Functional Analysis - The irreducible Minimum

1 Structures

1.1 Topological spaces

We will deal with various structures, in a sense, they build a certain hierachy. Topological spaces as a framework, then metric spaces, normed spaces, Banach spaces and conclusively Hilbert Spaces.

Definition 1.1. *A topological space is a pair (X, τ), consisting of a set X and a family $\tau \subset 2^X$ that we call a topology on X, such that*

1. $\emptyset, X \in \tau$.
2. *Stable under countable intersection of open sets.*
3. *Stable under arbitrary union of open sets.*

Remark 1.1. Thanks to the laws of De Morgan we easily conclude from the definition that arbitrary intersections and finite unions of closed sets are again closed.

Definition 1.2. *Let (X, τ) be a topological space and $A \subset X$ a subset. The closure $\overline{A}$ of A is defined by*

$$\overline{A} = \bigcap \{B \subset X : B \text{ is closed and } A \subset B\}$$

by definition, it is the smallest closed set that contains A. The set A is called dense in X if $\overline{A} = X$. The space X is called separable, if it contains a countable dense set. Moreover, the interior of A is defined through

$$A^\circ = \bigcup \{B \subset A : B \text{ is open}\}$$

in other words, A° is the largest open set contained in A. Finally, the boundary of A is defined as $\partial A = \overline{A} \setminus A^\circ$.

Remark 1.2. Evidently, the closure and interior of a set depend on the choice of the topology.

Definition 1.3. *Let (X, τ) be a topological space, $x \in X$. A set $U \subset X$ is called an open neighbourhood of x if $U \in \tau$ (i.e. U is open) and $x \in U$.*

Definition 1.4. *Let (X, τ) be a topological space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that x_n converges to $x \in X$ as $n \to \infty$, written $x_n \to X$, if*

For every open neighbourhood U of x, there exists $n_0 \in \mathbb{N} : x_n \in U \ \forall n \geq n_0$.

Definition 1.5. *Let $(X, \tau), (Y, \mathcal{S})$ be two topological spaces. A function $f : X \to Y$ is called continuous if*

$$f^{-1}(V) \in \tau, \ \forall V \in \mathcal{S}.$$

I.e., if the pre-image of every open set in Y is an open set in X.

Remark 1.3. In metric spaces, the notion of convergence completely characterizes the topology. This is however not necessarily true in topological spaces.

Definition 1.6. *A topological space (X, τ) is called Hausdorff if*

$$\forall x, y \in X, x \neq y \implies \exists U_x, U_y \in \tau \text{ with } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset$$

Definition 1.7. *A topological space (X, τ) is called compact if it is Hausdorff and if for every*

$$(U_\lambda)_{\lambda\in\Lambda} \text{ family in } \tau \text{ with } \bigcup_{\lambda\in\Lambda} U_\lambda = X$$
$$\implies \exists n \in \mathbb{N} \text{ and } \lambda_1, \ldots, \lambda_n \in \Lambda : \bigcup_{j=1}^{n} U_{\lambda_j} = X$$

i.e. if for every open covering of X there exists a finite sub-covering.

Theorem 1.1. *Let (X, τ) be a compact space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then there exists at least one accumulation point of x_n on X.*

Theorem 1.2 (Lemma von Urysohn)**.** *Let (X, τ) be a compact space and $A, B \subset X$ disjoint, non-empty, closed subsets of X. Then there exists a continuous function $g : X \to [0, 1]$ with $g(A) = 0$ and $g(B) = 1$.*

Remark 1.4. The Lemma of Urysohn is important because it helps us avoid some of the pathological situations (like only constant functions being continuous). In particular, the theorem implies that on compact spaces the topology is large enough for us to do some meaningful analysis on.

Definition 1.8. *Let K be a compact space. We define*

$$C_{\mathbb{K}}(K) := \{f : K \to \mathbb{K} \text{ cotninuous}\}$$

As a consequence of Urysohn's Lemma we can show that $C_{\mathbb{K}}(K)$ separates the points of K.

