1 Generalization of Hahn-Banach

Let X be a vector space over $\mathbb{R}$ and $Y \subset X$ a linear subspace. Let $p : X \to \mathbb{R}$ be a sublinear functional and $f : Y \to \mathbb{R}$ linear with $f \leq p$ on Y.

Consider now $G \subset \mathcal{L}(X) = \mathcal{L}(X, X)$ a subset of bounded linear operators with the properties that $\mathrm{id}_X \in G$ and for all $A, B \in G$, $AB \in G$ and moreover $AB = BA$. Assume that for all $A \in G$ we have $p(Ax) \leq p(x)$ for all $x \in X$, $Ay \in Y$ and $f(Ay) = f(y)$ for all $y \in Y$.

Claim: There exists $F : X \to \mathbb{R}$ linear with $F_{|Y} = f, F \leq p$ on X and $F(Ax) = F(x)$ for all $x \in X$ and $A \in G$.

Proof. Given the hint to consider $q(x) := \inf_{A_1,\dots,A_n} \frac{1}{n} p(A_1x + \cdots + A_nx)$ for $x \in X$. Here the infimum is taken over finitely many $A_i \in G$. Then it is clear that q is sublinear because all the $A_i : X \to X$ are linear and p itself is sublinear.

Moreover for $y \in Y$ and $A_1, \dots, A_n \in G$ where $n \in \mathbb{N}$ we have

$$f(A_1y + \cdots + A_ny) = f(A_1y) + \cdots + f(A_ny) = nf(y)$$

But also $f(A_1y+\cdots+A_ny) \leq p(A_1y+\cdots+A_ny)$ by assumption and thus we have $f(y) \leq \frac{1}{n}p(A_1y+\cdots+A_ny)$ for all $A_1, \dots, A_n \in G$. Taking the infimum we conclude that $f(y) \leq q(y)$ on Y.

By **Hahn-Banach Theorem** there exists $F : X \to \mathbb{R}$ linear with $F_{|Y} = f$ and $F(x) \leq q(x)$ for all $x \in X$. However since we also have that $p(A_1x + \cdots + A_nx) \leq p(A_1x) + \cdots + p(A_nx) \leq np(x)$ it also follows that $q \leq p$ on X which takes care of the first part of the claim.

Let now $A_1, \dots, A_n \in G$ be arbitrary, then we have that

$$\begin{aligned} q(Ax - x) &\leq \frac{1}{n} p(A_1(Ax - x) + \cdots + A_n(Ax - x)) \\ &= \frac{1}{n} p(A_1Ax + \cdots + A_nAx - (A_1x + \cdots + A_nx)) \\ &= \frac{1}{n} p(AA_1x + \cdots + AA_nx - (A_1x + \cdots + A_nx)) \end{aligned}$$

If we now apply this to the special case where $A_i = A^{i-1} \in G$ for $i = 1, \dots, n$ where $A^0 = id_X \in G$ we obtain from the emerging telescoping sum that for all $n \in \mathbb{N}$ we have

$$q(Ax - x) \leq \frac{1}{n} p(A^n x - x) \leq \frac{1}{n}(p(A^n x) + p(-x)) \leq \frac{1}{n}(p(x) + p(-x))$$

Passing this statement to the limit as $n \to \infty$ we obtain that $q(Ax - x) \leq 0$. This implies that $F(Ax - x) \leq 0$ because $F \leq q$ on X. Thus we've got the inequality

$$F(Ax) \leq F(x)$$

But since both F and $A \in G$ are linear we obtain that

$$-F(Ax) = F(-Ax) = F(A(-x)) \leq F(-x) = -F(x)$$

Multiplying by (-1) we get that $F(Ax) \geq F(x)$ and thus $F(Ax) = F(x)$. $\square$

2 Reflexivity

Let X and Y be Banach spaces with an isometric linear map $f : X \to Y^*$ such that $f^* : Y^{**} \to X^*$ is also isometric. Moreover, let X be reflexive. Show that there exists isometric isomorphisms $Y \cong X^*$ and $X \cong Y^*$.

