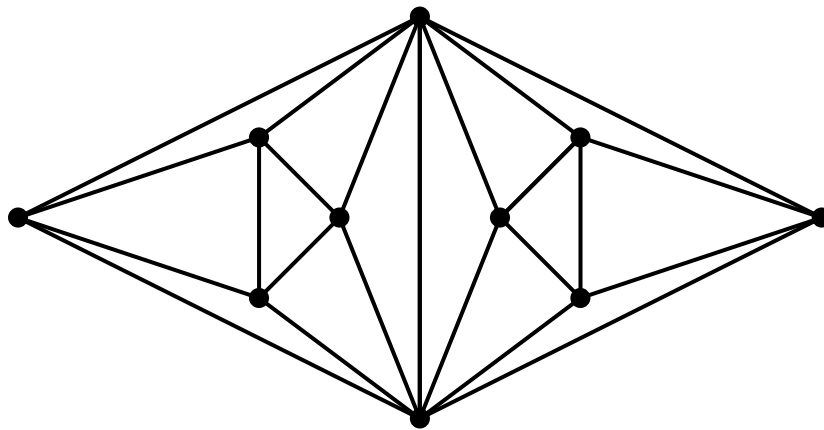


UNIVERSITY OF ZURICH



Graph Theory

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LaTeX by
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Graph Theory

Marco Bertenghi

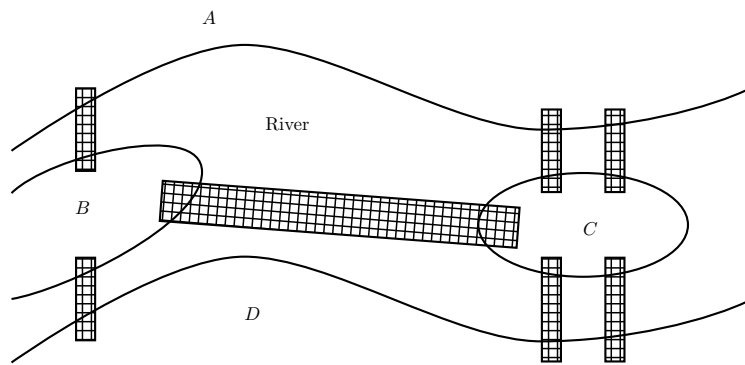


Figure 1: Königsberg's bridge problem

Königsberg's bridges problem: Is it possible to walk through town, starting and ending at the same point, so that we use each bridge exactly once? This problem, solved by Euler, is considered as the starting point of graph theory.

1 Basic concepts

Definition 1.1. A **graph** G is a pair $G = (V(G), E(G))$, or more simply $G = (V, E)$ where V is a finite set and E is a finite set, possibly empty, endowed with an application $f : E \rightarrow P_2(V) \cup P_1(V)$ where $P_2(V)$ denotes all subsets of V with two elements, and $P_1(V)$ all subsets containing only a single element of V . The elements of V are called **vertices**, the elements of E are called **edges**. The cardinality of V is called the **order** of G . The **size** of the graph is the cardinality of the edge set E . For $f(e) = \{u, v\}$, we say that u and v are **endvertices** of e and that the vertices u and v are **adjacent**. A **loop** is an edge e such that $f(e) \in P_1(V)$. If $f^{-1}(\{u, v\})$ is of cardinality 1, then we say that $f^{-1}(\{u, v\})$ is a **simple edge**. If $f^{-1}(\{u, v\})$ is of cardinality ≥ 2 , then we say that $f^{-1}(\{u, v\})$ is a **multiple edge**. A vertex which is the endpoint of no edge is called **isolated**.

Example 1.1.

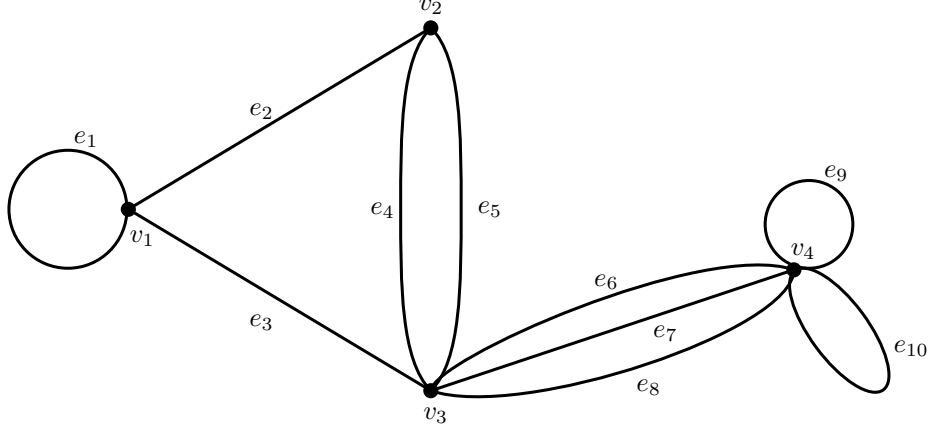


Figure 2: $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{e_1, e_2, e_3, \dots, e_{10}\}$

$f(e_1) = \{v_1\}$, $f(e_2) = \{v_1, v_2\}$, $f(e_3) = \{v_1, v_3\}$, $f(e_4) = f(e_5) = \{v_2, v_3\}$, $f(e_6) = f(e_7) = f(e_8) = \{v_3, v_4\}$, $f(e_9) = f(e_{10}) = \{v_4\}$. e_2 and e_3 are simple edges. $\{e_4, e_5\}$ and $\{e_6, e_7, e_8\}$ are multiple edges. e_1 is a simple loop. $\{e_9, e_{10}\}$ are multiple loops. v_5 is an isolated vertex.

Definition 1.2. A graph $G = (V; E)$ is called **simple** if it has no loops and no multiple edges. In this case, if $E \neq \emptyset$, $f : E \rightarrow P_2(V)$ is an injection and we identify e with $f(e)$. For notational convenience, instead of representing an edge as $\{u, v\}$, we denote it by uv .

Remark 1.1.

1. In a simple graph G with n vertices and m edges, we have

$$m \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$

2. In the literature, and in what follows, a simple graph is simply called a graph. Then a graph with multiple edges is called a **multigraph** and a graph with loops is called a **pseudograph**.

If in a graph we decide that the order of the vertices matters, then we make the following definition:

Definition 1.3. A **directed graph** (or **digraph**) is a pair $G = (V(G), A(G))$ or more simply $G = (V; A)$ where V is a non empty finite set and where A is a set, possibly empty, endowed with an application $f : A \rightarrow V \times V$.

This time we have $f(a) = (u, v)$ which is in general different from (v, u) . A directed graph is usually represented as follows:

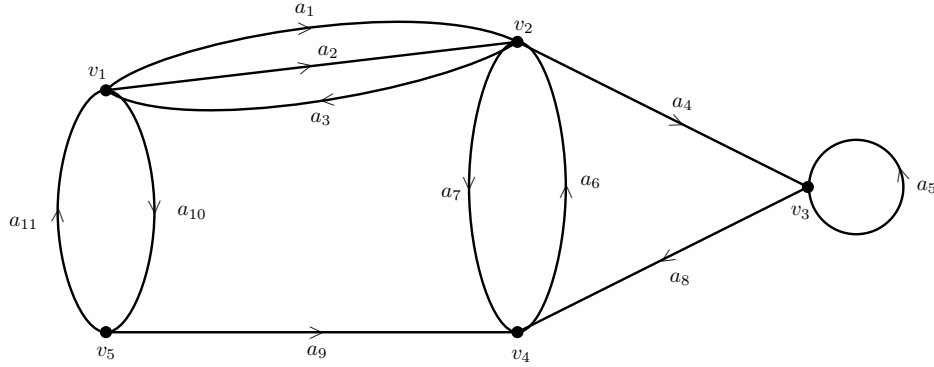


Figure 3: A directed graph.

Remark 1.2. Allowing edges to be arbitrary subsets of vertices (rather than just singletons or pairs) gives us a **hypergraph**:

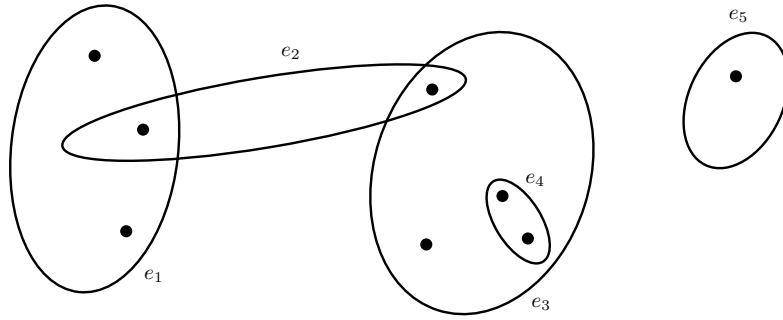
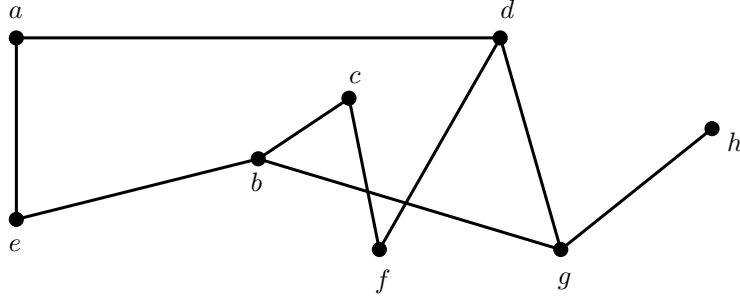


Figure 4: A hypergraph.

Definition 1.4. The **neighborhood** of a vertex v , denoted by $N(v)$, is the set of vertices adjacent to v : $N(v) = \{x \in V, vx \in E\}$. The **closed neighborhood** of a vertex v , denoted by $N[v]$, is the set $\{v\} \cup N(v)$. Given a set S , we define the neighborhood of S , denoted by $N(S)$, to be the union of neighborhoods of the vertices in S , i.e. $N(S) := \bigcup_{v \in S} N(v)$. Similarly we define the closed neighborhood of S , denoted by $N[S]$, as $S \cup N(S)$. The **degree** of v , denoted by $\deg(v)$ or $d(v)$, is the number of edges incident with v . In simple graphs we have $d(v) = |N[v]|$. The **maximum degree** of a graph G , denoted by $\Delta(G)$, or Δ , is defined as $\Delta(G) = \max\{d(v), v \in V(G)\}$. Similarly the **minimum degree** of a graph G , denoted by $\delta(G)$, is defined as $\delta(G) = \min\{d(v), v \in V(G)\}$. The **degree sequence** of a graph G of order n is the n -term sequence (usually written in increasing order) of the vertex degrees.

Example 1.2.



$N(d) = \{a, fg\}$, $N[d] = \{a, d, f, g\}$, $\Delta(g) = 3$, $\delta(G) = 1$. Degree sequence: $(1, 2, 2, 2, 2, 3, 3, 3)$.

Theorem 1.1. In a (loopless) graph, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently the number of vertices with odd degree is even.

Proof. In $\sum_{v \in V} d(v)$, we count each edge exactly twice. This entails that $\sum_{v \in V} d(v) = 2|E|$. \square

Definition 1.5. A sequence $D = (d_1, \dots, d_n)$ with $d_1 \leq d_2 \leq \dots \leq d_n$, is called **graphical** if there exists a graph G on n vertices such that D is its degree sequence.

Theorem 1.2. Let n be an integer ≥ 1 . Let $D = (d_1, \dots, d_n)$ be a sequence of integers such that $d_1 \leq d_2 \leq \dots \leq d_n$. There exists a graph G on n vertices such that D is its degree sequence iff the sequence $D' = (d'_1, \dots, d'_{n-1})$ defined by

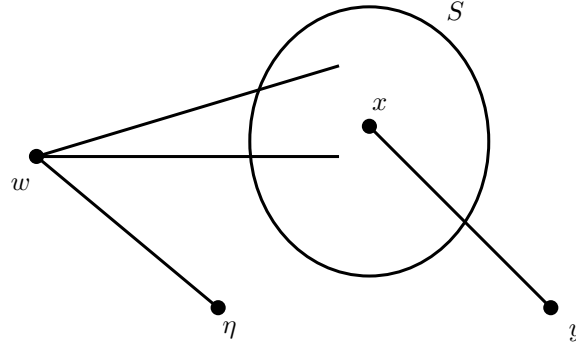
$$d'_i = \begin{cases} d_i, & \text{if } 1 \leq i < n - d_n \\ d_i - 1, & \text{otherwise} \end{cases}$$

is a graphical sequence.

Proof. (\Leftarrow) Let $D = (d_1, \dots, d_n)$ be such that D' is a graphical sequence. Then there exists a graph G' on $n - 1$ vertices such that D' is its degree sequence. Let us add a vertex v to G' , adjacent to the vertices of degree $d_{n-d_n} - 1, \dots, d_{n-1} - 1$. Thus the degree sequence of G is $(d'_1, \dots, d'_{n-d_n-1}, \dots, d_{n-1}, d_n) = (d_1, \dots, d_n)$.

(\Rightarrow) Let $G = (V, E)$ be a graph with degree sequence $D = (d_1, \dots, d_n)$. Note that d_n is the maximal degree of G . Let w be a vertex of G such that

$d(w) = d_n = \Delta$. Let S be the set of n vertices of degree d_{n-d_n}, \dots, d_n . If $N(w) = S$, then we can remove w and its adjacent edges and we obtain a graph G' whose degree sequence is D' . Let us assume that $N(w) \neq S$. As $|N(w)| = |S|$, there exists $x \in S \setminus N(w)$ and $\eta \in N(w) \setminus S$. It follows from the choice of S that $d(x) \geq d(\eta)$. Let us now note that there is a vertex $y \notin \{x, \eta, w\}$ such that y is adjacent to x , but not to η . Indeed, the number of neighbors of x , excluding η , is $d(x) = \epsilon$ where $\epsilon = 1$ if x is adjacent to η and 0 otherwise. The number of neighbors of η other than x and w is $d(\eta) = 1 - \epsilon$. Note that we have the existence of y because $d(x) \geq d(\eta)$.



We then consider the graph $G_1 = (V, E \setminus \{\{w, \eta\}, \{x, y\}\} \cup \{\{w, x\}, \{\eta, y\}\})$. We haven't changed the degree sequence but increased by 1 $|N(w) \cap S|$. Continuing this process, we obtain a graph with degree sequence D and $N(w) = S$. \square

Example 1.3. *Prove that $(1, 1, 1, 2, 2, 2, 3, 3, 3)$ is a graphical sequence. Let us apply the previous theorem.*

$$\begin{array}{lll}
\rightarrow (1, 1, 1, 2, 2, 1, 2, 2) & \text{that we rewrite} & (1, 1, 1, 1, 2, 2, 2, 2). \\
\rightarrow (1, 1, 1, 1, 2, 1, 1) & \text{that we rewrite} & (1, 1, 1, 1, 1, 2). \\
\rightarrow (1, 1, 1, 1, 0, 0) & \text{that we rewrite} & (0, 0, 1, 1, 1, 1). \\
\rightarrow (0, 0, 1, 1, 0)
\end{array}$$

where $(0, 0, 1, 1, 0)$ is simply the degree sequence of the empty graph on 3 vertices or a path on 2 vertices.

We now wish to introduce the concept of graph isomorphism. For this we introduce a few more definitions.

Definition 1.6. *Pairwise non-adjacent vertices or edges are called **independent**. More generally a set of vertices or edges is independent if no two of its elements are adjacent. Independent sets of vertices are also called **stable**.*

Definition 1.7. Let $G = (V, E)$ and $G' = (V', E')$ be two (simple) graphs. A map $f : V \rightarrow V'$ is a **homomorphism** from G to G' if it preserves the adjacency of vertices, that is $\{x, y\} \in E \rightarrow \{f(x), f(y)\} \in E'$. Then in particular for every $x' \in f(V)$, $f^{-1}(\{x'\})$ is an independent set of vertices in G . If φ is bijective and its inverse f^{-1} is also a homomorphism, so that for all $x, y \in V$ we have $xy \in E \iff f(x)f(y) \in E'$, then we call f an **isomorphism**, we say that G and G' are **isomorphic**, and write $G \simeq G'$. An isomorphism from G to itself is an **automorphism** of G . In general we do not distinguish between isomorphic graphs.

Remark 1.3. A class of graphs that are closed under isomorphisms is called a **graph property**. For example "containing a triangle" is a graph property. Indeed if G contains three pairwise adjacent vertices then so does every graph isomorphic to G .

Definition 1.8. A map taking graphs as arguments is called a **graph invariant** if it assigns equal values to isomorphic graphs. The number of vertices and the number of edges of a graph are two simple graph invariants. The degree sequence is another one.

Let us illustrate the above with examples.



Figure 5: $G \simeq G'$, $f(v_1) = w_1, f(v_2) = w_4, f(v_3) = w_2, f(v_4) = w_3$.

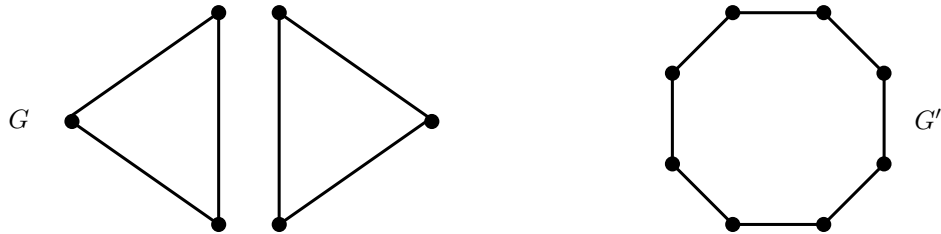


Figure 6: G and G' are not isomorphic.

G and G' are not isomorphic even though $|V(G)| = |V(G')|$ and $|E(G)| = |E(G')|$, and the degree sequence for both graphs is $(2, 2, 2, 2, 2, 2)$.

2 Special graphs

2.1 Complete graphs

A graph is said to be complete if every vertex is adjacent to every other vertex. A complete graph on n vertices is denoted by K_n , and sometimes called a n -clique.

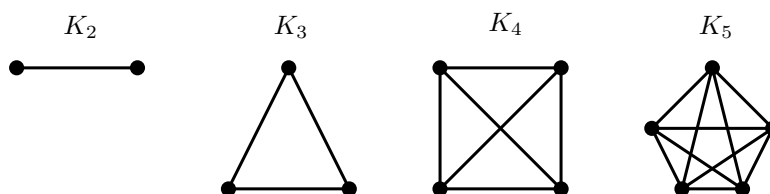


Figure 7: The figures above depict the complete graphs K_2, \dots, K_5 .