Corollary 1.1. *Let K be a compact space. Then $C_{\mathbb{K}}(K)$ separates the points of K. In other words, for every $x, y \in K$ with $x \neq y$, there exists $f \in C_{\mathbb{K}}(K)$ such that $f(x) \neq f(y)$.*

Proof. Since K is Hausdorff, for every $x \neq y$ we can find two open neighbourhoods U_x, U_y of x and y respectively with $U_x \cap U_y = \emptyset$. We can further find closed sets A, B with $A \subset U_x, B \subset U_y$ with $x \in A$ and $y \in B$. In particular we have $A \cap B = \emptyset$. Urysohn's Lemma then gives that there exists $f \in C_{\mathbb{K}}(K)$ with $f(x) = 0$ and $f(y) = 1$. $\square$

1.2 Metric Spaces

Definition 1.9. *A metric space is a pair (X, d) with X being an arbitrary set and a map $d : X \times X \to [0, \infty)$ called a metric on X, with the following properties*

1. *$d(x, y) = 0$ if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$.*
3. *(Δ-inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Remark 1.5.

1. Every metric space (X, d) is a topological space (X, τ_d) with topology τ_d inducedby the metric. The topology τ_d is defined by the condition that $A \in \tau_d$ if and only if for all $x \in A$, there exists $\epsilon > 0$ with $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\} \subset A$.

2. In contrast with general topological spaces, on metric spaces the notion of convergence characterizes the topology. That is we have the convenient characterizations

   (a) A set $A \subset X$ is closed if and only if for every sequence x_n in A with $x_n \to x$ in X, we have that $x \in A$.

   (b) $\overline{A} = \{x \in X : \exists (x_n)_{n\in\mathbb{N}} \text{ sequence in } A \text{ with } x_n \to x\}$.

   (c) $A \subset X$ is dense if and only if $\forall x \in X, \exists (x_n)_{n\in\mathbb{N}}$ sequence in A with $x_n \to x$.

   (d) The function $f : X \to Y$ between the two metric spaces $(X, d_1), (Y, d_2)$ is continuous at the point $x \in X$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$ we have $f(x_n) \to f(x)$.

Definition 1.10. *Let* (X, d) *be a metric space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *is called a Cauchy sequence (or is said to have the Cauchy property) if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$*. Every convergent sequence is indeed Cauchy. The metric space* (X, d) *is called complete, if every Cauchy sequence is convergent.*

1.3 Normed spaces

Definition 1.11. *A normed space is a pair* $(X, \|\cdot\|)$*, consisting of a vector space* V *over a field* $\mathbb{K}$ *(*$\mathbb{R}$ *or* $\mathbb{C}$*) and a map* $\|\cdot\| : V \to [0, \infty)$ *called a norm on* V *with the following properties*

1. $\|x\| = 0$ *if and only if* $x = 0$.
2. *(Homogenity)* $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{K}, x \in V$.
3. *(*Δ*-inequality)* $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

Remark 1.6. Every norm induces the metric $d(x, y) = \|x - y\|$, hence every normed space is also a metric space and therefore also a topological space.

Definition 1.12. *A normed space* $(X, \|\cdot\|)$ *is called complete, if* X*, equipped with the induced metric* $d(x, y) = \|x - y\|$*, is a complete metric space. A complete normed space is called a Banach space.*

Completeness is very important for analysis. It is not a coincidence that we always do analysis on $\mathbb{R}$ instead of $\mathbb{Q}$. For this reason, it is useful to have a general recipe to complete normed spaces.

Definition 1.13. *Let* $(X, \|\cdot\|)$ *be a normed space. A completion of* $(X, \|\cdot\|)$ *is a 3-tuple* $(Y, \|\cdot\|_Y, \phi)$ *consisting of a Banach space* $(Y, \|\cdot\|_Y)$ *and an isometric linear map* $\phi : X \to Y$*, with* $\overline{\phi(X)} = Y$.

Theorem 1.3. *Every normed space* $(X, \|\cdot\|)$ *has a completion, which is unique, up to linear isometric isomorphisms.*

Proof. The proof is constructive, we give a sketch. Let $\mathcal{C}_X$ denote the set of all Cauchy sequence on X. We can easily give this space the structure of a vector space over $\mathbb{K}$. Next we define the linear subspace $\mathcal{N}_X \subset \mathcal{C}_X$ consisting of all null-sequence on X, i.e.

$$\mathcal{N}_X := \{x = (x_n)_{n\in\mathbb{N}} \in \mathcal{C}_X : x_n \to 0\}$$

Moreover we define $Y := \mathcal{C}_X/\mathcal{N}_X$ as the quotient space of $\mathcal{C}_X$ w.r.t. the equivalence relation defined by $x \sim y :\iff x - y \in \mathcal{N}_X$. In other words, in Y, we identify Cauchy sequences whose difference converges to zero. Y is also a vector space over $\mathbb{K}$.

