ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 7

Solution 7-1

a) Clearly $\big(\mathcal{M}_{loc}\big)_{loc} \supseteq \mathcal{M}_{loc}$. To show the opposite, let $M \in \big(\mathcal{M}_{loc}\big)_{loc}$. We want to find a sequence of stopping times (R_n) converging to ∞ a.s. such that $M^{R_n} \in \mathcal{M}$ for each n. Since $M \in \big(\mathcal{M}_{loc}\big)_{loc}$, there exists a localizing sequence (T_n) such that $M^{T_n} \in \mathcal{M}_{loc}$ for each n. Thus, by definition of $\mathcal{M}_{loc}$, there exists for each n a localizing sequence $(S_{n,m})_m$ such that $(M^{T_n})^{S_{n,m}} \in \mathcal{M}$ for each m. For each n, choose m_n large enough such that

$$P\big[S_{n,m_n} < T_n \wedge n\big] \leq 2^{-n}$$

and then define

$$R_n := T_n \wedge \big(\inf_{k\geq n} S_{k,m_k}\big).$$

By definition (R_n) is a sequence of stopping times. Moreover, by Borel-Cantelli, R_n converges to ∞ a.s. like T_n as

$$P\big[R_n < T_n \wedge n\big] \leq \sum_{k=n}^{\infty} P\big[S_{k,m_k} < T_n \wedge n\big] \leq \sum_{k=n}^{\infty} P\big[S_{k,m_k} < T_k \wedge k\big] \leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1} < \infty.$$

Moreover, as every stopped martingale is a martingale, we obtain that for each n

$$M^{R_n} = \Big(\big(M^{T_n}\big)^{S_{n,m_n}}\Big)^{R_n} \in \mathcal{M}.$$

b) By definition of being a σ-martingale, $ZS = \int \psi\, dM$ for a local martingale M and an integrand $\psi \in L(M)$ with $\psi > 0$. We claim that it suffices to show that ZS has locally a lower bound in L^1. To prove this claim we first show that $\mathcal{M}_{loc} = \mathcal{H}^1_{loc}$. By definition, we have $\mathcal{M}_{loc} \supseteq \mathcal{H}^1_{loc}$. To show the reverse inclusion, let M be a local martingale with localizing sequence (σ_n). Moreover, for each n, consider the stopping times $S_n := \inf\big\{t \geq 0 \,\big|\, |M_t| \geq n\big\}$ and then define $\rho_n := \sigma_n \wedge S_n$. By construction, (ρ_n) is a sequence of stopping times converging to infinity and as a martingale is integrable, we have

$$\sup_{t\geq 0} |M^{\rho_n}_t| \leq n + |\Delta M_{\rho_n}| \in L^1(P),$$

which implies that $M \in \mathcal{H}^1_{loc}$ and thus $\mathcal{M}_{loc} = \mathcal{H}^1_{loc}$ as desired. Now, let (ρ_n) be a localizing sequence of M such that $M^{\rho_n} \in \mathcal{H}^1$ for each n. As in the proof of the Ansel-Stricker theorem (see Proposition 6.1 from the lecture), for each n, we define $\psi^n := \psi\, 1_{\{|\psi|\leq n\}}$ and $M^n := \int \psi^n\, dM^{\rho_n}$, which is in $\mathcal{H}^1$ by Proposition 5.8. By definition, $M^n \to \int \psi\, dM^{\rho_n}$ for the Eḿery topology, as well as $(\Delta M^n)^{\pm} \leq (\Delta \int \psi\, dM^{\rho_n})^{\pm}$. Assume for the moment that $\int \psi\, dM$ has locally a lower bound in L^1, which means that there exists a localizing sequence (τ_m) of stopping times and a sequence of random variables $(\gamma_m) \subseteq L^1(P)$ such that $(\int \psi\, dM)^{\tau_m} \geq \gamma_m$ for each m. In particular, we have $(\int \psi\, dM^{\rho_n})^{\tau_m} \geq \gamma_m$ for each n, m. Thus, all the assumptions of Lemma 6.2 are satisfied for each n, which implies that $\int \psi\, dM^{\rho_n} = (\int \psi\, dM)^{\rho_n} \in \mathcal{M}_{loc}$ for each n. In other words, $\int \psi\, dM \in \big(\mathcal{M}_{loc}\big)_{loc}$ which implies that $\int \psi\, dM \in \mathcal{M}_{loc}$ due to **a)**. So we have proved the claim. Therefore, it remains to show that $ZS = \int \psi\, dM$ has locally a lower bound in L^1. For that purpose, we let (T_n) be a localizing sequence for the local martingale Z and define for each $n \in \mathbb{N}$ the stopping times

