Functional Analysis

Benjamin Schlein

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These notes are a summary of the class “Functional analysis” taking place in Fall 2017 at the University of Zurich. They are taken from different sources, including the books:

- H.W. Alt, Lineare Funktionalanalysis, Springer.
- E.H. Lieb, M. Loss. Analysis. 2nd Edition. Graduate Studies in Mathematics (AMS).
- B. Bollobas. Linear Analysis. 2nd Edition. Cambridge Mathematical Textbooks, Cambridge University Press.

Inhaltsverzeichnis

1 Structures

In this chapter, we are going to introduce spaces and notions that play an important role in analysis.

We start with topological spaces, and then we continue with metric space, normed spaces, Banach spaces, to conclude with Hilbert spaces.

1.1 Topological Spaces

Definition 1.1.1. *A topological space is a pair (X, τ), consisting of a set X and a family $\tau \subset 2^X$ such that*

i) $\emptyset, X \in \tau$.

ii) $U_1, \ldots, U_n \in \tau$, $n \in \mathbb{N}$. *Then* $\bigcap_{i=1}^n U_i \in \tau$.

iii) Λ *an arbitrary set,* $U_\lambda \in \tau$, *for all* $\lambda \in \Lambda$. *Then* $\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$.

Remarks:

- $2^X = \{A : A \subset X\}$ is the power set of X, i.e. the set of all subsets of X.
- τ is called a topology on X. $A \subset X$ is called an open set if $A \in \tau$. $A \subset X$ is called closed, if $A^c = X \backslash A$, the complement of A, is open.
- ii) and iii) state that finite intersections and arbitrary unions of open sets are open again. With
$$\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$$
we conclude that arbitrary intersections and finite unions of closed sets are closed.

Examples:

- For any set X, $\tau = \{\emptyset, X\}$ and also $\tau = 2^X$ are always topologies (they are, respectively, the smallest and the largest topology on X).
- For $X = \mathbb{R}^n$, we say that $A \subset X$ is offen, if, for all $x \in A$, there exists $\varepsilon > 0$ with $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x - y| < \varepsilon\} \subset A$. This defines a topology τ, the standard topology on $\mathbb{R}^n$, satisfying the axioms i),ii),iii) in the definition above (Proof: exercise).
- For $X = \mathbb{R}$,
$$\tau = \{U \subset X : U = \emptyset \text{ oder } \quad U^c \text{ ist abzählbar}\}$$
defines another topology on X (Proof: exercise).

- (Subspace topology) Let (X, τ) be a topological space and $Y \subset X$. We define then the subspace topology
$$\tau_Y := \{Y \cap A : A \in \tau\}$$
With this definition, one can check that (Y, τ_Y) is again a topological space.

- (Product topology) Let (X, τ), $(Y, \mathcal{S})$ be topological spaces. On the product space $X \times Y = \{(x, y) : x \in X, y \in Y\}$ we can define the product topology τ_{prod} as follows: $W \in \tau_{\text{prod}}$ if and only if
$$\forall\, (x, y) \in W\ \exists U \in \tau, V \in \mathcal{S}\ :\ (x, y) \in U \times V \subset W$$
Equivalently, $W \in \tau_{\text{prod}}$, if W can be written as union of sets of the form $U \times V$, $U, V \in \tau$ (in this case, we say that sets of the form $U \times V$ are a basis for the topology τ_{prod}).

Next, we define the closure and the interior of a set $A \subset X$, for X a topological space.

Definition 1.1.2. *Let (X, τ) be a topological space, and $A \subset X$ a subset. The closure $\overline{A}$ of A is defined by*
$$\overline{A} = \bigcap \{B \subset X : B^c \in \tau \text{ and } A \subset B\}$$
In other words, $\overline{A}$ is the smallest closed set containing A. The set A is called dense in X if $\overline{A} = X$. The space X is called separable, if it contains a countable dense set.

Furthermore, the interior of A is defined through
$$\mathring{A} = \bigcup \{B \subset A : B \in \tau\}$$
In other words, $\mathring{A}$ is the largest open set contained in A. Finally, the boundary of A is defined as $\partial A = \overline{A} \backslash \mathring{A}$.

Remark: the definition of closure and interior depend on the choice of the topology. For example, if $\tau = \{\emptyset, X\}$, we have $\overline{A} = X$, $\mathring{A} = \emptyset$ for all $A \neq \emptyset, X$ (in particular, every non-empty set is dense). On the other hand, if $\tau = 2^X$ we have $\overline{A} = \mathring{A} = A$ for all $A \subset X$ (in this case, only the set X is dense in X and therefore X is separable if and only if it is countable).

The topology allows us to introduce the basic analytic notions, like convergence of sequences and continuity of functions, on topological spaces. To this end, we need to define first the idea of open neighbourhoods of points in X.

Definition 1.1.3. *Let (X, τ) be a topological space, $x \in X$. A set $U \subset X$ is an open neighbourhood of x if $U \in \tau$ and $x \in U$.*

For an arbitrary index set Λ, the family $\{U_\lambda\}_{\lambda \in \Lambda}$ of open neighbourhoods of x is called a neighbourhood-basis at $x \in X$ if
$$V \subset X \, \textit{open neighbourhood of } x \quad \Rightarrow \quad \exists \lambda \in \Lambda : U_\lambda \subset V$$

Remark: an equivalent definition of the closure $\overline{A}$ of a set $A \subset X$ is given in terms of open neighbourhoods as follows (proof: exercise):

$$\overline{A} = \{x \in X : U_x \cap A \neq \emptyset, \quad \text{for all open neighbourhoods } U_x \text{ of } x\}$$

Next, we define the notion of convergence of sequences on topological spaces.

Definition 1.1.4. *Let (X, τ) be a topological space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. We say that x_n converges to $x \in X$ as $n \to \infty$, written $x_n \to x$, if*

$$\forall \textit{ open neighbourhood } U \textit{ of } x\,,\ \exists\, n_0 \in \mathbb{N} : x_n \in U\ \forall\, n \geq n_0$$

Clearly, it is enough to check this condition for open neighbourhoods in a neighbourhood basis $\{U_\lambda\}_{\lambda\in\Lambda}$ at x.

Continuity and continuity at a point x are defined as follows:

Definition 1.1.5. *Let (X, τ), $(Y, \mathcal{S})$ be two topological spaces. A function $f : X \to Y$ is called continuous if*

$$f^{-1}(V) \in \tau,\ \forall\, V \in \mathcal{S},$$

i.e. if the pre-image of every open set in Y is an open set in X.

For $x \in X$, we say that f is continuous at x, if

$$\forall \textit{ open neighbourhood } V \textit{ of } f(x), \exists \textit{ open neighbourhood } U \textit{ of } x : f(U) \subset V.$$

Then, it is possible to check that f is continuous if and only if f is continuous at x, for all $x \in X$ (proof: exercise).

Warning: in metric spaces, the notion of convergence completely characterizes the topology. A subset A of a metric space X is closed if and only if it contains all limits of sequences in A. In other words, in this case, A is closed if and only if, for every sequence $x_n \in A$ with $x_n \to x$ in X, we have $x \in A$. This characterization of the topology is not true on general topological spaces. In general, on a topological space X we can only conclude that every closed set A contains all limits of sequences in A. The opposite implication is not always true. In other words, on topological spaces there may exist sets A that are not closed but, nevertheless, contain all limits of sequences in A.

As a consequence of this remark, it follows that, on general topological spaces, the closure of a set A is not the same as the set of all limits of sequences in A, as it is the case on metric spaces). On general topological spaces, we only have the inclusion $\overline{A} \supset \{x : \exists (x_n) \in A \text{ mit } x_n \to x\}$. In particular, A being dense in X does not imply for all $x \in X$ there exists a sequence x_n in A with $x_n \to x$, as it is the case on metric spaces.

Similarly, on general topological spaces, continuity cannot be characterized by the notion of convergence. If $f : X \to Y$ is continuous, it follows namely that $f(x_n) \to f(x)$ for all sequences x_n in X with $x_n \to x$. The converse is not correct; i.e. the condition $f(x_n) \to f(x)$ for all sequences x_n in X with $x_n \to x$ does not imply that f is continuous.

The important difference between general topological spaces and metric spaces is the fact that, on metric spaces, every point has a countable neighbourhood basis (take for example the basis $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$, consisting of open balls around x). If we assume that X is a topological space so that for every $x \in X$ there exists a countable neighbourhood basis $\{U_n\}_{n\in\mathbb{N}}$ at x, then the characterization of the topology through the notion of convergence, as we know it for metric spaces, holds true. In this case, the closure of a set A is the set of limits of sequences in A, a set A is dense in X if and only if every $x \in X$ can be written as limit of a sequence in A, a function f on X is continuous if and only if $f(x_n) \to f(x)$ for all sequences x_n in X with $x_n \to x$.

Comparison of topologies: Let τ_1, τ_2 be two topologies on a set X. We say that τ_1 is stronger as τ_2, or equivalentely that τ_1 is a refinement of τ_2, if $\tau_2 \subset \tau_1$. In this case, (X, τ_1) contains more open sets than (X, τ_2). As a consequence, it is more difficult for a sequence to converge in (X, τ_1) than in (X, τ_2). In other words,

$$x_n \to x \quad \text{w.r.t. } \tau_1 \quad \Rightarrow \quad x_n \to x \quad \text{w.r.t. } \tau_2$$

In particular, if $\tau = \{\emptyset, X\}$ every sequence converges towards every limit, while, if $\tau = 2^X$, $x_n \to x$ if and only if $x_n = x$ for all sufficiently large $n \in \mathbb{N}$.

Also the notion of continuity depends on the choice of the topology. If τ_1, τ_2 are two topologies on X and if τ_1 is stronger than τ_2, then continuity of $f : X \to Y$ w.r.t. τ_2 implies that f is continuous w.r.t. τ_1. On the other hand, if σ_1, σ_2 are two topologies on Y and if σ_1 is stronger than σ_2, then continuity w.r.t. σ_1 implies continuity w.r.t. σ_2. In particular, if $\tau = 2^X$ or if $\sigma = \{\emptyset, Y\}$, then every function $f : (X, \tau) \to (Y, \sigma)$ is continuous. On the other hand, if $\tau = \{\emptyset, X\}$ (and if the topology on Y can separate points) only constants functions are continuous.

The pathological example $\tau = \{\emptyset, X\}$ shows that to have a useful notion of convergence and of continuity, it is important to make sure that there are sufficiently many open sets (i.e. that the topology is large enough). To this end, we introduce the notion of Hausdorff spaces and of compact spaces.

Definition 1.1.6. *A topological space (X, τ) is called Hausdorff if*

$$x, y \in X,\, x \neq y \;\Rightarrow\; \exists\, U_x, U_y \in \tau \text{ with } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset$$

Among Hausdorff spaces, compact spaces play an especially important role.

Definition 1.1.7. *A topological space (X, τ) is called compact if it is Hausdorff and if*

$$\{U_\lambda\}_{\lambda\in\Lambda} \text{ family in } \tau \text{ wiit } \bigcup_{\lambda\in\Lambda} U_\lambda = X \;\Rightarrow\; \exists\, n \in \mathbb{N} \text{ and } \lambda_1, \dots, \lambda_n \in \Lambda : \bigcup_{j=1}^{n} U_{\lambda_j} = X,$$

i.e. if every open covering of X contains a finite sub-covering.

Remark: let (X,τ) be a compact space, $A \subset X$ closed. Let τ_A be the subspace topology induced by τ on A. Then (A,τ_A) is again a compact space (proof: exercise).

Warning: on metric spaces, the notions of covering- and of sequential compactness are equivalent. I.e. a subset A of a metric space X is covering compact (meaning that every open covering of A has a finite sub-covering) if and only if every sequence on A has a convergent subsequence. This is not true on general topological spaces. What is true on a general topological space X is that if X is compact (according to the definition given above, i.e. covering compact) then every sequence on X has at least one accumulation point (i.e. a point for which every neighbourhood contains infinitely many elements of the sequence).

Theorem 1.1.8. *Let (X,τ) be a compact space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then there exists at least one accumulation point of x_n on X.*

Proof. Suppose that x_n has no accumulation points in X. Then, for every $y \in X$ we can find an open neighbourhood U_y of y containing at most finitely many elements of x_n. $\{U_y\}_{y\in X}$ is an open covering of X. Hence there exists $m \in \mathbb{N}$ and $y_1, \dots, y_m \in X$, s.t. $U_{y_1} \cup \dots \cup U_{y_m} = X$. This is a contradiction to the fact that each U_y contains only finitely many points from x_n. $\square$

Next, we want to show that on compact spaces, continuous functions separate points (i.e. if X is a compact space and $x, y \in X$ with $x \neq y$, then there exists a continuous function $f : X \to \mathbb{R}$ with $f(x) = f(y)$). In particular, this implies that on compact spaces the topology is large enough to avoid some of the pathological situations considered above (like only constant functions being continuous). To reach this goal, we need few lemmas.

Lemma 1.1.9. *Let (X,τ) be a compact topological space. Let $x \in X$, U an open neighbourhood of x. Then there exists another open neighbourhood V of x with $\overline{V} \subset U$.*

Proof. For every $y \in X\backslash U$ we choose an open neighbourhood W_y of y and an open neighbourhood V_y of x wiit $W_y \cap V_y = \emptyset$. The family $\{\{W_y\}_{y\in X/U}, U\}$ is then an open covering of X. Hence, there exist $n \in \mathbb{N}$ and $y_1, \dots, y_n \in X\backslash U$ with

$$X = U \cup \bigcup_{j=1}^{n} W_{y_j} \, .$$

Let $W := \bigcup_{j=1}^n W_{y_j}$ and $V := \bigcap_{j=1}^n V_{y_j}$. Then V is an open neighbourhood of of x. Furthermore, W is open with $W \cap V = \emptyset$. It follows that W^c is closed, with $V \subset W^c$, and thus

$$\overline{V} \subset W^c \subset U$$

$\square$

Lemma 1.1.10. *Let (X,τ) be a compact space and $U, V \in \tau$ with $\overline{V} \subset U$. Then there exists $W \in \tau$ with $\overline{V} \subset W \subset \overline{W} \subset U$.*

Proof. For every $x \in \overline{V}$ there exists, from Lemma 1.1.9, an open neighbourhood $W_x \in \tau$ of x with $\overline{W}_x \subset U$. The family $\{\{W_x\}_{x\in\overline{V}}, X\backslash\overline{V}\}$ is then an open covering of X. Hence, there are $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \overline{V}$ with

$$(X\backslash\overline{V}) \cup \bigcup_{j=1}^{n} W_{x_j} = X$$

Let $W = \bigcup_{j=1}^{n} W_{x_j}$. The set W is open and contains $\overline{V}$. Thus, $\overline{W} \subset \bigcup_{j=1}^{n} \overline{W}_{x_j} \subset U$. □

Remark: A topological space (X, τ) with the property that, for every $U, V \in \tau$ with $\overline{V} \subset U$ there exists $W \in \tau$ with $\overline{V} \subset W \subset \overline{W} \subset U$ is called a normal space. Lemma 1.1.10 shows that every compact space is normal. Also next theorem can be extended to arbitrary normal Hausdorff spaces (compactness is only used here to guarantee normality).

Theorem 1.1.11 (Lemma von Urysohn). *Let (X, τ) be a compact space and $A, B \subset X$ disjoint, non-empty, closed subsets of X. Then there exists a continuous function $g : X \to [0, 1]$ with $A \subset g^{-1}(0)$, $B \subset g^{-1}(1)$ (we equip the interval $[0, 1] \subset \mathbb{R}$ with the standard topology).*

Proof. Choose $U_{1/2} \in \tau$ with $A \subset U_{1/2} \subset \overline{U}_{1/2} \subset B^c$ (here we use Lemma 1.1.10). Next, we choose $U_{1/4}, U_{3/4} \in \tau$, s.t.

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset B^c$$

By iteration, we define U_λ for all

$$\lambda \in \Lambda = \{m/2^n : m = 1, \dots, 2^n - 1 \quad \text{and} \quad n \in \mathbb{N}\}.$$

Furthermore, we set $U_1 := B^c$. We define a function $f : X \to [0; 1]$ by setting

$$f(x) = 1 \qquad \text{if } x \in X\backslash \bigcup_{\lambda\in\Lambda\cup\{1\}} U_\lambda$$

and

$$f(x) = \inf_{x\in U_\lambda} \lambda \qquad \text{if } x \bigcup_{\lambda\in\Lambda\cup\{1\}} U_\lambda$$

By construction, $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. We prove that f is continuous. In fact, fix $x \in X$ and $\varepsilon > 0$. Then we find an open neighbourhood U of x with $f(U) \subset (f(x) - \varepsilon; f(x) + \varepsilon)$. To this end, we distinguish three cases:

1) $x \in A$. We choose λ_0 with $0 < \lambda_0 < \varepsilon$. Then U_{λ_0} is an open neighbourhood of x and $f(U_{\lambda_0}) \subset (-\varepsilon, \varepsilon)$.

2) $x \in X\backslash \bigcup_\lambda U_\lambda$. Choose λ_0 with $1 - \varepsilon < \lambda_0 < 1$ and set $U = X\backslash\overline{U}_{\lambda_0}$. Then $f(U) \subset (1 - \varepsilon, 1 + \varepsilon)$.

3) $x \in U_\lambda \backslash A$ (for a $\lambda \in \Lambda$). Choose λ_0, λ_1 with $f(x) - \varepsilon < \lambda_0 < f(x) < \lambda_1 < f(x) + \varepsilon$ and set $U = U_{\lambda_1} \backslash \overline{U}_{\lambda_0}$. Then $f(U) \subset (\lambda_0; \lambda_1) \subset (f(x) - \varepsilon; f(x) + \varepsilon)$.

□

Definition 1.1.12. *Let K be a compact space. We define*

$$C_{\mathbb{K}}(K) = \{f : K \to \mathbb{K} \text{ continuous } \}$$

Here $\mathbb{K} = \mathbb{R}$ for real valued functions on K, or $\mathbb{K} = \mathbb{C}$ for complex-valued functions on K. In both cases, $\mathbb{K}$ is equipped with the standard topology.

It follows from Theorem 1.1.11 that $C_{\mathbb{K}}(K)$ separates the points of K.

Corollary 1.1.13. *Let K be a compact space. Then $C_{\mathbb{K}}(K)$ separates the points of K. In other words, for every $x, y \in K$, with $x \neq y$, there exists $f \in C_{\mathbb{K}}(K)$ with $f(x) \neq f(y)$.*

Proof. Since K is Hausdorff, for every $x \neq y$ we can find two open neighbourhoods U_x, U_y of x and, respectively, y, with $U_x \cap U_y = \emptyset$. From Lemma 1.1.9 there exist closed sets $A, B \subset K$ with $A \subset U_x$, $B \subset U_y$ and $x \in A$, $y \in B$. In particular, $A \cap B = \emptyset$. Theorem 1.1.11 implies that there exists $f \in C_{\mathbb{K}}(K)$ with $f(x) = 0$ and $f(y) = 1$. □

We are going to study properties of the function space $C_{\mathbb{K}}(K)$ in more details in Section 2.1 below.

1.2 Metric Spaces

Definition 1.2.1. *A metric space is a pair (X, d), with X a set and d a map $d : X \times X \to [0, \infty)$, called metric, with the following properties*

i) $d(x, y) = 0$ *if and only if* $x = y$.

ii) $d(x, y) = d(y, x)$.

iii) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Induced topology: Every metric space (X, d) is a topological space (X, τ_d) with the topology τ_d induced by the metric d. The topology τ_d is defined by the condition that $A \in \tau_d$ if and only, for all $x \in A$, there exists $\varepsilon > 0$ with $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\} \subset A$. It has been proven in previous classes that τ_d really satisfies the axioms of a topology.

Using the induced topology τ_d, it is possible to translate some properties of topological spaces into properties of metric spaces. For example, we say that a metric space (X, d) is compact if and only if (X, τ_d) is compact. Similarly, we say that (X, d) is separable if and only if (X, τ_d) is separable.

Examples:

- $X = \mathbb{K}^n$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and the metric

$$d(x, y) = \left(\sum_{j=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

is a metric space.

- The space of sequences

$$\ell^2(\mathbb{K}) = \{(x_1, x_2, \dots) : x_j \in \mathbb{K} \text{ for all } j \in \mathbb{N}, \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$$

with metric

$$d(x, y) = \left(\sum_{j=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$$

is a metric space.

- $X = \mathbb{K}^n$ with the discrete metric $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$ is also a metric space. The corresponding topology is $\tau_d = 2^X$.

Convergence of sequences and continuity on metric spaces: the topology induced by the metric can be used to define the notions of convergence and continuity. It turns out that, on a metric space (X, d), a sequence x_n converges to $x \in X$ if and only if

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : d(x_n, x) < \varepsilon \; \forall n \geq n_0.$$

In contrast with general topological spaces, on metric spaces the notion of convergence characterizes the topology. In other words, on a metric space we can decide whether a set is closed (or whether a set is open) just by knowing which sequences converge to which limit (a set A is closed if and only if for every sequence x_n in A with $x_n \to x$ in X, we have $x \in A$, i.e. if and only if A contains all limits of sequences in A). What makes this possible (as already explained in the section on topological spaces) is the fact that on metric spaces each point has a countable neighbourhood basis (given for example by the family $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$).

It follows that, on metric spaces,

$$\overline{A} = \{x \in X : \; \exists \, (x_n)_{n\in\mathbb{N}} \text{ sequence in } A \text{ with } x_n \to x \text{ as } n \to \infty\}$$

and therefore $A \subset X$ is dense if and only if

$$\forall \, x \in X, \exists \, (x_n)_{n\in\mathbb{N}} \text{ sequence in } A \text{ with } x_n \to x \text{ as } n \to \infty \, .$$

We leave the proof of these two statements as an exercise.

Also the notion of continuity can be expressed, on metric spaces, through convergence of sequences. Let $(X, d_1), (Y, d_2)$ be two metric spaces. The function $f : X \to Y$ is continuous at the point $x \in X$ if and only if

$$(x_n)_{n\in\mathbb{N}} \text{ Folge in } X \text{ mit } x_n \to x \;\Rightarrow\; f(x_n) \to f(x).$$

Moreover, $f : X \to Y$ is continuous, if and only if it is continuous at x, for all $x \in X$.

While every metric induces a topology (and therefore every metric space can be thought of as a topological space), given a topology τ on a set X it is not always possible to find a metric d on X so that $\tau = \tau_d$. In other words, not all topologies are metrizable. For example, it is clear that all topologies that are induced by a metric are Hausdorff. Hence, non-Hausdorff topologies are never metrizable. Take, for example, the topology

$$\tau = \{U \subset \mathbb{R} : U = \emptyset \text{ or } U^c \text{ is countable}\}$$

on $X = \mathbb{R}$. Then τ is not Hausdorff and therefore certainly not metrizable (proof: exercise).

Another property that metrizable topologies always have is the existence, at every point, of a countable neighbourhood basis (which is important, as we discussed above, to make sure that the topology is determined through the notion of convergence). The question of determining sufficient conditions for a topology to be metrizable is a difficult question. It turns out that additionally to being Hausdorff and having a countable neighbourhood basis at every point, some local form of compactness, called paracompactness, is needed for a topology to be metrizable. We will not discuss this issue in these notes.

Definition 1.2.2. *Let (X, d) be a metric space. A sequence (x_n) in X is called a Cauchy sequence (or it is said to have the Cauchy property) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Every convergent sequence is Cauchy. The metric space (X, d) is called complete, if every Cauchy sequence is convergent.*

Example: $\mathbb{Q}$ with $d(x, y) = |x - y|$ is a metric space, but it is not complete (take a sequence in $\mathbb{Q}$ that converge to $\sqrt{2}$ in $\mathbb{R}$; in $\mathbb{Q}$, this sequence is Cauchy, but it does not converge). $\mathbb{R}$ is instead a complete metric space (actually $\mathbb{R}$ is defined as the completion of $\mathbb{Q}$; the notion of completion will be explained later when we will discuss normed spaces). Also $\mathbb{C}$, equipped with the metric $d(x, y) = |x - y|$, is complete. It follows easily (since a sequence in $\mathbb{R}^n$ or in $\mathbb{C}^n$ converges if and only if its components converge) that $\mathbb{R}^n, \mathbb{C}^n$, equipped with $d(x, y) = \left(\sum_{j=1}^n |x_i - y_i|^2\right)^{1/2}$, are complete metric spaces.

Notice, that the Cauchy property can only be defined on metric space. It does not make sense to ask whether a sequence on a topological space is Cauchy.

1.3 Normed Spaces

Definition 1.3.1. *A normed space is a pair $(X, \|\cdot\|)$, consisting of a vector space X over a field $\mathbb{K}$ (as usual $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$), and a map $\|\cdot\| : X \to [0,\infty)$ with*

i) $\|x\| = 0$ *if and only if* $x = 0$.

ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{K}, x \in X$.

iii) $\|x + y\| \leq \|x\| + \|y\|$, *for all* $x, y \in X$.

The map $\|\cdot\|$ satisfying the axioms i), ii), iii), is called a norm on X. Every norm induces the metric $d(x, y) = \|x - y\|$; hence, every normed space is also a metric space and therefore also a topological space. Conversely, a metric d defined on a vector space X defines a norm (through $\|x\| = d(x, 0)$) if and only if it is homogeneous (i.e. $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $\lambda \in \mathbb{K}, x, y \in X$) and translation invariant (i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$). It is easy to check that the topology induced by the norm on the vector space X always has two important properties: the vector space operations (i.e. sum and scalar multiplication) are continuous and all one-point sets $\{x\}$ are closed. A topology defined on a vector space with these two properties (continuity of sum and scalar multiplication and closedness of one-point sets) is called a vector space topology. A vector space, equipped with a vector space topology, is called a topological vector space. Every normed space is automatically a topological vector space. Not every topological vector space is a normed space (because not every vector space topology is induced by a norm); it turns out that every "locally bounded" and "locally convex" topological vector space is normable. We will not go further in this direction.

Definition 1.3.2. *A normed space (X, d) is called complete, if X, equipped with the induced metric $d(x, y) = \|x - y\|$, is a complete metric space. A complete normed space is called a Banach space.*

Examples:

- $\mathbb{R}^n$, $\mathbb{C}^n$, equipped with the euclidean norm $\|x\|^2 = \sum_{j=1}^n |x_j|^2$, are examples of Banach spaces.
- Let K be a compact topological space. Then, as we will show in the next chapter, $C_{\mathbb{K}}(K)$, equipped with the norm $\|f\| := \max_{x \in K} |f(x)|$, is a Banach space.
- If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces, then so is also the product space $X \times Y = \{(x, y) : x \in X, y \in Y\}$, equipped with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$ (or with $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$). Proof: exercise.

- For $1 \leq p < \infty$, we define the spaces of sequences

$$\ell^p(\mathbb{K}) = \{x = (x_1, x_2, \dots,) : x_j \in \mathbb{K} \quad \forall j \text{ and } \sum_{j=1}^{\infty} |x_j|^p < \infty\} \tag{1.1}$$

equipped with the norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}$$

Then $\ell^p(\mathbb{K})$ is a Banach space, for all $1 \leq p < \infty$. In fact, it is easy to check that the map $\|.\|_p$ satisfies the axioms i) and ii). The triangle inequality holds true as well (proof: exercise). It remains to show that $\ell^p(\mathbb{K})$ is complete. To this end, let $x_n = (x_n^1, x_n^2, \dots)$ be a Cauchy sequence in $\ell^p(\mathbb{K})$. For every fixed $\ell \in \mathbb{N}$, we have

$$|x_n^\ell - x_m^\ell| \leq \|x_n - x_m\|_p \to 0$$

as $n, m \to \infty$. Hence, for every fixed $\ell \in \mathbb{N}$, x_n^ℓ defines a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, the limit

$$x^\ell = \lim_{n \to \infty} x_n^\ell$$

exists, for all $\ell \in \mathbb{N}$. We set $x = (x^1, x^2, \dots)$ and we claim that $x \in \ell^p(\mathbb{K})$. In fact, for any fixed $m \in \mathbb{N}$,

$$\sum_{j=1}^{m} |x^j|^p = \lim_{n \to \infty} \sum_{j=1}^{m} |x_n^j|^p \leq \limsup_{n \to \infty} \sum_{j=1}^{\infty} |x_n^j|^p = \limsup_{n \to \infty} \|x_n\|_p^p$$

Since the r.h.s. is finite (Cauchy sequences are always bounded), independently of $m \in \mathbb{N}$, we conclude that $x \in \ell^p(\mathbb{K})$. Furthermore, we claim that $x_n \to x$ in $\ell^p(\mathbb{K})$. In fact, for any fixed $m, r \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{\ell=1}^{m} |x^\ell - x_n^\ell|^p &\leq C_p \left(\sum_{\ell=1}^{m} |x^\ell - x_r^\ell|^p + \sum_{\ell=1}^{m} |x_r^\ell - x_n^\ell|^p \right) \\ &\leq C_p \left(\sum_{\ell=1}^{m} |x^\ell - x_r^\ell|^p + \|x_r - x_n\|_p^p \right) \end{aligned}$$

We let $r \to \infty$ (the left hand side is independent of r), keeping $m, n \in \mathbb{N}$ fixed. We obtain

$$\sum_{\ell=1}^{m} |x^\ell - x_n^\ell|^p \leq C_p \limsup_{r \to \infty} \|x_r - x_n\|_p^p$$

Since the right hand side does not depend on $m \in \mathbb{N}$, we can let $m \to \infty$. We find

$$\|x - x_n\|_p^p \leq C_p \limsup_{r\to\infty} \|x_r - x_n\|_p^p$$

The claim follows, because $\limsup_{r\to\infty} \|x_r - x_n\|_p^p \to 0$, as $n \to \infty$ (this follows from the Cauchy property). Hence, $\ell^p(\mathbb{K})$ is complete, as claimed.

Completion of normed spaces: Completeness is very important for analysis. It is not a coincidence that we always do analysis on $\mathbb{R}$ instead of $\mathbb{Q}$. For this reason, it is useful to have a general recipe to complete normed spaces.

Definition 1.3.3. *Let $(X, \|\cdot\|)$ be a normed space. A completion of $(X, \|\cdot\|)$ is a 3-tuple $(Y, \|\cdot\|_Y, \phi)$ consisting of a Banach space $(Y, \|\cdot\|_Y)$ and an isometric linear map $\phi : X \to Y$, with $\overline{\phi(X)} = Y$.*

Theorem 1.3.4. *Every normed space $(X, \|\cdot\|)$ has a completion, which is unique, up to linear isometric isomorphisms.*

Proof. We construct the completion explicitly. At the end, we show its uniqueness. Let $\mathcal{C}_X$ denote the set of all Cauchy sequences on X. For $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \mathcal{C}_X$ and $\lambda \in \mathbb{K}$, we define

$$x + y = (x_n + y_n)_{n\in\mathbb{N}}, \qquad \lambda x = (\lambda x_n)_{n\in\mathbb{N}}$$

With these operations, $\mathcal{C}_X$ has the structure of a vector space over $\mathbb{K}$. We define the linear subspace $\mathcal{N}_X \subset \mathcal{C}_X$ consisting of all null-sequences on X, i.e.

$$\mathcal{N}_X = \{x = (x_n)_{n\in\mathbb{N}} \in \mathcal{C}_X : x_n \to 0 \quad \text{as } n \to \infty\}$$

Moreover, we define $Y := \mathcal{C}_X/\mathcal{N}_X$ as the quotient space of $\mathcal{C}_X$ w.r.t. to the equivalence relation defined by $x \sim y :\Leftrightarrow x - y \in \mathcal{N}_X$. In other words, Y is the space of all equivalence classes

$$[x] = \{\tilde{x} = (\tilde{x}_n)_{n\in\mathbb{N}} : x_n - \tilde{x}_n \to 0 \quad \text{as } n \to \infty\}$$

(in Y, we identify Cuachy sequences whose difference converges to zero). Y is also a vector space over $\mathbb{K}$, w.r.t. the operations $[x] + [y] = [x + y]$ and $\lambda[x] = [\lambda x]$. We introduce now a norm over Y. To this end, we define the function $p : \mathcal{C}_X \to [0, \infty)$ through

$$p(x) = \lim_{n\to\infty} \|x_n\| \quad \text{ falls } x = (x_n)_{n\in\mathbb{N}}\,. \tag{1.2}$$

Notice that, for $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{C}_X$,

$$|\|x_k\| - \|x_\ell\|| \leq \|x_k - x_\ell\| \to 0$$

as $\ell, k \to \infty$. This implies in particular, that the limit in (1.2) is well defined. We set $\|[x]\|_Y := p(x)$ and we claim, that $\|\cdot\|_Y$ is a norm on Y. First of all, $\|\cdot\|_Y$ is well-defined because, if $x, y \in \mathcal{C}_X$ with $x \sim y$, then $x - y$ is a null-sequence and therefore $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$. Secondly, it is easy to check that $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = |\lambda| p(x)$ and that $p(x) = 0$ if and only if $x \in \mathcal{N}_X$. This implies that $\|\cdot\|_Y$ is a norm on Y. We also define the map $\phi : X \to Y$ through $\phi(z) = [(z, z, \dots)]$ (in other words, $\phi(z)$ denotes the equivalence class of all sequences on X that converge to z in the limit). The map ϕ is clearly linear and, since

$$\|\phi(z)\|_Y = \|z\|_X,$$

it defines an isometry. To show that $(Y, \|\cdot\|_Y, \phi)$ is a completion of $(X, \|\cdot\|_X)$, we still have to prove that $(Y, \|\cdot\|_Y)$ is complete and that $\phi(X)$ is dense in Y. Let us begin with the second statement. Let $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{C}_X$. For $\varepsilon > 0$, we find $k_0 \in \mathbb{N}$ with $\|x_\ell - x_k\| < \varepsilon$ for all $\ell, k \geq k_0$. Hence

$$\|\phi(x_{k_0}) - [x]\|_Y = \|[(x_{k_0} - x_n)_{n\in\mathbb{N}}]\|_Y = \lim_{n\to\infty} \|x_{k_0} - x_n\|_X \leq \varepsilon$$

In other words, for all $[x] \in Y$, we can find $\tilde{x} \in X$ with $\|\phi(\tilde{x}) - [x]\|_Y < \varepsilon$; this proves that $\overline{\phi(X)} = Y$.

A small remark (not important for the proof): If $(X, \|\cdot\|_X)$ is already complete, for every Cauchy sequence $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{C}_X$, the limit $x_\infty = \lim_{n\to\infty} x_n$ exists; in this case we have $\phi(x_\infty) = [x]$. Hence, in this case, ϕ is an isometric isomorphism and $(X, \|\cdot\|_X)$ is isometrically isomorph to its completion $(Y, \|\cdot\|_Y)$ (this remark goes already in the direction of the uniqueness, which will be discussed below).

Next, we show that $(Y, \|\cdot\|_Y)$ is always complete (even if X is not). Let $([x_\ell])_{\ell\in\mathbb{N}}$ be a Cauchy sequence in Y. Since $\overline{\phi(X)} = Y$, we can choose $z_\ell \in X$ with

$$\|\phi(z_\ell) - [x_\ell]\|_Y \leq 2^{-\ell}$$

The sequence $z = (z_\ell)_{\ell\in\mathbb{N}}$ is a Cauchy sequence on X, since

$$\begin{aligned}
\|z_\ell - z_m\|_X &= \|\phi(z_\ell - z_m)\|_Y \\
&= \|\phi(z_\ell) - \phi(z_m)\|_Y \\
&\leq \|\phi(z_\ell) - [x_\ell]\|_Y + \|[x_\ell] - [x_m]\|_Y + \|[x_m] - \phi(z_m)\|_Y \\
&\leq 2^{-\ell} + 2^{-m} + \|[x_\ell] - [x_m]\|_Y \to 0
\end{aligned}$$

as $\ell, m \to \infty$. We claim that $[x_\ell] \to [z]$ as $\ell \to \infty$. In fact, with $x_\ell = (x_\ell^1, x_\ell^2, \dots)$,

$$\begin{aligned}
\|[z] - [x_\ell]\|_Y &= \lim_{k\to\infty} \|z_k - x_\ell^k\|_X \\
&\leq \limsup_{k\to\infty} \|z_k - z_\ell\|_X + \limsup_{k\to\infty} \|z_\ell - x_\ell^k\|_X \\
&\leq \limsup_{k\to\infty} \|z_k - z_\ell\|_X + \|\phi(z_\ell) - [x_\ell]\|_Y \\
&\leq \limsup_{k\to\infty} \|z_k - z_\ell\|_X + 2^{-\ell} \to 0
\end{aligned}$$

when $\ell \to \infty$.

Finally, we prove the uniqueness of the completion, up to isometric isomorphisms. Let $(Y, \|\cdot\|_Y, \phi)$ and $(W, \|\cdot\|_W, \psi)$ be two completions of $(X, \|\cdot\|_X)$. For $y \in Y$ we choose a sequence $(x_k)_{k\in\mathbb{N}}$ in X, with $\phi(x_k) \to y$, as $k \to \infty$ (this is possible because $\overline{\phi(X)} = Y$). Since $\phi(x_k)$ is a Cauchy sequence on Y, it follows that $\psi(x_k)$ is a Cauchy sequence on W (since ϕ and ψ are isometric). From the completeness of W, there is $z \in W$ with $\psi(x_k) \to z$ as $k \to \infty$. The limit is clearly independent of the choice of the sequence $(x_k)_{k\in\mathbb{N}}$. Hence, the map $\xi : Y \to W$ defined through $\xi(y) = z$ is well-defined and linear. Since ϕ and ψ are isometries, also ξ is an isometry. We still have to show that ξ is surjective. However, this is clear, since for any given $z \in W$ we can find a sequence x_k in X with $\psi(x_k) \to w$. With $y := \lim_{k\to\infty} \phi(x_k) \in Y$, we have $\xi(y) = w$. We conclude that ξ is a linear isometric isomrphism between the two completions of $(X, \|\cdot\|_X)$. $\square$

Remarks: The completion of $\mathbb{Q}$ is $\mathbb{R}$. This construction can also be used to define the Lebesgue spaces $L^p(\Omega)$ with $\Omega \subset \mathbb{R}^n$; in fact, $L^p(\Omega)$ is a completion of the space $C(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ w.r.t. the norm

$$\|f\|_p = \left(\int_\Omega dx \, |f(x)|^p \right)^{1/p}$$

Also Sobolev spaces(which will be defined in the next Chapter) can be defined as completion of certain normed vector spaces.

1.4 Hilbert Spaces

Definition 1.4.1. *Let H be a vector space over the field $\mathbb{K}$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$). A scalar product (or an inner product) on H is a map $(\cdot,\cdot) : H \times H \to \mathbb{K}$ with the properties*

- $(z, x + \lambda y) = (z, x) + \lambda(z, y)$, *for all* $x, y, z \in H$, $\lambda \in \mathbb{K}$.
- $(x, y) = \overline{(y, x)}$ *for all* $x, y, \in H$.
- $(x, x) > 0$ *for all* $x \neq 0$.

Here we used the notation $\overline{\lambda}$ for the complex conjugated number to λ, if $\mathbb{K} = \mathbb{C}$, while $\overline{\lambda} = \lambda$, if $\mathbb{K} = \mathbb{R}$. A pair $(H, (\cdot,\cdot))$ consisting of a vector space H over $\mathbb{K}$ and of a scalar product $(\cdot,\cdot)$ is called a scalar product space or a pre-Hilbert space.

Remark: for $\mathbb{K} = \mathbb{C}$, we defined the scalar product to be linear in its second argument and anti-linear in its first argument. In many mathematics books, a different convention is used, and scalar products are defined to be linear in the first and anti-linear in their second argument.

The Cauchy-Schwarz inequality allows us to use the scalar product to define a norm on every pre-Hilbert space.

Lemma 1.4.2. *Let $(H, (\cdot,\cdot))$ be a pre-Hilbert space. Then*

$$|(x,y)|^2 \leq (x,x)(y,y)$$

for all $x, y \in H$.

Proof. It is enough to notice that, for every $t \in \mathbb{C}$,

$$0 \leq (x - ty, x - ty) = (x,x) - 2\mathrm{Re}\; t(x,y) + |t|^2(y,y)$$

In particular, choosing $t = (y,x)/(y,y)$ we obtain the desired bound. □

Corollary 1.4.3. *Let $(H, (\cdot,\cdot))$ be a pre-Hilbert space. Then*

$$\|x\| := \sqrt{(x,x)}$$

defines a norm on H.

Proof. The triangle inequality follows from Cauchy-Schwarz, since

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\;(x,y) \\
&\leq \|x\|^2 + \|y\|^2 + 2|(x,y)| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2
\end{aligned}$$

The other properties follows from the properties of scalar products. □

Definition 1.4.4. *A pre-Hilbert space is called a Hilbert space if H, equipped with the norm $\|x\| = \sqrt{(x,x)}$ induced by the scalar product, is a Banach space (i.e. if H is complete).*

Examples:

- The standard example is the space $H = \mathbb{R}^n$, equipped with the scalar product $(x,y) = \sum_{j=1}^n x_j y_j$, or $H = \mathbb{C}^n$, with $(x,y) = \sum_{j=1}^n \overline{x}_j y_j$.
- The space of square summable sequences $\ell^2(\mathbb{K})$, equipped with the scalar product

$$(x,y) = \sum_{j=1}^{\infty} \overline{x_j} y_j \tag{1.3}$$

is an example of an infinite dimensional Hilbert space. Notice that the scalar product is well-defined because,

$$\sum_{j=1}^n |x_j||y_j| \leq \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} \left(\sum_{j=1}^n |y_j|^2\right)^{1/2} \leq \|x\|_{\ell^2}\, \|y\|_{\ell^2}$$

for all $n \in \mathbb{N}$. It is then easy to check that (1.3) indeed satisfies the axioms of a scalar product. The norm induced by (1.3) coincides with the norm $\|\cdot\|_{\ell^2}$ that we defined in the previous section, where we also proved that $\ell^2(\mathbb{K})$, equipped with $\|.\|_2$, is complete; it follows that $\ell^2(\mathbb{K})$ is a Hilbert space. One can check that $\ell^2(\mathbb{K})$ is a separable Hilbert space. In fact, we will see that every separable Hilbert space can be identified with $\ell^2(\mathbb{K})$ (or with a subspace of $\ell^2(\mathbb{K})$)

We discussed in the previous section how to construct completion of normed spaces. We can apply the same procedure to complete pre-Hilbert spaces.

Theorem 1.4.5. *Let $(H, (\cdot,\cdot))$ be a pre-Hilbert space. Then its completion (i.e. the completion of the normed space $(H, \|.\|)$ with the norm $\|.\|$ induced by the scalar product) is naturally an Hilbert space (with respect to a scalar product that is compatible with the scalar product on H).*

Proof. Let $(\widetilde{H}, \|\cdot\|_{\widetilde{H}}, \phi)$ be the completion of the normed space $(H, \|.\|)$ equipped with the norm induced by $(\cdot,\cdot)$. For $u, v \in \widetilde{H}$, we can find sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ in H with $\phi(x_k) \to u$, $\phi(y_k) \to v$. Since ϕ is isometric, x_k, y_k are Cauchy sequences on X. We define

$$(u, v)_{\widetilde{H}} := \lim_{k\to\infty} (x_k, y_k) \tag{1.4}$$

The limit is well-defined, since $(x_k), (y_k)$ are Cauchy sequences (and thus, in particular, bounded sequences) and therefore

$$|(x_k, y_k) - (x_m, y_m)| \leq \|x_k - x_m\| \|y_k\| + \|x_m\| \|y_k - y_m\| \to 0$$

as $k, m \to \infty$. Moreover, one can check that i) the limit (1.4) does not depend on the choice of the sequences x_k, y_k, ii) (1.4) really satisfies the properties of a scalar product, iii) (1.4) induces the norm defined on $\widetilde{H}$, iv) if $u, u \in \phi(H)$ and $u = \phi(x)$, $v = \phi(y)$ for $x, y \in H$, then $(u, v)_{\widetilde{H}} = (x, y)$. □

Hilbert spaces have more structure than Banach space. Every Hilbert space is also a Banach space (with the norm induced by the scalar product) and therefore also a metric and a topological space. Not every Banach space is a Hilbert space, because not all norms on a vector space can be induced by a scalar product. There are therefore many results that hold for Hilbert spaces but in general not for Banach spaces; the next theorem is an example.

Theorem 1.4.6. *Let $(H, (\cdot,\cdot))$ be a Hilber space, $K \subset H$ a closed convex set in H and $x_0 \in H$. Then there exists a unique $y \in K$ such that*

$$\|x_0 - y\| = dist(x_0, K) \equiv \inf_{x\in K} \|x_0 - x\|$$

Proof. Let $d = \text{dist}(x_0, K)$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in K with $\|x_0 - y_n\| \to d$ as $n \to \infty$. With the identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

we conclude that

$$\begin{aligned}\|y_n - y_m\|^2 &= \|y_n - x_0 + x_0 - y_m\|^2 \\ &= 2\left(\|y_n - x_0\|^2 + \|y_m - x_0\|^2\right) - 4\left\|\frac{y_n + y_m}{2} - x_0\right\|^2 \\ &\leq 2\left(\|y_n - x_0\|^2 + \|y_m - x_0\|^2\right) - 4d^2\end{aligned} \tag{1.5}$$

where we used convexity of K to conclude that $\|(y_n + y_m)/2 - x_0\| \geq d$. The r.h.s. of (1.5) converges to zero, as $n, m \to \infty$. Therefore, y_n is a Cauchy sequence. Since H is complete K is closed, there exists $y \in K$ with $y_n \to y$. It follows that $d = \|x_0 - y\|$. If we assume that $y_1, y_2 \in K$ with $d = \|x_0 - y_1\| = \|x_0 - y_2\|$, the sequence $(z_n) = (y_1, y_2, y_1, y_2, \dots)$ is such that $\|z_n - x_0\| \to d$. From (1.5) it follows that (z_n) is a Cauchy sequence; this is obviously only possible if $y_1 = y_2$. □

As an application of the last theorem, we show that every Hilbert space H can be decomposed in the direct sum of an arbitrary closed subspace and of its orthogonal complement.

Theorem 1.4.7. *Let $(H, (\cdot,\cdot))$ be a Hilbert space and $M \subset H$ a linear closed subspace. Then the orthogonal complement $M^\perp$ of M, defined through*

$$M^\perp = \{x \in H : (x, m) = 0 \quad \forall \quad m \in M\},$$

is also a linear closed subspace of H and $H = M \oplus M^\perp$, meaning that $H = M + M^\perp$ and that $M \cap M^\perp = \{0\}$.

Proof. It is clear that $M^\perp$ is linear and closed (if x_k is a sequence in $M^\perp$ and $x_k \to x$ as $k \to \infty$, then $(x, m) = \lim_{k\to\infty}(x_k, m) = 0$, because $|(x - x_k, m)| \leq \|x - x_k\|\|m\| \to 0$; therefore $x \in M^\perp$). The fact that $M \cap M^\perp = \{0\}$ follows, because $(x, x) = 0$ implies that $x = 0$. It remains to show that $M + M^\perp = H$. To this end, fix $x \in H$. By Theorem 1.4.6, we find $z \in M$ such that $\text{dist}(M, x) = \|x - z\|$. We claim that $x - z \in M^\perp$. In fact, suppose $(x - z) \notin M^\perp$. Then there exists $\alpha \in M$ with $(x - z, \alpha) > 0$. For $t \in [-1, 1]$, let $z_t = z + t\alpha$. Then $z_t \in M$ for all t and

$$\|x - z_t\|^2 = \|x - z\|^2 + t^2\|\alpha\|^2 - 2t(x - z, \alpha) < \|x - z\|^2 = \text{dist}(x, M)$$

for $t > 0$ small enough. This is a contradiction to the definition of $\text{dist}(x, M)$. □

The notions of orthonormal systems and orthonormal basis (or Hilbert space basis) are very important when dealing with Hilbert spaces.

Definition 1.4.8. *An orthonormal system in $(H, (\cdot, \cdot))$ is a family $(x_\alpha)_{\alpha \in A} \subset H$ for an arbitrary index-set A with $(x_\alpha, x_\beta) = \delta_{\alpha,\beta}$.*

Orthonormal systems satisfy Bessel's inequality.

Lemma 1.4.9. *Let $(H, (\cdot, \cdot))$ be a pre-Hilbert space, $A \subset \mathbb{N}$, and $(x_n)_{n \in A}$ an orthonormal system (in this case, since $A \subset \mathbb{N}$, we also call the orthonormal system an orthonormal sequence). Then*

$$\sum_{\alpha \in A} |(x_\alpha, x)|^2 \leq (x, x) = \|x\|^2$$

Proof. Consider

$$\begin{aligned} 0 &\leq \left(x - \sum_{\alpha \in A} (x_\alpha, x) x_\alpha, x - \sum_{\beta \in A} (x_\beta, x) x_\beta \right) \\ &= (x, x) - 2 \sum_{\alpha \in A} |(x, x_\alpha)|^2 + \sum_{\alpha \in A, \beta \in A} (x, x_\alpha)(x_\beta, x)(x_\alpha, x_\beta) \\ &= (x, x) - \sum_{\alpha \in A} |(x, x_\alpha)|^2 \end{aligned}$$

□

Lemma 1.4.10. *Let H be a Hilbert space, $(x_n)_{n \in \mathbb{N}}$ an orthonormal system (orthonormal sequence) and $(\alpha_n)_{n \in \mathbb{N}}$ a sequence in $\mathbb{K}$. Then, we have*

i) $\sum_{k=1}^\infty \alpha_k x_k$ *converges if and only if* $\sum_{k=1}^\infty |\alpha_k|^2 < \infty$.

ii) $\| \sum_{k=1}^n \alpha_k x_k \|^2 = \sum_{k=1}^n |\alpha_k|^2$.

iii) *If* $\sum_{k=1}^\infty \alpha_k x_k$ *converges, then the limit is independent of the order of the terms. In other words, if* $\phi : \mathbb{N} \to \mathbb{N}$ *is a bijection, then*

$$\sum_{k=1}^\infty \alpha_{\phi(k)} x_{\phi(k)} = \sum_{k=1}^\infty \alpha_k x_k$$

Proof. Since H is complete, $\sum_{k=1}^\infty \alpha_k x_k$ converges if and only if the sequence of the partial sums is a Cauchy sequence. Since

$$\left\| \sum_{k=n}^m \alpha_k x_k \right\|^2 = \sum_{k=n}^m |\alpha_k|^2$$

we obtain i). ii) is a simple computation. To prove iii), let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then

$$\left\| \sum_{k=1}^{n} \alpha_{\phi(k)} x_{\phi(k)} \right\|^2 = \sum_{k=1}^{n} |\alpha_{\phi(k)}|^2$$

Since $\sum_{k=1}^{n} \alpha_k x_k$ converges, also $\sum_{k=1}^{n} |\alpha_k|^2$ converges. Hence,

$$\sum_{k=1}^{\infty} |\alpha_{\phi(k)}|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$$

Therefore, from i), we conclude that the sum $\sum_{k=1}^{\infty} \alpha_{\phi(k)} x_{\phi(k)}$ converges as well. Let

$$x = \sum_{k=1}^{\infty} \alpha_k x_k \qquad \text{und} \qquad z = \sum_{k=1}^{\infty} \alpha_{\phi(k)} x_{\phi(k)}$$

We claim that $x = z$. In fact

$$\|x - z\|^2 = \|x\|^2 + \|z\|^2 - 2\text{Re } (x, z) \tag{1.6}$$

where

$$\|x\|^2 = \|z\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \tag{1.7}$$

On the other hand,

$$(x, z) = \lim_{n,m \to \infty} \left(\sum_{k=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \alpha_{\phi(j)} x_{\phi(j)} \right)$$

For a given $m \in \mathbb{N}$, we choose $n \in \mathbb{N}$ so large, that $\phi(1), \dots, \phi(m) \subset \{1, \dots, n\}$. Then

$$\left(\sum_{k=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \alpha_{\phi(j)} x_{\phi(j)} \right) = \sum_{j=1}^{m} |\alpha_{\phi(j)}|^2$$

As $m \to \infty$, this converges to $\|z\|^2 = \|x\|^2$. From (1.6), (1.7), we get iii). □

So far, we considered orthonormal sequences, i.e. countable orthonormal systems. The next lemma allows us to extend properties of orthonormal sequences to general orthonormal systems with uncountably many elements.

Lemma 1.4.11. *Let $(H, (\cdot, \cdot))$ be a Hilbert, A an arbitrary set and $(x_\alpha)_{\alpha \in A}$ an orthonormal system. Then, for every $x \in H$ the set*

$$\theta_x = \{\alpha : (x, x_\alpha) \neq 0\}$$

is countable.