Next we want to introduce a norm on Y. To this end, we define the function $p : \mathcal{C}_X \to [0, \infty)$ through

$$p(x) = \lim_{n\to\infty} \|x_n\|$$

Thanks to the reverse triangle inequality we've got

$$|\|x_k\| - \|x_l\|| \leq \|x_k - x_l\| \to 0 \text{ as } k, l \to \infty$$

which shows that $(\|x_n\|)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}_X$. Thus the limit above is well-defined and finite. We now set

$$\|[x]\|_Y := p(x) = \lim_{n\to\infty} \|x_n\|$$

We then can verify that $\|\cdot\|_Y$ is indeed a norm on Y.

Next we define the map $\phi : X \to Y$ by $\phi(z) = [(z, z, \dots)]$, i.e. $\phi(z)$ denotes the equivalence class of all sequences on X that converge to z in the limit. This map is clearly linear and since

$$\|\phi(z)\|_Y = \|z\|_X$$

it defines an isometry. We now claim that $(Y, \|\cdot\|_Y, \phi)$ is a completion of $(X, \|\cdot\|_X)$. In order to show that we can show that $(Y, \|\cdot\|_Y)$ is always complete and that $\phi(X)$ is dense in Y. That is we want to show that for all $[x] \in Y$, we can find $\tilde{x} \in X$ with $\|\phi(\tilde{x}) - [x]\|_Y < \epsilon$. We start with the density and then use this to show that the space is complete.

Finally we can show that the uniqueness of the completion is up to isometric isomorphisms. $\square$

Remark 1.7. The completion of $\mathbb{Q}$ is $\mathbb{R}$. For $\Omega \subset \mathbb{R}^n$, the completion of the space $C(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ is $L^p(\Omega)$.

1.4 Hilbert Spaces

Definition 1.14. *Let H be a vector space over the field $\mathbb{K}$. A scalar product (or an inner product) on H is a map $(\cdot, \cdot) : H \times H \to \mathbb{K}$ with the properties*

1. $(z, x + \lambda y) = (z, x) + \lambda(z, y)$ *for all* $x, y, z \in H, \lambda \in \mathbb{K}$.
2. $(x, y) = \overline{(y, x)}$.
3. $(x, x) > 0$ *for all* $x \neq 0$.

A pair $(H, (\cdot, \cdot))$ consisting of a vector space H over $\mathbb{K}$ and a scalar product $(\cdot, \cdot)$ is called a pre-Hilbert space.

Remark 1.8. We defiend the scalar product to be linear in its second argument and anti-linear in its first argument.

Lemma 1.1 (Cauchy Schwarz Inequality)**.** *Let* $(H, (\cdot, \cdot))$ *be a pre-Hilbert space. Then*

$$|(x, y)|^2 \leq (x, x)(y, y).$$

Remark 1.9. The Cauchy-Schwarz inequality allows us to use the scalar product to define a norm on every pre-Hilbert space.

Corollary 1.2. *Let* $(H, (\cdot, \cdot))$ *be a pre-Hilbert space. Then*

$$\|x\| := \sqrt{(x, x)}$$

defines a norm on H*.*

Remark 1.10. The triangle inequality follows from the Cauchy-Schwarz inequality. The remaining properties follow from the properties of scalar products.

Definition 1.15. *A pre-Hilbert space is called a Hilbert space if* H*, equipped with the norm* $\|x\| = \sqrt{(x, x)}$ *induced by the scalar product, is a Banach space (i.e. if* H *is complete).*

Remark 1.11.

1. Every pre-Hilbert space can be completed into a Hilbert space.

2. Every Hilbert space is a metric space and therefore a topological space. However, clearly, not every Banach space is a Hilbert space, simply because not all norms on a vector space can be induced by a scalar product.

