have that $y_n \rightarrow x \in M$, henceforth $[x] = 0$.

Next we have

$$\begin{aligned} \|[x] + [y]\|_{X/M} &= \|[x + y]\|_{X/M} = \inf_{v \in M} \|x + y - v\| \\ &= \inf_{v \in M} \|x - v + y - v + v\| \\ &\leq \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| + \inf_{v \in M} \|v\| \\ &= \inf_{v \in M} \|x - v\| + \inf_{v \in M} \|y - v\| \\ &= \|[x]\|_{X/M} + \|[y]\|_{X/M} \end{aligned}$$

Finally we have for all $\lambda \in \mathbb{K}$

$$\begin{aligned} \|\lambda[x]\|_{X/M} &= \|[\lambda x]\|_{X/M} = \inf_{y \in M} \|\lambda x - y\| \\ &= \inf_{\lambda y \in M} \|\lambda x - \lambda y\| \\ &= \inf_{u \in M} \|\lambda x - \lambda u\| \\ &= \inf_{u \in M} |\lambda| \|x - u\| = |\lambda| \|[x]\|_{X/M} \end{aligned} \tag{*}$$

Where we used in (*) that $\varphi_\lambda : M \rightarrow M$ given by $\varphi_\lambda(x) = \lambda x$ is a bijection for all $\lambda \in \mathbb{K}$. Thus we have shown that $\|\cdot\|_{X/M}$ is a norm on X/M . \square

b) Show that the projection $\pi : X \rightarrow X/M$, $x \mapsto [x]$ is continuous.

Proof. Evidently, the function π is linear because we have by definition that

$$\begin{aligned}[x + y] &= [x] + [y] \\ \lambda[x] &= [\lambda x]\end{aligned}$$

For all $x, y \in X, \lambda \in \mathbb{K}$. Moreover we have

$$\|\pi(x)\|_{X/M} = \|[x]\|_{X/M} = \inf_{y \in M} \|x - y\| \leq \inf_{y \in M} \|x\| + \underbrace{\inf_{y \in M} \|y\|}_{=0} = \|x\|_X$$

That is π is a bounded linear operator and thus continuous. \square

c) Show that X/M is complete.

Proof. Recall from Exercise Sheet 1 Exercise 4: Normed spaces and Banach spaces where we have shown

Theorem: Let $(X, \|\cdot\|)$ be a normed space. $(X, \|\cdot\|)$ is a Banach space if and only if, all absolutely convergent series are also convergent, i.e. if

$$\sum_{n=0}^{\infty} \|x_n\| < \infty \implies \sum_{n=0}^{\infty} x_n < \infty$$

Thanks to the above theorem in order to prove that X/M is a Banach space it is enough to show that every series in X/M that converges absolutely also converges in X/M .

To this extent let us take an arbitrary sequence $([x_n])_{n \in \mathbb{N}}$ in X/M such that its series is absolutely convergent, i.e. we have

$$\sum_{n=0}^{\infty} \|[x_n]\|_{X/M} < \infty$$

By definition of the quotient norm (via the inf) we have for all $n \in \mathbb{N}$ the existence of some $y_n \in M$ such that $\|x_n - y_n\| \leq \|[x_n]\|_{X/M} + 1/2^n$. Since we've already established the convergence of the right hand side, the convergence of the left hand side is immediate by domination. But X is a Banach space, thus it follows from the Theorem above that

$$\sum_{n=0}^{\infty} (x_n - y_n) = x < \infty$$

We now claim that the series over the sequence $([x_n])_{n \in \mathbb{N}}$ in X/M converges to $[x] \in X/M$.

Indeed, let us consider for $N \in \mathbb{N}$

$$\left\| [x] - \sum_{n=0}^N [x_n] \right\|_{X/M} = \left\| [x - \sum_{n=0}^N x_n] \right\|_{X/M} \leq \left\| x - \sum_{n=0}^N x_n \right\|_X \rightarrow 0$$

as $N \rightarrow \infty$. Where the last inequality just follows by the very definition of the quotient norm. \square

d) Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$.

We define the kernel of T , $\ker T := \{x \in X : Tx = 0\}$ and the range of T , $\text{ran}(T) := \{Tx : x \in X\} \subset Y$. Let $\iota : \text{ran} T \rightarrow Y, x \mapsto x$ be the inclusion map. Show that then there exists a bijective operator $\hat{T} \in \mathcal{L}(X/\ker T, \text{ran} T)$ with $T = \iota \hat{T} \pi$ and $\|\hat{T}\| = \|T\|$.

Proof. We have indeed that $\ker T \subset X$ is a closed linear subspace. Now we define

$$\hat{T} : X/\ker T \rightarrow \text{ran} T, [x] \mapsto T(x)$$

We notice that that \hat{T} is well-defined. Indeed, let $[x], [y] \in X/\ker T$ be such that $[x] = [y]$, this is the case if and only if $x \sim y$ i.e. $x - y \in \ker T$, or equivalently $x \in y + \ker T$. This entails that we can write $x = y + t$ for some $t \in \ker T$. Thus we get

$$\hat{T}([x]) = T(x) = T(y + t) = T(y) + \underbrace{T(t)}_{=0} = T(y) = T([y])$$

Which shows that \hat{T} is well-defined. Moreover, \hat{T} is by definition (defined through T) surjective onto the range of T . Furthermore \hat{T} is injective, let $[x], [y] \in X/\ker T$ be such that

$$\begin{aligned} \hat{T}([x]) = \hat{T}([y]) &\iff T(x) = T(y) \implies T(x - y) = 0 \\ &\iff x - y \in \ker T \\ &\iff x \sim y \\ &\iff [x] = [y] \end{aligned}$$

The linearity of \hat{T} follows immediatly by the linearity of π and the linearity of T . Finally, we have that π is continuous, and by assumption T is continuous too.

