ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 8

Solution 8-1

a) $d\langle B, W\rangle_t = \rho dt$ because $d\langle W, W'\rangle_t = 0$, and

$$\begin{aligned}\langle S, Y\rangle_t &= \left\langle \int \sigma(u, S_u, Y_u) dW_u, \int a(u, Y_u) dB_u \right\rangle_t \\ &= \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) d\langle W, B\rangle_u = \int_0^t \sigma(u, S_u, Y_u) a(u, Y_u) \rho du.\end{aligned}$$

b) Let Z^Q be the density process of $Q \approx P$. Note that $\mathcal{F}_0$ is trivial and Z^Q is continuous since the filtration is generated by (W, W'). Defining L^Q by

$$L^Q = \int \frac{1}{Z^Q} dZ^Q$$

we have $Z^Q = \mathcal{E}(L^Q)$. By the Kunita–Watanabe decomposition, L^Q is given by

$$L^Q = \int \gamma^Q \sigma dW + N^Q$$

with $N^Q \in \mathcal{M}_{0,loc}(P)$ and $\langle N^Q, \int \sigma dW\rangle = 0$. By Bayes' rule, Q is an ELMM for S iff $Z^Q S \in M_{loc}(P)$. By the product rule, we obtain

$$\begin{aligned}d(Z_t^Q S_t) &= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + d\left\langle L^Q, \int \sigma dW\right\rangle_t\right) \\ &= Z_t^Q \sigma_t dW_t + S_t dZ_t^Q + Z_t^Q \left(\mu_t dt + \gamma_t^Q \sigma_t^2 dt\right),\end{aligned}$$

yielding $Z^Q S \in \mathcal{M}_{loc}(P)$ if and only if $\gamma^Q = -\frac{\mu_t}{\sigma_t^2}$. Therefore the equivalent local martingale measures Q are parametrized via

$$Z^Q = \mathcal{E}\left(-\int \frac{\mu}{\sigma} dW + N^Q\right).$$

Since the filtration is generated by (W, W'), we can apply the martingale representation theorem to write N^Q as

$$N^Q = \int \psi dW + \int \nu dW',$$

where ψ and ν are some predictable processes. As $\langle N^Q, \int \sigma dW\rangle = 0$, it follows that $0 = \int \psi_t \sigma_t \, dt$ and hence $\psi = 0$ so that we finally obtain

$$Z^Q = \mathcal{E}\left(-\int \lambda dW + \int \nu dW'\right), \tag{1}$$

where $\lambda = \mu/\sigma$ and ν is some predictable process.

c) By Girsanov, (W^Q, W'^Q), defined by $W^Q = W + \int \lambda dt$ and $W'^Q = W' - \int \nu dt$, is a 2-dimensional Q-Brownian motion. Plugging this into the SDEs for S and Y gives

$$dS_t = \mu_t dt + \sigma_t(dW_t^Q - \lambda_t dt) = \sigma_t W_t^Q$$

and

$$\begin{aligned} dY_t &= b_t dt + a_t\rho(dW_t^Q - \lambda_t dt) + a_t\sqrt{1-\rho^2}(dW_t'^Q + \nu_t dt) \\ &= \big(b_t + a_t(\sqrt{1-\rho^2}\ \nu_t - \rho\lambda_t)\big)dt + a_t dB^Q \end{aligned}$$

for the Q-Brownian motion $B_t^Q = \rho W_t^Q + \sqrt{1-\rho^2}\ W_t'^Q$.

Solution 8-2

a) Since Z is an exponential Lévy process, it is a P-martingale if and only if it is integrable with mean 1. First, note that

$$E[\exp(\phi(Y_1))] = \int_{\mathbb{R}} \frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{\nu}}{d\nu}(x)\, d\nu(x) = \frac{\widetilde{\lambda}}{\lambda}\int_{\mathbb{R}} d\widetilde{\nu}(x) = \frac{\widetilde{\lambda}}{\lambda}. \tag{2}$$

Moreover, for $a \in \mathbb{C}$ and $t \in [0,T]$,

$$E\left[a^{N_t}\right] = \sum_{k=0}^{\infty} \frac{(a\lambda t)^k}{k!}\exp(-\lambda t) = \exp((a-1)\lambda t). \tag{3}$$

Fix $t \in [0,T]$. Then by independence of N and the Y_k and by the above,

$$\begin{aligned} E[Z_t] &= \exp((\lambda - \widetilde{\lambda})t)E\left[E[\exp(\phi(Y_1)]^{N_t}\right] = \exp((\lambda-\widetilde{\lambda})t)E\left[(\widetilde{\lambda}/\lambda)^{N_t}\right] \\ &= \exp((\lambda-\widetilde{\lambda})t)\exp((\widetilde{\lambda}/\lambda - 1)\lambda t) = 1. \end{aligned} \tag{4}$$

b) First, X has clearly RCLL paths under Q.