Proof. Assume that θ_x is uncountable. Then we could find $N \in \mathbb{N}$ such that also

$$\theta_x^N = \{\alpha : |(x, x_\alpha)| \geq 1/N\}$$

is uncountable (because otherwise also $\theta_x = \bigcup_{N\in\mathbb{N}} \theta_x^N$, as countable union of countable sets, would be countable). In this case, we could find $\ell \in \mathbb{N}$ with $\ell N^{-2} \geq (x, x) + 1$, and ℓ indices $\alpha_1, \dots, \alpha_\ell$ in θ_x^N. Then Bessel's inequality for the finite orthonormal system $(x_{\alpha_j})_{j=1}^\ell$ would imply that

$$(x, x) \geq \sum_{j=1}^{\ell} |(x, x_{\alpha_j})|^2 \geq \ell N^{-2} \geq (x, x) + 1$$

giving a contradiction. □

Orthonormal systems are very useful because it is very easy to project orthogonally onto the subspace that they span.

Lemma 1.4.12. *Let $(H, (\cdot, \cdot))$ be a Hilbert space, A an arbitray set and $(x_\alpha)_{\alpha\in A}$ an orthonormal system in H. Then $\sum_{\alpha\in A}(x_\alpha, x)x_\alpha$ converges for every $x \in H$ and the linear map $\phi : H \to H$ defined through $\phi(x) = \sum_{\alpha\in A}(x_\alpha, x)x_\alpha$ is the continuous projection onto*

$$M := \overline{\mathrm{span}\{x_\alpha : \alpha \in A\}}$$

along its orthogonal complement $M^\perp$ (we say that ϕ is the orthogonal projection onto M). In particular, for $x \in \overline{\mathrm{span}\{x_\alpha : \alpha \in A\}}$, we find

$$x = \sum_{\alpha\in A}(x_\alpha, x)x_\alpha \tag{1.8}$$

Proof. For $x \in H$, it follows from Lemma 1.4.11 and Lemma 1.4.9 that

$$\sum_{\alpha\in A} |(x_\alpha, x)|^2 \leq \|x\|^2 < \infty$$

Lemma 1.4.10 implies that

$$\sum_{\alpha\in A}(x_\alpha, x)x_\alpha$$

converges. Hence, the map ϕ is well-defined. To show continuity of ϕ we notice that

$$\|\phi(x)\|^2 = \left\|\sum_{\alpha\in A}(x_\alpha, x)x_\alpha\right\|^2 = \sum_{\alpha\in A} |(x_\alpha, x)|^2 \leq \|x\|^2$$

Hence, since ϕ is linear, we obtain

$$\|\phi(x) - \phi(y)\| = \|\phi(x - y)\| \leq \|x - y\|$$

To prove that ϕ is a projection, we notice that, for $x \in \text{span}\{x_\alpha : \alpha \in A\}$,

$$x = \sum_{j=1}^{n} b_j x_j$$

A simple computation then shows that $\phi(x) = x$. By continuity of ϕ, we conclude that $\phi(x) = x$ for all x in the closure of $\text{span}\{x_\alpha : \alpha \in A\}$. On the other hand, by definition of ϕ, $\phi(x) \in \overline{\text{span}\{x_\alpha : \alpha \in A\}}$ for all $x \in H$. This shows that $\phi \circ \phi = \phi$ and therefore that ϕ is a projection. For x in the orthogonal complement of $\overline{\text{span}\{x_\alpha : \alpha \in A\}}$, we clearly have $(x, x_\alpha) = 0$ for all $\alpha \in A$; therefore $\phi(x) = 0$. □

In particular, if

$$M = \text{span}\{x_\alpha : \alpha \in A\}$$

is dense in H, i.e if $\overline{M} = H$, the expression (1.8) gives a representation of every vector in H. In this case, we say that $(x_\alpha)_{\alpha \in A}$ is a Hilbert space basis.

Definition 1.4.13. *Let H be a Hilbert space. A Hilbert space basis is an orthonormal system $(x_\alpha)_{\alpha \in A}$ with*

$$\overline{\mathit{span}\{x_\alpha : \alpha \in A\}} = H$$

Example: $H = \ell^2(\mathbb{K})$. Let $e_k = (0, \dots, 0, 1, 0, \dots)$ with 1 at the k-th position. Then $(e_k)_{k \in \mathbb{N}}$ is a Hilbert space basis

There are many equivalent characterizations for Hilbert space bases.

Theorem 1.4.14. *Let H ne a Hilbert space, and $(x_\alpha)_{\alpha \in A}$ an orthonormal systems. The following statements are then equivalent:*

i) (x_α) is a Hilbert space basis.

ii) $x = \sum_\alpha (x_\alpha, x) x_\alpha$, for all $x \in H$.

iii) $\|x\|^2 = \sum_\alpha |(x_\alpha, x)|^2$, for all $x \in H$.

iv) $(x_\alpha, x) = 0$ for all $\alpha \in A$ implies that $x = 0$.

v) $(x_\alpha)_{\alpha \in A}$ is a maximal orthonormal system in the sense of inclusions.

Proof. The theorem follows from the implications:

i) ⇒ ii) follows from Lemma 1.4.12.

ii)⇒ iii) Lemma 1.4.10.

iii) $\Rightarrow$ iv) clear.

iv) $\Rightarrow$ v) Suppose that $(x_\alpha)_{\alpha\in A}$ is not maximal. Then there exists $x_\infty \in H$ with $\|x_\infty\| = 1$ and $(x_\alpha, x_\infty) = 0$ for all $\alpha \in A$. iv) implies that $x_\infty = 0$, in contradiction with $\|x_\infty\| = 1$.

v) $\Rightarrow$ i) Let $M = \overline{\text{span}\{x_\alpha : \alpha \in A\}}$. If $M \neq H$, then $M^\perp \neq \{0\}$, and $H = M \oplus M^\perp$. Choose $x_\infty \in M^\perp$ with $\|x_\infty\| = 1$. Then $(x_\alpha)_{\alpha\in A} \cup x_\infty$ is an orthonormal systems in contradiction with the maximality of $(x_\alpha)_{\alpha\in A}$. Hence $M = H$.

$\square$

Using the maximality properties of orthonormal bases with respect to inclusions, i.e. point v) in Theorem 1.4.14, it follows easily from the Lemma of Zorn that every pre-Hilbert space has a Hilbert space basis. Recall the statement of the Lemma of Zorn: Suppose that a partially ordered set P has the property that each chain (i.e. each totally ordered subset of P) has an upper bound. Then P contains at least one maximal element.

In particular, it follows that every separable Hilbert space admits a countable orthonormal basis. In fact, suppose that H is a separable Hilbert space. Let $\{a_j\}_{j\in\mathbb{N}}$ be a countable set, dense in H, and assume, on the other hand, that $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis of H. We proceed by contradiction and assume that A is uncountable. Since $\|e_\alpha - e_\beta\| = \sqrt{2}$ for all $\alpha \neq \beta$, the balls $B_{1/2}(e_\alpha)$, centered around the basis vectors e_α, are pairwise disjoint. Since a_j is dense, each ball $B_{1/2}(e_\alpha)$ must contain at least one vector a_j, and since the balls are disjoint, the a_j must all be different. Since there are uncountably many balls, there must be also uncountably many a_j, in contradiction with the separability assumption. Conversely, it is easy to check that if a Hilbert space has a countable orthonormal basis, it is separable (proof: exercise).

The observation that separable Hilbert spaces have countable orthonormal bases can be used to identify separable Hilbert spaces with the sequence space $\ell^2(\mathbb{K})$.

Theorem 1.4.15. *Let H be an infinite dimensional separable Hilbert space over $\mathbb{K}$. Then there exists a linear Isomorphismum $\phi : H \to \ell^2(\mathbb{K})$ with*

$$(\phi(x), \phi(y))_{\ell^2} = (x, y)_H$$

for all $x, y \in H$ (in particular, the isomorphism is isometric).

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a Hilbert space basis. Since H is separable, A is countable. Since H has infinite dimension, $|A| = \infty$, and therefore we can assume $A = \mathbb{N}$. Define $\phi : H \to \ell^2(\mathbb{K})$ through

$$\phi(x_k) = e_k = (0, \dots, 0, 1, 0, \dots)$$

for all $k \in \mathbb{N}$, and through linear extension. It is easy to check that ϕ has the desired properties.

$\square$

2 Function Spaces

2.1 Continuous Functions on Compact Spaces

In this section, K denotes a compact space and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We consider the space

$$\mathcal{C}_{\mathbb{K}}(K) = \{f : K \to \mathbb{K} : f \text{ is continuous}\}$$

By continuity, we mean here continuity with respect to the standard topology on $\mathbb{K}$.

Lemma 2.1.1. *Let $f \in \mathcal{C}_{\mathbb{K}}(K)$. Then f is bounded. Moreover, if $\mathbb{K} = \mathbb{R}$, its supremum and infimum are attained; in other words, if $\mathbb{K} = \mathbb{R}$, there exist*

$$x_1, x_2 \in K \quad \textit{with} \quad f(x_1) = \sup_{x \in K} f(x), \quad f(x_2) = \inf_{x \in K} f(x)$$

Proof. For $x \in K$, let V_x be an open neighbourhood of x, with

$$f(V_x) \subset B_1(f(x))$$

where $B_1(f(x))$ is the open ball of radius 1 around $f(x)$ (V_x exists because of the continuity of f at x). Then $\bigcup_{x \in K} V_x = K \Rightarrow \exists\, x_1, \ldots, x_n \in K$ with $\bigcup_{j=1}^{n} V_{x_j} = K$.
This implies that

$$\sup_{x \in K} |f(x)| = \max_{j=1,\ldots,n} \sup_{x \in V_{x_j}} |f(x)| \leq \max_{j=1,\ldots,n} (|f(x_j)| + 1) < \infty.$$

Now, let $\mathbb{K} = \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in K with $f(x_n) \to \sup_{y \in K} f(y) =: s$. Since K is compact, $(x_n)_{n \in \mathbb{N}}$ must have an accumulation point $x \in K$ (see Theorem 1.1.8). In other words, for every open neighbourhood U of x, there are infinitely many $n \in \mathbb{N}$ with $x_n \in U$. We claim that $f(x) = s$. If not, there is $\varepsilon > 0$ with $f(x) < s - 2\varepsilon$, and an open neighbourhood V of x with $f(V) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \subset (-\infty, s - \varepsilon)$. Since $f(x_n) \to s$, this contradicts the fact that there are infinitely many n with $x_n \in V$. Similarly, we can show that the infimum is attained. □

Definition 2.1.2. *For $f \in \mathcal{C}_{\mathbb{K}}(K)$ let*

$$\|f\| := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|$$

It is simple to check that $\|\cdot\|$ defines a norm on $\mathcal{C}_{\mathbb{K}}(K)$. Hence, the pair $(\mathcal{C}_{\mathbb{K}}(K), \|\cdot\|)$ is a normed space.

Theorem 2.1.3. $(\mathcal{C}_{\mathbb{K}}(K), \|\cdot\|)$ *is a Banach space.*

Proof. Let f_n be a Cauchy sequence. Then, for any $x \in K$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\| \to 0$$

as $n, m \to \infty$. It follows that $f_n(x)$ is a Cauchy sequence on $\mathbb{K}$ and therefore that $f_n(x)$ converges. We define

$$f(x) := \lim_{n\to\infty} f_n(x)$$

Notice that

$$|f(x)| \le \limsup_{n\to\infty} |f_n(x)| \le \limsup_{n\to\infty} \|f_n\| < \infty.$$

since Cauchy sequences are always bounded. Moreover, since, for every $x \in K$,

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \le \limsup_{m\to\infty} \|f_n - f_m\|$$

we conclude that

$$\sup_{x\in K} |f(x) - f_n(x)| \le \limsup_{m\to\infty} \|f_m - f_n\| \to 0 \tag{2.1}$$

as $n \to \infty$. We still have to show that f is continuous. Fix $x \in K$ and $\varepsilon > 0$. We need to show that there exists an open neighbourhood U of x in K with $f(U) \subset B_\varepsilon(f(x)) = \{\lambda \in \mathbb{K} : |\lambda - f(x)| < \varepsilon\}$. From (2.1) we find $n \in \mathbb{N}$, with $\sup_{z\in K} |f_n(z) - f(z)| \le \varepsilon/3$. By continuity of f_n, we find a neighbourhood U of x in K with $f_n(U) \subset B_{\varepsilon/3}(f_n(x))$. Hence, for $y \in U$,

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< 2\sup_{z\in K} |f(z) - f_n(z)| + \varepsilon/3 \le \varepsilon \end{aligned}$$

This proves that $f(U) \subset B_\varepsilon(f(x))$ and implies, that $f \in \mathcal{C}_{\mathbb{K}}(K)$. From (2.1) we conclude that f_n converges in $\mathcal{C}_{\mathbb{K}}(K)$. □

It is important to notice that $\mathcal{C}_{\mathbb{K}}(K)$ does not only have the structure of a Banach space; it is also an algebra, if we define the multiplication of $f, g \in \mathcal{C}_{\mathbb{K}}(K)$ pointwise through $(f \cdot g)(x) = f(x) \cdot g(x)$, using the product defined on the field $\mathbb{K}$. A Banach space which is also an algebra is called a Banach algebra; $\mathcal{C}_{\mathbb{K}}(K)$ is a Banach algebra.

Our next goal is to show the Stone-Weierstrass Theorem which allows us for example to approximate continuous functions on compact subsets of $\mathbb{R}^n$ though series of polynomals.

Definition 2.1.4. *$\mathbb{A} \subset \mathcal{C}_{\mathbb{K}}(K)$ is called a subalgebra if $\mathbb{A}$ is a linear subspace of $\mathcal{C}_{\mathbb{K}}(K)$ and if, for every $f, g \in \mathbb{A}$, also $f \cdot g \in \mathbb{A}$. We say that the subalgebra $\mathbb{A}$ separates the points of K, if*

$$\forall\, x, y \in K \text{ with } x \neq y, \exists f \in \mathbb{A} : f(x) \neq f(y).$$

It follows from Corollary 1.1.13 that $\mathcal{C}_{\mathbb{K}}(K)$ is a subalgebra separating the points of K.

Theorem 2.1.5 (Stone-Weierstrass Theorem, $\mathbb{K} = \mathbb{R}$)**.** *Let $\mathbb{A}$ be a subalgebra of $\mathcal{C}_{\mathbb{R}}(K)$ separating the points of K. Then we have either $\overline{\mathbb{A}} = \mathcal{C}_{\mathbb{R}}(K)$ or there exists a unique $x_0 \in K$ so, that $\overline{\mathbb{A}} = \{f \in \mathcal{C}_{\mathbb{R}}(K) : f(x_0) = 0\}$*

Example: Let $K \subset \mathbb{R}^n$ compact and $\mathbb{A}$ be the set of all polynomials in the variables $x_1, \dots, x_n$. In other words

$$\mathbb{A} = \left\{ p(\mathbf{x}) = \sum_{\alpha : |\alpha| \leq m} a_\alpha \mathbf{x}^\alpha, \qquad \text{for a } m \in \mathbb{N} \text{ and } a_\alpha \in \mathbb{R} \right\}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. It is easy to check that $\mathbb{A}$ is a subalgebra of $\mathcal{C}_{\mathbb{K}}$, separating the points of K and that, in this case, $\overline{\mathbb{A}} = \{f \in \mathcal{C}_{\mathbb{K}}(K) : f(x_0) = 0\}$ cannot hold true. Hence, the Stone-Weierstrass Theorem implies that $\overline{\mathbb{A}} = \mathcal{C}_{\mathbb{K}}(K)$. In other words, continuous functions on compact sets can be approximated by sequences of polynomials; by definition of the norm on $\mathcal{C}_{\mathbb{K}}(K)$, the approximation is even uniform (i.e. the approximating sequence of polynomials converges uniformly to the given continuous function).

Proof. We assume first, that

$$\forall\, x \in K, \exists\, f \in \mathbb{A} \text{ with } f(x) \neq 0.$$

Under this assumption, we are going to show that $\overline{\mathbb{A}} = \mathcal{C}_{\mathbb{R}}(K)$. To this end, we proceed in several steps.

- *Step 1*: Let $x_1, x_2 \in K, x_1 \neq x_2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

  $$\exists\, f \in \mathbb{A} \text{ with } f(x_1) = \alpha_1, f(x_2) = \alpha_2.$$

  Proof. Since $\mathbb{A}$ separates the points of K,

  $$\exists\, g \in \mathbb{A} \text{ with } g(x_1) \neq g(x_2)$$

  Without loss of generality, we can assume that $g(x_1) = 1$ and $g(x_2) =: \alpha \neq 1$. We distinguish two cases. Case 1: $\alpha \neq 0$. We consider the ansatz $f = \lambda g + \mu g^2$ for $\lambda, \mu \in \mathbb{R}$. Clearly $f \in \mathbb{A}$ satisfies the requirements, if

  $$\alpha_1 = \lambda + \mu \qquad \text{and} \qquad \alpha_2 = \lambda\alpha + \mu\alpha^2$$

  Since

  $$\det \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^2 \end{pmatrix} = \alpha^2 - \alpha \neq 0$$

it is always possible to find λ, μ with $f(x_1) = \alpha_1, f(x_2) = \alpha_2$. Case 2: $\alpha = 0$. We choose $h \in \mathbb{A}$ with $h(x_2) = 1$ and we consider the ansatz $f = \lambda g + \mu h$. f satisfies the requirements, if

$$\alpha_1 = \lambda + \mu \cdot h(x_1) \quad \text{and} \quad \alpha_2 = \mu$$

The linear system can be solved choosing $\mu = \alpha_2$ and $\lambda = \alpha_1 - \alpha_2 h(x_1)$. □

- *Step 2*: $\overline{\mathbb{A}}$ is a subalgebra. Proof: exercise.
- *Step 3*: $f \in \mathbb{A} \Rightarrow |f| \in \overline{\mathbb{A}}$.

Proof. Without loss of generality we can assume that $\max_{x \in K} |f(x)| \leq 1$. We write $|f| = \sqrt{f^2}$ and we approximate $\sqrt{s}$ uniformly on $[0,1]$ through polynomials. In other words, we find a sequence $p_n^{(s)}$ of polynomials on $[0,1]$ with $p_n(0) = 0$ and

$$\sup_{s \in [0,1]} |p_n(s) - \sqrt{s}| \to 0$$

as $n \to \infty$. With such a sequence p_n, the claim follows because

$$f \in \mathbb{A} \Rightarrow p_n(f^2) \in \mathbb{A} \, \text{for all } n$$

and because $p_n(f^2) \to |f|$ as $n \to \infty$ (the assumption $p_n(0) = 0$ is important, because otherwise it is not clear that $p_n(f^2) \in \mathbb{A}$). Since $\overline{\mathbb{A}}$ is closed, we conclude that $|f| \in \overline{\mathbb{A}}$.

To define polynomials $p_n(s)$, we set $p_1(s) = 0$ and, recursively,

$$p_{n+1}(s) = p_n(s) + \frac{1}{2}(s - p_n^2(s)) \tag{2.2}$$

Notice that $p_n(0) = 0 \, \forall n$. Moreover

$$\begin{aligned} (\sqrt{s} - p_{n+1}(s)) &= (\sqrt{s} - p_n(s)) - \frac{1}{2}(\sqrt{s} - p_n(s))(\sqrt{s} + p_n(s)) \\ &= (\sqrt{s} - p_n(s))(1 - \frac{1}{2}(\sqrt{s} + p_n(s))) \end{aligned} \tag{2.3}$$

We have

$$0 \leq (\sqrt{s} - p_1(s)) \leq 1 \quad \text{and} \quad 0 \leq p_1(s) \leq 1$$

Inductively, we obtain that, for all $n \in \mathbb{N}$,

$$0 \leq (\sqrt{s} - p_n(s)) \leq 1 \quad \text{and} \quad 0 \leq p_n(s) \leq 1$$

(the first inequality follows from (2.3), the inequality $p_n(s) \geq 0$ follows instead from (2.2)). Furthermore,

$$p_{n+1}(s) - p_n(s) = \frac{1}{2}(s - p_n^2(s)) \geq 0$$

Hence, $p_1(s) \leq p_2(s) \leq \ldots \leq 1$. This implies that the sequence $p_n(s)$ is monotonically increasing and bounded. Therefore, for arbitrary $s \in [0,1]$ fixed, there exists the limit $p_\infty(s) = \lim_{n\to\infty} p_n(s)$. From (2.2), it follows that $p_\infty(s) = \sqrt{s}$.

We still have to show that the convergence is uniform. Fix $\varepsilon > 0$. For $s \in [0,1]$, we find $n_{\varepsilon,s} \in \mathbb{N}$, so that

$$p_n(s) \geq \sqrt{s} - \frac{\varepsilon}{2} \qquad \text{for all } n \geq n_{\varepsilon,s}$$

We also find an open neighbourhood $U(s)$ of s in $[0,1]$ with

$$p_{n_{\varepsilon,s}}(t) \geq \sqrt{t} - \varepsilon \qquad \text{for all } t \in U(s)\,.$$

(of course, the neighbourhood depends on the choice of ε). Since $\{U(s) : s \in [0,1]\}$ is an open covering of $[0,1]$ and since $[0,1]$ is compact, there exist $s_1, \ldots, s_\ell \in [0,1]$ with

$$\bigcup_{j=1}^{\ell} U(s_j) = [0,1]$$

We set $\bar{n}_\varepsilon := \max_{j=1,\ldots\ell} n_{\varepsilon,s_j}$. For $n \geq \bar{n}_\varepsilon$ we have

$$p_n(t) \geq \sqrt{t} - \varepsilon \qquad \text{for all } t \in [0,1]$$

Hence

$$\sup_{t\in[0,1]} |p_n(t) - \sqrt{t}| \leq \varepsilon \qquad \forall\, n \geq \bar{n}_\varepsilon\,.$$

□

- *Step 4*: For $f, g \in \overline{\mathbb{A}}$, we have $\min\{f,g\}, \max\{f,g\} \in \overline{\mathbb{A}}$.

  Proof. The claim follows from step 3, noticing that

$$\min\{f,g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

$$\max\{f,g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f+g|$$

□

- *Step 5*: Let $g \in \mathcal{C}_{\mathbb{R}}(K)$ and $\varepsilon > 0$. Then

$$\forall\, x \in K\; \exists\, f_x \in \overline{\mathbb{A}} \text{ with } f_x(x) = g(x) \text{ and } f_x(y) \le g(y) + \varepsilon \quad \forall\, y \in K \tag{2.4}$$

Proof. Fix $x \in K$. For $y \in K$ we find by Step 1 $f_{x,y} \in \mathbb{A}$ with

$$f_{x,y}(x) = g(x) \qquad \text{and} \qquad f_{x,y}(y) = g(y)$$

Now, let

$$U_{x,y} := \{z \in K \,:\, f_{x,y}(z) < g(z) + \varepsilon\}$$

Since $y \in U_{x,y}$ and since $U_{x,y}$ is open, $\{U_{x,y}\}_{y \in K}$ provides an open covering of K. Hence, there exist $y_1, \ldots y_n \in K$ with $U_{x,y_1} \cup \cdots \cup U_{x,y_n} = K$. Set

$$f_x := \min\{f_{x,y_1}, f_{x,y_2}, \ldots f_{x,y_n}\}$$

Then $f_x \in \overline{\mathbb{A}}$ by Step 4,

$$f_x(x) = g(x), \quad \text{and } f_x(z) \le g(z) + \varepsilon \quad \text{for all } z \in K.$$

□

- *Step 6*: Let $g \in \mathcal{C}_{\mathbb{R}}(K)$ and $\varepsilon > 0$. Then there exists $f \in \overline{\mathbb{A}}$ with $\|f - g\| = \sup_{z \in K} |f(z) - g(z)| < \varepsilon$.

Proof. From Step 5 we have

$$\forall\, x \in K,\, \exists\, f_x \in \overline{\mathbb{A}} \quad \text{with} \quad f_x(x) = g(x) \quad \text{and} \quad f_x(z) \le g(z) + \varepsilon \quad \text{for all } z \in K\,.$$

For arbitrary $x \in K$ we define

$$V_x := \{z \in K : f_x(z) > g(z) - \varepsilon\}$$

V_x is an open neighbourhood of x and $\{V_x\}_{x \in K}$ is an open covering of K. Hence, we can find $x_1, \ldots, x_m \in K$ with

$$V_{x_1} \cup \ldots \cup V_{x_m} = K$$

We set

$$f := \max\{f_{x_1}, \ldots, f_{x_m}\}.$$

Then $f \in \overline{\mathbb{A}}$ (Step 4) and

$$|f(x) - g(x)| < \varepsilon \forall x \in K.$$

□

This proves the theorem under the additional assumption that $\forall\, x \in K\ \exists f \in \mathbb{A}$ with $f(x) \neq 0$.

In the following, we assume that there exists $x_0 \in K$ with $f(x_0) = 0$, for all $f \in \mathbb{A}$. We want to prove that

$$\overline{\mathbb{A}} = \{f \in \mathcal{C}_{\mathbb{K}}(K) : f(x_0) = 0\}$$

Clearly, $f(x_0) = 0$ for all $f \in \overline{\mathbb{A}}$ and therefore

$$\overline{\mathbb{A}} \subset \{f \in \mathcal{C}_{\mathbb{K}}(K) : f(x_0) = 0\}$$

On the other hand, let $g \in \mathcal{C}_{\mathbb{R}}(K)$ with $g(x_0) = 0$. We want to show that $g \in \overline{\mathbb{A}}$. To this end, we consider

$$\mathbb{B} = \{f + \lambda : f \in \mathbb{A}, \lambda \in \mathbb{R}\}$$

It is easy to check that $\mathbb{B}$ is a subalgebra of $\mathcal{C}_{\mathbb{R}}(K)$, separating the points of K and so, that

$$\forall\, x \in K, \exists\, h \in \mathbb{B} \quad \text{with} \quad h(x) \neq 0.$$

From the first part of the proof, we find $\overline{\mathbb{B}} = \mathcal{C}_{\mathbb{R}}(K)$. In particular, for a given $\varepsilon > 0$ we find $\lambda \in \mathbb{R}, f \in \mathbb{A}$ with

$$\|g - (\lambda + f)\| = \sup_{x \in K} |g(x) - f(x) - \lambda| < \varepsilon\,.$$

Since $g(x_0) = f(x_0) = 0$, we obtain that

$$|\lambda| = |g(x_0) - f(x_0) - \lambda| < \varepsilon$$

and therefore that

$$\|f - g\| < 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $g \in \overline{\mathbb{A}}$. Hence

$$\overline{\mathbb{A}} = \{f \in \mathcal{C}_{\mathbb{R}}(K) : f(x_0) = 0\}$$

The uniqueness of x_0 follows from the assumption, that $\mathbb{A}$ separates the points of K. □

For the case $\mathbb{K} = \mathbb{C}$, we have to assume additionally that the subalgebra remains closed with respect to complex conjugation.

Theorem 2.1.6. *(Stone-Weierstrass, $\mathbb{K} = \mathbb{C}$) Let $\mathbb{A}$ be a subalgebra of $\mathcal{C}_{\mathbb{C}}(K)$, separating the points K and so that $\bar{f} \in \mathbb{A}$, $\forall f \in \mathbb{A}$. Then we either have $\bar{\mathbb{A}} = \mathcal{C}_{\mathbb{C}}(K)$ or*

$$\exists!\, x_0 \in K \quad \textit{mit} \quad \bar{\mathbb{A}} = \{f \in \mathcal{C}_{\mathbb{C}}(K) : f(x_0) = 0\}.$$

Proof. Let $\mathbb{A}_\mathbb{R} := \mathcal{C}_\mathbb{R}(K) \cap \mathbb{A}$. $\mathbb{A}_\mathbb{R}$ is a subalgebra of $\mathcal{C}_\mathbb{R}(K)$. Since $\bar{f} \in \mathbb{A}$, for all $f \in \mathbb{A}$, we find

$$\text{Re } f = \frac{1}{2}(f + \bar{f}), \quad \text{Im } f = \frac{1}{2i}(f - \bar{f}) \in \mathbb{A}$$

Hence $f \in \mathbb{A}$ implies that Re f, Im $f \in \mathbb{A}_\mathbb{R}$. Since $\mathbb{A}$ separates the points of K, so does $\mathbb{A}_\mathbb{R}$. From the Stone-Weierstrass Theorem for $\mathbb{K} = \mathbb{R}$ it follows that either $\mathcal{C}_\mathbb{R}(K) = \bar{\mathbb{A}}_\mathbb{R}$ or $\exists\, x_0 \in K$ with

$$\overline{\mathbb{A}}_\mathbb{R} = \{f \in \mathcal{C}_\mathbb{R}(K) : f(x_0) = 0\} \tag{2.5}$$

In the case $\bar{\mathbb{A}}_\mathbb{R} = \mathcal{C}_\mathbb{R}(K)$, we obtain

$$\mathcal{C}_\mathbb{C}(K) = \mathcal{C}_\mathbb{R}(K) + i\mathcal{C}_\mathbb{R}(K) = \overline{\mathbb{A}}_\mathbb{R} + i\overline{\mathbb{A}}_\mathbb{R} = \overline{\mathbb{A}_\mathbb{R} + i\mathbb{A}_\mathbb{R}} = \bar{\mathbb{A}} \tag{2.6}$$

In the case (2.5),

$$\begin{aligned} \overline{\mathbb{A}} &= \overline{\mathbb{A}_\mathbb{R} + i\mathbb{A}_\mathbb{R}} = \overline{\mathbb{A}_\mathbb{R}} + i\overline{\mathbb{A}_\mathbb{R}} \\ &= \{f \in \mathcal{C}_\mathbb{R}(K) : f(x_0) = 0\} + i\{f \in \mathcal{C}_\mathbb{R}(K) : f(x_0) = 0\} \\ &= \{f \in \mathcal{C}_\mathbb{C}(K) : f(x_0) = 0\} \end{aligned} \tag{2.7}$$

□

In applications, we will mostly consider functions defined on subsets of $\mathbb{R}^n$. The standard topology on $\mathbb{R}^n$ induces a topology on every subset $A \subset \mathbb{R}^n$. It follows that the notion of continuity of $\mathbb{K}$-valued functions is well-defined on every A and we can always define the space

$$\mathcal{C}_\mathbb{K}(A) := \{f : A \to \mathbb{K} \,:\, f \text{ continuous}\}$$

If A is not compact, however, the expression $\sup_{x \in A} |f(x)|$ does not define a norm on A, since the supremum may be infinity. It turns out that if there exists a sequence $(K_i)_{i \in \mathbb{N}}$ of compact subsets of $\mathbb{R}^n$ with $A = \bigcup_{i \in \mathbb{N}} K_i$, such that $\emptyset \neq K_i \subset K_{i+1} \subset A \;\forall\, i \in \mathbb{N}$ and with

$$\forall x \in A \;\exists \delta > 0, i \in \mathbb{N} \,:\, B_\delta(x) \cap A \subset K_i$$

then there is a natural definition for a metric on $\mathcal{C}_\mathbb{K}(A)$ (the Fréchet metric), but we will not go in this direction in this class. Instead, for non-compact $A \subset \mathbb{R}^n$, we will simply consider $\mathcal{C}_\mathbb{K}(A)$ as a vector space over $\mathbb{K}$, without a norm.

For functions on open subsets $\Omega \subset \mathbb{R}^n$, we can also introduce the notion of differentiability and of derivative for functions on Ω, as it was done in Analysis 1 and 2. We define then the space

$$\mathcal{C}^m_\mathbb{K}(\Omega) := \{f : \Omega \to \mathbb{K} \,:\, f \text{ is } m \text{ times continuously differentiable}\}$$

We also define the space

$$\begin{aligned}\mathcal{C}^m_{\mathbb{K}}(\overline{\Omega}) := \{ & f : \Omega \to \mathbb{K} \,:\, f \text{ is } m \text{ times continuously differentiable in } \Omega \text{ and} \\ & \partial^{\underline{\alpha}} f \text{ can be continuously extended to } \overline{\Omega}\,, \forall\, \underline{\alpha} \in \mathbb{N}^n : |\underline{\alpha}| \leq m \}\end{aligned}$$

If the set $\Omega \subset \mathbb{R}^n$ is bounded, $\mathcal{C}^m_{\mathbb{K}}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|f\| = \sum_{|\underline{\alpha}| \leq m} \|\partial^{\underline{\alpha}} f\|_{\mathcal{C}_{\mathbb{K}}(\bar{\Omega})}$$

We can also define the space of smooth functions on an open set $\Omega \subset \mathbb{R}^n$ by setting

$$\mathcal{C}^\infty_{\mathbb{K}}(\Omega) := \bigcap_{m \in \mathbb{N}} \mathcal{C}^m_{\mathbb{K}}(\Omega) \qquad \text{und } \mathcal{C}^\infty_{\mathbb{K}}(\overline{\Omega}) := \bigcap_{m \in \mathbb{N}} \mathcal{C}^m_{\mathbb{K}}(\overline{\Omega})$$

For a function $f : \Omega \to \mathbb{K}$, we define the support of f by

$$\text{supp}(f) := \{x \in \Omega : f(x) \neq 0\}$$

The subspaces of $\mathcal{C}^m_{\mathbb{K}}(\Omega), \mathcal{C}^\infty_{\mathbb{K}}(\Omega)$, consisting of functions with compact support are denoted by

$$\mathcal{C}^m_{c,\mathbb{K}}(\Omega) := \{f \in \mathcal{C}^m_{\mathbb{K}}(\Omega) : \text{supp}(f) \quad \text{kompakt}\}$$

$$\mathcal{C}^\infty_{c,\mathbb{K}}(\Omega) := \{f \in \mathcal{C}^\infty_{\mathbb{K}}(\Omega) : \text{supp}(f) \quad \text{kompakt}\}$$

(The subscript $\mathbb{K}$ will be often removed from the notation).

For several reasons, for example to solve partial differential equations, it will be often important to consider also functions with singularities, that are not contained in the spaces $\mathcal{C}^m(\Omega)$. Such functions are considered in the next two sections.

2.2 Lebesgue Spaces

In this section, we recall the definition and the main properties of L^p-spaces. Since these properties have been already proven in the class Analysis 3, we will not repeat their proofs. For more details on L^p spaces, one can either look at my script on Analysis 3 (in German) or one can check the book "Analysis", by Lieb and Loss.

Let (Ω, Σ, μ) be a measure space (i.e. let Ω be a set, Σ a σ-algebra over Ω and μ a σ-additive measure on Σ). We define the space

$$\widetilde{L}^p(\Omega, d\mu) = \{f : \Omega \to \mathbb{C} : f \text{ measurable }, \int_\Omega |f|^p d\mu < \infty\}$$

Since $|\alpha + \beta|^p \leq 2^p(|\alpha|^p + |\beta|^p), \forall \alpha, \beta \in \mathbb{C}$, the set $\widetilde{L}^p(\Omega, d\mu)$ is clearly a vector space.

For $f \in \widetilde{L}^p(\Omega, d\mu)$, we define

$$\|f\|_p := \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \tag{2.8}$$

Then

- $\|\lambda f\|_p = |\lambda| \|f\|_p, \forall \lambda \in \mathbb{C}, \forall f \in \widetilde{L}^p(\Omega, d\mu)$
- $\|f\|_p = 0 \Leftrightarrow f(x) = 0$ für fast alle $x \in \Omega$.
- $\|f + g\|_p \leq \|f\|_p + \|g\|_p, \forall f, g \in f \in \widetilde{L}^p(\Omega, d\mu)$

The first two properties are clear. The proof of the triangle inequality (called here Minkowski inequality) follows from Hölder's inequality.

Theorem 2.2.1 (Hölder's inequality)**.** *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* (Ω, Σ, μ) *be a measure space and* $f \in L^p(\Omega), g \in L^q(\Omega)$*. Then*

$$\left|\int_\Omega fg d\mu\right| \leq \int_\Omega |f||g| d\mu \leq \|f\|_p \cdot \|g\|_q$$

Notice that $\|.\|_p$ does not define a norm on $\widetilde{L}^p(\Omega, \mathcal{A}, \mu)$ because there are $f \neq 0$ in $\widetilde{L}^p(\Omega, \mathcal{A}, \mu)$ with $\|f\|_p = 0$. To solve this problem, we identify functions that are almost everywhere the same. More precisely, on $\widetilde{L}^p(\Omega, \mathcal{A}, \mu)$ we define the equivalence relation $f \sim g :\Leftrightarrow f(x) = g(x)$ almost everywhere with respect to μ. In the equivalence class

$$[f] = \{g \in \widetilde{L}^p(\Omega, \mathcal{A}, \mu) : g(x) = f(x) \text{ almost everwhere}\}$$

we identify all functions, coinciding with f almost everywhere. Then we define the space

$$L^p(\Omega, \mathcal{A}, \mu) := \widetilde{L}^p(\Omega, \mathcal{A}, \mu)/\sim = \left\{[f] : f \in \widetilde{L}^p(\Omega, \mathcal{A}, \mu)\right\}. \tag{2.9}$$

On $L^p(\Omega, \mathcal{A}, \mu)$,

$$\|[f]\|_p := \|f\|_{\widetilde{L}^p}$$

defines a norm. Hence $(L^p(\Omega, , \mathcal{A}, \mu), \|.\|_p)$ is a normed space.

For $p = \infty$, we also define

$$\widetilde{L}^\infty(\Omega, \mathcal{A}, \mu) = \{f : \Omega \to \mathbb{C} : f \text{ measurable and } \exists K < \infty : |f(x)| \leq K \text{ for almost all } x \in \Omega\}.$$

For $f \in \widetilde{L}^\infty(\Omega, \mathcal{A}, \mu)$, let

$$\|f\|_\infty := \inf\{K > 0 : |f(x)| < K \text{ almost everywhere}\}$$

$\|f\|_\infty$ is called the "essential supremum" of f on Ω. Like for $p < \infty$, $\|.\|_\infty$ does not define a norm on $\widetilde{L}_\infty(\Omega, \mathcal{A}, \mu)$ (because there are $f \neq 0$ with $\|f\|_\infty = 0$). For this reason, we define

$$L^\infty(\Omega, \mathcal{A}, \mu) := \widetilde{L}^\infty(\Omega, \mathcal{A}, \mu)/\sim = \{[f] : f \in \widetilde{L^\infty}(\Omega, \mathcal{A}, \mu)\} \tag{2.10}$$

Then $(L^\infty(\Omega, \mathcal{A}, \mu), \|\cdot\|)$ is also a normed space.

By definition, elements of $L^p(\Omega, \mathcal{A}, \mu), 1 \leq p \leq \infty$ are not functions, but instead equivalence classes of functions. Nevertheless, we will often speak about functions $f \in L^p(\Omega)$. What we mean, when we say that a function $f \in L^p(\Omega, \mathcal{A}, \mu)$, is that f is a representative of an equivalence class in $L^p(\Omega, \mathcal{A}, \mu)$.

Theorem 2.2.2 (Fischer-Riesz Theorem)**.** *Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. Then $L^p(\Omega, \mathcal{A}, \mu)$ is complete.*

It follows that $(L^p(\Omega, \mathcal{A}, \mu), \|\cdot\|_p)$ is a Banach space, for all $1 \leq p \leq \infty$. For $p = 2$, we can also define a scalar product on $L^2(\Omega, \mathcal{A}, \mu)$ by setting

$$(f, g)_{L^2} = \int \overline{f(x)} g(x) d\mu(x) \tag{2.11}$$

It is easy to check that $(.,.)_{L^2}$ is indeed a scalar product on $L^2(\Omega, \mathcal{A}, \mu)$ and that $\|f\|_2 = (f, f)_{L^2}^{1/2}$ for all $f \in L^2(\Omega, \mathcal{A}, \mu)$ (in other words, the scalar product defined in (2.11) induces the norm on L^2). Hence, $L^2(\Omega, \mathcal{A}, \mu)$ is also a Hilbert space.

For $p \neq 2$, the norm $\|.\|_p$ on $L^p(\Omega, \mathcal{A}, \mu)$ cannot be induced by a scalar product. This follows for example from the observation that the parallelogram identity

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2 \tag{2.12}$$

is only true if $p = 2$.

For $p \neq 2$, the identity (2.12) is replaced by the so-called Hanner's inequalities.

Lemma 2.2.3 (Hanner'sche inequalities)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f, g \in L^p(\Omega, \mathcal{A}, \mu)$. For $1 \leq p \leq 2$, we have*

$$\|f + g\|_p^p + \|f - g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + |\|f\|_p - \|g\|_p|^p \tag{2.13}$$

and

$$(\|f + g\|_p + \|f - g\|_p)^p + |\|f + g\|_p - \|f - g\|_p|^p \leq 2^p(\|f\|_p^p + \|g\|_p^p) \tag{2.14}$$

For $2 \leq p \leq \infty$, on the other hand, both inequalities are reversed.

A consequence of the Hanner's inequality, is the fact that, for all $1 < p < \infty$, the Banach space $L^p(\Omega, \mathcal{A}, \mu)$ is strictly convex (the unit ball is strictly convex). Another consequence, of the strictly convexity or directly of Hanner's inequalities, is the following theorem (which extends the corresponding results for Hilbert spaces that was proven in Theorem 1.4.6).

Theorem 2.2.4. *Let (Ω, Σ, μ) be a measure space, $1 < p < \infty$, $K \subset L^p(\Omega)$ closed and convex. Let $f \in L^p(\Omega)$. Then there exists a unique $h \in K$ with $\|f-h\|_p = dist(f, K) = \inf_{g\in K} \|f-g\|_p$.*

With Theorem 2.2.4, we can show the duality between L^p and L^q, if $1/p + 1/q = 1$. Let us recall the definition of the dual of a Banach space (we will come back to this point later, in Section 4).

Definition 2.2.5. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$. We define the dual space X^* of X as the space of all continuous linear functionals on X, i.e.*

$$X^* := \{f : X \to \mathbb{K} : \ f \text{ is linear and continuous}\}$$

The space X^* is always a Banach space, with respect to the norm

$$\|f\|_{X^*} := \sup_{x\in X:\|x\|\leq 1} |f(x)|$$

The following theorem determines the dual space of $L^p(\Omega, \mathcal{A}, \mu)$; it shows that every continuous linear functional L on $L^p(\Omega, \mathcal{A}, \mu)$ for $1 \leq p < \infty$ has the form

$$L(f) = \int_\Omega g(x)f(x)d\mu(x)$$

for a $g \in L^q(\Omega, \mathcal{A}, \mu)$, if $1/p + 1/q = 1$.

Theorem 2.2.6. *Let $1 \leq p < \infty$, $1 < q \leq \infty$ with $1/p + 1/q = 1$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $p = 1$, we assume additionally the measure space $(\Omega, \mathcal{A}, \mu)$ to be σ-finite. Then, the map*

$$\begin{array}{rcl} \phi : L^q(\Omega, \mathcal{A}, \mu) & \to & (L^p(\Omega, \mathcal{A}, \mu))^* \\ f & \to & \phi_f \end{array}$$

with

$$\phi_f(g) = \int_\Omega \overline{f(x)}g(x)d\mu(x)$$

for all $g \in L^p(\Omega, \mathcal{A}, \mu)$ is an anti-linear isometric isomorphism. In other words, $L^p(\Omega, \mathcal{A}, \mu)^* \simeq L^q(\Omega, \mathcal{A}, \mu)$.

Our last goal in this section is to show that smooth functions are dense in L^p, for all $1 \leq p < \infty$. To simplify matters, we focus here on $\Omega = \mathbb{R}^n$ with $\mathcal{A}$ being either the Borel or the Lebesgue σ-algebra and $\mu = \lambda_n$ being Lebesgue measure. We use the shorthand notation $L^p(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n, \mathcal{A}, \lambda_n)$. In this case, next theorem shows how to approximate L^p-functions with sequences of $\mathcal{C}^\infty$-functions.

Theorem 2.2.7. *Let $j \in L^1(\mathbb{R}^n)$ with $\int j\, d\lambda_n(x) = 1$. For $\varepsilon > 0$ let $j_\varepsilon(x) = \varepsilon^{-n} j(x/\varepsilon)$ so that $\int j_\varepsilon d\lambda_n = 1$ for all $\varepsilon > 0$. Let $f \in L^p(\mathbb{R}^n)$ for a $1 \leq p < \infty$ and set $f_\varepsilon = f * j_\varepsilon$. Then $f_\varepsilon \in L^p(\mathbb{R}^n)$ with $\|f_\varepsilon\|_p \leq \|j\|_1 \|f\|_p$ and $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. If $j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $D^\alpha f_\varepsilon = (D^\alpha j_\varepsilon) * f$ for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.*

Remarks:

1) It is easy to check that the theorem holds true for $f \in L^p(\Omega)$ and arbitrary $\Omega \subset \mathbb{R}^n$ measurable.

2) It is in general impossible to approximate $f \in L^\infty(\mathbb{R}^n)$ by sequences of smooth (or even just continuous) functions with respect to the L^∞ norm.

Sometimes it is also important to approximate functions in $L^p(\mathbb{R}^n)$ through functions in $C_c^\infty(\mathbb{R}^n)$, having compact support. More generally, it may be important to approximate functions in $L^p(\Omega)$, for an open set $\Omega \subset \mathbb{R}^n$, by functions in $C_c^\infty(\Omega)$, having compact support inside Ω. That this is always possible is a consequence of the next lemma.

Lemma 2.2.8. *Let $\Omega \subset \mathbb{R}^n$ open, $K \subset \Omega$ compact. Then there exists $J_K \in C_c^\infty(\Omega)$ with $0 \leq J_K(x) \leq 1$ for all $x \in \Omega$, $J_K(x) = 1$ for all $x \in K$. Therefore, there exists a sequence $(g_j)_{j\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ with $0 \leq g_j(x) \leq 1$ for all $j \in \mathbb{N}$, and $\lim_{j\to\infty} g_j(x) = 1$ for all $x \in \Omega$. Hence, if $(f_j)_{j\in\mathbb{N}}$ is a sequence in $C^\infty(\Omega)$ with $f_j \to f$ in $L^p(\Omega)$, $1 \leq p < \infty$, then $g_j f_j \in C_c^\infty(\Omega)$ and $g_j f_j \to f$ in $L^p(\Omega)$. Theorem 2.2.7 implies therefore that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for all $1 \leq p < \infty$.*

Proof. Since K is compact, there exists $\varepsilon > 0$ with

$$\{x : |x - y| \leq 2\varepsilon \text{ for a } y \in K\} \subset \Omega$$

Let

$$K_+ := \{x : |x - y| \leq \varepsilon \text{ for a } y \in K\}$$

Then $K_+ \subset \Omega$ is also compact. Let $j \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp} j \subset B_1(0)$, $0 \leq j(x) \leq 1$, $\int j dx = 1$, and set $J_K := j_\varepsilon * \chi_{K_+}$. Then $0 \leq J_K(x) \leq 1$, J_K has compact support in Ω, $J_K = 1$ on K. To construct the sequence g_i, we consider a sequence of compact sets $K_1 \subset K_2 \subset \dots$, such that, for all $x \in \Omega$, there exists $m \in \mathbb{N}$ with $x \in K_m$. Then we define $g_i = J_{K_i}$. The convergence $g_i f_i \to f$ in $L^p(\Omega)$ follows easily from the dominated convergence theorem. □

An application of the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ is the fact that $L^p(\mathbb{R}^n)$ is separable, for all $1 \leq p < \infty$.

Theorem 2.2.9. *For every measurable $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \infty$, the Banach space $L^p(\Omega)$ is separable.*

Proof. It is enough to consider the case $\Omega = \mathbb{R}^n$. We show that there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ with $\varphi_j \in L^p(\mathbb{R}^n)$ with the following property: for all $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ there exists $j \in \mathbb{N}$ with $\|f - \varphi_j\| < \varepsilon$.

For every $j \geq 1$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, we define

$$\Gamma_{m,j} := \left\{ x \in \mathbb{R}^n : \frac{m_i}{2^j} < x_i \leq \frac{m_{i+1}}{2^j}, \text{ for all } i = 1, \dots, n \right\}$$

$\Gamma_{m,j}$ is a cube with length 2^{-j}. For every $j \in \mathbb{N}$, we define

$$\mathcal{F}_j := \left\{ f : \mathbb{R}^n \to \mathbb{C} \text{ so that } \exists\, m^{(1)}, \dots, m^{(N)} \in \mathbb{Z}^n \text{ with } \; f(x) = 0 \text{ for all } x \in \left(\bigcup_{i=1}^n \Gamma_{j,m^{(i)}} \right)^c \right.$$
$$\left. \text{and } f(x) = C_{j,i}, \;\text{ for all } x \in \Gamma_{j,m^{(i)}}, \text{ for constants } C_{j,i} \in \mathbb{Q} + i\mathbb{Q} \right\}$$

The family $\mathcal{F}_j$ is countable, for all $j \in \mathbb{N}$. Let $\mathcal{F} = \cup_{j\geq 1} \mathcal{F}_j$. Also $\mathcal{F}$ is countable, as countable union of countable sets. We claim, that $\mathcal{F}$ is dense in $L^p(\mathbb{R}^n)$. Fix $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$. From Lemma 2.2.8 there exists $g \in C_c^\infty(\mathbb{R}^n)$ with $\|f - g\|_p \leq \varepsilon/2$. Let $K > 0$ be so large, that $\text{supp} g \subset B_K(0)$. Since g is continuous and it has compact support, g must be uniformly continuous. Hence, there exists $\delta > 0$ with

$$|g(x) - g(y)| \leq \frac{\varepsilon}{4|B_K(0)|^{1/p}}$$

for all $x, y \in \mathbb{R}^n$ with $|x - y| \leq \delta$. We choose $j \in \mathbb{N}$ with $2^{-j}\sqrt{n} \leq \delta$ and we set, for $m \in \mathbb{Z}^n$,

$$h(x) = \frac{1}{2^{nj}} \int_{\Gamma_{j,m}} g(y) dy$$

for all $x \in \Gamma_{j,m}$. Then h is a step-function and

$$|h(x) - g(x)| \leq \frac{\varepsilon}{4|B_K(0)|^{1/p}}$$

for all $x \in \mathbb{R}^n$. h is constant in every cube $\Gamma_{j,m}$ and it is different from zero only in finitely many cubes. Therefore, we can find another step-function $\widetilde{h}$, which is again constant in every cube $\Gamma_{j,m}$ and different than zero only in finitely many cubes, but which now takes values in $\mathbb{Q} + i\mathbb{Q}$, so that

$$|\widetilde{h}(x) - g(x)| \leq \frac{\varepsilon}{2|B_K(0)|^{1/p}}$$

Then $\widetilde{h} \in \mathcal{F}_j$ and

$$\|\widetilde{h} - g\|_p^p = \int dx\, |\widetilde{h}(x) - g(x)|^p = \int_{|x|\leq K} dx\, |\widetilde{h}(x) - g(x)|^p \leq (\varepsilon/2)^p$$

Therefore $\|f - \widetilde{h}\|_p \leq \varepsilon$. $\square$

2.3 Sobolev Spaces

Functions in L^p-spaces are characterized by their integrability properties. In Analysis, however, it is often important to consider derivatives. The notion of derivative does not combine very well with L^p spaces; functions in L^p spaces are typically not differentiable. For this reason, we are going to introduce a weaker notion of derivative; Sobolev spaces consists then of L^p-functions, whose weak derivative is again an L^p-function. Sobolev spaces play a very important role in modern analysis and in particular in the study of solutions of partial differential equations.