Proof. First consider the diagram below, seriously, it's pretty sweet.

$$\begin{tikzcd}
Y^* \arrow[d, hook, "J_{Y^*}"'] & X \arrow[l, hook', "f"'] \arrow[r, "g"] \arrow[d, hook, two heads, "J_X"'] & Y \arrow[d, hook, "J_Y"] & \\
Y^{***} & X^{**} \arrow[l, "f^{**}"'] \arrow[r, "g^{**}"] & Y^{**} \arrow[r, hook, "f^*"] & X^*
\end{tikzcd}$$

In the previous exercise sheet I have already shown that the above diagram commutes, i.e. we have $g^{**} \circ J_X = J_Y \circ g$ under the assumption that g is a linear map. *(A similar proof can be found in the proof of Thm 4.3.3.)*

Now we recall that J_X, J_Y are linear isometries and quite generally, the composition of (linear) isometries is again an (linear) isometry. Moreover, isometries are always continuous and injective. Finally, if we have a bijective isometry, then quite trivially the inverse of said isometry is again an isometry.

Recall from **Thereom 4.3.4.** That for a Banach Space X we have X is reflexive if and only if X^* is reflexive. We will need that later.

Also recall from **Exercise Sheet 6 Exercise 5** that if X, Y are Banach spaces and $T : X \to Y$ is a continuous isomorphism then X is reflexive if and only if Y is reflexive.

Claim 1: Let $f : X \to Y$ be an linear, continuous and injective map, then $f^* : Y^* \to X^*$ is surjective.

Proof of Claim 1: Let $x^* \in X^*$ be arbitrary. Since f is linear $\mathrm{Im}(f) \subset Y$ is a linear subspace. We define

$$\Psi : \begin{cases} \mathrm{Im}(f) & \longrightarrow \mathbb{K} \\ f(x) & \longmapsto \Psi(f(x)) = x^*(x) \end{cases}$$

We notice that Ψ is well defined because f is assumed to be injective, in particular f has a 1 to 1 correspondence to its image. Clearly Ψ is linear, because both f and x^* are linear. Further we have

$$\|\Psi(f(x))\| \leq \|x^*\|\|x\| \text{ for all } x \in X$$

Hence we have that Ψ is continuous and thus $\Psi \in \mathrm{Im}(f)^*$. By **Corollary 4.2.7. to Hahn Banach** we have that Ψ extends continuously and linearly on Y i.e. $\Psi \in Y^*$. But then by definition of the adjoint we have that

$$f^*(\Psi) = \Psi \circ f = x^*$$

which entails that f^* is surjective as claimed. □

Applying this result now to our $f : X \to Y^*$ which is a linear isometry we obtain that $f^* : Y^{**} \to X^*$ is surjective. Moreover, by assumption f^* is an linear isometry we now have that f^* is in fact an continuous isomorphism between Y^{**} and X^* and since X^* is reflexive, so is Y^{**}.

Since by assumption f^* is an linear isometry (in particular injective) we also get that $f^{**} : X^{**} \to Y^{***}$ is surjective.
But now we have that $J_{Y^*} \circ f = f^{**} \circ J_X : X \to Y^{***}$ is surjective as the composition of surjective maps *(recall that X is reflexive by assumption)*, thus J_{Y^*} has to be surjective too. This entails that Y^* is reflexive and this is the case if and only if Y is reflexive, i.e. J_Y is an isometric isomorphisms.

This now shows that $f^* \circ J_Y : Y \to X^*$ is an isometric isomorphism.

In order to show that $X \cong Y^*$ we start with another claim.

Claim 2: Let $f : X \to Y$ be an isometric isomorphism. Then $f^* : Y^* \to X^*$ is a linear isometry. In Particular by the first **Claim 1** it follows that f^* is an isometric isomorphism.