2.2 Empty graphs

The empty graph on n vertices, denoted by E_n , is the graph of order n where E is the empty set.

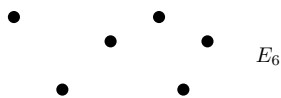


Figure 8: Empty graph on 6 vertices.

2.3 Complements

Given a graph G , the complement of G , denoted by \overline{G} , is the graph whose vertex set is the same as that of G , and whose edge set consists of all the edges that are **not** in G .

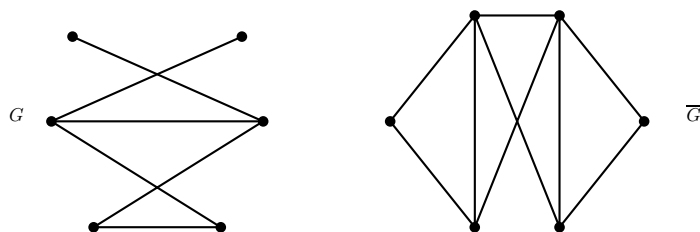


Figure 9: G and its complement \overline{G} .

2.4 Regular graphs

A graph G is regular if every vertex has the same degree. G is said to be regular of degree r or r -regular if $d(v) = r$ for all $v \in V(G)$. Complete graphs are regular of degree $n - 1$, and empty graphs are regular of degree 0.

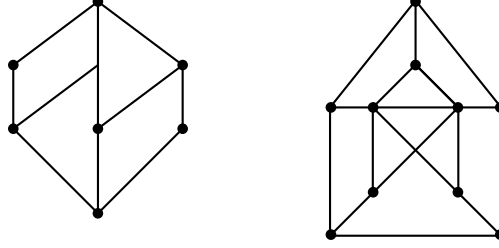


Figure 10: The picture on the right is a 3-regular Petarsen graph.

Since $\sum_{v \in V(G)} d(v) = 2|E|$, we have $k|V| = kn = 2|E|$. So both r and n cannot be odd.

Theorem 2.1. *Let r and n be integers, such that $0 \leq r \leq n - 1$. Then there exists an r -regular graph of order n iff at least one of the integers r or n is even.*

Proof. Let r and n be given, $0 \leq r \leq n - 1$, such that one of them is even. We are going to construct a graph $H_{n,r}$ of order n which is r -regular. Let us enumerate the vertices of $H_{n,r}$ as v_0, \dots, v_{n-1} and let us put them in order on a circle. If r is even, we note $r = 2k$ for some integer k . For each vertex v_i , we draw an edge with v_{i+2}, \dots, v_{i+k} and v_{i-k}, \dots, v_{i-2} ($i+1, \dots, i+k, i-1, \dots, i-k$ are defined modulo n). $H_{n,r}$ is then r -regular. Now let us assume that r is odd, and n is even. Let $r = 2k - 1$ and $n = 2l$. As before, we draw an edge between the k following vertices as well as the k preceding vertices, and draw an additional edge to v_{i+l} . The graph $H_{n,r}$ obtained this way is r -regular.

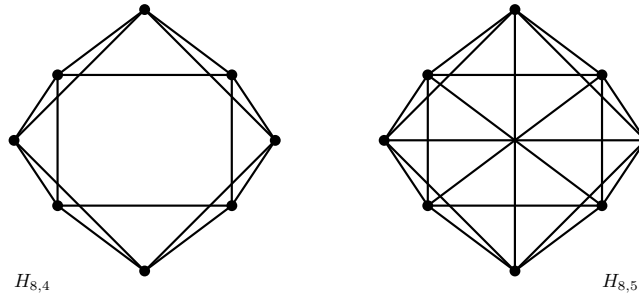


Figure 11: The Harary's graphs $H_{8,4}$ and $H_{8,5}$.

□

2.5 Cycles

For $n \geq 3$, the n -cycle, noted C_n is the (simple) graph of order n such that $V(C_n) = \{v_1, \dots, v_n\}$ and $E(C_n) = \{\{v_i, v_{i+1}\}, i = 1, \dots, n-1\} \cup \{v_n, v_1\}$.

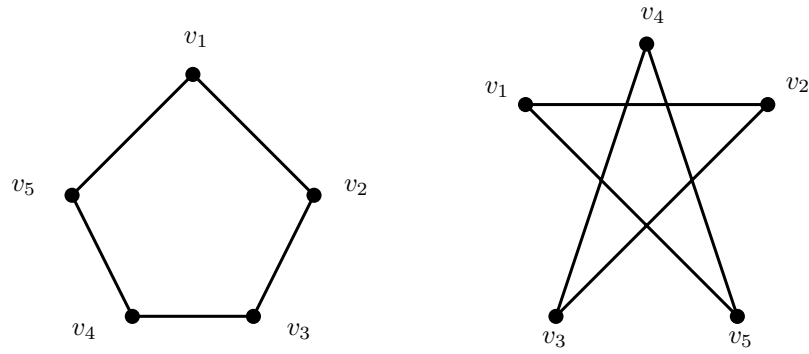


Figure 12: Cycles of order 5.

2.6 Paths

For $n \geq 1$, the graph P_n , called n -path, is the graph of order $n+1$ such that $V(P_n) = \{v_0, v_1, \dots, v_n\}$, $E(P_n) = \{\{v_i, v_{i+1}\}, i = 0, \dots, n-1\}$.

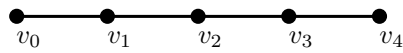


Figure 13: The path P_4 .

2.7 Subgraphs

A graph H is a **subgraph** of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. in this case we write $H \subset G$ and we say that G contains H . In a graph where vertices and edges are unlabelled, we say that $H \subset G$ if the vertices, could be labelled in such a way that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

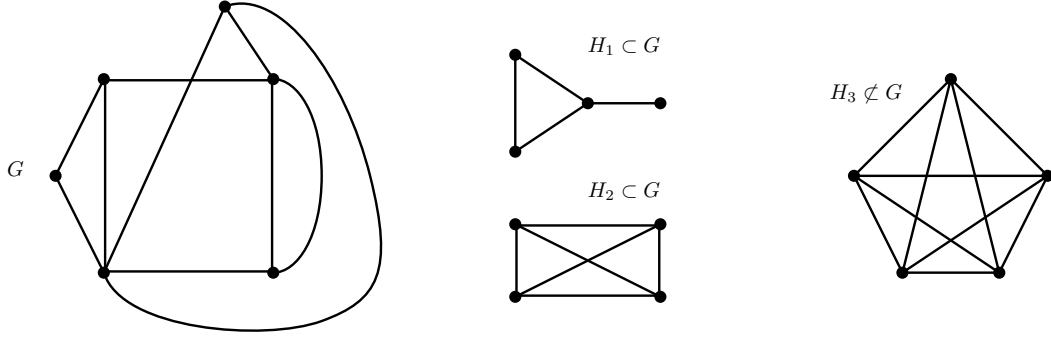


Figure 14: The graph G with its subgraphs H_1, H_2 and a non-subgraph H_3 .

2.8 Induced Subgraphs

Given a graph G and a subset S of the vertex set, the **subgraph of G induced by S** , denoted by $\langle S \rangle$, is the subgraph with vertex set S and with edge set $\{uv, u \in S, v \in S, uv \in E(G)\}$. So $\langle S \rangle$ contains all vertices of S and all edges of G whose end vertices are both in S . Sometimes $\langle S \rangle$ is also denoted as $G[S]$.

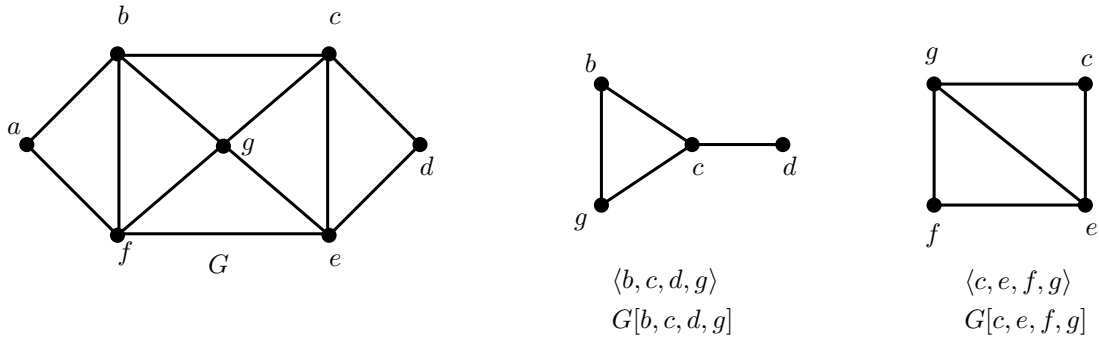


Figure 15: Example of induced subgraphs.

2.9 k -partite graphs

Let $G = (V, E)$ be a graph of order n and let $k \leq n$. The graph G is called k -partite if there exists a partition of $V = V_1 \cup \dots \cup V_k$ such that for every $1 \leq i \leq k$, $\langle V_i \rangle$ (or $G[V_i]$) is the empty graph. Such a graph is sometimes noted as $G = (V_1, \dots, V_k; E)$. When $k = 2$, the graph is called **bipartite**. Note that if G is k -partite, then it is also h -partite for $k \leq h \leq n$.

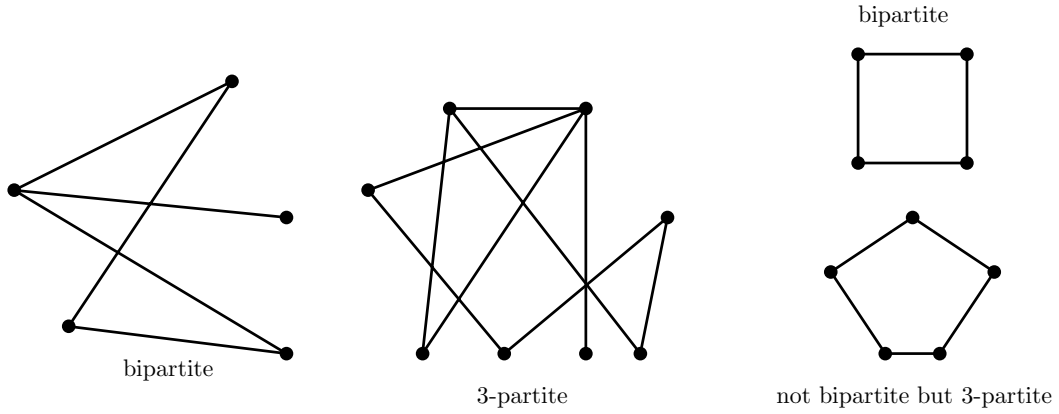


Figure 16: Example of bipartite and 3-partite graphs.

A k -partite graph $G(V_1, \dots, V_k; E)$ is called complete k -partite graph if $\{xy\} \in E$ whenever $x \in V_i, y \in V_j, i \neq j$. If for all $1 \leq i \leq k, |V_i| = n_i$, then we note the graph K_{n_1, \dots, n_k} .

Example 2.1.

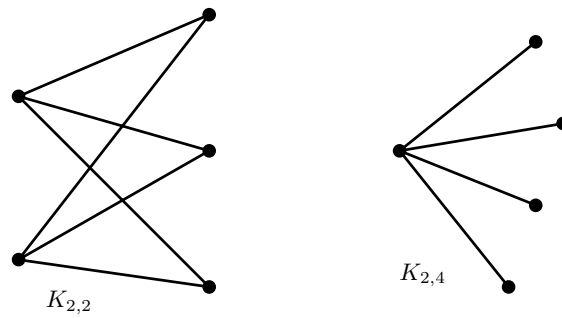


Figure 17: Complete k -partite graphs

3 Connectivity

We start this section with some fundamental definitions.

Definition 3.1. A **walk** in a graph is a sequence of (not necessarily distinct) vertices v_1, v_2, \dots, v_k such that $v_i v_{i+1} \in E$ for $i = 1, \dots, k-1$. If the vertices in a walk are distinct, then the walk is called a **path**. If the edges in a random walk are distinct, then the walk is called a **trail**. In this way, every path is a trail, but not every trail is a path. A **closed path** or **cycle** is a path v_1, \dots, v_k , $k \geq 3$, together with the edge $v_k v_1$. Similarly a trail that begins and ends at the same vertex is called a **closed trail**, or **circuit**. The **length** of a walk (or path, or trail, or cycle, or circuit) is its number of edges, counting repetitions.

Example 3.1.

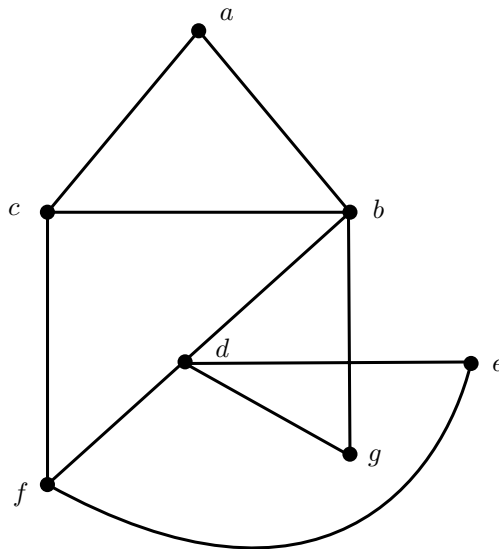


Figure 18: This figure is to help motivate the above fixed definitions.

- a, c, f, c, b, d is a walk of length 5.
- b, a, c, b, a is a trail of length 4.
- d, g, b, a, c, f is a path of length 5.
- g, d, b, c, a, b, g is a circuit.
- e, d, b, a, c, f, e is a cycle.

Theorem 3.1. *In a simple graph, every $u - v$ walk contains a $u - v$ path ($u \neq v$).*

First proof. Let W be a $u - v$ walk in G with maximal length. Then W is a path. Indeed, if this were not the case, and if x is a vertex which is repeated twice, by removing the walk between u and x we would obtain a shorter walk which contradicts the fact that W is of maximal length. \square

Second proof. Let W be a $u - v$ walk. We prove this theorem by induction on the length of W . If W is of length 1 or 2, then it is a path.

$< k \rightarrow k$. Say that W is $u = w_0, w_1, \dots, w_{k-1}, w_k = v$. If the vertices are distinct, then W itself is a $u - v$ path. If not, let j be the smallest integer such that $w_j = w_r$ for some $r > j$. Let W_j be the walk $u = w_0, w_1, \dots, w_j, w_{r+1}, \dots, w_k = v$. This walk has length $< k$ we can apply the induction assumption. \square

Corollary 3.1. *Let G be a simple graph and let u and v be two distinct vertices of G . The following are equivalent:*

1. *There exists a $u - v$ walk in G .*
2. *There exists a $u - v$ in G .*
3. *There exists a $u - v$ path in G .*

Definition 3.2. Given a graph G , and a vertex $v \in V(G)$, we let $G - v$ denote the graph obtained from G by removing v and all edges incident with v in G . If S is a set of vertices, we let $G - S$ denote the graph, obtained by removing all vertices of S and all associated incident edges of G . If e is an edge of G , then $G - e$ is the graph obtained by removing only the edge e . If T is a set of edges, then $G - T$ is the graph obtained by deleting each edge of T from G .

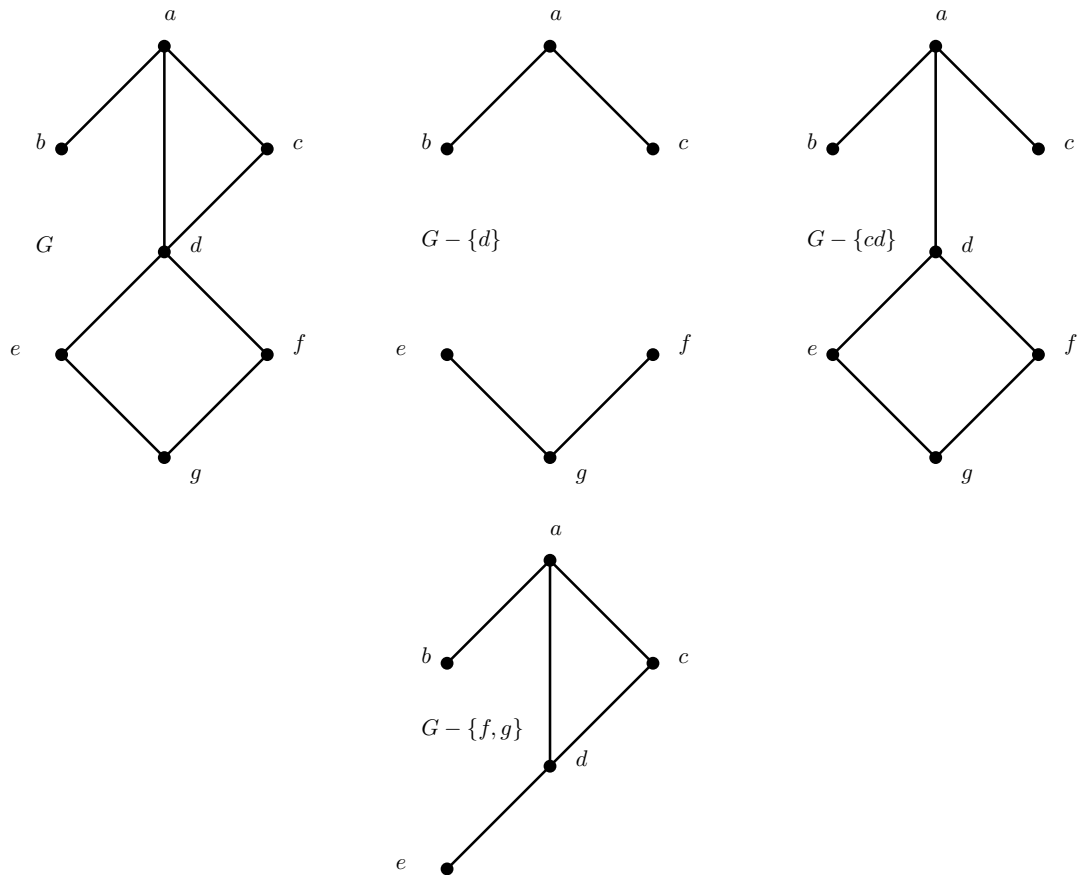


Figure 19: Example motivating the above definitions.

Definition 3.3. A graph G is **connected** if every pair of vertices can be joined by a path (or equivalently by a walk).