Let $\Omega \subset \mathbb{R}^n$ open and $\Sigma \subset 2^\Omega$ be the Borel σ-algebra over Ω. On Σ, we consider Lebesgue mass dx. We define the normed space

$$X = \{f \in C^\infty(\Omega) : \|f\|_X < \infty\}$$

with

$$\|f\|_X = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}$$

where the sum runs over all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_{i=1}^n \alpha_i$ and where $D^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$. It is easy to check that X is indeed a normed space (i.e. that $\|.\|_X$ defines a norm on X). On the other hand, it is also easy to check that X, equipped with this norm, is not complete. To this end, it is enough to find a sequence of C^∞ functions, whose L^p limit is not in C^∞. We define $\widetilde{X}$ as the completion of the normed space $(X, \|.\|_X)$. We would like to find a good characterization of $\widetilde{X}$, which is abstractly defined as the quotient space $\widetilde{X} = \mathcal{C}_X / \mathcal{N}_X$, where $\mathcal{C}_X$ and $\mathcal{N}_X$ are spaces of Cauchy- and, respectively, of null-sequences on X. Let $[(f_j)_{j\in\mathbb{N}}] \in \widetilde{X}$, i.e. let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in X. Then, by definition of the norm on X, $(D^\alpha f_j)_{j\in\mathbb{N}}$ defines a Cauchy sequence on $L^p(\Omega)$, for all $\alpha \in \mathbb{N}$ with $|\alpha| \leq m$, since

$$\|D^\alpha f_j - D^\alpha f_\ell\|_{L^p} \leq \|D^\alpha f_j - D^\alpha f_\ell\|_X \to 0$$

as $j, \ell \to \infty$. Since $L^p(\Omega)$ is complete, there exists $f^{(\alpha)} \in L^p(\Omega)$ with $D^\alpha f_j \to f^{(\alpha)}$ as $j \to \infty$. For any $\xi \in C_c^\infty(\Omega)$ we obtain that

$$\int_\Omega D^\alpha \xi \, f^{(0)} \, dx = \lim_{j\to\infty} \int_\Omega D^\alpha \xi \, f_j \, dx = (-1)^{|\alpha|} \lim_{j\to\infty} \int_\Omega \xi \, D^\alpha f_j dx = (-1)^{|\alpha|} \int \xi \, f^{(\alpha)} dx$$

This identity gives a relation between $f^{(\alpha)}$ and $f^{(0)}$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$. In other words, the sequence $[(f_j)_{j\in\mathbb{N}}]$ can be identified with a $f^{(0)} \in L^p(\Omega)$ having the following properties: for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$ there exists $f^{(\alpha)} \in L^p(\Omega)$, with

$$\int_\Omega D^\alpha \xi \, f^{(0)} \, dx = (-1)^{|\alpha|} \int_\Omega \xi f^{(\alpha)} \, dx$$

for all $\xi \in C_c^\infty(\Omega)$. The norm of $[(f_j)_{j\in\mathbb{N}}]$ is given by

$$\|[(f_j)_{j\in\mathbb{N}}]\|_{\widetilde{X}} = \lim_{j\to\infty} \|f_j\|_X = \sum_{|\alpha|\leq m} \|f^{(\alpha)}\|_{L^p}$$

Motivated by this remark, we define the Sobolev space of order $m \in \mathbb{N}$ with exponent $1 \leq p \leq \infty$ as

$$\begin{aligned} H^{m,p}(\Omega) = \Big\{ f \in L^p(\Omega) : \text{ for all } \alpha \in \mathbb{N}^n \text{ with } 1 \leq |\alpha| \leq m \text{ there exists } f^{(\alpha)} \in L^p(\Omega) \\ \text{such that } \int_\Omega D^\alpha \xi \, f \, dx = (-1)^{|\alpha|} \int_\Omega \xi \, f^{(\alpha)} \, dx \quad \text{ for all } \xi \in C_c^\infty(\Omega) \Big\} \end{aligned}$$

We equip $H^{m,p}(\Omega)$ with the norm

$$\|f\|_{H^{m,p}} := \sum_{|\alpha|\leq m} \|f^{(\alpha)}\|_{L^p}$$

For $f \in H^{m,p}(\Omega)$ we call the functions $f^{(\alpha)}$, $1 \leq |\alpha| \leq m$ the weak derivatives of f. We use the notation $f^{(\alpha)} = \partial^\alpha f$. The fact that the weak derivatives of a function $f \in H^{m,p}(\Omega)$ are unique follows from the next lemma.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ open and $f \in L^p(\Omega, dx)$, $1 \leq p \leq \infty$, with*

$$\int \xi \, f \, dx = 0 \qquad \textit{für alle } \xi \in C_c^\infty(\Omega)$$

Then $f = 0$.

Proof. We consider first $p < \infty$. Let $\delta > 0$ and $\Omega_\delta = \Omega \backslash \overline{B_\delta(\partial\Omega)}$. Then Ω_δ is open. We prove that $f|_{\Omega_\delta} = 0$. Since $\delta > 0$ is arbitrary, it follows that $f = 0$ on Ω. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$, $\int \varphi = 1$, $\operatorname{supp} \varphi \subset B_1(0)$. For $0 < \varepsilon < \delta$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ and $f_\varepsilon = \varphi_\varepsilon * f$ (here, we extend f to $\mathbb{R}^n$ through the definition $f = 0$ on Ω^c). From Theorem 2.2.7 we obtain that $f_\varepsilon|_{\Omega_\delta} \to f|_{\Omega_\delta}$ in $L^p(\Omega_\delta)$ as $\varepsilon \to 0$ and that $f_\varepsilon|_{\Omega_\delta} \in C^\infty(\Omega_\delta)$. On the other hand, for all $\xi \in C_c^\infty(\Omega_\delta)$ we have

$$\int \xi f_\varepsilon \, dx = \int \xi(\varphi_\varepsilon * f) \, dx = \int (\xi * \varphi_\varepsilon) f \, dx = 0 \tag{2.15}$$

because $\xi * \varphi_\varepsilon \in C_c^\infty(\Omega)$, for $0 < \varepsilon < \delta$. Since $f_\varepsilon|_{\Omega_\delta}$ is a smooth function, (2.15) implies that $f_\varepsilon|_{\Omega_\delta} = 0$. In fact, if for example $\text{Re } f_\varepsilon(x_0) = a > 0$ for a $x_0 \in \Omega_\delta$, we find $\kappa > 0$ with $B_\kappa(x_0) \subset \Omega_\delta$ and with $\text{Re } f_\varepsilon(x) > a/2$ for all $x \in B_\kappa(x_0)$. For positive $\xi \in C_c^\infty(\Omega_\delta)$ with $\operatorname{supp}\xi \subset B_\kappa(x_0)$, we would have

$$\text{Re} \int \xi f_\varepsilon dx > 0$$

in contradiction to (2.15). $f_\varepsilon|_{\Omega_\delta} = 0$ for all $\varepsilon > 0$ and $f_\varepsilon|_{\Omega_\delta} \to f|_{\Omega_\delta}$ in $L^p(\Omega_\delta)$ imply also $f|_{\Omega_\delta} = 0$.

For $p = \infty$, we choose $\delta > 0$ and a bounded open set $\widetilde{\Omega} \subset \Omega_\delta$. Since $\widetilde{\Omega}$ is bounded, it follows from $f|_{\widetilde{\Omega}} \in L^\infty(\widetilde{\Omega})$ that $f|_{\widetilde{\Omega}} \in L^p(\widetilde{\Omega})$, for all $1 \le p < \infty$. The argument above can be used to show again that $f|_{\widetilde{\Omega}} = 0$. Since $\widetilde{\Omega} \subset \Omega_\delta$ bounded and open is arbitrary, it follows that $f|_{\Omega_\delta} = 0$. Since $\delta > 0$ is arbitrary, it follows that $f = 0$ on Ω. $\square$

If $f \in L^p(\Omega) \cap C^m(\Omega)$, the classical derivatives $D^\alpha f$ coincide with the weak derivatives; this can be shown using integration by parts. The advantage of the weak derivatives is the fact that they exist for a much larger class of functions.

The completeness of L^p spaces implies also the completeness of Sobolev spaces; this is the content of the next theorem.

Theorem 2.3.2. *For all $m \in \mathbb{N}$, $1 \le p \le \infty$, $H^{m,p}(\Omega)$ is a Banach space.*

Proof. The fact that $\|.\|_{H^{m,p}}$ defines a norm is a consequence of the fact that $\|.\|_{L^p}$ is a norm. We have to show that $H^{m,p}$ is complete. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $H^{m,p}(\Omega)$. Then f_j is clearly also a Cauchy sequence in $L^p(\Omega)$; hence there exists $f \in L^p(\Omega)$ with $f_j \to f$ in $L^p(\Omega)$, as $j \to \infty$. Moreover, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$, also $\partial^\alpha f_j$ is a Cauchy sequence in $L^p(\Omega)$. Hence, there exists $g_\alpha \in L^p(\Omega)$ with $\partial^\alpha f_j \to g_\alpha$ in $L^p(\Omega)$. We claim that $f \in H^{m,p}(\Omega)$ and that $\partial^\alpha f = g_\alpha$. In fact, we observe that, for an arbitrary $\xi \in C_c^\infty(\Omega)$,

$$\int_\Omega D^\alpha \xi \, f dx = \lim_{j\to\infty} \int_\Omega D^\alpha \xi \, f_j dx = (-1)^{|\alpha|} \lim_{j\to\infty} \int_\Omega \xi \, \partial^\alpha f_j dx = (-1)^{|\alpha|} \int_\Omega \xi \, g_\alpha dx$$

$\square$

Observe that we used the definition of the completion $\widetilde{X}$ to motivate the definition of the Sobolev space $H^{m,p}(\Omega)$. We have not proven, so far, that $\widetilde{X}$ can be identified with $H^{m,p}(\Omega)$. All we have shown above is the fact that $\widetilde{X}$ can be identified with a subspace of $H^{m,p}(\Omega)$ through the linear map $\phi : \widetilde{X} \to H^{m,p}(\Omega)$ defined through $\phi([(f_j)_{j\in\mathbb{N}}]) = L^p - \lim_{j\to\infty} f_j$.

In the next theorem we show that, for $1 \le p < \infty$, the map ϕ is indeed surjective and therefore that $\widetilde{X} \simeq H^{m,p}(\Omega)$. For $p = \infty$, on the other hand, $\phi(\widetilde{X})$ is a strict subspace of $H^{m,\infty}(\Omega)$ (we already discussed this issue in the case $m = 0$; the space of smooth functions $\mathcal{C}^\infty(\Omega)$, or even the space of continuous functions $\mathcal{C}(\Omega)$, is not dense in $L^\infty(\Omega)$).

Theorem 2.3.3. *Let $f \in H^{m,p}(\Omega)$, $1 \le p < \infty$, $m \in \mathbb{N}$. Then there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $H^{m,p}(\Omega) \cap C^\infty(\Omega)$ with $\|f_j - f\|_{H^{m,p}} \to 0$ as $j \to \infty$.*

Remarks:

- The completeness of the spaces $H^{m,p}(\Omega)$ for $1 \le p < \infty$ is also a consequence of this theorem (because the theorem allows us to identify $H^{m,p}(\Omega)$ with the completion $\widetilde{X}$ which is, by definition, complete).

- THe theorem shows that, for all $1 < p < \infty$, $H^{m,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm of $H^{m,p}$. Sometimes, it is important to consider the completion of the space $C_c^\infty(\Omega)$ (consisting of smooth functions on Ω, with compact supported contained in Ω) with respect to the same norm. For $1 \le p < \infty$ we define the space

  $$H_0^{m,p}(\Omega) = \{f \in H^{m,p}(\Omega) : \exists f_k \in C_c^\infty(\Omega) \text{ with } \|f - f_k\|_{H^{m,p}} \to 0 \text{ als } k \to \infty\}$$

  By definition $H_0^{m,p}(\Omega)$ is a closed subspace of $H^{m,p}(\Omega)$. If $\Omega = \mathbb{R}^n$, then $H_0^{m,p}(\mathbb{R}^n) = H^{m,p}(\mathbb{R}^n)$. On the other hand, if $\Omega \ne \mathbb{R}^n$ and $m \ge 1$, then $H_0^{m,p}(\Omega)$ is strictly contained in $H^{m,p}(\Omega)$. Recall that, for $1 \le p < \infty$, the space $C_0^\infty(\Omega)$ and $C^\infty(\Omega)$ have the same completion with respect to the L^p-norm, namely the space L^p itself. This is no longer true for Sobolev spaces of order $m \ge 1$. The reason is that the price for approximating a function on Ω through a function with compact support is small in the L^p-norm but, on the other hand, it is large in the $H^{m,p}$-norm (if $m \ge 1$), because derivatives become large close to the boundary. The issue at the boundary is also the reason why the proof of Theorem 2.3.3 is much more difficult than the proof of the corresponding result for L^p spaces (Theorem 2.2.7).

To prove Theorem 2.3.3, we first need some preparation.

Definition 2.3.4. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \ge 0$, $\int \varphi = 1$ and $\operatorname{supp} \varphi \subset B_1(0)$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\int \varphi_\varepsilon = 1$ for all $\varepsilon > 0$ and $\int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon \to 0$ for all $\delta > 0$. Such a family $(\varphi_\varepsilon)_{\varepsilon>0}$ is called a standard Dirac sequence (φ_ε converges to a Dirac δ-function).*

Lemma 2.3.5. *Let $\Omega \subset \mathbb{R}^n$ open, $K \subset \mathbb{R}^n$ compact with*

$$B_\delta(K) = \{x \in \mathbb{R}^n : d(x, K) < \delta\} \subset \Omega$$

for a $\delta > 0$. Then there exists a cutoff function $\eta \in C_c^\infty(\Omega)$ with $0 \le \eta \le 1$, $\eta = 1$ on K, $\operatorname{supp} \eta \subset B_\delta(K)$, $|D^\alpha \eta| \le C_\alpha \delta^{-|\alpha|}$, for all $\alpha \in \mathbb{N}^n$.

Proof. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard Dirac sequence. Then

$$\eta := \varphi_{\delta/4} * \chi_{B_{\delta/4}(K)}$$

has the desired properties (see Theorem 2.2.7). □

We need cutoff functions to localize $f \in H^{m,p}(\Omega)$ on subsets of Ω. To this end, we introduce the notion of partition of the identity.

Definition 2.3.6. *Let $S \subset \mathbb{R}^n$, $S \ne \emptyset$, $N \subset \mathbb{N}$.*

- *$(U_i)_{i\in N}$ is an open covering of S, if U_i are non-empty open sets with $S \subset \cup_{i\in N} U_i$.*

- *The covering is called locally finite, if for every* $x \in \cup_{i\in N} U_i$ *there exists* $\varepsilon > 0$ *such that*

$$\{i : B_\varepsilon(x) \cap U_i \neq \emptyset\}$$

  is finite.

- $(\eta_j)_{j\in N}$ *is called a partition of the identity on* S *corresponding to a locally finite open covering* $(U_i)_{i\in N}$ *of* S*, if*

$$\eta_j \in C_c^\infty(U_j),\ \eta_j \geq 0,\ \textit{and} \sum_{j\in N} \eta_j(x) = 1 \quad \textit{for all } x \in S$$

  (in the sum there are only finitely many terms different from zero).

In the next lemma, we show the existence of partitions of the identity.

Lemma 2.3.7. *Let* $\Omega \subset \mathbb{R}^n$ *be open and let* $K_j \subset U_j \subset \bar{U}_j \subset \Omega$ *for all* $j \in \mathbb{N}$ *with* $K_j, \bar{U}_j$ *compact, such that* $(U_j)_{j\in\mathbb{N}}$ *is a locally finite open covering of* Ω *and* $K_i \cap K_j = \emptyset$ *for all* $i \neq j$*. Then there exists a partition of the identity* $(\eta_j)_{j\in\mathbb{N}}$ *on* Ω *corresponding to the open covering* $(U_j)_{j\in\mathbb{N}}$ *and having the additional property, that* $\eta_j(x) = 1$ *for all* $x \in K_j$*. Here some or even all* K_j *are allowed to be empty.*

Proof. We define the sets $V_j = U_j \setminus \cup_{i\neq j} K_i$. Then (since the sets K_i are pairwise disjoint) $K_j \subset V_j \subset U_j$ and $V_j \cap K_i = \emptyset$ for all $i \neq j$. We claim now that $(V_j)_{j\in\mathbb{N}}$ is a locally finite covering of Ω. Since $(U_j)_{j\in\mathbb{N}}$ is locally finite, it follows that

$$\{i \in \mathbb{N} : U_i \cap \overline{U}_j \neq \emptyset\}$$

is finite. Otherwise, we could find a sequence $(x_i)_{i\in\mathbb{N}}$ in $\overline{U}_j$ such that each x_i is contained in a different U_i. Since $\overline{U}_j$ is compact, we could find an accumulation point $x \in \overline{U}_j$ of the sequence $(x_i)_{i\in\mathbb{N}}$. In this case, every neighbourhood of x would have non-empty intersection with infinitely many U_i, contradicting the assumption that $(U_i)_{i\in\mathbb{N}}$ is locally finite.

It follows that $K_i \cap \overline{U}_j \neq \emptyset$ for at most finitely many $i \in \mathbb{N}$, let's say for $i = 1, 2, \dots, m_j$. Therefore

$$V_j = U_j \setminus \cup_{i=1,\dots,m_j} K_i$$

is non-empty and open. Moreover $(V_j)_{j\in\mathbb{N}}$ defines a covering of Ω because, for every $x \in \Omega$ there exists j with $x \in U_j$; hence, either $x \in V_j$ or $x \in K_i \subset V_i$ for a $i \neq j$. The fact that $(V_j)_{j\in\mathbb{N}}$ is locally finite follows from $V_i \subset U_i$ and since $(U_i)_{i\in\mathbb{N}}$ is locally finite.

Next, we construct an open covering $(W_j)_{j\in\mathbb{N}}$ of Ω with $K_j \subset W_j \subset \overline{W}_j \subset V_j$ and so that, for all $j \in \mathbb{N}$ there exists $\delta_j > 0$ with $B_{\delta_j}(\overline{W}_j) \subset V_j$. We define W_m recursively. Assume we already constructed open sets $W_1, \dots, W_{m-1}$ with $K_j \subset W_j \subset \overline{W}_j \subset B_{\delta_j}(\overline{W}_j) \subset V_j$, for a $\delta_j > 0$, and with

$$\Omega \subset \cup_{j\leq m-1} W_j \cup \cup_{j\geq m} V_j$$

Then

$$\partial V_m \subset \cup_{j\leq m-1} W_j \cup \cup_{j>m} V_j \, .$$

Since ∂V_m is compact and the right hand side is open, there exists $\delta_m > 0$ with

$$\overline{B_{\delta_m}(\partial V_m)} \subset \cup_{j\leq m-1} W_j \cup \cup_{j>m} V_j \, .$$

We set $W_m = V_m \backslash \overline{B_{\delta_m}(\partial V_m)}$. Since $V_m \neq \emptyset$, we can choose δ_m so small that $W_m \neq \emptyset$. Then

$$V_m \subset W_m \cup \overline{B_{\delta_m}(\partial V_m)} \subset \cup_{j\leq m} W_j \cup \cup_{j>m} V_j$$

and

$$B_{\delta_m}(\overline{W_m}) \subset V_m$$

By Lemma 2.3.5 there exists $\widetilde{\eta}_j \in C_c^\infty(V_j)$; with $0 \leq \widetilde{\eta}_j \leq 1$ and $\widetilde{\eta}_j|_{\overline{W}_j} = 1$. Since $\cup_{j\geq 1} W_j = \Omega$, we find $\sum_{j\in\mathbb{N}} \widetilde{\eta}_j(x) > 0$ for all $x \in \Omega$, where, in the sum, only finitely terms are different than zero. The functions

$$\eta_j(x) = \frac{\widetilde{\eta}(x)}{\sum_j \widetilde{\eta}_j(x)}$$

have the desired properties. $\square$

To approximate functions in $H^{m,p}(\Omega)$, we show first, how to approximate them locally. Afterwards, we use an appropriate partition of the identity to reduce the global problem to the local analysis.

Lemma 2.3.8. *Let $\Omega \subset \mathbb{R}^n$ open, $f \in H^{m,p}(\Omega)$, $1 \leq p < \infty$. We choose a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ and we set*

$$(T_\varepsilon f)(x) = \int_\Omega \varphi_\varepsilon(x-y) f(y) dy = (\varphi_\varepsilon * \chi_\Omega f)(x)$$

For an open set $D \subset \Omega$ with $\delta = dist(D, \partial\Omega) > 0$, we find $T_\varepsilon f \in H^{m,p}(D) \cap C^\infty(D)$ for all $0 < \varepsilon < \delta$ and $T_\varepsilon f \to f$ as $\varepsilon \to 0$ in $H^{m,p}(D)$.

Proof. By Theorem 2.2.7, $T_\varepsilon f \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha T_\varepsilon f(x) = \int dy \, D^\alpha \varphi_\varepsilon(x-y) f(y) = (-1)^{|\alpha|} \int dy \, D_y^\alpha(\varphi_\varepsilon(x-y)) f(y)$$

For $x \in \Omega$ with $\text{dist}(x, \partial\Omega) > \varepsilon$, the function $y \to \varphi_\varepsilon(x-y)$ is contained in the space $C_c^\infty(\Omega)$. Since $f \in H^{m,p}(\Omega)$, we obtain

$$D^\alpha T_\varepsilon f(x) = \int dy \, \varphi_\varepsilon(x-y) \, \partial^\alpha f(y) = T_\varepsilon(\partial^\alpha f)(x) \tag{2.16}$$

Hence, $D^\alpha T_\varepsilon f$ and $T_\varepsilon(\partial^\alpha f)$ are equal on D. In other words, $f \in H^{m,p}(\Omega)$ implies that $f \in H^{m,p}(D)$ and thus that $\partial^\alpha f \in L^p(D)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$. This implies, by Theorem 2.2.7, that $T_\varepsilon(\partial^\alpha f) \in L^p(D)$ and therefore, by (2.16), that $\partial^\alpha T_\varepsilon f = D^\alpha T_\varepsilon f \in L^p(D)$, for all $|\alpha| \leq m$. Hence, $T_\varepsilon f \in H^{m,p}(D)$. Moreover

$$\|T_\varepsilon f - f\|_{H^{m,p}} \leq \|T_\varepsilon f - f\|_p + \sum_{|\alpha|\leq m} \|\partial^\alpha T_\varepsilon f - \partial^\alpha f\|_p = \|T_\varepsilon f - f\|_p + \sum_{|\alpha|\leq m} \|T_\varepsilon(\partial^\alpha f) - \partial^\alpha f\|_p \to 0$$

as $\varepsilon \to 0$, from Theorem 2.2.7. $\square$

We are now ready to show Theorem 2.3.3.

Proof of Theorem 2.3.3. It is always possible to find a locally finite open covering $(U_j)_{j\in\mathbb{N}}$ of Ω such that $\overline{U}_k \subset \Omega$ is compact for all $k \in \mathbb{N}$. In fact, if $\Omega = \mathbb{R}^n$, we can choose a family of open rectangles

$$\{(i_1 - 1, i_1 + 1) \times \cdots \times (i_n - 1, i_n + 1) : i_1, \ldots, i_n \in \mathbb{Z}\} .$$

If $\partial\Omega \neq \emptyset$, we set

$$\widetilde{U}_k = \left\{x \in \Omega : 2^{k-1} < \operatorname{dist}(x, \partial\Omega) < 2^{k+1}\right\}$$

for all $k \in \mathbb{Z}$. The sets $\widetilde{U}_k$ do not need to be bounded, but then we can take intersections with the rectangles considered above. I.e. we can consider sets of the form

$$\left\{\widetilde{U}_k \cap (i_1 - 1, i_1 + 1) \times \cdots \times (i_n - 1, i_n + 1) : k, i_1, \ldots, i_n \in \mathbb{Z}\right\} .$$

This gives a locally finite covering of Ω with the property that $\overline{U}_j \subset \Omega$ is compact for all j. Let now $(\eta_k)_{k\in\mathbb{N}}$ denote a corresponding partition of the identity. By Lemma 2.3.8, we find, for all $k \in \mathbb{N}$, $f_{k,\varepsilon} \in C^\infty(U_k)$ with

$$\|f - f_{k,\varepsilon}\|_{H^{m,p}(U_k)} \leq \frac{\varepsilon}{\|\eta_k\|_{C^m(\overline{\Omega})} + 1}$$

because $\operatorname{dist}(U_k, \Omega) > \delta_k$ for sufficiently small $\delta_k > 0$. We define

$$f_\varepsilon := \sum_{k\in\mathbb{N}} \eta_k f_{k,\varepsilon}$$

Then

$$f - f_\varepsilon = \sum_{k\in\mathbb{N}} \eta_k (f_{k,\varepsilon} - f)$$

where, locally in Ω, only finitely many terms are different from zero. We compute the weak derivative of the terms in the sum. To this end, we use the product rule. For arbitrary open $\Omega \subset \mathbb{R}^n$, $f \in H^{1,p}(\Omega)$, $\eta \in C^\infty(\Omega)$, we have $\eta f \in H^{1,p}(\Omega)$ and

$$\partial_j(\eta f) = D_i \eta \, f + \eta \partial_i f \tag{2.17}$$

where $D_i \eta$ and $\partial_i f$ are the classical derivative of η and, respectively, the weak derivative of f. To prove (2.17) we notice that, for all $\xi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega D_i \xi \, \eta f &= \int_\Omega D_i(\xi\eta) \, f - \int_\Omega \xi \, (D_i \eta) \, f \\
&= - \int_\Omega \xi\eta \, \partial_i f - \int_\Omega \xi (D_i \eta) \, f \\
&= - \int_\Omega \xi \, (\eta \partial_i f + D_i \eta f)
\end{aligned}$$

where we used the fact that $\xi\eta \in C_c^\infty(\Omega)$. Inductively, it follows that $f \in H^{m,p}(\Omega), \eta \in C^\infty(\Omega)$ implies that $f\eta \in H^{m,p}(\Omega)$ and

$$\partial^\alpha f\eta = \sum_{\beta \le \alpha} \binom{\beta}{\alpha} \partial^{\alpha-\beta} \eta \, \partial^\beta f$$

We apply this result to the difference $f - f_\varepsilon$. We find

$$\partial^\alpha f - \partial^\alpha f_\varepsilon = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \sum_k \partial^{\alpha-\beta} \eta_k \, \left(\partial^\beta f - \partial^\beta f_{k,\varepsilon} \right)$$

Hence

$$\| \partial^\alpha f - \partial^\alpha f_\varepsilon \|_p \le C \sum_k \| \eta_k \|_{C^m(\overline{\Omega})} \, \| f - f_{\varepsilon,k} \|_{H^{m,p}(U_k)} \le C\varepsilon$$

for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$. Thus

$$\| f - f_\varepsilon \|_{H^{m,p}} \le C\varepsilon$$

□

An application of the density of $C^\infty(\Omega)$ in $H^{m,p}(\Omega)$ is the extension of the product rule to products of Sobolev functions.

Theorem 2.3.9. *Let $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$ and $p' \ge 1$, so that $1/p + 1/p' = 1$. Let $f \in H^{m,p}(\Omega), g \in H^{m,p'}(\Omega)$. Then $fg \in H^{m,1}(\Omega)$ and*

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g$$

Proof. We can assume that $p < \infty$ (otherwise $p' < \infty$). We assume moreover that $m = 1$ and we choose a sequence $f_k \in C^\infty(\Omega) \cap H^{1,p}(\Omega)$ with $f_k \to f$ in $H^{1,p}(\Omega)$. Then, for $\xi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int D_i\xi\, fg &= \lim_{k\to\infty} \int D_i\xi\, f_k g = \lim_{k\to\infty} \int D_i(\xi f_k)g - \int (D_i f_k)\xi g \\ &= -\lim_{k\to\infty} \int (\xi f_k \partial_i g + (D_i f_k)\xi g) = -\int (\xi f \partial_i g + \partial_i f_k \xi g)\end{aligned}$$

Recursively, we can also consider the case $m > 1$. $\square$

3 Compactness

3.1 Compact Sets in Metric Spaces

Recall that a topological space (X, τ) is called compact if, for every family $(U_i)_{i\in I}$ with $U_i \in \tau$ for all $i \in I$ and $\cup_{i\in I} U_i = X$ there exist $i_1, \dots, i_n \in I$ with $\cup_{j=1}^n U_{i_j} = X$. A metric space (X, d) is called compact, if (X, τ_d) is a compact topological space, where τ_d is the topology on X induced by the metric d. A set $A \subset X$ is called compact if (A, d) is a compact metric space. In other words, $A \subset X$ is compact if for every open covering $(U_i)_{i\in I}$ of A (i.e. every family of open sets $(U_i)_{i\in I}$ with $A \subset \cup_{i\in I} U_i$), there exist $i_1, \dots, i_n \in I$ with $A \subset \cup_{j=1}^n U_{i_j}$. We will also use the notion of pre-compactness.

Definition 3.1.1. *Let (X, d) be a metric space. A subset $A \subset X$ is called pre-compact if, for all $\varepsilon > 0$, A has a finite covering with ε-balls, i.e. if there are $x_1, \dots, x_n \in X$ with $\cup_{i=1}^n B_\varepsilon(x_i) \supset A$.*

Remarks: Let (X, d) be a metric space. Then

- Subsets of pre-compact sets are pre-compact.
- $A \subset X$ pre-compact implies that A is bounded.
- $A \subset X$ pre-compact implies that $\overline{A}$ is closed and pre-compact.

Theorem 3.1.2. *Let (X, d) be a metric space. For $A \subset X$, the following statements are equivalent:*

i) A is compact.

ii) A is sequentially compact, i.e. every sequence in A has a convergent subsequence.

iii) (A, d) is complete and A is pre-compact.

Proof. i) $\Rightarrow$ ii). Let A compact and $(x_k)_{k\in\mathbb{N}}$ be a sequence in A. We proved in Theorem 1.1.8 that $(x_k)_{k\in\mathbb{N}}$ has at least one accumulation point $x \in A$. Every open neighbourhood of x contains infinitely many points from the sequence $(x_k)_{k\in\mathbb{N}}$. For all $n \in \mathbb{N}$, we find thus $k_n \in \mathbb{N}$ with $x_{k_n} \in B_{1/n}(x)$. The subsequence x_{k_n} converges therefore towards x.

ii) $\Rightarrow$ iii). Let A be sequentially compact and $(x_k)_{k\in\mathbb{N}}$ a Cauchy sequence in A. From ii) there exists a subsequence x_{i_j} and $x \in A$ with $x_{i_j} \to x$ as $j \to \infty$. Then $d(x_\ell, x) \leq d(x_\ell, x_{i_j}) + d(x_{i_j}, x)$ for all $j \in \mathbb{N}$. Hence $d(x_\ell, x) \leq \limsup_{j\to\infty} d(x_\ell, x_{i_j})$. The r.h.s. converges towards zero, as $\ell \to \infty$, since x_ℓ is a Cauchy sequence. Therefore $x_\ell \to x$. We proved that A is complete. To show pre-compactness, we assume there exists $\varepsilon > 0$, so that A has no finite covering with ε-balls. Under this assumption, we can inductively construct a sequence $x_j \in A$ with

$$x_{j+1} \in A \setminus \cup_{j=1}^{j} B_\varepsilon(x_j)\,.$$

The sequence (x_j) has no accumulation points, in contradiction to ii).

iii) $\Rightarrow$ i). Let $(U_j)_{j\in J}$ be an open covering of A. We set

$$\mathcal{B} = \{B \subset A : I \subset J, B \subset \cup_{i\in I} U_i \;\Rightarrow\; |I| = \infty\}$$

($\mathcal{B}$ is the family of subsets of A, having no finite sub-covering). For all $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $x \in X$ with $B \cap B_\varepsilon(x) \in \mathcal{B}$. In fact, from iii) there exists $n_\varepsilon \in \mathbb{N}$ with $x_1, \dots, x_{n_\varepsilon} \in X$ and with $A \subset \cup_{i=1}^{n_\varepsilon} B_\varepsilon(x_i)$. Therefore we have $B = \cup_{i=1}^{n_\varepsilon} B \cap B_\varepsilon(x_i)$ and there exists $j \in \{1, 2, \dots, n_\varepsilon\}$ with $B \cap B_\varepsilon(x_j) \in \mathcal{B}$. Let us assume that $A \in \mathcal{B}$. We set $B_1 = A$ and we find $x_2 \in X$ with

$$B_2 = B_{1/2}(x_2) \cap A \in \mathcal{B}$$

Furthermore, there exists $x_3 \in X$ with

$$B_3 = B_{1/3}(x_3) \cap B_2 \in \mathcal{B}$$

Iteratively, we find, for all $k \in \mathbb{N}$, $x_k \in X$ with

$$B_k = B_{1/k}(x_k) \cap B_{k-1} \in \mathcal{B}$$

Choosing a sequence $y_k \in B_k$ for all $k \in \mathbb{N}$, we obtain, for $\ell \geq k$, $d(y_k, y_\ell) \leq 2/k$. Therefore $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in A. From iii), there exists $y \in A$ with $y_k \to y$. Since $(U_j)_{j\in\mathbb{N}}$ is a covering of A, there exists i_0 with $y \in U_{i_0}$. Thus,

$$B_k \subset B_{1/k}(x_k) \subset B_{2/k}(y_k) \subset B_{(2/k)+d(y_k,y)}(y) \subset U_{i_0}$$

for k large enough, because $(2/k) + d(y_k, y) \to 0$ as $k \to \infty$ and because U_{i_0} is open. □

Consequences: Let (X, d) be a metric space. Then

- $A \subset X$ compact $\Rightarrow$ A closed (since A complete implies that A is closed).
- If X is complete, then $A \subset X$ is pre-compact if and only if $\overline{A}$ is compact (since close subsets of complete spaces are complete).

For subsets of finite dimensional normed spaces, a set A is pre-compact if and only if it is bounded; this also implies that A is compact if and only if it is closed and bounded. To prove this claim, we first show that all norms on finite dimensional vector spaces are equivalent.

Lemma 3.1.3. *Let X be a finite dimensional vector space over $\mathbb{K}$. Then all norms on X are equivalent. In other words, if $\|.\|_1, \|.\|_2$ are two norms on X, then there exists $C > 0$ with*

$$\frac{1}{C}\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2$$

for all $x \in X$.

Proof. Let $n = \dim X < \infty$ and $\{e_1, \ldots, e_n\}$ a basis of X. Every $x \in X$ has a unique representation as $x = \sum_{j=1}^n x_j e_j$. Then

$$\|x\|_\infty := \max_{j=1,\ldots,n} |x_i|$$

defines a norm on X. Now let $\|.\|$ be any other norm on X. We have

$$\|x\| \leq \sum_{j=1}^n |x_j| \, \|e_j\| \leq \left(\sum_{j=1}^n \|e_j\| \right) \|x\|_\infty \tag{3.1}$$

Assume that the converse estimate does not hold true. Then there exists a sequence $x^{(k)} \in X$ with $\|x^{(k)}\|_\infty = 1$ for all $k \in \mathbb{N}$ and $\|x^{(k)}\| \to 0$ as $k \to \infty$. Then $|x_i^{(k)}| \leq 1$ for all $i = 1, \ldots, n$ and all $k \in \mathbb{N}$. Since $B_1^{\mathbb{K}}(0)$ is pre-compact in $\mathbb{K}$, it follows that there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ and $\xi_1, \ldots, \xi_n \in \mathbb{K}$ such that

$$x_i^{(k_j)} \to \xi_i \qquad \text{as } j \to \infty,$$

for all $i = 1, \ldots, n$. With $\xi = \sum_{j=1}^n \xi_j \, e_j$, we obtain

$$\|x^{(k_j)} - \xi\|_\infty \to 0 \tag{3.2}$$

as $j \to \infty$. In particular $\|\xi\|_\infty = 1$. On the other hand (3.1) and (3.2) imply that

$$\|x^{(k_j)} - \xi\| \to 0$$

as $k_j \to \infty$. Therefore $\|\xi\| = 0$ and $\xi = 0$; this is a contradiction to $\|\xi\|_\infty = 1$. $\square$

Corollary 3.1.4. *Every finite dimensional subspace of a normed space is complete and therefore closed.*

Proof. Let $Y \subset X$ be a closed linear subspace with $\dim Y = n < \infty$ and with basis $\{e_1, \dots, e_n\}$. On Y, the norms $\|.\|_X$ and

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j| \qquad \text{if } x = \sum_{j=1}^n x_j e_j$$

are equivalent by Lemma 3.1.3. It is therefore enough to show completeness w.r.t. the norm $\|.\|_\infty$. Let $x^{(k)}$ be a Cauchy sequence on Y. Then $x_i^{(j)}$ is a Cauchy sequence on $\mathbb{K}$, and therefore it converges. There exist therefore $\xi_1, \dots, \xi_n \in \mathbb{K}$ with $x_i^{(k)} \to \xi_i$ as $k \to \infty$, for all $i = 1, \dots, n$. Then, setting $\xi = \sum_{j=1}^n \xi_j e_j$, we find $\|x^{(k)} - \xi\|_\infty \to 0$ as $k \to \infty$. □

We are now ready to characterize compactness on finite dimensional spaces.

Theorem 3.1.5 (Heine-Borel's Theorem)**.** *Let X be a finite dimensional normed space. Then $A \subset X$ is compact if and only if A is bounded and closed.*

Proof. The implication "⇒" follows from Theorem 3.1.2. To show the implication "⇐", it is enough to show that every bounded subset of a finite dimensional space is pre-compact. Let $\{e_1, \dots, e_n\}$ be a basis of X. Then we define the norm

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j| \qquad (\text{for } x = \sum_{j=1}^n x_j e_j)$$

on X. Since all norms are equivalent, it is enough to prove the claim w.r.t. the norm $\|.\|_\infty$. To this end, it is enough to show that, for all $R > 0, \varepsilon > 0$, there exists $n(R, \varepsilon) \in \mathbb{N}$ and $x^{(1)}, \dots, x^{(n(R,\varepsilon))} \in X$ with

$$B_R(0) \subset \cup_{j=1}^{n(R,\varepsilon)} B_\varepsilon(x^{(j)}).$$

Let $m = \lceil (R/\varepsilon) \rceil$ be the smallest integer larger than R/ε. Then

$$\begin{aligned} B_R(0) &= \{x : \|x\|_\infty < R\} = \cap_{\ell=1}^n \{x \in X : |x_\ell| < R\} \\ &\subset \cap_{\ell=1}^n \cup_{j=-m}^{m-1} \{x \in X : j_\ell \varepsilon \le x_\ell < (j_\ell + 1)\varepsilon\} \\ &\subset \cup_{j_1,\dots,j_n=-m}^m \cap_{\ell=1}^n \{x \in X : |x_\ell - j_\ell \varepsilon| < \varepsilon\} \\ &= \cup_{j_1,\dots,j_n=-m}^m \{x \in X : \max_{\ell=1,\dots,n} |x_\ell - j_\ell \varepsilon| < \varepsilon\} \\ &= \cup_{j_1,\dots,j_n=-m}^m B_\varepsilon(\sum_{\ell=1}^n \varepsilon j_\ell e_\ell) \end{aligned}$$

□

Conversely, it turns out that if every closed and bounded subset A of a normed space X is compact, then X must have finite dimension.

Proposition 3.1.6. *Let X be a normed space. Then $\overline{B_1(0)}$ is compact if and only if* $\dim X < \infty$.

Proof. "$\Leftarrow$": follows from Theorem 3.1.5. "$\Rightarrow$": Let $y_1, \dots, y_n \in X$ with

$$\overline{B_1(0)} \subset \bigcup_{j=1}^{n} B_{1/2}(y_j) \tag{3.3}$$

Let $Y = \operatorname{span}\{y_1, \dots, y_n\}$. From Corollary 3.1.4 it follows that Y is closed. Assume that $Y \neq X$. Then we claim that, for all $\theta \in (0,1)$, there exists $x_\theta \in X \backslash Y$ with $\|x_\theta\| = 1$ and $\operatorname{dist}(x, Y) \geq \theta$. Assuming the claim to hold true, it is easy to find a contradiction to (3.3), choosing $\theta > 1/2$. To prove the claim, choose $x \in X \backslash Y$. Since Y is closed, $\operatorname{dist}(x, Y) > 0$. Thus there exists $y_\theta \in Y$ with

$$0 < \|x - y_\theta\| \leq \frac{1}{\theta} \operatorname{dist}(x, Y)$$

We put $x_\theta = (x - y_\theta)/\|x - y_\theta\|$. Then, for all $y \in Y$,

$$\|x_\theta - y\| = \frac{1}{\|x - y_\theta\|} \, \|x - (y_\theta + \|x - y_\theta\| y)\| \geq \frac{\operatorname{dist}(x, Y)}{\|x - y_\theta\|} \geq \theta$$

$\square$

3.2 Compact Subsets of $\mathcal{C}(K)$

In Section 3.1 we found a simple characterization of compact sets in finite dimensional normed spaces; if dim $X < \infty$, $A \subset X$ is compact if and only if A is closed and bounded. In this and in the next section, we consider examples of infinite dimensional spaces, and we try to find a similar simple characterization of compact subsets. In the current section, we consider the space $\mathcal{C}(K)$ of continuous functions on a compact space K, equipped with the norm

$$\|f\| = \sup_{x \in K} |f(x)|$$

We drop here the subscript $\mathbb{K}$ in the notation for $\mathcal{C}(K)$ but, like in Section 2.1, functions in $\mathcal{C}(K)$ can take values in $\mathbb{R}$ or in $\mathbb{C}$.

To describe compact subsets of $\mathcal{C}(K)$, we need to introduce the notion of equicontinuity.

Definition 3.2.1. *$S \subset \mathcal{C}(K)$ is called equicontinuous at $x \in K$ if*

$$\forall \varepsilon > 0 \, \exists U_x \text{ open enighbourhood of } x \text{ in } K : |f(x) - f(y)| \leq \varepsilon, \; \forall y \in U_x, \forall f \in S$$

$S \subset \mathcal{C}(K)$ is called equicontinuous, if S is equicontinuous at x, for all $x \in K$.

With this definition, we are ready to describe compact subsets of $\mathcal{C}(K)$.

Theorem 3.2.2 (Arzelá-Ascoli)**.** *A subset $S \subset \mathcal{C}(K)$ is pre-compact if and only if S is bounded and equicontinuous.*

Remark: $S \subset \mathcal{C}(K)$ is pre-compact if and only if $\overline{S}$ is compact. Hence, from Theorem 3.2.2 it follows that S is compact if and only if S is closed, bounded and equicontinuous.

Proof. " $\Rightarrow$" : Fix $\varepsilon > 0$. S has a finite covering with ε-balls. In other words, there exist $f_1, \dots, f_n \in S$ with $S \subset \cup_{j=1}^n B_\varepsilon(f_j)$. Hence, for all $f \in S$, there exist $j \in \{1, \dots, n\}$ with $f \in B_\varepsilon(f_j)$, and therefore

$$\|f\| \leq \|f_j\| + \varepsilon$$

This proves that

$$\sup_{f \in S} \|f\| \leq \max_{j=1,\dots,n} \|f_j\| + \varepsilon$$

and also that S is bounded. Now, fix $x \in K$. For $i = 1, \dots, n$ there exists an open neighbourhood U_i of x in K, so that

$$|f_i(x) - f_i(y)| \leq \varepsilon \quad \forall\, y \in U_i$$

(by continuity of f_i). Now, let $U := \cap_{i=1}^n U_i$. Then U is an open neighbourhood of $x \in K$ and, for all $f \in S$, for all $y \in U$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq 2\|f - f_i\| + |f_i(x) - f_i(y)| \leq 3\varepsilon \end{aligned}$$

for appropriate choice of $i \in \{1, \dots, n\}$. This shows the equicontinuity of S.

"$\Leftarrow$" : Fix $\varepsilon > 0$. For $x \in K$, let U_x be an open neighbourhood of x, so that

$$|f(x) - f(y)| \leq \varepsilon \quad \forall\, f \in S, y \in U_x$$

Then $\{U_x\}_{x \in K}$ is an open covering of K. Hence, there exist $x_1, \dots, x_n \in K$ with $K = \cup_{j=1}^n U_{x_j}$. Since S is bounded,

$$R = \{(f(x_1), \dots, f(x_n)) : f \in S\}$$

is a bounded subset of $\mathbb{C}^n$, equipped with the max-norm. Therefore, R is pre-compact in $\mathbb{C}^n$. In other words, there are $f_1, \dots, f_m \in S$, with

$$R \subset \bigcup_{i=1}^m B_\varepsilon((f_i(x_1), \dots, f_i(x_n))$$

We claim that

$$S \subset \bigcup_{i=1}^m B_{3\varepsilon}(f_i) \tag{3.4}$$

Since $\varepsilon > 0$ is arbitrary, (3.4) implies the pre-compactness of S. To show (3.4) we notice that, for given $f \in S$, there exists $j \in \{1, \dots, m\}$ with

$$\|(f(x_1), \dots, f(x_n)) - (f_j(x_1), \dots, f_j(x_n))\|_{\mathbb{C}^n} = \max_{i=1,\dots,n} |f_j(x_i) - f(x_i)| < \varepsilon \tag{3.5}$$

For arbitrary $x \in K$, there is $i \in \{1, \dots, n\}$ with $x \in U_{x_i}$. Therefore (with j chosen so that (3.5) holds true), we find

$$|f(x) - f_j(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| \leq 3\varepsilon$$

Hence $\|f - f_j\| \leq 3\varepsilon$ and $f \in B_{3\varepsilon}(f_j)$. □

3.3 Compact Subset of L^p-Spaces

We can ask the same question as in Section 3.2 but now for subsets of L^p-spaces. In this case, the description of compact subsets is given by the following Theorem of M. Riesz.

Theorem 3.3.1 (Riesz's Theorem). *Let $1 \leq p < \infty$, $A \subset L^p(\mathbb{R}^n)$. Then A is pre-compact if and only if A is bounded,*

$$\sup_{f \in A} \|f(. + h) - f\|_p \to 0 \qquad \text{as } h \to 0 \tag{3.6}$$

and

$$\sup_{f \in A} \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \to 0 \qquad \text{as } R \to \infty\,. \tag{3.7}$$

Remark: for $f \in L^p(\mathbb{R}^n)$ fixed, we have

$$\|f(. + h) - f\|_p \to 0 \qquad \text{as } h \to 0 \tag{3.8}$$

$$\|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \to 0 \qquad \text{as } R \to \infty \tag{3.9}$$

In Theorem 3.3.1 these conditions are non-trivial, because they have to hold uniformly in $f \in A$. The convergence (3.9) follows from dominated convergence, since $f(x)\mathbf{1}(|x| > R) \to 0$ pointwise as $R \to \infty$ and since $|f(x)\mathbf{1}(|x| > R)| \leq |f(x)|$ uniformly in R. To show (3.8), we choose a sequence $f_j \in C_c^\infty(\mathbb{R}^n)$ with $\|f - f_j\|_p \to 0$ as $j \to \infty$. Then

$$\begin{aligned}\|f(. + h) - f\|_p &\leq \|f(. + h) - f_j(. + h)\|_p + \|f_j(. + h) - f_j\|_p + \|f_j - f\|_p \\ &\leq 2\|f - f_j\|_p + C_j \sup_{x \in \mathbb{R}^n} |f_j(x + h) - f_j(x)|\end{aligned}$$

where we used the fact that f_j has compact support to estimate the L^p-norm through the L^∞-norm. Now, for given $\varepsilon > 0$, choose $j \in \mathbb{N}$ so large that $\|f - f_j\| \leq \varepsilon/3$. For j fixed, we find then (since the f_j are uniformly continuous) $\delta > 0$ so small that

$$C_j \sup_{x \in \mathbb{R}^n} |f_j(x + h) - f_j(x)| < \varepsilon/3$$

for all $|h| \le \delta$. Then $\|f(.+h) - f\|_p \le \varepsilon$ for all $|h| \le \delta$. This shows (3.8).

Remark: The theorem holds true for subsets of $L^p(\mathbb{R}^n)$. There are extensions to subsets of $L^p(\Omega)$, for measurable $\Omega \subset \mathbb{R}^n$, but they are non-trivial and we will not discuss them here.

Proof. "$\Rightarrow$": For arbitrary $\varepsilon > 0$ there are $g_1, \ldots, g_n \in L^p(\mathbb{R}^n)$ with $A \subset \cup_{j=1}^n B_\varepsilon(g_j)$. For $f \in A$ we find $f \in B_\varepsilon(g_j)$ for an appropriate $j \in \{1, \ldots, n\}$. Therefore

$$\sup_{f \in A} \|f\|_p \le \varepsilon + \max_{j=1,\ldots,n} \|g_j\| \tag{3.10}$$

$$\sup_{f \in A} \|f(.+h) - f\|_p \le 2\varepsilon + \max_{j=1,\ldots,n} \|g_j(.+h) - g_j\|_p \tag{3.11}$$

$$\sup_{f \in A} \|f\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \le \varepsilon + \max_{j=1,\ldots,n} \|g_j\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \tag{3.12}$$

The r.h.s. of (3.11) and (3.12) are certainly smaller than 3ε, respectively, 2ε, if $|h|$ is small enough, respectively, if R is large enough.

"$\Leftarrow$" : Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard Dirac sequence and R_ε with $R_\varepsilon \to \infty$ as $\varepsilon \to 0$. For $f \in A$, let

$$T_\varepsilon f(x) = \int_{B_{R_\varepsilon}(0)} \varphi_\varepsilon(x-y) f(y) dy = \left(\varphi_\varepsilon * \mathbf{1}_{B_{R_\varepsilon}(0)} f\right)(x)$$

From Theorem 2.2.7, $T_\varepsilon f \in L^p(\mathbb{R}^n)$ and

$$\begin{aligned}(T_\varepsilon f - f)(x) &= \int \varphi_\varepsilon(x-y)(f(y) - f(x)) \mathbf{1}_{B_{R_\varepsilon}(0)}(y) - \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} \varphi_\varepsilon(x-y) f(x) dy \\ &= \int \varphi(y) \left(f(x+\varepsilon y) - f(x)\right) \mathbf{1}(|x+\varepsilon y| \le R_\varepsilon) - \int dy \varphi(y) f(x) \mathbf{1}(|x+\varepsilon y| \ge R_\varepsilon)\end{aligned}$$

Since $|x+\varepsilon y| \ge R_\varepsilon$ implies that $|x| \ge R_\varepsilon - \varepsilon$, we conclude that

$$|T_\varepsilon f(x) - f(x)| \le \int dy \varphi(y) |f(x+\varepsilon y) - f(x)| + |f(x)| \mathbf{1}(|x| \ge R_\varepsilon - \varepsilon)$$

This implies, with Hölder's inequality, that

$$\begin{aligned}\|T_\varepsilon f - f\|_p^p &\lesssim \int dx dy\, \varphi(y) |f(x+\varepsilon y) - f(x)|^p + \|f\|^p_{L^p(\mathbb{R}^n \setminus B_{R_\varepsilon - \varepsilon}(0))} \\ &\le \sup_{|h| \le \varepsilon} \|f(.+h) - f\|_p^p + \|f\|^p_{L^p(\mathbb{R}^n \setminus B_{R_\varepsilon - \varepsilon}(0))} \\ &\le \left(\sup_{|h| \le \varepsilon} \sup_{f \in A} \|f(.+h) - f\|^p + \sup_{f \in A} \|f\|^p_{L^p(\mathbb{R}^n \setminus B_{R_\varepsilon - \varepsilon}(0))} \right)^p =: \kappa_\varepsilon^p\end{aligned} \tag{3.13}$$

From the assumptions, $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, $T_\varepsilon f \in C_c^\infty(B_{R_\varepsilon+\varepsilon}(0))$ and there exists a constant $C(\varepsilon)$, depending of $\varepsilon > 0$ but not of $f \in A$, with

$$\begin{aligned} \|T_\varepsilon f\|_\infty &\leq \|\varphi_\varepsilon\|_{p'}\|f\|_p \leq C(\varepsilon) \\ \|\nabla T_\varepsilon f\|_\infty &\leq \|\nabla\varphi_\varepsilon\|_{p'}\|f\|_p \leq C(\varepsilon) \end{aligned}$$

where $1/p + 1/p' = 1$. Therefore, the set

$$B = \{T_\varepsilon f : f \in A\}$$

is a bounded and equicontinuous subset of $\mathcal{C}(\overline{B_{R_\varepsilon+\varepsilon}(0)})$, for every $\varepsilon > 0$. The equicontinuity follows from

$$|T_\varepsilon f(x) - T_\varepsilon f(y)| \leq \|\nabla T_\varepsilon f\|_\infty |x-y| \leq C(\varepsilon)|x-y|$$

for all $f \in A$. From Theorem 3.2.2, B is pre-compact. In other words, for every $\delta > 0$ there exist $g_1, \dots, g_n \in \mathcal{C}(\overline{B_{R_\varepsilon+\varepsilon}(0)})$, with

$$B \subset \bigcup_{j=1}^n B_\delta(g_j)$$

w.r.t. the $\mathcal{C}$-norm. Since the L^p-norm on $\overline{B_{R_\varepsilon+\varepsilon}(0)}$ can be estimated by the $\mathcal{C}$-norm (with a bound depending of ε), we find that

$$B \subset \bigcup_{j=1}^n B^{L^p}_{\rho(\delta,\varepsilon)}(g_j)$$

where $\rho(\delta,\varepsilon) > 0$ is such that $\rho(\delta,\varepsilon) \to 0$ as $\delta \to 0$, for every fixed $\varepsilon > 0$ (convergence is not uniform in $\varepsilon > 0$). From (3.13) we find

$$A \subset \bigcup_{j=1}^n B_{\kappa_\varepsilon+\rho(\delta,\varepsilon)}(g_j)$$

(where the balls are defined through the L^p-norm). Hence, for given $\nu > 0$, we choose $\varepsilon > 0$ so that $\kappa_\varepsilon < \nu/2$, and then, with a fixed $\varepsilon > 0$, we choose $\delta > 0$ so small that $\rho(\delta,\varepsilon) < \nu/2$. Thus, we find $g_1, \dots, g_n \in L^p(\mathbb{R}^n)$, with

$$A \subset \bigcup_{j=1}^n B_\nu(g_j)$$

□

4 Linear Operators and Functionals on Normed Spaces

4.1 Continuous Operators

Definition 4.1.1. *Let X, Y be two normed spaces. A (linear) operator $T : X \to Y$ is a linear map. A continuous operator $T : X \to Y$ is a continuous linear map. An operator $T : X \to Y$ is called bounded, if there is a constant $C > 0$ with*

$$\|Tx\|_Y \leq C\|x\|_X \quad \forall\, x \in X$$

We will often denote the norms $\|.\|_X$ and $\|.\|_Y$ simply by $\|.\|$; it should always be clear which norm is meant.

