ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 13

Solution 13-1

Denote by $Z = (Z_t)_{t\in[0,T]}$ the density process process of Q with respect to P.

a) The second claim follows directly from the first claim together with the fact that $zZ_T = z\frac{dQ}{dP} \in \mathcal{D}(z)$ since $Z \in \mathcal{Z}(1)$ and the fact that the function J is decreasing. So it remains to show the first claim. Seeking a contradiction, suppose there exists $h \in \mathcal{D}(z)$ such that $A := \{h > zZ_T\}$ has $P[A] > 0$. Set $a = Q[A] > 0$ and define the Q-martingale $M = (M_t)_{t\in[0,T]}$ by $M_t := E_Q[1_A \mid \mathcal{F}_t]$. Then M is nonnegative and $M_0 = a$ by the fact that $\mathcal{F}_0$ is P-trivial. By the predictable representation property of S under Q, there exists $H \in L(S)$ such that $M = a + H \bullet S$. Thus, $M \in \mathcal{V}(a)$. Now, on the one hand, by the definition of $\mathcal{D}(z)$, there exists a supermartingale $\widetilde{Z} \in \mathcal{Z}(z)$ with $h \leq \widetilde{Z}_T$. Therefore,

$$E[M_T h] \leq E[M_T \widetilde{Z}_T] \leq E[M_0 \widetilde{Z}_0] = az. \tag{1}$$

On the other hand,

$$E[Z_T M_T] = E_Q[M_T] = M_0 = a. \tag{2}$$

Thus, we arrive at the contradiction

$$0 \geq E[M_T(h - zZ_T)] = E[1_{\{h>zZ_T\}}(h - zZ_T)] > 0. \tag{3}$$

b) Note that $0 \leq z_0 < \infty$ and $j(z) < \infty$ on (z_0, ∞) by the first part of Theorem 11.10 (which does not assume that $AE_{+\infty}(U) < 1$). Moreover, recall that the function J is strictly decreasing, strictly convex and in C^1 on $(0, \infty)$.

First, define the function $g : (z_0, \infty) \to [-\infty, 0]$ by

$$g(s) = E[Z_T J'(sZ_T)]. \tag{4}$$

This is well defined as $Z_T > 0$ P-a.s. and $J' < 0$. Moreover, it is increasing as J' is increasing. Thus if $g(s_0) > -\infty$ for some $s_0 > z_0$, it follows by dominated convergence that it is continuous on (s_0, ∞).

Next, for $z_1, z_2 \in (z_0, \infty)$, $z_1 < z_2$, the fundamental theorem of calculus gives

$$J(z_2 Z_T) - J(z_1 Z_T) = \int_{z_1}^{z_2} Z_T J'(sZ_T)\, ds. \tag{5}$$

Now, the left-hand side of (5) is integrable by assumption. Thus, the right-hand side is so, too, and since $J' < 0$, the integrand on the right-hand side is strictly negative, and Fubini's theorem gives

$$j(z_2) - j(z_1) = \int_{z_1}^{z_2} g(s)\, ds. \tag{6}$$

In particular, the function g is finite a.e. on (z_0, ∞), and thus continuous and finite on (z_0, ∞). Now the claim follows from the fundamental theorem of calculus.

c) First, Proposition 11.14 in the lecture notes shows that $f \in \mathcal{C}(x)$ if and only if

$$\sup_{h\in\mathcal{D}(1)} E[fh] \leq x. \tag{7}$$

By part **a)**, this is equivalent to

$$E[fZ_T] \leq x. \tag{8}$$

Now, by part **b)** and the choice of z_x,

$$E[f^* Z_T] = E[-J'(z_x Z_T) Z_T] = -j'(z_x) = x, \tag{9}$$

and so $f^* \in \mathcal{C}(x)$.