Proposition 3.1. Let G be a simple graph and let \sim be the binary relation defined on $V(G)$ by $u \sim v$ iff there exists a $u - v$ walk in G . We then have

1. \sim is an equivalence relation on $V(G)$.
2. The equivalence classes of $V(G)$ form a partition of $V(G)$ and the obtained subgraphs by these equivalence classes are the connected components of G .
3. Every connected component is a connected graph.
4. The graph G is connected iff it has a unique connected component.
5. If G is not connected and if u and v are in two distinct connected components, then there exists a $u - v$ walk.

Proposition 3.2. The vertices of a connected graph G can always be enumerated, say v_1, \dots, v_n , so that $G_i = G[v_1, \dots, v_i]$ is connected for every $i = 1, \dots, n$.

Proof. Pick any vertex as v_1 , and assume inductively that v_2, \dots, v_i have been chosen for some $i < |G|$. Now pick a vertex $v \in G - \{v_1, \dots, v_i\}$. As G is connected, it contains a $u - v_1$ path P . Choose as v_{i+1} the last vertex of P in $G - \{v_1, \dots, v_i\}$, then v_{i+1} has a neighbour in G_i . The results then follows by induction. \square

Definition 3.4. If the deletion of a vertex v from G causes the number of connected components of G to increase, then v is called a **cut vertex**. In the example above (see Figure 19) d is a cut vertex, whereas c is not. Similarly an edge e in G is said to be a bridge if the graph $G - e$ has more components than G . In the example of Figure 19, ab is the only bridge. A proper subset S of vertices of a graph G is called a **vertex cut** (or simply a **cutset**) if the graph $G - S$ is disconnected.

Definition 3.5. Let $G = (V, E)$ be a (simple) graph. Let $A, B \subset V$ and let $X \subset V$ or $X \subset E$, $X \neq V$ or X possibly empty. Assume that every $A - B$ path in G contains a vertex or an edge from X . Then we say that X **separates** the sets A and B in G . We say that X separates two vertices a and b if it separates the sets $\{a\}, \{b\}$ but $a, b \notin X$. We say that X separates G if X separates some two vertices in G . A separating set of vertices is called a **separator**.

Definition 3.6. G is called **k -connected** if $|G| > k$ and $G - X$ is connected for every set $X \subset V$ with $|X| < k$. In other words, no two vertices of G are separated by strictly less than k vertices. Equivalently, G is k -connected iff every separator of G has at least k -elements. Every graph is 0-connected and the 1-connected graphs are the non-trivial connected graphs. The greatest integer K such that G is k -connected is the **connectivity** $K(G)$ of G . If $|G| > 1$ and $G - F$ is connected for every set $F \subset E$ of fewer than λ edges, then G is called **λ -edge-connected**. The greatest integer λ such that G is λ -edge-connected is the **edge connectivity** $\lambda(G)$ of G . In particular, $\lambda(G) = 0$ iff G is disconnected.

Remark 3.1. A k -connected graph is also h -connected for $h \leq k$.

Example 3.2.

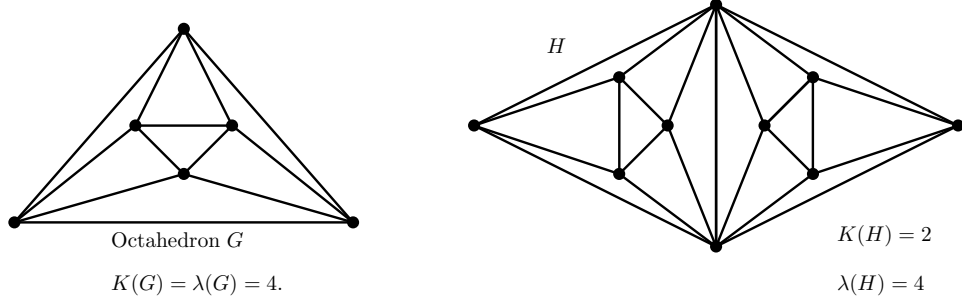


Figure 20: Introducing the connectivity constants K and λ .

Lemma 3.1. Let G be a k -connected graph, with $k \geq 1$ and let $e \in E(G)$. Then the graph $G - \{e\}$ is $(k - 1)$ -connected.

Proof. Note that $K(G - e) \leq K(G)$. If $K(G - e) = K(G)$, then the result is trivial. If $G = K_n$, then $K(G) = n - 1$ and $K(G - e) = n - 2 = K(G) - 1$ and the result follows again. Let us thus assume that $K(G - e) < K(G)$ and $G \neq K_n$. Then $K(G) \leq n - 2$ and $K(G - e) \leq n - 3$. Let S be a not separating set of $G - e$ with minimal cardinality, which is not separating for G . The graph $G - e \setminus S$ is not connected and each vertex of e belongs to different connected components of $G - e \setminus S$. But since $|S| \leq n - 3$, then one of the connected components has at least an element $\eta \neq x, \eta \neq y$, where $e = xy$. Let us assume that η and x are in the same connected component. Then $S \cup \{x\}$ is separating G . Thus $K(G) \leq K(G \setminus \{e\}) + 1$ so $K(G - e) \geq K(G) - 1 \geq k - 1$. \square

Proposition 3.3. *If G is a graph of order ≥ 2 , then we have $K(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. The only non trivial inequality to prove is $K(G) \leq \lambda(G)$. If G is not connected, then $K(G) = \lambda(G) = 0$. Let $n \geq 2$. If $G = K_n$, then $K(G) = \lambda(G) = n - 1$. Let us thus assume that G is connected with $G \neq K_n$. Let X be a minimal separating set of edges, $\lambda(G) = |X| \leq \delta(G) \leq n - 2$. Since X is minimal, $G - X$ has exactly two connected components, G_1 and G_2 , and all edges from X have one adjoint in G_1 and the other one in G_2 . Let $k = |E(G_1)| \geq 1$ and $n - k = |E(G_2)| \geq 1$. If every vertex of G_1 is adjacent to every vertex of G_2 , then $|X| = k(n - k) = (k - 1)(n - k - 1) + n - 1 \geq n - 1$ which contradicts an assumption that $|X| \leq n - 2$. Hence there exists a vertex $u \in V(G_1), v \in V(G_2)$ which are not adjacent. Let T be the set consisting of neighbours of u in G_2 and all vertices in G_1 which has a neighbour in G_2 . There is no $u - v$ walk in $G \setminus T$ by construction. Hence T is a separator of G . Thus $K(G) \leq |T| \leq |X| = \lambda(G)$. \square

Example 3.3. *The previous inequalities can be strict as shown in the following example:*

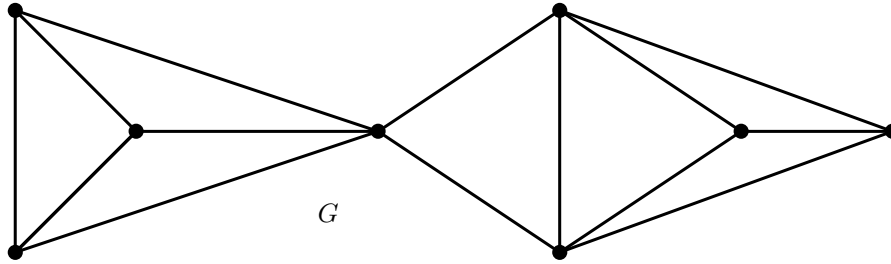


Figure 21: H is 1-connected with $K(G) = 1$, $\lambda(G) = 2$ and $\delta(G) = 3$.

Definition 3.7. *We say that two $(u - v)$ -walks are **internally disjoint** if u and v are the only common vertices of the two walks. If $W \subset V(G)$ separates the vertices u and v , then two $(u - v)$ -walks which are internally disjoint intersect W at a distinct vertices. Thus the number of (u, v) -walks that are internally disjoint is at most $|W|$.*

Theorem 3.2 (Menger). *Let u and v be two distinct and non adjacent vertices of a simple graph G . Then the minimal cardinality of a set of vertices that separates u and v is equal to the maximum number of internally disjoint (u, v) -walks.*

Corollary 3.2 (Whitney). *Let G be a simple graph, $|G| \geq 2$. The graph G is k -connected iff any two vertices of G can be formed by at least k walks which are internally disjoint. In particular, the graph G is 2-connected iff two arbitrary vertices of G belong to the same cycle.*

Proof. For $k = 1$ the result is immediate. Let us assume $k \geq 1$.

" \implies " Let G be k -connected and let u and v be two vertices of G . Let us first assume that u and v are not adjacent. Let W be a set of vertices that separates u and v , of minimal cardinality. We have $|W| \geq K(G) \geq k$. The result follows from Menger's theorem. Let us now assume that u and v are adjacent, and let $e = uv$. Then $G - e$ is $(k - 1)$ -connected. Let U be a (u, v) -separator of minimal cardinality in $G - e$. We then have $|U| \geq K(G \setminus \{e\}) \geq k - 1$. Hence $G \setminus \{e\}$ contains at least $k - 1$ (u, v) -walks which are internally disjoint. Then we add e and obtain the desired result for the graph G .

" \impliedby " Let us assume that for every $\{u, v\}$, vertices in G , G contains at least k (u, v) -walks that are internally disjoint. Let W be a separator of minimal cardinality. Then $|W| \geq K(G)$. Let x and y be two vertices belonging to distinct connected components of $G - W$. Then W is a (x, y) -separator and there are at least $k(x, y)$ -walks that are internally disjoint, thus $|W| \geq k$ and G is k -connected. \square

4 Counting in graphs

One can naturally ask, given a graph G , if it contains a cycle, a path of a certain length, some specific subgraphs, etc. We now give a few elementary results in this direction.

Theorem 4.1. *Let $k \in \mathbb{N}$. Let G be a simple graph such that $\delta(G) \geq k$. Then there is a path of length $\geq k$ in G .*

Proof. If $k = 0$, the result is clear. Let us thus assume $k > 0$. Let v_0, v_1, \dots, v_h be a path of maximal length h in G . Assume that $h < k$. We would then have $h < k \leq \delta(G) \leq d(v_0)$. Then there would exist a vertex w adjacent to v_0 which is not in G , contradicting the fact that was chosen to be of maximal length. \square

Theorem 4.2. *Let G be a simple graph such that $\delta(G) \geq 2$. Then G contains a cycle.*

Proof. Let $G = (V, E)$, such that $\delta(G) \geq 2$. If all edges are loops, then there is 1 cycle. Otherwise there exists $e \in E$ such that $e = uv$ and uv is a path. Let (v_0, \dots, v_h) be a path of maximal length h , $d(v_h) \geq 2$. There exists an edge $f \neq e_h = v_{h-1}v_h$ such that $f = v_hw$. The path v_0, \dots, v_h is of maximal length, thus $w \in \{v_0, \dots, v_{h-2}\}$. Let i be such that $w = v_i$. Then $(v_i, \dots, v_h, f, w = v_i)$ is a cycle. \square

Theorem 4.3. *Let $k \geq 2$ and let G be a simple graph such that $\delta(G) \geq 2$. Then G contains a cycle of length $\geq k + 1$.*

Proof. From Theorem 4.1, we know that G contains a path of length k . Let v_0, \dots, v_h be a path of maximal length in G . All neighbors of v_0 are necessarily in $\{v_1, \dots, v_h\}$. Let l be the largest index i such that v_i is a neighbor of v_0 . Then $k \leq d(v_0) \leq l$ and v_0v_1, \dots, v_lv_0 is a cycle of length $l + 1 \geq k + 1$. \square

Theorem 4.4. *Let G be a simple graph such that $\delta(G) \geq 3$. Then:*

1. *The graph G contains a cycle with two non consecutive vertices which are adjacent in G .*
2. *The graph G has a cycle of even length.*

Proof.

1. It follows from Theorem 4.1 that G contains a path of length ≥ 3 . Let v_0, v_1, \dots, v_h be a path of maximal length in G . Then all neighbors of v_0 are in $\{v_0, \dots, v_k\}$. But $d(v_0) \geq \delta(G) \geq 3$. Let us note $v_{i_1}, v_{i_2}, v_{i_3}$ three neighbors of v_0 , $1 \leq i_1 < i_2 < i_3 < k$. Then $C = v_0 v_1 \dots v_{i_1} \dots v_{i_2} \dots v_{i_3} \dots v_0$ is a cycle contained in G and (v_0, v_{i_1}) is an edge between two non-consecutive vertices in C .
2. With C as above, if it is of even length then were done. Otherwise, one of v_0, \dots, v_{i_3} or v_{i_3}, \dots, v_0 is of odd length and adding the edge $\{v_0, v_{i_2}\}$ makes it a cycle of even length.

□

Theorem 4.5. *Let G be a simple graph of order ≥ 2 . Then G is bipartite iff it does not contain a cycle/circuit of odd length.*

Proof. Let $G = (V_1, v_2; E)$ be a bipartite graph. If G does not contain a circuit, then it does not have a circuit of odd length. Otherwise let $(v_0, e_1, v_1, \dots, e_k, v_k = v_0)$ be a circuit of length k , with $v_0 \in V_2$. Then the vertices with even index belong to v_2 , and these with odd index to V_1 . In particular $v_k = v_0 \in V_2$ which shows that k is even.

For the converse, we proceed with an induction on the number of edges $m = |E(G)|$.

$P(m)$: every simple graph G of order ≥ 2 without circuits of odd length and with m edges is bipartite.

For $m = 0$, the graph G has no edge and is bipartite, i.e. $P(0)$ is true. Let $m \geq 1$ and assume that $P(m - 1)$ is true. Let G be a graph of order ≥ 2 , with m edges and without circuits of odd length.

Case 1: $\delta(G) = 1$. The graph G has at least one vertex u with degree 1. Let v be the unique neighbor of u and not $e = uv$. It follows from the induction assumption that $G - e$ is bipartite, with $V = V_1 \cup V_2$. Let us assume that $v \in V_1$. If $u \in V_2$, then the graph G is bipartite with the partition $V = V_1 \cup V_2$. If $u \in V_1$, then G is bipartite with the partition $V = (V_1 - \{u\}) \cup (V_2 \cup \{u\})$.

Case 2: $\delta(G) \geq 2$. The graph G contains a cycle $(v_0, e_1, \dots, e_k, v_k = v_0)$. By assumption this cycle is of even length, i.e. k is even. It follows from the

induction assumption that $G \setminus \{e_1\}$ is bipartite, with $V = V_1 \cup V_2$. $G \setminus \{e_1\}$ contains the path (v_1, \dots, v_k) . Let us assume that $v_1 \in V_1$. Then for i even we have $v_i \in V_2$, in particular $v_k \in V_2$ and G is thus bipartite with the partition $V = V_1 \cup V_2$. \square

A typical question in graph theory is the following one: Given a graph G of order n and $p \leq n$, how many p -cliques does G contain? Let us start with the case of $p = 3$, i.e. with the case of triangles.

Lemma 4.1. *Let H be a simple graph on $2m$ vertices, with $m \geq 2$, such that $|E(H)| \geq m^2 + 1$. Then H contains a triangle.*

Proof. We proceed with an induction on m . If $m = 2$, then H is a subgraph of K_4 with at least 5 edges. Theorem 4.5 shows that H is not bipartite if it contains an odd cycle. This odd cycle has to be a triangle as H has only 4 vertices.

$< m \rightarrow m$. Let u and v be two adjacent vertices of H . If $d(u), d(v) > 2m$, then $H - \{u, v\}$ will have at least $m^2 + 1 - (2m - 1)$ edges. But $m^2 + 1 - (2m - 1) = m^2 - 2m + 2 = (m - 1)^2 + 1$. By the induction assumption such a graph contains a triangle. \square

Theorem 4.6. *Let H be a simple graph on $2m$, with $m \geq 2$, vertices such that $|E(H)| \geq m^2 + 1$. Then H contains at least m triangles.*

Proof. Without loss of generality we can assume that $|E(H)| = m^2 + 1$ and we prove the theorem with an induction on m . For $m = 2$, H is a subgraph of H_4 with one edge missing. Therefore it contains two triangles.

$< m \rightarrow m$. Lemma 4.1 shows that H contains at least one triangle uvw . We need to find $m - 1$ other triangles. We will distinguish three cases based on the number of edges connecting outside vertices to the vertices of the triangle uvw . We claim that if the number of all these edges is $2m - 3 + x$, where $x \geq 1$, then there are x triangles formed by two vertices of the triangle uvw , and the third vertex that comes from outside the triangle. Indeed if such an outside vertex is connected to two vertices of uvw , then it forms a triangle with them. As there are $2m - 3$ outside vertices the claim follows by the pigeon hole principle.

1. if $x \geq m - 1$, then the claim is obvious.
2. $1 \leq x < m - 1$, then the total number of edges between uvw and the outside vertices is at most $(2m - 3) + m - 2 = 3m - 5$. As uvw contains

three edges, it follows that there are at least $m^2 + 1 - (3m - 5) - 3 = m^2 - 3m + 3 = (m - 1)(m - 2) + 1$ edges in the subgraph induced by all $(2m - 3)$ outside vertices. If we omit then the vertex z which has the smallest degree, it follows that we get a graph H' on $2m - 4$ vertices that has at least

$$(m - 1)(m - 2) \frac{2m - 4}{2m - 3} = (m - 2)^2 \frac{2m - 2}{2m - 3} > (m - 2)^2 \text{ edges.}$$

So H' has at least $(m - 2)^2 + 1$ edges. Therefore the induction assumption implies that there are at least $m - 2$ triangles in H' . There are also $x \geq 1$ triangles that we had removed and also the initial triangle uvw , so in total there are at least $m - 2 + x - 1 \geq m$ triangles, so we have at least $m - 1$ other triangles.

3. Last we consider the case when the number of edges connecting outside vertices to uvw is $\leq 2m - 3$. We can always assume that there is at least one such edge, otherwise H' has $m^2 - 2$ edges, and adding any vertex of uvw to H' gives a graph on $2m - 2$ vertices with $m^2 - 2 \geq (m - 1)^2 + 1$ edges and the proof follows by induction. That said, the number of edges of H' is at least $m^2 + 1 - (2m - 3) - 3 = (m - 1)^2$. Adding a vertex of uvw which is adjacent to at least one vertex in H' gives a graph on $2m - 2$ vertices and at least $(m - 2)^2 + 1$ edges, and we conclude again with the induction assumption.

□

Here is a simple result

Proposition 4.1. *Let G be a simple bipartite graph on n vertices. Then G has at most $n^2/4$ edges if n is even and at most $(n^2 - 1)/4$ edges if n is odd.*

Proof. We have $V = V_1 \cup V_2$ with $|V_1| = a$ and $|V_2| = n - a$. Thus, $K_{a, n-a}$ has $a(n - a)$ edges. If we maximize this with respect to a the desired result follows. □

We have also the follow result concerning more generally p -cliques.