Proposition 4.1.2. *Let X, Y be normed spaces and $T : X \to Y$ a linear operator. The following statements are equivalent:*

1) T is continuous.

2) T is bounded.

3) T is continuous at $x = 0$.

Beweis. 1) $\Rightarrow$ 3) is clear.

3) $\Rightarrow$ 2): For $\varepsilon > 0$ there exists $\delta > 0$ with $T(\overline{B_\delta(0)}) \subset \overline{B_\varepsilon(0)}$ ($T(0) = 0$ because of linearity). Hence, $\|Tx\| \leq \varepsilon$ for all $x \in X$ with $\|x\| \leq \delta$. Therefore, if $x \neq 0$,

$$\|Tx\| = \left\|T\left(\frac{\delta\|x\|}{\delta\|x\|}x\right)\right\| = \frac{\|x\|}{\delta}\left\|T\left(\delta\frac{x}{\|x\|}\right)\right\| \leq \frac{\varepsilon}{\delta}\|x\|$$

2) $\Rightarrow$ 1): Choose $x_0 \in X$. Then

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq C\|x - x_0\|$$

$\square$

Definition 4.1.3. *Let X, Y be normed spaces and $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X to Y. $\mathcal{L}(X, Y)$ has the structure of a vector space over $\mathbb{K}$ with the operations*

$$(T + S)(x) := T(x) + S(x) \qquad \text{for all } x \in X \quad \text{and all } T, S \in \mathcal{L}(X, Y)$$

and

$$(\lambda T)(x) := \lambda T(x) \qquad \text{for all } x \in X \quad \text{and all } T \in \mathcal{L}(X, Y)$$

If $X = Y$, we can also define the product of two operators $T, S : X \to X$ through $TS(x) = T(S(x))$ for all $x \in X$. In this case, $\mathcal{L}(X) \equiv \mathcal{L}(X, X)$ is not only a vector space over $\mathbb{K}$, instead it is also a non-commutative algebra.

For $T \in \mathcal{L}(X, Y)$, we define

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| \tag{4.1}$$

From (4.1.2), we have $\|T\| < \infty$, for all $T \in \mathcal{L}(X, Y)$. It is simple to check, that (4.1) defines a norm on $\mathcal{L}(X; Y)$. Hence $(\mathcal{L}(X, Y), \|.\|)$ is a normed space.

Proposition 4.1.4. *Let X be a normed space and Y a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space.*

Proof. Let (T_ℓ) be a Cauchy sequence on $\mathcal{L}(X, Y)$ and $x \in X$. Then, since

$$\|T_\ell x - T_k x\| \leq \|T_\ell - T_k\| \|x\|,$$

$T_\ell x$ is a Cauchy sequence in Y. Since Y is complete, there exists

$$Tx := \lim_{\ell \to \infty} T_\ell x$$

The map $T : X \to Y$ is clearly linear. Moreover, $T \in \mathcal{L}(X, Y)$ since

$$\|Tx\| = \lim_{\ell \to \infty} \|T_\ell x\| \leq \limsup_{\ell \to \infty} \|T_\ell\| \|x\| \leq C \|x\| .$$

For $\|x\| \leq 1$, we find

$$\|Tx - T_\ell x\| = \lim_{k \to \infty} \|T_k x - T_\ell x\| \leq \limsup_{k \to \infty} \|T_k - T_\ell\|$$

Hence

$$\|T - T_\ell\| \leq \limsup_{k \to \infty} \|T_k - T_\ell\| \to 0$$

as $\ell \to \infty$, since T_k is a Cauchy sequence. □

Proposition 4.1.4 implies that $\mathcal{L}(X, Y)$ is a Banach space, if Y is a Banach space, independently of the completeness of X.

Definition 4.1.5. *Let X be a normed space over the field $\mathbb{K}$. We define the (topological) dual space X^* of X through $X^* = \mathcal{L}(X; \mathbb{K})$. Elements of X^* are continuous linear functionals X (i.e. continuous linear operators on X with values on $\mathbb{K}$). Since $\mathbb{K}$ is complete, it follows from (4.1.4) that X^* is always a Banach space, independently of the completeness of X.*

Example: take $X = \mathbb{R}^n$, for $n \in \mathbb{N}$. Every linear functional on X has the form

$$L(x_1, \dots, x_n) = \sum_{j=1}^{n} \ell_i x_i$$

for coefficients $\ell_1, \dots, \ell_n \in \mathbb{K}$. In other words, we can identify the dual space X^* with X.

Example: as established in Section 2.2, for all $1 \le p < \infty$, we have, with $1 < q \le \infty$ such that $1/p + 1/q = 1$,

$$L^p(\Omega, \mathcal{A}, \mu)^* = L^q(\Omega, \mathcal{A}, \mu)$$

Example: in particular it follows from the last example that the dual space of $L^2(\Omega, \mathcal{A}, \mu)$ can be identified with $L^2(\Omega, \mathcal{A}, \mu)$. We say that $L^2(\Omega, \mathcal{A}, \mu)$ is self dual. In fact, it turns out that every Hilbert space H is self dual. We will show in Section **??** that the map $\phi : H \to H^*$ defined through $\phi(u) = \phi_u$ where $\phi_u \in H^*$ is given by $\phi_u(v) = \langle u, v \rangle$ defines a linear (if $\mathbb{K} = \mathbb{R}$ or an antilinear (if $\mathbb{K} = \mathbb{C}$) isometric isomrphism between H and H^*.

4.2 The Hahn-Banach Theorem and its Applications

On L^p spaces, it is easy to construct continuous linear functionals. For abstract normed spaces, the existence of (sufficiently many) continuous linear functionals follows from the Hahn-Banach theorem, which is the subject of this section.

We will need here the Lemma of Zorn. Le us recall again the statement of the Lemma of Zorn. A partial order on a set P is a relation $\preceq$ on P with the following properties: for $a, b, c \in P$, $a \preceq b, b \preceq a \Rightarrow a = b$ (antisymmetry), $a \preceq a$ (reflexivity), $a \preceq b$, $b \preceq c \Rightarrow a \preceq c$ (transitivity). A subset M of a partially ordered set P is called totally ordered (or a chain)if for all $a, b \in M, a \neq b$ we have either $a \preceq b$ or $b \preceq a$. $b \in P$ is called an upper bound for $M \subset P$ if $a \preceq b$ for all $a \in M$. $b \in P$ is called a maximal element in P, if there is no $a \in P$ with $a \neq b$ and $b \le a$.

Lemma 4.2.1 (Lemma of Zorn)**.** *Let P be a partially ordered set, so that every totally ordered subset of P has an upper bound. Then P contains at least one maximal element.*

One can show that the Lemma of Zorn is equivalent to the axiom of choice, stating that for every family $(S_i)_{i \in I}$ of non-empty sets there exists a family $(x_i)_{i \in I}$ with $x_i \in S_i$ for all $i \in I$.

To state the Hahn-Banach Theorem, we will need the following definition.

Definition 4.2.2. *Let X be a vector space over $\mathbb{R}$. A map $p : X \to \mathbb{R}$ is called a sublinear functional i) $p(\lambda x) = \lambda p(x)$ for all $x \in X$, $\lambda \ge 0$, and ii) $p(x + y) \le p(x) + p(y)$, for all $x, y \in X$.*

Together with the Lemma of Zorn, the next lemma plays an importantn role in the proof of the Hanhn-Banach theorem.

Lemma 4.2.3. *Let X be a vector space over $\mathbb{R}$, $M \subset X$ a linear subspace, $p : X \to \mathbb{R}$ a sublinear functional, $f : M \to \mathbb{R}$ a linear functional, $x_0 \in X \backslash M$. Assume $f(x) \leq p(x)$ for $x \in M$. Then there exists $F : \widetilde{M} := M + \mathbb{R}\, x_0 \to \mathbb{R}$ linear, with $F(x) \leq p(x)$ for all $x \in \widetilde{M}$ and with $F|_M = f$.*

Proof. Let $y', y'' \in M$. Then

$$f(y') - f(y'') = f(y' - y'') \leq p(y' - y'') = p(y' + x_0 - x_0 - y'') \leq p(y' + x_0) + p(-x_0 - y'')$$

Hence

$$-f(y'') - p(-x_0 - y'') \leq -f(y') + p(y' - x_0) \tag{4.2}$$

for all $y', y'' \in M$. We put

$$s_1 := \sup_{y'' \in M} \left(-f(y'') - p(-x_0 - y'')\right)$$
$$s_2 := \inf_{y' \in M} \left(-f(y') + p(y' + x_0)\right)$$

Then $-\infty < s_1 \leq s_2 < \infty$. We choose $c_0 \in [s_1, s_2]$ and we define $F : \widetilde{M} \to \mathbb{R}$ through $F(m + tx_0) = f(m) + tc_0$, for all $m \in M$. Then F is linear, $F|_M = f$, and $F(m) \leq p(m)$ for all $m \in M$. We claim that $F(m + tx_0) \leq p(m + tx_0)$ for all $m \in M$, $t \in \mathbb{R}$. In fact, for $t > 0$,

$$F(m + tx_0) = tF(m/t + x_0) = t\,(f(m/t) + c_0) \leq tp(m/t + x_0) = p(m + tx_0)$$

since

$$c_0 \leq s_2 \leq p(x_0 + y) - f(y)$$

for all $y \in M$ (in particular for $y = m/t$). On the other hand, if $t < 0$, we have

$$F(m + tx_0) = -tF(-m/t - x_0) = (-t)(-f(m/t) - c_0) \leq (-t)p(-m/t - x_0) = p(m + tx_0)$$

where we used the fact that

$$c_0 \geq s_1 \geq -p(-x_0 - y) - f(y)$$

for all $y \in M$ (in particular, for $y = m/t$). Hence, we proved that $F(m + tx_0) \leq p(m + tx_0)$ for all $m \in M$, $t \in \mathbb{R}$. □

We are now ready to state and prove the Hahn-Banach theorem

Theorem 4.2.4 (Hahn-Banach, vector spaces over $\mathbb{R}$)**.** *Let X be a vector space over $\mathbb{R}$, $M \subset X$ a linear subspace and $f : M \to \mathbb{R}$ linear. Let $p : X \to \mathbb{R}$ be a sublinear functional with $f(x) \leq p(x)$ for all $x \in M$. Then there exists $F : X \to \mathbb{R}$ linear with $F|_M = f$ and $F(x) \leq p(x)$ for all $x \in X$.*

Proof. Let

$$\mathcal{F} = \Big\{(\widetilde{M}, \widetilde{f}) : \widetilde{M} \subset X \text{ linear with } \widetilde{M} \supset M, \widetilde{f} \text{ a linear functional on } \widetilde{M},$$
$$\text{with } \widetilde{f}|_M = f \text{ and } \widetilde{f}(x) \leq p(x) \text{ for all } x \in \widetilde{M}\Big\}$$

$\mathcal{F} \neq \emptyset$, since $(M, f) \subset \mathcal{F}$. On $\mathcal{F}$ we define a partial order by setting $(\widetilde{M}, \widetilde{f}) \preceq (\widetilde{N}, \widetilde{g})$ if $\widetilde{M} \subset \widetilde{N}$ and $\widetilde{g}|_{\widetilde{M}} = \widetilde{f}$. We claim that every totally ordered subset of $\mathcal{F}$ has an upper bound. In fact, let $\mathcal{G} = \{(M_i, g_i) : i \in I\}$ be a totally ordered subset of $\mathcal{F}$. We define then

$$\widetilde{M} = \bigcup_{i \in I} M_i$$

and $\widetilde{g} : \widetilde{M} \to \mathbb{R}$, so that for $x \in M_i$, $\widetilde{g}(x) = g_i(x)$. Then $\widetilde{g}$ is linear. Here we use the fact that $\mathcal{G}$ is totally ordered (for given $x, y \in \widetilde{M}$ and $\lambda \in \mathbb{R}$ there exists, since $\mathcal{G}$ is totally ordered, $i \in I$ with $x, y, \lambda y \in M_i$; hence $\widetilde{g}(x + \lambda y) = \widetilde{g}(x) + \lambda \widetilde{g}(y)$ because of the linearity of g_i). Moreover $\widetilde{g}|_M = f$ and $\widetilde{g}(x) \leq p(x)$ for all $x \in \widetilde{M}$. Thus, $(\widetilde{M}, \widetilde{g}) \in \mathcal{F}$ is an upper bound for $\mathcal{G}$. The Lemma of Zorn implies that there exists a maximal element (N, F) in $\mathcal{F}$. If $N \neq X$, we choose $x_0 \in X \backslash N$ and we apply Lemma 4.2.3. The result is that there exists G with $(N + \mathbb{R}x_0, G) \in \mathcal{F}$, $G|_N = F$. Hence $(N, F) \leq (N + \mathbb{R}x_0, G)$ and $(N, F) \neq (N + \mathbb{R}x_0, G)$. This is a contradiction to the maximality of (N, F). Thus, $N = X$ and $F : X \to \mathbb{R}$ is linear with $F|_M = f$ and $F(x) \leq p(x)$ for all $x \in X$. $\square$

To extend the theorem to vector spaces over $\mathbb{C}$, we introduce the notion of seminorm.

Definition 4.2.5. *Let X be a vector space over $\mathbb{K}$ and $q : X \to \mathbb{R}$ a map with the properties $q(\alpha x) = |\alpha| q(x)$ for all $\alpha \in \mathbb{K}$, $x \in X$, and with $q(x + y) \leq q(x) + q(y)$ for all $x, y \in X$. Then we call q a seminorm.*

Theorem 4.2.6 (Hahn-Banach, general version, for real and complex vector spaces)**.** *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $q : X \to \mathbb{R}$ a seminorm, $M \subset X$ a linear subspace and $f : M \to \mathbb{K}$ a linear functional with $|f(x)| \leq q(x)$ for all $x \in M$. Then there exists a linear functional $F : X \to \mathbb{K}$ with $F|_M = f$ and $|F(x)| \leq q(x)$ for all $x \in X$.*

Proof. If $\mathbb{K} = \mathbb{R}$, we find from Theorem 4.2.4 $F : X \to \mathbb{R}$ linear with $F|_M = f$ and with $F(x) \leq q(x)$ for all $x \in X$. Then

$$-F(x) = F(-x) \leq q(-x) = q(x) \quad \Rightarrow \quad |F(x)| \leq q(x)$$

So, let us now assume that $\mathbb{K} = \mathbb{C}$. We can consider X and M as vector spaces over $\mathbb{R}$. Notice that $\mathrm{Re}\ f : M \to \mathbb{R}$ is $\mathbb{R}$-linear with $|\mathrm{Re}\ f(x)| \leq |f(x)| \leq q(x)$. We apply the statement of the theorem for vector spaces over $\mathbb{R}$ and we find $G : X \to \mathbb{R}$, $\mathbb{R}$-linear with $G|_M = \mathrm{Re}\ f$ and with $|G(x)| \leq q(x)$ for all $x \in X$. We define now $F(x) = G(x) - iG(ix)$. Then F ois clearly $\mathbb{R}$-linear. However, since

$$F(ix) = G(ix) - iG(i^2x) = iG(x) + G(ix) = i(G(x) - iG(ix)) = iF(x)$$

F is also $\mathbb{C}$-linear. Moreover, for $x \in M$, Re $F(x) = G(x) =$ Re $f(x)$ and

$$\text{Im } F(x) = -G(ix) = -\text{Re } f(ix) = -\text{Re } if(x) = \text{Im } f(x)$$

In other words, $F|_M = f$. We still have to show that $|F(x)| \leq q(x)$ for all $x \in X$. For given $x \in X$, we write $F(x) = re^{i\theta}$. Then $e^{-i\theta}F(x)$ is real and therefore

$$|F(x)| = |e^{-i\theta}F(x)| = |F(e^{-i\theta}x)| = |G(e^{-i\theta}x)| \leq q(e^{-i\theta}x) = q(x)$$

□

Hanhn-Banach's theorem is used to construct linear functional. We list now some consequences.

Corollary 4.2.7. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$, $M \subset X$ a linear subspace and $f \in M^*$. Then there exists $F \in X^*$ with $F|_M = f$ and $\|F\|_{X^*} = \|f\|_{M^*}$.*

Proof. We define $q(x) = \|f\|_{M*}\|x\|$. Then q is clearly a seminorm, with $|f(x)| \leq q(x)$ for all $x \in M$. Theorem 4.2.6 implies the existence of a linear functional $F : X \to \mathbb{K}$ with $F|_M = f$ and $|F(x)| \leq q(x)$ for all $x \in X$. This implies that $|F(x)| \leq \|f\|_{M^*}\|x\|$ for all $x \in X$ and therefore, that $\|F\|_{X^*} \leq \|f\|_{M^*}$. On the other hand

$$\|f\|_{M^*} = \sup_{x \in M, \|x\| \leq 1} |f(x)| = \sup_{x \in M, \|x\| \leq 1} |F(x)| \leq \sup_{x \in X, \|x\| \leq 1} |F(x)| = \|F\|_{X^*}$$

Hence $\|F\|_{X^*} = \|f\|_{M^*}$. □

Corollary 4.2.8. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$, $y \in X \backslash \{0\}$. Then there exists $f \in X^*$ with $\|f\| = 1$ and $f(y) = \|y\|$.*

Proof. Define $g : \mathbb{K} \cdot y =: M \to \mathbb{K}$ through $g(ty) = t\|y\|$. Then $g \in M^*$ with $\|g\|_{M^*} = 1$, $g(y) = \|y\|$. From Corollary 4.2.7 there exists $f \in X^*$ with $f|_M = g$ (hence, in particular with $f(y) = \|y\|$) and $\|f\|_{X^*} = \|g\|_{M^*} = 1$. □

Corollary 4.2.9. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$, $Z \subset X$ a linear subspace and $y \in X \backslash Z$. Let $d = dist(y, Z) = \inf_{z \in Z} \|z - y\| > 0$. Then there exists $F \in X^*$ with $\|F\| = 1$, $F|_Z = 0$, $F(y) = d$.*

Proof. Let $M = Z + \mathbb{K}y$ and define $f : M \to \mathbb{K}$ through $f(z + \alpha y) = \alpha d$, for all $\alpha \in \mathbb{K}$. f ist clearly linear. We claim that $\|f\| = 1$. In fact, since for $\alpha \in \mathbb{K}$ and $z \in Z$, we have $-z/\alpha \in Z$, we conclude that

$$|f(z + \alpha y)| = |\alpha| d \leq |\alpha| \|y - (-z/\alpha)\| = \|\alpha y + z\|$$

In other words, $\|f\|_{M^*} \leq 1$. On the other hand, let $(z_n) \in Z$ be a sequence with $\|y - z_n\| \to d$. Then

$$d = f(y - z_n) \leq \|f\|_{M^*} \|y - z_n\|$$

for all $n \in \mathbb{N}$. As $n \to \infty$, we find $\|f\|_{M^*} \geq 1$. Summarizing $f \in M^*$, with $f|_Z = 0$ and $\|f\|_{M^*} = 1$. From Corollary 4.2.7 there exists $F \in X^*$ with $\|F\| = 1$, $F|_M = f$ (in particular $F(y) = d$ and $F|_Z = 0$). $\square$

Other consequences:

- Let X be a normed space. Then X^* separates the points of X, i.e. for every $x_1, x_2 \in X$, with $x_1 \neq x_2$, there exists $f \in X^*$ with $f(x_1) \neq f(x_2)$. In fact, for given $x_1, x_2 \in X$ with $x_1 \neq x_2$, set $y = x_2 - x_1 \neq 0$. Then, from Corollary 4.2.8, there is $f \in X^*$ with $f(y) = \|y\| \neq 0$, implying $f(x_1) \neq f(x_2)$.

- Let X be a normed space. Then, for $x \in X$, we have

$$\|x\| = \sup_{f \in X^* : \|f\| \leq 1} |f(x)| \tag{4.3}$$

  In fact, on the one hand,

$$|f(x)| \leq \|f\| \|x\| \leq \|x\|$$

  for all $f \in X^*$ with $\|f\| \leq 1$. This proves that

$$\|x\| \geq \sup_{f \in X^* : \|f\| \leq 1} |f(x)|$$

  On the other hand, from Corollary 4.2.8, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. This implies that

$$\|x\| \leq \sup_{f \in X^* : \|f\| \leq 1} |f(x)|$$

We introduce now the notion of adjoint operator.

Definition 4.2.10. *Let X, Y be two normed spaces and $T : X \to Y$ a continuous linear operator. The adjoint operator to T, denoted by T^*, is a linear map $T^* : Y^* \to X^*$, defined by $T^*(f) = f \circ T$, for all $f \in Y^*$*

Notice that, if $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, we have $(ST)^* = T^*S^*$. In fact, for given $f \in Z^*$,

$$((ST)^*f = f \circ ST = (f \circ S) \circ T = T^*(f \circ S) = T^*S^*f$$

Proposition 4.2.11. *Let $T \in \mathcal{L}(X,Y)$. Then $T^* \in \mathcal{L}(Y^*,X^*)$, with $\|T^*\| = \|T\|$.*

Proof. The linearity of T^* is clear. Furthermore

$$\begin{aligned}
\|T^*\| &= \sup_{f\in Y^*,\|f\|\leq 1} \|T^*f\| \\
&= \sup_{f\in Y^*,\|f\|\leq 1} \sup_{x\in X,\|x\|\leq 1} |T^*f(x)| \\
&= \sup_{f\in Y^*,\|f\|\leq 1} \sup_{x\in X,\|x\|\leq 1} |f(Tx)| \\
&\leq \sup_{f\in Y^*,\|f\|\leq 1} \sup_{x\in X,\|x\|\leq 1} \|f\|\|T\|\|x\| \\
&\leq \|T\|
\end{aligned}$$

In particular, T^* is continuous. To show the inequality $\|T^*\| \geq \|T\|$ we fix $\varepsilon > 0$. Then we find $x_0 \in X$ with $\|x_0\| \leq 1$ and $\|Tx_0\| \geq \|T\| - \varepsilon$. By Corollary 4.2.8 there exists $f_0 \in Y^*$ with $\|f_0\| = 1$ and $f_0(Tx_0) = \|Tx_0\|$. We conclude that

$$\begin{aligned}
\|T\| \leq \|Tx_0\| + \varepsilon = f_0(Tx_0) + \varepsilon &\leq \sup_{\|f\|\leq 1} |f(Tx_0)| + \varepsilon \\
\leq \sup_{\|f\|\leq 1} \sup_{\|x\|\leq 1} |f(Tx)| + \varepsilon &= \|T^*\| + \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\|T\| \leq \|T^*\|$. □

4.3 Reflexive normed spaces

Let X be a normed space. Apart from the dual space X^* also the bidual space $X^{**} = (X^*)^*$ plays an important role. For $x \in X$ we define the linear functional $\widetilde{x} : X^* \to \mathbb{K}$ through

$$\widetilde{x}(f) := f(x)\,.$$

We notice that $\widetilde{x}$ is continuous, since

$$|\widetilde{x}(f)| = |f(x)| \leq \|f\|\|x\| \tag{4.4}$$

Hence $\widetilde{x} \in X^{**}$. The map $J_X : X \to X^{**}$, defined by $J_X(x) = \widetilde{x}$, is call the canonical inclusion of X in X^{**}. J_X ist clearly linear.

Theorem 4.3.1. *Let X be a normed vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The map $J_X : X \to X^{**}$ is a linear isometry.*

Proof. From (4.4), we have $\|\widetilde{x}\| \le \|x\|$. On the other hand, for given $x \in X$, $x \neq 0$, there exists from Corollary 4.2.8, $x_0^* \in X^*$ with $\|x_0^*\| = 1$, $x_0^*(x) = \|x\|$. Hence,

$$\|\widetilde{x}\| = \sup_{\|x\|=1} |\widetilde{x}(x^*)| \ge |\widetilde{x}(x_0^*)| = \|x\|$$

and $\|\widetilde{x}\| = \|x\|$. □

Definition 4.3.2. *A normed space $(X, \|.\|)$ is called reflexive, if the map J_X is surjective (and therefore an isometric isomorphism).*

Remark: the space X^{**} is always a Banach space. Hence X reflexive implies that X is a Banach space, i.e. that X is complete.

Theorem 4.3.3. *Let X be a reflexive Banach space and $M \subset X$ a close linear subspace. Then M is reflexive, as well.*

Proof. Let $J_M : M \to M^{**}$, $J_X : X \to X^{**}$ the canonical inclusion. Let $j : M \to X$ be the inclusion of M in X (in other words, $j(x) = x$, for all $x \in M$) and $j^{**} : M^{**} \to X^{**}$ be the bi-adjoint map corresponding to j. Notice that $j^* : X^* \to M^*$ is given by $j^*(f) = f|_M$ for all $f \in X^*$. Then $j^{**}(m^{**})(f) = m^{**}(f|_M)$. Wwe show that $j^{**} \circ J_M = J_X \circ j$. In fact, for arbitrary $m \in M$ and $x^* \in X^*$,

$$(j^{**}(J_M(m)))(x^*) = (J_M(m))(j^*(x^*)) = (J_M(m))(x^* \circ j) = (x^* \circ j)(m) = J_X(j(m))(x^*)$$

Now let $m^{**} \in M^{**}$. We have to find $m \in M$ with $J_M(m) = m^{**}$. Since J_X is surjective, we find $x \in X$ with $J_X x = j^{**}(m^{**})$. We claim that $x \in M$. Assume $x \notin M$. Then, by Corollary 4.2.9, there exists $x^* \in X^*$ with $x^*|_M = 0$ and $x^*(x) = 1$. Therefore

$$1 = x^*(x) = (J_X x)(x^*) = (j^{**}(m^{**}))(x^*) = m^{**}(j^*(x^*)) = 0$$

since $j^*(x^*) = x^*|_M = 0$. Hence, as claimed, $x \in M$. Next, we prove that $J_M(x) = m^{**}$. In fact, for $m^* \in M^*$, we find by Corollary 4.2.7 $x^* \in X^*$ with $x^*|_M = m^*$. Therefore

$$\begin{aligned}(J_M x)(m^*) &= m^*(x) = x^*(x) = (J_X x)(x^*) \\ &= (j^{**}(m^{**}))(x^*) = m^{**}(j^*(x^*)) = m^{**}(x^* \circ j) = m^{**}(m^*)\end{aligned}$$

Hence $J_M(x) = m^{**}$. □

Theorem 4.3.4. *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.*

Beweis. Suppose X is reflexive. We show that X^* is reflexive. To this end, we consider the canonical inclusion $J_{X^*} : X^* \to X^{***}$ and we show that it is surjective. In fact, let $x_0^{***} \in X^{***}$.

Then we define $x_0^* := x_0^{***} \circ J_X \in X^*$. Then we set, for arbitrary $x^{**} \in X^{**}$, $x = J_X^{-1}(x^{**})$ (so that $J_X(x) = x^{**}$; this is possible since X is reflexive). We find

$$\begin{aligned}(J_{X^*}x_0^*)(x^{**}) &= x^{**}(x_0^*) = x^{**}(x_0^{***} \circ J_X) = (J_X x)(x_0^{***} \circ J_X)\\ &= (x_0^{***} \circ J_X)(x) = x_0^{***}(x^{**})\end{aligned}$$

Hence X^* is reflexive.

Now, suppose X^* is reflexive. If X is not reflexive, with Corollary 4.2.9 we could find a $x_0^{**} \in X^{**} \backslash J_X(X)$ and $x_0^{***} \in X^{***}$ with $x_0^{***}|_{J_X(X)} = 0$ and $x_0^{***}(x_0^{**}) = 1$ (here we use the fact that $J_X(X)$ is closed). Then we find $x_0^* \in X^*$ with $J_{X^*}(x_0^*) = x_0^{***}$. Since $x_0^{***} \neq 0$, we would also have $x_0^{**} \neq 0$. For $x \in X$ we would have

$$0 = x_0^{***}(J_X(x)) = (J_{X^*}(x_0^*))(J_X(x)) = (J_X(x))(x_0^*) = x_0^*(x)$$

Hence $x_0^* = 0$, giving a contradiction. $\square$

Examples:

- Let (Ω, Σ, μ) be a measure space. For $1 < p < \infty$, $L^p(\Omega)$ is reflexive. This follows from the fact that $\phi : L^{p'}(\Omega) \to (L^p(\Omega))^*$, defined in Theorem 2.2.6, is an isometric isomorphism. In fact, if $G \in (L^p(\Omega))^{**} \simeq (L^{p'}(\Omega))^*$, there is $g \in L^p(\Omega)$ with

  $$G(f) = \int fg d\mu \qquad \text{for all } f \in L^{p'}(\Omega)$$

  We claim that $J_{L^p}(g) = G$. In fact, for $f \in L^{p'}(\Omega)$, we have

  $$(J_{L^p}g)(f) = f(g) = \int fg d\mu = G(f)$$

- On the other hand, in general, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive (there are exceptions, depending on the choice of Ω, $\mathcal{A}$ and μ; for example if Ω consists of finitely many points, then $L^1 = L^\infty$ are finite dimensional vector spaces, and therefore certainly reflexive). If $L^1(\Omega)$ was reflexive, then $(L^1)^{**}(\Omega)$ would also be separable. This would imply (proof: exercise) that $(L^1)^*(\Omega)$ is also separable, but $(L^1)^*(\Omega) = L^\infty(\Omega)$, which is typically not separable. From Theorem 4.3.4 also $(L^1)^*(\Omega) = L^\infty(\Omega)$ is not reflexive.

4.4 Hilbert Space Methods

Hilbert spaces can be identified with their dual spaces.

Theorem 4.4.1 (Riesz's Representation Theorem). *Let $(H, (.,.))$ be an Hilbert space. The map $R_H : H \to H^*$ defined through $(R_H(u))(v) = (u, v)$ is an anti-linear isometric isomorphism.*

Proof. • R_H is well defined because $R_H(u)$ is clearly linear and, by

$$|(R_H(u))(v)| = |\langle u, v\rangle| \leq \|u\|\|v\|, \tag{4.5}$$

also continuous.

• R_H is anti-linear, because of the anti-linearity of the inner product in the first argument.

• R_H is isometric (and thus injective). From (4.5), we have $\|R_H(u)\| \leq \|u\|$. On the other hand, $R_H(u) = \|u\|^2$ implies that $\|R_H u\| \geq \|u\|$.

• R_H is surjektive. Let $L \in H^*$ and set $M = \ker L = \{u \in H : L(u) = 0\}$. M is clearly a linear subspace of H. Since L is continuous, M is closed. Moreover, from Theorem 1.4.7, $H = M \oplus M^\perp$. W.l.o.g. we can assume that $L \neq 0$. Then $\dim M^\perp = 1$. In fact, if $u, v \in M^\perp$, we obtain

$$L(L(v)u - L(u)v) = 0 \quad \Rightarrow \quad L(v)u - L(u)v \in M \cap M^\perp \quad \Rightarrow \quad L(v)u = L(u)v\,.$$

Thus $M^\perp = \mathbb{K}\,v$, for an appropriate $v \in H\backslash\{0\}$. We claim that

$$L(u) = \left(\frac{\overline{L(v)}}{\|v\|^2}v, u\right) \tag{4.6}$$

for all $u \in H$, i.e. $L = R_H(\overline{L(v)}/\|v\|^2 v)$. To show (4.6), we write $u = \widetilde{u} + \lambda v$, for $\widetilde{u} \in M$ and $\lambda \in \mathbb{K}$. Then

$$\left(\frac{\overline{L(v)}}{\|v\|^2}v, u\right) = \left(\frac{\overline{L(v)}}{\|v\|^2}v, \lambda v\right) = \lambda L(v) = L(u)\,.$$

□

It follows from Theorem 4.4.1 that H^* has the structure of a Hilbert space (by definition, H^* has the structure of a Banach space). For $f, g \in H^*$ we define

$$\langle f, g\rangle_{H^*} := \left\langle R_H^{-1}g, R_H^{-1}f\right\rangle_H$$

Then it is simple to check that $\langle ., .\rangle$ is a scalar product and that

$$\|f\|_{H^*}^2 = \|R_H^{-1}f\|_H^2 = \langle R_H^{-1}f, R_H^{-1}f\rangle_H = \langle f, f\rangle_{H^*}$$

since R_H and thus also R_H^{-1} are isometric.

Corollary 4.4.2. *Every Hilbert space is reflexive.*

Proof. We claim that $J_H = R_{H^*} \circ R_H$. Then J_H, as composition of two surjective maps is again surjective. Let $u \in H$ and $f \in H^*$. Then

$$\begin{aligned}((R_{H^*} \circ R_H)(u))\,(f) &= (R_{H^*}(R_H u))(f) = \langle R_H u, f\rangle_{H^*} \\ &= \langle R_H^{-1}f, u\rangle_H = (R_H(R_H^{-1}f))(u) = f(u) = (J_H u)(f)\end{aligned}$$

□

A consequence of Riesz's representation theorem which is very useful in applications to the study of partial differential equations is the following result of Lax-Milgram.

Satz 4.4.3 (Lax-Milgram)**.** *Let H be a Hilbert space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and let $a : H \times H \to \mathbb{K}$ be a sesquilinear form (i.e. a is linear in the second argument and antilinear in the first argument). There are constants $0 < c_0 \le C_0 < \infty$ with*

$$\begin{aligned} |a(x,y)| &\le C_0 \|x\| \, \|y\| \qquad \textit{for all } x, y \in H \quad \textit{(continuity)} \\ \textit{Re } a(x,x) &\ge c_0 \|x\|^2 \qquad \textit{for all } x \in H \quad \textit{(coercivity)} \end{aligned}$$

Then there exists a linear operator $A : H \to H$ with

$$a(x,y) = (Ax, y) \qquad \textit{for all } x, y \in H$$

Moreover $A \in \mathcal{L}(H)$ is invertible, with

$$\|A\| \le C_0 \qquad \textit{and} \qquad \|A^{-1}\| \le c_0$$

Proof. For all fixed $x \in H$, the map $y \to a(x,y)$ is in H^* with

$$\|a(x,.)\|_{H^*} \le C_0 \|x\|_H$$

From Theorem 4.4.1 there exists, for arbitrary $x \in H$, exactly one $A(x) \in H$, with $a(x,y) = (A(x), y)$ for all $y \in H$, and with $\|A(x)\| = \|a(x,.)\|_{H^*} \le C_0 \|x\|$. Moreover

$$\begin{aligned} (A(x_1 + \lambda x_2), y) &= a(x_1 + \lambda x_2, y) = a(x_1, y) + \overline{\lambda} a(x_2, y) \\ &= (A(x_1), y) + \overline{\lambda}(A(x_2), y) \\ &= (A(x_1) + \lambda A(x_2), y) \end{aligned}$$

for all $y \in H$. Hence, $A(x_1 + \lambda x_2) = A(x_1) + \lambda A(x_2)$ and thus $A \in \mathcal{L}(H)$ with $\|A\| \le C_0$. Furthermore

$$c_0 \|x\|^2 \le \text{Re } a(x,x) = \text{Re } (Ax, x) \le |(Ax, x)| \le \|Ax\| \|x\| \quad \Rightarrow \quad \|Ax\| \ge c_0 \|x\| \tag{4.7}$$

for all $x \in H$. This implies that ker $A = \{0\}$ and thus that A is injective. It remains to prove that A is surjective. To this end, we claim first that Ran $A = \{Ax : x \in H\}$ is closed. In fact, if $Ax_k \to y$ in H, then Ax_k is a Cauchy sequence. From (4.7) also x_k is a Cauchy sequence on H; therefore there is $x \in H$, with $x_k \to x$. Because of the continuity of A, we obtain that $y = Ax$ and thus that $y \in$ Ran A. From Theorem 1.4.7, $H =$ Ran $A \oplus$ (Ran $A)^\perp$. If Ran $A \neq H$, then $x_0 \in$ (Ran $A)^\perp, x_0 \neq 0$ with $(y, x_0) = 0$ for all $y \in$ Ran A. In particular $(Ax_0, x_0) = 0$. This implies however that Re $(Ax_0, x_0) = 0$ and contradicts, because of the coercivity, the assumption $x_0 \neq 0$. Hence, Ran $A = H$ and A is bijective. Thus A^{-1} is well defined and linear. Moreover, from (4.7)

$$c_0 \|A^{-1} x\| \le \|A(A^{-1}x)\| = \|x\|$$

for all $x \in H$, we conclude that $\|A^{-1}\| \le 1/c_0$. □

We show next how to use the Theorem of Lax-Milgram to prove the existence and uniqueness of solutions of certain partial differential equations.

Abstract setting: let $a : H \times H \to \mathbb{K}$ as in Theorem 4.4.3 and fix $x^* \in H^*$. The unique solution of the problem

$$a(x,y) = x^*(y) \quad \text{for all } y \in H \tag{4.8}$$

is given by $x = A^{-1}R_H^{-1}x^*$. In fact, with $x = A^{-1}R_H^{-1}x^*$, we have

$$a(x,y) = (Ax,y) = (AA^{-1}R_H^{-1}x^*,y) = (R_H^{-1}x^*,y) = x^*(y)$$

for all $y \in H$. The problem (4.8) is stable in the following sense: for two given data $x_1^*, x_2^* \in H^*$, we can estimate the distance between the corresponding solutions $x_1 = A^{-1}R_H^{-1}x_1^*$, $x_2 = A^{-1}R_H^{-1}x_2^*$ through

$$\|x_1 - x_2\|_H = \|A^{-1}R_H^{-1}(x_1^* - x_2^*)\| \le \frac{1}{c_0}\|x_1^* - x_2^*\|_{H^*}$$

We notice that, if $a(.,.)$ is also antisymmetric (i.e. if $a(x,y) = \overline{a(y,x)}$, for all $x,y \in H$) and therefore defines a scalar product on H, then the solution $x = A^{-1}R_H^{-1}x^*$ of the problem $a(x,y) = x^*(y)$ for all $y \in H$, is also the absolute minimum of the functional

$$F(y) = \frac{1}{2}a(y,y) - \text{Re } x^*(y)$$

In fact, for $x = A^{-1}R_H^{-1}x^*$ and for arbitrary $y \in H$, we have

$$\begin{aligned} F(y) - F(x) &= \frac{1}{2}\left(a(y,y) - a(x,x)\right) - \text{Re } x^*(y-x) \\ &= \frac{1}{2}\left(a(y,y) - a(x,x)\right) - \text{Re } \left(a(x,y) - a(x,x)\right) \\ &= \frac{1}{2}\left(a(y,y) + a(x,x) - a(x,y) - a(y,x)\right) \\ &= \frac{1}{2}a(x-y,x-y) \ge \frac{c_0}{2}\|x-y\|^2 \end{aligned}$$

We apply now this general theory to solve so called elliptic boundary value problems.

Elliptic boundary value problem: Let $\Omega \subset \mathbb{R}^n$ be open and bounded; we consider functions on Ω with values in $\mathbb{K} = \mathbb{R}$. We look for a function $u \in C^2(\Omega)$, solving the partial differential equation

$$-\sum_{i=1}^n \partial_i \sum_{j=1}^n a_{ij}\partial_j u + bu + f = 0 \tag{4.9}$$

on Ω. Here $b, f \in C(\Omega)$ have values in $\mathbb{R}$ and the coefficients $a_{ij} = a_{ij}(x) \in C^1(\Omega)$ are real valued and satisfy the positivity condition

$$\sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \geq c_0 |z|^2 \quad \text{für alle } z \in \mathbb{R}^n, x \in \Omega \tag{4.10}$$

We say that the matrix $(a_{ij}(x))$ satisfying (4.10) is uniformly elliptic. Eq. (4.9) is a generalization of the Poisson equation $(-\Delta + b)u = -f$ which corresponds to the choice $a_{ij} = \delta_{ij}$.

Boundary conditions: To find a unique solution of (4.9), we have to specify boundary conditions. There are two types of boundary conditions that are particularly important for applications in physics; Dirichlet and Neumann boundary conditions.

- *Dirichlet boundary conditions:* in this case, we look for $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving (4.9) in Ω and so that $u = g$ on $\partial\Omega$, for given $g \in C(\partial\Omega)$.
- *Neumann boundary conditions:* in this case, we look for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solving (4.9) in Ω and so that
$$-\sum_{i=1}^{n} \nu_i \sum_{j=1}^{n} a_{ij} \partial_j u = g \tag{4.11}$$
on $\partial\Omega$ for a given $g \in C(\partial\Omega)$. Here $\nu = (\nu_1, \dots, \nu_n)$ denotes the outwards pointing normal to $\partial\Omega$.

In Dirichlet boundary value problems, we specify the values of u on the boundary $\partial\Omega$. In Neumann boundary value problems, we specify the values of the normal derivatives of u on $\partial\Omega$.

Reduction to homogeneous problem: a first condition to find a solution of the Dirichlet problem (4.9) with $u = g$ on $\partial\Omega$ is the existence of a $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ with $u_0 = g$ on $\partial\Omega$ (u_0 is a C^2-extension of g on Ω). Then $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the Dirichlet boundary value problem if and only if $\widetilde{u} = u - u_0$ solves the problem

$$-\sum_{i=1}^{n} \partial_i \sum_{j=1}^{n} a_{ij} \partial_j \widetilde{u} + b\widetilde{u} + \widetilde{f} = 0$$

on Ω with $\widetilde{f} = f + bu_0 - \sum_{i=1}^{n} \partial_i \sum_{j=1}^{n} a_{ij} \partial_j u_0$ and with the homogeneous boundary conditions $\widetilde{u} = 0$ on $\partial\Omega$.

Analogously, a first condition for the existence of a solution of the Neumann problem (4.9) with the boundary condition (4.11) is the existence of $u_0 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with

$$-\sum_{i=1}^{n} \nu_i \sum_{j=1}^{n} a_{ij} \partial_j u_0 = g$$

on $\partial\Omega$. Then u solves the Neumann boundary value problem if and only if $\widetilde{u} = u - u_0$ is a solution of

$$-\sum_{i=1}^n \partial_i \sum_{j=1}^n a_{ij}\partial_j \widetilde{u} + b\widetilde{u} + \widetilde{f} = 0$$

with $\widetilde{f} = f + bu_0 - \sum_{i=1}^n \partial_i \sum_{j=1}^n a_{ij}\partial_j u_0$ and with homogeneous boundary condition

$$-\sum_{i=1}^n \nu_i \sum_{j=1}^n a_{ij}\partial_j \widetilde{u} = 0$$

on $\partial\Omega$. We have schon, that it is enough to solve the boundary value problems with homogeneous boundary conditions.

Integral formulation of boundary value problems: We assume that u is a solution of the boundary value problem (4.9), wither with Dirichlet or with Neumann boundary conditions. Then for any $\xi \in C_0^\infty(\Omega)$,

$$\int \xi \left[\sum_{i=1}^n \partial_i \sum_{j=1}^n a_{ij}\partial_j u + bu + f \right] = 0$$

Integration by parts gives

$$\int \left(\sum_{i=1}^n \partial_i \xi \sum_{j=1}^n a_{ij}\partial_j u + \xi(bu + f) \right) = 0 \tag{4.12}$$

for all $\xi \in C_0^\infty(\Omega)$. Conversely, if $u \in C^2(\Omega)$ satisfies (4.12) for all $\xi \in C_0^\infty(\Omega)$, then u solves (4.9) in Ω. This can be shown easily, using integration by parts. If u solves the problem with with homogeneous Neumann boundary conditions, then (4.12) holds true for all $\xi \in C^\infty(\overline{\Omega})$. In fact, if we define $F = (F_1, \dots, F_n)$, with $F_i = \sum_j a_{ij}\partial_j u$, we find

$$\begin{aligned}\int_\Omega \xi \sum_i \partial_i \sum_j a_{ij}\partial_j u &= \int_\Omega \xi \nabla \cdot F = \int_\Omega \nabla \cdot (\xi F) - \int_\Omega F \cdot \nabla\xi = \int_{\partial\Omega} \xi F \cdot \nu - \int_\Omega F \cdot \nabla \xi \\ &= -\int_\Omega F \cdot \nabla \xi\end{aligned}$$

where we used the Nuemann boundary condition $F \cdot \nu = 0$ on $\partial\Omega$. Conversely, if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves (4.12) for all $\xi \in C^\infty(\overline{\Omega})$, then u is automatically a solution of (4.9) with homogeneous Neumann boundary conditions. In fact, if (4.12) holds for all $\xi \in C_0^\infty(\Omega)$

we obtain that u solves (4.9) in Ω. Then, for arbitrary $\xi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned}
0 &= \int \sum_{i=1}^n \partial_i \xi \sum_{j=1}^n a_{ij} \partial_j u + \xi(bu + f) \\
&= \int_\Omega \xi \left[\sum_i \partial_i \sum_j a_{ij} \partial_j u + bu + f \right] + \int_{\partial\Omega} \xi \left[\sum_{i=1}^n \nu_i \sum_{j=1}^n a_{ij} \partial_j u \right] \\
&= \int_{\partial\Omega} \xi \left[\sum_{i=1}^n \nu_i \sum_{j=1}^n a_{ij} \partial_j u \right]
\end{aligned}$$

implying $\sum_{i=1}^n \nu_i \sum_{j=1}^n a_{ij} \partial_j u = 0$ on $\partial\Omega$.

To use Lax-Milgram's Theorem, we want to rewrite (4.12) as

$$\int \sum_{ij} a_{ij}\, \partial_i \xi\, \partial_j u + b\xi u = -\int f\xi \tag{4.13}$$

and to consider the l.h.s. as a sesquilinear form on an appropriate Hilbert space. Since in (4.13), u, ξ and their derivatives $\partial_i \xi$, $\partial_j u$ appear, the appropriate Hilbert space to solve homogeneous Dirichlet problems is the completion of $C_0^\infty(\Omega)$ in the $H^1(\Omega) = H^{1,2}(\Omega)$ norm, i.e. the space $H_0^1(\Omega) = H_0^{1,2}(\Omega)$. On the other hand, for Neumann boundary value problems, the appropriate Hilbert space is the completion of $C^\infty(\overline{\Omega})$ in the H^1 norm, i.e. the space $H^1(\Omega) = H^{1,2}(\Omega)$. We give therefore a new formulation of the boundary value problems.

Weak solutions of boundary value problems: Let $\Omega \subset \mathbb{R}^n$ open and bounded, $b \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $a_{ij} \in L^\infty(\Omega)$, so that the condition (4.10) holds for almost all $x \in \Omega$. We call u a weak solution of the homogeneous Dirichlet problem, if $u \in H_0^1(\Omega)$ and

$$\int \left(\sum_{i=1}^n \partial_i \xi \sum_{j=1}^n a_{ij} \partial_j u + (bu + f)\xi \right) = 0 \tag{4.14}$$

for all $\xi \in H_0^1(\Omega)$ (since $u \in H^1(\Omega)$, it is enough to show (4.14) for all $\xi \in C_0^\infty(\Omega)$). We call$u$ a weak solution of the homogeneous Neumann problem, if $u \in H^1(\Omega)$ and

$$\int \left(\sum_{i=1}^n \partial_i \xi \sum_{j=1}^n a_{ij} \partial_j u + (bu + f)\xi \right) = 0 \tag{4.15}$$

for all $\xi \in H^1(\Omega)$ (equivalently, for all $\xi \in C^\infty(\overline{\Omega})$).

Relationship between weak and classical solutions of boundary value problems: So far, we argued that every classical solution of the original boundary value problem is also a weak solution. It is natural to ask whether weak solutions, satisfying the Dirichlet/Neumann boundary value problems in a weak sense, are also classical solutions. To prove that this is the case, one needs to know that weak solutions are regular enough (to be able to integrate by parts). Indeed, assuming regularity of the coefficients a_{ij}, b, f, it is possible to show regularity of weak solutions of boundary value problems through the so called theory of elliptic regularity. In this class, we will not discuss this important theory and we will instead focus on establishing existence and uniqueness of weak solutions of boundary value problems. You should keep in mind, however, that once the existence and the uniqueness of weak solution is known, one can typically conclude existence and uniqueness of classical solutions of boundary value problems proving, by means of elliptic regularity, that weak solutions are regular and therefore also classical solutions.

Existence and uniqueness of weak solutions of boundary value problems: making use of Theorem 4.4.3, we prove, first of all, existence and uniqueness of weak solutions of Neumann boundary value problems.

Theorem 4.4.4. *We assume that all conditions above are satisfied. Furthermore, we assume that there is $b_0 > 0$ with $b(x) \geq b_0$ for almost every $x \in \Omega$. Then there exists exactly one weak solution $u \in H^1(\Omega)$ of the Neumann boundary value problem (**??**). There exists $C > 0$ with $\|u\|_{H^1(\Omega)} \leq C\|f\|_2$.*

Proof. For $u, v \in H^1(\Omega)$ let

$$a(u,v) = \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_i u \partial_j v + \int_\Omega buv$$

a is bilinear (and thus sesquilinear, since $\mathbb{K} = \mathbb{R}$), with

$$|a(u,v)| \leq \sum_{ij} \|a_{ij}\|_\infty \|\partial_i u\|_2 \|\partial_j v\|_2 + \|b\|_\infty \|u\|_2 \|v\|_2 \leq \left(\sum_{ij} \|a_{ij}\|_\infty + \|b\|_\infty \right) \|u\|_{H^1} \|v\|_{H^1}$$

Moreover

$$a(u,u) \geq c_0 \int_\Omega \sum_{j=1}^n |\partial_j u|^2 + b_0 \int |u|^2 \geq \min(c_0, b_0) \|u\|_{H^1}^2$$

For $v \in H^1(\Omega)$, let

$$F(v) = -\int_\Omega fv$$

Then F is linear and $|F(v)| \leq \|f\|_2 \|v\|_{H^1}$. Hence $F \in (H^1(\Omega))^*$. From (4.8), we conclude that the problem

$$a(u,v) = F(v) \quad \text{for all } v \in H^1(\Omega)$$

has a unique solution $u \in H^1(\Omega)$ with

$$\|u\|_{H^1} \leq \frac{1}{\min(c_0, b_0)} \|F\| \leq \frac{1}{\min(c_0, b_0)} \|f\|_2$$

$\square$

For weak solution of Dirichlet problems, the same result holds true. In fact, for Dirichlet problems, we do not need to assume that b is strictly positive, $b \geq 0$ is enough. This is a consqeuence of the following Poincaré inequality.

Proposition 4.4.5 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ open and bounded. Then there exists C_0 (depending on Ω) with*

$$\int_\Omega |u|^2 dx \leq C_0 \int_\Omega |\nabla u|^2 dx$$

for all $u \in H_0^1(\Omega)$.

Proof. It is enough to consider $u \in C_0^\infty(\Omega)$. For $n = 1$, let $\Omega \subset [a, b]$. Then $u(a) = 0$, and

$$|u(x)|^2 = |u(x) - u(a)|^2 = \left(\int_a^x u'(y) dy \right)^2 \leq |x - a| \int_a^x |u'(y)|^2 dy \leq |b - a| \int_a^b |u'(y)|^2 dy$$

for all $x \in [a, b]$. Hence

$$\int_\Omega |u(x)|^2 dx \leq \int_a^b |u(x)|^2 dx \leq (b-a)^2 \int_a^b |u'(y)|^2 dy = (b-a)^2 \int_\Omega |u'(y)|^2 dy$$

If $n > 1$, we find $a, b \in \mathbb{R}$ with $\Omega \subset [a, b] \times \mathbb{R}^{n-1}$. As above, we find

$$\int_a^b |u(x_1, \mathbf{x})|^2 dx_1 \leq (b-a)^2 \int_a^b |\partial_1 u(x_1, \mathbf{x})|^2 dx_1$$

for all $\mathbf{x} = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. The claim follows integrating over $\mathbf{x}$. $\square$

Theorem 4.4.6. *We assume that all conditions in the definition of weak solutions of Dirichlet boundary value problem are satisfied. We assume furthermore that $b(x) \geq 0$ for almost every $x \in \Omega$. Then there exists exactly one weak solution $u \in H_0^1(\Omega)$ of the Dirichlet problem. There is a constant $C > 0$ with $\|u\|_{H^1(\Omega)} \leq C\|f\|_2$.*

Proof. We proceed as in the proof of Theorem 4.4.4; $a(u, v)$ and $F(v)$ are just defined on $H_0^1(\Omega)$, instead of $H^1(\Omega)$. The only difference is the fact that, now,

$$a(u, u) \geq c_0 \int_\Omega \sum_{j=1}^n |\partial_j u|^2 dx = c_0 \int_\Omega |\nabla u|^2 dx \geq c\|u\|_{H^1}^2$$

where the bound

$$\|u\|_{H^1}^2 = \int |\nabla u|^2 dx + \int |u|^2 dx \leq C \int |\nabla u|^2 dx$$

for all $u \in H_0^1(\Omega)$ is used ; see Proposition 4.4.5. $\square$

5 Baire Category Theorem and Consequences

5.1 Theorems of Baire and of Banach-Steinhaus

Lemma 5.1.1. *Let (X, τ) be a topological space. The following statements are equivalent.*

1) *Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of closed set in X. If the interior of each A_i is empty, then also the interior of $\cup_{j=1}^{\infty} A_i$ is empty.*

2) *Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of open sets in XR. If each B_i is dense in X, then also $\cap_{j=1}^{\infty} B_j$ is dense in X.*

Proof. The equivalence follows from the remark that a set is dense if and only if its complement has an empty interior. □

Definition 5.1.2. *A topological space (X, τ) is called a Baire's space, if condition 1) oder condition 2) (and thus both) are satisfied.*

Theorem 5.1.3 (Baire)**.** *Every complete metric space is a Baire space.*

Proof. Let (X, d) be a complete metric space, with $X \neq \emptyset$. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of open dense sets in X. We have to show that $L := \cap_{j=1}^{\infty} B_j$ is dense, as well. To this end, we show that for every non-emtpy open set $G \subset X$, we have $G \cap L \neq \emptyset$.