Theorem 1.4. *Let* $(H, (\cdot, \cdot))$ *be a Hilbert space,* $K \subset H$ *a closed convex set in* H *and* $x_0 \in H$*. Then there exists a unique* $y \in K$ *such that*

$$\|x_0 - y\| = dist(x_0, K) := \inf_{x \in K} \|x_0 - x\|$$

As an application of this theorem, we show that every Hilbert space H can be decomposed in the direct sum of an arbitrary closed subspace and of its orthogonal complement.

Theorem 1.5. *Let* $(H, (\cdot, \cdot))$ *be a Hilbert space and* $M \subset H$ *a closed linear subspace. Then the orthogonal complement* $M^\perp$ *of* M*, defined through*

$$M^\perp := \{x \in H : (x, m) = 0, \text{ for all } m \in M\}$$

is also a linear closed subspace of H *and* $H = M \oplus M^\perp$*, meaning that* $H = M + M^\perp$ *and* $M \cap M^\perp = \{0\}$*.*

Proof. Clearly $M^\perp$ is linear. In order to see that it is closed take x_n to be a sequence in $M^\perp$ such that $x_n \to x$ in H. Then we have

$$(x, m) = \lim_{n\to\infty} (x_n, m) = 0$$

because thanks to the Cauchy-Schwarz inequality we have

$$|(x - x_n, m)| \le \|x - x_n\| \|m\| \to 0 \text{ as } n \to \infty$$

in particular $|(x - x_n, m)| = |(x, m) - (x_n, m)| \to 0$ and it follows that $x \in M^\perp$, i.e. $M^\perp$ is closed.

Moreover, the fact that $M \cap M^\perp = \{0\}$ follows, because $(x, x) = 0$ implies that $x = 0$. Thus it only rmeains to show that $M + M^\perp = H$. To this end, we fix $x \in H$. Since $M \subset H$ is a closed linear subspace (and therefore in particular convex) we can apply the previous theorem to find $z \in M$ such that $\mathrm{dist}(x, M) = \|x - z\|$.

We now claim that $x - z \in M^\perp$, which gives that $x = z + (x - z)$ is the desired decomposition of H into M and $M^\perp$. Lets assume for contradiction that $(x - z) \notin M^\perp$. Then there exists $\alpha \in M$ with $(x - z, \alpha) > 0$. For $t \in [-1, 1]$ let $z_t = z + t\alpha$. Then we have $z_t \in M$ for all t and

$$\|x - z_t\|^2 = \|x - z\|^2 + t^2\|\alpha\|^2 - 2t(x - z, \alpha) < \|x - z\|^2 = \mathrm{dist}(x, M)$$

for $t > 0$ small enough. But this condradicts the definition of $\mathrm{dist}(x, M)$. $\square$

Definition 1.16. *An orthonormal system in $(H, (\cdot,\cdot))$ is a family $(x_\alpha)_{\alpha\in A} \subset H$ for an arbitrary index-set A with $(x_\alpha, x_\beta) = \delta_{\alpha,\beta}$. In the case when $A = \mathbb{N}$, we also call the orthonormal system an orthonormal sequence.*

Lemma 1.2. *Let H be a Hilbert space, $(x_n)_{n\in\mathbb{N}}$ an orthonormal system (orthonormal sequence) and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{K}$. Then we have*

1. *$\sum_{k=1}^\infty \alpha_k x_k$ converges if and only if $\sum_{k=1}^\infty |\alpha_k|^2$ converges.*
2. *$\|\sum_{k=1}^n \alpha_k x_k\|^2 = \sum_{k=1}^n |\alpha_k|^2$.*
3. *If $\sum_{k=1}^\infty \alpha_k x_k$ converges, then the limit is independent of the order of the terms.*

Lemma 1.3. *Let $(H, (\cdot,\cdot))$ be a Hilbert space, A an arbitrary set an $(x_\alpha)_{\alpha\in A}$ an orthonormal system in H. Then $\sum_{\alpha\in A}(x_\alpha, x)x_\alpha$ converges for every $x \in H$. Moreover, the linear map $\phi : H \to H$ defined through $\phi(x) = \sum_{\alpha\in A}(x_\alpha, x)x_\alpha$ is the continuous projection onto*

$$M := \overline{\mathit{span}\{x_\alpha : \alpha \in A\}} \text{ along its orthogonal complement } M^\perp$$

In particular for $x \in M$, we find that $x = \sum_{\alpha\in A}(x_\alpha, x)x_\alpha$.