We have seen that $\|[x]\|_{X/\ker T} \leq \|x\|_X$ and by the continuity of T we obtain for some $C > 0$ that

$$\|\hat{T}([x])\|_Y = \|T(x)\|_Y \leq C\|x\|_X$$

let us now set $\tilde{C} := \max(1, C)$. This yields that

$$\|\hat{T}([x])\|_Y \leq C\|x\|_X \leq \tilde{C}\|x\|_X = \tilde{C}\|x - 0\|_X$$

Since $0 \in \ker T$ we conclude that

$$\|\hat{T}([x])\|_Y \leq \inf_{t \in \ker T} \|x - t\| = \|x\|_{X/\ker T}$$

showing that indeed $\hat{T} \in \mathcal{L}(X/\ker T, \text{ran} T)$.

Further, by the very definition of \hat{T} we have that $T = \iota \hat{T} \pi$.

Lastly, we observe that since $0 \in \ker T$ we have that if $\|x\|_X \leq 1$ for an arbitrary $x \in X$, then $\|x - 0\|_X = \|x\|_X \leq 1$ and thus by definition of the infimum as the greatest lower bound we must also have $\inf_{y \in \ker T} \|x - y\| \leq 1$.

Conversely, if $\inf_{y \in \ker T} \|x - y\| \leq 1$. Then again by definition of the greatest upper bound we must also have that $\|x - 0\| = \|x\| \leq 1$. This shows that

$$\begin{aligned} \|\hat{T}\| &= \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|\hat{T}([x])\|_Y = \sup_{\substack{[x] \in X/\ker T \\ \|[x]\|_{X/\ker T} \leq 1}} \|T(x)\|_Y \\ &= \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|T(x)\|_Y = \|T\| \end{aligned}$$

□

2 Reflexivity and weak convergence

a) Let X be a normed space and X^* its dual. Let $(L_n)_{n \in \mathbb{N}}$ be a sequence in X^* and $(x_n)_{n \in \mathbb{N}}$ a sequence in X . Let $L_n \xrightarrow{*} L$ in X^* with respect to the weak-* topology and $x_n \rightarrow x$ with respect to the norm in X . Show that if X is reflexive, then $L_n(x_n) \rightarrow L(x)$ as $n \rightarrow \infty$.

Proof. We recall that the space X^{**} is always a Banach space, since by assumption X is reflexive we have $X \cong X^{**}$, i.e. X is a Banach space as well. Thus we know that the reflexivity of X is equivalent to saying that X^* is reflexive.

Claim: Let X be a reflexive Banach space, then weak-* convergence of a sequence $(f_n)_{n \in \mathbb{N}}$ in X^* implies weak convergence.

Proof of Claim: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in X^* that converges towards $f \in X^*$ w.r.t. τ_W^* , this is the case if and only if $f_n(x) \rightarrow f(x)$ for all $x \in X$ as $n \rightarrow \infty$.

We want to show that $f_n \rightharpoonup f$ in X^* , that is by **Lemma 6.2.1.** equivalent to showing that $\lambda(f_n) \rightarrow \lambda(f)$ as $n \rightarrow \infty$ for all $\lambda \in X^{**}$.

Since X is reflexive, the canonical inclusion $J_X : X \rightarrow X^{**}$ is surjective, that is to say that for every $\lambda \in X^{**}$ we find an $x \in X$ such that $J_X(x) = \lambda$. But then by the definition of the canonical inclusion we have

$$\lambda(f_n) = J_X(x)(f_n) = f_n(x) \rightarrow f(x) = J_X(x)(f) = \lambda(f)$$

which gives the weak convergence. \square

Next we recall that weakly convergent sequences are always bounded. By our efforts above we now have for the sequence $(L_n)_{n \in \mathbb{N}}$ in X^* that $L_n \xrightarrow{*} L$ in X^* implies that $L_n \rightharpoonup L$ in X^* and the sequence $(L_n)_{n \in \mathbb{N}}$ is bounded. Let $M > 0$ be a constant such that $\|L_n\|_{X^*} \leq M$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} |L_n(x_n) - L(x)| &= |L_n(x_n) - L_n(x) + L_n(x) - L(x)| \\ &\leq |L_n(x_n - x)| + |L_n(x) - L(x)| \\ &\leq \|L_n\|_{X^*} \|x_n - x\|_X + |L_n(x) - L(x)| \\ &\leq M \|x_n - x\|_X + |L_n(x) - L(x)| \end{aligned}$$

Letting $n \rightarrow \infty$ the right hand side converges to zero since $x_n \rightarrow x$ with respect to the norm in X and L_n converges in the sense of weak-* converges towards L (see characterization at begin of proof of claim). \square