Since B_1 is open and dense, $G \cap B_1 \neq \emptyset$ and open. Hence, we can find $\varepsilon_1 \in (0, 1]$ and $x_1 \in X$ such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset B_1 \cap G.$$

Since B_2 is open and dense, $B_{\varepsilon_1}(x_1) \cap B_2$ is also non-empty and open. Hence, we can find $\varepsilon_2 \in (0, 1/2]$ and $x_2 \in X$ with

$$\overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap B_2$$

Iteratively, we find $\varepsilon_n \in (0, 1/n]$ and $x_n \in X$ with

$$\overline{B_{\varepsilon_n}(x_n)} \subset B_{\varepsilon_{n-1}}(x_{n-1}) \cap B_n$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X; in fact, for every $m \geq n$, we have $B_{\varepsilon_m}(x_m) \subset B_{\varepsilon_n}(x_n)$ and therefore $d(x_n, x_m) \leq \varepsilon_n \leq 1/n \to 0$ as $n, m \to \infty$. Since X is complete, the limit $x = \lim_{n\to\infty} x_n$ exists. From $x_k \in \overline{B_{\varepsilon_n}(x_n)}$ for all $k \geq n$, we deduce that $x \in \overline{B_{\varepsilon_n}(x_n)}$ for all $n \in \mathbb{N}$. Hence

$$x \in G \cap \bigcap_{j=1}^{\infty} B_j = G \cap L$$

and therefore $G \cap L \neq \emptyset$. □

As first application of Baire's Theorem, we prove the Theorem of Banach-Steinhaus.

Theorem 5.1.4 (Banach-Steinhaus). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subset \mathcal{L}(X,Y)$. Assume that for all $x \in X$ there exists $c_x \geq 0$ with*

$$\sup_{T\in\mathcal{F}} \|Tx\| \leq c_x.$$

Then there exists $c \geq 0$ with

$$\sup_{T\in\mathcal{F}} \|T\| \leq c\,.$$

Proof. For $k \in \mathbb{N}$ let

$$A_k := \{x \in X : \|Tx\| \leq k \quad \text{for all } T \in \mathcal{F}\}$$

Then A_k is closed for all $k \in \mathbb{N}$. In fact, if $(x_j)_{j\in\mathbb{N}}$ is a sequence in A_k with $x_j \to x$ as $j \to \infty$, then, for every $T \in \mathcal{F}$,

$$\|Tx\| \leq \|Tx_j\| + \|T(x - x_j)\| \leq k + \|T\|\|x - x_j\|$$

for arbitrary $j \in \mathbb{N}$. Hence, for the given $T \in \mathcal{F}$ and for arbitrary $\varepsilon > 0$ we can choose $j \in \mathbb{N}$ so large that $\|x - x_j\| \leq \varepsilon/\|T\|$. Then $\|Tx\| \leq k + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\|Tx\| \leq k$. This holds for all $T \in \mathcal{F}$; therefore

$$\sup_{T\in\mathcal{F}} \|Tx\| \leq k$$

In other words, $x \in A_k$ and A_k is closed. We have

$$\bigcup_{k=1}^{\infty} A_k = X$$

From Theorem 5.1.3, there exists $k_0 \in \mathbb{N}$ with $\mathring{A}_{k_0} \neq \emptyset$. We find therefore $x_0 \in X$, $\varepsilon_0 > 0$ with $\overline{B_{\varepsilon_0}(x_0)} \subset \mathring{A}_{k_0}$. Hence, for any $y \in X$ with $\|y\| \leq \varepsilon_0$, we have

$$\|Ty\| = \|T(y + x_0) - Tx_0\| \leq \|T(y + x_0)\| + \|Tx_0\| \leq k_0 + c_{x_0}$$

for all $T \in \mathcal{F}$. We conclude that

$$\sup_{\|y\|\leq 1} \|Ty\| \leq \frac{k_0 + c_{x_0}}{\varepsilon_0}$$

for all $T \in \mathcal{F}$. $\square$

5.2 The open mapping and the closed graph theorems

Lemma 5.2.1. *Let X be a Banach space, Y a normed space, $r, s > 0$ and $T \in \mathcal{L}(X,Y)$ with $\overline{T(B_r^X(0))} \supset B_s^Y(0)$. Then $T(B_r^X(0)) \supset B_{s/2}^Y(0)$.*

Proof. We assume that $r = s = 1$ (otherwise, we consider the map λT, for appropriate $\lambda > 0$). Because of linearity, we find

$$\overline{T(B_r^X(0))} \supset B_r^Y(0) \qquad \text{for all } r > 0. \tag{5.1}$$

We choose $y_0 \in B_{1/2}^Y(0)$. From (5.1) (with $r = 1/2$) we find $x_1 \in B_{1/2}^X(0)$ with $\|y_0 - Tx_1\| \leq 1/4$. Since now $y_1 = Tx_1 - y_0 \in B_{1/4}^Y(0)$, from (5.1) (with $r = 1/4$) we can find $x_2 \in B_{1/4}^X(0)$ with

$$\|(y_0 - Tx_1) - Tx_2\| \leq 1/8$$

Inductively, we find $x_k \in B_{2^{-k}}^X(0)$ with

$$\|T(x_1 + \cdots + x_k) - y_0\| \leq 2^{-k-1}$$

Since $\sum_{k\geq 1} \|x_k\| < 1$, the limit

$$x = \lim_{n\to\infty} \sum_{k=1}^n x_k$$

exists in X (using the completeness of X). Then $\|x\| < 1$ and $Tx = y_0$. This proves that $B_{1/2}^Y(0) \subset T(B_1^X(0))$. □

Lemma 5.2.1, together with Baire's theorem, implies the open mapping theorem.

Theorem 5.2.2 (Open mapping theorem)**.** *Let X, Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$ be surjective (i.e. $T(X) = Y$). Then $T(U)$ is open in Y, for all open set $U \subset X$.*

Proof. Because of the linearity of T it is enough to show that there exists $r > 0$ with $T(B_1^X(0)) \supset B_r^Y(0)$. Since

$$Y = T(X) = \bigcup_{n=1}^\infty T(B_n^X(0)) = \bigcup_{n=1}^\infty nT(B_1^X(0)) = \bigcup_{n=1}^\infty n\overline{T(B_1^X(0))}$$

Theorem 5.1.3 implies that there exists at least one $n \in \mathbb{N}$ such that $n\overline{T(B_1^X(0))}$ has a non-empty interior. Hence, $\overline{T(B_1^X(0))}$ has a non-empty interior. Therefore, there exist $y_0 \in Y$ and $\varepsilon_0 > 0$ with

$$B_{\varepsilon_0}(y_0) \subset \overline{T(B_1^X(0))}$$

Hence

$$
\begin{aligned}
B_{\varepsilon_0}(0) \subset B_{\varepsilon_0}(y_0) - B_{\varepsilon_0}(y_0) &\subset \overline{T(B_1^X(0))} - \overline{T(B_1^X(0))} = \overline{T(B_1^X(0))} + \overline{T(-B_1^X(0))} \\
&= \overline{T(B_1^X(0))} + \overline{T(B_1^X(0))} \subset \overline{T(B_1^X(0)) + T(B_1^X(0))} = \overline{T(B_2^X(0))}
\end{aligned}
$$

where we used the linearity of T, the fact that $-B_1^X(0) = B_1^X(0)$ and the continuity of the sum. Again by linearity of T we conclude that $B^Y_{\varepsilon_0/2}(0) \subset \overline{T^X(B_1(0))}$. Lemma 5.2.1 implies that $B^Y_{\varepsilon_0/4}(0) \subset T_X(B_1(0))$. □

An equivalent formulation of the open mapping theorem is given by the following inverse map theorem.

Theorem 5.2.3 (Inverse Mapping Theorem)**.** *Let X, Y be Banach spaces, $T : X \to Y$ a continuous linear bijection. Then $T^{-1} \in \mathcal{L}(Y, X)$ is also linear and continuous, i.e. there exists $c > 0$ with*

$$\frac{1}{c}\|x\| \le \|Tx\| \le c\|x\|$$

for all $x \in X$.

The statement of the theorem is non-trivial, despite the assumption that T is invertible, because there could still be a sequence x_k with $\|x_k\| = 1$ for all $k \in \mathbb{N}$ and $\|Tx_k\| \to 0$ as $k \to \infty$. In this case, T^{-1} would exist, but it would not be continuous. The inverse mapping theorem shows that this is not possible.

Proof. T^{-1} is well-defined and linear. From Theorem 5.2.2 there exists $r > 0$ with $T(B_1^X(0)) \supset B_r^Y(0)$, i.e. with $T^{-1}(B_r^Y(0)) \subset B_1^X(0)$. This implies that, for arbitrary $y \in Y$,

$$\|T^{-1}y\| = \frac{2\|y\|}{r}\left\|T^{-1}\frac{ry}{2\|y\|}\right\| \le \frac{2}{r}\|y\|$$

Hence, T^{-1} is bounded and continous. □

Corollary 5.2.4. *Let $\|.\|_1$ and $\|.\|_2$ be two norms on a vector space X over $\mathbb{K}$, so that $(X, \|.\|_1)$ and $(X, \|.\|_2)$ are Banach spaces. Assume that $\|x\|_1 \le c\|x\|_2$ for all $x \in X$. Then there exists also $d > 0$ with $\|x\|_2 \le d\|x\|_1$ for all $x \in X$ (hence, the two norms are equivalent).*

Proof. From the assumption $\|x\|_1 \le c\|x\|_2$ for all $x \in X$, the identity map $\mathrm{id} : (X, \|.\|_2) \to (X, \|.\|_1)$ defined by $\mathrm{id}(x) = x$, is a continuous linear map. Hence, also the inverse $\mathrm{id}^{-1} : (X, \|.\|_1) \to (X, \|.\|_2)$ must be continuous and bounded. □

Another consequence of the inverse mapping theorem is the following closed graph theorem.

Theorem 5.2.5 (Closed Graph Theorem)**.** *Let X, Y be Banach spaces, $T : X \to Y$ linear. The following statements are equivalent.*

1) graph(T) $= \{(x, Tx) \in X \times Y : x \in X\}$ is closed in $X \times Y$ (w.r.t. the norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$).

2) T is continuous.

Proof. "2) $\Rightarrow$ 1)": Let $(x,y) \in \overline{\text{graph}(T)}$. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X with $(x_k, Tx_k) \to (x,y)$. Hence $x_k \to x$ in X and $Tx_k \to y$ in Y. Since T is continuous, we have $y = Tx$ and $(x,y) \in \text{graph}(T)$. Therefore graph($T$) is closed.

"1) $\Rightarrow$ 2)": Define $\phi : X \to X \times Y$ through $\phi(x) = (x, Tx)$. The image of ϕ is given by

$$R(\phi) = \{\phi(x) : x \in X\} = \text{graph}(T)$$

and, by assumption, is a closed subset of $X \times Y$. Hence $R(\phi)$ is a Banach space w.r.t. the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ and the map $\phi : X \to R(\phi)$ is a linear bijection between Banach spaces. Also the inverse map $\phi^{-1} : R(\phi) \to X$, defined by $\phi^{-1}(x, Tx) = x$, is a linear bijection. Since

$$\|\phi^{-1}(x, Tx)\|_X = \|x\|_X \leq \|(x, Tx)\|_{X\times Y}$$

ϕ^{-1} is a continuous linear bijection. The inverse mapping theorem, Theorem 5.2.3, implies that also ϕ is continuous. Since the projection $p : X \times Y \to X$, defined by $p(x,y) = y$ is clearly continuous, we conclude that $T = p \circ \phi$ is continuous, as well. $\square$

Direct sums of closed subspaces: Let X be a Banach space and A, B closed linear subspaces with $X = A + B$ and $A \cap B = \{0\}$. Then, for every $x \in X$, there exist unique $a \in A, b \in B$ with $x = a + b$. In other words, the map $\phi : A \times B \to X$, defined through $\phi(a,b) = a + b$, is a linear bijection. The triangle inequality also implies that ϕ is continuous. Under the assumption that A, B are closed, ϕ is a continuous linear isomorphism between two Banach spaces. The inverse mapping theorem implies that $\phi^{-1} : X \to A \times B$ is continuous, as well. Since the maps $p_1 : A \times B \to A$, $p_2 : A \times B \to B$, defined through $p_1(a,b) = a$, $p_2(a,b) = b$ are also continuous, it follows that the maps $P_A : X \to A$ and $P_B : X \to B$, defined through $P_A(x) = a$ and $P_B(x) = b$ if $x = a + b$ are continuous.

Definition 5.2.6. *Let A, B be linear closed subspaces of a Banach space X with $A \cap B = \{0\}$ and $A + B = X$. We say that X is the (topological) direct sum of A and B and we write $X = A \oplus B$. In this case, B is known as the topological complement to A. As explained above, it follows from the inverse map theorem that P_A and P_B are continuous (X is said to be the algebraic direct sum of two linear subspaces A, B if $X = A + B$ and $A \cap B = \{0\}$; in this case, A, B do not need to be closed and the projections P_A, P_B are in general not continuous).*

The maps P_A, P_B are examples of projections.

Definition 5.2.7. *Let X be a Banach space. A continuous map $P : X \to X$ with $P^2 = P$ is called a projection on X. If P is a projection, then also $1 - P$ is a projection.*

In infinite dimensional Banach spaces it is not clear that a given closed subspace has a topological complement. Every projection on X allows us to write X as a topological direct sum.

Theorem 5.2.8. *Let X be a Banach space and $P \in \mathcal{L}(X)$ a projection. Then $X = P(X) \oplus (1-P)(X)$.*

Proof. We set $A = \text{Ran } P = \{Px : x \in X\}$ and $B = \text{Ran } (1-P) = \{(1-P)x : x \in X\}$. Then

- A is closed. In fact, if $(y_k)_{k\in\mathbb{N}}$ is a sequence in A with $y_k \to y$, then $y_k = Px_k$ for appropriate $x_k \in X$. Hence
$$Py_k = P^2 x_k = Px_k = y_k \qquad \text{for all } k \in \mathbb{N}\,.$$
If we let $k \to \infty$, we find (since P is continuous) $Py = y$ and thus $y \in A$.
- B is closed. Same argument as above, with P replaced by $Q = 1 - P$.
- $X = A + B$. This is clear, since $x = Px + x - Px = Px + (1-P)x$.
- $A \cap B = \{0\}$. Here we use the fact that $x = Px$, if $x \in A$ (because $x = Py$ implies that $Px = P^2 y = Py = x$) and $x = (1-P)x$ if $x \in B$. Thus $x = Px = P(1-P)x = 0$ if $x \in A \cap B$.

□

It follows from the last theorem that finite dimonsional subspaces always have a topological complement.

Theorem 5.2.9. *Let X be a Banach space, $A \subset X$ a finite dimensional linera subspace. Then there exists a closed subspace $B \subset X$ with $X = A \oplus B$.*

Proof. Since dim $A < \infty$, A is closed. Let $x_1, \dots, x_n$ be a basis for A and let $x_1^*, \dots, x_n^*$ be the dual basis in A^* (the linear functional x_j^* on A is defined by the condition that $x_j^*(x_i) = \delta_{ij}$). Hahn-Banach implies that there exists a continuous linear extension of x_j^* on X, for $j = 1, \dots, n$. We denote the extension again with $x_1^*, \dots, x_n^*$. Then, we define $P : X \to X$ through $P(x) = \sum_{j=1}^n x_j^*(x) x_j$. P is clearly linear and it is also continuous, since
$$\|Px\| \leq \sum_{j=1}^n \|x_j^*\| \|x_j\| \|x\| \leq \left(\sum_{j=1}^n \|x_j^*\| \|x_j\| \right) \|x\|$$
Moreover, we have
$$P \circ P(x) = P\left(\sum_{j=1}^n x_j^*(x) x_j \right) = \sum_{i=1}^n \sum_{j=1}^n x_j^*(x) x_i^*(x_j) x_i = \sum_{j=1}^n x_j^*(x) x_j = P(x)$$
Hence P is a projection. Since Ran $P = A$, the claim follows from Theorem 5.2.8 with $B =$ Ran $(1-P)$. □

6 Weak Topologies on normed spaces

6.1 The weak and the weak-* topologies

Let X be a set, $\mathcal{F}$ a family of maps $f : X \to Y_f$, where Y_f is a topological space. On X, we introduce the topology $\tau_{\mathcal{F}}$, defined as the smallest topology on X so that all functions in $\mathcal{F}$ are continuous. To this end, we notice that, if $(\tau_\lambda)_{\lambda\in\Lambda}$ is a family of topologies on X, then also $\tau = \cap_{\lambda\in\Lambda}\tau_\lambda$ is a topology on X (proof: exercise). Thus, we can define

$$\mathcal{S} = \left\{ f^{-1}(V) : V \subset Y_f \text{ open, } f \in \mathcal{F} \right\}$$

and

$$\tau_{\mathcal{F}} = \bigcap \{\tau : \tau \text{ is a topology on } X \text{ and } \mathcal{S} \subset \tau\}$$

Then $\tau_{\mathcal{F}}$ is the smallest topology, with the property that all maps in $\mathcal{F}$ are continuous.

Proposition 6.1.1. *Let X be a set and $\mathcal{F}$ a family of maps $f : X \to Y_f$, with Y_f Hausdorff for every $f \in \mathcal{F}$. $\mathcal{F}$ separates the points of X, ie. for every $x, y \in X$, $x \not y$, there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$). Then $(X, \tau_{\mathcal{F}})$ is a Hausdorff space.*

Proof. Choose $x \neq y$ in X. Then we find $f \in \mathcal{F}$ with $f(x) \neq f(y)$. Since Y_f is Hausdorff, we find also open neighbourhoods U_x, U_y of $f(x)$ and $f(y)$ in Y_f with $U_x \cap U_y = \emptyset$. Then $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are $\tau_{\mathcal{F}}$-open neighbourhoods of x and y with $f^{-1}(U_x) \cap f^{-1}(U_y) = \emptyset$. □

A useful characterization of sets in $\tau_{\mathcal{F}}$ is given by the following lemma.

Lemma 6.1.2. *Let X be a set and $\mathcal{F}$ a family of maps $f : X \to Y_f$, so that Y_f is a topological space, for all $f \in \mathcal{F}$. Then*

$$\begin{aligned}\tau_{\mathcal{F}} = \big\{ &\textit{arbitrary unions of finite intersections of sets} \\ &\textit{having the form } f^{-1}(U), \textit{ with } U \subset Y_f \textit{ open and } f \in \mathcal{F} \big\}\end{aligned}$$

Proof. Exercise. □

We apply now this construction to normed spaces.

Definition 6.1.3. *Let X be a normed space, X^* its (topological) dual space and X^{**} the bidual space. The smallest topology on X, with the property that all $f \in X^*$ are continuous is called the weak topology on X; it is denoted by τ_W.*

Since X^* is again a normed space, we can introduce also on X^* the weak topology, defined as the smallest topology with the property that all $f \in X^{**}$ are continuous. On X^* we define also the weak- topology, denoted by τ_W^*, defined as the smallest topology such that all $f \in J_X(X) \subset X^{**}$ are continuous. Here $J_X : X \to X^{**}$ is the natural inclusion of X into X^{**}, defined in Section 4.3.*

Remarks:

- On X we have: $\tau_W \subset \tau_{\|.\|}$. This follows because, by definition $f \in X^*$ is always continuous w.r.t. $\tau_{\|.\|}$ and because τ_W is the *smallest* topology on X such that all $f \in X^*$ are continuous.

- On X^* we have: $\tau_W^* \subset \tau_W \subset \tau_{\|.\|}$. This follows because $J_X(X) \subset X^{**}$.

- Since X^* separates the points of X, it follows from Prop. 6.1.1 that (X, τ_W) is a Hausdorff space.

- $J_X(X)$ separates the points of X^*. In fact, for given $f, g \in X^*$ with $f \neq g$ there exists $x \in X$ with $f(x) \neq g(x)$ and therefore with $J_X(x)(f) \neq J_X(x)(g)$. It follows that also (X^*, τ_W^*) is a Hausdorff space.

- For an arbitrary normed space X, the spaces (X, τ_W), (X^*, τ_W), (X^*, τ_W^*) are topological vectors spaces. In other words, τ_W, τ_{W^*} define vector space topologies, i.e. topologies with the additional properties that $\{x\}$ is closed, for all $x \in X$ or, respectively, in X^*, and that vector space operations are continuous. The topologies τ_W and τ_W^* are even locally convex vector space topologies, meaning that they have a neighbourhood basis at 0 consisting of convex sets (proof: exercise).

  For an arbitrary topological vector space (X, τ), it is possible to define the dual space $(X, \tau)^*$ as the space of all linear functional on X that are continuous w.r.t. to τ. We have $(X, \tau_W)^* = X^* \equiv (X, \tau_{\|.\|})^*$. In fact, every $f \in (X, \tau_{\|.\|})^*$ is, by definition, continuous w.r.t. τ_W. On the other hand, $\tau_W \subset \tau_{\|.\|}$ implies that every τ_W-continuous functional is also $\tau_{\|.\|}$-continuous. Analogously, $(X^*, \tau_W)^* = X^{**} \equiv (X^*, \tau_{\|.\|})^*$ and $(X^*, \tau_W^*) = J_X(X) \subset X^{**}$.

- If X is reflexive, then $J_X(X) = X^{**}$ and $\tau_W = \tau_W^*$ as topologies on X^*.

For infinite dimensional vector spaces norm- and weak topologies are not equivalent.

Lemma 6.1.4. *Let X be a normed space. Then $\tau_W = \tau_{\|.\|}$ if and only if $\dim X < \infty$. If $\dim X = \infty$, then every τ_W-open neighbourhood of 0 contains an infinitely dimensional linear subspace of X.*

Proof. “ $\Rightarrow$ ”: Let $W \subset X$ be a τ_W-open neighbourhood of 0. Then (from Lemma 6.1.2) there exist $f_1, \dots, f_k \in X^*$ with

$$V := \{x \in X : |f_i(x)| < 1 \text{ for all } i = 1, \dots, k\} \subset W$$

Hence, the linear subspace $N = \{x \in X : f_i(x) = 0 \text{ for } i = 1, \dots, k\} \subset W$. Clearly, we have $\dim N = \infty$ if and only if $\dim X = \infty$. Hence, if $\dim X = \infty$, W contains an infinite dimensional linear subspace of X. This shows, in particular, that $B_1(0) = \{x \in X : \|x\| < 1\} \notin \tau_W$, and thus that $\tau_W \neq \tau_{\|.\|}$.

" $\Leftarrow$": Without loss of generality, we can assume $X = \mathbb{K}^n$, with $\|x\| = \max_{i=1,\dots,n} |x_i|$, for $x = (x_1, \dots, x_n)$. We define $f_i(x_1, \dots, x_n) = x_i$. Then $f_i \in X^*$ for all $i = 1, \dots, n$ and

$$B_\varepsilon(0) \supset \{x \in X : |f_i(x)| < \varepsilon \text{ for all } i = 1, \dots, n\}$$

for all $\varepsilon > 0$. Hence $\tau_W \supset \tau_{\|.\|}$ and thus $\tau_W = \tau_{\|.\|}$. □

6.2 The notion of weak convergence

Let X be a normed space. As every topology, the weak topology τ_W induces a notion of convergence of sequences in X. A sequence $(x_k)_{k\in\mathbb{N}}$ converges towards $x \in X$ w.r.t. τ_W, if, for every τ_W-open neighbourhood V of x there exists $k_0 \in \mathbb{N}$ with $x_k \in V$ for all $k \geq k_0$. In this case, we say that x_k converges weakly towards x; we write $x_k \rightharpoonup x$. Analogously, on X^* we can define the notion of convergence w.r.t. τ_W^*.

Remarks:

- From the definition of τ_W, it is simple to check that $x_k \rightharpoonup x$, if, for every τ_W-open neighbourhood V of 0, there exists $k \in \mathbb{N}$ with $x_k \in x + V$ for all $k \geq k_0$. A similar statement holds for the weak-* convergence.
- Since (X, τ_W) and (X^*, τ_W^*) are both Hausdorff, weak and weak-* limits are always unique, if they exist.
- Since $\tau_W \subset \tau_{\|.\|}$, $x_k \to x$ w.r.t. $\tau_{\|.\|}$ implies that $x_k \rightharpoonup x$. Analogously, norm convergence and weak convergence imply weak-* convergence on X^*.

Lemma 6.2.1. *Let X be a normed space. For a sequence $(x_k)_{k\in\mathbb{N}}$, we have $x_k \rightharpoonup x$ if an only if $f(x_k) \to f(x)$ for all $f \in X^*$.*

Proof. " $\Rightarrow$ ": is a consequence of the continuity of f. In fact, for a given $f \in X^*$ and $\varepsilon > 0$, set $V = \{x \in E : |f(x)| < \varepsilon\}$. V is a τ_W-open neighbourhood of 0. Hence, there exists $k_0 \in \mathbb{N}$ with $x_k \in x + V$ for all $k \geq k_0$. Therefore, $|f(x_k) - f(x)| = |f(x_k - x)| \leq \varepsilon$ for all $k \geq k_0$.

" $\Leftarrow$": from Prop. 6.1.2 there exists a basis of τ_W-open neighbourhoods of 0 consisting of sets of the form

$$V = \{x \in E : |f_i(x)| < 1, \text{for } i = 1, \dots, n\} = \bigcap_{i=1}^n f_i^{-1}(B_1^{\mathbb{K}}(0))$$

Hence, we find $k_0 \in \mathbb{N}$ with

$$|f_i(x_k) - f_i(x)| < 1 \quad \text{for all } k \geq k_0 \text{ and for } i = 1, \dots, n,$$

Then $x_k \in x + V$ for all $k \geq k_0$. □

Remarks:

- A similar characterization holds true for the weak-* convergence on X^*; a sequence $(f_i)_{i\in\mathbb{N}}$ in X^* converges towards $f \in X^*$ w.r.t. τ_W^* if and only if $f_i(x) \to f(x)$ for all $x \in X$. The proof is similar as the proof of Lemma 6.2.1.

- Since, in general, the topological space (X, τ_W) is not metrizable, one cannot characterize the weak topology through the notion of weak convergence. Consider, for example, the space
$$\ell^1(\mathbb{K}) = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{K}, \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$
with the norm
$$\|x\|_1 = \sum_{j=1}^{\infty} |x_j|$$
As we already discussed, $\ell^1(\mathbb{K})$ is a Banach space. It turns out (proof;exercise) that, for a sequence $(x^{(k)})_{k\in\mathbb{N}}$ on $\ell^1(\mathbb{K})$, $x^{(k)} \to x$ if and only if $x^{(k)} \rightharpoonup x$, ie. the norm-topology and the weak-topology on $\ell^1(\mathbb{K})$ induce the same notion of convergence. Nevertheless, since $\dim \ell^1(\mathbb{K}) = \infty$, we know from Lemma 6.1.4 that $\tau_W \neq \tau_{\|.\|}$.

Despite the fact that in general weak convergent sequences do not converge w.r.t. the norm topology, they are always bounded. This is a consequence of the Banach-Steinhaus theorem.

Proposition 6.2.2. *Let X be a normed space, $(x_k)_{k\in\mathbb{N}}$ a sequence on X with $x_k \rightharpoonup x$. Then $(x_k)_{k\in\mathbb{N}}$ is bounded and*
$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|$$

Proof. By assumption, $f(x_k) \to f(x)$ for all $f \in X^*$. Hence, for every $f \in X^*$ there exists $c_f > 0$ with
$$|J(x_k)(f)| = |f(x_k)| \leq c_f$$
for all $k \in \mathbb{N}$. From Theorem 5.1.4 (using the fact that X^* is always a Banach space) there exists $c > 0$ with $\|J(x_k)\| \leq c$ for all $k \in \mathbb{N}$. This implies that $\|x_k\| \leq c$ for all $k \in \mathbb{N}$.

From Hahn-Banach there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. Hence,
$$\|x\| = f(x) = \lim_{k\to\infty} f(x_k) \leq \liminf_{k\to\infty} \|f\|\|x_k\| = \liminf_{k\to\infty} \|x_k\|$$

□

Example: For $1 < p < \infty$, consider the space
$$\ell^p(\mathbb{K}) = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{K}, \sum_{j=1}^{\infty} |x_j|^p < \infty\}$$

equipped with the norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}$$

We know that $\ell^p(\mathbb{K})$ is a Banach space. For $1 < p < \infty$, we have $x^{(n)} \rightharpoonup x$ in $\ell^p(\mathbb{K})$ if and only if the sequence $x^{(n)}$ is bounded (i.e. $\|x^{(n)}\|_p < C$, for all $n \in \mathbb{N}$) and for all $i \in \mathbb{N}$, $x_i^{(n)} \to x_i$ as $n \to \infty$ (proof: exercise).

Remark: From Prop. 6.2.2 it follows that the norm of a weak limit is always smaller than the limit of the norm. In general (but $X = \ell^1(\mathbb{K})$ is an exception), one cannot expect that the norm of a weak limit is the same as the limit of the norm (this is instead of course correct for norm limits). As an example, consider the sequence $x_k = (0, \dots, 0, 1, 0, \dots)$ (with the entry 1 in the k-th coordinate) in $\ell^p(\mathbb{K})$. On the one hand, $\|x_k\|_p = 1$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$. On the other hand, for $1 < p < \infty$ we have $x_k \rightharpoonup 0$.

6.3 Weak-* topology and compactness

For finite dimensional normed spaces, it is easy to characterize compact sets; a set M is compact (covering and sequentially compact) if and only if M is bounded and closed. For infinite dimensional normed spaces, this is not true. In this case, the closed ball $\overline{B_1(0)}$ is neither covering nor sequentially compact. Weak topologies are important because often they "restore" the compactness (covering and sequential) of the closed ball. In this section, we show the covering and, assuming additionally separability, the sequential compactness of the closed unit ball in X^*, w.r.t. the weak-* topology. In the next section, we prove then the compactness of the unit ball in X w.r.t. the weak topology of X, under the assumption that X is reflexive.

To prove the compactness of the closed unit ball in X^* w.r.t. τ_W^* (Theorem of Banach-Alaoglu), we need few definitions and some preparatory lemma.

Definition 6.3.1. *Let S be a set. A family $\mathcal{F}$ of subsets of S has a finite character, if, for every $A \subset S$,*

$$A \in \mathcal{F} \quad \Leftrightarrow \quad \text{every finite subset of } A \text{ is in } \mathcal{F}$$

The next result follows from the Lemma of Zorn (and thus from the axiom of choice).

Lemma 6.3.2 (Tukey's Lemma)**.** *Let $\mathcal{F}$ be a family of sets with finite character and $F \in \mathcal{F}$. Then there exists a maximal element of $\mathcal{F}$, containing F.*

Proof. Let $\mathcal{F}_0 = \{A \in \mathcal{F} : F \subset A\}$. On $\mathcal{F}_0$ we introduce the order relation $A \preceq B$ if and only if $A \subset B$. This relation defines a partial order on $\mathcal{F}_0$. If $\mathcal{C} \subset \mathcal{F}_0$ is a totally ordered subset, then $D = \cup_{C \in \mathcal{C}} C \in \mathcal{F}_0$ (because every finite subset of D must be subset of a single $C_0 \in \mathcal{C}$ and therefore, it must be contained in $\mathcal{F}$; this shows that $D \in \mathcal{F}$ and, since $F \subset D$ is clear, that $D \in \mathcal{F}_0$). D is clearly an upper bound for $\mathcal{C}$. Zorn's Lemma implies that $\mathcal{F}_0$ has a maximal element M. M contains F and it is also a maximal element for $\mathcal{F}$ (because $\widetilde{M} \in \mathcal{F}$ with $\widetilde{M} \supset M$ implies that $\widetilde{M} \supset F$ and thus that $\widetilde{M} \in \mathcal{F}_0$, in contradiction to the maximality of M). $\square$

Definition 6.3.3. *Let S be a set and $\mathcal{F}$ a family of subsets of S. We say that $\mathcal{F}$ has the finite intersection property if $\bigcap_{i=1}^n F_i \neq \emptyset$ for all $F_1, \dots, F_n \in \mathcal{F}$, $n \in \mathbb{N}$.*

Remark: Let X be a topological space. Then X is compact, if and only if the intersection of all elements of a family $\mathcal{F}$ consisting of closed subsets of X having the finite intersection property is not empty. In fact, let X be non-compact. Then there is a family $\mathcal{G}$ of open subsets of X, so that $X = \cup_{G \in \mathcal{G}} G$ and $X \not\subset \cup_{i=1}^n G_i$ for all $G_1, \dots, G_n \in \mathcal{F}$. Then $\mathcal{F} = \{G^c : G \in \mathcal{G}\}$ is a family of closed subsets of X with the finite intersection property so that $\cap_{F \in \mathcal{F}} F = \emptyset$. On the other hand, if there exists a family $\mathcal{F}$ of closed subsets of X with the finite intersection property, such that $\cap_{F \in \mathcal{F}} F = \emptyset$, then we can find an open covering of X given by $\mathcal{G} = \{F^c : F \in \mathcal{F}\}$ with no finite subcovering. In this case, X is not compact.

An equivalent characterisation is as follows: A topological space X is compact if and only if for every family $\mathcal{A}$ of subsets of X with the finite intersection property, we have $\cap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$. We are going to use this characterisation of compactness in the following theorem.

Theorem 6.3.4 (Tychonov Theorem)**.** *Let $(X_\lambda)_{\lambda \in \Lambda}$ be a family of compact sets. Then the product $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact, w.r.t. to the product topology defined on X.*

Bemerkung: The product topology on X is generated by product sets having the form

$$\prod_{\lambda \in \Lambda} U_\lambda = \{(x_\lambda)_{\lambda \in \Lambda} : x_\lambda \in U_\lambda\},$$

with $U_\lambda \subset X_\lambda$ open for all $\lambda \in \Lambda$. Equivalently, the product topology is the smallest topology on X, so that all projections $p_\lambda : X \to X_\lambda$, defined through $p((x_\lambda)_{\lambda \in \Lambda}) = x_\lambda$, are continuous.

Proof. Let $\mathcal{A}$ a family of subsets of X with the finite intersection property. We show that $\cap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$.

To this end, consider the set $\mathcal{F}$, consisting of all families $\mathcal{B}$ of subsets of X with the finite intersection property. In particular, $\mathcal{A} \in \mathcal{F}$. Notice that $\mathcal{F}$ has finite character (because whether or nor a family $\mathcal{A}$ has the finite intersection property only depends on subfamilies of $\mathcal{A}$ containing finitely many subsets).

From Lemma 6.3.2, there exists a maximal family $\mathcal{B}$ of subsets of X with the finite intersection property, containing $\mathcal{A}$. Since

$$\bigcap_{B \in \mathcal{B}} \bar{B} \subset \bigcap_{A \in \mathcal{A}} \bar{A}$$

it is enough to show that $\bigcap_{B\in\mathcal{B}} \bar{B} \neq \emptyset$.

The maximality of $\mathcal{B}$ implies that if a set $B \subset X$ has non-empty intersection with every set in $\mathcal{B}$, then $B \in \mathcal{B}$ (otherwise, the family $\widetilde{\mathcal{B}}$, consisting of all sets in $\mathcal{B}$ and of B, would have again the finite intersection property, contradicting the maximality of $\mathcal{B}$). In particular, if $A \supset B$ for a $B \in \mathcal{B}$, then $A \in \mathcal{B}$. Moreover, again from maximality, finite intersections of sets in $\mathcal{B}$ must also be contained in $\mathcal{B}$.

For every $\lambda \in \Lambda$, the family $\{p_\lambda(B) : B \in \mathcal{B}\}$ has the finite intersection property (because $x \in \cap_{i=1}^n B_i$ implies that $p_\lambda(x) \in \cap_{i=1}^n p_\lambda(B_i)$; therefore $\cap_{i=1}^n B_i \neq \emptyset$ implies that $\cap_{i=1}^n p_\lambda(B_i) \neq \emptyset$). Since X_λ is compact, there exists $x_\lambda \in X_\lambda$ with

$$x_\lambda \in \bigcap_{B\in\mathcal{B}} \overline{p_\lambda(B)}$$

Now, let U_λ be an arbitrary open neighbourhood of x_λ in X_λ. Then $p_\lambda^{-1}(U_\lambda) \cap B \neq \emptyset$ for all $B \in \mathcal{B}$ (because $x_\lambda \in \overline{p_\lambda(B)}$ implies that $U_\lambda \cap p_\lambda(B) \neq \emptyset$ and thus that $p^{-1}(U_\lambda) \cap B \neq \emptyset$). Hence, $p_\lambda^{-1}(U_\lambda) \in \mathcal{B}$ for every $\lambda \in \Lambda$. As noticed above, the maximality of $\mathcal{B}$ als implies that

$$\bigcap_{j=1}^n p_{\lambda_j}^{-1}(U_{\gamma_j}) \in \mathcal{B} \tag{6.1}$$

for arbitrary $\{\lambda_1, \dots, \lambda_n\} \in \Gamma$ and open neighbourhoods U_{λ_j} of x_{λ_j}.

Let $x = (x_\lambda)_{\lambda\in\Lambda} \in X$. We claim that

$$x \in \bigcap_{B\in\mathcal{B}} \overline{B}\,.$$

In fact, for every neighbourhood U of x there exist, by definition of the product topology, finitely many $\lambda_1, \dots, \lambda_n \in \Lambda$ and opne neighbourhoods U_{λ_j} of x_{λ_j} with

$$\bigcap_{j=1}^n p_{\lambda_j}^{-1}(U_{\lambda_j}) \subset U$$

This follows from the remark that the product topology on X is exactly the smallest topology such that all maps $p_\lambda : X \to X_\lambda$ are continuous. From Lemma 6.1.2, we know that there exists a neighbourhood basis at $x \in X$, consisting of finite intersections of sets of the form $p_\lambda^{-1}(U_\lambda)$, where U_λ is an open neighbourhood of $p_\lambda(x) = x_\lambda$.

From (6.1) we conclude, by maximality of $\mathcal{B}$, that $U \in \mathcal{B}$. This implies that $U \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. Since U is an arbitrary open neighbourhood of x, we obtain that $x \in \overline{B}$ for all $B \in \mathcal{B}$ and thus that $x \in \cap_{B\in\mathcal{B}}\overline{B}$. $\square$

We apply Tychonov Theorem to show the τ_W^*-compactness of the closed unit ball in X^* In the following we use the notation $k_X = \{x \in X : \|x\| \leq 1\}$ to denote the closed unit ball in the Banach space X.

Theorem 6.3.5 (Banach-Alaoglu)**.** *Let X be a normed space and X^* its dual space. Then the close unit ball $k_{X^*} = \{f \in X^* : \|f\| \leq 1\}$ is compact, w.r.t. the weak-* topology τ_W^*.*

Proof. For $x \in X$, let

$$D_x = \{\alpha \in \mathbb{K} : |\alpha| \leq \|x\|\} .$$

For every $x \in X$, D_x is a compact space (with the topology induced by $\mathbb{K}$). We set $P = \prod_{x\in X} D_x$, and equip it with the product topology τ_P. We will consider the projections $p_x : P \to D_x$. Theorem 6.3.4 implies that (P, τ_P) is compact.

Elements of P are families $(\alpha_x)_{x\in X}$ and they can be thought of as maps $\varphi : X \to \mathbb{K}$ with $\varphi(x) = \alpha_x$. Then, we have $|\varphi(x)| \leq \|x\|$ for all $x \in X$. Hence, the closed unit ball $k_{X^*} = X^* \cap P$. We show that k_{X^*} is a closed subset of (P, τ_P), w.r.t. the topology τ_P. This implies that k_{X^*} is compact w.r.t. τ_P (as closed subset of a compact space). Moreover, we show that $\tau_P|_{k_{X^*}} \supset \tau_W^*|_{k_{X^*}}$; this implies that k_{X^*} is compact also w.r.t. τ_W^* (if a topology contains more open sets, it is more difficult for a set to be compact).

To show that k_{X^*} is closed in (P, τ_P) we define, for $x, y \in X$, $\lambda \in \mathbb{K}$ a map $\sigma_{x,y,\lambda} : P \to \mathbb{K}$ through

$$\sigma_{x,y,\lambda}(\varphi) = \varphi(x + \lambda y) - \varphi(x) - \lambda\varphi(y)$$

Since $\sigma_{x,y,\lambda} = p_{x+\lambda y} - p_x - \lambda p_y$, we obtain that $\sigma_{x,y,\lambda}$ is continuous w.r.t. τ_P. Since $\{0\} \subset \mathbb{K}$ is closed, it follows that $\sigma^{-1}_{x,y,\lambda}(\{0\})$ is closed in P w.r.t. τ_P. But then also

$$k_{X^*} = \bigcap_{x,y\in X,\lambda\in\mathbb{K}} \sigma^{-1}_{x,y,\lambda}(\{0\})$$

is closed w.r.t. τ_P.

We still have to show that $\tau_W^*|_{k_{X^*}}$ is rougher as $\tau_P|_{k_{X^*}}$, i.e. that $\tau_W^*|_{k_{X^*}} \subset \tau_P|_{k_{X^*}}$. A neighbourhoods basis at $\varphi \in k_{X^*}$ w.r.t. the topology τ_W^* consists of sets of the form

$$V = \{\varphi + f \in k_{X^*} : |f(x_i)| < 1 \text{ for } i = 1, \ldots, n\}$$

because $f(x_i) = J_X(x_i)(f)$ and

$$\{f \in X^* : |J_X(x_i)(f)| < 1 \text{ for all } i = 1, \ldots, n\} = \bigcap_{i=1}^{n} (J_X(x_i))^{-1}(B_1(0))$$

is an element of a typical τ_W^*-neighbourhoods basis at 0 in X^*. Hence,

$$V - \varphi = \{f \in k_{X^*} : |f(x_i)| < 1, i = 1, \ldots, n\} = \bigcap_{i=1}^{n} p^{-1}_{x_i}(B_1(0))$$

is also open w.r.t. τ_P, because $B_1(0)$ is open in $\mathbb{K}$ and p_{x_i} is continuous w.r.t. τ_P, for all $i = 1, \dots, n$. This proves that $\tau_W^*|_{k_{X^*}} \subset \tau_P|_{k_{X^*}}$ (the same argument shows that actually $\tau_W^*|_{k_{X^*}} = \tau_P|_{k_{X^*}}$). □

Since in general the weak-* topology is not metrizable, the notion of sequentially compactness is not equivalent to the notion of compactness (i.e. covering compactness). In fact,in general the unit ball in X^* is not sequentially compact w.r.t. τ_W^*. The next theorem shows, however, that the unit ball in X^* is sequentially compact, if X is separable (in fact, in this case it turns out that the τ_W^* topology is metrizable, on every bounded subset of X^*; however, we will not show this fact here).

Theorem 6.3.6. *Let X be a separable normed space. Then the closed unit ball k_{X^*} in X^* is sequentially compact, w.r.t. the τ_W^* topology.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. Let $(x_k^*)_{k\in\mathbb{N}}$ be a sequence in X^* with $\|x_k^*\| \leq 1$ for all $k \in \mathbb{N}$ (i.e. x_k^* is a sequence in k_{X^*}). Since

$$|x_k^*(x_1)| \leq \|x_1\|$$

the sequence $x_k^*(x_1)$ is bounded in $\mathbb{K}$. Hence, there exists α_k and a subsequence $n_{1,k} \in \mathbb{N}$ with $x_{n_{1,k}}^*(x_1) \to \alpha_1$ as $k \to \infty$. Now, $x_{n_{1,k}}^*(x_2)$ is also a bounded sequence in $\mathbb{K}$; hence there exist $\alpha_2 \in \mathbb{K}$ and a subsequence $n_{2,k}$ of $n_{1,k}$, such that $x_{n_{2,k}}^*(x_j) \to \alpha_j$, for $j = 1, 2$. Iteratively, we find for all $\ell \in \mathbb{N}$, $\alpha_\ell \in \mathbb{K}$ and a subsequence $n_{\ell,k}$, so that $x_{n_{\ell,k}}^*(x_j) \to \alpha_j$, for all $j = 1, \dots, \ell$. Consider the diagonal subsequence $m_\ell = n_{\ell,\ell}$ for $\ell \in \mathbb{N}$. Then we have

$$x_{m_\ell}^*(x_j) \to \alpha_j$$

as $\ell \to \infty$ for all $j \in \mathbb{N}$. For arbitrary $x \in X$, we have

$$\begin{aligned} |x_{m_\ell}^*(x) - x_{m_r}^*(x)| &\leq (\|x_{m_\ell}^*\| + \|x_{m_r}^*\|)\|x - x_j\| + |x_{m_\ell}^*(x_j) - x_{m_r}^*(x_j)| \\ &\leq 2\|x - x_j\| + |x_{m_\ell}^*(x_j) - x_{m_r}^*(x_j)| \,. \end{aligned}$$

For a given $\varepsilon > 0$, we can therefore find $j \in \mathbb{N}$ with $\|x - x_j\| \leq \varepsilon/4$. Then

$$|x_{m_\ell}^*(x) - x_{m_r}^*(x)| \leq \varepsilon/2 + |x_{m_\ell}^*(x_j) - x_{m_r}^*(x_j)| \leq \varepsilon$$

for $\ell, r \in \mathbb{N}$ sufficiently large. Hence, the limit

$$\lim_{\ell\to\infty} x_{m_\ell}^*(x) =: x^*(x)$$

exists for all $x \in X$. It is easy to check that x^* is linear and that

$$|x^*(x)| \leq \limsup_{\ell\to\infty} \|x_{m_\ell}^*\|\|x\| \leq \|x\|,$$

Hence, $x^* \in k_{X^*}$. Since $J_X(x)(x_{m_\ell}^*) = x_{m_\ell}^*(x) \to x^*(x) = J_X(x)(x^*)$ for all $x \in X$, it follows that $x_{m_\ell}^* \to x^*$ w.r.t the topology τ_W^*. □

6.4 Reflexivity and compactness

In Section 6.3 we studied the compactness and the sequential compactness of the closed unit ball in X^* w.r.t. the τ_W^* topology. In this section, we consider reflexive Banach spaces and we show that, under appropriate assumptions, the closed unit ball in X is compact and sequential compact w.r.t. the τ_W topology.

Proposition 6.4.1. *Let X be a Banach space. Then $J_X(\overline{B_1(0)})$ is dense in $k_{X^{**}}$. w.r.t. the τ_W^* topology.*

Proof. Let $x^{**} \in k_{X^{**}}$. For $\varepsilon > 0$ and $f_1, \dots, f_k \in X^*$ we have to find $x \in k_X$ with

$$\sum_{j=1}^{n} |(J(x) - x^{**})(f_i)|^2 < \varepsilon$$

Without loss of generality, we can assume that the functionals $f_1, \dots, f_k$ are linear independent. We define $\phi : X \to \mathbb{K}^n$ through $\phi(x) = (f_1(x), \dots, f_n(x))$. Then ϕ is linear and continuous. Let $N = \ker \phi$. On X we consider the equivalence relation $x \sim y :\Leftrightarrow x - y \in N$. Because of the linear independence of the f_j, the quotient space $X/N = \{[x] : x \in X\}$ has dimension n. Let $[x_1], \dots, [x_n]$ be a basis of X/N. With $B = \text{span}\,\{x_1, \dots, x_n\}$, we have clearly $X = N \oplus B$. Then it is simple to check that $\phi(x_1), \dots \phi(x_n)$ is a basis for $\mathbb{K}^n$. This implies that ϕ is surjective. We can define a map $\hat{\phi} : X/N \to \mathbb{K}^n$ through $\hat{\phi}([x]) = \phi(x)$. $\hat{\phi}$ is well-defined, linear, continuous and a bijection. Hence, we find a unique $\hat{x} \in X/N$ with $\hat{\phi}(\hat{x}) = (x^{**}(f_1), \dots, x^{**}(f_n))$. For all $x \in \hat{x}$ (hence, for all $x \in X$ in the equivalence class $\hat{x}$, we have $\phi(x) = (f_1(x), \dots, f_n(x)) = (x^{**}(f_1), \dots, x^{**}(f_n))$, and thus

$$\sum_{j=1}^{n} |f_j(x) - x^{**}(f_j)|^2 = 0$$

We claim that

$$\inf_{x \in \hat{x}} \|x\| \leq \|x^{**}\| \leq 1. \tag{6.2}$$

Then, for an arbitrary $\delta > 0$, we can find $x_0 \in \hat{x}$ with $\|x_0\| \leq (1-\delta)^{-1}$, such that $(1-\delta)x_0 \in k_X$ and

$$\sum_{j=1}^{n} |f_j((1-\delta)x_0) - x^{**}(f_j)|^2 = \delta^2 \sum_{j=1}^{n} |f_j(x_0)|^2 \leq \frac{\delta^2}{(1-\delta)^2} \sum_{j=1}^{n} \|f_j\|^2$$

Choosing $\delta > 0$ small enough, we obtain that $\delta^2/(1-\delta^2) \sum_{j=1}^{n} \|f_j\|^2 < \varepsilon$, as desired.

It remains to show (6.2). To this end, we notice that

$$\|[x]\|_{X/N} := \inf_{y \sim x} \|y\|$$

defines a norm on X/N; then $(X/N, \|.\|_{X/N})$ is a normed space (actually, even a Banach space). From Hahn-Banach, there exists $\psi \in (X/N)^*$ with $\|\psi\| = 1$ and $\psi(\hat{x}) = \|\hat{x}\|$.

For an arbitrary $\beta \in (\mathbb{K}^n)^* \simeq (\mathbb{K})^n$, given through $\beta(\alpha) = \sum_{i=1}^n \beta_i \alpha_i$, for appropriate coordinates $(\beta_1, \dots, \beta_n) \in \mathbb{K}^n$, we have

$$x^{**}(\beta \circ \phi) = x^{**}(\sum_{j=1}^n \beta_j f_j) = \sum_{j=1}^n \beta_j x^{**}(f_j) = \beta \circ \hat{\phi}(\hat{x})$$

In particular for $\beta = \psi \circ \hat{\phi}^{-1} \in (\mathbb{K}^n)^*$, we find

$$x^{**}(\beta \circ \phi) = \beta \circ \hat{\phi}(\hat{x}) = \psi(\hat{x}) = \|\hat{x}\|$$

Thus

$$\|\hat{x}\| \leq |x^{**}(\beta \circ \phi)| \leq \|x^{**}\| \|\psi\| \|\hat{\phi}^{-1} \circ \psi\| \leq \|x^{**}\|$$

where we used the fact that $\hat{\phi}^{-1} \circ \psi(x) = [x]$ and therefore, that

$$\|\hat{\phi}^{-1} \circ \psi(x)\| = \inf_{y \in [x]} \|y\| \leq \|x\| \quad \Rightarrow \quad \|\hat{\phi}^{-1} \circ \psi\| \leq 1$$

This shows (6.2). □

Corollary 6.4.2. *Let X be a Banach space. Then X is reflexive if and only if k_X is compact w.r.t. the τ_W topology.*

Proof. " $\Rightarrow$": Let X be reflexive. Then X^* is reflexive as well and, on X^{**}, $\tau_W = \tau_W^*$. Hence $k_{X^{**}}$ is compact w.r.t. τ_W. Next we observe that the map $J_X^{-1} : (X^{**}, \tau_W) \to (X, \tau_W)$ is continuous. In fact, for arbitrary $x_1^*, \dots, x_n^* \in X^*$, we have

$$\begin{aligned} J_X^{-1} \{x^{**} \in X^{**} : |J_{X^*}(x_i^*)(x^{**})| < 1 \text{ for all } i = 1, \dots, n\} \\ = \{x \in X : |x_i^*(x)| < 1 \text{ for all } i = 1, \dots, n\} \end{aligned} \tag{6.3}$$

This follows from the fact that $(J_{X^*} x_i^*)(Jx) = (J_X x)(x_i^*) = x_i^*(x)$. (6.3) implies the continuity of J^{-1}, because the sets on the r.h.s. are a typical elements of a basis of τ_W-open neighbourhoods of 0 in X^{**} (because J_{X^*} is a bijection), while sets on the l.h.s. are typical elements of a basis of τ_W-open neighbourhoods of 0 in X. Since continuous functions map compact sets into compact sets (proof: exercise), $J^{-1}(k_{X^{**}}) = k_X$ is compact w.r.t. τ_W.