Remark 1.12. In particular, if $M = \text{span}\{x_\alpha : \alpha \in A\}$ is dense in H, i.e. if $\overline{M} = H$, then gives us the previous Lemma a representation for every vector $x \in H$. In this case, we say that $(x_\alpha)_{\alpha \in A}$ is a Hilbert space basis.

Definition 1.17. *Let H be a Hilbert space. A Hilbert space basis is an orthonormal system $(x_\alpha)_{\alpha \in A}$ with*

$$\overline{\mathit{span}\{x_\alpha : \alpha \in A\}} = H$$

Theorem 1.6 (Characterizations of Hilbert space bases)**.** *Let H be a Hilbert space, and $(x_\alpha)_{\alpha \in A}$ an orthonormal system. Then the following statements are equivalent:*

1. *$(x_\alpha)_{\alpha \in A}$ is a Hilbert space basis.*
2. *$x = \sum_{\alpha \in A} (x_\alpha, x) x_\alpha$, for all $x \in H$.*
3. *$\|x\|^2 = \sum_\alpha |(x_\alpha, x)|^2$ for all $x \in H$.*
4. *$(x_\alpha, x) = 0$ for all $\alpha \in A$ implies that $x = 0$.*
5. *$(x_\alpha)_{\alpha \in A}$ is a maximal orthonormal system in the sense of inclusions.*

Remark 1.13. Using the maximality property, i.e. point 5 in the theorem above, it follows easily from the Lemma of Zorn that every pre-Hilbert space has a Hilbert space basis. In particular, it follows that every separable Hilbert space admits a countable orthonormal basis. The observation that separable Hilbert spaces have countable orthonormal bases can be used to identify separable Hilbert spaces with the sequence space $\ell^2(\mathbb{K})$.

Theorem 1.7. *Let H be an infinite dimensional separable Hilbert space over $\mathbb{K}$. Then there exists a linear Isomorphism $\phi : H \to \ell^2(\mathbb{K})$ with*

$$(\phi(x), \phi(y))_{\ell^2} = (x, y)_H$$

for all $x, y \in H$. In particular the isomorphism is isometric.

2 Function Spaces

2.1 Continuous Functions on Compact Spaces

Lemma 2.1. *Let $f \in C_{\mathbb{K}}(K)$ where K is compact. Then f is bounded and its supremum and infimum are attained.*

This allows us to define the sup/max norm of f on the space $C_{\mathbb{K}}(K)$.

Definition 2.1. *For $f \in C_{\mathbb{K}}(K)$ let*

$$\|f\| := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|$$

It is then simple to check that $\|\cdot\|$ defines a norm on $C_{\mathbb{K}}(K)$ (well-defined because of the above Lemma). Hence, the pair $(C_{\mathbb{K}}(K), \|\cdot\|)$ is a normed space.

Theorem 2.1. $(C_{\mathbb{K}}(K), \|\cdot\|)$ *is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $C_{\mathbb{K}}(K)$. Then, trivially we have for any $x \in K$ that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \to 0 \text{ as } n, m \to \infty$$

It follows that $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy Sequence in $\mathbb{K}$, but since $\mathbb{K}$ is complete we conclude that $f_n(x)$ converges. Let $f(x) := \lim f_n(x)$ denote its limit. We easily note that

$$|f(x)| \leq \limsup_{n \to \infty} |f_n(x)| \leq \limsup_{n \to \infty} \|f_n\|.$$

Moreover, since for every $x \in K$, we have

$$|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \leq \limsup_{m \to \infty} \|f_n - f_m\|$$

we conclude that

$$\sup_{x \in K} |f(x) - f_n(x)| \leq \limsup_{m \to \infty} \|f_m - f_n\| \to 0 \text{ as } n \to \infty.$$

since this convergences in uniform over all $x \in K$, we conclude that f is indeed continuous, thus our Cauchy sequence f_n converges to $f \in C_{\mathbb{K}}(K)$ and our space is complete. $\square$

Definition 2.2. $\mathbb{A} \subset C_{\mathbb{K}}(K)$ *is called a subalgebra if* $\mathbb{A}$ *is a linear subspace of* $C_{\mathbb{K}}(K)$ *and if, for every* $f, g \in \mathbb{A}$*, we also have* $f \cdot g \in \mathbb{A}$*. We say that the subalgebra* $\mathbb{A}$ *separates the points of* K*, if*

$$\forall x, y \in K \text{ with } x \neq y, \exists f \in \mathbb{A} : f(x) \neq f(y).$$

Remark 2.1. We already know that if $\mathbb{A} = C_{\mathbb{K}}(K)$ then indeed $C_{\mathbb{K}}(K)$ is a subalgebra separating the points of K.