Next, we show that under Q, X has stationary increments and $X_t - X_s$ is independent of $\mathcal{F}_s$ for all $0 \le s < t \le T$. So fix $0 \le s < t \le T$ and let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Since X is a Lévy process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ under P, it follows that the process $\widetilde{X} = (\widetilde{X}_u)_{u\in[0,T-s]}$ defined by $\widetilde{X}_u := X_{u+s} - X_s$ is independent of $\mathcal{F}_s$ and equal in distribution to $(X_u)_{u\in[0,T-s]}$ under P. Using this, the fact that

$$\begin{aligned} \frac{Z_t}{Z_s} &= \exp\left(\sum_{s<u\le t}\phi(\Delta X_u) + (\lambda-\widetilde{\lambda})(t-s)\right) \\ &= \exp\left(\sum_{0<u\le t-s}\phi(\Delta \widetilde{X}_u) + (\lambda-\widetilde{\lambda})(t-s)\right), \end{aligned} \tag{5}$$

the fact that Z is the density process of Q with respect to P on $(\mathcal{F}_t)_{t\in[0,T]}$ by part **a)** and Bayes' theorem, gives

$$\begin{aligned} E_Q[g(X_t - X_s) \mid \mathcal{F}_s] &= E\left[\frac{Z_t}{Z_s}g(X_t - X_s)\,\middle|\, \mathcal{F}_s\right] \\ &= E\left[\left(\sum_{0<u\le t-s}\phi(\Delta\widetilde{X}_u) + (\lambda-\widetilde{\lambda})(t-s)\right)g(\widetilde{X}_{t-s})\right)\,\middle|\,\mathcal{F}_s\right] \\ &= E\left[\left(\sum_{0<u\le t-s}\phi(\Delta X_u) + (\lambda-\widetilde{\lambda})(t-s)\right)g(X_{t-s})\right)\right] \\ &= E[Z_{t-s}g(X_{t-s})] = E_Q[g(X_{t-s})]. \end{aligned} \tag{6}$$

So X is a Lévy process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ under Q. In order to show that it is even a compound Poisson process with rate $\widetilde{\lambda}$ and jump distribution $\widetilde{\nu}$, we calculate the characteristic function of X_1 under Q to determine its law. To this end, let $v \in \mathbb{R}$. First, note that

$$E[\exp(ivY_1 + \phi(Y_1))] = \int_{\mathbb{R}} \exp(ivx) \frac{\widetilde{\lambda}}{\lambda} \frac{d\widetilde{\nu}}{d\nu}(x)\, d\nu(x) = \frac{\widetilde{\lambda}}{\lambda} \int_{\mathbb{R}} \exp(ivx)\, d\widetilde{\nu}(x). \tag{7}$$

Using this, the independence of N and the Y_k under P and (3), gives

$$\begin{aligned} E_Q[\exp(ivX_1)] &= E[Z_1 \exp(ivX_1)] = \exp(\lambda - \widetilde{\lambda}) E\left[E[\exp(ivY_1 + \phi(Y_1))]^{N_1}\right] \\ &= \exp(\lambda - \widetilde{\lambda}) \exp\left(\left(\frac{\widetilde{\lambda}}{\lambda} \int_{\mathbb{R}} \exp(ivx)\, d\widetilde{\nu}(x) - 1\right) \lambda\right) \\ &= \exp\left(\widetilde{\lambda} \int_{\mathbb{R}} (\exp(ivx) - 1)\, d\widetilde{\nu}(x)\right). \end{aligned} \tag{8}$$

Solution 8-3

a) $\underline{(\Rightarrow)}$: Since $S \in \mathcal{M}_\sigma(P)$, there exists $H > 0$ and a local P-martingale M with $H \in L(M)$ such that $S = \int H\, dM$. For each m, define $D_m := \{|H| \le m\} \in \mathcal{P}$. Then $(D_m) \uparrow \Omega \times [0,\infty)$ and $1_{D_m} \in L(S)$ for each m being predictable and locally bounded. Moreover, by associativity, we have $1_{D_m} \bullet S = (1_{D_m} H) \bullet M$. Thus, we conclude from Exercise **7-1 c)** that $1_{D_m} \bullet S \in \mathcal{M}_{loc}(P)$ for each m.
$\underline{(\Leftarrow)}$: First, observe that there exists a countable partition (B_m) of $\Omega \times [0,\infty)$ lying in $\mathcal{P}$. Indeed, define recursively $B_1 := D_1$, $B_2 := D_2 \setminus B_1$, $B_m = D_m \setminus (\cup_{i=1}^{m-1} B_i)$. Since $B_m \subseteq D_m$, we obtain that $1_{B_m} \bullet S = 1_{B_m} \bullet (1_{D_m} \bullet S)$ Thus, we deduce from Exercsie **6-1 c)** that $1_{B_m} \bullet S \in \mathcal{M}_{loc}(P)$ for each m. Since $\mathcal{M}_{loc} = \mathcal{H}^1_{loc}$, there exists for each m a localizing sequence $(\tau_{m,j})_{j\in\mathbb{N}}$ with $(1_{B_m} \bullet S)^{\tau_{m,j}} \in \mathcal{H}^1$ for each j. For each m, choose $j(m)$ such that

