ETH Zürich HS 2014
D-MATH
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Mathematical Finance

Exercise Sheet 11

Please hand in until Wednesday, December 3rd, 12:00 in your assistant's box in HG G 52.1

Exercise 11-1

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space on $[0, T]$ with $\mathcal{F}_0$ trivial, and consider a bank account $B \equiv 1$ and a discounted stock price $S = (S_t)_{0 \le t \le T}$ which is an $\mathbb{R}^d$-valued semimartingale. Moreover, assume that $\mathbb{P}_\sigma \neq \emptyset$. An *American option* is described by its (discounted) payoff process $U = (U_t)_{0 \le t \le T}$, a nonnegative $\mathbb{F}$-adapted RCLL process. The interpretation of U is that the owner of the American option gets U_τ as payoff if he decides to exercise his option at time τ. The natural selling price for an American option $U = (U_t)_{0 \le t \le T}$ at time t is

$$\overline{V}_t := \operatorname*{ess\,sup}_{Q \in \mathbb{P}_\sigma, \tau \in \mathcal{T}_{t,T}} E_Q[U_\tau \,|\, \mathcal{F}_t],$$

where $\mathcal{T}_{t,T}$ is the set of stopping times taking values in $[t, T]$. Assume that $\sup_{Q \in \mathbb{P}_\sigma, \tau \in \mathcal{T}_{0,T}} E_Q[U_\tau] < \infty$.

a) Show that $\overline{V}$ is a Q-supermartingale for each $Q \in \mathbb{P}_\sigma$.
Hint: Argue similarly to the European option case.

b) Show that $\overline{V}$ is the smallest RCLL adapted process $V' \ge U$ which is a Q-supermartingale for each $Q \in \mathbb{P}_\sigma$.
Hint: You can use that $\overline{V}$ admits an RCLL version and use that version.

c) Assume that we are in a finite discrete-time setting. Define the process $(J_k)_{k=0,1,\dots,T}$ by

$$J_k := \begin{cases} \max\big(U_k, \operatorname{ess\,sup}_{Q \in \mathbb{P}} E_Q[J_{k+1} \,|\, \mathcal{F}_k]\big) & \text{if } k < T, \\ U_T & \text{if } k = T. \end{cases}$$

Moreover, define

$$\overline{V}_k := \operatorname*{ess\,sup}_{Q \in \mathbb{P}, \tau \in \mathcal{T}^d_{k,T}} E_Q[U_\tau \,|\, \mathcal{F}_k],$$

where $\mathcal{T}^d_{k,T}$ is the set of discrete stopping times taking values in $\{k, k+1, \dots, T\}$. Show that $J = \overline{V}$.
Hint: You can use that the result in **a)** also holds true for $\overline{V}$ in the discrete-time setting.

Exercise 11-2

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space on $[0, T]$ with $\mathcal{F}_0$ trivial, and consider a strictly positive, not necessarily constant bank account $\widetilde{B}$ and a undiscounted stock price $\widetilde{S} = (\widetilde{S}_t)_{0 \le t \le T}$ which is an $\mathbb{R}^d$-valued semimartingale. Moreover, assume that $\mathbb{P}_\sigma \neq \emptyset$. An American option is described by its undiscounted payoff process $\widetilde{U} = (\widetilde{U}_t)_{0 \le t \le T}$ which is a nonnegative, $\mathbb{F}$-adapted RCLL process. The natural selling price for an American option $\widetilde{U} = (\widetilde{U}_t)_{0 \le t \le T}$ at time t is

$$\widetilde{V}_t := \widetilde{B}_t \operatorname*{ess\,sup}_{Q \in \mathbb{P}_\sigma, \tau \in \mathcal{T}_{t,T}} E_Q\left[\frac{\widetilde{U}_\tau}{\widetilde{B}_\tau} \,\middle|\, \mathcal{F}_t\right],$$

where $\mathcal{T}_{t,T}$ is the set of stopping times taking values in $[t, T]$. Assume that $\sup_{Q \in \mathbb{P}_\sigma, \tau \in \mathcal{T}_{0,T}} E_Q\big[\frac{\widetilde{U}_\tau}{\widetilde{B}_\tau}\big] < \infty$.