Theorem 4.7 (Turán). *Let $p \geq 2$ be an integer and let G be a simple graph on n vertices, which does not contain a p -clique. Then, if we note $|E(G)| = m$, we have*

$$m \leq \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2}.$$

Proof. We will proceed with an induction on n , with $p \geq 2$ fixed. For $n = 1$, the result is trivial.

$< n \rightarrow n$.

Case 1: If $n < p$. Then $p \geq n + 1$ and $1 - \frac{1}{p-1} \geq \frac{n-1}{n}$ and

$$\left(1 - \frac{1}{p-1}\right) \frac{n^2}{2} \geq \frac{n(n-1)}{2} \geq n.$$

Case 2: $n \geq p$. $G = K_n$. We can introduce the set $P_2(V) \setminus E = \{e_1, \dots, e_k\}$. Let $G_0 = G$ and for all $j \in \{1, \dots, k\}$ we set $G_j = (V, E \cup \{e_1, \dots, e_j\})$. G_0 does not contain a p -clique and G_k is the n -clique. So the set $\{j \in \{0, \dots, k-1\} \mid G_j \text{ does not contain a } p\text{-clique}\}$ is bounded and has a greatest element q . G_{q-1} contains a p -clique. G_q is obtained from G_{q-1} by removing an edge and this edge belongs to the p -clique of G_{q-1} . Consequently G_q contains a $p-1$ -clique, that we note H . Let H' be the subgraph of G_q induced by $V(G_j) \setminus V(H)$. Let $E(H, H')$ be the set of edges which have one endpoint in H and one adjacent in H' . We have,

$$m \leq |E(G)| \leq |E(G_q)| = |E(H)| + |E(H')| + |E(H, H')|.$$

it follows from the induction assumption that

$$|E(H')| \leq \left(1 - \frac{1}{p-1}\right) \frac{(n-p+1)^2}{2}.$$

Moreover $|E(H)| = \frac{(p-1)(p-2)}{2}$. Since G_q does not contain a p -clique, each vertex of H' is adjacent to at most $p-2$ vertices of H . Hence we have $|E(H, H')| \leq (p-2)(n-p+1)$. Consequently,

$$m \leq \frac{(p-1)(p-2)}{2} + \left(1 - \frac{1}{p-1}\right) \frac{(n-p+1)^2}{2} + (p-2)(n-p+1).$$

Let us use the fact that $p-2 = (p-1)\left(1 - \frac{1}{p-1}\right)$, to obtain,

$$\begin{aligned} m &\leq \frac{1}{2} \left(1 - \frac{1}{p-1}\right) ((p-1)^2 + (n-p+1)^2 + 2(p-1)(n-p+1)) \\ &\leq \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2}. \end{aligned}$$

□

We continue by introducing some geometric motivated definitions:

Definition 4.1. Let G be a connected graph. The distance between two vertices u and v of G , noted $d(u, v)$, is the shortest (u, v) -path. A (u, v) -path of minimal distance is called a (u, v) -**geodesic**. It can be easily checked that d defines a distance. For a given vertex v , the **eccentricity** of v , denoted by $\text{ecc}(v)$, is defined to be the greatest distance from v to any other vertex in G . That is $\text{ecc}(v) = \max_{x \in V(G)} d(v, x)$. In a connected graph, the **radius** of G , denoted $\text{rad}(G)$, is the value of the smallest eccentricity. Similarly, the **diameter** of G , denoted $\text{diam}(G)$, is the value of the greatest eccentricity. The **center** of the graph G is the set of vertices δ such that $\text{ecc}(\delta) = \text{rad}(G)$. The **periphery** of G is the set of vertices u such that $\text{ecc}(u) = \text{diam}(G)$.

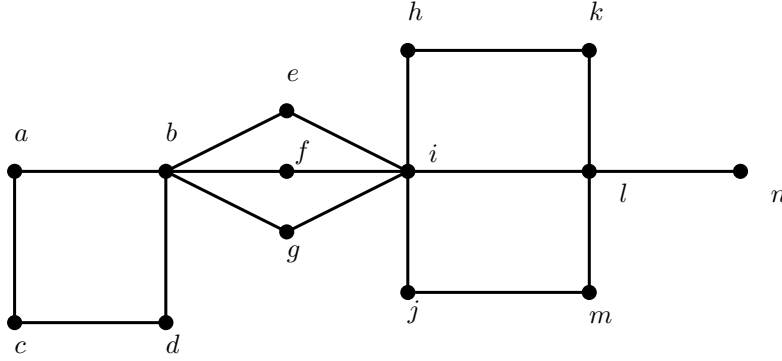


Figure 22: See discussion below.

In the figure above we have: $d(b, k) = 4$, $d(c, m) = 6$, $\text{ecc}(a) = 5$, $\text{ecc}(c) = \text{ecc}(h) = \text{ecc}(m) = \text{ecc}(n) = 6$, $\text{rad}(G) = 3$, $\text{center} = \{e, f, g\}$, $\text{diam}(G) = 6$, $\text{periphery} = \{c, h, m, n\}$.

Remark 4.1. For a cycle or a complete graph, the radius and the diameter are equal.

Theorem 4.8. For any connected graph G , $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.

Proof. Clearly $\text{rad}(G) \leq \text{diam}(G)$. Let $u, v \in V$ such that $d(u, v) = \text{diam}(G)$. Let c be in the center of G . Then $\text{diam}(G) = d(u, v) \leq d(u, c) \leq d(c, v) \leq 2\text{ecc}(c) = 2\text{rad}(G)$. \square

Remark 4.2. We can extend these definitions to any type of simple graph. If u and v are in different connected components, we set $d(u, v) = \infty$.

Theorem 4.9. *Every graph G is the center of some graph.*

Proof. We construct a new graph by adding four vertices (w, x, y, z) to G along with the following edges: $\{wx, yz\} \cup \{xa, a \in V(G)\} \cup \{yb, b \in V(G)\}$.

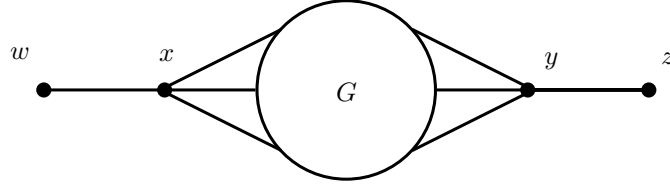


Figure 23: The new constructed graph.

Now $\text{ecc}(w) = \text{ecc}(z) = 4$, $\text{ecc}(y) = \text{ecc}(x) = 3$ and for every $x \in V(G)$, $\text{ecc}(x) = 2$. Therefore G is the center of this new constructed graph. \square

Theorem 4.10. *Let $G = (V, E)$ be a graph of order ≥ 3 . Then G is connected iff it has two distinct vertices u and v , such that $G - u$ and $G - v$ are connected.*

Proof. (\Rightarrow): Let $k = \text{diam}(G)$ and let u and v be such that $d(u, v) = k$. Let $P = u_0 \dots u_k$ be a (u, v) -geodesic ($u_0 = u, u_k = v$). Let us show that $G - u$ is connected. By definition of diameter, every $x \in V(G)$ is in P . Let $\{x, y\}$ be a pair of vertices in $G - u$. There exists a path P' in G joining x to y . If u is not in P' , then P' is also a path in $G - u$ as well. If not, the predecessor of u in P' is a vertex $v_i \in P$ and the successor of $u \in P'$ is a vertex $v_j \in P$. We now take the part in P' from x to v_i , then the part v_i to $v_j \in P$, and v_j to y in P' . Hence there exists a path from x to y in $G - u$ and $G - u$ is connected. We show in exactly the same way that $G - v$ is connected.

(\Leftarrow): Let x, y be two vertices of G . case1: $\{x, y\} \neq \{u, v\}$, for instance $u \neq x, u \neq y$. Then x and y are in $G - u$. As $G - u$ is connected, there is a path from x to y in $G - u$ and hence in G .

case2: $\{x, y\} = \{u, v\}$. For instance $x = u, y = v$. $|G| \geq 3$. Hence there exists $w \in V(G)$, $w \neq u, w \neq v$. $G - v$ is connected, so there exists a (u, w) -walk in P' and similarly there exists a (v, w) -walk in P'' . Then $P' \cup P''$ is a (u, v) -walk in G . \square

Theorem 4.11. *Let G be a simple graph, $|G| = n$, $|E(G)| = m$, such that G has q connected components. Then we have:*

$$n - q \leq m \leq \frac{(n - q)(n - q + 1)}{2}$$

Proof. We are going to show that $n - q \leq m$. It suffices to show it for $q = 1$ (i.e. a connected graph): the general case follows by addition of each connected component. The property is obvious for $n = 1$ and $n = 2$.

Suppose $n \geq 3$. We show that $n - 1 \rightarrow n$. Let $G = (V, E)$ be a graph of order n . Let $v \in V$ be such that $G \setminus v$ is connected (such a vertex exists according to the previous theorem, i.e. Theorem 4.10). We then have $|E(G \setminus v)| = m' \leq m - 1$.

It follows from the induction assumption that $m' \geq (n - 1) - 1$ which implies that $m \geq n - 1$. Next we show that

$$m \leq \frac{(n - q)(n - q - 1)}{2},$$

with an induction on q . For $q = 1$, the statement is obviously true. Let us thus assume $q \geq 2$. We show $q - 1 \rightarrow q$. Let $G(V, E)$ be a graph with n vertices, q connected components and m edges. Let $G_1 = (V_1, E_1)$ be a connected component of G of order n_1 . The graph $G - V_1$ has $n - n_1$ vertices and $q - 1$ connected components.

We then have:

$$\begin{aligned} |E(G_1)| &\leq \frac{n_1(n_1 - 1)}{2}, \\ |E(G - V_1)| &\leq \frac{((n - n_1) - (q - 1))((n - n_1) - (q - 1) + 1)}{2} \end{aligned}$$

and $m = |E(G)| = |E(G_1)| + |E(G - V_1)|$. Hence we obtain:

$$\begin{aligned} m &\leq \frac{n_1(n_1 - 1)}{2} + \frac{(n - n_1 - q + 1)(n - n_1 - q + 2)}{2} \\ &\leq \frac{(n - q)(n - q + 1)}{2} - (n_1 - 1)(n - q - n_1 + 1). \end{aligned}$$

But $n_1 \geq 1$ and $n \geq m + q - 1$ because each connected component must have at least one vertex. \square

Next we show how one can naturally associate a matrix to a graph.

Definition 4.2. Let $G = (V, E)$ be a general graph and note $V(G) = \{v_1, \dots, v_n\}$. The **adjacency matrix** of G is the $n \times n$, noted $A_G = A = (a_{ij})_{1 \leq i, j \leq n}$ defined as

$$a_{ij} = \begin{cases} \# \text{ edges that are incident to } v_i \text{ and } v_j, & i \neq j \\ \# \text{ of loops incident to } v_i, & \text{otherwise} \end{cases}$$

Remark 4.3. Note that A is always symmetric and that for a simple graph, the entries are either 0 or 1.

From now on, unless stated otherwise, we focus on simple graphs.

Example 4.1.

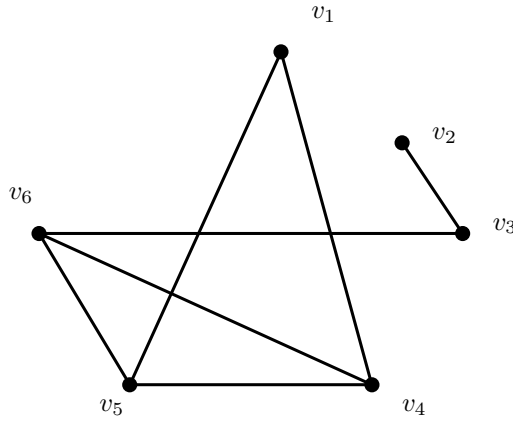


Figure 24: A graph G on 6 vertices with adjacency matrix A given below.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Theorem 4.12. Let $G = (V, E)$ be a (simple) graph of order n and let $k \in \mathbb{N}$. Let A be the adjacency matrix of G , corresponding to $V = \{v_1, \dots, v_n\}$. Then the number of walks of length k between v_i and v_j is the (i, j) -th entry of the matrix A^k .

Proof. We note $A = (a_{ij})$ and $A^k = (a_{ij}^k)$. We prove the result with an induction on k . For $k = 1$, the property follows from the definition of an adjacency matrix.

$k \rightarrow k + 1$. A walk of length $(k + 1)$ between v_i and v_j is the union of a walk of length k between v_i and another vertex v_h , and an edge between v_h and v_j . We deduce from the induction assumption that the number of walks of length $k + 1$ between v_i and v_j is :

$$\sum_{h=1}^n a_{ih}^{(k)} a_{hj} = a_{ij}^{(k+1)}.$$

□

Example 4.2. In the case of Example 4.1, we have $a_{1,6}^{(3)} = 1$.

Theorem 4.13. Let G be a simple graph on n vertices, and let A be its adjacency matrix. Then G is connected iff $(I + A)^{n+1}$ consists of strictly positive entries.

Proof. We know that if there is a walk from v_i to v_j in G , then there is a path too. The length of a path in G is at most $n - 1$. Therefore G is connected iff for any pair of vertices v_i and v_j there is a positive integer $k \leq n - 1$ such that $a_{ij}^{(k)} > 0$. Moreover we have

$$(I + A)^{n+1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A^k,$$

and the statement follows. □

More generally let us note $S_k = I + A + \dots + A^k$. The entries of I and A are either 0 or 1. So the entries of S_k are positive integers. In particular $(S_k)_{i,j} \leq (S_{k+1})_{i,j}$.

Theorem 4.14. Let G be a connected graph with vertices labelled v_1, v_2, \dots, v_n and let A be its corresponding adjacency matrix.

1. If k is the smallest positive integer such that the j -th row of S_k contains no zeroes, then $\text{ecc}(v_j) = k$.
2. If r is the smallest positive integer such that all entries of at least one row of S_n are > 0 , then $\text{rad}(G) = r$.
3. If m is the smallest integer such that all entries of S_n are positive, then $\text{diam}(G) = m$.

5 Coloring

Definition 5.1. Let G be a simple graph and let $k \in \mathbb{N}, k > 0$. A **k -coloring** of the vertices of G is an application $g : V(G) \rightarrow \{1, \dots, k\}$ such that if $\{u, v\} \in E$, then $g(u) \neq g(v)$. A graph for which a k -coloring exists is said to be **k -colorable**. Note that a graph of order n is always n -colorable. Hence for every simple graph G , there exists a smallest integer k such that G is k -colorable. This number is called the **chromatic number** of G and is noted $\chi(G)$.

Remark 5.1. A graph G is k -colorable iff it is k -partite.

It is not an easy problem to find in practice the chromatic number of G . We now propose an algorithm, called the **greedy algorithm**, which allows us to find a coloring for a given graph G .

Algorithm: The algorithm proceeds as follows: we consider an enumeration v_1, \dots, v_n of the vertices of G . We associate the color 0 to v_1 . Then we associate color 1 by going through the list v_2, \dots, v_n to all vertices, which are not adjacent to a vertex which is already colored. If after this process, there are vertices which are not colored yet, we associate color 2 to the first non colored vertex and proceed as before, until all vertices are colored.

Remark 5.2. This algorithm provides us with a k -coloring with $k \geq \chi(G)$. But it does not necessarily give χ . Let us for instance consider P_5 as depicted in the figure below:

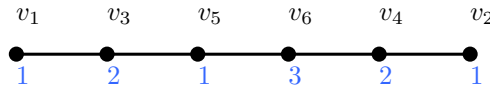


Figure 25: The greedy algorithm gives $h = 3$ but $\chi = 2$.

Theorem 5.1. Let G be a simple graph with maximal degree $\Delta(G)$ and chromatic number $\chi(G)$. Then we have:

$$\chi(G) \leq \Delta(G) + 1.$$

Proof. If G is colored following the greedy algorithm, with k colors, then the last colored vertex is adjacent to one vertex of the $k - 1$ first colors, for otherwise it would have been colored with one of these colors. The degree of these vertices is $\geq k - 1$. Hence $\Delta(G) \geq k - 1 \geq \chi(G) - 1$. \square

Remark 5.3. Note that the k -cycle C_k , with k odd, and the complete graph K_n satisfy $\chi(C_k) = \Delta(C_k) + 1$ and $\chi(K_n) = \Delta(K_n) + 1$. In fact, it can be shown that these are the only graphs with this property. For all other types of graphs, we have $\chi(G) \leq \Delta(G)$.

6 Trees

Definition 6.1. A **forest** is a graph without cycles (or *acyclic*). A **tree** is a connected and acyclic graph. Note that the connected components of a forest are trees. In general, in a graph, a vertex of degree 1 is called a **leaf**.

Example 6.1.

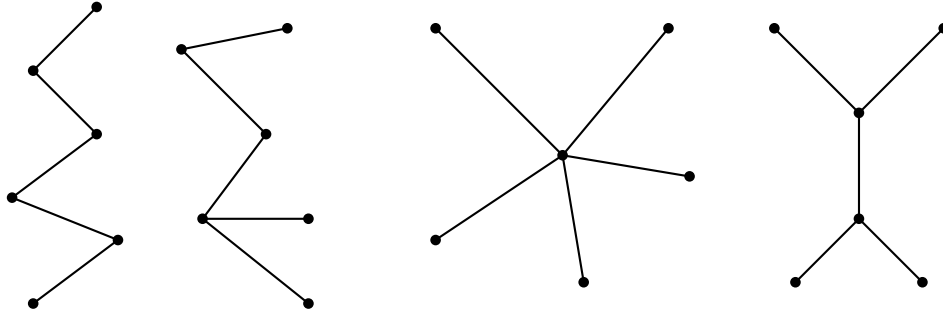


Figure 26: Trees on six vertices.

Theorem 6.1. Let $G = (V, E)$ be a forest, such that $E \neq \emptyset$. Then G has at least two leaves.

Proof. Let $C = (v_0, e_1, \dots, e_k, v_k)$ be a path of maximal length in G . Since $E \neq \emptyset$, $k \geq 1$. Assume that $d(v_k) > 1$. Since G is acyclic, there is a vertex $v \notin \{v_0, \dots, v_{k-1}\}$ which is adjacent to v_k . But this contradicts the maximality of C . Hence $d(v_k) = 1$. Similarly we have $d(v_0) = 1$. \square

Corollary 6.1. A tree $G = (V, E)$, with $|V| \geq 2$, has at least two leaves.

Lemma 6.1. A tree of order ≥ 2 is bipartite, and also 2-colorable.