$\sigma_n := \{t \geq 0 \,|\, |S_t| \geq n\}$ and $\widehat{T}_n := \{t \geq 0 \,|\, |Z_t| \geq n\}$. Consider the sequence of stopping times (τ_n) defined by $\tau_n := T_n \wedge \widehat{T}_n \wedge \sigma_n$. By construction, (τ_n) converges to ∞ a.s. Moreover, as S is continuous and $\sup_{t \geq 0} |Z_t^{\tau_n}| \leq n + \Delta |Z_{\tau_n}| \in L^1(P)$, we obtain for each n that $(ZS)^{\tau_n} \geq -n(n + \Delta |Z_{\tau_n}|) \in L^1$.

c) First, we show that $H \in L(M)$. $H \in lb\mathcal{P}$ implies the existence of a localizing sequence (τ_m) such $\forall m: \;\; H_m := H\, 1_{(\!(0,\tau_m]\!]} \in b\mathcal{P} \subseteq L(M)$. Thus, for all m, we have for $H_m^n := H\, 1_{\{|H| \leq n\}}\, 1_{(\!(0,\tau_m]\!]}$ that $(H_m^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in the Émery topology. Moreover, $H_m = H$ on $(\!(0,\tau_m]\!]$ and thus $(H_m^n \bullet S)_{n \in \mathbb{N}} = (H^n \bullet S)_{n \in \mathbb{N}}$ on $(\!(0,\tau_m]\!]$ (where $H^n := H\, 1_{\{|H| \leq n\}}$). Since $P[\tau_m < \infty] \to 0$, we conclude that $(H^n \bullet S)_{n \in \mathbb{N}}$ is Cauchy in the Émery topology and thus $H \in L(M)$. Now, let (τ_n) be a localizing sequence such that H^{τ_n} is bounded (by C_n) for each n and let (ρ_n) be a localizing sequence for the local martingale M. Moreover, let $\widehat{\rho}_n := \inf\left\{t \geq 0 \,\middle|\, |M_t| \geq n\right\}$. Set $T_n := \tau_n \wedge \rho_n \wedge \widehat{\rho}_n$. By construction, (T_n) is a sequence of stopping times converging to infinity. Moreover, we see that $(\Delta \int H\, dM)^{T_n} = (H \Delta M)^{T_n} \geq -C_n(n + \Delta |M_{T_n}|) \in L^1(P)$, which means that the jumps of $\int H\, dM$ have locally a lower bound in L^1, which is equivalent that $\int H\, dM$ has locally a lower bound in L^1 (see Exercise **5-1 b)**). Thus, we conclude that $\int H\, dM \in \mathcal{M}_{loc}$ directly from the arguments used in **b)**.

Solution 7-2

a) For $n \in \mathbb{N}$, define the stopping time $\tau_n := \inf\{t > 0 : X_t < 1/n\}$. Then by right-continuity of X, $X_{\tau_n} \leq 1/n$ on $\{\tau_n < \infty\}$ for $n \in \mathbb{N}$. Hence, by the optional stopping theorem, for all $n \in \mathbb{N}$,

$$E[X_t\, 1_{\{\tau_n \leq t\}}] \leq E[X_{\tau_n}\, 1_{\{\tau_n \leq t\}}] \leq 1/n, \quad t \geq 0. \tag{1}$$

Since $\tau_0 = \lim_{n \to \infty} \tau_n$ P-a.s., nonnegativity of X and dominated convergence give

$$E[X_t\, 1_{\{\tau_0 \leq t\}}] = 0, \quad t \geq 0. \tag{2}$$

This implies that $X_t = 0$ on $\{\tau_0 \leq t\}$ P-a.s. for each $t \geq 0$, and right-continuity of X establishes the claim.

b) First, note that since X is a strictly positive local martingale, it is a strictly positive supermartingale by Fatou's lemma and hence $X_- > 0$ P-a.s. by part **a)**. This implies that the process $\frac{1}{X_-}$ is well-defined. Since it is adapted and left-continuous, it is in addition predictable and locally bounded. Hence by the hint, the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t := \int_0^t \frac{1}{X_{s-}}\, dX_s, \quad t \geq 0, \tag{3}$$

is well defined and a local martingale. Moreover, associativity of the stochastic integral gives

$$\int_0^t X_{s-}\, dM_s = \int_0^t \frac{X_{s-}}{X_{s-}}\, dX_s = X_t - X_0 = X_t - 1, \quad t \geq 0. \tag{4}$$

This shows existence of M.