" $\Leftarrow$": it follows from Prop. 6.4.1 that $J_X(k_X)$ is τ_W^*-dense in $k_{X^{**}}$. The map $J_X : (X, \tau_W) \to (X^{**}, \tau_W^*)$ is continuous, because, for arbitrary $x_1^*, \dots, x_n^* \in X^*$, we have

$$J_X^{-1} \{x^{**} \in X^{**} : |x^{**}(x_i^*)| < 1, i = 1, \dots, n\} = \{x \in X : |x_i^*(x)| < 1, i = 1, \dots, n\}$$

where the set on the r.h.s. is a typical element of a basis of τ_X^*-open neighbourhoods of 0 in X^{**}, while a set on the l.h.s. is a typical element of a basis of τ_W-open neighbourhoods of 0. Since k_X is compact, also $J_X(k_X)$ is compact w.r.t. τ_W^*. Hence $J_X(k_X)$ is τ_W^*-dense and τ_W^*-closed in $k_{X^{**}}$. Thus $J(k_X) = k_{X^{**}}$ and from (anti)-linearity, $J_X(X) = X^{**}$. Thus J_X is surjective and X is reflexive. □

For reflexive Banach spaces, k_X is not only compact, it is also sequentially compact, w.r.t. the topology τ_W.

Theorem 6.4.3. *Let X e a reflexive Banach space. Then the closed unit ball k_X is sequentially compact w.r.t. the topology τ_W. In other words, every bounded sequence in X has a τ_W-convergent subsequence*

To prove Theorem 6.4.3, we are going to use the following proposition.

Proposition 6.4.4. *Let X be a reflexive Banach space. Then X is τ_W complete, in the following sense: Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X, so that, for every τ_W-open neighbourhood V of 0 there exists $N \in \mathbb{N}$ with $x_n - x_m \in V$ for all $n, m > N$; in this case, we say that x_k is a τ_W-Cauchy sequence. Then x_n converges, w.r.t. the topology τ_W.*

Proof. Let (x_n) be a τ_W-Cauchy sequence. For arbitrary $f \in X^*$ and for arbitrary $\varepsilon > 0$, the set $\{x \in X : |f(x)| < \varepsilon\}$ is a τ_W-open neighbourhood of 0. Hence, there is $N \in \mathbb{N}$ with $|f(x_n) - f(x_m)| = |f(x_n - x_m)| < \varepsilon$, for all $n, m > N$. Therefore $f(x_n)$ is a Cauchy sequence in $\mathbb{K}$ and $f(x_n)$ converges, for every $f \in X^*$. Hence, for every $f \in X^*$ there exists $c_f > 0$ with $|J(x_n)(f)| = |f(x_n)| \leq c_f$ for all $n \in \mathbb{N}$. Banach-Steinhaus implies that the sequence x_n is bounded, i.e. that there exists $C > 0$ with $\|x_n\| \leq C$ for all $n \in \mathbb{N}$. We define $\lim_{n\to\infty} f(x_n) =: T(f)$. $T : X^* \to \mathbb{K}$ is linear and continuous, since

$$|T(f)| \leq \limsup_{n\to\infty} \|f\|\|x_n\| \leq C\|f\|$$

In other words, $T \in X^{**}$. Since X is reflexive, there exists $x \in X$ with $T = J_X(x)$. Then $f(x_n) \to f(x)$ for all $f \in X^*$ and thus $x_n \rightharpoonup x$. □

Proof of Theorem 6.4.3. Let (x_k) be a sequence in k_X and set $E := \overline{\text{span}\{x_k : k \in \mathbb{N}\}}$. Then E is a closed linear and separable subspace of X. From Theorem 4.3.3, E is also reflexive. Hence $E^{**} = J_E(E)$ is separable, and E^* is separable (see exercises). Let $(f_k)_{k\in\mathbb{N}}$ be a dense sequence in E^*. As in the proof of Theorem 6.3.6, we can find a subsequence x_{m_ℓ}, such that $f_j(x_{m_\ell})$ converges, in $\mathbb{K}$, as $\ell \to \infty$, for all $j \in \mathbb{N}$. This implies in turns, similarly as in the proof of Theorem 6.3.6, that $f(x_{m_\ell})$ converges for all $f \in E^*$, and therefore also for all $f \in X^*$ (because $f|_E \in E^*$ for all $f \in X^*$). This implies that x_{m_ℓ} is a τ_W-Cauchy sequence in the sense of Proposition 6.4.4. This implies that x_{m_ℓ} converges, w.r.t. the τ_W topology. □

6.5 Weak topology and convexity

A weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X has one additional property, that is often useful; if x_n converges weakly, there exists namely another sequence, consisting of convex combinations of the elements of x_n, converging strongly (i.e. w.r.t. the norm topology). This result is know as the Lemma of Mazur. To prove it, we will make use of the following separation theorem.

Theorem 6.5.1 (Separation theorem). *Let X be a normed space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), $M \subset X$ closed and convex and $x_0 \in X \backslash M$. Then there exists $F \in X^*$ and $\alpha \in \mathbb{R}$ with*

$$\textit{Re } F(x) \leq \alpha \quad \textit{for all } x \in M \textit{ and } \quad \textit{Re } F(x_0) > \alpha$$

In other words, M and x_0 are divided by the hyperplane Re $F(x) = \alpha$.

Proof. We consider first the case $\mathbb{K} = \mathbb{R}$. W.l.o.g. we assume that 0 is contained in the interior of M. If this is not the case, we choose $\widetilde{x} \in M$ and we set $\widetilde{x}_0 = x_0 - \widetilde{x}$ and $\widetilde{M} = \overline{B_r(M - \widetilde{x})}$ for $0 < r < \text{dist}(M, x_0)$. Then is 0 clearly in the interior of $\widetilde{M}$. From the existence of $F \in X^*$ and $\widetilde{\alpha} \in \mathbb{R}$ with $F(x) \leq \widetilde{\alpha}$ for all $x \in \widetilde{M}$ and $F(\widetilde{x}_0) > \widetilde{\alpha}$, it follows that $F(x_0) > \widetilde{\alpha} + F(\widetilde{x})$ and that

$$F(x) \leq \widetilde{\alpha} + F(\widetilde{x})$$

for all $x \in \overline{B_r(M)}$, in particular, for all $x \in M$.

From the assumption that 0 is contained in the interior of M, there exists $\rho > 0$ with $B_\rho(0) \subset M$. We define then the Minkowski functional

$$p(x) = \inf\{r > 0 : x/r \in M\}$$

for all $x \in X$. Clearly, $p(x) \geq 0$ for all $x \in X$. Since $\rho x/\|x\| \in B_\rho(0) \subset M$, we find

$$p(x) \leq \frac{1}{\rho}\|x\|$$

for all $x \in X$. Furthermore, $p(x) \leq 1$ for all $x \in M$, $p(x_0) > 1$ and $p(ax) = ap(x)$ for all $a \geq 0$. We have

$$p(x + y) \leq p(x) + p(y)$$

for all $x, y \in X$. This follows from the convexity of M, since

$$\frac{x}{r}, \frac{y}{s} \in M \quad \Rightarrow \quad \frac{x+y}{r+s} = \frac{r}{r+s}\frac{x}{r} + \frac{s}{r+s}\frac{y}{s} \in M\,.$$

Hence, p is a sublinear functional, as introduced in Definition 4.2.2.

We define now $f : \mathbb{R} \cdot x_0 \to \mathbb{R}$ (notice that $x_0 \neq 0$, since we assumed that $0 \in M$) through

$$f(\lambda x_0) = \lambda p(x_0)$$

Then $f(\lambda x_0) = p(\lambda x_0)$ for all $\lambda \geq 0$ and $f(\lambda x_0) \leq 0 \leq p(\lambda x_0)$ for all $\lambda < 0$. Hence $f(x) \leq p(x)$ for all $x \in \mathbb{R} \cdot x_0$. From the Hahn-Banach theorem (Theorem 4.2.4), there exists $F : X \to \mathbb{R}$ linear with $F|_{\mathbb{R}x_0} = f$ and $F(x) \leq p(x)$ for all $x \in X$. Therefore $F(x) \leq 1$ for all $x \in M$, $F(x_0) = p(x_0) > 1$. Moreover,

$$F(x) \leq p(x) \leq \frac{1}{\rho}\|x\|$$

and

$$-F(x) = F(-x) \leq p(-x) \leq \frac{1}{\rho}\| - x\| = \frac{1}{\rho}\|x\|$$

Thus $|F(x)| \leq (1/\rho)\|x\|$ for all $x \in X$ and $F \in X^*$.

In the case $\mathbb{K} = \mathbb{C}$, we consider X as a real vector space $X_\mathbb{R}$. Proceeding as above, we find $F_\mathbb{R} \in X_\mathbb{R}^*$ $\mathbb{R}$-linear. Then we can define $F(x) = F_\mathbb{R}(x) - iF_\mathbb{R}(ix)$. It is easy to check that F is a $\mathbb{C}$-linear functional on X, with all desired properties. $\square$

Theorem 6.5.1 implies that convex sets are closed with respect to the norm topology, if and only if they are closed w.r.t. the topology τ_W.

Theorem 6.5.2. *Let X be a normed space over $\mathbb{K}$ and $A \subset X$ convex. Then $\overline{A}^{\tau_{\|.\|}} = \overline{A}^{\tau_W}$, i.e. the closure of A w.r.t. the norm topology is the same as the closure of A w.r.t. the weak topology.*

Proof. We consider the case $\mathbb{K} = \mathbb{R}$. Since $\tau_W \subset \tau_{\|.\|}$ is clearly $\overline{A}^{\tau_W} \supset \overline{A}^{\tau_{\|.\|}}$. We have to show that $\overline{A}^{\tau_W} \subset \overline{A}^{\tau_{\|.\|}}$. Let $x_0 \in \overline{A}^{\tau_W}$. We assume that $x_0 \notin \overline{A}^{\tau_{\|.\|}}$. Then, since $\overline{A}^{\tau_{\|.\|}}$ is closed w.r.t. $\tau_{\|.\|}$ and it is convex (it is easy to check that the norm closure of a convex set is convex again), it follows from Theorem 6.5.1, that there are $f \in X^*$ and $\alpha \in \mathbb{R}$ with $f(x) \leq \alpha$ for all $x \in \overline{A}^{\tau_{\|.\|}}$ and with $f(x_0) > \alpha$. In particular, $f(x) \leq \alpha$ for all $x \in A$, or, equivalently, $A \subset f^{-1}((-\infty, \alpha])$. Since

$$\left(f^{-1}((-\infty, \alpha])\right)^c = \{x \in X : f(x) > \alpha\} = f^{-1}((\alpha, \infty)) \in \tau_W$$

the set $f^{-1}((-\infty, \alpha])$ is τ_W-closed and thus $\overline{A}^{\tau_W} \subset f^{-1}((-\infty, \alpha])$ (because the closure of A w.r.t. τ_W is the intersection of all closed sets, containing A). This is however a contradiction to $f(x_0) > \alpha$. The case $\mathbb{K} = \mathbb{C}$ can be handled similarly. $\square$

Remark: Theorem 6.5.2 implies also that every convex norm-closed set $A \subset X$ is sequentially closed, w.r.t. the topology τ_W (meaning that for every sequence x_k in A, with $x_k \rightharpoonup x$ as $k \to \infty$, we have $x \in A$). This is a consequence of the remark that every closed sets is also sequentially closed (while the converse is not true, on general topological spaces).

As Corollary to Theorem 6.5.2 we show the Lemma of Mazur.

Corollary 6.5.3 (Mazur's Lemma). *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space $(X, \|.\|)$, converging weakly towards x. Then there exists a sequence $(y_k)_{k\in\mathbb{N}}$, so that every y_k is a finite convex combination of the elements x_n (i.e. for every $k \in \mathbb{N}$, we can write $y_k = \sum_{n\geq 1} t_n^{(k)} x_n$, where, for every fixed $k \in \mathbb{N}$, there are only finitely many $n \in \mathbb{N}$ with $t_n^{(k)} \neq 0$ and $\sum_{n\in\mathbb{N}} t_n^{(k)} = 1$), and $y_k \to x$ w.r.t. to the norm topology.*

Proof. Let A be the convex hull of the sequence (x_n) (i.e., A is the intersection of all convex sets containing the sequence x_n). Then $\overline{A}^{\tau_{\|\cdot\|}} = \overline{A}^{\tau_W}$. Since $x_n \in A \subset \overline{A}^{\tau_W}$ for all n, and from $x_n \rightharpoonup x$ it follows that $x \in \overline{A}^{\tau_W} = \overline{A}^{\tau_{\|\cdot\|}}$. Hence, there exists a sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k \in A$ and $y_k \to x$, because the norm-closure of A is exactly the set of all norm limits of sequences in A (notice again, that, in general, this property does not hold true for the weak topology). $\square$

7 Sobolev Embedding Theorems

7.1 Rellich's Embedding

Theorem 7.1.1 (Rellich's Embedding in $H_0^{m,p}$). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 \leq p < \infty$, $m \geq 1$. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H_0^{m,p}(\Omega)$ and $u \in H_0^{m-1,p}(\Omega)$ with $u_k \to u$ weakly in $H_0^{m-1,p}(\Omega)$, as $k \to \infty$. Then $u_k \to u$ strongly in $H_0^{m-1,p}(\Omega)$ as $k \to \infty$.*

Proof. It is enough to consider $m = 1$. For $m > 1$, we can apply the result with $m = 1$ to the weak derivatives $\partial^\alpha u_k$ and to $\partial^\alpha u$, for all $|\alpha| \leq m - 1$ (strong/weak convergence in $H^{m-1,p}$ means strong/weak convergence of all weak derivatives of order smaller or equal m in L^p). We extend u_k, u by defining it to vanish on $\mathbb{R}^n \backslash \Omega$. Then $u_k, u \in H^{1,p}(\mathbb{R}^n)$, with compact support in Ω. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard Dirac sequence. Then $\varphi_\varepsilon * u_k \in C_0^\infty(\mathbb{R}^n)$ for all $\varepsilon > 0$, and

$$\varphi_\varepsilon * u_k \to \varphi_\varepsilon * u \qquad \text{for } k \to \infty \text{ in } L^p(\mathbb{R}^n) \tag{7.4}$$

In fact, for arbitrary $x \in \mathbb{R}^n$, we have $\varphi_\varepsilon(x - .) \in L^{p'}(\mathbb{R}^n)$, with $1/p + 1/p' = 1$. Since $u_k \to u$ weakly in $L^p(\mathbb{R}^n)$, it follows that

$$\varphi_\varepsilon * u_k(x) = \int dy \varphi_\varepsilon(x - .) u_k(y) \to \int dy \varphi_\varepsilon(x - .) u(y) = \varphi_\varepsilon * u(x)$$

for all $x \in \mathbb{R}^n$. Since $\varphi_\varepsilon * (u_k - u)$ is compactly supported in $B_\varepsilon(\Omega)$, and since

$$|\varphi_\varepsilon * (u_k - u)(x)| \leq C_\varepsilon \|u_k - u\|_p \leq C_\varepsilon \left(\|u_k\|_p + \|u\|_p\right) \leq C_\varepsilon$$

Hence, dominated convergence implies (7.4).

Moreover, for all $v \in H_0^{1,p}(\mathbb{R}^n)$ we have

$$\|v - \varphi_\varepsilon * v\|_p \leq \varepsilon \|\nabla v\|_p \tag{7.5}$$

To show (7.5) it is enough to consider $v \in C_0^\infty(\mathbb{R}^n)$ (an arbitrary $v \in H_0^{1,p}$ can be approximated by a sequence in $C_0^\infty(\mathbb{R}^n)$). For $v \in C_0^\infty(\mathbb{R}^n)$, we have

$$v(x) - \varphi_\varepsilon * v(x) = \int dy\, \varphi_\varepsilon(y)\, (v(x) - v(x-y)) = \int dy\, \varphi_\varepsilon(y) \int_0^1 ds\, \nabla v(x - sy) \cdot y$$

and thus, with Hölder (since $\|\varphi_\varepsilon\| = 1$ and supp $\varphi_\varepsilon \subset B_\varepsilon(0)$),

$$\begin{aligned}
\|v - \varphi_\varepsilon * v\|_p^p &= \int dx \left| \int dy\, \varphi_\varepsilon(y) \int_0^1 ds\, \nabla v(x - sy) \cdot y \right|^p \\
&\leq \int dx \left(\int dy |\varphi_\varepsilon(y)| \right)^{p/p'} \left(\int dy |\varphi_\varepsilon(y)| \left| \int ds \nabla v(x - sy) \cdot y \right|^p \right) \\
&\leq \int dy |\varphi_\varepsilon(y)| \int dx \left| \int ds \nabla v(x - sy) \cdot y \right|^p \\
&\leq \varepsilon^p \sup_{h \in \text{supp } \varphi_\varepsilon} \| \int ds |\nabla v(. - sh)| \|_p^p \\
&\leq \varepsilon^p \|\nabla v\|_p^p
\end{aligned}$$

From (7.5), we find

$$\begin{aligned}
\|u_k - u\|_p &\leq \|u_k - u_k * \varphi_\varepsilon\|_p + \|u_k * \varphi_\varepsilon - u * \varphi_\varepsilon\|_p + \|u * \varphi_\varepsilon - u\|_p \\
&\leq \varepsilon \left(\|\nabla u_k\|_p + \|\nabla u\|_p \right) + \|u_k * \varphi_\varepsilon - u * \varphi_\varepsilon\|_p
\end{aligned}$$

Since u_k is bounded in $H^{1,p}(\Omega)$, we obtain

$$\|u_k - u\|_p \leq C\varepsilon + \|\varphi_\varepsilon * u - u\|_p + \|(u_k - u) * \varphi_\varepsilon\|_p$$

For fixed $\varepsilon > 0$, the last term vanishes as $k \to \infty$ by (7.4). Therefore,

$$\lim_{k \to \infty} \|u_k - u\|_p \leq C\varepsilon + \|\varphi_\varepsilon * u - u\|_p$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $u_k \to u$ in $L^p(\Omega)$, because $\|\varphi_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \to 0$. $\square$

We would like to show Rellich's Embedding Theorem for sequences in $H^{m,p}(\Omega)$, which do not necessarikly vanish close to the boundary. To this end, we need some assumptions on the regularity of the boundary. In particular, we will assume that the domain Ω has a Lipschitz boundary. We say that an open and bounded set $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary if $\partial\Omega$ can be covered through finitely many open sets $U_1, \dots, U_\ell$ so that, for all $j = 1, \dots, \ell$, $\partial\Omega \cap U_j$ is the graph of a Lipschitz continuous function and $\Omega \cap U_j$ is located on one side of the graph. The precise definition is as follows.

Definition 7.1.2 (Lipschitz boundary)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that Ω has a Lipschitz boundary if there exists $\ell \in \mathbb{N}$ and, for all $j = 1, \dots, \ell$, there exist a euclidean basis $e_1^j, \dots, e_n^j$ of $\mathbb{R}^n$, a reference point $y_j \in \mathbb{R}^{n-1}$, $r_j, h_j > 0$ and a Lipschitz continuous function $g_j : \mathbb{R}^{n-1} \to \mathbb{R}$, so that, with the notation $\hat{x}_n^j = (x_1^j, \dots, x_{n-1}^j)$ for $x = \sum_{i=1}^n x_i^j e_i^j$, and with*

$$U_j = \{x \in \mathbb{R}^n : |\hat{x}_n^j - y_j| < r_j, |x_n^j - g_j(\hat{x}_n^j)| < h_j\}$$

we have $\partial\Omega \subset \bigcup_{j=1}^{\ell} U_j$ and, for all $x \in U_j$,

$$\begin{aligned} x_n^j = g_j(\hat{x}_n^j) \qquad &\Rightarrow \qquad x \in \partial\Omega, \\ 0 < x_n^j - g_j(\hat{x}_n^j) < h_j \qquad &\Rightarrow \qquad x \in \Omega, \\ -h_j < x_n^j - g_j(\hat{x}_n^j) < 0 \qquad &\Rightarrow \qquad x \notin \Omega \end{aligned}$$

In this case, we can also find an open set U_0 with $\overline{U}_0 \subset \Omega$, so that $\overline{\Omega} \subset \bigcup_{j=0}^{\ell} U_j$.

To this open covering of Ω we can associate a partition of the identity $\eta_0, \dots, \eta_\ell$ with $0 \leq \eta_j \leq 1$ and $\eta_j \in C_0^\infty(U_j)$ for all $j = 0, \dots, \ell$, and with $\sum_{j=1}^{\ell} \eta_j = 1$ on $\overline{\Omega}$. For $u \in H^{m,p}(\Omega)$ we have $u = \sum_{j=1}^{\ell} \eta_j u$, where $\eta_0 u \in H^{m,p}(\Omega)$ with compact support in Ω, and, for all $j = 1, \dots, \ell$, $\eta_j u \in H^{m,p}(\Omega_j)$, with

$$\Omega_j := \{x \in \mathbb{R}^n : 0 < x_n^j - g_j(\hat{x}_n^j)\}$$

where $(\eta_j u)(x) = 0$ if $|\hat{x}_n^j - y_j| \geq r_j$ or $x_n^j - g_j(\hat{x}_n^j) \geq h_j$.

Theorem 7.1.3 (Rellich's Embedding Theorem for $H^{m,p}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with Lipschitz boundary, $1 \leq p < \infty$, $m \in \mathbb{N}\backslash\{0\}$. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^{m,p}(\Omega)$ and $u \in H^{m-1,p}(\Omega)$, with $u_k \rightharpoonup u$ weakly in $H^{m-1,p}(\Omega)$, as $k \to \infty$. Then $u_k \to u$ strongly in $H^{m-1,p}(\Omega)$, as $k \to \infty$.*

Proof. Like in Theorem 7.1.1, we can assume that $m = 1$. We use the partition $\eta_0, \dots, \eta_\ell$ introduced in Defintion 7.1.2. We set $u_k^j := \eta_j u_k$ and $u^j := \eta_j u$. We have to show that $u_k^j \to u^j$ strongly in $L^p(\Omega_j)$. For $j = 0$, the convergence follows from Theorem 7.1.1. Also for $j \geq 1$, we can proceed as in the proof of Theorem 7.1.1, if we could choose the standard-Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ so that, for all $x \in \Omega_j$, the function $y \to \varphi_\varepsilon(x - y)$ had compact support in Ω_j. With the definition of Ω_j given in Def. 7.1.2, this means that

$$x_n^j > g_j(\hat{x}_n^j), \varphi_\varepsilon(x - y) \neq 0 \quad \Rightarrow \quad y_n^j > g_j(\hat{y}_n^j) \tag{7.6}$$

Let $\lambda > 0$ be the Lipschitz constant of the function g_j. We claim that (7.6) holds true, if we choose the sequence φ_ε so that

$$\varphi_\varepsilon(z) \neq 0 \qquad \Rightarrow \qquad z_n^j < -\lambda|\hat{z}_n^j| \tag{7.7}$$

In other words, we choose φ_ε so that its support is contained in a cone below the origin, with opening angle θ with $\tan\theta = 1/\lambda$. In fact, if (7.7) holds true, the conditions $x_n^j > g_j(\hat{x}_n^j)$ and $\varphi_\varepsilon(x-y) \neq 0$, imply that

$$g_j(\hat{y}_n^j) = g_j(\hat{x}_n^j) + g_j(\hat{y}_n^j) - g_j(\hat{x}_n^j) < x_n^j + \lambda|\hat{y}_n^j - \hat{x}_n^j| < x_n^j - (x_n^j - y_n^j) = y_n^j\,.$$

To make sure that (7.7) is satisfied, we can simply choose $\varphi \in C^\infty(\{z \in B_1(0) : z_n^j < -\lambda|\hat{z}_n^j|\})$ and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. $\square$

The case $p=\infty$ plays a special role, because $H^{m,\infty}(\Omega)$ can be identified with $C^{m-1,1}(\overline{\Omega})$ through a continuous bijection with continuous inverse. As a consequence, Rellich's Embedding Theorem follows, in this case, from the Arzela-Ascoli Theorem. Here

$$\begin{aligned} C^{k,1}(\overline{\Omega}) =\{ & f : \overline{\Omega} \to \mathbb{K} : f \text{ is } k \text{ times continuously differentiable and the derivatives } \partial^\alpha f\} \\ & \text{with } |\alpha| = k \text{ are Lipschitz continuous}\} \end{aligned}$$

equipped with the norm

$$\|f\|_{C^{k,1}} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{C^0} + \sum_{|\alpha|=k} \text{Lip }(\partial^\alpha f)$$

(where $\|f\|_{C^0} = \sup_{x\in\overline{\Omega}} |f(x)|$) defines the Banach space of k-times differentiable functions with Lipschitz continuous derivatives. Here Lip (f) denotes the Lipschitz constant of the Lipschitz function f (i.e. the smallest constant $K>0$ with $|f(x)-f(y)| \leq K|x-y|$ for all $x,y \in \overline{\Omega}$).

Theorem 7.1.4. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Then the embedding*

$$Id\ : C^{k,1}(\overline{\Omega}) \to H^{k+1,\infty}(\Omega)$$

is well-defined, continuous and a bijection with continuous inverse, in the following sense: for every $u \in H^{k+1,\infty}(\Omega)$ there exists exactly one $\widetilde{u} \in C^{k,1}(\overline{\Omega})$ with $\widetilde{u} = u$ almost everywhere in Ω.

To show this theorem, we will make use of the following lemma.

Lemma 7.1.5. *Let Ω be open, bounded, connected, with Lipschitz boundary. For all $x_0, x_1 \in \Omega$ there exists a curve $\gamma \in C^\infty([0,1],\Omega)$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$, so that*

$$L(\gamma) = \int_0^1 |\gamma'(t)| \leq \sup_{0\leq t\leq 1} |\gamma'(t)| \leq C_\Omega\, |x_1 - x_0|$$

for a constant C_Ω, depending only on Ω.

Proof. It is enough to construct $\gamma \in C^{0,1}([0,1],\Omega)$ with Lipschitz constant Lip $(\gamma) \leq C|x_1 - x_0|$ and with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. In fact, with such a γ, we can define $\gamma(t) := x_0$, for all $t < 0$, $\gamma(t) := x_1$, for all $t > 1$, and $\gamma_\varepsilon := \varphi_\varepsilon * \gamma$ with a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon > 0}$. Then, $\gamma_\varepsilon \in C^\infty(\mathbb{R},\Omega)$,

$$\|\gamma_\varepsilon'\|_\infty \leq \text{Lip }(\gamma_\varepsilon) \leq \text{Lip }(\gamma) \leq C|x_1 - x_0|,$$

$\gamma_\varepsilon(-\varepsilon) = x_0$ and $\gamma_\varepsilon(1+\varepsilon) = x_1$. Using an affine linear transformation of $[-\varepsilon, 1+\varepsilon]$ into $[0,1]$, we obtain a curve with the desire properties.

To construct the curve $\gamma \in C^{0,1}([0,1],\Omega)$ with the properties listed at the beginning of the proof, we consider a covering $U_1, \dots, U_\ell$ of $\partial\Omega$ defined in Def. 7.1.2. We choose, for every $j = 1, \dots, \ell$, $z_j \in U_j \cap \Omega$. Moreover, we choose an open set $D \subset \Omega$ with $\overline{D} \subset \Omega$ and such that $z_1, \dots z_\ell \in D$ and $\overline{\Omega} \subset \overline{D} \cup \bigcup_{j=1}^{\ell} U_j$. We cover $\overline{D}$ through finitely many balls $U_j = B_\rho(z_j) \subset \Omega$, with $j = \ell+1, \dots, k$ for appropriate $z_{\ell+1}, \dots, z_k \in \overline{D}$. Between two arbitrary z_j, z_m we find a Lipschitz-curve γ_{jm} with Lipschitz constant λ_{jm}. We denote $\lambda = \max_{j,m} \lambda_{jm}$ (there are only finitely pairs j, m). The constant λ depends only on Ω. We consider three cases:

- $x_0, x_1 \in U_j$ for a $j > \ell$. Then U_j is a ball and we can define $\gamma(t) = tx_0 + (1-t)x_1$. This curve has the desired properties, since $|\gamma'(t)| = |x_1 - x_0|$.

- $x_0, x_1 \in U_j$ for a $j \leq \ell$. We set

  $$\gamma(t) = \tau\left((1-t)\tau^{-1}(x_0) + t\tau^{-1}(x_1)\right)$$

  where $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

  $$\tau(z) = \sum_{i=1}^{n-1} z_i e_i^j + (z_n + g_j(\hat{z}_n))e_n^j\,.$$

  Notice that $\tau^{-1}(U_j) = B_{r_j}(y_j) \times (-h_j, h_j)$ is convex. Then, we have

  $$\text{Lip }(\gamma) \leq \text{Lip }(\tau)|\tau^{-1}(x_1) - \tau^{-1}(x_0)| \leq \text{Lip }(\tau)\text{Lip }(\tau^{-1})|x_1 - x_0| \leq C_j|x_1 - x_0|$$

- $x_0 \in U_j$ and $x_1 \in U_m \backslash U_j$, for $m \neq j$. Then $|x_1 - x_0| > c$, for a constant $c > 0$ depending only on Ω (and on the choice of the sets $U_1, \dots, U_k$). We can connect the points x_0 und x_1 through a curve γ, consisting of three parts: the first part connects $\gamma(0) = x_0$ with $\gamma(1/3) = z_j$, the second part $\gamma(1/3) = z_j$ with $\gamma(2/3) = z_m$ and the third part $\gamma(2/3) = z_m$ with $\gamma(1) = x_m$. The Lipschitz constant of this curve is bounded, up to a universal constant, by the sum of the Lipschitz constants of the three curves. The Lipschitz constant of the first and the last parts are bounded (like in the first two cases considered above) by the distances $|x_j - z_j|$ and $|x_m - z_m|$, hence by diam (U_j) and diam (U_m). The Lipschitz constant of the second part is bounded by the constant λ

defined above. Hence, the Lipschitz constant of the curve we constructed can be bounded by a constant $\widetilde{C}_\Omega$ depending only on Ω. Therefore

$$\text{Lip }(\gamma) \leq \widetilde{C}_\Omega \leq \frac{\widetilde{C}_\Omega}{c}|x_1 - x_0|$$

□

We can now show Theorem 7.1.4.

Proof of Theorem 7.1.4. Let first $k = 0$ and $u \in C^{0,1}(\overline{\Omega})$. Then, for arbitrary $\xi \in C_0^\infty(\Omega)$

$$\int_\Omega u(x)\,\frac{\xi(x+he_i) - \xi(x)}{h}\,dx \to \int_\Omega u(x)\partial_i\xi(x)dx$$

as $h \to 0$. Since, on the other hand,

$$\left|\int_\Omega u(x)\,\frac{\xi(x+he_i) - \xi(x)}{h}\,dx\right| = \left|\int_\Omega \xi(x)\,\frac{u(x-he_i) - u(x)}{h}\,dx\right| \leq \text{Lip }(u)\,\int_\Omega |\xi(x)|dx$$

we conclude that

$$\left|\int_\Omega u(x)\partial_i\xi(x)dx\right| \leq \text{Lip }(u)\,\|\xi\|_1$$

for all $\xi \in C_0^\infty(\Omega)$. We define the linear functional

$$F(\xi) = \int_\Omega u(x)\partial_i\xi(x)dx$$

for all $\xi \in C_0^\infty(\Omega)$. Since $|F(\xi)| \leq \text{Lip }(u)\|\xi\|_1$ and since $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, we can extend F as a continuous linear functional on $L^1(\Omega)$, with $\|F\| \leq \text{Lip }(u)$. Since $(L^1(\Omega))^* = L^\infty(\Omega)$, there exists $g \in L^\infty(\Omega)$ with $\|g\|_\infty = \|F\| \leq \text{Lip }(u)$ and with

$$\int_\Omega u(x)\partial_i\xi(x)dx = \int_\Omega g(x)\xi(x)dx$$

This means that $u \in H^{1,\infty}(\Omega)$, with $\|\partial_i u\|_\infty \leq \text{Lip }(u) \leq \|u\|_{C^{0,1}}$ for all $i = 1,\dots,n$. Since also $\|u\|_\infty \leq \|u\|_{C^{1,0}}$, it follows that the embedding is well-defined and continuous. For $k > 0$ we can apply the same argument to the functions $\partial^\alpha u \in C^{0,1}(\overline{\Omega})$ with $|\alpha| = k$ (here $\partial^\alpha u$ are, for $|\alpha| \leq k$, the classical derivatives of u, which coincide with the weak derivatives); this shows that $\partial^\alpha u \in H^{1,\infty}(\Omega)$, for all $|\alpha| = k$ and therefore, that $u \in H^{k+1,\infty}(\overline{\Omega})$.

We show the surjectivity of the embedding and the continuity of the inverse. Consider first $k = 0$ and $u \in H^{1,\infty}(\Omega)$. Let $u_\varepsilon = \varphi_\varepsilon * u$ for a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. Now let $x_0, x_1 \in \overline{\Omega}$ and $\gamma \in C^\infty([0,1],\Omega)$ be a curve as in Lemma 7.1.5. Then

$$u_\varepsilon(x_0) - u_\varepsilon(x_1) = \int_0^1 dt\,\frac{d}{dt}u_\varepsilon(\gamma(t)) = \int_0^1 dt\,\nabla u_\varepsilon(\gamma(t))\,\cdot\gamma'(t)$$

and thus

$$|u_\varepsilon(x_0) - u_\varepsilon(x_1)| \leq \int_0^1 dt\, |\nabla u_\varepsilon(\gamma(t))|\, |\gamma'(t)| \leq \|\nabla u_\varepsilon\|_\infty L(\gamma)$$

We obtain

$$|u_\varepsilon(x_0) - u_\varepsilon(x_1)| \leq C_\Omega \, \|\nabla u\|_\infty \, |x_1 - x_0|$$

Since $u_\varepsilon \to u$ in $L^p(\Omega)$ for every $p < \infty$, there is a subsequence such that $u_\varepsilon(x) \to u(x)$ almost everywhere. We find therefore that for almost all $x_0, x_1 \in \Omega$

$$\frac{|u(x_0) - u(x_1)|}{|x_0 - x_1|} \leq C_\Omega \, \|\nabla u\|_\infty$$

Modifying u on a set with measure zero we conclude that $u \in C^{0,1}(\Omega)$, with

$$\text{Lip } (u) \leq C_\Omega \|\nabla u\|_\infty$$

For $k > 0$, take $u \in H^{k+1,\infty}$. With the same arguments as above we can show that $v_\alpha := \partial^\alpha u \in C^{0,1}(\overline{\Omega})$ for all $|\alpha| \leq k$, with Lipschitz constant bounded by $\|u\|_{H^{k+1,\infty}}$. For $|\alpha| = k$ this is already enough. For $|\alpha| < k$, we still have to prove that $\partial^\alpha u$ is $(k - |\alpha|)$-times differentiable (in the classical sense). To this end, we notice that, for $|\alpha| = k - 1$, the weak derivatives of v_α are given by $\partial_i v_\alpha = v_{\alpha + e_i} \in C^0(\overline{\Omega})$. Consider now the sequence $\varphi_\varepsilon * v_\alpha$. We have $\varphi_\varepsilon * v_\alpha \to v_\alpha$ uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$. Moreover, the classical derivatives

$$\partial_i(\varphi_\varepsilon * v_\alpha) = (\partial_i \varphi_\varepsilon) * v_\alpha = \varphi_\varepsilon * \partial_i v_\alpha = \varphi_\varepsilon * v_{\alpha + e_i} \to v_{\alpha + e_i}$$

also uniformly. This implies that $v_\alpha \in C^1(\overline{\Omega})$. In fact,

$$\varphi_\varepsilon * v_\alpha(x_1) - \varphi_\varepsilon * v_\alpha(x_0) = \int_0^1 dt\, (\nabla \varphi_\varepsilon * v_s)(x_t) \cdot (x_1 - x_0)$$

with $x_t = tx_0 + (1-t)x_1$. Hence

$$\begin{aligned} |\, \varphi_\varepsilon * v_\alpha(x_1) - & \varphi_\varepsilon * v_\alpha(x_0) - \sum_{i=1}^n (x_1 - x_0)_i \, \partial_i(\varphi_\varepsilon * v_\alpha)(x_0)| \\ & \leq \int_0^1 dt \;\; |\nabla \varphi_\varepsilon * v_\alpha(x_t) - \nabla \varphi_\varepsilon * v_\alpha(x)| \, |x_1 - x_0| \\ & \leq \left(2\|\nabla \varphi_\varepsilon * v_\alpha - v_{\alpha+e_i}\|_{C^0} + \sup_{0 \leq t \leq 1} |v_{\alpha+e_j}(x_t) - v_{\alpha+e_j}(x_0)| \right) |x_0 - x_1| \end{aligned}$$

and thus, if we let $\varepsilon \to 0$,

$$\left| v_\alpha(x_1) - v_\alpha(x_0) - \sum_{i=1}^n (x_1 - x_0)_i \, v_{\alpha+e_i}(x_0) \right| \leq \sup_{0 \leq t \leq 1} |v_{\alpha+e_i}(x_t) - v_\alpha(x)| \, \cdot |x_0 - x_1|$$

Since $v_{\alpha+e_i}$ is continuous, the r.h.s. is $o(|x_0 - x_1|)$. This implies that v_α is differentiable (in the classical sense) and that $v_\alpha \in C^1(\overline{\Omega})$, for all $|\alpha| \leq k-1$. Iteratively, we can show that $v_\alpha \in C^{k-|\alpha|}(\overline{\Omega})$ for all $|\alpha| \leq k$. □

Corollary 7.1.6 (Rellich's Embedding Theorem for $H^{m,\infty}(\Omega)$)**.** *Let Ω be open, bounded, with Lipschitz boundary. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^{m,\infty}(\Omega)$ and $u \in H^{m-1,\infty}(\Omega)$ with $u_k \to u$ weakly in $H^{m-1,\infty}(\Omega)$, as $k \to \infty$. Then $u_k \to u$ strongly in $H^{m-1,\infty}(\Omega)$.*

Proof. Let $m = 1$. From Theorem 7.1.4, the sequence $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $C^{0,1}(\overline{\Omega})$. This implies obviously that (u_k) is a bounded sequence in $C^0(\overline{\Omega})$. Moreover,

$$|u_k(x) - u_k(y)| \leq \text{Lip }(u_k)|x-y| \leq C|x-y|$$

for all $x, y \in \overline{\Omega}$, because $\text{Lip }(u_k) \leq \|u_k\|_{C^{0,1}} \leq C$ for all $k \in \mathbb{N}$ (because the sequence is bounded in $C^{0,1}(\overline{\Omega})$). Hence, the sequence (u_k) is equicontinuous on $\overline{\Omega}$. The Theorem of Arzela-Ascoli (i.e. Theorem 3.2.2) implies that every subsequence of u_k contains a convergent subsequence in $C^0(\overline{\Omega})$ and therefore also in $L^\infty(\Omega)$. Since $u_k \rightharpoonup u$ weakly in $L^\infty(\Omega)$, every accumulation point of u_k must coincide with u. A compact sequence (i.e. a sequence with the property that every subsequence contains a convergent subsubsequence) with at most one accumulation point is also convergent (proof: exercise). □

7.2 Boundary Values Sobolev Functions

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Functions in $L^p(\Omega)$ are defined only up to a subset with measure zero. In general, it does not make sense, in general, to restrict a function $u \in L^p(\Omega)$ to the boundary $\partial\Omega$. In this section, we will see that, if Ω has Lipschitz boundary, it is possible to define the boundary value of Sobolev functions in $H^{m,p}(\Omega)$ (with $m \geq 1$). This observation is very important in the study of (weak) boundary value problems that play an important role in physics and other sciences.

First of all, we have to define integration on the boundary of a domain with Lipschitz boundary.

Definition 7.2.1. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with Lipschitz boundary. We say, that a function $f : \partial\Omega \to \mathbb{K}$ is measurable/integrable, if, for all $j = 1, \ldots, \ell$, the function*

$$y \to (\eta_j f)\left(\sum_{i=1}^{n-1} y_i e_i^j + g_j(y) e_n^j\right)$$

defined on the ball $B_{r_j}^{\mathbb{R}^{n-1}}(y_j)$ is measurable/integrable with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$ (Def. 7.1.2 defines the basis $e_1^j, \ldots e_n^j$, the radius $r_j > 0$, the Lipschitz functions g_j, and *the partition η_j). The integral of f on $\partial\Omega$ is then defined by*

$$\int_{\partial\Omega} f d^{n-1}x := \sum_{j=1}^{\ell} \int_{\partial\Omega} (\eta_j f)\, d^{n-1}x$$

and, if $\operatorname{supp} f \subset U_j$,

$$\int_{\partial\Omega} f d^{n-1}x := \int_{\mathbb{R}^{n-1}} f\left(\sum_{i=1}^{n-1} y_i e_i^j + g_j(y) e_n^j\right) \sqrt{1 + |\nabla g_j(y)|^2}\, dy$$

Here ∇g_j is the weak gradient of the Lipschitz function g_j. Notice, that, since $g_j \in C^{0,1}(\mathbb{R}^{n-1})$, the function g_j is locally in $H^{1,\infty}$ by Theorem 7.1.4 (in particular, this implies that we do not need to take into account the factor $\sqrt{1+|\nabla g_j|^2}$ when we check whether the function is integrable).

This definition defines an extension of integration on differentiable manifolds. One can check that this definition is independent of the choice of the local coordinates on $\partial\Omega$ and of the choice of the partition $\eta_1, \dots, \eta_\ell$. To prove this fact, one can approximate g (and also f) through differentiable functions and one can use the normal definition of integrals on surfaces. We skip the details (which can be found in the book by Alt).

Definition 7.2.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For $1 \le p \le \infty$, we define*

$$L^p(\partial\Omega) = \{f : \partial\Omega \to \mathbb{K} : f \text{ (Borel) measurable and} \|f\|_{L^p(\partial\Omega)} < \infty\}$$

where

$$\|f\|_{L^p(\partial\Omega)} = \int_{\partial\Omega} |f|^p d^{n-1}x$$

for $1 \le p < \infty$, and

$$\|f\|_{L^\infty(\partial\Omega)} = ess \sup_{\partial\Omega} |f|$$

For a Borel set $E \subset \partial\Omega$ we define the measure

$$\mu(E) = \int_{\partial\Omega} \chi_E d^{n-1}x$$

Then $L^p(\partial\Omega)$ is exactly the L^p space, defined on the measure space $(\partial\Omega, \mathcal{B}(\partial\Omega), \mu)$. In particular, it follows that $L^p(\partial\Omega)$ is a Banach space and, that $\{f|_{\partial\Omega} : f \in C^\infty(\mathbb{R}^n)\}$ is dense in $L^p(\partial\Omega)$.

Theorem 7.2.3. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary and $1 \leq p \leq \infty$. Then there exists exactly one continuous linear map $S : H^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the trace operator, so that $Su = u|_{\partial\Omega}$ for all $u \in H^{1,p}(\Omega) \cap C^0(\overline{\Omega})$.*

Remark: Su is called the (weak) boundary value of u on $\partial\Omega$.

Proof. For $p = \infty$ the space $H^{1,\infty}(\Omega)$ is embedded in $C^{0,1}(\overline{\Omega})$, from Theorem 7.1.4. In this case, the theorem is trivial. Now, let $1 \leq p < \infty$ and $u \in H^{1,p}(\Omega)$. Let $U_1, \dots, U_\ell$ be a covering of $\partial\Omega$, as given in Definition 7.1.2, and let $\eta_1, \dots, \eta_\ell$ be a corresponding partition of the identity (also given in Definition 7.1.2). Then $v := \eta^j u \in H^{1,p}(\Omega_j)$ (the domains Ω_j are defined in Def. 7.1.2) and, for a $\delta > 0$, we have

$$v(x) = 0 \qquad \text{for } |\hat{x}_n^j - y_j| \geq r_j - \delta \qquad \text{and for } x_n^j - g_j(\hat{x}_n^j) \geq h_j - \delta$$

For $0 < s < h_j$, we define $v_s : \mathbb{R}^{n-1} \to \mathbb{R}$ through

$$v_s(y) = v(y, g_j(y) + s), \qquad \text{wobei } (y, g_j(y) + s) = \sum_{i=1}^{n-1} y_i e_i^j + (g_j(y) + s) e_n^j$$

By Fubini, the functions v_s are measurable for almost all s (because the map $(y, h) \to (y, g_j(y) + h)$ from $\mathbb{R}^n$ into $\mathbb{R}^n$ is Lipschitz and thus it maps measurable functions in measurable functions). Moreover, we have $v_s = 0$ for $s \geq h_j - \delta$. By approximation of v through functions $w_k \in H^{1,p}(\Omega_j) \cap C^\infty(\Omega_j)$, we can show that, for almost all $s_1, s_2 > 0$ and for almost all $y \in \mathbb{R}^{n-1}$,

$$v_{s_2}(y) - v_{s_1}(y) = v(y, g_j(y) + s_2) - v(y, g_j(y) + s_1) = \int_{g_j(y)+s_1}^{g_j(y)+s_2} \partial_{e_n^j} v(y, h) dh$$

With $D := B_{r_j}(y_j)$ and with Hölder's inequality, we find (for $s_1 < s_2$)

$$\begin{aligned} \int_D |v_{s_2}(y) - v_{s_1}(y)|^p d^{n-1}y &\leq (s_2 - s_1)^{p-1} \int_D d^{n-1}y \int_{g_j(y)+s_1}^{g_j(y)+s_2} dh \, |\partial_{e_n^j} v(y, h)|^p \\ &\leq (s_2 - s_1)^{p-1} \int_{x \in \Omega_j : s_1 < x_n^j - g_j(\hat{x}_n^j) < s_2} |\nabla v|^p d^n x \end{aligned} \tag{7.8}$$

In other words,

$$\|v_{s_2} - v_{s_1}\|_{L^p(D)} \leq |s_1 - s_2|^{1-1/p} \|\nabla v\|_{L^p(\{x \in \Omega_j : s_1 < x_n^j - g_j(\hat{x}_n^j) < s_2\})}$$

Since the norm on the r.h.s. converges to 0, as $s_1, s_2 \to 0$, the functions v_s define a Cauchy sequence in $L^p(\mathbb{R}^{n-1})$, as $s \to 0$. Hence, we can find $v_0 \in L^p(\mathbb{R}^{n-1})$ with $v_s \to v_0$ as $s \to 0$ in

$L^p(\mathbb{R}^{n-1})$. We define now $(S_jv)(y, g_j(y)) \equiv (S_jv)(\sum_{i=1}^{n-1} y_i e_i^j + g_j(y)e_j^n) := v_0(y)$. Then, from Definition 7.2.1, $S_jv \in L^p(\partial\Omega)$ with

$$\|S_jv\|_{L^p(\partial\Omega} \leq C_j \|v_0\|_{L^p(D)}$$

for a constant C_j, depending on g_j. If we choose $h_j - \delta < s_0 < h_j$, we obtain $v_{s_0} = 0$. Therefore

$$\begin{aligned}\|S_jv\|_{L^p(\partial\Omega)} &\leq C_j\|v_0\|_{L^p(D)} = C_j\|v_0 - v_{s_0}\|_{L^p(D)} = C_j \lim_{s\to 0} \|v_s - v_{s_0}\|_{L^p(D)} \\ &\leq C_j \lim_{s\to 0} |s - s_0|^{1-1/p} \|\nabla v\|_{L^p(\{x\in\Omega_j : s< x_n^j - g_j(\hat{x}_n^j) < s_0\})} \\ &\leq C_j |s_0|^{1-1/p} \|\nabla v\|_{L^p(\Omega_j)} \leq C\|u\|_{H^{1,p}(\Omega_j)}\end{aligned}$$

for a constant C depending on the partition functions η_j. For arbitrary $u \in H^{1,p}(\Omega)$ we can then define $Su := \sum_{j=1}^k S_j(\eta_j u)$. Clearly $Su = u|_{\partial\Omega}$, if u on $\overline{\Omega}$ is continuous. This shows the existence of S, as a continuous linear map. The uniqueness follows from Lemma 7.2.4 below, where we show the fact that $H^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ is dense in $H^{1,p}(\Omega)$. $\square$

Lemma 7.2.4. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary and $1 \leq p < \infty$, $m \geq 1$. Then*

$$\{u|_\Omega : u \in C_0^\infty(\mathbb{R}^n)\} \text{ is dense in } H^{m,p}(\Omega)$$

In particular, it follows that $H^{m,p}(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $H^{m,p}(\Omega)$, since $u|_{\overline{\Omega}} \in H^{m,p}(\Omega) \cap C^\infty(\overline{\Omega})$ for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $u \in H^{m,p}(\Omega)$. We split $u = \sum_{j=0}^\ell \eta_j u$ (with the partition η_j from Def. 7.1.2). For $j = 0$, we choose a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. Then $\varphi_\varepsilon * (\eta_0 u) \in C_0^\infty(\Omega)$ for $\varepsilon > 0$ small enough and $\varphi_\varepsilon * (\eta_0 u) \to \eta_0 u$ as $\varepsilon \to 0$ in $H^{m,p}(\Omega)$. For $j \geq 1$, let Ω_j and $e_1^j, \dots, e_n^j$ be as in Def. 7.1.2. For $\delta > 0$, we set

$$v_\delta(x) := (\eta_j u)(x + \delta e_n^j) \qquad \text{for } x \in \Omega_\delta^j,$$

$\Omega_\delta^j := \{x \in \mathbb{R}^n : |\hat{x}_n^j - y_j| < r_j \text{ and } -\delta < x_n^j - g_j(\hat{x}_n^j) < h_j\}$

Then $v_{\delta,\varepsilon} := \varphi_\varepsilon * (\chi_{\Omega_\delta^j} v_\delta) \in C_0^\infty(\mathbb{R}^n)$ and the restriction to Ω, given by $v_{\delta,\varepsilon}|_\Omega = \varphi_\varepsilon * v_\delta$ for $\varepsilon > 0$ small enough (so small that $x \in \Omega_j$ and $\varphi_\varepsilon(y - x) \neq 0$ imply that $y \in \Omega_\delta^j$; this condition can be reached if $(1 + \text{Lip }(g_j))\varepsilon < \delta$), is in $H^{m,p}(\Omega)$ with

$$v_{\varepsilon,\delta} \to \eta_j u \text{in } H^{m,p}(\Omega)$$

if first $\varepsilon \to 0$ and then $\delta \to 0$. In fact, if $\varepsilon \to 0$, $v_{\varepsilon,\delta} \to v_\delta$ on Ω, because the function is defined away from the boundary $\partial\Omega_\delta^j$; as then $\delta \to 0$, $v_\delta \to v$ on Ω w.r.t. the $H^{m,p}(\Omega)$-norm (the weak derivatives of v_δ are simply given by the shift of the weak derivatives of v; the problem reduces to establishing the convergence of the shifted function v_δ towards v in L^p; this is left as an exercise). This proves that $\eta_j u$ can be approximated in the $H^{m,p}(\Omega)$-norm through restriction of functions in $C_0^\infty(\mathbb{R}^n)$; the same result holds for u as well. $\square$

With the trace operator $S : H^{1,p}(\Omega) \to L^p(\partial\Omega)$ we can define the boundary values of a Sobolev function (on domains with Lipschitz boundary). It is not difficult to guess that $Su = 0$ if and only if $u \in H_0^{1,p}(\Omega)$, i.e.if and only if u can be approximated by a sequence in $C_0^\infty(\Omega)$, vanishing close to the boundary. This is the content of Theorem 7.2.6 below. To prove this theorem, we need first the following lemma.

Lemma 7.2.5. *Let $1 \le p < \infty$, $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz continuous,*

$$\Omega_\pm = \{(y, h) \in \mathbb{R}^n : \pm(h - g(y)) > 0\}$$

and $u : \mathbb{R}^n \to \mathbb{R}$ with $u|_{\Omega_+} \in H^{1,p}(\Omega_+)$ and $u|_{\Omega_-} \in H^{1,p}(\Omega_-)$. Moreover, let $S_\pm$ be trace operators associated with the domains $\Omega_\pm$. Then

$$u \in H^{1,p}(\mathbb{R}^n) \Leftrightarrow S_+(u) = S_-(u)$$

Remark: the trace operator S is defined in Theorem 7.2.3 only for bounded domains with Lipschitz boundary (because only for such domains we defined in Def. 7.1.2 the necessary covering and partition). However, it is clear that exactly the same construction works well also for the domains $\Omega_\pm$.