Proof. Follows from the fact that a graph without odd cycles is bipartite. \square

Next we state in a single theorem a lost of fundamental properties of trees.

Theorem 6.2. *Let $G = (V, E)$ be a graph with n vertices and m edges. The following properties are equivalent:*

1. *The graph G is a tree.*
2. *The graph G is connected and $m = n - 1$.*
3. *The graph G is acyclic and $m = n - 1$.*
4. *For every pair of vertices $\{x, y\}$, there exists a unique (x, y) -path.*
5. *The graph G is connected and for every $e \in E$, $G - e$ is not connected.*
6. *The graph G is acyclic, and for all pairs of vertices $\{x, y\}$, which are non adjacent, $(V, E \cup \{x, y\})$ has a cycle.*

Proof. i) \implies ii) and iii). We prove with an induction on $m = n - 1$. For $n = 1$, G is an isolated vertex and the statement is obvious.

$n - 1 \implies n$. Let G be a tree on n -vertices. G has at least two leaves. Let v_0 be a leaf. Then $G - v_0$ has $n - 1$ vertices and is a tree. It follows from the induction assumption that it has $n - 2$ edges. So G has $n - 1$ vertices.

ii) \implies i) We shall again proceed with an induction. We first note that for $n \geq 2$. such a graph has at least one leaf. Such a graph does not have a vertex of degree 0 because it is connected, on the other hand if all degrees are ≥ 2 , then $\sum d(v) \geq 2n$, but we know that $\sum d(v) = 2|E| = 2n - 2$ and this is a contradiction.

For $n = 1$, the graph is just a single vertex and is thus a tree.

$n \geq 2, n - 1 \rightarrow n$. Let v_0 be a leaf of G . $G - v_0$ is connected and has $n - 2$ edges. So by the induction assumption, it is a tree. Consequently G is a tree.

i) \implies iv) Let G be a tree and $\{x, y\} \subset V$. G is connected, so there is a (x, y) -path. Let us assume that there exists two such (x, y) -paths, say C_1 and C_2 such that $C_1 \neq C_2$. Then there exists at least one edge in C_1 and not in C_2 . For instance $e = \{z, t\}$. The subgraph of G , union of C_1 and C_2 minus e would be connected and would contain a (z, t) -path. Adding e to this path we would obtain a cycle. Which is a contradiction.

iv) \implies i) It follows from iv) that G is connected. If G contains a cycle this cycle would contain at least two different vertices and have two distinct (x, y) -paths.

iv) \implies v) Let E be a vertex, $e = \{x, y\}$. Then (x, e, y) is a (x, y) -path, and by assumption, it is the unique (x, y) -path. The graph $G - e$ does not have (x, y) -paths and hence is not connected.

iv) \implies vi) We have already seen that iv) \implies G is acyclic. Let x, y be two non adjacent vertices of G . There exists a unique (x, y) -path. If we add $\{x, y\}$, then we have a (unique) cycle.

vi) \implies i) We only need to prove that vi) implies that G is connected. Let $x, y \in V(G)$. If x and y are adjacent, then there is a (x, y) -path. If x and y are not adjacent, then adding $\{x, y\}$ creates a cycle and we then have a (x, y) -path in G . \square

Recall that a spanning subgraph of a graph G is a subgraph of G with vertex set $V(G)$.

Definition 6.2. A **spanning tree** of a graph G is a spanning subgraph of G which is also a tree.

Theorem 6.3. A graph G is connected iff it has a spanning tree.

Proof. Let G' be a spanning subgraph of G , which is connected and minimal in the sense that if $e' \in E(G')$, then $G' - e'$ is not connected. It follows from Theorem 6.2 that G' is a tree. The converse is obvious (by truncation). \square

Corollary 6.2. Let F be a forest on n vertices with k connected components. Then F has $n - k$ edges.

Proof. This follows immediately by using a simply counting argument on the number of vertices. \square

Now we would like to ask how many trees there are on n vertices. We denote $[n] := \{1, 2, \dots, n\}$ as the vertex set.

Theorem 6.4 (Cayley's formula). *For $n \geq 1$, the number of all trees with vertex set $[n]$ is denoted as A_n and it holds that*

$$A_n = n^{n-2}.$$

This is a fundamental result of graph theory and there exists many proofs for this result. We shall give three of them here.

Proof 1. Take all A_n trees on $[n]$ and in each of them, choose two vertices, which are also not have to be different, and call one of them Start and the other one End. Do this in all possibly n^2 -ways for each tree. Call the $n^2 A_n$ objects obtained this way **doubly rooted trees**.

We are going to show that the number of doubly rooted trees on $[n]$ is n^n by constructing a bijection from the set of all functions from $[n]$ to $[n]$ to that of doubly rooted trees on $[n]$. Let $f : [n] \rightarrow [n]$ be a function. Let $C \subset [n]$ be the subset of elements $x \in [n]$ which are part of a cycle under the action of f , that is, there is a positive i such that $f^i(x) = x$. Let $C = \{c_1 < c_2 < \dots < c_k\}$. Now let $d_i = f(c_i)$ and write the integers d_1, d_2, \dots, d_k in order to the nodes of a tree consisting of one of k vertices. Also we mark d_1 with Start and d_k by End.

Finally, if $j \in [n]$, but $j \notin C$, then join the vertex j to the vertex $f(j)$ by an edge. This way we always get a tree. Indeed, we get a connected graph as the Start-End tree is connected to all vertices and we get a cycle free graph as the only cycles created by f involved elements from C are the vertices F in G , and C corresponds to a single line. The tree is doubly rooted.

To see that this is a bijection, take a doubly rooted tree on $[n]$. For vertices j not in the Start-End line, define $f(j)$ to be the first neighbor of j in the unique path from j to the Start-End line. For the vertices on the $S - E$ line, define f so that the image of the i -th smallest of them is the one that is in the i -th position from Start. \square

In order to get a better understanding of the previous proof, we will work through an explicit example below.

Example 6.2. Let $n = 8$ and $f : [8] \rightarrow [8]$ be defined by:

$$f(1) = 3, f(2) = 4, f(3) = 1, f(4) = 5, f(5) = 5, f(6) = 7, f(7) = 8, f(8) = 6.$$

$C = \{1, 3, 5, 6, 7, 8\}$. So the function f creates the cycles $(1\ 3)$, (5) and $(6\ 7\ 8)$.

So we can graphically represent the action of f as:

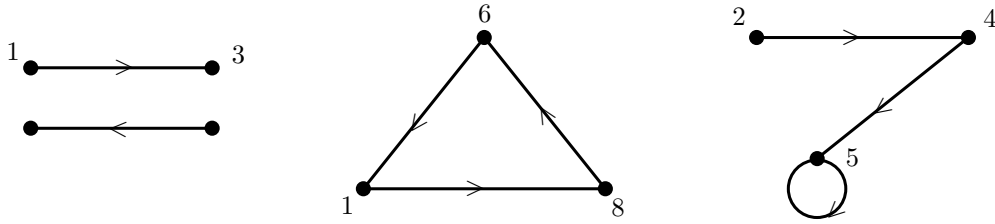


Figure 27: Graphical representation of the action of f .

We thus have $d_1 = 3, d_2 = 1, d_3 = 5, d_4 = 7, d_5 = 8, d_6 = 6$. Therefore one Start-End line will contain the integers 3, 1, 5, 7, 8, 6.

As $f(2) = 4$ and $f(4) = 5$, we connect the vertex 2 to 4 and the vertex 4 to 5. And we obtain the doubly rooted tree depicted in the figure below.

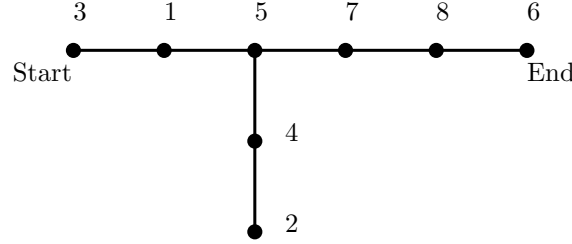


Figure 28: The doubly rooted tree obtained by our process.

Exercise:

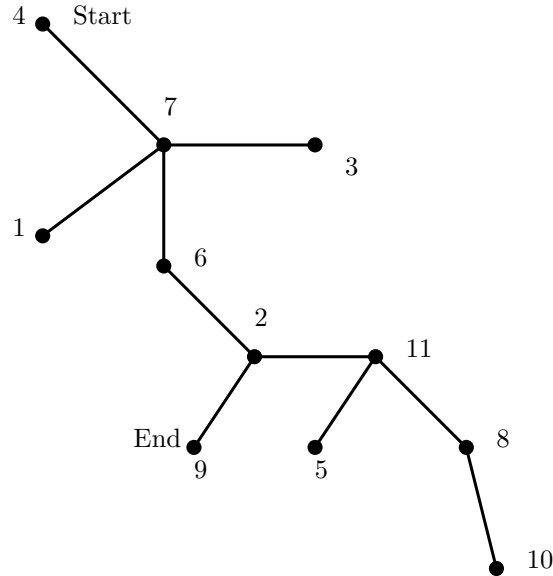


Figure 29: What is the function f associated with the above tree?

The answer is given by: $f(1) = 7, f(2) = 4, f(3) = 7, f(4) = 7, f(5) = 11, f(6) = 6, f(7) = 2, f(8) = 11, f(9) = 9, f(10) = 8, f(11) = 2$.

Definition 6.3. To the analogy of doubly rooted trees, we can define **rooted trees**, which are trees with one vertex called the root. SO the number of rooted trees on $[n]$ is n^{n+1} . A **rooted forest** is a forest in which each component is a rooted tree.

Corollary 6.3. For all positive integers n , the number of rooted forests on $[n]$ is $(n+1)^{n+1}$.

Proof. Take a rooted forest on $[n]$ and join all roots to the next vertex $n+1$. This transforms the original rooted forest to an unrooted tree on $[n+1]$. This map is a bijection: given a tree on $[n+1]$, we can simply mark all neighbours of $n+1$ as roots, then delete all edges adjacent to $n+1$, to get the original rooted forest back. So the set of all rooted forests on $[n]$ is in bijection with that of all trees on $[n+1]$ and therefore there are $(n+1)^{n+1}$ of them. \square

Now we provide another proof of Cayley's Theorem, due to Pitman as a problem with solution.

Problem: Pitman's proof of Cayley's theorem. Let F be a rooted forest on n vertices and view F as a directed graph, in which all edges are directed from the roots. If F' is another rooted forest, then we say that F contains F' if F contains F' as a directed graph. Clearly, in that case F has less components than F' . We say that F_1, F_2, \dots, F_k is a **refining sequence** if for all $i \in [k]$, F_i is a rooted forest on $[n]$ having i components, and F_i contains F_{i+1} . Now fix F_k .

- (a) Find the number $N^*(F_k)$ of refining sequences ending in F_k .
- (b) Find the number $N(F_k)$ of rooted trees containing F_k .
- (c) Deduce Cayley's formula.

We now provide detailed solutions to the above questions to conclude the proof:

- (a) Let us build a refining sequence starting from F_k . First we need to choose F_{k-1} by adding an edge e to F_k . The starting vertex of e can be any of the n vertices, but the ending vertex of e , however must be the root of one of the components of F_k not containing e . Therefore we have $n(k+1)$ choices for e , and thus we have “chains for F_{k-1} ”. Repeating the argument, we have $n(k-2)$ choices for F_{k-2} (for each choice of F_{k-1}), and so on. Therefore, repeating the argument $k-1$ times, we get

$$N^*(F_k) = n^{k-1}(k-1)!$$

- (b) If F_1 is a rooted tree containing F_k , then F_1 has $k-1$ more edges than F_k . We can remove those $(k-1)$ edges in $(k-1)!$ different ways, then obtaining

$$N^*(F_k) = (k-1)!N(F_k).$$

It follows from the above that $N(F_k) = n^{k-1}$.

- (c) Choose $k = n$. Then F_k is made of rooted vertices and all rooted trees contain F_n . Then $N(F_n) = n^{n-1}$ is the number of all rooted trees on $[n]$. The number of unrooted trees on $[n]$ is therefore n^{n-2} .

□

Exercise: Find a formula for the number of rooted forests on $[n]$ having k components.

Proof. We have just shown that $N^*(F_n) = n^{n-1}(n-1)!$. Now let $N^{**}(F_n)$ be the number of the refining sequences F_1, \dots, F_n where the k -th term is F_k . There are $N^*(F_k)$ choices for the sequence F_1, F_2, \dots, F_k , then there are $(n-k)!$ ways of removing the $(n-k)$ remaining edges. This shows that

$$N^{**}(F_k) = N^*(F_k)(n-k)! = n^{k-1}(k-1)!(n-k)!$$

This number does not depend on the choice of F_k . On the other hand, each refining sequence F_1, F_2, \dots, F_k contains exactly all rooted forests of k components. Therefore the number of rooted forests on $[n]$ with k components is the number of all refining sequences divided by the number of refining sequences in which a sequence F_k occurs.

$$\frac{n^{n-1}(n-1)!}{n^n(k-1)!(n-k)!} = \binom{n}{k} k n^{n-k-1}.$$

□

Prüfer's method: Given a tree T with vertices labelled 1 to n .

1. Let $i = 0$, and $T_0 = T$.
2. Find the leaf on T_i with the smallest label and it v .
3. Record in the sequence the label of v 's neighbor.
4. Remove v from T_i to create a new " T_{i+1} ".
5. If $T_{i+1} = K_2$, then stop. Otherwise increment i by 1 and go to step 2.

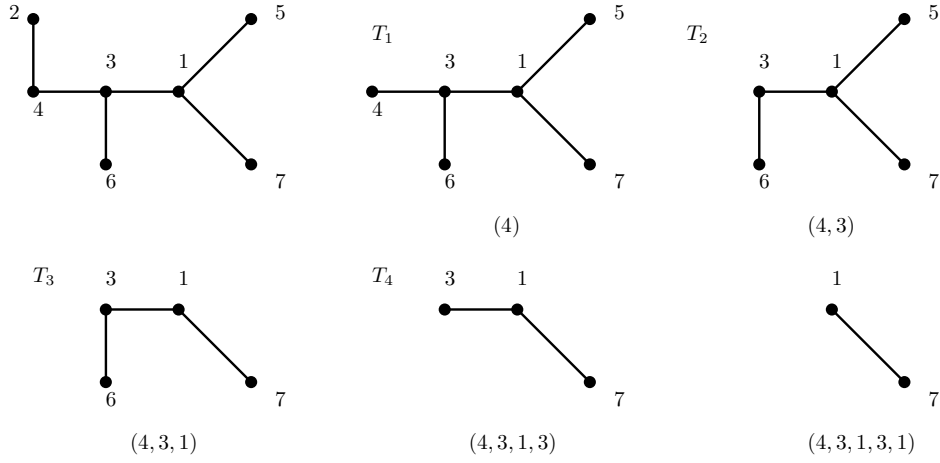


Figure 30: A worked example of Prüfer's method.

It can be shown that each vertex v appears in the sequence exactly $d(v) - 1$ times. Conversely one can associate a tree to such a sequence.

Given a sequence $\delta = a_1, a_2, \dots, a_k$ of entries from $\{1, \dots, k-1\}$.

1. Draw $k + 1$ vertices, label them v_1, \dots, v_{k+1} . Let $S = \{1, \dots, k + 1\}$.
2. Let $i = 0$, $\delta_0 = \delta$, $S_0 = S$.
3. Let j be the smallest number in S_i that does not appear in δ_i .
4. Place an edge between vertex v_j and the vertex whose subscript appears first in v_i .
5. Remove the first number in δ_i to create a new sequence δ_{i+1} . Remove j from S_i to create a new set S_{i+1} .
6. If $\delta_{i+1} = \emptyset$, place an edge between the two vertices whose subscripts are in S_{i+1} and stop. Otherwise increment i by 1 and return to step 3.

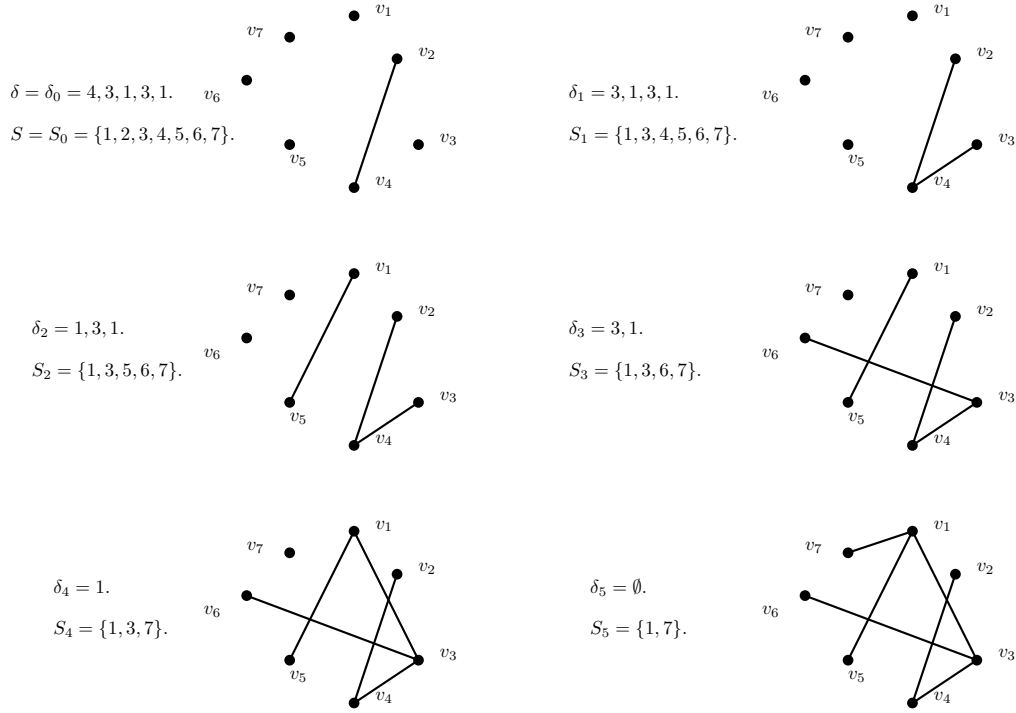


Figure 31: The previous algorithm explained.

So we need to check that the Prüfer method provides us with a bijection. We take $\delta = (a_1, a_2, \dots, a_{n-2})$ and we show that there exists a unique tree T on $[n]$ whose Prüfer case is δ . Note that the elements of $[n]$ that are not in δ must precisely be the

However in this case the degree of j in the final 2-element tree is one, while its degree in the original tree was at least 2, as j was not a leaf. So again at some point a vertex was cut off from j , putting j into δ . So δ tells us what the leaves of the original tree were; denote them by b_1, \dots, b_n in increasing order. We know that first we have to cut off the leaf with the smallest label. Therefore we are reconstructing the line by joining b_1 to a_1 , as a_1 is by definition the single neighbor of the smallest leaf. How do we know whether a_1 became a leaf after the first step? If it does not occur in delta any additional times, as in this case nothing else can be cut off from it.