To establish uniqueness, suppose that $\widetilde{M}$ is a local martingale null at 0 such that $X = \mathcal{E}(\widetilde{M})$. Then associativity of the stochastic integral together with the definition of the stochastic exponential give

$$\widetilde{M}_t = \int_0^t \frac{1}{X_{s-}} X_{s-}\, dM_s = \int_0^t \frac{1}{X_{s-}}\, dX_s = M_t, \quad t \geq 0. \tag{5}$$

Solution 7-3

a) We first show that S fails NA. Consider the strategy $\vartheta = -1_{\rrbracket \frac{1}{2}, 1 \rrbracket}$. Then $V_\cdot(\vartheta) \geq -1$ on $[0,1]$ since

$$V_t(\vartheta) = \vartheta \bullet S_t = -\int_{(\frac{1}{2}, t\wedge\gamma)} 2\,du + 1_{\{\frac{1}{2} < \gamma \leq t\}} = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} \text{ or } \gamma \leq \frac{1}{2}, \\ -2(t - \frac{1}{2}) & \text{if } \frac{1}{2} < t < \gamma, \\ 1 - 2(\gamma - \frac{1}{2}) & \text{if } \frac{1}{2} < \gamma \leq t. \end{cases}$$

Thus, $\vartheta \in \Theta^1$ and $V_1(\vartheta) \in L^0_+(\mathcal{F}_1) \setminus \{0\}$.
To show that NUPBR holds, we show that $\mathcal{G}^1$ is bounded in L^1 which implies that $\mathcal{G}^1$ is bounded in L^0. So let $\vartheta \in \Theta^1$. We first observe that

$$\mathcal{F}_t = \sigma\big(S_u;\, u \leq t\big) = \sigma\Big(\big\{\{\gamma < u\};\, u \leq t\big\} \cup \Omega\Big) = \sigma\big(\gamma \wedge u;\, u \leq t\big).$$

In particular, we see that on $\llbracket 0, \gamma))$, every adapted process is deterministic. Moreover, if the process is additionally left-continuous, it is even deterministic on $\llbracket 0, \gamma \rrbracket$. Thus, by a monotone class argument, it is not difficult to show that there exists a deterministic Borel-measurable function $h : (0,1] \to \mathbb{R}$ such that $\vartheta 1_{\rrbracket 0, \gamma \rrbracket} = h 1_{\rrbracket 0, \gamma \rrbracket}$. This yields for $t \in [0,1]$

$$V_t(\vartheta) = \vartheta \bullet S_t = \int_{(0, t\wedge\gamma)} 2\vartheta_u\,du - \vartheta_\gamma 1_{\{\gamma \leq t\}} = \int_{(0,t\wedge\gamma)} 2h(u)\,du - h(\gamma) 1_{\{\gamma \leq t\}} \geq -1 \tag{6}$$

by 1-admissibility of ϑ. As γ is uniformly distributed on $(0,1)$ we have $P[\gamma > t] > 0$ for all $t \in [0,1)$ and thus by (6),

$$\int_0^t 2h(u)\,du \geq -1, \qquad t \in [0,1], \tag{7}$$

$$\int_0^t 2h(u)\,du - h(t) \geq -1, \qquad \text{for a.e. } t \in (0,1). \tag{8}$$

Note that (8) remains valid when replacing h by h^+. A version of Gronwall's inequality therefore implies that $h^+(t) \leq 1 + 2t\exp(2t) < 19$ for a.e. $t \in (0,1)$. This together with (7) implies that $\int_0^1 |h(u)|\,du < 38.5$. Thus,

$$E[|\vartheta \bullet S_T|] = E\left[\left|\int_0^\gamma 2h(t)\,dt - h(\gamma)\right|\right] \leq E\left[\int_0^1 2|h(u)|\,du\right] + E[|h(\gamma)|] = 3\int_0^1 |h(u)|\,du < 120.$$