Next, fix $f \in \mathcal{C}(x)$. We may assume without loss of generality that $E[U(f)] > -\infty$. By the fact that $f^* > 0$ P-a.s. and U is in C^1 and strictly concave on $(0, \infty)$,

$$U(f) - U(f^*) \le U'(f^*)(f - f^*), \tag{10}$$

where the equality is strict on $\{f \neq f^*\}$. Taking expectations and using the fact that $U'(-J') = \mathrm{id}$ and (8) and (9) gives

$$E[U(f) - U(f^*)] \le E[U'(f^*)(f - f^*)] = z_x E[Z_T(f - f^*)] \le 0. \tag{11}$$

If $f = f^*$ P-a.s., then both inequalities are trivially equalities, and if $P[f \neq f^*] > 0$, then the first inequality is strict.

Solution 13-2

The discounted stock price process S^1 satisfies the SDE

$$dS_t^1 = S_t^1((\mu - r)\, dt + \sigma\, dW_t) = S_t^1 \sigma(\lambda\, dt + dW_t), \tag{12}$$

where $\lambda := \frac{\mu - r}{\sigma}$ denotes the market price of risk. We know from Chapter 3 in the lecture that S has a unique equivalent martingale measure $Q \approx P$ on $\mathcal{F}_T$ given by

$$\frac{dQ}{dP} = \mathcal{E}(-\lambda W)_T. \tag{13}$$

Moreover, elementary analysis gives $J(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}}$ and $J'(z) = -z^{-\frac{1}{1-\gamma}}$.

a) Fix $z > 0$. Then by Exercise **13-1 a)** and the fact that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$,

$$\begin{aligned}
j(z) &= E\left[\frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \left(\mathcal{E}(-\lambda W)_T\right)^{-\frac{\gamma}{1-\gamma}}\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma} W_T + \frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma} T\right)\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma} W\right)_T\right] \\
&= \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) < \infty.
\end{aligned} \tag{14}$$

b) First, note that $j(z) < \infty$ for some $z \in (0, \infty)$ implies that

$$u(x) \le j(z) + zx < \infty, \quad x \in (0, \infty). \tag{15}$$

Next, fix $x > 0$. Then by Exercise **13-1 b)** and part **a)**,

$$\begin{aligned}
f_x^* &= -J'\left(z_x \frac{dQ}{dP}\right) = z_x^{-\frac{1}{1-\gamma}} \left(\mathcal{E}(-\lambda W)_T\right)^{-\frac{1}{1-\gamma}} \\
&= -j'(z_x) \exp\left(-\frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2} T\right) \exp\left(\frac{\lambda}{1-\gamma} W_T + \frac{1}{2}\frac{\lambda^2}{1-\gamma} T\right) \\
&= x \exp\left(\frac{\lambda}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2}{(1-\gamma)^2} T\right) \\
&= x \mathcal{E}\left(\frac{\lambda}{1-\gamma} R\right)_T.
\end{aligned} \tag{16}$$

c) Fix $x > 0$. By the definition of the stochastic exponential,

$$
\begin{aligned}
f_x^* &= x\left(1 + \int_0^T \mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma}\,dR_t\right) \\
&= x + \int_0^T x\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_t \frac{\lambda}{1-\gamma}\frac{1}{\sigma S_t}\,dS_t.
\end{aligned}
\tag{17}
$$

This gives the first claim. Using again that $\mathcal{E}(aW)$ is a P-martingale for all $a \in \mathbb{R}$ gives

$$
\begin{aligned}
u(x) = E\left[U(f_x^*)\right] &= \frac{x^\gamma}{\gamma} E\left[\left(\mathcal{E}\left(\frac{\lambda}{1-\gamma}R\right)_T\right)^\gamma\right] \\
&= \frac{x^\gamma}{\gamma} E\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}(W_T + \lambda T) - \frac{1}{2}\frac{\lambda^2\gamma}{(1-\gamma)^2}T\right)\right] \\
&= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right) E\left[\mathcal{E}\left(\frac{\lambda\gamma}{1-\gamma}W\right)_T\right] \\
&= \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\lambda^2\gamma}{1-\gamma}T\right).
\end{aligned}
\tag{18}
$$

This establishes the second claim.

d) We deduce from **b)** and **c)** that

$$
\begin{aligned}
\vartheta_t^x &= \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t, \\
V_t(x, \vartheta^x) &= x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu - r}{\sigma}R\right)_t
\end{aligned}
$$

Therefore, we obtain directly that

$$
\pi_t^* := \frac{\vartheta_t^x S_t^1}{V_t(x, \vartheta^x)} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.
$$