So if a_1 does not occur in S after the first position, then in the second step we join $\min(a_1, b_2)$ to a_2 by an edge. Otherwise we simply join b_2 to a_2 by an edge. In general, at the i^{th} step, we have to find the minimal element b_j that has not been attached to any edge and the minimal element a_k that has not been added to an edge yet, and does not occur in S anywhere after position a_{i+1} . Then we join $\min(a_k, b_j)$ to a_i by an edge.

Any connected graph G has at least one spanning tree. In general, a connected graph will have many spanning trees. For instance K_n has n^{n-2} spanning trees.

Definition 6.4. Let G be a connected simple graph. Let $w : E(G) \rightarrow \mathbb{R}_+$ be a function. The function w is usually called the **weight function** and $w(e)$ is called the **weight** of e , while for a tree T , $\sum_{e \in T} w(e)$ is called the weight of T .

Problem: Find the spanning tree T of G so that $\sum_{e \in T} w(e)$ is minimal.

We cannot attack this problem by enumerating all possible spanning trees and compute their respective weight. For instance take $n = 20$, $G = K_n$. Then we would have to compute the total weight of $20^{18} > 2.5 \times 10^{23}$ spanning trees. If one computer would could handle 10^9 trees per second, then it would still take 2.5×10^{14} seconds, i.e. more than 91 years!

So instead, we try a smart algorithm. Take the edge with the smallest weight (or one of them if there are several), and put it in T . Second, take the edge that has the smallest weight among them not in T , and add it to T . In the third step and all subsequent steps, we have to be careful. We want to make sure that by adding an edge, we will not create a cycle.

In general in the i -th step of the algorithm, we look for the edge e_i that has the following properties:

1. the edge e_i is not yet in T .
2. if we add e_i to T , the obtained graph does not contain a cycle
3. the weight of e_i is minimal among all edges that have properties 1), and 2).

When we found the edge e_i , we add it to T . It is clear that we can continue this procedure until T has $n - 1$ edges. This algorithm is called **Kruskal's algorithm** Will it give us the minimum weight spanning tree?

Lemma 6.2. Let F and F' be two forests on the same vertex set, and let F have less edges than F' . Then F' has an edge e that can be added to F so that $F' \cup e$ is still a forest.

Proof. Assume that there is no such edge $e \in F'$. Then adding any edge of F' to F would create a cycle in F . So all edges of F' are between two vertices of the same connected component of F . Therefore F' has at least as

many components as F . This is a contradiction to the assumption that F' has more edges than F . \square

Theorem 6.5. *Kruskal's algorithm always finds the minimum weight spanning tree.*

Proof. Assume that Kruskal's algorithm produces a spanning tree T while G has a spanning tree H whose total weight is less than T . Let h_1, \dots, h_{n-1} be the edges of H so that $w(h_1) \leq w(h_2) \leq \dots \leq w(h_{n-1})$. Similarly let t_1, \dots, t_{n-1} be the edges of T such that $w(t_1) \leq w(t_2) \leq \dots \leq w(t_{n-1})$. Let i be the smallest integer such that

$$\sum_{j=1}^i w(h_j) < \sum_{j=1}^i w(t_j).$$

Such an i exists by assumption on H . It is also clear that $i > 1$, because $w(t_1)$ is minimal. Since

$$\sum_{j=1}^i w(h_j) < \sum_{j=1}^i w(t_j), \text{ and } \sum_{j=1}^{i-1} w(h_j) \geq \sum_{j=1}^{i-1} w(t_j),$$

we must have $w(h_i) < w(t_i)$. Let T_{i-1} be the forest the Kruskal's algorithm produced in $i-1$ steps, and let H_i be the forest formed by the edges h_1, \dots, h_i . We apply the previous lemma to T_{i-1} and H_i we see that is an edge h_j with $j \leq i$ that can be added to T_{i-1} without forming a cycle. However $h_j \leq h_i < t_i$. So at step i , the Kruskal's algorithm could not add t_i to T_{i-1} as t_i did not have minimal weight among the edges that could be added to T_{i-1} without forming a cycle. This is a contradiction. \square

Example 6.3.

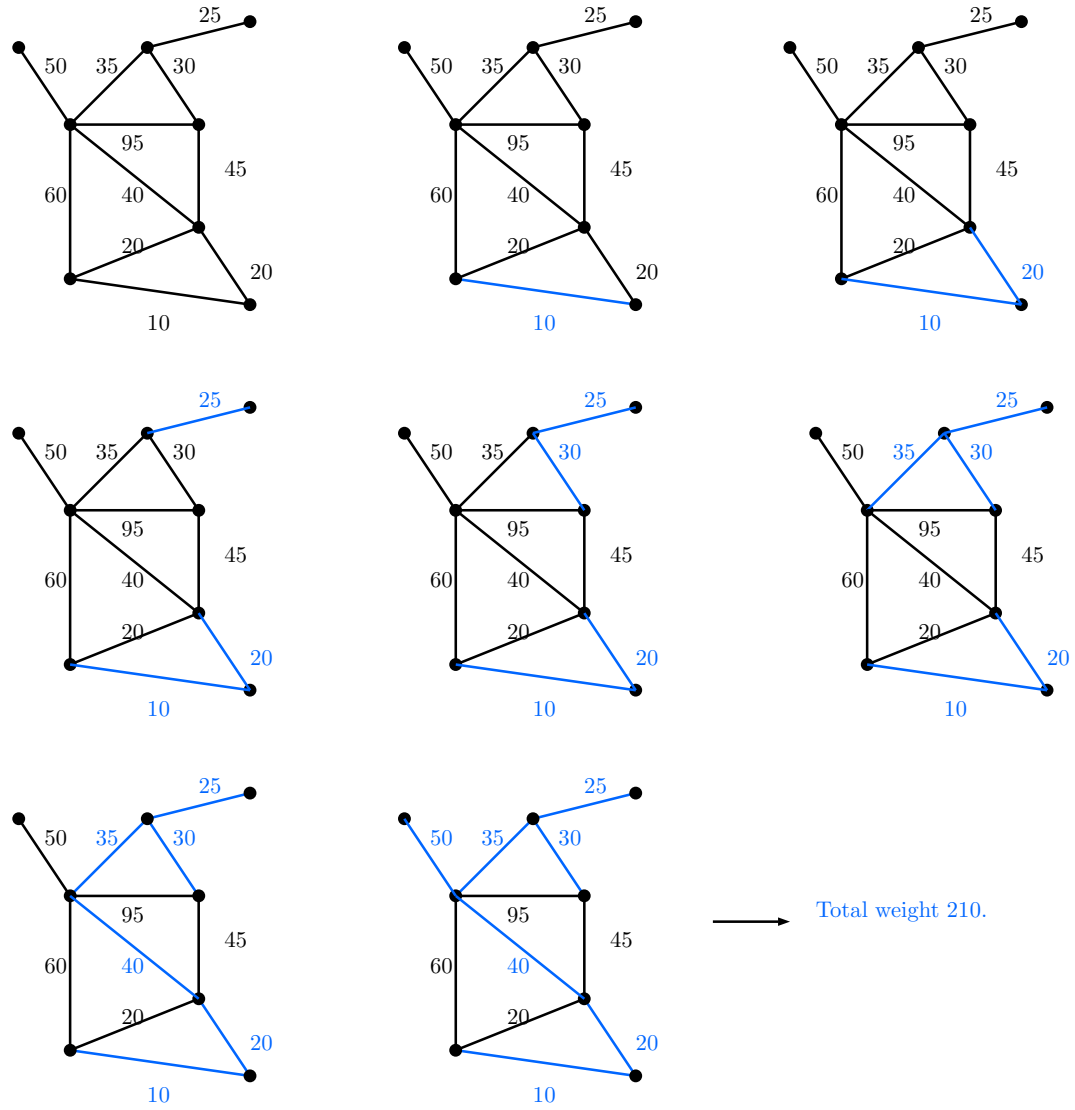


Figure 32: A worked example of Kruskal's algorithm.

Theorem 6.6 (Matrix Tree Theorem). *Let G be a connected simple graph with vertices $\{c_1, \dots, v_n\}$. Define $D = \text{diag}(d(v_i))_{1 \leq i \leq n}$. and let A be the adjacency matrix of G . Then the number of spanning trees of G is equal to the value of any cofactor of the matrix $D - A$.*

Remark 6.1. Recall that if $M = (m_{ij})_{1 \leq i, j \leq n}$ is a matrix, then the (i, j) cofactor of M is defined as $(-1)^{i+j} \det(M(i, j))$ where $M(i, j)$ is the matrix obtained from M by deleting its i -th row and its j -th column.

Example 6.4.

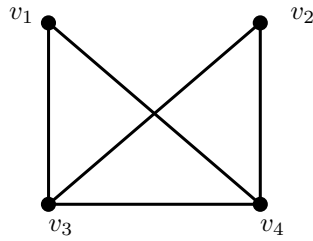


Figure 33: A graph on 4 vertices to motivate the matrix tree theorem.

With the graph as depicted above we have:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad D = \text{diag}(2, 2, 3, 3), \quad D - A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

The cofactor of $(1, 1)$ is given by

$$\det \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} = 8.$$

Exercise: Use the matrix tree theorem to prove that the number of trees on n vertices is n^{n-2} .

Proof.

$$\det \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & \ddots & -1 & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \dots & -1 & n-1 \end{pmatrix} = n^{n-2}.$$

□

Exercise: Find the number of spanning trees of $K_{m,n}$.
Answer: $m^{(n-1)} \cdot n^{(m-1)}$

7 Eulerian trails and Hamiltonian cycles

Definition 7.1. *If a trail in a graph G includes every edge of G , then that trail is said to be an **Eulerian trial**. Similarly an **Eulerian circuit** in a graph is a circuit that includes every edge of the graph. A graph that contains an Eulerian circuit is called an **Eulerian graph**.*

Example 7.1. *The cycles C_n are Eulerian graphs. But of course the paths P_n are not.*

Theorem 7.1. *For a connected graph G , the following statements are equivalent:*

1. G is Eulerian.
2. Every vertex of G has even degree.
3. The edges of G can be partitioned into edge disjoint cycles.

Proof. 1 \implies 2. Let C be an Eulerian circuit. Let v be an arbitrary vertex of G . Every time the circuit enters v or an edge, it must leave on a different edge, there must then be an even number of edges incident with v and hence $d(v)$ is even.

2 \implies 1. Take any vertex v , and start walking along an edge e_1 to the other endpoint a_1 of that edge, then walk along a new edge e_2 that starts in a_1 . Continue this way using new edges at each step, until a closed trail C_1 is found. As G is finite, such a trail will always be formed. The first closed trail will be found when we first visit a vertex already visited.

If $C_1 = G$ we are done. If not, since G is connected, we can have a vertex v in C_1 so that C_1 does not contain all edges adjoint to v . Let us now unite all edges of C_1 from G . We get a graph in which again all vertices have even degree. Starting at v , let us take another closed trail C_2 in the remaining graph. We can then unite C_1 and C_2 into one closed trail in G . Indeed if we start walking by w_1 , we can stop at v , walk through C_2 and complete our trail by using the remaining part of C_1 . If the new trail $C_1 \cup C_2$ contains all edges of G , we are done. If not, then let us omit $C_1 \cup C_2$ from G and find a new closed trail C_3 in the remaining graph. As G has a finite number of edges, the procedure has to stop after a finite number of steps. Therefore after a finite number of steps $C_1 \cup C_2 \cup \dots \cup C_k$ will be a closed trail containing all edges of G . \square

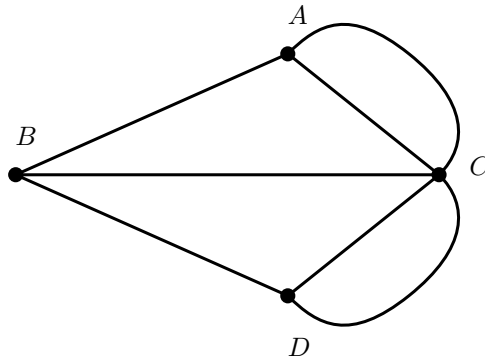


Figure 34: The graphs of the Königsberg bridges.

We cannot walk through all bridges of Königsberg so that we start and end where we started, and use each bridge exactly once.

Corollary 7.1. *Let G be a connected graph. Then G has an Eulerian trail starting at vertex u and ending at a different vertex v iff u and v have odd degree, and all other vertices of G have even degree.*

Proof. Add a new edge joining u and v , and call the new graph obtained H . Then H has a Eulerian circuit iff G has an Eulerian trail from u to v . \square

Remark 7.1. The proof we have for Theorem 7.1 allows us to find an Eulerian circuit.

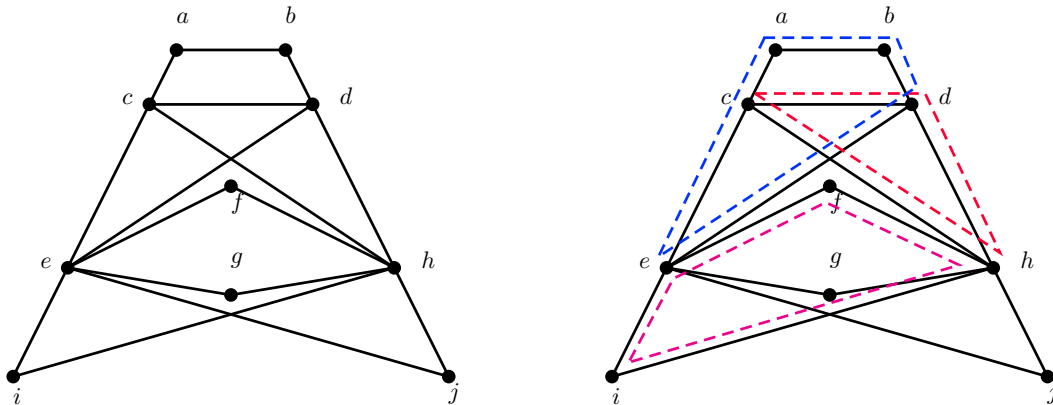


Figure 35: Shows how to find an Eulerian circuit.

Definition 7.2. Let G be a graph. If a path P spans the vertices of G , i.e. $V(P) = V(G)$, then P is said to be a **Hamiltonian path**. Any graph G containing a Hamiltonian path is called **traceable**. If a cycle spans the vertices of a graph G , such a cycle is called a **Hamiltonian cycle** and any graph containing a Hamiltonian cycle is called a **Hamiltonian graph**.

Remark 7.2. Hamiltonian graphs are clearly traceable, but the converse is not always true.

Theorem 7.2. Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

Proof. Let us assume that G does not contain a Hamiltonian cycle. Let us add to G a new edge as long as we can without creating a Hamiltonian cycle. When we stop we have a graph G' in which all vertices have degree $\geq n/2$, there is no Hamiltonian cycle, but adding one new edge would create a Hamiltonian cycle.

Let x and y be two vertices in G' that are not connected by an edge. If adding the edge xy would create a Hamiltonian cycle, it follows that G' has a Hamiltonian path P that starts at x and ends at y . Let $x = z_1, z_2, \dots, z_{n-1}, z_n = y$ be the vertices of this path. Therefore the pigeonhole principle implies that there must be an index i so that $2 \leq i \leq n$, where xz_i is an edge and where $z_{i-1}y$ is also an edge. Indeed, if this was not the case, then for any neighbor z_{i-1} of y on P , z_i is not a neighbor of x . \square

We have the following more general theorem:

Theorem 7.3. Let G be a simple graph of order ≥ 3 . Assume that for all $(u, v) \in V(G) \times V(G)$ such that $\{u, v\} \in E(G)$ we have $d(u) + d(v) \geq n$. Then G is Hamiltonian.

Proof. We start as in the previous proof with a maximal Hamiltonian graph and prove that in this case there exists u, v non adjacent such that $d(u) \leq d(v) \leq n - 1$. There exists a non adjacent pair of vertices u, v , such that $G + \{u, v\}$ is Hamiltonian. Let C be a Hamiltonian cycle of $G + \{u, v\}$. $C - \{u, v\}$ is a Hamiltonian path in G . Let us note (v_1, \dots, v_n) for this path, with $v_1 = u, v_n = v$. Let $i \in \{1, \dots, n - 1\}$ be such that $\{v_i, v_{i+1}\}$ and $\{v_n, v_i\} \in E(G)$. Then this would be a Hamiltonian cycle, which is not possible.

Let us thus set

$$I = \{i \in \{1, \dots, m-1\} \mid \{v_1 v_{i-1}\} \in E(G)\}.$$

$$J = \{i \in \{1, \dots, n-1\} \mid \{v_n, v_i\} \in E(G)\}.$$

It follows from the previous remark that $I \cap J = \emptyset$. Therefore $|I| + |J| = |I \cup J| \leq n-1$. We note that $|I| = d(u)$ and $|J| = d(v)$, thus $d(u) \leq d(v) \leq n-1$. \square

Remark 7.3. Note that the above theorem does not admit a converse. For instance C_n for $n \geq 5$ is Hamiltonian but $d(u) = d(v) = 4 < n = 5$.

Proposition 7.1. *Let G be a simple graph of order n with two non adjacent vertices, such that $d(u) = d(v) \geq n$. Then $G + \{u, v\}$ is Hamiltonian iff G is Hamiltonian.*

Proof. If G is Hamiltonian, obviously $G + \{u, v\}$ is also Hamiltonian. Let us assume that G is not Hamiltonian. and that there exists two non adjacent vertices u, v such that $G + \{u, v\}$ is Hamiltonian. But then every Hamiltonian cycle would include $\{u, v\}$ and then G would have a Hamiltonian path with endpoints u and v . So as in the previous proof, we get $d(u) = d(v) \leq n-1$, which is a contradiction. \square

Proposition 7.2. *Let G be a simple graph of order $n \geq 3$. Assume that $|E(G)| \geq \binom{n-1}{2} + 2$. Then G is Hamiltonian.*

Proof. For $n = 3$ the result is obvious. Let us assume that $n \geq 4$. let us assume that G is a graph with $\binom{n-1}{2} + 2$ edges, but that it is not Hamiltonian. Let u and v be two non adjacent vertices of G . Let $G' = G[V \setminus \{u, v\}]$. We have:

$$|E(G')| = \binom{n-1}{2} + 2 - d(u) - d(v)$$

$$|V(G')| = n - 2.$$

We have $|E(G')| \leq \binom{n-2}{2}$ which implies that:

$$d(u) + d(v) \geq \binom{n-1}{2} - \binom{n-2}{2} + 2 = \binom{n-2}{1} + 2 = n,$$

which is a contradiction. \square

Exercise: Prove that the Petersen graph is not Hamiltonian.