b) To prove that S satisfies NA, let $\vartheta \in \Theta^1$ with $\vartheta \bullet S_T \in L^0_+(\mathcal{F}_T)$. As in **a)**, there exists a measurable function $h : (0, \frac{1}{2}] \to \mathbb{R}$ such that $\vartheta 1_{\rrbracket 0,\gamma \rrbracket} = h 1_{\rrbracket 0, \gamma \rrbracket}$. Moreover, by hypothesis,

$$0 \leq V_{\frac{1}{2}}(\vartheta) = \vartheta \bullet S_{\frac{1}{2}} = \int_0^\gamma h(u)\,dS_u + h(\gamma)\Delta S_\gamma = \int_0^\gamma h(u)\left(1 - \frac{4}{3}u\right)du - h(\gamma)\frac{2}{3}(1-2\gamma)(1-\gamma)1_{\{\gamma < \frac{1}{2}\}}. \tag{9}$$

We have $P[\gamma > t] > 0$ for all $t \in [0, \frac{1}{2})$ and thus by (9),

$$\int_0^t h(u)\left(1 - \frac{4}{3}u\right)du - h(t)\frac{2}{3}(1-2t)(1-t) \geq 0 \text{ for a.e. } t \in \left(0, \frac{1}{2}\right), \tag{10}$$

$$\int_0^{\frac{1}{2}} h(u)\left(1 - \frac{4}{3}u\right)du \geq 0. \tag{11}$$

As for (8), note that (10) remains valid when replacing h by h^+. So the version of Gronwall's inequality given in the hints implies that $h^+(t) = 0$ for a.e. $t \in (0, \frac{1}{2})$. This together with (11) and the fact that $1-\frac{4}{3}u \geq 0$ for $u \in (0, \frac{1}{2})$ shows that $h(t) = 0$ for a.e. $t \in (0, \frac{1}{2})$, and so $V_{\frac{1}{2}}(\vartheta) = \vartheta \bullet S_{\frac{1}{2}} = 0$ P-a.s.

To argue that NUPBR fails, we show that $\mathcal{G}^1$ is not bounded in probability. For that purpose, we set for $n \geq 2$

$$\vartheta_t^{(n)} = \frac{3}{2}\frac{1}{1-2t}1_{(\ell(n),r(n)]}(t), \quad 0 \leq t \leq \frac{1}{2}, \tag{12}$$

where $\ell(n) = \frac{1}{2}\left(1 - \frac{3}{2n}\right)$ and $r(n) = \frac{1}{2}\left(1 - \frac{3}{2n}\exp(-4n)\right)$. Note that $0 < \ell(n) < r(n) < \frac{1}{2}$. We choose $r(n) < \frac{1}{2}$ to get the integrability condition $\vartheta^{(n)} \in L(S)$. To show that $\vartheta^{(n)} \in \Theta^1$, since S is increasing on $[\![0, \gamma[\![$ and $\vartheta^{(n)} \geq 0$, it suffices to check that $V_t(\vartheta^{(n)}) \geq -1$ P-a.s. on $\{\ell(n) < \gamma \leq t \leq r(n)\}$. So fix $n \geq 2$. Then on $\{\ell(n) < \gamma \leq t \leq r(n)\}$,

$$\begin{aligned} V_t(\vartheta^{(n)}) = \vartheta^{(n)} \bullet S_t &= \int_{\ell(n)}^{\gamma} \frac{3}{2}\frac{1}{1-2t}\left(1 - \frac{4}{3}t\right)\,dt - \frac{3}{2}\frac{1}{1-2\gamma}\frac{2}{3}(1-2\gamma)(1-\gamma) \\ &= \gamma - \ell(n) + \frac{1}{4}\log\frac{1-2\ell(n)}{1-2\gamma} - (1-\gamma) \geq -1. \end{aligned}$$

Moreover on $\{\gamma = \frac{1}{2}\}$, for each $n \geq 2$,

$$V_{\frac{1}{2}}(\vartheta^{(n)}) = \vartheta^{(n)} \bullet S_{\frac{1}{2}} = \int_{\ell(n)}^{r(n)} \frac{3}{2}\frac{1}{1-2t}\left(1 - \frac{4}{3}t\right)\,dt = r(n) - \ell(n) + \frac{1}{4}\log\frac{1-2\ell(n)}{1-2r(n)} \geq n$$

by the choice of $\ell(n)$ and $r(n)$. Since $P[\gamma = \frac{1}{2}] = \frac{1}{2}$, this implies that $\mathcal{G}^1$ is not bounded in probability.