Theorem 2.2 (Stone-Weierstrass Theorem, $\mathbb{K} = \mathbb{R}$)**.** *Let* $\mathbb{A}$ *be a subalgebra of* $C_{\mathbb{K}}(K)$ *separating the points of* K*. Then we have either* $\overline{\mathbb{A}} = C_{\mathbb{K}}(K)$ *or there exists a unique point* $x_0 \in K$ *such that* $\overline{\mathbb{A}} = \{f \in C_{\mathbb{K}}(K) : f(x_0) = 0\}$*.*

Remark 2.2. The Stone-Weierstrass Theorem allows us for example to approximate continuous functions on compact subsets of $\mathbb{R}^n$ through series of polynomials, see next example.

Example 2.1. *Let* $K \subset \mathbb{R}^n$ *be compact and* $\mathbb{A}$ *be the set of all polynomials in the variables* $x_1, \dots, x_n$*, i.e.*

$$\mathbb{A} = \left\{ p(x) = \sum_{\alpha : |\alpha| \leq n} b_\alpha x^\alpha, \;\; m \in \mathbb{N}, \;\; b_\alpha \in \mathbb{R} \right\}$$

It is then easy to check that $\mathbb{A}$ *is a subalgebra of* $C_{\mathbb{K}}(K)$*, separating the points of* K*, moreover that* $\{f \in C_{\mathbb{K}}(K) : f(x_0) = 0\}$ *cannot hold true, because evidently we have* $1 \in \mathbb{A}$ *and if we were in the second case, then there would exists* $x_0 \in K$ *such that* $0 = 1(x_0) = 1$ *which is a contradiction. Thus by Stone-Weierstrass we concludethat* $\overline{\mathbb{A}} = C_{\mathbb{K}}(K)$*.*

Theorem 2.3 (Stone-Weierstrass, $\mathbb{K} = \mathbb{C}$)**.** *Let* $\mathbb{A}$ *be a subalgebra of* $C_{\mathbb{C}}(K)$*, separating the points of* K *and such that for all* $f \in \mathbb{B}$ *we have* $\overline{f} \in \mathbb{A}$*. Then we either have* $\overline{\mathbb{A}} = C_{\mathbb{C}}(K)$ *or*

$$\exists! x_0 \in K \text{ with } \overline{\mathbb{A}} = \{f \in C_{\mathbb{C}}(K) : f(x_0) = 0\}.$$

2.2 Lebesgue Spaces

This should already be known, we will only repeat the most relevant results without much explanation in this section. We work on some measure space (Ω, Σ, μ) and consider the space

$$\tilde{L}^p(\Omega, d\mu) = \{f : \Omega \to \mathbb{R} : f \text{ measurable}, \int_\Omega |f|^p d\mu < \infty\}$$

Theorem 2.4 (Hölder's inequality)**.** *Let* $1 \leq p, q \leq \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* (Ω, Σ, μ) *be a measure space and* $f \in L^p(\Omega), g \in L^q(\Omega)$*, then* $fg \in L^1(\Omega)$ *and*

$$\left| \int_\Omega f g d\mu \right| \leq \int_\Omega |f||g| d\mu \leq \|f\|_p \cdot \|g|_q$$

Theorem 2.5 (Fischer-Riesz)**.** *Let* (Ω, Σ, μ) *be a measure space and* $1 \leq p \leq \infty$*. Then* $L^p(\Omega, \Sigma, \mu)$ *is complete.*

Remark 2.3. For $p = 2$ the Lebesgue space L^2 is even a Hilbert space with scalar product on $L^2(\Omega, \Sigma, \mu)$ given by

$$(f, g)_{L^2} = \int \overline{f(x)} g(x) d\mu(x)$$

an exercise from class entails that this is on the only Lebesgue Space that is a Hilbert Space, i.e. only for the case $p = 2$.