Proof of Claim 2: We have by definition

$$\|f^*(y^*)\|_{X^*} = \|y^* \circ f\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |y^*(f(x))|$$

Since $f : X \to Y$ is an isomorphism we can find for all $y \in Y$ with $\|y\| \leq 1$ an $x \in X$ such that $f(x) = y$, but then we have because f is also isometric that

$$\|y\|_Y = \|f(x)\|_Y = \|x\|_X \leq 1$$

Similarly if $x \in X$ such that $\|x\| \leq 1$ then also $\|f(x) = y\| = \|x\| \leq 1$. Thus we obtain that

$$\|f^*(y^*)\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |y^*(f(x))| = \sup_{\substack{y \in Y \\ \|y\| \leq 1}} y^*(y)| = \|y^*\|_{Y^*}$$

That is f^* is an isometry $\square$

Now if we bring this all together, we have that $J_X : X \to X^{**}$ and $J_{Y^*} : Y^* \to Y^{**}$ are isometric isomorphisms, moreover, by our efforts above we have that $f^{**} : X^{**} \to Y^{***}$ is an isometric isomorphism.

Remark: Alternatively we could also have said that because $f^{**} \circ J_X = J_{Y^*} \circ f$, it follows immediately that f^{**} is an isometry, because $f^{**} = J_{Y^*} \circ f \circ (J_X)^{-1}$ is the composition of linear isometries.

We conclude that $(J_{Y^*})^{-1} \circ f^{**} \circ J_X : X \to Y^*$ is an isometric isomorphism, that is $X \cong Y^*$ which concludes the proof. $\square$

3 Hellinger-Toeplitz Theorem

Let H be a Hilber space and $A : H \to H$ linear and symmetric, i.e.

$$\langle y, Ax \rangle = \langle Ay, x \rangle \text{ for all } x, y \in H.$$

Show that then A is bounded.

Proof. We first recall **Theorem 5.1.4.**

Theorem 5.1.4. (Banach-Steinhaus): Let X be a Banach space, Y a normed space and $\mathcal{F} \subset \mathcal{L}(X, Y)$. Assume that for all $x \in X$ there exists $c_x \geq 0$ such that

$$\sup_{T \in \mathcal{F}} \|Tx\| \leq c_x$$

Then there exists $c \geq 0$ with

$$\sup_{T \in \mathcal{F}} \|T\| \leq c$$

We take $X = H$ (recall that every Hilbert Space is also a Banach Space) and $Y = \mathbb{K}$. For $y \in H$ we define

$$f_y : H \to \mathbb{K}, \; f_y(x) := \langle Ay, x \rangle$$

Then we have that f_y is linear because by our definition of the scalar product we have that $\langle \cdot, \cdot \rangle$ is linear in its second argument. Moreover the function f_y is continuous, because it's defined as an inner product. Thus we have that $f_y \in \mathcal{L}(H, \mathbb{K})$.

Let us now define

$$\mathcal{F} := \{f_y \in \mathcal{L}(X, \mathbb{K}) \mid \|y\| = 1\} \subset \mathcal{L}(X, \mathbb{K})$$

Let now $f_y \in \mathcal{F}$ be arbitrary. Then for $x \in H$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned} \|f_y(x)\| = |\langle Ay, x \rangle| = |\langle y, Ax \rangle| \leq \|y\| \|Ax\| = \|Ax\| =: c_x \\ \implies \sup_{f_y \in \mathcal{F}} \|f_y(x)\| \leq c_x \end{aligned}$$

By **Banach-Steinhaus** there exists $c \geq 0$ such that $\sup_{f_y \in \mathcal{F}} \|f_y\| \leq c$

We now have established the existence of a constant $c \geq 0$ such that

$$\sup_{f_y \in \mathcal{F}} \sup_{\|x\|=1} \|f_y(x)\| \leq c$$

Now we obtain that

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \|x\| \langle A(x/\|x\|), Ax \rangle = \|x\| f_{x/\|x\|}(Ax)$$

We have for $y = x/\|x\|$ that $\|y\| = 1$ and we use the bound that we established through the Banach-Steinhaus theorem, i.e.

$$f_{x/\|x\|}(Ax) \leq c\|Ax\|, \text{ where } c = \sup_{f_y \in \mathcal{F}} \|f_y\|$$