Solution 13-3

a) Suppose that the numéraire portfolio N^* exists and that $\ell < \infty$. By the hint,

$$
\log N_T^* \geq \log N_T + 1 - \frac{N_T}{N_T^*} \text{ for all } N \in \mathcal{N}. \tag{19}
$$

For $N \equiv 1$, using that $E\left[\frac{1}{N_T^*}\right] \leq 1$ by the definition of the numéraire portfolio, this gives $\log^- N_T^* \in L^1(P)$. Now let $N \in \mathcal{N}$ with $\log^- N_T \in L^1(P)$. Then using that $E\left[\frac{N_T}{N_T^*}\right] \leq 1$ by the definition of the numéraire portfolio gives

$$
\ell \geq E[\log N_T^*] \geq E[\log N_T] + 1 - E\left[\frac{N_T}{N_T^*}\right] \geq E[\log N_T]. \tag{20}
$$

Thus N^* is the growth optimal portfolio.

b) Let $\epsilon \in (0, 1)$ and set $N^\epsilon := \epsilon N + (1-\epsilon)N^{\log}$. Then $N^\epsilon \in \mathcal{N}$ as $\mathcal{N}$ is clearly convex. Moreover, since

$$
\log N_T^\epsilon \geq \log(1-\epsilon) + \log N_T^{\log} \tag{21}
$$

and $\log N_T^{\log} \in L^1(P)$, it follows first that $\log^- N_T^\epsilon \in L^1(P)$ and then $\log N_T^\epsilon \in L^1(P)$ as $\ell < \infty$. Thus by optimality of $N^{\log}$ and the fact that $\log x \leq x - 1$ for $x > 0$,

$$0 \leq E[\log N_T^{\log} - \log N_T^\epsilon] = E\left[\log \frac{N_T^{\log}}{N_T^\epsilon}\right] \leq E\left[\frac{N_T^{\log} - N_T^\epsilon}{N_T^\epsilon}\right] = E\left[\frac{\epsilon(N_T^{\log} - N_T)}{N_T^\epsilon}\right]. \tag{22}$$

Thus,

$$0 \leq E\left[\frac{N_T^{\log} - N_T}{N_T^\epsilon}\right] = E\left[\frac{N_T^{\log} - N_T}{(1-\epsilon)N_T^{\log} + \epsilon N_T}\right]. \tag{23}$$

Moreover, for $\epsilon \leq 1/2$, the fact that $N_T^{\log}, N_T > 0$ P-a.s. gives

$$\frac{N_T^{\log} - N_T}{(1-\epsilon)N_T^{\log} + \epsilon N_T} \leq \frac{N_T^{\log}}{(1-\epsilon)N_T^{\log}} \leq 2\frac{N_T^{\log}}{N_T^{\log}} = 2 \;\; P\text{-a.s.} \tag{24}$$

Thus, we can apply (reverse) Fatou's lemma to (23) for $\epsilon \downarrow 0$ and get

$$0 \leq E\left[\frac{N_T^{\log} - N_T}{N_T^\epsilon}\right] = E\left[\frac{N_T^{\log} - N_T}{(1-\epsilon)N_T^{\log} + \epsilon N_T}\right] \leq E\left[\frac{N_T^{\log} - N_T}{N_T^{\log}}\right],$$

which implies that

$$E\left[\frac{N_T}{N_T^{\log}}\right] \leq E\left[\frac{N_T^{\log}}{N_T^{\log}}\right] = 1. \tag{25}$$

c) First, we show that $\widetilde{\vartheta} \in L(S)$. By definition, $\widetilde{\vartheta}$ is predictable. Moreover, observe that if H^1, H^2 are both in $L(S)$, then $H^1 + H^2 \in L(S)$. Furthermore, if $H \in L(S)$ and G is predictable, locally bounded, then $GH \in L(S)$, too. Thus, we see from the structure of $\widetilde{\vartheta}$ that it is in $L(S)$. Next, for $u \in [0, s]$ and $u \in (s, T]$ on A^c,