Definition 2.3. *Let* $(X, \|\cdot\|)$ *be a normed space over* $\mathbb{K}$*. We define the dual space* X^* *of* X *as the space of all continuous linear functionals on* X*, i.e.*

$$X^* := \{f : X \to \mathbb{K} \mid f \text{ is linear and continuous}\}$$

We will discuss this space in much more detail later on, for now we note that X^* *is always a Banach space w.r.t. the norm*

$$\|f\|_{X^*} := \sup_{x \in X : \|x\| = 1} |f(x)|$$

Theorem 2.6 (Duality of L^p spaces)**.** *Let* $1 \leq p < \infty, 1 < q \leq \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space. If* $p = 1$*, we assume additionally that the measure space* $(\Omega, \mathcal{A}, \mu)$ *is* σ*-finite. Then the map*

$$\phi : \begin{cases} L^q(\Omega, \mathcal{A}, \mu) & \longrightarrow (L^p(\Omega, \mathcal{A}, \mu))^* \\ f & \longmapsto \phi_f \end{cases}$$

with

$$\phi_f(g) = \int_\Omega \overline{f(x)} g(x) d\mu(x)$$

for all $g \in L^p(\Omega, \mathcal{A}, \mu)$ *is an anti-linear isometric isomorphism. In other words, we have*

$$L^p(\Omega, \mathcal{A}, \mu)^* \cong L^q(\Omega, \mathcal{A}, \mu)$$

Remark 2.4. The theorem determines the dual space of $L^p(\Omega, \mathcal{A}, \mu)$, it shows that every continuous linear functional L on $L^p(\Omega, \mathcal{A}, \mu)$ for $1 \leq p < \infty$ has the form

$$L(f) = \int_\Omega \overline{g(x)} f(x) d\mu(x)$$

for a $g \in L^q(\Omega, \mathcal{A}, \mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Our last goal in this section is to show that smooth functions are dense in L^p, for all $1 \leq p < \infty$. For simplicity we work on $\mathbb{R}^n$, then the next theorem shows how to approximate L^p-functions with sequences of C^∞-functions.

Theorem 2.7. *Let $j \in L^1(\mathbb{R}^n)$ with $\int j d\lambda_n(x) = 1$. For $\epsilon > 0$ we let $j_\epsilon(x) = \epsilon^{-1} j(x/\epsilon)$ so that $\int j_\epsilon d\lambda_n = 1$ for all $\epsilon > 0$. Let now $f \in L^p(\mathbb{R}^n)$ for a $1 \leq p < \infty$ and set $f_\epsilon = f * j_\epsilon$. Then $f_\epsilon \in L^p(\mathbb{R}^n)$ with $\|f_\epsilon\|_p \leq \|j\|_1 \|f\|_p$ and $f_\epsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$. If $j \in C_c^\infty(\mathbb{R}^n)$, then we have $f_\epsilon \in C^\infty(\mathbb{R}^n)$ and $D^\alpha f_\epsilon = (D^\alpha j_\epsilon) * f$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.*

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ open, $K \subset \Omega$ compact. Then there exists $J_K \in C_c^\infty(\Omega)$ with $0 \leq J_K(x) \leq 1$ for all $x \in \Omega$, $J_K(x) = 1$ for all $x \in K$. Therefore, there exists a sequence $(g_j)_{j \in \mathbb{N}}$ in $C_C^\infty(\Omega)$ with $0 \leq g_j(x) \leq 1$ for all $j \in \mathbb{N}$ and $\lim g_j(x) = 1$ for all $x \in \Omega$. Hence if $(f_j)_{j \in \mathbb{N}}$ is a sequence in $C^\infty(\Omega)$ with $f_j \to f$ in $L^p(\Omega)$ for $1 \leq p < \infty$, then $g_j f_j \in C_c^\infty(\Omega)$ and $g_j f_j \to f$ in $L^p(\Omega)$. The Theorem above implies therefore that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \leq p < \infty$.*

As an application of the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ we have the following theorem.

Theorem 2.8. *For every measurable $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \infty$, the Banach space $L^p(\Omega)$ is seperable.*