Proof. "$\Rightarrow$": Let $u \in H^{1,p}(\mathbb{R}^n)$ and set $u_s(y) = u(y, g(y) + s)$ for $s \in \mathbb{R}$. Then we have (see (7.8))

$$\int_{\mathbb{R}^{n-1}} |u_\varepsilon(y) - u_{-\varepsilon}(y)|^p dy \le (2\varepsilon)^{p-1} \int_{\mathbb{R}^{n-1}} \int_{g(y)-\varepsilon}^{g(y)+\varepsilon} |\nabla u(y, h)|^p dy dh \to 0$$

as $\varepsilon \to 0$. This proves that $S_+(u) = S_-(u)$, by the definition of the trace operators $S_\pm$.

"$\Leftarrow$": We define the external normal vector ν associated with the domain Ω_+ in the point $(y, g(y)) \in \partial\Omega_+$ through

$$\nu_+(y, g(y)) := (1 + |\nabla g(y)|^2)^{-1/2} \sum_{i=1}^{n-1} (\partial_i g(y) e_i - e_n)$$

Analogously, we can define the external normal vector ν_- associated to Ω_-: clearly, we have $\nu_+ = -\nu_-$. We use now the weak version of Gauss Theorem: let $u \in H^{1,1}(\Omega_+)$, then, for all $i = 1, \dots, n$,

$$\int_{\Omega_+} \partial_i u \, d^n x = \int_{\partial\Omega_+} S_+(u) \nu_{+,i} d^{n-1} x \tag{7.9}$$

where $\nu_{+,i}$ is the i-th component of the external normal vector ν_+. Similarly, we have weak Gauss theorem for the domain Ω_-. To prove this identity, one can approximate g through continuous functions and one can apply the classical Theorem of Gauss; we skip the details.

Now, let $u : \mathbb{R}^n \to \mathbb{R}$, with $u|_{\Omega_+} \in H^{1,p}(\Omega_+)$ and $u|_{\Omega_-} \in H^{1,p}(\Omega_-)$, with $S_+(u) = S_-(u)$. Then $S_+(u)\nu_+ + S_-(u)\nu_- = 0$ on the graph of g. Hence, for arbitrary $\xi \in C_0^\infty(\mathbb{R}^n)$, since $S_\pm(\xi u) = \xi S_\pm(u)$,

$$\begin{aligned}
0 &= \int_{\text{graph(g)}} \xi \left(S_+(u)\nu_+ + S_-(u)\nu_-\right) d^{n-1}x \\
&= \int_{\partial\Omega_+} \xi S_+(u)\nu_+ d^{n-1}x + \int_{\partial\Omega_-} \xi S_-(u)\nu_- d^{n-1}x \\
&= \int_{\Omega_+} \nabla(u\xi) d^n x + \int_{\Omega_-} \nabla(u\xi) d^n x \\
&= \int_{\Omega_+} \left(\nabla\xi u + \xi\nabla u\right) d^n x + \int_{\Omega_-} \left(\nabla\xi u + \xi\nabla u\right) d^n x \\
&= \int_{\mathbb{R}^n} \nabla\xi u + \int_{\mathbb{R}^n} \xi g
\end{aligned}$$

where g is defined on Ω_+ as the weak gradient of $u \in H^{1,p}(\Omega_+)$ and on Ω_- as the weak gradient of $u \in H^{1,1}(\Omega_-)$. Since clearly $g \in L^p(\mathbb{R}^n)$, we proved that $u \in H^{1,p}(\mathbb{R}^n)$, with $\nabla u = g$. □

Theorem 7.2.6. *Let Ω be open, bounded, with Lipschitz boundary and $1 \leq p < \infty$. Then, with the trace operator S defined as in Theorem 7.2.3,*

$$H_0^{1,p}(\Omega) = \{u \in H^{1,p}(\Omega) : S(u) = 0\}$$

Proof. Let $u \in H_0^{1,p}(\Omega)$. Then we can approximate u in the $H^{1,p}$ norm through a sequence $u_i \in C_0^\infty(\Omega)$. Clearly $S(u_i) = 0$ for all i. Since S is continuous on $H^{1,p}$, it follows that $S(u) = \lim_{i\to\infty} S(u_i) = 0$.

Let now $u \in H^{1,p}(\Omega)$ with $Su = 0$. Let $\eta_0, \dots, \eta_\ell$ be a partition of the identity as in Def. 7.1.2. Then $S(\eta_j u) = \eta_j S(u) = 0$ for $j = 1, \dots, \ell$. We define moreover $v_j(x) := (\eta_j u)(x)$ for $x \in \Omega_j$ and $v_j(x) = 0$ otherwise (see Def. 7.1.2 for the definition of Ω_j). By Lemma 7.2.5, we have $v_j \in H^{1,p}(\mathbb{R}^n)$. Hence, for $\delta > 0$, also $v_{\delta,j}(x) = v_j(x - \delta e_n^j)$ is in $H^{1,p}(\mathbb{R}^n)$ and $v_{\delta,j} \to v_j$ in $H^{1,p}(\mathbb{R}^n)$, as $\delta \to 0$. Therefore, also

$$u_\delta = \eta_0 u + \sum_{j=1}^{\ell} v_{\delta,j} \to u$$

converges in $H^{1,p}(\Omega)$, as $\delta \to 0$. However, since u_δ has compact support within Ω, it can be approximated by means of convolution with functions in $C_0^\infty(\Omega)$ (in the $H^{1,p}$ norm). □

We show next a useful theorem concerning the extension of Sobolev function to domains with Lipschitz boundary.

Theorem 7.2.7 (Extension Theorem). *Let $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^n$ open, bounded, with Lipschitz boundary and let $\delta > 0$. Then there exists a continuous linear extension operator*

$$E : H^{1,p}(\Omega) \to H_0^{1,p}(B_\delta(\Omega))$$

with $(Eu)|_\Omega = u$ for all $u \in H^{1,p}(\Omega)$.

Proof. Using the partition of the identity $\eta_0, \dots, \eta_\ell$ in Def. 7.1.2, it is enough to solve the problem locally, close to the boundary. Let $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz continuous function, $\Omega_+ = \{(y,h) \in \mathbb{R}^n : h > g(y)\}$ and $\Omega_- = \{(y,h) \in \mathbb{R}^n : h < g(y)\}$. We construct a continuous linear operator $E : H^{1,p}(\Omega_+) \to H^{1,p}(B_\delta(\Omega_+))$ with $Eu|_{\Omega_+} = u$ for all $u \in H^{1,p}(\Omega_+)$. To this end, we choose a cutoff function $\eta \in C^\infty(\mathbb{R}^n)$ with $\eta = 1$ on $B_{\delta/2}(\Omega_+)$ and $\eta = 0$ on $\mathbb{R}^n \backslash B_\delta(\Omega_+)$. Then we define $Eu := \eta\widetilde{u}$, with $\widetilde{u}(y,h) := u(y,h)$ for all $h > g(y)$, and $\widetilde{u}(y,h) = u(y, 2g(y) - h)$ for $h < g(y)$. For $p = \infty$, Theorem 7.1.4 implies that $u \in C^{0,1}(\Omega_+)$ and $\widetilde{u}$ defines a $C^{0,1}$-extension of u. For $p < \infty$, we have $\widetilde{u} \in H^{1,p}(\Omega_-)$ with $\|\widetilde{u}\|_{L^p(\Omega_-)} = \|u\|_{L^p(\Omega_+)}$ and

$$\|\nabla\widetilde{u}\|_{L^p(\Omega_-)} \le (2 + \text{Lip }(g))\|\nabla u\|_{L^p(\Omega_+)}$$

This follows from

$$\begin{aligned} (\partial_n\widetilde{u})(y,h) &= -(\partial_n u)(y, 2g(y) - h) \\ (\partial_i\widetilde{u})(y,h) &= (\partial_i u)(y, 2g(y) - h) + 2(\partial_i g)(y)(\partial_n u)(y, 2g(y) - h) \end{aligned} \tag{7.10}$$

For differentiable g, this is just the usual chain rule. For g Lipschitz continuous, (7.10) is a weak chain rule (in this case, $\partial_i g$ is the weak derivative of g), which can be proven through approximation of g with differentiable functions. Hence $Eu|_{\Omega_-} \in H^{1,p}(\Omega_-)$ with $\|Eu\|_{H^{1,p}(\Omega_-)} \le C\|u\|_{H^{1,p}(\Omega_+)}$. From definition of $\widetilde{u}$, we have $Eu|_{\Omega_+} = u \in H^{1,p}(\Omega_+)$. Furthermore $S_+(Eu) = S_-(Eu)$ (because the definition of Eu is symmetric w.r.t. the boundary, at least close to the boundary where $\eta = 1$). It follows from Lemma 7.2.5 that $Eu \in H^{1,p}(\mathbb{R}^n)$ with $\|Eu\|_{H^{1,p}(\mathbb{R}^n)} \le C\|u\|_{H^{1,p}(\Omega_+)}$. □

7.3 Sobolev Inequalities and Sobolev Embeddings

Sobolev inequalities are very important in modern analysis, in particular in the study of partial differential equations.

Theorem 7.3.1 (Sobolev). *Let $n \in \mathbb{N}$, $1 \le p < n$ and $q = np/(n-p)$. Let $u \in L^s(\mathbb{R}^n)$ for an $s \in [1, \infty)$ with weak gradient $\nabla u \in L^p(\mathbb{R}^n)$. Then $u \in L^q(\mathbb{R}^n)$, with*

$$\|u\|_q \le q\frac{n-1}{n}\|\nabla u\|_p$$

Remark: for $n = 1$ the theorem cannot be applied. In this case, however, we have the estimate $\|u\|_\infty \le \|\nabla u\|_1$ for all $u \in L^s(\mathbb{R}^n)$ for an $s \in [1; \infty)$ with $\nabla u \in L^1(\mathbb{R}^n)$.

Proof. The proof is divided into three steps.

Step 1: it is enough to show the theorem for $u \in C^\infty(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ (in this case, ∇u is the classical gradient of u).

Assuming the theorem holds true on $L^s(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, then we can approximate an arbitrary $u \in L^s(\mathbb{R}^n)$ with $\nabla u \in L^p(\mathbb{R}^n)$ through a sequence $u_\varepsilon = u * \varphi_\varepsilon \in L^s \cap C^\infty(\mathbb{R}^n)$, with φ_ε a standard Dirac sequence. Then $u_\varepsilon \to u$ in $L^s(\mathbb{R}^n)$ and

$$\nabla u_\varepsilon = u * \nabla \varphi_\varepsilon = \nabla u * \varphi_\varepsilon \to \nabla u$$

in $L^p(\mathbb{R}^n)$, as $\varepsilon \to 0$. For $\varepsilon, \delta > 0$, we have therefore

$$\|u_\varepsilon - u_\delta\|_q \leq q\frac{n-1}{n}\|\nabla u_\varepsilon - \nabla u_\delta\|_p \to 0$$

as $\varepsilon, \delta \to 0$. Hence u_ε is a Cauchy sequence in $L^q(\mathbb{R}^n)$. Therefore, there exists $\widetilde{u} \in L^q(\mathbb{R}^n)$ with $u_\varepsilon \to \widetilde{u}$ in $L^q(\mathbb{R}^n)$. Since, on the other hand, $u_\varepsilon \to u$ in $L^s(\mathbb{R}^n)$, we can find a subsequence ε_j with $u_{\varepsilon_j}(x) \to u(x)$ and $u_\varepsilon(x) \to \widetilde{u}(x)$ almost everywhere. This implies that $\widetilde{u}(x) = u(x)$ almost everywhere and thus that

$$\|u\|_q = \|\widetilde{u}\|_q = \lim_{\varepsilon \to 0} \|u_\varepsilon\|_q \leq q\frac{n-1}{n} \limsup_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_p = q\frac{n-1}{n}\|\nabla u\|_p \,.$$

Step 2: The theorem holds true for $p = 1$.

From Step 1, it is enough to show that

$$\|u\|_{n/(n-1)} \leq \|\nabla u\|_1$$

for all $u \in C^\infty(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ with $\nabla u \in L^1(\mathbb{R}^n)$. For $i = 1, \dots, n$, Fubini implies that the map $\xi \to u(x_1, \dots, x_{i-1}, \xi, u_{i+1}, \dots, x_n) = u(x', \xi)$ is in $L^s(\mathbb{R})$ for almost all $x' \in \mathbb{R}^{n-1}$. Here we introduced the notation $x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ and $(x', \xi) = (x_1, \dots, x_{i-1}, \xi, x_{i+1}, \dots, x_n)$. We find a sequence $\xi_k \to \infty$ with $u(x', \xi_k) \to 0$. Thus

$$u(x) = u(x', \xi_k) - \int_{x_i}^{\xi_k} \partial_i u(x', \xi) d\xi$$

and

$$|u(x)| \leq \int_{-\infty}^{\infty} d\xi |\partial_i u(x', \xi)|$$

This proves the remark after Theorem for the case $n = 1$. We obtain

$$|u(x)|^{n/(n-1)} \leq \prod_{i=1}^n \left(\int d\xi_i |\partial_i u| \right)^{1/(n-1)}$$

Integrating over x_1 gives

$$\begin{aligned}
&\int dx_1\, |u(x)|^{n/(n-1)} \\
&\leq \left(\int_{\mathbb{R}} |\partial_1 u|\right)^{1/(n-1)} \int dx_1 \prod_{i=2}^{n} \left(\int d\xi_i |\partial_i u|\right)^{1/(n-1)} \\
&\leq \left(\int_{\mathbb{R}} |\partial_1 u|\right)^{1/(n-1)} \prod_{i=2}^{n} \left(\int_{\mathbb{R}^2} d\xi_1 d\xi_i\, |\partial_i u(\xi_1, x_2, \dots, x_{i-1}, \xi_i, x_{i+1}, \dots, x_n)|\right)^{1/(n-1)}
\end{aligned}$$

Here we used the generalized Hölder inequality $\|f_1 \dots f_m\|_1 \leq \|f_1\|_{p_1} \dots \|f_m\|_{p_m}$, if $1/p_1 + \dots 1/p_m = 1$. Integrating over x_2 we find, similarly,

$$\begin{aligned}
&\int dx_1 dx_2\, |u|^{n/(n-1)} \\
&\leq \left(\int_{\mathbb{R}^2} |\partial_2 u| d\xi_1 d\xi_2\right)^{1/(n-1)} \int dx_2 \left(\int_{\mathbb{R}} |\partial_1 u| d\xi_1\right)^{1/(n-1)} \prod_{i=3}^{n} \left(\int_{\mathbb{R}^2} d\xi_1 d\xi_i |\partial_i u|\right)^{1/(n-1)} \\
&\leq \left(\int_{\mathbb{R}^2} |\partial_2 u| d\xi_1 d\xi_2\right)^{1/(n-1)} \left(\int_{\mathbb{R}} |\partial_1 u| d\xi_1 d\xi_2\right)^{1/(n-1)} \prod_{i=3}^{n} \left(\int_{\mathbb{R}^3} d\xi_1 d\xi_2 d\xi_i |\partial_i u|\right)^{1/(n-1)}
\end{aligned}$$

After n iterations, we obtain

$$\int_{\mathbb{R}^n} dx_1 \dots dx_n\, |u|^{n/(n-1)} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_i u| d\xi_1 \dots d\xi_n\right)^{1/(n-1)} \leq \left(\int |\nabla u| d\xi_1 \dots d\xi_n\right)^{n/(n-1)}$$

d.h. $\|u\|_{n/(n-1)} \leq \|\nabla u\|_1$.

Step 3: Extension to arbitrary $1 < p < n$.

Notice, that, for $p > 1$, $q(n-1)/n = p(n-1)/(n-p) \leq p > 1$. Hence, if $u \in C^1(\mathbb{R}^n)$, then also $|u|^{q(n-1)/n} \in C^1(\mathbb{R}^n)$ and

$$\nabla |u|^{q(n-1)/n} = q\frac{n-1}{n} |u|^{q(n-1)/n-1} \mathrm{Re}\, \frac{\overline{u}}{|u|} \nabla u$$

Thus we find, from Step 2,

$$\begin{aligned}
\|u\|_q^{q(n-1)/n} = \| |u|^{q(n-1)/n} \|_{n/(n-1)} \leq \|\nabla |u|^{q(n-1)/n}\|_1 &= q\frac{n-1}{n} \| |u|^{q(n-1)/n-1} |\nabla u| \|_1 \\
&\leq q\frac{n-1}{n} \| |u|^{q(n-1)/n-1} \|_{p'} \|\nabla u\|_p
\end{aligned} \tag{7.11}$$

where $p' \geq 1$ is such that $1/p + 1/p' = 1$. From $q = np/(n-p)$, it follows that $p' = qn/(qn - q - n)$. Therefore

$$\| |u|^{q(n-1)/n-1} \|_{p'} = \|u\|_q^{qn-q-n} n$$

From (7.11) with $q(n-1)/n - (qn-n-q)/n = 1$, we find

$$\|u\|_q \leq q\frac{n-1}{n}\|\nabla u\|_p \,.$$

$\square$

Remarks:

- For $u \in H^{1,p}(\mathbb{R}^n)$, the assumption of the theorem is satisfied. Hence every $u \in H^{1,p}(\mathbb{R}^n)$ is also in $L^q(\mathbb{R}^n)$ with
$$\|u\|_q \leq q\frac{(n-1)}{n}\|\nabla u\|_p$$
if $q = np/(n-p)$. This implies that every $u \in H^{1,p}(\mathbb{R}^n)$ is also in $L^q(\mathbb{R}^n)$, with
$$\|u\|_q \leq C\|u\|_{H^{1,p}}$$
for all $p \leq q \leq np/(n-p)$ and for a constant $C > 0$, depending only on n, p. This is a consequence of the bound $\|u\|_q \leq C(\|u\|_p + \|u\|_{q*})$, with $q* = np/(n-p)$, which by Hölder is valid for all $p \leq q \leq q*$ (by interpolation).

- *Extension to domains* $\Omega \subset \mathbb{R}^n$. For arbitrary $\Omega \subset \mathbb{R}^n$ open, Theorem 7.3.1 implies that every $u \in H_0^{1,p}(\Omega)$ is also in $L^q(\Omega)$ with $\|u\|_q \leq q((n-1)/n)\|\nabla u\|_p$. For functions that do not vanish close to the boundary, hence, for $u \in H^{1,p}(\Omega)$ instead of $H_0^{1,p}(\Omega)$, the extension of the Sobolev inequality is more complicated and require regularity of the boudary. Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded and with Lipschitz boundary. Let $n \in \mathbb{N}$, $1 \leq p < n$. Then, every $u \in H^{1,p}(\mathbb{R}^n)$ is also in $L^q(\Omega)$ with
$$\|u\|_q \leq C\|u\|_{H^{1,p}(\Omega)} \tag{7.12}$$
for all $1 \leq q \leq pn/(n-p)$ and for a constant $C > 0$, depending only on p, n and on Ω (here $q \geq 1$ instead of $q \geq p$ is allowed, because Ω is bounded). In fact, from Theorem 7.2.7, every $u \in H^{1,p}(\Omega)$ has an extension $Eu \in H_0^{1,p}(B_\delta(\Omega))$ (for an arbitrary $\delta > 0$). Then
$$\|u\|_{L^q(\Omega)} \leq \|Eu\|_{L^q(B_\delta(\Omega))} \leq C\|Eu\|_{H^{1,p}(B_\delta(\Omega))} \leq C\|u\|_{H^{1,p}(\Omega)}$$
from the continuity of the extension operator E.

- *Extension to higher derivatives.* Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary. Let $k \geq 0$, $m \geq k$, $n \in \mathbb{N}$, $p \geq 1$ with $kp < n$. Then every $u \in H^{m,p}(\Omega)$ is also in $H^{m-k,q}(\Omega)$, with
$$\|u\|_{H^{m-k,q}} \leq C\|u\|_{H^{m,p}}$$

for all $1 \leq q \leq np/(n-kp)$ and for a constant $C > 0$, depending only on n, p, k, m and Ω. For $k = 0$ there is nothing to show, since $q \leq p$ and the L^q norm is controlled by the L^p norm, since $|\Omega| < \infty$. For $k > 0$, we have

$$\|u\|_{H^{m-k,q}} \leq \sum_{|\alpha| \leq m-k} \|\partial^\alpha u\|_q \leq \sum_{|\alpha| \leq m-k} (\|\partial^\alpha u\|_p + \|\partial^\alpha u\|_{q*})$$

with $q* = np/(n-kp)$ (here we used again interpolation). Let $p* = np/(n-(k-1)p)$. Then $p \leq p* \leq q*$ and $q* = np*/(n-p*)$. Thus, the standard Sobolev inequality implies that

$$\|u\|_{H^{m-k,q}} \leq C \sum_{|\alpha| \leq m-(k-1)} (\|\partial^\alpha u\|_p + \|\partial^\alpha u\|_{p*})$$

for an appropriate constant $C > 0$. After k iterations, we find

$$\|u\|_{H^{m-k,q}} \leq \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p \leq \|u\|_{H^{m,p}}$$

Theorem 7.3.2 (Soboleb Embedding, $kp < n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary, $k \geq 0$, $m \geq k$ in $\mathbb{N}$, $p \geq 1$ with $kp < n$. Then the embedding $Id : H^{m,p}(\Omega) \to M^{m-k,q}(\Omega)$ is well-defined and continuous, for all $1 \leq q \leq np/(n-kp)$. Furthermore, if $1 \leq q < np/(n-kp)$ and $k \geq 1$, then the embedding is also compact. In other words, every bounded sequence in $H^{m,p}(\Omega)$ has a subsequence that converges in $H^{m-k,q}(\Omega)$.*

Proof. We already proved that the embedding is well-defined and continuous, if $1 \leq q \leq np/(n-kp)$. We show now that the embedding is also compact, if $1 \leq q < np/(n-kp)$ and $k \geq 1$. We assume first that $p > 1$. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a bounded sequence in $H^{m,p}(\Omega)$. Then $\partial^\alpha u_\ell$ are bounded sequences in $L^p(\Omega)$, for all $|\alpha| \leq m$. Since $L^p(\Omega)$ is reflexive, there is a weak convergent subsequence. In other words, there is a subsequence $\ell_j \to \infty$, as $j \to \infty$, and $u_\alpha \in L^p(\Omega)$ for all $|\alpha| \leq m$, with $\partial^\alpha u_{\ell_j} \to u_\alpha$ weakly in $L^p(\Omega)$, as $j \to \infty$. It is simple to prove that $u := u_0 \in H^{m,p}(\Omega)$ with $\partial^\alpha u = u_\alpha$ for all $|\alpha| \leq m$ and that $u_{\ell_j} \to u$ weakly in $H^{m,p}(\Omega)$. This implies of course that $u_{\ell_j} \to u$ weakly in $H^{m-1,p}(\Omega)$ and hence, from Theorem 7.1.3 (Rellich'sche Embedding Theorem), that $u_{\ell_j} \to u$ strongly in $H^{m-1,p}(\Omega)$. In other words, this implies that $\partial^\alpha u_{\ell_j} \to \partial^\alpha u$ strongly in $L^p(\Omega)$, for all $|\alpha| \leq m-1$. Now, for an appropriate

$\beta \in (0,1)$,

$$\begin{aligned}
\|u_{\ell_j} - u\|_{H^{m-k,q}} &\leq \sum_{|\alpha|\leq m-k} \|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_q \\
&\leq \sum_{|\alpha|\leq m-k} \|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_p^\beta \, \|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_{np/(n-pk)}^{1-\beta} \\
&\leq \|u_{\ell_j} - u\|_{H^{m-k,np/(n-kp)}}^{1-\beta} \sum_{|\alpha|\leq m-k} \|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_p^\beta \\
&\leq \|u_{\ell_j} - u\|_{H^{m,p}}^{1-\beta} \sum_{|\alpha|\leq m-k} \|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_p^\beta
\end{aligned}$$

The right hand side converges to 0, as $\|\partial^\alpha u_{\ell_j} - \partial^\alpha u\|_p \to 0$ for all $|\alpha| \leq m-k$, and because $\|u_{\ell_j} - u\|_{H^{m,p}} \leq C$, uniformly in ℓ_j, since, by assumption, the sequence u_ℓ is bouded in $H^{m,p}$. If $p = 1$, $(u_\ell)_{\ell\in\mathbb{N}}$ is limited in $H^{m,1}(\Omega)$. This implies that u_ℓ is bounded in $H^{m-1,\widetilde{p}}(\Omega)$, for all $1 < \widetilde{p} \leq np/(n-p)$. Since $L^{\widetilde{p}}$ is reflexive, for $\widetilde{p} > 1$, this implies that there is a subsequence ℓ_j and a $u \in H^{m-1,\widetilde{p}}$ with $u_{\ell_j} \to u$ weakly in $H^{m-1,\widetilde{p}}$. This implies also that $u_{\ell_j} \to u$ weakly in $H^{m-1,1}(\Omega)$ (because $|\Omega| < \infty$). Since u_{ℓ_j} is bounded in $H^{m,1}(\Omega)$, it follows from Rellich Embedding Theorem (Theorem 7.1.3), that $u_{\ell_j} \to u$ strongly in $H^{m-1,1}(\Omega)$. This implies as above that $u_{\ell_j} \to u$ strongly in $H^{m-k,q}$, for all $q < n/(n-k)$. □

So far, we considered the case $kp < n$. What happens if $kp \geq n$?

Theorem 7.3.3. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary, $k, m \in \mathbb{N}$ with $k \geq 0$, $m \geq k$, and $p \geq 1$.*

- *If $kp = n$, the embedding $Id : H^{m,p}(\Omega) \to H^{m-k,q}(\Omega)$ is well-defined and continuous for all $1 \leq q < \infty$.*
- *If $kp > n$, the embedding $Id : H^{m,p}(\Omega) \to H^{m-k,q}(\Omega)$ is well-defined and continuous, for all $1 \leq q \leq \infty$.*

Proof. In the case $kp = n$ we find, for a given $q < \infty$, a $p^* \geq 1$ with $kp^* < n$ and with $q = np^*/(n-kp^*)$. Then $p^* < p$ and thus, for arbitrary $u \in H^{m,p}$,

$$\|u\|_{H^{m-k,q}} \leq C\|u\|_{H^{m,p^*}} \leq C\|u\|_{H^{m,p}}$$

where the first inequality follows from Theorem 7.3.2 .

Consider the case $kp > n$. As usual, we can restrict our attention to the case $k = m = 1$. The general statement follows from iterative application of the inequality with $k = m = 1$. Since $|\Omega| < \infty$, we only need to show that

$$\|u\|_\infty \leq C\|\nabla u\|_p \tag{7.13}$$

for all $n < p < \infty$ and for all $u \in C_0^\infty(\Omega)$, for a constant C, depending only on n, p and Ω (since Ω has a Lipschitz boundary, every $u \in H^{1,p}(\Omega)$ can be continuously extended to $u \in H_0^{1,p}(B_\delta(\Omega))$).

Let $R = \text{diam}\ \Omega$ (this is the largest distance between two points in Ω). Then, for all $x_0 \in \Omega$,

$$u(x_0) = -\int_0^R \frac{d}{ds} u(x_0 + s\xi) = -\int_0^R \nabla u(x_0 + s\xi) \cdot \xi ds$$

for all $\xi \in \partial B_1(0)$. Hence

$$|u(x_0)| \leq \int_0^R ds\, |\nabla u(x_0 + s\xi)|$$

for all $\xi \in \partial B_1(0)$. Integration over ξ on the sphere gives

$$\begin{aligned}\sigma_n |u(x_0)| &\leq \int_{S^{n-1}} d\xi \int_0^R ds\, |\nabla u(x_0 + s\xi)| \\ &= \int_{B_R(x_0)} \frac{|\nabla u(x)|}{|x - x_0|^{n-1}} dx \leq \left(\int_{B_R(x_0)} \frac{1}{|x - x_0|^{p'(n-1)}} \right)^{1/p'} \|\nabla u\|_p\end{aligned}$$

where $p' = p/(p-1) < n/(n-1)$. Hence, the integral is finite and (7.13) is proven. □

Remarks:

- In the case $kp = n$, the embedding $\text{Id} : H^{m,p}(\Omega) \to H^{m-k,q}(\Omega)$ is also compact, for all $k \geq 0$, $m \geq k$, $1 \leq q < \infty$, if $k \geq 1$. As in the proof of Theorem 7.3.3, this can be shown through choice of $p^* \geq 1$ with $kp^* < n$ and $q < np^*/(n - kp^*)$ and through application of Theorem 7.3.2 with this p^*.

- In the case $kp = n$, the embedding $\text{Id} : H^{m,p}(\Omega) \to H^{m-k,q}(\Omega)$ is for the case $k \geq 0$, $m \geq k$ and $q = \infty$ only well-defined in the case $n = 1$ (for $n \geq 2$ there are counter-examples; for examples, if $n = p = 2$, $k = m = 1$, the function $u(x) = \log|\log|x||$ is not bounded on $\Omega = B_1(0) \subset \mathbb{R}^2$; it has however weak gradient in $L^2(\Omega)$; this was shown in the exercises).

- In the case $kp > n$, we have more information. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary. Let $m, k \in \mathbb{N}$, with $k \geq 0$, $m \geq k$. Let $p \geq 1$ and $0 < \alpha < 1$ with $n/p \leq k - \alpha$. Then the embedding $\text{Id} : H^{m,p}(\Omega) \to C^{m-k,\alpha}(\Omega)$ is well-defined and continuous. In other words, if $kp > n$, then functions in $H^{m,p}(\Omega)$ are not only bounded, but instead also Hölder continuous, with exponent $\alpha \leq k - n/p$. If $k \geq 1$ and $n/p < k - \alpha$, the embedding is even compact.

7.4 Application: Ground State of Quantum Systems

A particle in the n-dimensional space can be described in quantum mechanics through a wave function $\psi \in L^2(\mathbb{R}^n)$. The absolute value square $|\psi(x)|^2$ is interpreted as probability density to find the particle close to the point $x \in \mathbb{R}^n$. For this reason, wave functions are always normalized, so that $\|\psi\|_2 = 1$. The energy of the particle is given by the functional

$$\varepsilon(\psi) = \int dx\, |\nabla\psi(x)|^2 + \int dx\, V(x)|\psi(x)|^2$$

where $V \in L^s_{\text{loc}}(\mathbb{R}^n)$, for a $1 \leq s \leq \infty$, is an external potential acting on the particle. An important question is whether ε attains a minimum on the unit sphere $\{\psi \in L^2(\mathbb{R}^n) : \|\psi\|_2 = 1\}$ (the minimum of ε is called the ground state energy, the corresponding ψ, if it exists, is known as the ground state).

To answer this question, we first have to understand, under which conditions on V is ε bounded below. It is simple to check that not every V leads to an energy ε bounded below.

Consider, for example, for $n = 3$, the potential $V(x) = -|x|^{-5/2}$. Then $V \in L^s_{\text{loc}}(\mathbb{R}^3)$, for all $s < 6/5$. For every $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\|\psi\|_2 = 1$ and for $\lambda > 0$ we set

$$\psi_\lambda(x) = \lambda^{-3/2}\psi(x/\lambda)$$

Then $\|\psi_\lambda\|_2 = 1$ for all $\lambda > 0$ and

$$\begin{aligned}
\varepsilon(\psi_\lambda) &= \int |\nabla\psi_\lambda(x)|^2 dx - \int |x|^{-5/2}|\psi_\lambda(x)|^2 \\
&= \lambda^{-2}\int |\nabla\psi(x)|^2 dx - \lambda^{-5/2}\int dx\, |x|^{-5/2}\, |\psi(x)|^2
\end{aligned}$$

For $\lambda \to 0$ we notice that the second term dominates and that the energy takes arbitrarily large negative values. In this case, $\varepsilon(\psi)$ is not bounded below and the minimum cannot be attained. The followsin theorem provides sufficient conditions to make sure that the energy is bounded below. We use the notation

$$T(\psi) = \int dx\, |\nabla\psi(x)|^2$$

for the kinetic energy of the particle.

Theorem 7.4.1. *Assume that $V \in L^\infty(\mathbb{R}^n) + L^{n/2}(\mathbb{R}^n)$, if $n \geq 3$, that $V \in L^\infty(\mathbb{R}^n) + L^{1+\varepsilon}(\mathbb{R}^n)$, if $n = 2$, for an arbitrary $\varepsilon > 0$, and that $V \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ if $n = 1$. Then there exist constants $C, D > 0$ with*

$$\varepsilon(\psi) \geq CT(\psi) - D\|\psi\|^2$$

In particular

$$E_0 := \inf\{\varepsilon(\psi) : \|\psi\|_2 = 1\} > -\infty$$

Remark: Here $V \in L^{p_1} + L^{p_2}$ means that there are $V_1 \in L^{p_1}$ and $V_2 \in L^{p_2}$ such that $V = V_1 + V_2$.

Proof. We consider only the case $n \geq 3$ (the other cases can be handled analogously). By assumption, we have $V_1 \in L^\infty$, $V_2 \in L^{n/2}$ with $V = V_1 + V_2$. We claim that, for arbitrary $\delta > 0$ there exists $W_1 \in L^\infty$, $W_2 \in L^{n/2}$ with $V = W_1 + W_2$ and $\|W_2\|_{n/2} \leq \delta$. In fact, since $|V_2(x)|^{n/2}\chi(|V_2(x)| \geq \mu) \leq |V_2(x)|^{n/2}$ for all $x \in \mathbb{R}^n$ and since $|V_2(x)|^{n/2}\chi(|V_2(x)| \geq \mu) \to 0$ for almost all $x \in \mathbb{R}^n$, as $\mu \to \infty$, it follows from dominated convergence that

$$\int |V_2(x)|^{n/2}\chi(|V_2(x)| \geq \mu) \to 0$$

as $\mu \to \infty$. Hence, there exists $\mu_0 > 0$ large enough with

$$\int |V_2(x)|^{n/2}\chi(|V_2(x)| \geq \mu) \leq \delta^{n/2}$$

Then $W_2(x) = V_2(x)\chi(|V_2(x)| \geq \mu_0)$ and $W_1(x) = V_1(x) + V_2(x)\chi(|V_2(x)| \leq \mu_0)$ have the desired properties. Thus

$$\begin{aligned}
\varepsilon(\psi) &= \int |\nabla\psi|^2 dx + \int V|\psi|^2 \\
&= \|\nabla\psi\|_2^2 + \int W_1(x)|\psi(x)|^2 + \int W_2(x)|\psi(x)|^2 \\
&\geq \|\nabla\psi\|_2^2 - \|W_1\|_\infty\|\psi\|_2^2 - \|W_2\|_{n/2}\|\psi\|_{2n/(n-2)}^2 \\
&\geq (1-2\delta)\|\nabla\psi\|_2^2 - \|W_1\|_\infty\|\psi\|_2^2
\end{aligned}$$

where in the last inequality we used Theorem 7.3.1 (Sobolev inequality). The theorem follows by choosing $\delta < 1/2$. $\square$

For example, for $n = 3$, we can apply the last theorem to the hydrogen atom, where $V(x) = -1/|x| = -\chi(|x| \leq 1)/|x| - \chi(|x| \geq 1)/|x| \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, for all $p < 3$. As a consequence of the last theorem, the quantum mechanical energy of the hydrogen atom is bounded below. The negative potential energy is compensated by the positive kinetic energy, so that the total energy is always bounded below. In other words, to localize the electron close to the nucleus (i.e. to make the potential energy very negative), we have to pay a prize in terms of kinetic energy (this is a formulation of Heisenberg's uncertainty principle). This explains the stability of the hydrogen atom in quantum mechanics.

We want next to find conditions that guarantee the existence of a minimum of the energy. We will make use to the following theorem.

Theorem 7.4.2. *Let $V : \mathbb{R}^n \to \mathbb{R}$, with $V \in L^\infty(\mathbb{R}^n) + L^{n/2}(\mathbb{R}^n)$, if $n \geq 3$, $V \in L^\infty(\mathbb{R}^n) + L^{1+\varepsilon}(\mathbb{R}^n)$, if $n = 2$, and $V \in L^\infty(\mathbb{R}^n) + L^1(\mathbb{R}^n)$, if $n = 1$. We assume moreover that $V \in L^\infty(\mathbb{R}^n \backslash B_R(0))$ for sufficiently large $R > 0$, with $\|V\|_{L^\infty(\mathbb{R}^n \backslash B_R(0))} \to 0$ as $R \to \infty$. The potential energy*

$$P(\psi) = \int dx V(x) |\psi(x)|^2$$

is then weakly continuous in $H^1(\mathbb{R}^n)$. In other words, if $\psi_j \to \psi$ weakly in $H^1(\mathbb{R}^n)$, then $P(\psi_j) \to P(\psi)$ as $j \to \infty$ (Notice that weak continuity is stronger than stron continuity!)

Proof. We consider the case $n \geq 3$, the other cases can be handled analogously. Let ψ_j be a sequence in $H^{1,2}(\mathbb{R}^n)$ with $\psi_j \to \psi$ weakly in $H^{1,2}(\mathbb{R}^n)$. Then the sequence ψ_j is bounded in $H^{1,2}(\mathbb{R}^n)$, i.e. $\|\psi_j\|_{H^{1,2}} \leq C$ für alle $j \in \mathbb{N}$. Since

$$\int_{|x| \geq R} V(x) |\psi_j(x)|^2 \leq \|V\|_{L^\infty(B_R^c(0))} \|\psi_j\|_2^2 \leq C \|V\|_{L^\infty(B_c^R(0))} \to 0$$

uniformly in j, it is enough to show that

$$\int \chi_{B_R(0)}(x) V(x) |\psi_j(x)|^2 \to \int \chi_{B_R(0)}(x) V(x) |\psi(x)|^2$$

as $j \to \infty$, for an arbitrary, but fixed, $R > 0$. We write now $V(x) = V_1(x) + V_2(x)$, with $V_1 \in L^{n/2}(\mathbb{R}^n)$ and $V_2 \in L^\infty(\mathbb{R}^n)$. For $\delta > 0$ we set

$$V_{1,\delta}(x) = \begin{cases} V_1(x) & \text{falls } |V_1(x)| \leq 1/\delta \\ 0 & \text{sonst} \end{cases}$$

and $V_\delta = V_{1,\delta} + V_2$. Then $|V_{1,\delta}(x)| \leq |V_1(x)|$ for all $\delta > 0$, and $V_{1,\delta}(x) \to V_1(x)$ almost everywhere. Dominated convergence implies that

$$\int |V(x) - V_\delta(x)|^{n/2} dx = \int |V_1(x) - V_{1,\delta}(x)|^{n/2} dx \to 0 \qquad \text{als } \delta \to 0$$

and therefore

$$\begin{aligned} \left| \int \chi_{B_R(0)}(x) \; (V_\delta(x) - V(x)) \; |\psi_j(x)|^2 \right| &\leq \int |V_\delta(x) - V(x)| \; |\psi_j(x)|^2 dx \\ &\leq \|\psi_j\|_{2n/n-2}^2 \int \; |V_\delta(x) - V(x)|^{n/2} \; dx \\ &\leq \|\psi_j\|_{H^{1,2}}^2 \int |V_\delta(x) - V(x)|^{n/2} \, dx \to 0 \end{aligned}$$

This means that it is enough to show that

$$\int \chi_{B_R(0)} V_\delta |\psi_j|^2 \to \int \chi_{B_R(0)} V_\delta |\psi|^2$$

as $j \to \infty$, for all fixed $\delta, R > 0$. To this end, notice that $\psi_j \to \psi$ weakly in $H^{1,2}(B_R(0))$. This implies, by the Sobolev Embedding Theorem, that $\psi_j \to \psi$ strongly in $L^q(B_R(0))$, for all $1 \leq q < 2n/(n-2)$. Hence, $|\psi_j|^2 \to |\psi|^2$ strongly in $L^{q/2}(B_R(0))$ for all $1 \leq q < 2n/(n-2)$. In particular $|\psi_j|^2 \to |\psi|^2$ strongly in $L^1(B_R(0))$. Hence,

$$\left| \int \chi_{B_R(0)} V_\delta \left(|\psi_j|^2 - |\psi|^2 \right) \right| \leq \|V_\delta\|_\infty \left\| |\psi_j|^2 - |\psi|^2 \right\|_{L^1(B_R(0))} \to 0$$

as $j \to \infty$. □

Now, we can show the existence of a minimum.

Theorem 7.4.3. *Let $V : \mathbb{R}^n \to \mathbb{R}$ with $V \in L^\infty(\mathbb{R}^n) + L^{n/2}(\mathbb{R}^n)$, if $n \geq 3$, $V \in L^\infty(\mathbb{R}^n) + L^{1+\varepsilon}(\mathbb{R}^n)$, if $n = 2$, and $V \in L^\infty(\mathbb{R}^n) + L^1(\mathbb{R}^n)$, if $n = 1$. Moreover, let $V \in L^\infty(\mathbb{R}^n \backslash B_R(0))$ for R large enough, with $\|V\|_{L^\infty(B_R^c(0))} \to 0$ as $R \to \infty$. We assume that*

$$E_0 = \inf \left\{ \varepsilon(\psi) : \psi \in H^{1,2}(\mathbb{R}^n), \|\psi\|_2 = 1 \right\} < 0\,.$$

Then there exists $\psi_0 \in H^{1,2}(\mathbb{R}^n)$, with $\|\psi_0\|_2 = 1$ and $\varepsilon(\psi_0) = E_0$. ψ_0 satisfies (in a weak sense, after integration against a smooth test function) the Schrödinger equation

$$(-\Delta + V)\psi_0 = E_0 \psi_0$$

Proof. Let ψ_j be a sequence in $H^{1,2}(\mathbb{R}^n)$ with $\|\psi_j\|_2 = 1$ and $\varepsilon(\psi_j) \to E_0$ as $j \to \infty$. Since

$$\varepsilon(\psi_j) \geq \frac{1}{2}\|\nabla \psi_j\|_2^2 - C$$

it follows from Theorem 7.4.1, that $\|\nabla \psi_j\|_2$ is bounded. Hence, $\|\psi_j\|_{H^{1,2}} \leq C$ for all j. This implies that there exists a subsequence n_j and $\psi_0 \in H^{1,2}(\mathbb{R}^n)$ with $\psi_{n_j} \to \psi_0$ weakly in $H^{1,2}(\mathbb{R}^n)$ (in other words $\psi_{n_j} \to \psi_0$ weakly in $L^2(\mathbb{R}^n)$ and $\nabla\psi_{n_j} \to \nabla\psi_0$ weakly in $L^2(\mathbb{R}^n)$). Since in the weak limit the norm can only get smaller, we obtain

$$\|\psi_0\|_2 \leq 1, \quad \text{and} \quad \|\nabla\psi_0\|_2 \leq \liminf_{j\to\infty} \|\nabla\psi_{n_j}\|_2$$

From Theorem 7.4.2, we have $P(\psi_0) = \lim_{j\to\infty} P(\psi_{n_j})$. This implies that

$$E_0\|\psi_0\|_2^2 \leq \varepsilon(\psi_0) = \|\nabla\psi_0\|_2^2 + P(\psi_0) \leq \liminf_{j\to\infty} \left(\|\nabla\psi_{n_j}\|_2^2 + P(\psi_{n_j}) \right) = \liminf_{j\to\infty} \varepsilon(\psi_{n_j}) = E_0$$

Since $E_0 < 0$, we find $\|\psi_0\|_2 \geq 1$. This means that $\|\psi_0\|_2 = 1$ and $\varepsilon(\psi_0) = E_0$. To show that ψ_0 satisfies the Schrödinger equation, we consider the variation of ψ_0. For $\delta \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R}^n)$, let $\psi_\delta = \psi_0 + \delta f$ and $R(\delta) = \varepsilon(\psi_\delta)/\|\psi_\delta\|_2^2$. Then $R(\delta)$ has a minimum in $\delta = 0$. Hence, since R is differentiable in $\delta = 0$, we must assume that

$$0 = \frac{dR(\delta)}{d\delta}|_{\delta=0} = \frac{d\varepsilon(\psi_\delta)}{d\delta}|_{\delta=0} - E_0 \frac{d\|\psi_\delta\|_2^2}{d\delta}$$

A simple computation shows that

$$\frac{d\varepsilon(\psi_\delta)}{d\delta}|_{\delta=0} = 2\mathrm{Re} \int dx \left(\nabla \bar{f} \cdot \nabla \psi_0 + V \bar{f} \psi_0 \right)$$

and that

$$\frac{d\|\psi_\delta\|_2^2}{d\delta}|_{\delta=0} = 2\mathrm{Re} \int dx \bar{f} \psi_0$$

Thus

$$\mathrm{Re} \int \left[(-\Delta + V - E_0) \bar{f} \right] \psi_0 = 0$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. If we replace f by if we conclude that

$$\int \left[(-\Delta + V - E_0) \bar{f} \right] \psi_0 = 0$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. This shows that ψ_0 solves the Schrödinger equation (in the weak sense). □

Remark: the wave function ψ_0 minimizing the energy is known as the ground state of the system. It is unique, under general assumptions on V (of course, up to multiplication with a phase). ψ_0 is an eigenvector of the operator $-\Delta + V$ (known as the Hamilton operator), associated with the smallest eigenvalue. The eigenvalue equation

$$(-\Delta + V)\psi = E\psi$$

can also have more solutions, associated with $E > E_0$ (these solutions are known as the excited states). Also these states can be characterized by a variational principle.

8 Spectral Theorem for Compact Operators

In this section we consider bounded linear operators defined on a Banach space over $\mathbb{C}$.

8.1 The spectrum of bounded operators

Let X be a Banach space $\mathbb{C}$.

Definition 8.1.1. *Let $T \in \mathcal{L}(X)$. We define the resolvent set*

$$\rho(T) = \{\lambda \in \mathbb{C} : ker\ (\lambda\, Id\ - T) = \{0\} and\ Ran\ (\lambda Id\ - T) = X\}$$

i.e. $\rho(R)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda Id\ - T$ is invertible. The spectrum of T is then defined as $\sigma(T) = \mathbb{C} \backslash \rho(T)$. The spectrum $\sigma(T)$ can be decomposed into the point spectrum

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : ker\ (\lambda Id\ - T) \neq \{0\}\}\ ,$$

the continuous spectrum

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} : ker\ (\lambda Id\ - T) = \{0\},\ Ran\ (\lambda Id\ - T) \neq X,\ aber\ \overline{Ran\ (\lambda Id\ - T)} = X\right\}\ ,$$

and the residual spectrum

$$\sigma_r(T) == \left\{\lambda \in \mathbb{C} : ker\ (\lambda Id\ - T) = \{0\}, \overline{Ran\ (\lambda Id\ - T)} \neq X\right\}$$

Remarks:

- For $\lambda \in \rho(T)$, the operator $(\lambda - T)$ is bijective and invertible. It follows from the inverse function theorem that the inverse $(\lambda - T)^{-1}$ is automatically continuous and thus contained in the space $\mathcal{L}(X)$. The function $R_T(\lambda) = (\lambda - T)^{-1} \in \mathcal{L}(X)$, defined on $\rho(T)$, is called the resolvent of T (here and later we use the notation $\lambda - T$ to indicate the operator $\lambda \mathrm{Id}\ - T$).
- $\lambda \in \sigma_p(T)$ if and only if there exists $x \neq 0$ with $Tx = \lambda x$. Such an x is called an eigenvector of T associated with the eigenvalue λ. ker $(\lambda - T)$ is called the eigen space of T with eigenvalue λ. Eigen spaces are invariant with respect to T.

Theorem 8.1.2. *The resolvent set $\rho(T)$ is open and the resolvent $R_T(\lambda)$ is an analytic function on $\rho(T)$ with values in $\mathcal{L}(X)$, such that $\|R_T(\lambda)\| \geq 1/dist(\lambda, \sigma(T))$.*

Remark: a map $F : D \to Y$, where $D \subset \mathbb{C}$ is open and Y a Banach space, is called analytic if, for all $\lambda \in D$, there exists a ball $B_{r_0}(\lambda) \subset D$ such that

$$F(\mu) = \sum_{j=0}^{\infty} A_n(\mu - \lambda)^n$$

where $A_n \in Y$ for all n, and the series is absolutely convergent.

To prove Theorem 8.1.2, we will use the following lemma.

Lemma 8.1.3 (Neumann Series)**.** *Let $T \in \mathcal{L}(X)$, with $\|T\| < 1$. Then $1-T$ is bijective, with $(1-T)^{-1} \in \mathcal{L}(X)$ and*

$$(1-T)^{-1} = \sum_{j=0}^{\infty} T^n$$

Proof. Let $S_\ell = \sum_{n=0}^{\ell} T^n$. Then, for $k < \ell$,

$$\|S_\ell - S_k\| \leq \sum_{n=k+1}^{\ell} \|T^n\| \leq \sum_{n=k+1}^{\ell} \|T\|^n \to 0$$

as $\ell, k \to \infty$. In other words, S_ℓ is a Cauchy sequence and therefore it converges. Let $S = \lim_{\ell\to\infty} S_\ell$. Then

$$(1-T)S = \lim_{\ell\to\infty} (1-T)S_\ell = \lim_{\ell\to\infty} 1 - T^{\ell+1} = 1$$

because $\|T^{\ell+1}\| \leq \|T\|^{\ell+1} \to 0$ as $\ell \to \infty$. □

We can now show Theorem 8.1.2.

Proof of Theorem 8.1.2. Let $\lambda \in \rho(T)$. For $\mu \in \mathbb{C}$ we have

$$(\lambda - \mu) - T = (\lambda - T)(1 - \mu(\lambda - T)^{-1})$$

If now $|\mu| \leq \|(\lambda - T)^{-1}\|^{-1}$, then $\|\mu(\lambda - T)^{-1}\| < 1$, and therefore $(\lambda - \mu) - T$ is invertible, with

$$[(\lambda - mu) - T]^{-1} = \left(1 - \mu(\lambda - T)^{-1}\right)^{-1} (\lambda - T)^{-1} = \sum_{n=0}^{\infty} \mu^n \left((\lambda - T)^{-1}\right)^{n+1}$$

This proves that $\rho(T)$ is open and that $\|R_T(\lambda)\| \geq 1/\text{dist}\,(\lambda, \sigma(T))$. □

The following theorem localizes the spectrum of T.

Theorem 8.1.4 (Spectral radius)**.** *$\sigma(T)$ is a compact non-empty set (if $X \neq \{0\}$) and*

$$\sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{m\to\infty} \|T^m\|^{1/m} \leq \|T\|$$

Proof. For $|\lambda| > \|T\|$, the operator $1 - T/\lambda$ is invertible. Hence $\lambda - T$ is invertible with

$$(\lambda - T)^{-1} = \lambda^{-1} (1 - T/\lambda)^{-1} = \sum_{n=0}^{\infty} T^n/\lambda^{n+1} \tag{8.14}$$

and $\lambda \in \rho(T)$. This already shows that $\sigma(T) \subset B_{\|T\|}(0)$. If $\sigma(T) = \emptyset$, then $R_\lambda(T)$ would be an analytic function on $\mathbb{C}$ with $\|R_T(\lambda)\| \to 0$, as $\lambda \to \infty$ (from (8.14)). Liouville's Theorem implies that $R_T(\lambda) = 0$, a contradiction. Hence $\sigma(T) \neq \emptyset$.

We prove now that $r \leq \liminf_{m\to\infty} \|T^m\|^{1/m}$. To this end, we notice that

$$\lambda^m - T^m = (\lambda - T)p_m(T) = p_m(T)(\lambda - T)$$

with $p_m(T) = \sum_{j=0}^{m-1} \lambda^{m-j-1} T^j$. It follows that $\lambda \in \sigma(T)$ implies that $\lambda^m \in \sigma(T^m)$. Hence $\lambda \in \sigma(T)$ implies $\lambda^m \in \sigma(T^m)$ and thus $|\lambda|^m \leq \|T^m\|$ and $|\lambda| \leq \|T^m\|^{1/m}$. Therefore

$$r = \sup_{\lambda\in\sigma(T)} |\lambda| \leq \lim_{m\to\infty} \|T^m\|^{1/m}.$$

We still have to prove that $r \geq \liminf_{m\to\infty} \|T^m\|^{1/m}$. To this end, we observe that the integral

$$I_j := \frac{1}{2\pi i} \int_{\partial B_s(0)} \lambda^j \, (\lambda - T)^{-1} d\lambda$$

is independent of s, if $s > r$. For $s > \|T\|$, we have, however,

$$(\lambda - T)^{-1} = \sum_{n\geq 0} \frac{T^n}{\lambda^{n+1}}$$

and therefore

$$I_j = \sum_{n\geq 0} T^n \frac{1}{2\pi i} \int_{\partial B_s(0)} \lambda^{j-n-1} d\lambda = T^j$$

Hence

$$\|T^j\| \leq s^{j+1} \sup \left\|(\lambda - T)^{-1}\right\|$$

and

$$\|T^j\|^{1/j} \leq s \left(s \sup_{|\lambda|=s} \|(\lambda - T)^{-1}\| \right)^{1/j}$$

for all $s > r$. This implies that $\liminf_{j\to\infty} \|T^j\|^{1/j} \leq s$ for all $s > r$ and thus that $\liminf_{j\to\infty} \|T^j\|^{1/j} \leq r$. $\square$

To conclude this section, we study the relation between the spectrum of $T \in \mathcal{L}(X)$ and the spectrum of the adjoint operator $T^* \in \mathcal{L}(X^*)$, defined in Def. 4.2.10 through $(T^* f)(x) = f(Tx)$ for all $f \in X^*$, $x \in X$.