$$\widetilde{N}_u = 1 + \widetilde{\vartheta} \bullet S_u = 1 + \vartheta^1 \bullet S_u = N_u^1 > 0 \;\; P\text{-a.s.}, \tag{26}$$

and for $u \in (s, T]$ on A,

$$\begin{aligned}\widetilde{N}_u = 1 + \widetilde{\vartheta} \bullet S_u &= 1 + \widetilde{\vartheta} \bullet S_s + \int_s^u \widetilde{\vartheta}_t \, dS_t = N_s^1 + \frac{N_s^1}{N_s^2}\int_s^u \vartheta_t^2 \, dS_t \\ &= N_s^1 + \frac{N_s^1}{N_s^2}(N_u^2 - N_s^2) = \frac{N_s^1}{N_s^2}N_u^2 > 0 \;\; P\text{-a.s.}\end{aligned} \tag{27}$$

This shows that $\widetilde{N} = 1 + \widetilde{\vartheta} \bullet S \in \mathcal{N}$.

d) We claim that $N^{\log}$ is the numéraire portfolio. Seeking a contradiction, suppose there exist $0 \leq s < t \leq T$ and $N \in \mathcal{N}$ such that the set

$$A := \{E[N_t/N_t^{\log}|\mathcal{F}_s] > N_s/N_s^{\log}\}$$

has $P[A] > 0$. Define

$$\widehat{N} := 1_{[\![0,s]\!]}N^{\log} + 1_{]\!]s,t]\!]}\left(1_A\frac{N_s^{\log}}{N_s}N + 1_{A^c}N^{\log}\right) + 1_{]\!]t,T]\!]}\left(1_A\frac{N_s^{\log}}{N_s}\frac{N_t}{N_t^{\log}} + 1_{A^c}\right)N^{\log}. \tag{28}$$

We claim that $\widehat{N} \in \mathcal{N}$. Indeed, set

$$N^1 := 1_{[\![0,s]\!]}N^{\log} + 1_{]\!]s,T]\!]}\left(1_A\frac{N_s^{\log}}{N_s}N + 1_{A^c}N^{\log}\right) \quad \text{and} \quad N^2 := N^{\log}. \tag{29}$$

Then $N^1, N^2 \in \mathcal{N}$ by part **c)** and trivially. Now, set

$$\widetilde{N} := 1_{[\![0,t]\!]} N^1 + 1_{]\!]t,T]\!]} \left(1_A \frac{N_t^1}{N_t^2} N^2 + 1_{A^c} N^1 \right). \tag{30}$$

Then $\widetilde{N} \in \mathcal{N}$ by part **c)** and since $A \in \mathcal{F}_s \subset \mathcal{F}_t$. Moreover, for $u \in [0,s]$ and $u \in (s,T]$ on A^c,

$$\widetilde{N}_u = N_u^1 = N_u^{\log} = \widehat{N}_u \;\; P\text{-a.s.}, \tag{31}$$

for $u \in (s,t]$ on A,

$$\widetilde{N}_u = N_u^1 = \frac{N_s^{\log}}{N_s} N_u = \widehat{N}_u \;\; P\text{-a.s.}, \tag{32}$$

and for $u \in (t,T]$ on A,

$$\widetilde{N}_u = \frac{N_t^1}{N_t^2} N_u^2 = \frac{\frac{N_s^{\log}}{N_s} N_t}{N_t^{\log}} N_u^{\log} = \widehat{N}_u \;\; P\text{-a.s.} \tag{33}$$

Thus $\widetilde{N} = \widehat{N} \in \mathcal{N}$.

Finally, by part **b)**, by the tower property of conditional expectations and by the definition of A,

$$\begin{aligned} 1 \geq E\left[\frac{\widetilde{N}_T}{N_T^{\log}} \right] &= E\left[1_A \frac{N_s^{\log}}{N_s} \frac{N_t}{N_t^{\log}} + 1_{A^c} \right] \\ &= E\left[1_A \frac{N_s^{\log}}{N_s} E\left[\frac{N_t}{N_t^{\log}} \,\middle|\, \mathcal{F}_s \right] + 1_{A^c} \right] > E[1_A + 1_{A^c}] = 1. \end{aligned} \tag{34}$$

Thus, we arrive at a contradiction.