Thus we have

$$\|Ax\|^2 \leq c\|x\|\|Ax\| \implies \|Ax\| \leq c\|x\|$$

That is A is bounded. $\square$

4 Exercise 5: Bilinear functionals

Let X be a normed vector space over $\mathbb{K}$. A bilinear functional on X is a map $B : X \times X \to \mathbb{K}$ such that for all $x, y \in X$ the maps $B(x, \cdot) : X \to \mathbb{K}$ and $B(\cdot, y) : X \to \mathbb{K}$ are linear functionals on X.

a) Let X be a Banach space and B a bilinear functional on X which is continuous in each variable separately, i.e. for every fixed $x, y \in X$, the maps $B(x, \cdot)$ and $B(\cdot, y)$ are continuous. Show that there exists a constant $C > 0$ such that $|B(x, y)| \leq C\|x\|\|y\|$ for all $x, y \in X$. Conclude that B is continuous with respect to the norm $\|(x, y)\| := \|x\| + \|y\|$ on $X \times X$.

Proof. We define

$$\mathcal{F} := \{B(\cdot, y) : \|y\| = 1\}$$

By definition $B(\cdot, y)$ is linear because it's a bilinear functional. Further by assumption it is continuous in it's first argument, thus we have for all $x \in X$ that

$$|B(x, y)| \leq c_y\|x\|$$

Hence we indeed have that $\mathcal{F} \subset \mathcal{L}(X, \mathbb{K})$. Further, because B is also linear in its second argument we also have for any $x \in X$ and any $y \in X$ with $\|y\| = 1$ that

$$|B(x, y)| \leq c_x\|y\| = c_x$$

Thus we conclude that

$$\sup_{T \in \mathcal{F}} |T(x)| \leq c_x$$

Thanks to Banach-Steinhaus we now know that there exists a constant $c \geq 0$ such that

$$\sup_{T \in \mathcal{F}} \|T\| \leq c$$

Let now $x, y \in X$ be arbitrary.

$$\begin{aligned} |B(x, y)| &= \|x\|\|y\||B(x/\|x\|, y/\|y\|)| \leq \|x\|\|y\| \sup_{\|\xi\|=1} |B(\xi, y/\|y\|)| \\ &\leq \sup_{\|\zeta\|=1} \sup_{\|\xi\|=1} |B(\xi, \zeta)| = \|x\|\|y\| \sup_{\|\zeta\|=1} |B(\cdot, \zeta)| \leq c\|x\|\|y\| \end{aligned}$$

Which takes care of the first part of the claim.

Next we need to show that with respect to the norm $\|(x, y)\| := \|x\| + \|y\|$ on $X \times X$ the function B is continuous.

To this extent consider the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ and assume that this sequence converges to $(x, y) \in X \times X$. Then by definition of the norm on $X \times X$ we have that

$$0 \overset{n\to\infty}{\longleftarrow} \|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\|$$

Which shows that $x_n \to x$ and $y_n \to Y$ in X. In particular the sequence y_n is bounded as a convergent sequence in X, i.e. there exists $M \geq 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Finally we obtain that

$$\begin{aligned}
|B(x_n, y_n) - B(x, y)| &= |B(x_n, y_n) - B(x, y_n) + B(x, y_n) - B(x, y)| \\
&= |B(x_n - x, y_n) + B(x, y_n - y)| \\
&\leq |B(x_n - x, y_n)| + |B(x, y_n - y)| \\
&\leq c\|x_n - x\|\|y_n\| + c\|x\|\|y_n - y\| \\
&\leq cM\|x_n - x\| + \|x\|\|y_n - y\| \to 0 \text{ as } n \to \infty
\end{aligned}$$

This shows that that B is continuous. □

b) Let $\mathcal{P}$ be the vector space of real polynomials in one variable, equipped with the norm $\|p\| = \int_0^1 |p(t)|dt$ for $p \in \mathcal{P}$. Let

$$B(p, q) = \int_0^1 p(t)q(t)dt$$

Show that B is a (real valued) bilinear functional on $\mathcal{P}$ which is continuous variable separately, but that B is not continuous on $\mathcal{P} \times \mathcal{P}$.