Proposition 8.1.5. *Let $T \in \mathcal{L}(X)$. Then $\sigma(T) = \sigma(T^*)$.*

Proof. It is enough to show that, for $T \in \mathcal{L}(X)$, T is invertible if and only if T^* is invertible. Then

$$\begin{aligned} \lambda \in \rho(T) \quad &\Leftrightarrow \quad \lambda - T \text{ invertible} \quad \Leftrightarrow \quad (\lambda - T)^* \text{ invertible} \\ &\Leftrightarrow \quad \lambda - T^* \text{ invertible} \quad \Leftrightarrow \quad \lambda \in \rho(T^*) \end{aligned}$$

and therefore $\lambda \in \sigma(T)$ if and only if $\lambda \in \sigma(T^*)$. To prove that T is invertible if and only if T^* is invertible, we assume first that T is invertible. Then there exists $T^{-1} \in \mathcal{L}(X)$ with $1_X = TT^{-1} = T^{-1}T$. Hence,

$$1_{X^*} = (T^{-1})^* \, T^* = T^* \, (T^{-1})^*$$

This means that T^* is invertible, and that $(T^*)^{-1} = (T^{-1})^*$. Let T^* be invertible. We claim that T is invertible. Top show this claim, we observe that, for an arbitrary $x \in X$, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$ (from Corollary 4.2.8). Hence

$$\|x\| = f(x) = \left(T^*(T^*)^{-1}f\right)(x) = \left((T^*)^{-1}f\right)(Tx) \leq \|(T^*)^{-1}f\| \, \|Tx\| \leq \|(T^*)^{-1}\| \, \|Tx\| \tag{8.15}$$

This implies that T is injective. We still have to show that Ran $T = X$. To this end, we notice that

$$\begin{aligned} \{0\} = \ker T^* &= \{f \in X^* : T^*f = 0\} = \{f \in X^* : T^*f(x) = 0 \text{ for all } x \in X\} \\ &= \{f \in X^* : f(Tx) = 0 \text{ for all } x \in X\} = \{f \in X^* : f(y) = 0 \text{ for all } y \in \text{Ran } (T)\} \end{aligned}$$

This implies that $\overline{\text{Ran } T} = X$ (otherwise we could find $f \in X^*$ with $f \neq 0$ but $f|_{\text{Ran } T} = 0$; this can be achieved for example using Corollary 4.2.9). For an arbitrary $y \in X$ we find a sequence x_k in X with $Tx_k \to y$ as $k \to \infty$. The bound (8.15) implies that

$$\|x_k - x_\ell\| \leq C\|Tx_k - Tx_\ell\| \to 0$$

as $k, \ell \to \infty$, because Tx_k converges. Hence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Therefore the limit $x = \lim_{k\to\infty} x_k$ exists. The continuity of T implies that $Tx = y$ and therefore that $y \in \text{Ran } T$. Hence Ran $T = X$ and T is invertible. □

8.2 Compact Operators on Banach Spaces

If dim $X < \infty$ the spectrum of $T \in \mathcal{L}(X)$ consists only of finitely many eigenvalues. If instead dim $X = \infty$, the spectrum can be more complicated. In this section, we consider a class of bounded operators, known as compact operators, whose spectrum is similar to the case dim $X < \infty$.

Definition 8.2.1. *Let X, Y be Banach spaces and $B_X = \{x \in X : \|x\| < 1\}$ the open unit ball in X. An operator $T \in \mathcal{L}(X,Y)$ is called compact, if TB_X is pre-compact, i.e. if for all $\varepsilon > 0$, TB_X has a finite cover with ε-balls. In other words, the operator T is compact if $\overline{TB_X}$ is compact. Equivalently, T is compact if, for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence Tx_n in Y has a convergent subsequence. We denote by $\mathcal{K}(X,Y)$ the set of all compact operators from X to Y. We write $\mathcal{K}(X) = \mathcal{K}(X,X)$.*

Examples:

- We say that the operator $T \in \mathcal{L}(X,Y)$ has finite rank, if $\dim TX < \infty$. Every finite rank operator is compact. In fact, if x_n is a bounded sequence in X, then Tx_n is a bounded sequence in Ran T. Since dim Ran $T < \infty$, it follows that Tx_n has a convergent subsequence (Ran T is always closed, if dim Ran $T < \infty$).

- In particular, every continuous linear functional $f \in X^*$ compact (because $\dim f(X) \leq 1$).

- Let $I = [0,1]$ and $K : I \times I \to \mathbb{C}$ continuous. For $f \in C(I)$ let

$$(Tf)(x) = \int dy K(x,y) f(y)$$

  This defines a linear operator $T : C(I) \to C(I)$ ($K(x,y)$ is called the integral kernel of T). T is bounded, with $\|T\| \leq \sup_x \int dy\,|K(x,y)|$, because

$$\|Tf\|_{C(I)} = \sup_{x\in I} |(Tf)(x)| \leq \|f\| \sup_x \int dy\,|K(x,y)| \tag{8.16}$$

  We claim that T is also compact. From the Theorem of Arzela Ascoli (Theorem 3.2.2) it is enough to show that $TB_{C(I)}$ is uniformly bounded and equicontinuous. The boundedness follows from (8.16) (with $\|f\| \leq 1$). The equicontinuity follows because, for arbitrary $x_1, x_2 \in I$,

$$\begin{aligned}|Tf(x_1) - Tf(x_2)| &\leq \int dy |K(x_1,y) - K(x_2,y)||f(y)| \leq \|f\| \int dy\,|K(x_1,y) - K(x_2,y)| \\ &\leq \int dy\,|K(x_1,y) - K(x_2,y)|\end{aligned}$$

  for all $f \in B_{C(I)}$. In other words, $|Tf(x_1) - Tf(x_2)| \to 0$, as $x_1 - x_2 \to 0$, uniformly in $f \in B_{C(I)}$.

- Sobolev Embedding: Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary. $k, m \in \mathbb{N}$, $k \geq 1$, $m \geq k$. $p \geq 1$, $kp < n$, $1 \leq q < np/(n-kp)$. Then the embedding Id $: H^{m,p}(\Omega) \to H^{m-k,q}(\Omega)$ is a compact operator, i.e. Id $\in \mathcal{K}(H^{m,p}(\Omega), H^{m-k,q}(\Omega))$.

In the following theorem, we prove some important properties of compact operators (in particular, we show that $\mathcal{K}(X,Y)$ is a 2-sided $*$-ideal in $\mathcal{L}(X,Y)$).

Theorem 8.2.2. *Let X, Y, Z be Banach spaces.*

a) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$.

b) $T \in \mathcal{K}(X,Y)$, $S \in \mathcal{L}(Y,Z)$, $R \in \mathcal{L}(Z,X)$. Then $ST \in \mathcal{K}(X,Z)$, $TR \in \mathcal{K}(Z,Y)$.

c) $T \in \mathcal{L}(X,Y)$. Then $T \in \mathcal{K}(X,Y)$ if and only if $T^* \in \mathcal{K}(Y^*, X^*)$.

Proof. a) Let $T, S \in \mathcal{K}(X,Y)$, $\lambda \in \mathbb{C}$, and $(x_n)_{n\in\mathbb{N}}$ a bounded sequence in X. Then x_n has a subsequence x_{n_j} such that Tx_{n_j} converges. Since x_{n_j} is again bounded, there exists a sub-subsequence $x_{n_{j_i}}$ such that $Tx_{n_{j_i}}$ and also $Sx_{n_{j_i}}$ converge. Then $(T+\lambda S)(x_{n_{j_i}})$ converges. Hence, $\mathcal{K}(X,Y)$ is a linear subspace. Now let T_n be a sequence in $\mathcal{K}(X,Y)$, with $T_n \to T$ in $\mathcal{L}(X,Y)$. We show that $T \in \mathcal{K}(X,Y)$, by proving the pre-compactness of TB_X. Fix $\varepsilon > 0$, and $n \in \mathbb{N}$ so large, that $\|T_n - T\| < \varepsilon$. Since T_n is compact, there exist $x_1, \dots, x_m \in B_X$ such that $T_n B_X$ is covered by the ε-balls around $T_n x_1, \dots, T_n x_m$. In other words, for every $x \in B_X$ there is $j \in \{1, \dots, m\}$ such that

$$\|T_n x - T_n x_j\| \leq \varepsilon. \tag{8.17}$$

We claim that the 3ε-balls around $Tx_1, \dots, Tx_m$ provide also a covering of TB_X. In fact, for an arbitrary $x \in B_X$, we can choose $j \in \{1, \dots, m\}$ as in (8.17). Then

$$\|Tx - Tx_j\| \leq \|Tx - T_n x\| + \|T_n x - T_n x_j\| + \|T_n x_j - Tx_j\| \leq 2\|T_n - T\| + \|T_n x - T_n x_j\| \leq 3\varepsilon$$

This implies that TB_X is pre-compact, i.e. that $T \in \mathcal{K}(X)$.

b) Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a subsequence n_j such that Tx_{n_j} converges in Y. Then STx_{n_j} converges in Z, for every bounded operator $S \in \mathcal{L}(Y,Z)$. Analogously, if $(z_n)_{n\in\mathbb{N}}$ is bounded in Z, then Rz_n is bounded in X. Therefore, there exists a subsequene n_j, such that TRz_{n_j} converges.

c) Let $T \in \mathcal{L}(X,Y)$ be compact, i.e. $K = \overline{TB_X} \subset Y$ is compact. For $f \in Y^*$, let $Rf = f|_K$ be the restriction of f on K. Then $Rf \in C(K)$ and $\sup_{y\in K} |f(y)| \leq \|f\|\|T\|$ (becuase $\|y\| \leq \|T\|$ for every $y \in K = \overline{TB_X}$). In other words, the map $R : Y^* \to C(K)$ is linear and bounded, with $\|R\| \leq \|T\|$ (with respect to the sup-norm on $C(K)$). For $f \in Y^*$, we have

$$\|T^* f\| = \sup_{x\in B_X} |T^* f(x)| = \sup_{x\in B_X} |f(Tx)| = \sup_{y\in TB_X} |f(y)| = \sup_{y\in K} |f(y)| = \|Rf\|_{C(K)}$$

The map $T^*f \to Rf$ defines also an isometric isomorphism between $T^*B_{Y^*}$ and $RB_{Y^*} \subset C(K)$. To prove that $T^*B_{Y^*}$ is pre-compact, it is enough to show that RB_{Y^*} is pre-compact in $C(K)$. To this end, we use the Theorem of Arzela-Ascoli (Theorem 3.2.2). The boundedness of RB_{Y^*} follows because $\|R\| \leq \|T\|$. To show the equicontinuity, we notice that, for all $y_1, y_2 \in K$,

$$|Rf(y_1) - Rf(y_2)| \leq |f(y_1) - f(y_2)| = |f(y_1 - y_2)| \leq \|f\|\|y_1 - y_2\| \leq \|y_1 - y_2\|,$$

where the right hand side is independent of $f \in B_{Y^*}$. Now let $T^* \in \mathcal{K}(Y^*, X^*)$. From what we already proved, we know that $T^{**} \in \mathcal{K}(X^{**}, Y^{**})$, i.e. that $T^{**}B_{X^{**}}$ is pre-compact in Y^{**}. The canonical embedding $J : X \to X^{**}$ allows us to consider B_X as a subset of $B_{X^{**}}$. On B_X, the map T^{**} acts like T, since

$$T^{**}(J_X(x))(f) = (J_X(x))(T^*f) = T^*f(x) = f(Tx) = J_Y(Tx)(f)$$

In other words, $T^{**} \circ J_X = J_Y \circ T$. We conclude that $J_Y(TB_X) = T^{**}(J_X B_X) \subset T^{**}(B_{X^{**}})$ is pre-compact. Since J_Y is isometric, it follows that TB_X is pre-compact. Hence, T is compact.

□

We consider now the spectrum of compact operators. We will use the following observation: Let X be a Banach space and $Y \subset X$ a proper closed linear subspace of X. For every $\theta \in (0, 1)$ there is $x_\theta \in X \backslash Y$ with

$$\|x_\theta\| = 1 \text{ und } \mathrm{dist}(x_\theta, Y) \geq \theta. \tag{8.18}$$

The proof of this remark can be found in the proof of Theorem 3.1.6. We will also use the notion of Fredholm operator.

Definition 8.2.3. *Let X, Y be Banach spaces. $A \in \mathcal{L}(X, Y)$ is called a Fredholm operator if dim ker $(A) < \infty$, Ran (A) is closed and codim Ran $(A) < \infty$. The index of A is then defined as ind $(A) =$ dim ker (A) - codim Ran (A).*

Remark: the codimension codim Z of a closed subspace $Z \subset X$ is the dimension of the quotient space $X/\sim$, where $\sim$ is the equivalence relation $x \sim y :\Leftrightarrow \quad x - y \in Z$. If Z has codimension $m < \infty$, then there exist linear independent $x_1, \dots, x_m \in X$ with $X = Z \oplus$ span $(x_1, \dots, x_m)$ (Proof: exercise)

Theorem 8.2.4. *Let $T \in \mathcal{K}(X)$. Then $A = 1 - T$ is a Fredholm operator with index 0.*

Proof. The proof is divided into 5 steps.

Step 1: dim ker $A < \infty$.

Let $Ax = 0$. Then $Tx = x$. Hence $B_1(0) \cap \ker A \subset T(B_1(0))$. Since T is compact, $T(B_1(0))$ is pre-compact. Hence $B_1(0) \cap \ker A$ is precompact. From Theorem 3.1.6 this is only possible if dim ker $A < \infty$.

Step 2: Ran A is closed.

Let $x \in \overline{\text{Ran } A}$ and $Ax_n \to x$ as $n \to \infty$. We can assume that

$$\|x_n\| \le 2d_n \qquad \text{where } d_n = \text{dist}(x_n, \ker (A))$$

If this is not the case, we can find $a_n \in \ker (A)$ with $\|x_n - a_n\| \le 2d_n$, and we replace x_n by $\widetilde{x}_n = x_n - a_n$; this is possible because $A\widetilde{x_n} = Ax_n \to x$ as $n \to \infty$. We assume first that there exists a subsequence n_j with $d_{n_j} \to \infty$, as $j \to \infty$. We set $y_j = x_{n_j}/d_{n_j}$. Then $Ay_j = Ax_{n_j}/d_{n_j} \to 0$ as $j \to \infty$ (because $Ax_{n_j} \to x$). Since y_j is a bounded sequence, there exists a subsequence i_j of n_j and $y \in X$ with $Ty_{i_j} \to y$ as $j \to \infty$. This implies that

$$y_{i_j} = Ay_{i_j} + Ty_{i_j} \to y$$

as $j \to \infty$. Since A is continuous, we must have $Ay = 0$. Hence $y \in \ker A$ and

$$\|y_j - y\| = \frac{1}{d_{n_j}} \left\| x_{n_j} - d_{n_j} y \right\| \ge \frac{\text{dist } (x_{n_j}, \ker A)}{d_{n_j}} = 1$$

in contradiction to $y_{i_j} \to y$. It follows therefore that the sequence d_n is bounded, and thus x_n as well. Thus there is a subsequence x_{n_j} and $z \in X$ with $Tx_{n_j} \to z$ as $j \to \infty$.This implies that

$$x = \lim_{j\to\infty} Ax_{n_j} = \lim_{j\to\infty} A(Ax_{n_j} + Tx_{n_j}) = A(x + z)$$

and therefore that $x \in \text{Ran } A$.

Step 3: $\ker A = \{0\}$ implies that $\text{Ran } A = X$.

Assume that $\ker A = \{0\}$ and that there exists $x \in X \backslash \text{Ran } A$. Then $A^n x \in \text{Ran } A^n \backslash \text{Ran } A^{n+1}$. In fact, if $A^n x = A^{n+1} y$ for a $y \in X$, then $A(A^{n-1}x - A^n y) = 0$ and therefore, since $\ker A = \{0\}$, $A^{n-1}x = A^n y$. Repeating this argument n times, we obtain $x = Ay$, in contradiction to the assumption $x \notin \text{Ran } A$. Furthermore $\text{Ran } A^{n+1}$ is closed. In fact,

$$A^{n+1} = (1 - T)^{n+1} = 1 + \sum_{k=1}^{n+1} \binom{n+1}{k} (-T)^k$$

has again the form 1−compact (because products of compact operators is again compact). Step 2 implies that $\text{Ran } A^{n+1}$ is closed. Therefore $\text{dist}(A^n x, \text{Ran } A^{n+1}) > 0$, and there is $a_{n+1} \in \text{Ran } A^{n+1}$ with

$$0 < \|A^n x - a_{n+1}\| \le 2\text{dist } (A^n x, \text{Ran } A^{n+1})$$

Now let

$$x_n := \frac{A^n x - a_{n+1}}{\|A^n x - a_{n+1}\|} \in \text{Ran } A^n$$

Then we have dist $(x_n, \text{Ran } A^{n+1}) \geq 1/2$, because for every $y \in \text{Ran } A^{n+1}$

$$\|x_n - y\| = \frac{\|A^n x - (a_{n+1} + \|A^n x - a_{n+1}\| y)\|}{\|A^n x - a_{n+1}\|} \geq \frac{\text{dist } (A^n x, \text{Ran } A^{n+1})}{\|A^n x - a_{n+1}\|} \geq \frac{1}{2}$$

For $m > n$ we have $Ax_n + x_m - Ax_m \in \text{Ran } A^{n+1}$ and therefore (since $T = 1 - A$)

$$\|Tx_n - Tx_m\| = \|x_n - (Ax_n + x_m - Ax_m)\| \geq \text{dist } (x_n, \text{Ran } A^{n+1}) \geq 1/2$$

Hence, the sequence Tx_n has no convergent subsequence, although x_n is a bounded sequence in X; this is a contradiction to the compactness of T.

Step 4: codim Ran $A \leq$ dim ker A.

From Step 1 we know that dim ker $A < \infty$. Let $x_1, \dots, x_n$ be a basis of ker A. Then there are $f_1, \dots, f_n \in X^*$ with $f_i(x_\ell) = \delta_{i\ell}$, for all $i, \ell \in \{1, \dots, n\}$.

If the claim was wrong, there would be linearly independent $y_1, \dots y_n$ in X, such that

$$\text{span } \{y_1, \dots, y_n\} \oplus \text{Ran } A$$

would be a proper subspace of X (with this notation, we mean, in particular, that span $\{y_1, \dots, y_n\} \cap \text{Ran } A = \{0\}$).

We define

$$\widetilde{T}x := Tx + \sum_{k=1}^{n} f_k(x) y_k$$

Then $\widetilde{T} \in \mathcal{K}(X)$ (as sum of finitely many compact operators) and ker $\widetilde{A} = \{0\}$,where $\widetilde{A} = 1 - \widetilde{T}$. In fact, $\widetilde{A}x = 0$ implies $\widetilde{T}x = x$ and thus $Ax = 0$ (i.e. $Tx = x$) and $f_k(x) = 0$ for all $k = 1, \dots, n$ (since span$\{y_1, \dots, y_n\} \cap \text{Ran } A = \{0\}$). $Ax = 0$ (i.e. $x \in$ ker A) implies that $x = \sum_{k=1}^n \alpha_k x_k$. Hence $0 = f_k(x) = \alpha_k$ for all $k = 1, \dots, n$ and $x = 0$. This proves that ker $\widetilde{A} = \{0\}$. Step 3 implies that Ran $\widetilde{A} = X$. Since however $\widetilde{A}x = Ax - \sum_{k=1}^n f_k(x) y_k$, $X = \text{Ran } \widetilde{A} \subset \text{Ran } A \oplus \text{span } \{y_1, \dots, y_n\}$ in contradiction to the assumption that Ran $A \oplus$ span $\{y_1, \dots, y_n\}$ is a proper subspace.

Step 5: dim ker $A \leq$ codim Ran A.

Let $A^* = 1 - T^* \in \mathcal{L}(X^*)$ be the adjoint of A. T^* is again compact (by Theorem 8.2.2). hence, Step 4 implies that codim Ran $A^* \leq$ dim ker A^*. The claim follows if we prove that

$$\begin{aligned} &1)\text{dim ker } A \leq \text{dim ker } A^{**} \\ &2)\text{dim ker } A^* = \text{codim Ran } A \end{aligned}$$

In fact, from codim Ran $A^* \leq$ dim ker A^*, 1) and 2), we find

$$\text{dim ker } A \leq \text{dim ker } A^{**} = \text{codim Ran } A^* \leq \text{dim ker } A^* = \text{codim Ran } A$$

To show 1) we notice that $A^{**} \circ J_X = J_X \circ A$. Since J_X is injective, we find ker $A =$ ker $J_X \circ A$. Therefore dim ker $A =$ dim ker $J_X \circ A =$ dim ker $A^{**} \circ J_X \leq$ dim ker A^{**}, again because of the injectivity of J_X. To prove 2) we observe that

$$\begin{aligned}\text{ker } A^* &= \{f \in X^* : A^*f(x) = 0 \text{ for all } x \in X\} = \{f \in X^* : f(Ax) = 0 \text{ für alle } x \in X\} \\ &= \{f \in X^* : f(y) = 0 \text{ for all } y \in \text{Ran } A\}\end{aligned}$$

It follows that dim ker $A^* =$ codim Ran A. □

We use the last theorem to describe the spectrum of compact operators.

Theorem 8.2.5 (Spectreal theorem for compact operators; Riesz-Schauder). *Let $T \in \mathcal{K}(X)$. Then*

- *The set $\sigma(T)\backslash\{0\}$ consists of countably (finite or infinite) many eigenvalues with 0 as the only possible accumulation point.*
- *For $\lambda \in \sigma(T)\backslash\{0\}$ we have dim ker $(\lambda - T) < \infty$ and*

  $$1 \leq n_\lambda = \max\left\{n \in \mathbb{N} : \text{ker } (\lambda - T)^{n-1} \neq \text{ker } (\lambda - T)^n\right\} < \infty$$

  n_λ is called the order of λ, dim ker $(\lambda - T)$ the multiplicity of λ.
- *For $\lambda \in \sigma(T)\backslash\{0\}$ we have*

  $$X = \text{Ran } (\lambda - T)^{n_\lambda} \oplus \text{ker } (\lambda - T)^{n_\lambda}$$

  Both subspaces are closed and inviariant w.r.t. T and dim ker $(\lambda - T)^{n_\lambda} < \infty$.
- *$\lambda \in \sigma(T)\backslash\{0\}$. Hence $\sigma(T|_{\text{Ran } (\lambda - T)^{n_\lambda}}) = \sigma(T)\backslash\{\lambda\}$.*
- *For $\lambda \in \sigma(T)\backslash\{0\}$, let E_λ be the projection on ker $(\lambda - T)^{n_\lambda}$ along Ran $(\lambda - T)^{n_\lambda}$. Then $E_\lambda E_\mu = \delta_{\lambda,\mu} E_\lambda$, for all $\lambda, \mu \in \sigma(T)\backslash\{0\}$.*

Proof.

- Let $\lambda \neq 0$, $\lambda \notin \sigma_p(T)$. Then ker $(\lambda - T) = \{0\}$, and thus ker $(1 - T/\lambda) = \{0\}$. Theorem 8.2.4 implies that Ran $(1 - T/\lambda) = X$, and therefore that $\lambda \in \rho(T)$. This proves that $\sigma(T)\backslash\{0\} \subset \sigma_p(T)$. If $\sigma(T)\backslash\{0\}$ is not finite, we choose a sequence $\lambda_n \in \sigma(T)\backslash\{0\}$ of pairwise different eigenvalues with eigenvectors $e_n \neq 0$. We set $X_n =$ span $\{e_1, \ldots, e_n\}$. The eigenvectors are linearly independent. Otherwise there would be a $1 < k \leq n$ with $e_k = \sum_{j=1}^{k-1} \alpha_i e_i$ and with already linearly independent $e_1, \ldots, e_{k-1}$. Then we had

  $$0 = Te_k - \lambda_k e_k = \sum_{i=1}^{k-1} \alpha_i(\lambda_i - \lambda_k)e_i$$

Since $(\lambda_i - \lambda_k) \neq 0$, for all $i = 1, \dots, k-1$, we must have $\alpha_i = 0$ for all $i = 1, \dots, k-1$. This implies that $e_k = 0$ and gives a contradiction. It follows that X_{n-1} is a proper subspace of X_n. From (8.18) there is $x_n \in X_n$ with $\|x_n\| = 1$ and dist $(x_n, X_{n-1}) \geq 1/2$. Since $x_n = \alpha_n e_n + \widetilde{x}_n$ with $\alpha_n \in \mathbb{C}$ and $\widetilde{x}_n \in X_{n-1}$, because of the T-invariance of X_{n-1} if follows that

$$Tx_n - \lambda_n x_n = T\widetilde{x}_n - \lambda_n \widetilde{x}_n \in X_{n-1}$$

This implies that

$$\frac{1}{\lambda_n}(Tx_n - \lambda_n x_n) - \frac{1}{\lambda_m} T x_m \in X_{n-1}$$

for all $m < n$. Hence

$$\left\| T\frac{x_n}{\lambda_n} - T\frac{x_m}{\lambda_m} \right\| = \left\| x_n + \frac{1}{\lambda_n}(Tx_n - \lambda_n x_n) - \frac{1}{\lambda_m} T x_m \right\| \geq 1/2$$

for all $m < n$. This means that the sequence $T(x_n/\lambda_n)$ has no convergent subsequence. Since T is compact, it follows that (x_n/λ_n) has no bounded subsequences. Hence, $\|x_n\|/|\lambda_n| \to \infty$ as $n \to \infty$. Since however $\|x_n\| = 1$ for all n, we must have $\lambda_n \to 0$. Zero is thus the only possible accumulation point of $\sigma(T)$. $\sigma(T)\backslash B_r(0)$ is finite for every $r > 0$. This implies that $\sigma(T)\backslash\{0\} = \cup_{n\in\mathbb{N}} \sigma(T)\backslash B_{1/n}(0)$, as countable union of finite sets, is countable.

- Let $\lambda \neq 0$. If dim ker $(\lambda - T) = \infty$, then the unit ball $K := B_1(0) \cap \ker(\lambda - T)$ is not pre-compact. Hence, there is $\varepsilon > 0$, so that K has no finite covering through ε-balls. We choose $x_1 \in K$, and then iteratively $x_k \in K \backslash \cup_{j=1}^{k-1} B_\varepsilon(x_j)$. Then $\|x_k - x_\ell\| \geq \varepsilon$ for all $k, \ell \in \mathbb{N}$, and thus

$$\|Tx_k - Tx_\ell\| = |\lambda| \|x_k - x_\ell\| \geq \varepsilon|\lambda|$$

for all $k, \ell \in \mathbb{N}$. This implies, that the sequence Tx_k has no convergent subsequence, although x_k is bounded, contradiction the compactness of T. We show now that $1 \leq n_\lambda < \infty$. Let $A = \lambda - T$. Then ker $A^{n-1} \subset$ ker A^n for every $n \in \mathbb{N}$. Let us assume that ker A^{n-1} is not a proper subspace of ker A^n, for all $n \in \mathbb{N}$. Then we could find, from (8.18), a sequence x_n with $x_n \in \ker A^n$, $\|x_n\| = 1$ and dist$(x_n, \ker A^{n-1}) \geq 1/2$. Then, for $m < n$,

$$Ax_n + \lambda x_m - Ax_m \in \ker A^{n-1}$$

and thus

$$\|Tx_n - Tx_m\| = \|\lambda x_n - (Ax_n + \lambda x_m - Ax_m)\| \geq |\lambda|/2$$

This means Tx_n has no convergent subsequence, although x_n is limited. This contradicts the compactness of T. Hence, there exists $n_0 \in \mathbb{N}$ with ker $A^{n_0 - 1} =$ ker A^{n_0}. For $m > n_0$, we have

$$x \in \ker A^m \Rightarrow A^{m-n_0} x \in \ker A^{n_0} = \ker A^{n_0 - 1}$$

This implies $A^{m-1}x = 0$ and thus $x \in \ker A^{m-1}$. Inductively, we find $\ker A^m = \ker A^{n_0-1}$ for all $m \geq n_0$. Therefore $n_\lambda < \infty$. Since $\ker A \neq 0$ it follows that $n_\lambda \geq 1$.

- Let $A = \lambda - T$. We have
$$\ker A^{n_\lambda} \oplus \text{Ran } A^{n_\lambda} \subset X$$
because, if $x \in \ker A^{n_\lambda} \cap \text{Ran } A^{n_\lambda}$, then $A^{n_\lambda}x = 0$, $x = A^{n_\lambda}y$ and therefore $A^{2n_\lambda}y = 0$. Hence $y \in \ker A^{2n_\lambda} = \ker A^{n_\lambda}$, and this implies $x = A^{n_\lambda}y = 0$. Moreover, we observe that
$$A^{n_\lambda} = \lambda^{n_\lambda}1 + \sum_{k=1}^{n_\lambda} \binom{n_\lambda}{k} \lambda^{n_\lambda-k}(-T)^k$$
Hence $A^{n_\lambda}/\lambda^{n_\lambda}$ has the form $1-$ compact. In other words,
$$\text{codim Ran } A^{n_\lambda} = \dim \ker A^{n_\lambda} < \infty$$
This implies that
$$X = \ker A^{n_\lambda} \oplus \text{Ran } A^{n_\lambda}$$
The T-invariance of $\ker A^{n_\lambda}$ and of $\text{Ran } A^{n_\lambda}$ is a consequence of the fact, that T and A commute.

- Let $T_\lambda = T|_{\text{Ran } A^{n_\lambda}}$. Then $T_\lambda \in \mathcal{K}(\text{Ran } A^{n_\lambda})$, where $\text{Ran } A^{n_\lambda}$ is closed and therefore a Banach space.
$$\ker (\lambda - T_\lambda) = \ker A \cap \text{Ran } A^{n_\lambda} = \{0\}$$
Since T_λ is compact (and therefore $\lambda - T_\lambda$ is a Fredholm operator) it follows also, that Ran $(\lambda - T_\lambda) = \text{Ran } A^{n_\lambda}$. Hence $\lambda \in \rho(T)$. It remains to prove that $\sigma(T_\lambda)\backslash\{\lambda\} = \sigma(T)\backslash\{\lambda\}$. Let $\mu \in \mathbb{C}\backslash\{\lambda\}$. Then $\ker A^{n_\lambda}$ is invariant w.r.t. $\mu - T$. Furthermore, $\mu - T = \mu - \lambda + A$ on $\ker A^{n_\lambda}$ is injective. In fact, if $x \in \ker (\mu - \lambda + A) \cap \ker A^{n_\lambda}$, then, on the one hand, $Ax = (\lambda - \mu)x$ and, on the other hand, $A^{n_\lambda}x = 0$. This implies that $(\lambda - \mu)A^{n_\lambda - 1}x = A^{n_\lambda}x = 0$ and thereofer $A^{n_\lambda-1}x = 0$. Inductively, we find $x = 0$. Since $\dim \ker A^{n_\lambda} < \infty$ we find that $(\mu - T)$ is bijective on $\ker A^{n_\lambda}$. Since $X = \ker A^{n_\lambda} \oplus \text{Ran } A^{n_\lambda}$, this means that $\mu \in \rho(T)$ if and only if $\mu \in \rho(T_\lambda)$ and therefore, that $\sigma(T)\backslash\{0\} = \sigma(T_\lambda)\backslash\{0\}$.

- Let $\lambda, \mu \in \sigma(T)\backslash\{0\}$ be different, $A_\lambda = \lambda - T$, $A_\mu = \mu - T$. Let $x \in \ker A_\mu^{n_\mu}$. There exist then $y \in \ker A_\lambda^{n_\lambda}$ and $z \in \text{Ran } A_\lambda^{n_\lambda}$ with $x = y + z$. Since $\ker A_\lambda^{n_\lambda}$ and $\text{Ran } A_\lambda^{n_\lambda}$ is invariant w.r.t. the action of $A_\mu^{n_\mu}$, it follows that $A_\mu^{n_\mu}y \in \ker A_\lambda^{n_\lambda}$, and $A_\mu^{n_\mu}z \in \text{Ran } A_\lambda^{n_\lambda}$. Hence,
$$0 = A_\mu^{n_\mu}x = A_\mu^{n_\mu}y + A_\mu^{n_\mu}z,$$
implies that $A_\mu^{n_\mu}y = A_\mu^{n_\mu}z = 0$. A_μ is a bijection on $\ker A_\lambda^{n_\lambda}$. Therefore $A_\mu^{n_\mu}$ is bijective on $\ker A_\lambda^{n_\lambda}$ and we find $y = 0$. Thus $x \in \text{Ran } A_\lambda^{n_\lambda}$. We have shown that
$$\ker A_\mu^{n_\mu} \subset \text{Ran } A_\lambda^{n_\lambda}$$

i.e. that Ran $E_\mu \subset \ker E_\lambda$, and $E_\lambda E_\mu = 0$.

□

The restriction of T on the subspace $\ker(\lambda - T)^{n_\lambda}$ can be represented, w.r.t. an appropriate basis of the finite dimensional space $\ker(\lambda - T)^{n_\lambda}$, by a matrix in the Jordan normal form

$$\begin{pmatrix} \lambda & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix} \tag{8.19}$$

As a consequence, if $\dim X < \infty$ (in this case, every operator is compact), the operator T can be represented through a block-diagonal matrix whose blocks have the form (8.19), with different λ's.

The case $\dim X = \infty$ is more complicated (although we restricted our attention to the relatively simple class of compact operators). The space $\ker(\lambda - T)^{n_\lambda}$ can be factorized out like in the finite dimensional case, for all $\lambda \in \sigma(T)\backslash\{0\}$. At the end, we are left with an operator, whose spectrum consists only of the point $\{0\}$. $\{0\}$ is in general, no eigenvalue (but, since the sequence of eigenvalues converges towards zero, and since the spectrum is close, we always have $0 \in \sigma(T)$). Then in general it is not clear, whether T has other invariant subspaces. This question becomes easier for normal compact operators on Hilbert spaces. This is the content of the next section.

8.3 Normal compact operators

Let X be a Banach space and $T \in \mathcal{L}(X)$. We recall again that the adjoint operator $T^* \in \mathcal{L}(X^*)$ is defined through $T^* f(x) = f(Tx)$ for all $f \in X^*$, $x \in X$. We know that $\|T^*\| = \|T\|$, $(T + \lambda S)^* = T^* + \lambda S^*$, $(TS)^* = S^* T^*$, $\sigma(T^*) = \sigma(T)$, and that T is compact if and only if T^* is compact.

If H is a Hilbert space, then we can identify H^* with H through the map $R_H : H \to H^*$. Here, R_H is defined through $(R_H(x))(y) = \langle x, y \rangle$, where $\langle ., . \rangle$ denotes the inner product on H. R_H is an anti-linear isometric bijection between H and H^*. The adjoint operator can therefore be interpreted as a map in $\mathcal{L}(H)$. In other words, we can define the adjoint to $T \in \mathcal{L}(H)$ as

$$T^\dagger = R_H^{-1} T^* R_H$$

$T^\dagger \in \mathcal{L}(H)$ is characterized by the equation

$$\begin{aligned} R_H T^\dagger(x) = T^* R_H(x) \quad &\Leftrightarrow \quad (R_H T^\dagger x)(y) = (R_H x)(Ty) \;\text{ for all } y \in H \\ &\Leftrightarrow \quad \langle T^\dagger x, y \rangle = \langle x, Ty \rangle \;\text{ for all } y \in H \end{aligned}$$

for all $x \in H$. D.h. $T^\dagger \in \mathcal{L}(H)$ is the adjoint to T if and only if $\langle x, Ty \rangle = \langle T^* x, y \rangle$, for all $x, y \in H$.

It is easy to derive, from the properties of $T^* \in \mathcal{L}(H^*)$, the corresponding properties of $T^\dagger \in \mathcal{L}(H)$. For $T, S \in \mathcal{L}(H)$, $\lambda \in \mathbb{C}$ we have for example $(T + \lambda S)^\dagger = T^\dagger + \bar{\lambda} S^\dagger$ (because R_H is anti-linear). Moreover $(TS)^\dagger = S^\dagger T^\dagger$, $\|T^\dagger\| = \|T\|$, T is comapct if and only if $T^\dagger$ is compact and $\sigma(T^\dagger) = \overline{\sigma(T)}$.

Lemma 8.3.1. *Let H be a Hilbert space, $T \in \mathcal{L}(H)$. Then ker $T^\dagger$ = (Ran $T)^\perp$ and Ran $T^\dagger$ = (ker $T)^\perp$.*

Proof. We notice easily that

$$\begin{aligned} x \in \ker T^\dagger \quad &\Leftrightarrow \quad T^\dagger x = 0 \quad \Leftrightarrow \quad \langle T^\dagger x, y \rangle = 0 \text{ for all } y \in H \\ &\Leftrightarrow \quad \langle x, Ty \rangle = 0 \text{ for all } y \in H \quad \Leftrightarrow \quad x \perp \text{Ran } T. \end{aligned}$$

□

Definition 8.3.2. *An operator $T \in \mathcal{L}(H)$ is called self-adjoint if $T^\dagger = T$, i.e. if $\langle x, Ty \rangle = \langle Tx, y \rangle$, for all $x, y \in H$. $T \in \mathcal{L}(H)$ is called normal if T and $T^\dagger$ commute, i.e. if $TT^\dagger = T^\dagger T$. Every self-adjoint operator is also normal.*

A couple important properties of normal operators are collected in the next theorem.

Theorem 8.3.3. *Let $T \in \mathcal{L}(H)$.*

a) T normal implies that $T - \lambda$ is normal, for all $\lambda \in \mathbb{C}$.

b) T normal if and only if $\|Tx\| = \|T^\dagger x\|$ for all $x \in H$.

c) T normal implies that ker $(\lambda - T)$ = ker $(\overline{\lambda} - T^\dagger)$.

Proof. a) We have $(\lambda - T)^\dagger = \overline{\lambda} - T^\dagger$. It is easy to check that, if T and $T^\dagger$ commute, then also $\lambda - T$ and $\overline{\lambda} - T^\dagger$ commute.

b) We assume first that T is normal. Then

$$\langle Tx, Tx \rangle = \langle x, T^\dagger T x \rangle = \langle x, TT^\dagger x \rangle = \langle T^\dagger x, T^\dagger x \rangle = \|T^\dagger x\|^2$$

for all $x \in H$. On the other hand, from

$$4\text{Re } \langle a, b \rangle = \|a + b\|^2 - \|a - b\|^2$$

we find

$$4\text{Re } (Tx, Ty) = \|T(x + y)\|^2 - \|T(x - y)\|^2$$

and analogously

$$4\text{Re }(T^\dagger X, T^\dagger y) = \|T^\dagger(x+y)\|^2 - \|T^\dagger(x-y)\|^2$$

$\|Tz\| = \|T^\dagger z\|$ for all $z \in H$ implies that Re $(Tx, Ty) =$ Re $(T^\dagger x, T^\dagger y)$ for all $x, y \in H$. The substitution of y through iy implies that $(Tx, Ty) = (T^\dagger x, T^\dagger y)$, and therefore that $((TT^\dagger - T^\dagger T)x, y) = 0$ for all $x, y \in H$. Hence $TT^\dagger = T^\dagger T$.

c) The operator $\lambda - T$ is normal from a). Moreover, b) implies that $\|(\lambda - T)x\| = \|(\overline{\lambda} - T^\dagger)x\|$ for all $x \in H$. This implies in particular, that ker $(\lambda - T) =$ ker $(\overline{\lambda} - T^*)$. □

Another important property of normal operators is proved in the following lemma.

Lemma 8.3.4. *Let $T \in \mathcal{L}(H)$ be normal. Then $\|T^m\| = \|T\|^m$ for all $m \in \mathbb{N}$.*

Proof. It is enough to show that

$$\|T^m\| \geq \|T\|^m. \tag{8.20}$$

For $m = 0, 1$ the claim is trivial. We show now (8.20) by induction over $m \in \mathbb{N}$. To this end, we observe that

$$\begin{aligned}\|T^m x\|^2 &= \langle T^m x, T^m x\rangle = \langle T^* T^m x, T^{m-1} x\rangle \leq \|T^* T^m x\| \, \|T^{m-1} x\| \\ &= \|T^{m+1} x\| \, \|T^{m-1} x\| \leq \|T^{m+1}\| \, \|T^{m-1}\| \|x\|^2 \leq \|T^{m+1}\| \, \|T\|^{m-1} \|x\|^2\end{aligned}$$

Das zeigt, dass $\|T^m\|^2 \leq \|T\|^{m-1} \|T^{m+1}\|$, and therefore, frmo the induction assumption $\|T^m\| \geq \|T\|^m$, we also find $\|T^{m+1}\| \geq \|T\|^{m+1}$. □

The following corollary is a simple consequence of the last lemma, combined with Theorem 8.1.4.

Corollary 8.3.5 (Spectral radius for normal operators)**.** *Let $T \in \mathcal{L}(H)$ be normal. Then*

$$\sup_{\lambda \in \sigma(T)} |\lambda| = \|T\|$$

Example: Let H be a separable Hilbert space, $\{e_k\}_{k \in \mathbb{N}}$ an orthonormal system in H, $(\lambda_k)_{k \in \mathbb{N}}$ a sequence in $\mathbb{C}$, with $|\lambda_k| \leq r$ for all $k \in \mathbb{N}$. Then

$$Tx = \sum_{k \in \mathbb{N}} \lambda_k \langle e_k, x\rangle e_k$$

defines a normal operator $T \in \mathcal{L}(X)$, with $\|T\| \leq r$ (since $\|Tx\|^2 = \sum_{k \in \mathbb{N}} |\lambda_k|^2 \, |(e_k, x)|^2 \leq r^2 \|x\|^2$). A simple computation shows that

$$T^* y = \sum_{n \in \mathbb{N}} \overline{\lambda}_n \, \langle e_n, y\rangle e_n$$

and that

$$TT^*x = T^*Tx = \sum_{n\in\mathbb{N}} |\lambda_n|^2 \langle e_n, x\rangle e_n$$

and athus that T is normal. T is compact if and only if $\lambda_k \to 0$ as $k \to \infty$. In fact, if $\lambda_k \to 0$ as $k \to \infty$, we define

$$T_n x = \sum_{k=1}^{n} \lambda_k \langle e_k, x\rangle e_k$$

Then T_n is compact for all $n \in \mathbb{N}$ (since T_n has finite rank) and

$$\|(T_n - T)x\|^2 = \sum_{k\geq n} |\lambda_k|^2 \, |\langle e_k, x\rangle|^2 \leq \sup_{k\geq n} |\lambda_k|^2 \, \|x\|^2$$

This implies that

$$\|T - T_n\| \leq \sup_{k\geq n} |\lambda_k| \to 0$$

as $n \to \infty$. Therefore also T in compact (the compact operators are closed) . On the other hand, if there is a subsequence λ_{k_j} with $|\lambda_{k_j}| \geq \delta$ for all $j \in \mathbb{N}$, then we would have

$$\|Te_{k_i} - Te_{k_j}\|^2 = \|\lambda_{k_i} e_{k_i} - \lambda_{k_j} e_{k_j}\|^2 = \left(|\lambda_{k_i}|^2 + |\lambda_{k_j}|^2\right) \geq 2\delta^2$$

for every i, j. Then, we had that Te_{k_j} is no convergent subsequence, altough e_{k_j} are bounded.

The next theorem is a complete spectral theorem for compact normal operators; it proves that all compact normal operators have the form of the operator in the example.

Theorem 8.3.6 (Spectral theorem for normal compact operators)**.** *Let H be a Hilbert space over $\mathbb{C}$, $T \in \mathcal{K}(H)$ be also normal, $T \neq 0$. Then:*

- *There exists an orthonormal system $(e_k)_{k\in N}$ in H and a sequence $(\lambda_k)_{k\in N}$ in $\mathbb{C}$ with $N \subset \mathbb{N}$, such that $\lambda_k \neq 0$ for all $k \in N$ and*

  $$Te_k = \lambda_k e_k \qquad \textit{for all } k \in N$$
  $$\sigma(T)\backslash\{0\} = \{\lambda_k : k \in N\}$$

  If $|N| = \infty$, then $\lambda_k \to 0$ (notice that in this statement, the λ_k do not need to be all different).

- *For all $k \in N$, we have*

  $$n_{\lambda_k} = \max\{n \in \mathbb{N} : \mathit{ker}\ (\lambda_k - T)^{n-1} \neq \mathit{ker}\ (\lambda_k - T)^n\} = 1$$

- *We can decompose*

  $$X = \mathit{ker}\ T \oplus \overline{\mathit{span}\{e_k : k \in N\}}$$

  and the two spaces are orthogonal to eachother.

- *For all $x \in X$, we have*
$$Tx = \sum_{k\in N} \lambda_k \langle e_k, x\rangle e_k$$
i.e. $T = \sum_{k\in N} \lambda_k P_k$, *where P_k is the orthogonal projection onto e_k.*

Proof. We know from the spectral theorem for compact operators, Theorem 8.2.5, that $\sigma(T)\backslash\{0\} = \{\lambda_j : j \in \widetilde{N}\}$, where λ_j is an eigenvalue with dim ker $(\lambda_j - T) < \infty$ and where $\lambda_j \to 0$, as $j \to \infty$, if $|\widetilde{N}| < \infty$ (here all λ_j are different). We denote by $E_j :=$ ker $(\lambda_k - T)$, for all $k \in \widetilde{N}$ (then dim $E_k < \infty$, for all $k \in \widetilde{N}$). We define also $E_0 =$ ker T, $\lambda_0 = 0$. From Theorem 8.3.3, we have $E_k =$ ker $(\overline{\lambda}_k - T^\dagger)$ for all $k \neq 0$. For $x \in E_k$ and $y \in E_\ell$ we have therefore $T^\dagger x = \overline{\lambda}_k x$ and $Ty = \lambda_\ell y$. Hence
$$\lambda_\ell \langle x, y\rangle = \langle x, Ty\rangle = \langle T^\dagger x, y\rangle = \lambda_k \langle x, y\rangle$$
This proves that $E_k \perp E_\ell$ for all $k \neq \ell$, $k, \ell \in \widetilde{N} \cup \{0\}$. We claim that
$$X = \overline{\bigoplus_{k\in\widetilde{N}\cup\{0\}} E_k}$$
In fact, let $Y = \left(\oplus_{k\in\widetilde{N}\cup\{0\}} E_k\right)^\perp$. Then Y is invariant w.r.t. T, since $y \in Y$ and $x \in E_k$, $k \in N \cup \{0\}$ imply that
$$\langle x, Ty\rangle = \langle T^\dagger x, y\rangle = \lambda_k \langle x, y\rangle = 0$$
because $Y \ni y \perp E_k$. Now let $T_0 = T|_Y$ be the restriction of T on Y. T_0 is then compact and normal. Hence, there exists $\lambda \in \sigma(T_0)$ with $|\lambda| = \|T_0\|$. If $T_0 \neq 0$, we would have $\lambda \neq 0$. In this case, λ would also be an eigenvalue of T, i.e. we would have $\lambda = \lambda_k$, fof an appropriate $k \in \widetilde{N}$, and $Y \cap E_k \neq \{0\}$, in contradiction to the definition of Y. Hence $T_0 = 0$, and $Y \subset$ ker $T = E_0$. In other words, $Y \subset E_0 \cap E_0^\perp = \{0\}$, hence $Y = \{0\}$. This proves that
$$X = \overline{\oplus_{k\in\widetilde{N}\cup\{0\}} E_k}$$
Let now, for arbitrary $k \in \widetilde{N} \cup \{0\}$, Q_k be the orthogonal projection onto E_k. Then, for all $x \in H$,
$$x = \sum_{k\in\widetilde{N}\cup\{0\}} Q_k x$$
and therefore
$$Tx = \sum_{k\in\widetilde{N}} TQ_k x = \sum_{k\in\widetilde{N}} \lambda_k Q_k x \tag{8.21}$$

Now let $d_k = \dim E_k$ and $(e_1^{(k)}, \ldots, e_{d_k}^{(k)})$ be an orthonormal basis of E_k, for all $k \in \widetilde{N}$. Then $Q_k = \sum_{j=1}^{d_k} P_{e_j^{(k)}}$ and

$$Tx = \sum_{k \in \widetilde{N}} \lambda_k \sum_{j=1}^{d_k} \langle e_j^{(k)}, x \rangle e_j^{(k)} = \sum_{k \in N} \lambda_k \langle e_k, x \rangle e_k$$

Here we change to a new counting of the eigenvalues and of the eigenvectors (now, eigenvalues are allowed to repeat). From the representation (8.21) it follows in particular that $\ker\,(\lambda_k - T)^2 = E_k$, because

$$(T - \lambda_k)x = \sum_{j \in \widetilde{N} \cup \{0\}} (\lambda_j - \lambda_k) Q_j x$$

and therefore

$$(T - \lambda_k)^2 x = \sum_{j \in \widetilde{N} \cup \{0\}} (\lambda_j - \lambda_k)^2 Q_j x$$

This implies that

$$\|(T - \lambda_k)^2 x\|^2 = \sum_{j \in \widetilde{N} \cup \{0\}} |\lambda_k - \lambda_j|^4 \|Q_j x\|^2$$

and also, that $x \in \ker\,(\lambda_k - T)^2$ if and only if $Q_j x = 0$ for all $j \neq k$, i.e. if and only if $x \in E_k$. □

Remarks:

- Compared with the case of general compact operators, there are two important consequences from the assumption that T is also normal, which allow us in Theorem 8.3.6 to obtain a complete diagonal decomposition of T. The first important consequence is the observation that

$$\ker\,(\lambda - T)^2 = \ker\,(\lambda - T)$$

  for all $\lambda \in \mathbb{C}$. In fact, if $x \in \ker\,(\lambda - T)^2$, then $(\lambda - T)x \in \ker\,(\lambda - T)$ and therefore $(\lambda - T)x \in \ker\,(\overline{\lambda} - T^\dagger)$. Hence, $(\overline{\lambda} - T^\dagger)(\lambda - T)x = 0$, and in particular

$$0 = \langle x, (\overline{\lambda} - T^\dagger)(\lambda - T)x \rangle = \|(\lambda - T)x\|^2$$

  Hence, $x \in \ker\,(\lambda - T)$. This implies (see Theorem 8.2.5), that for every eigenvalue $\lambda_k \in \sigma(T)\backslash\{0\}$, the order $n_{\lambda_k} = 1$ and therefore $X = \ker\,(\lambda_k - T) \oplus \mathrm{Ran}\,(\lambda_k - T)$. The restriction of T on $\ker\,(\lambda_k - T)$ acts as the identity times λ_k. The restriction of T on $\mathrm{Ran}\,(\lambda_k - T)$ is again a compact normal operator and therefore can be further decomposed. The second important consequence of the fact that T is normal is the identity $\sup_{\lambda \in \sigma(T)} |\lambda| = \|T\|$ (which follows from $\|T^m\| = \|T\|^m$ for normal operators). This formula implies that the only normal operator with $\sigma(T) = \{0\}$ is $T = 0$. This proves that after separation of all eigenspaces $\ker\,(\lambda_k - T)$ there is nothing left.

- If T is self-adjoint and compact, then Theorem 8.3.6 holds true with $(\lambda_k)_{k\in N}$ ea sequence in $\mathbb{R}$ (instead of $\mathbb{C}$). In this case, $\sigma(T) \subset [-\|T\|, \|T\|]$. Either $\|T\|$ or $-\|T\|$ is, in this case, an eigenvalue of T. The fact that the eigenvalues are real follows because, for $\lambda \in \mathbb{C}$ an eigenvalue of T and $x \neq 0$ a corresponding eigenvector, we find

$$\lambda\langle x, x\rangle = \langle x, Tx\rangle = \langle Tx, x\rangle = \overline{\lambda}\langle x, x\rangle$$

Hence $\lambda = \overline{\lambda}$, and thereofer $\lambda \in \mathbb{R}$.

- It is possible to prove a spectral theorem for more general, nor necessarily compact (and not necessarily bounded), self-adjoint (or normal) operators. In general, the spectrum will not consists only of eigenvalues; it will also contain a continuous part. For a self-adjoint operator T, we can define a spectral measure $dE_T(\lambda)$ such that $T = \int \lambda dE_T(\lambda)$. For compact operators, $dE_T(\lambda) = \sum_{k\in N} \delta(\lambda - \lambda_k) P_{E_k} d\lambda$, where $E_k = \ker\,(\lambda_k - T)$.