Mathematical Finance
Exercise Sheet 2

Please hand in until Wednesday, October 1st, 12:00 in your assistant's box in HG G 52.1

Exercise 2-1

Consider the multiperiod Cox–Ross–Rubinstein binomial model for the bank account $\tilde{B}$ and stock price process $\tilde{S}$ with parameters r (i.e., $\tilde{B}_n = (1+r)^n$) and $u > d > -1$. That is, the discounted stock price $S = \tilde{S}/\tilde{B}$ is given by

$$S_k = s_0 \prod_{j=1}^{k} R_j, \quad k = 0, 1, \ldots, T, \;\; s_0 > 0,$$

where the R_j are i.i.d. under P with

$$P\left[R_j = \frac{1+u}{1+r}\right] = p = 1 - P\left[R_j = \frac{1+d}{1+r}\right]$$

for some $p \in (0,1)$.

a) If $u \leq r$ or $d \geq r$, find an arbitrage opportunity, i.e., a self-financing strategy φ with $V_0(\varphi) = 0$, $V_T(\varphi) \geq 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$.
Hint: Compare the behaviour of $\tilde{S}$ and $\tilde{B}$.

b) If $u > r > d$, find a probability measure $Q \approx P$ such that S is a Q-martingale.

Together with the results from the lecture, this shows that the CRR model is arbitrage-free if and only if $u > r > d$.

Exercise 2-2

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space and $\bar{S} = (1, S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ a general continuous-time financial market with time horizon $T > 0$.

a) Show that if there exists an (elementary) arbitrage opportunity

$$\vartheta = \sum_{k=1}^{N} h_k 1_{]\!]\tau_{k-1},\tau_k]\!]} \in \mathbf{b}\mathcal{E},$$

then there also exists a "one-step buy-and-hold" arbitrage opportunity $\vartheta^* = h1_{]\!]\sigma_0,\sigma_1]\!]} \in \mathbf{b}\mathcal{E}$.

b) Assume that S is a semimartingale and satisfies NA. Prove that if $\vartheta \in \Theta_{adm}$ satisfies $G_T(\vartheta) \geq -c$ P-a.s. for some $c \geq 0$, then $G_.(\vartheta) \geq -c$ P-a.s.
Hint: Use that if ϑ is S-integrable, and $C \in \mathcal{P}$ is a predictable set, then $1_C\,\vartheta$ is S-integrable, too.

Please turn the page

Exercise 2-3

Consider a probability space with discrete-time filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ and an $\mathbb{R}^d$-valued local martingale $X = (X_t)_{t\in\mathbb{N}}$ null at 0. Prove the following:

a) If all the X_t are integrable, then X is a true martingale.

b) The following sharpening is also true: For any $T \in \mathbb{N}$ we have that if X_T^- is integrable, then $(X_t)_{t=0,1,\dots,T}$ is a true martingale. In particular, if X is bounded from below, then X is a true martingale.
Hint: First show via localization that X^- has the submartingale property. Deduce integrability of each X_t^- and hence of X_t^+.

c) If ϑ is any $\mathbb{R}^d$-valued predictable process with $\vartheta_0 := 0$, then the real-valued discrete-time integral $Y = \int \vartheta \, dX$, defined by $Y_t = \sum_{k=1}^{t} \vartheta_k (X_k - X_{k-1})$, is again a local martingale.

Note: All these assertions are false in continuous time in general.

Exercise sheets and further information are also available on:

http://www.math.ethz.ch/education/bachelor/lectures/hs2014/math/mf/uebungen