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a) The American call option with strike $\widetilde{K} > 0$ is given by $\widetilde{U}_t := (\widetilde{S}_t - \widetilde{K})^+$. Assume that $\mathcal{P}_\sigma = \{Q^*\}$ with $S := \frac{\widetilde{S}}{\widetilde{B}}$ being a true Q^*-martingale. Moreover, assume that $\widetilde{B}$ is increasing, i.e. the bank account has a nonnegative interest rate. Show that the American call option price $\widetilde{V}^{Am}$ equals the European call option price given by

$$\widetilde{V}_t^{Eu} := \widetilde{B}_t \, E_{Q^*}\left[\frac{(\widetilde{S}_T - \widetilde{K})^+}{\widetilde{B}_T} \,\middle|\, \mathcal{F}_t\right].$$

Remark: an example of a market which satisfies the condition in **a)** is e.g. the CRR binomial model with parameters $u > r > d > -1$.
Remark: In general, one cannot replace the call by the put option in **a)**. I.e., in general, the American put is worth strictly more than the European put. This is the goal of **b)** and **c)**.

Consider the CRR binomial model with parameters $u > r > d > -1$. Let $\widetilde{V}_0^{Am}$ be the (undiscounted) price of an American put option $(\widetilde{K} - \widetilde{S})^+$ at time $t = 0$, where $\widetilde{S}$ is the (undiscounted) stock price process.

b) Consider $\widetilde{V}_0^{Am}$ as a function of the initial stock price $x = \widetilde{S}_0$. For which values of x should an investor keep the put in the portfolio during the first time-step? (Equivalently, when is $\widetilde{V}_0^{Am}$ greater that the inner value $(\widetilde{K} - x)^+$ of the put?)
Hint: Use Exercises **9-2 b)** and **11-1 c)**.

c) Show that $\widetilde{V}_0^{Am}$ can be strictly greater than the price $\widetilde{V}_0^{Eu}$ of a European put option with the same strike.

Exercise 11-3

Consider a stock price model with finite time horizon T, where the (discounted) price process S is driven by a Brownian motion W and an independent Poisson process N^λ with intensity λ, that is

$$S_t = S_0 e^{\sigma W_t + a N_t^\lambda},$$

with respect to a filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Here $\sigma > 0$ and $a \neq 0$ are some constants.

a) Prove that Z given by

$$Z_t = \left.\frac{dQ^{\widetilde{\lambda}}}{dP}\right|_{\mathcal{F}_t} = \exp\left(\beta\sigma W_t + \ln\left(\frac{\widetilde{\lambda}}{\lambda}\right) N_t^\lambda - \frac{1}{2}\beta^2\sigma^2 t + (\lambda - \widetilde{\lambda})t\right),$$

with

$$\beta = -\widetilde{\lambda}\frac{e^a - 1}{\sigma^2} - \frac{1}{2},$$

defines an equivalent martingale measure $Q^{\widetilde{\lambda}}$ for every $\widetilde{\lambda} > 0$.

b) Prove that the superreplication price $\pi_s((S_T - K)^+)$ of a call option is S_0.
Hint: Show that $\lim_{\widetilde{\lambda}\to\infty} E_{Q^{\widetilde{\lambda}}}[(S_T - K)^+] = S_0$. To this end, denote by $D_\mu(x) := \nu(X \le x)$ when $X \sim \text{Poi}(\mu)$ under a probability measure ν and denote by $\widehat{D}_\mu(x) := \nu(X < x)$. You can use the fact that for any constant $c \in \mathbb{R}$,

$$\lim_{\mu\to\infty} D_\mu(c + \alpha\mu) = \lim_{\mu\to\infty} \widehat{D}_\mu(c + \alpha\mu) = \begin{cases} 1 & \text{if } \alpha > 1, \\ \frac{1}{2} & \text{if } \alpha = 1, \\ 0 & \text{if } 0 \le \alpha < 1. \end{cases}$$