$$P[\tau_{m,j(m)} < \infty] \le 2^{-m},$$

and then define the stopping time $T_m := \inf_{k\ge m} \tau_{k,j(k)}$. By construction, we have $T_m < \tau_{m,j(m)}$ thus we see that such that $(1_{B_m} \bullet S)^{T_m} \in \mathcal{H}^1$ for each m. Moreover (T_m) converges to infinity P-a.s., as

$$P[T_m < \infty] \le \sum_{k=m}^{\infty} P[\tau_{k,j(k)} < \infty] \le 2^{-m+1}.$$

Now, for each m define $A_m := B_m \cap [\![0, \tau_m]\!]$. By construction, (A_m) is a partition of $\Omega \times [0,\infty)$ lying in $\mathcal{P}$. Moreover, $1_{A_m} \bullet S \in \mathcal{H}^1$ for each m. Let $(c_m) \subseteq \mathbb{R}$ be a strictly positive, decreasing sequence converging to zero such that $\sum_{m=1}^{\infty} c_m \|1_{A_m} \bullet S\|_{H^1} < \infty$. Then, define

$$H := \sum_{m=1}^{\infty} \frac{1}{c_m} 1_{A_m}, \qquad M := \sum_{m=1}^{\infty} c_m (1_{A_m} \bullet S),$$

where $M \in \mathcal{H}^1$ is defined as the limit of the sequence $X_n := \sum_{m=1}^{n} c_m(1_{A_m} \bullet S)$, which is Cauchy in $\mathcal{H}^1$. Then, we see that $H \in L(M)$ as for $n_k := \sup\{n : c_n \ge \frac{1}{k}\}$, we have

$$H 1_{\{|H|\le k\}} \bullet M = \Big(\sum_{m=1}^{n_k} 1_{A_m}\Big) \bullet S,$$

and (A_m) is a partition of $\Omega \times [0,\infty)$. Moreover, since (n_k) tends to infinity, we obtain by uniqueness that $S = H \bullet M$ and hence $S \in \mathcal{M}_\sigma(P)$.

b) Let $Z = \mathcal{E}(N)$ for some $N \in \mathcal{M}_{loc}(P)$. Then, by the product rule, we see that

$$ZS = Z_- \bullet S + S_- \bullet Z + [Z, S] = S_- \bullet Z + Z_- \bullet \big(S + [S, N]\big).$$

Thus, we deduce from Exercise Exercise **7-1 c)** that

$$ZS \in \mathcal{M}_\sigma(P) \iff S + [S, N] \in \mathcal{M}_\sigma(P).$$

Now, using again the product rule, we obtain that

$$Z\,G(\vartheta) = Z_- \bullet G(\vartheta) + G(\vartheta)_- \bullet Z + [Z, G(\vartheta)] = G(\vartheta)_- \bullet Z + \vartheta \bullet \big(Z_- \bullet (S + [S, N])\big).$$

Now, from Exercise **7-1 c)**, we deduce that $G(\vartheta)_- \bullet Z \in \mathcal{M}_{loc}(P)$. Thus, we have

$$Z\,G(\vartheta) \in \mathcal{M}_{loc}(P) \iff \vartheta \bullet \big(Z_- \bullet (S + [S, N])\big) \in \mathcal{M}_{loc}(P).$$

Since $ZS \in \mathcal{M}_\sigma(P)$, we know that $S + [S, N] = H \bullet M$ for some $M \in \mathcal{M}_{loc}(P)$ hand $H > 0$ lying in $L(M)$. Thus, we see that

$$\vartheta \bullet \big(Z_- \bullet (S + [S, N])\big) = (\vartheta Z_- H) \bullet M.$$

Moreover, since ϑ is admissible, $Z\,G(\vartheta)$ is bounded from below and thus

$$(\vartheta Z_- H) \bullet M = Z\,G(\vartheta) - G(\vartheta)_- \bullet Z$$

has locally a lower bound in $L^1(P)$. Thus, we deduce from the proof of Exercise **7-1 c)** that $(\vartheta Z_- H) \bullet M \in \mathcal{M}_{loc}(P)$ which implies that $Z\,G(\vartheta) \in \mathcal{M}_{loc}(P)$. The supermartingale property now follows directly from Fatou's lemma as $Z\,G(\vartheta)$ is bounded from below.