Next, before concluding this section, we shall mention how some of our results are changed in the case of directed graphs. Recall that a directed graph $G = (V(G), A(G))$ is endowed with $\varphi : A \rightarrow V \times V$.

Example 7.2.

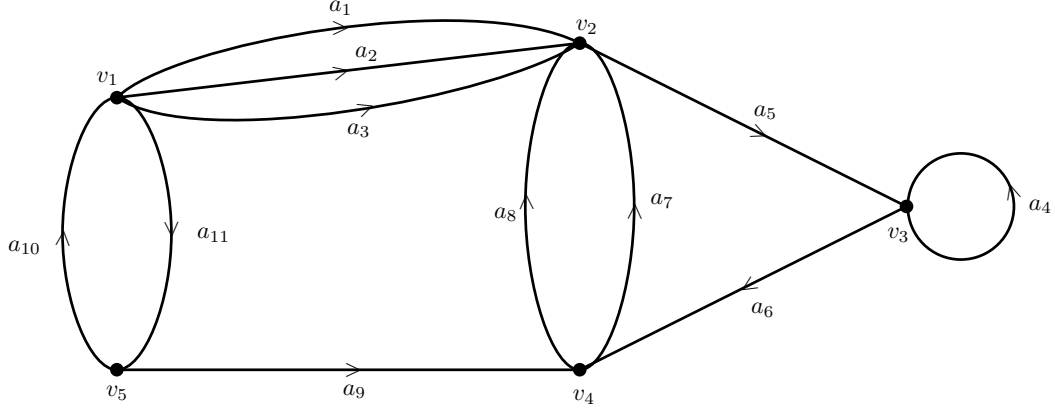


Figure 36: A directed graph on 5 vertices.

Definition 7.3. Let $G = (V, A)$ be a directed graph. Let us note $V(G) = \{v_1, \dots, v_n\}$. The **adjacency matrix** of G , is the matrix $M = (m_{ij})_{1 \leq i, j \leq n}$ where m_{ij} is the number of edges e such that $\varphi(e) = (u_i, v_j)$.

Example 7.3. The adjacency matrix of the above graph is given by:

$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Definition 7.4. The definitions of walks, trails, paths, cycles, circuits, etc. can be obviously adapted. The **in-degree** of a vertex of a directed graph, denoted by v^- , is the number of edges that end at that vertex. The **out-degree** of a vertex, denoted by d^+ , is the number of edges that start at that vertex. A directed graph is called **balanced** if for each vertex v of the graph, we have $d^+(v) = d^-(v)$.

Proposition 7.3. Let $G = (V, A)$ be a directed graph with adjacency matrix $M = (m_{ij})$, then:

$$|A(G)| = \sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = \sum_{i,j=1}^n m_{ij}.$$

Proof. Follows from the following simple remarks:

$$\begin{aligned} d^+(v_i) &= \sum_{j=1}^n m_{ij} \implies \sum_{i=1}^n d^+(v_i) = \sum_{i=1}^n \sum_{j=1}^n m_{ij}. \\ d^-(v_j) &= \sum_{i=1}^n a_{ij} \text{ and } \sum_{j=1}^n \sum_{i=1}^n d^-(v_j) = \sum_{j=1}^n \sum_{i=1}^n m_{ij}. \end{aligned}$$

□

Definition 7.5. A directed graph G is called **strongly connected** if for all $(u, v) \in V(G)^2$, there exists a path from u to v .

Remark 7.4. Define $u \sim v$ iff there exists (u, v) -path and there exists a (v, u) -path. Then \sim is an equivalence relation.

Theorem 7.4. A directed graph G has a closed Euclidian trail iff it is balanced and strongly connected.

Proof. First we prove that the conditions are necessary. As a closed Eulerian trail leaves each vertex as many times as it enters it, G must be balanced. Similarly we have a trail from any vertex to any vertex implies that G is strongly connected. To see that these conditions are sufficient, we copy the proof of the wanted case by replacing edges with directed edges. □

Definition 7.6. If we direct each edge of a complete graph on n vertices, then the resulting directed graph is called a **tournament**.

Theorem 7.5. All tournaments have a Hamiltonian path.

First proof. Let us prove the claim by induction on n , the number of vertices of a tournament T . If $n = 2$, then the statement is obvious. Now assume that we know the statement is true for all tournament having $n - 1$ vertices. Let T be any tournament on n vertices. Let v be any vertex. Let $T' = T - v$. By the induction hypothesis T' has a Hamiltonian path $h = h_1 h_2 \dots h_{n-1}$. If there is an index i such that $h_i v$ is an edge and $v h_{i+1}$ is an edge, then we insert v between h_i and h_{i+1} . If no such edge exists then there must exist an index k such that $0 \leq k \leq n - 1$ and $v h_k$ is an edge and for all $j > k$, $h_j v$ is an edge. Therefore either $v h_k$ is an edge or $h_{n-1} v$ is an edge. So we can fix v either to the front of the end of h . □

Second proof. Let $C = (v_0, \dots, v_q)$ be a path of maximal length in $T = (V, A)$ and let us assume that there exists $v \in V$ such that $v \notin C$. Since C is maximal $(v, v_0) \notin A$ and thus $(v_0, v) \in A$. Similarly we have $(v, v_q) \in A$. Let i be the smallest index such that $(v, v_i) \in A$. Then $i > 0$. Let us then replace (v_{i-1}, v_i) by $(v_{i-1}, v_1 v_i)$. We then have a path which is longer than C , thus a contradiction. Consequently C is a Hamiltonian path. \square

Definition 7.7. A tournament $T = (V, A)$ is called **transitive** if for $(u, v) \in A$ and $(v, w) \in A$ we also have $(u, w) \in A$.

Proposition 7.4. A tournament is transitive iff it does not contain a closed trail (circuit).

Proof. " \implies " Assume that T is transitive and has a circuit $C = (v_1, v_2, \dots, v_n, v_1)$. We have for $(v_1, v_2) \in A$ and $(v_2, v_3) \in A$ also that $(v_1, v_3) \in A$. Continuing we see that $(v_1, v_n) \in A$, which is not possible because (v_n, v_1) is in A already. " \impliedby " Let $(u, v) \in A$ and $(u, w) \in A$ for a tournament T . Since T does not have a cycle $(w, u) \notin A$. Hence $(u, w) \in A$. \square

Theorem 7.6. Let T be a tournament which is strongly connected, with $|T| \geq 3$. Let $v \in V(T)$, then for all $k \in \{3, \dots, n\}$, T contains a circuit of length k which also contains v .

Proof. Let $v \in V(T)$. We prove this result with an induction on k . Let $k = 3$. We note $V^+(v)$, the set of successors of v and $V^-(v)$ the set of predecessors of v . We have $V^+ \cap V^- = \emptyset$ and $V^+ \neq \emptyset$, $V^- \neq \emptyset$ because T is a strongly connected tournament. There exists an edge $(w, w') \in A$, with $w \in V^+$ and $w' \in V^-$. We obtain a circuit of length 3 by connecting (v, w, w') which contains v .

Let us assume that the result is established for $k \in \{3, \dots, n-1\}$ and let us prove it for $k+1$. Let $C = (v_0, v_1, \dots, v_k = v_0)$ be a circuit of length k that contains v . There exists a vertex of T that is not in V . Then we distinguish between two cases:

Case 1: There exists $u \in V(T) \setminus V(C)$ and $i, j \in \{1, \dots, k\}$ with $i \neq j$ such that $(v_i, u) \in A$, $(u, v_j) \in A$. Without loss of generality, we may take $i = 1$. Then $(v_1, u) \in A$ and $(v_j, u) \notin A$. Hence the set $\{h \in \{1, \dots, j-1\} \mid (v_h, u) \in A\}$ has a greatest element h_0 . We deduce that $(v_{h_0}, u) \in A$, $(v_{h_0-1}, u) \notin A$ and that $(u, v_{h_0+1}) \in A$. We replace (v_{h_0}, v_{h_0-1}) with (v_{h_0}, u, v_{h_0-1}) and we obtain a new circuit of length $k+1$.

Case 2: For all $u \in V(T) \setminus V(C)$, either $(u, v_i) \in A$ for all $i \in \{1, \dots, k\}$ or $(v_i, u) \in A$ for all $i \in \{1, \dots, k\}$. Consider the sets:

$$R = \{u \in V(T) \setminus V(C) : \forall i \in \{1, \dots, k\}, (u, v_i) \in A\}$$

$$S = \{u \in V(T) \setminus V(C) : \forall i \in \{1, \dots, k\}, (v_i, u) \in A\}.$$

As T is strongly connected, $R \neq \emptyset$ and $S \neq \emptyset$, and there exists $r \in R, s \in S$ with $(s, r) \in A$. We then replace (v_0, v_1, v_2) with (v_0, s, r, v_2) and we obtain a circuit of length $k + 1$. \square

8 Matchings

Definition 8.1. Let $G = (V, E)$ be a simple graph. A subset $M \subset E$ of edges is called a **matching** if no pair of edges shares a common vertex. Given a matching M in a graph G , the vertices belonging to the edges of M are said to be **saturated** by M (or M -saturated). The other vertices are M -unsaturated. If M saturates every vertex of G , then M is said to be a **perfect matching**.

Example 8.1.

1.

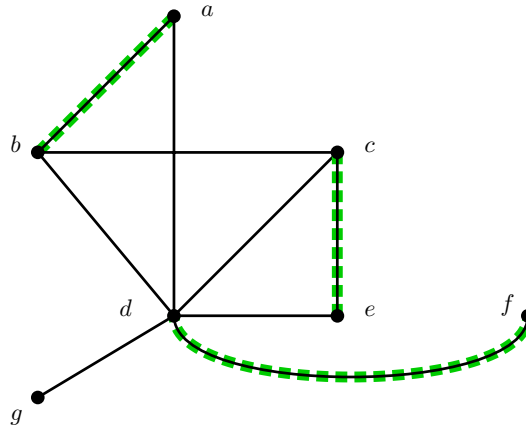


Figure 37: See description below.

$M_1 = \{ab, ce, df\}$ is a matching (depicted in green in the above figure). Moreover, $M_2 = \{cd, ab\}$ is also a matching, $M_3 = \{df\}$ is a matching. a, b, c, d are M_2 -saturated and e, f, g are M_2 -unsaturated. The only M_1 -unsaturated vertex is g .

2.

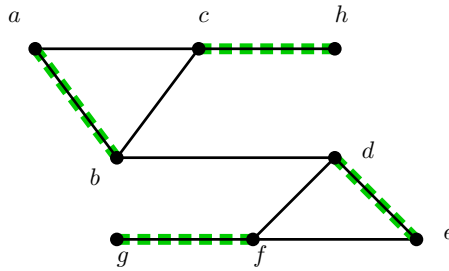


Figure 38: See description below.

$M = \{ab, ch, de, fg\}$ is a perfect matching.

Definition 8.2. A **maximum matching** in a graph is a matching that has the largest possible cardinality. We denote the cardinality of a maximal matching as $\nu(G)$. A **minimal matching** is a matching that cannot be enlarged by addition of an edge.

Example 8.2.

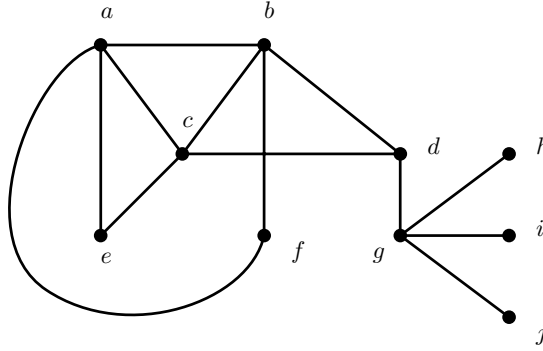


Figure 39: Depiction of maximal matchings.

$M_1 = \{ae, bf, cd, gh\}$ is a maximum matching. $M_2 = \{dg, af, bc\}$ is a maximal matching but not a minimal.

Definition 8.3. Given a matching M , an **M -alternating path** in G is a path where the path alternates between M -edges and non M -edges. An **M -augmenting path** is an M -alternating path whose both end vertices are M -unsaturated.

Example 8.3.

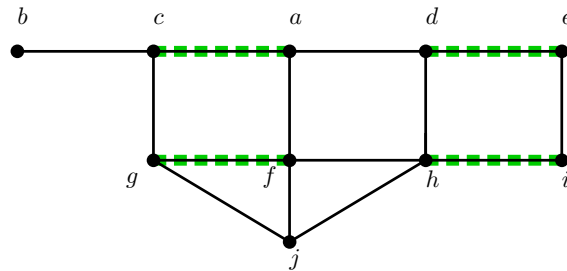


Figure 40: M -alternating path and M -augmenting path.

c, a, d, e, i is an M -alternating path, whereas j, g, f, a, c, b is an M -augmenting path.

Theorem 8.1 (Berge). *Let M be a matching in a graph G . M is maximal iff G contains no M -augmenting paths.*

Proof. First assume that M is a maximum matching, and suppose that $P = v_1v_2, \dots, v_k$ is an M -augmenting path. Then it must be even and the edges $v_2v_3, v_4v_5, \dots, v_{k-2}v_{k-1}$ are edges of M , while $v_1v_2, v_3v_4, \dots, v_{k-1}v_k$ are not edges of M .

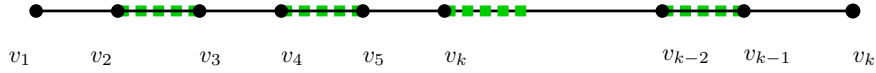


Figure 41: The situation as described above.

But then if we define M_1 to be:

$$M_1 := (M \setminus \{v_2v_3, \dots, v_{k-2}v_{k-1}\}) \cup \{v_1v_2, \dots, v_{k-1}v_k\},$$

then M_1 is a matching that contains one more edge than M . This is a contradiction and hence G contains augmenting paths.

For the converse, assume that G has no M -augmenting paths, and suppose that M' is a matching that is larger than M . We define a subgraph H of G as follows: $V(H) = V(G)$ and $E(H) = M \Delta M' = (M' \setminus M) \cup (M \setminus M')$. Since each of the vertices of G has at most one edge from M and at most one edge from M' it follows that $\Delta(H) \leq 2$. This implies that each connected component of H is either a single vertex, a path, or a cycle. If a component is a cycle, then it must be an even cycle, since the edges alternate between M -edges and M' -edges. Since $|M'| > |M|$ there must be at least one component of H that is a path that begins and ends on the edges from M' . But this is an M -augmenting path, which is a contradiction. Therefore no such matching M' can exist and M is maximal. \square

Definition 8.4. If G is a bipartite graph. $G = (X, Y; E)$, we say that X can be **matched into** Y if there exists a matching in G that saturates the vertices of X . A set T of vertices in a graph G is said to **cover** the edges of G if every edge of G is incident with at least one vertex of T . Such a set T is called an **edge cover** of G .

Example 8.4.

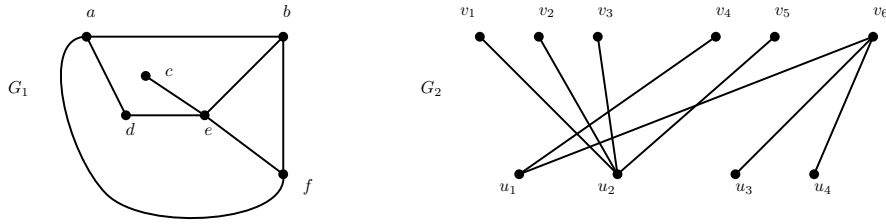


Figure 42: Two edge covers.

In G_1 , the set $\{b, d, e, a\}$ is an edge cover, as is the set $\{a, e, f\}$. In G_2 , $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is an edge cover as is the set $\{v_2, v_6, v_1\}$.

What is the size of a minimum edge cover? We note $\tau(G)$ the size of a minimum edge cover.

Lemma 8.1. Recall that for a graph G , $\nu(G)$ is the size of a maximal matching and $\tau(G)$ is the size of a minimum edge cover. We have $\nu(G) \leq \tau(G)$.

Proof. An edge cover T contains one end vertex at least of each edge e of a matching M in G . To each edge e of M , we can associate one of its endpoints v_e . As M is a matching for $e \neq f$, $v_e \neq v_f$. We thus have

$$|T| \geq |\{v_e : e \in M\}| = |M|.$$

Since this is true for any edge cover T and any matching M it follows that $\tau(G) \geq \nu(G)$. □

Example 8.5.

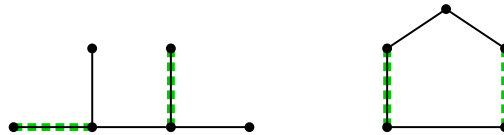


Figure 43: See Lemma above.

Theorem 8.2 (König-Egerváry). *Let $G = (U, V; E)$ be a bipartite graph. We have $\tau(G) = \nu(G)$.*

Proof. Let M be a maximum matching of $G = (U, V; E)$. For $X \subset V(G)$, we note $N_M(X)$ the neighbors of elements in X , in the subgraph $(U, V; M)$. We partition U as $U_1 \cup U_2 \cup U_3$ and $V = V_1 \cup V_2 \cup V_3$ as follows:

$$U_1 = \{u \in U : u \text{ is not } M\text{-saturated}\}.$$

$$V_1 = \{v \in V : v \text{ is not } M\text{-saturated}\}.$$

$$U_2 = \{u \in U \setminus U_1 : \exists u' \in V_1 \text{ and a } (u', u) \text{ - path which is } M\text{-saturated}\}.$$

$$V_2 = N_M(U_2).$$

$$U_3 = U \setminus (U_1 \cup U_2).$$

$$V_3 = N_M(U_3).$$

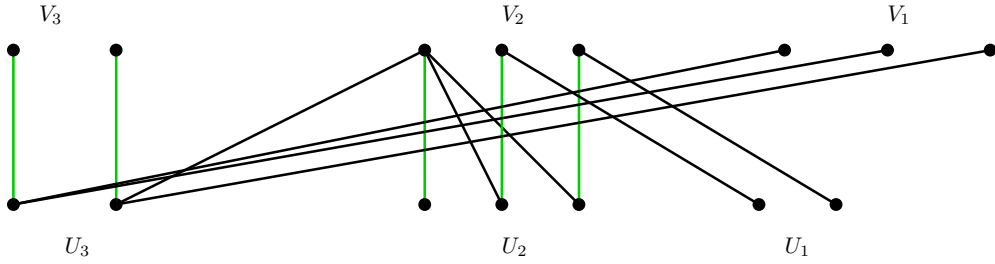


Figure 44: The partition of U and V .

We first show that $Q = U_3 \cup V_2$ is a edge cover. If Q were not an edge cover, then there would exist an edge $\{u, v\}$, with $u \in U_1 \cup U_2$ and $v \in V_3 \cup V_2$. If $u \in U_2$, we note P_u a path from $u' \in V_1$ to u . If $u \in U_1$, we just note P_u to the empty walk (u) .

Case 1: $v \in V_1$. In this case $P' = P_u \cup \{u, v\}$ would be M -augmenting and Berge's theorem implies that M is not maximal, which is a contradiction.

Case 2: $v \in V_3$. Let $v_m \in N_M(v)$, $P' = P_u \cup \{u, v\} \cup \{v, v_m\}$. Then (v', v_m) would be M -alternating, $u' \in V_1$, we have v_m should be in U_2 and not in V_3 . Contradiction.

Thus Q is an edge covering of G . Next we note that $|V_2| = |N_M(U_2)| = |U_2|$. Hence $|Q| = |V_2 \cup U_3| = |V_2| + |U_3| = |U_2| + |V_3| = |M|$. Now, using the previous lemma (Lemma 8.1), we see that $|M| \leq \nu(G) \leq \tau(G) \leq |Q| = |M|$ and thus $\nu(G) = \tau(G)$. \square

We now state a result that is not necessarily graph theoretical.

Definition 8.5. Let $\mathcal{A} = (A_i)_{i \in I}$ be a family of non empty sets. Let $S = \cup_{i \in I} A_i$. A **system of representations** of the family \mathcal{A} , noted **SR**, is a family of elements $(x_i)_{i \in I}$ of S , such that $x_i \in A_i$ for all $i \in I$. A **system of distinct representations** of the family \mathcal{A} , noted **SDR**, is a family $(x_i)_{i \in I}$ of SR, such that $x_i \neq x_j$ if $i \neq j$.

Example 8.6. $A_1 = \{2, 8\}$, $A_2 = \{8\}$, $A_3 = \{5, 7\}$, $A_4 = \{2, 4, 8\}$, $A_5 = \{2, 4\}$. The family $\mathcal{A} = (A_i)_{i=1}^5$, has a SDR, namely $\{2, 8, 7, 4\}$. The family $\mathcal{A}' = (A_1, A_2, A_4, A_5)$ does not have a SDR.

Theorem 8.3 (Hall). A finite family $\mathcal{A} = (A_i)_{i \in I}$ of non empty sets has a SDR iff \mathcal{A} satisfies Hall's property:

$$\forall J \subset I, |A_J| = \left| \bigcup_{i \in J} A_i \right| \geq |J|.$$

Proof. For all $J \subset I$, we note $A_J = \cup_{i \in J} A_i$.

" \implies " Let us assume that the family \mathcal{A} has a SDR $(x_i)_{i \in I}$. For $J \subset I$, A_J contains $\{x_i, i \in J\}$ which is by definition of cardinality J , thus $|A_J| \geq |J|$.

" \impliedby " Let us now assume that $\forall J \subset I, |A_J| \geq |J|$.

If for each $i \in I$, $A_i = \{x_i\}$, then $(x_i)_{i \in I}$ is a SDR. Otherwise, $\exists i_0 \in I$ such that A_{i_0} contains at least two distinct elements x and y . Let $B_{i_0} = A_{i_0} \setminus \{x\}$ and $C_{i_0} = A_{i_0} \setminus \{y\}$. Let \mathcal{B} and \mathcal{C} be the families obtained by replacing A_{i_0} with respectively B_{i_0} and C_{i_0} . We next show that one of \mathcal{B} and \mathcal{C} satisfies Hall's criterion.

If not, there would exists J and M in $I \setminus \{i_0\}$, such that

$$|J \cup \{i_0\}| > |B_{i_0} \cup A_J| \implies |J| \geq |B_{i_0} \cup A_J|,$$

and

$$|K| \geq |C_{i_0} \cup A_J|.$$

Note $S = B_{i_0} \cup A_J$, $T = C_{i_0} \cup A_M$. We thus have $|J| \geq |S| + |M| \geq |T|$. But $S \cup T = B_{i_0} \cup C_{i_0} \cup A_J \cup A_M = A_{i_0} \cup A_{J \cup M}$, and it follows from Hall's condition that $|S \cup T| \geq 1 + |J \cup M|$.

Moreover $S \cap T = (B_{i_0} \cup A_J) \cap (C_{i_0} \cup A_M) = A_J \cap A_M$. Hence it follows again from Hall's condition that $|S \cap T| \geq |J \cap M|$. Consequently

$|J|+|M| \geq |S|+|M| = |S \cup T|+|S \cap T| \geq 1+|J \cup M|+|J \cap M| = 1+|J|+|M|$, which is a contradiction. Thus either \mathcal{B} or \mathcal{C} satisfies Hall's condition. The set I being finite, representing the process we obtain a family of sets with one element satisfying Hall's condition. Such a family is a SDR for \mathcal{A} . \square

Now we give a graph version of Hall's theorem.

Theorem 8.4. *Let $G = (A, B; E)$ be a bipartite graph. Then G has a perfect matching iff*

1. $|A| = |B|$.
2. For all $X \subset A$, $|N(X)| \geq |X|$.

Proof. " \implies " If G has a perfect matching M , then for all $X \subset A$, $|N(X)| \geq |N_M(X)| = |X|$. In particular $|A| = |N_M(A)| = |B|$.

" \impliedby " Let Q be an edge cover of minimal cardinality. Let $A_1 = Q \cap A$, $B_1 = Q \cap B$ and $A_2 = A \setminus A_1$. As Q is an edge cover we have $N(A_2) \subset B_1$ and so $|B_1| \geq |N(A_2)| \geq |A_2|$. By the König-Egerváry theorem, we obtain that

$$\nu(G) = \tau(G) = |Q| = |A_1 \cup B_1| = |A_1| + |B_1| \geq |A_1| + |A_2| = |A|.$$

We have then proven that G has a matching M for which all vertices of A are saturated. So $|A| < |B|$ and M is a perfect matching. \square

Corollary 8.1. *A bipartite regular graph admits a perfect matching.*

Proof. Let $G = (A, B; E)$ be bipartite and k -regular. For all $X \subset A$, we have

$$k|X| = \sum_{u \in X} d(u) = E|G[X \cup N(X)]| \leq \sum_{v \in N(X)} d(v) = k|N(X)|,$$

thus $|X| \leq |N(X)|$ for all $X \subset A$. Similarly we show for all $X \subset B$ that $|X| \leq |N(X)|$. Take then $X = A$ and B to obtain $|A| = |B|$. \square

As an application we study the marriage problem. Let us consider a set of 5 boys $\{b_1, \dots, b_5\}$ and a set of 5 girls $\{g_1, \dots, g_5\}$. Each boy g_i makes a list of his favourite girls.

$$b_1 = \{g_1, g_2, g_3\}, b_2 = \{g_4, g_5\}, b_3 = \{g_4\}, b_4 = \{g_5\}, b_5 = \{g_2, g_5\}.$$

Is it possible that each boy can marry one of his favourite girls?. Let $G = (A, B; E)$. $A = \{b_1, \dots, b_5\}$, $B = \{g_1, \dots, g_5\}$ and $\{b_i, g_i\}$ is an $??$ iff g_i is on the list of favourite girls of b_i . The problem has a solution iff G has a perfect

matching. But for $A' = \{b_2, b_3, b_4\}$, $N(A') = \{g_4, g_5\}$ and $|A'| > |N(A')|$ so there is no perfect matching. There also exists a necessary and sufficient condition for the existence of perfect matching for a generic graph. If G is a graph and $S \subset V(G)$ then we denote $c_0(G - S)$ the number of connected components of $G - S$ that have an odd number of vertices.

Theorem 8.5 (Tutte's theorem). *A graph G has a perfect matching iff for all subsets S of the vertex set of G , we have $c_0(G - S) \leq |S|$.*

9 Planar graphs

Definition 9.1. Let G be a graph that can be drawn on a plane surface so that no of its edges intersect. Then G is called a **planar graph**. Let G be a planar graph. Then the edges of G partition the plane into regions, we will call these regions the **faces** of G .

Example 9.1.

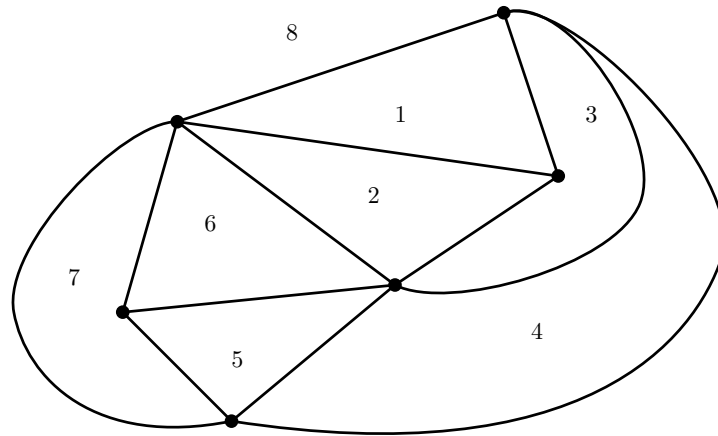


Figure 45: Example of a planar graph as defined above.

Theorem 9.1 (Euler's theorem on Planar graphs). Let G be a connected planar graph with V vertices, E edges and F faces. Then

$$V + F = E + 2.$$

Proof. We prove the statement by induction on E , the number of edges of G . If $E = 1$, then G is either a tree of one edge, $V = 2, F = 1$, and the statement is true, or G is the one vertex graph with a loop, thus $V = 1, F = 2$, and the statement is true again.

$E - 1 \rightarrow E$. Assume that G has E edges. We distinguish two cases. If we can omit an edge e from G so that the new graph G' is still connected, then e is in a cycle in G , and therefore from ??? his faces on the two sides of e in G . Then G' has $E - 1$ edges, V vertices and F_1 faces. Therefore $V + F - 1 = E - 1 + 2$ which gives $V + F = E + 2$. If there is no e with the mentioned property, then G is a cycle-free connected graph, that is a tree. Then $V = E + 1$. On the other hand $F = 1$ So the claim is true. \square

Example 9.2. The graph $K_{3,3}$ is not planar.

Remark 9.1. The previous definitions and results are intuitively *simple*, but we need to define them more rigorously.

Definition 9.2. A graph G is called planar if it has a representation in \mathbb{R}^2 in such a way that its edges are **simple curves** which do not intersect except at their endpoints. So in a planar graph the edges can intersect only at vertices.

Remark 9.2. Note that a planar representation of a graph is not unique in general.

Example 9.3.

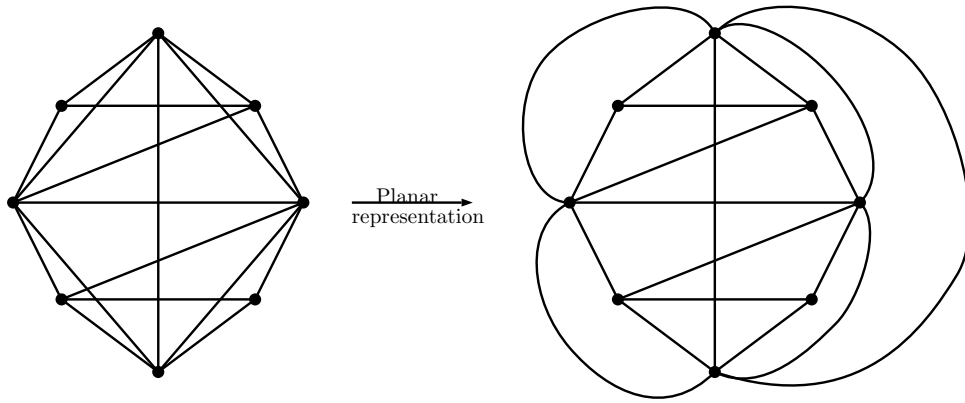


Figure 46: Example of a planar graph as defined above.

We shall use in what follows **Jordan's theorem**: A simple closed curve splits the plane in two domains (that is an open and connected set in \mathbb{R}^2), one of them being bounded and the other is unbounded. The curve is the common boundary to these two domains. In the representation of a planar graph G , the complements of edges given a partition of the plane \mathbb{R}^2 into domains, called the faces of the graph G . By definition, a planar representation of a graph has one unbounded face, which we shall call the **exterior face** or the **infinite face** of G .

A face f of a connected graph has as boundary a closed walk called the boundary of f . The length of the face f is noted $l(f)$ and is the length of this closed walk.

Example 9.4.

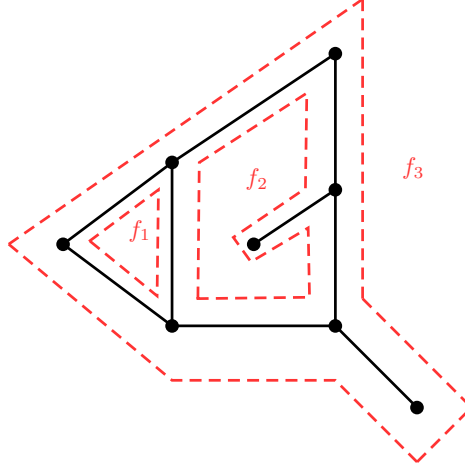


Figure 47: Three faces f_1, f_2, f_3 , with $l(f_1) = 3$, $l(f_2) = 7$, $l(f_3) = 8$.

Remark 9.3. A forest is a planar graph with a single face (the infinite face).

Example 9.5. The graph $K_{3,3}$ is not planar. Indeed, suppose that $K_{3,3}$ has a planar representation. If $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ is the partition of $V(K_{3,3})$, let $(a_1, b_1, a_2, b_2, a_1)$ be the 4-cycle of $K_{3,3}$. It follows from the Jordan's theorem that C partitions the plane into two domains. If b_3 is inside (resp. outside C), a_3 cannot be either inside, nor outside.

Example 9.6. K_5 is not planar. If K_5 had a planar representation, let C be the 5-cycle of K_5 . It follows from Jordan's theorem that C splits \mathbb{R}^2 into two domains. The 5 diagonals of C are edges of K_5 . Among these 5 diagonals, at most 4 can be drawn inside C and 2 outside C . Hence we cannot draw the last diagonal without crossing another diagonal.

Remark 9.4. Euler's formula for connected graphs is $n - m + r = 2$, where $n = |V|$, $m = |E|$, r is the number of faces.

As a consequence of Euler's formula we have:

Corollary 9.1. The number of faces of a planar graph does not depend on the chosen representation.

Lemma 9.1. *Let $G = (V, E)$ be a planar graph. We note F the set of its faces. We then have*

$$2|E| = \sum_{f \in F} l(f),$$

where $l(f)$ is the length of the face f .

Corollary 9.2. *Let G be a simple planar graph, of order $n \geq 3$, with $|E| = m$. We have:*

$$m \leq 3n - 6.$$

if in addition G does not contain a triangle, then

$$m \leq 2n - 4.$$

Proof. We can assume G is connected (otherwise we add edges). We have

$$2m = \sum_{f \in F} l(f).$$

Since G is simple, and $n \geq 3$, we have $l(f) \geq 3$. Hence $2m \geq 3|F| = 3(2 - n + m)$ which gives $3n \geq m + 6$. If G does not have a triangle, then $2m \geq 4|F| = 4(2 - n + m)$ which gives $2n \geq m + 4$. \square

Corollary 9.3. *A simple planar graph G satisfies $\delta(G) \leq 5$. If in addition it does not contain a triangle, then $\delta(G) \leq 3$.*

Proof. We can again assume that G is connected. If $\delta(G) \geq 4$, then $2m \geq 6n$. But from the previous corollary $3n - 6 \geq m \geq 3n$ which is a contradiction. If G does not have a triangle we have $2m \geq 4n$ and $2n - 5 \geq m \geq 2n$ which is again a contradiction. \square

Definition 9.3. *A polyhedron is a solid in \mathbb{R}^3 whose boundary is a union of polygons, called faces. The faces intersect at vertices and the lines that join two vertices are called edges. A polyhedron is called **regular**, if all its faces are regular polygons, i.e. all its faces have the same number of edges, all vertices are contained in the same number d of edges (d is called the degree of the polyhedron), all edges have the same length, all angles within the faces are equal, and all angles between the faces are equal.*

Example 9.7. *The cube is a regular polyhedron.*

In combinatorics, we discussed the conditions on the length of the edges, and the sign of angles, but we keep the graph theoretical conditions that each face is a cycle with l edges and each vertex has degree d . We will show, using Euler's formula, that there are only 5 regular and convex polyhedra.

Theorem 9.2. *Let P be a convex polyhedron with V vertices, F faces and E edges. Then $V + F - E = 2$.*

Next it is easy to see that in any convex polyhedra V , $3F \leq 2E$ and $3V \leq 2E$. From Corollary 9.3 we have $E \leq 3V - 6$. Similarly $E \leq 3F - 6$.

Lemma 9.2. *All convex polyhedra have at least one face that has at most five edges.*

Proof. We know that $E \leq 3F - 6$ and

$$\sum_{i=1}^F f_i = 2E \leq 6F - 12.$$

Therefore if $f_i \geq 6 \forall i$, $\sum_{i=1}^F f_i \geq 6F$, which is a contradiction. □

Similarly, we establish

Lemma 9.3. *All convex polyhedra have at least one vertex that is contained in at most five edges.*

The previous lemma shows that in regular polyhedra, the degree d of each vertex can only be 3, 4 or 5. Same thing for l . So there are $3 \times 3 = 9$ cases to check. Then one can show that:

1. $d = 3$, $l = 3$, $F = 5$, $V = 4$ which is a tetrahedron.
2. $d = 3$, $l = 4$, $F = 6$, $V = 8$ which is a cube.
3. $d = 3$, $l = 5$, $F = 12$, $V = 20$ which is a dodecahedron.
4. $d = 4$, $l = 3$, $F = 8$, $E = 12$, $V = 6$ which is octahedron.
5. $d = 5$, $l = 3$, $F = 20$, $E = 30$, $V = 12$ which is a icosahedron.

Theorem 9.3. *There are five regular polyhedra: The tetrahedron, the cube, the dodecahedron, the octahedron, the icosahedron.*