Mathematical Finance
Exercise Sheet 12

Please hand in until Wednesday, December 10th, 12:00 in your assistant's box in HG G 52.1

Exercise 12-1

Consider a utility function $U : (0, \infty) \to \mathbb{R}$ and denote its indirect utility or value function by $u(x) := \sup_{\vartheta \in \mathcal{A}(x)} \mathbb{E}\left[U\left(V_T^{x,\vartheta}\right)\right]$, where

$$V^{x,\vartheta} := x + G(\vartheta), \qquad \mathcal{A}(x) := \left\{\vartheta \,\middle|\, \vartheta \in \Theta^x_{adm} \text{ with } U^-\left(V_T^{x,\vartheta}\right) \in L^1(P)\right\}.$$

Prove the following properties of u:

a) u is increasing and concave on $(0, \infty)$.

b) If $u(x_0) < \infty$ for some $x_0 > 0$, then $u(x) < \infty$ for all $x > 0$.

c) If $u(x_0) < \infty$ for some $x_0 > 0$, then $U^+\left(V_T^{x,\vartheta}\right) \in L^1(P)$ for all $x > 0$ and $\vartheta \in \mathcal{A}(x)$.

Exercise 12-2

Let $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,1}, P)$ be a filtered probability space and $\bar{S} = (1, S_k^1, \dots, S_k^d)_{k=0,1}$ a one-period model. Assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and that S is *non-redundant* in the sense that for each $\vartheta \in \mathbb{R}^d$, we have $\vartheta^{tr} \Delta S_1 = 0$ P-a.s. if and only if $\vartheta = 0$. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function (without Inada conditions), i.e., U is strictly increasing, strictly concave and in C^1. Moreover, set $U(0) := \lim_{t \downarrow\downarrow 0} U(t) \in [-\infty, \infty)$ and $U(\infty) := \lim_{t \uparrow\uparrow \infty} U(t) \in (-\infty, +\infty]$, and note that U is strictly increasing, continuous and concave on $[0, \infty)$. For $x \geq 0$, set

$$\mathcal{A}(x) := \{\vartheta \in \mathbb{R}^d : x + \vartheta^{tr} \Delta S_1 \geq 0 \;\; P\text{-a.s.}\},$$
$$u(x) := \sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta^{tr} \Delta S_1)],$$

where $E[U(x + \vartheta^{tr} \Delta S_1)] := -\infty$ if $U(x + \vartheta^{tr} \Delta S_1)^- \notin L^1(P)$.

a) Fix $x \geq 0$. Show that the set $\mathcal{A}(x)$ is compact if and only if S satisfies NA.

Hint: For "$\Leftarrow$", argue by contradiction and assume that there exists a sequence $(\vartheta_n)_{n \in \mathbb{N}}$ in $\mathcal{A}(x) \setminus \{0\}$ such that $\lim_{n \to \infty} \|\vartheta_n\|_\infty = +\infty$. For $n \in \mathbb{N}$, set $\eta_n := \frac{\vartheta_n}{\|\vartheta_n\|_\infty}$, consider the sequence $(\eta_n)_{n \in \mathbb{N}}$ and use non-redundancy of S.

b) Suppose that $S_1^i \in L^1(P)$ for $i \in \{1, \dots, d\}$ and $U(\infty) = +\infty$. Fix $x > 0$. Show that $u(x) < \infty$ if and only if S satisfies NA.

Hint: For "$\Leftarrow$", construct $Y \in L^1(P)$ such that $U(x + \vartheta^{tr} \Delta S_1) \leq Y$ P-a.s. for all $\vartheta \in \mathcal{A}(x)$ using concavity of U and part (a).

c) Suppose that $S_1^i \in L^1(P)$ for $i \in \{1, \dots, d\}$ and that S satisfies NA. Fix $x > 0$. Show that there is a unique $\vartheta^* \in \mathcal{A}(x)$ such that

$$E[U(x + (\vartheta^*)^{tr} \Delta S_1)] = u(x) < \infty.$$

Hint: Use parts **a)** and **b)** and Fatou's lemma. Moreover, use without proof that U is strictly concave on $[0, \infty)$ in case that $U(0) > -\infty$.

Please turn the page

Exercise 12-3

Consider the same setup and notation as in Exercise **12-2**. Assume that $U(0) > -\infty$, that S satisfies NA and that $S_1^i \in L^1(P)$ for $i \in \{1, \dots, d\}$. Fix $x > 0$ and assume that the unique $\vartheta^* \in \mathcal{A}(x)$ satisfying $E[U(x + (\vartheta^*)^{tr} \Delta S_1)] = u(x) < \infty$ is in the interior of $\mathcal{A}(x)$.

a) Fix $z \geq 0$. Using only the concavity property, show that the function

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\},$$

is decreasing.

Remark: This shows in particular that $U'(0) := \lim_{h \downarrow\downarrow 0} \frac{U(h)-U(0)}{h} \in (0, +\infty]$ is well defined.

b) Show that $U'\left(x + (\vartheta^*)^{tr} \Delta S_1\right) < \infty$ P-a.s., that

$$U'\left(x + (\vartheta^*)^{tr} \Delta S_1\right) \Delta S_1^i \in L^1(P), \quad i \in \{1, \dots, d\},$$

and derive the *first-order condition*

$$E[U'(x + (\vartheta^*)^{tr} \Delta S_1) \Delta S_1^i] = 0, \quad i \in \{1, \dots, d\}.$$

Hint: Let $\eta \in \mathbb{R}^d \setminus \{0\}$, and consider the limit

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{U(x + (\vartheta^* + \epsilon\eta)^{tr} \Delta S_1) - U(x + (\vartheta^*)^{tr} \Delta S_1)}{\epsilon}$$

using part **a)**.

c) Show that there exists an equivalent martingale measure $Q \approx P$ on $\mathcal{F}_1$ for S with density

$$\frac{dQ}{dP} = \frac{U'(x + (\vartheta^*)^{tr} \Delta S_1)}{E[U'(x + (\vartheta^*)^{tr} \Delta S_1)]}.$$

Remark: The above result is a constructive proof of the Dalang–Morton–Willinger theorem in our setup.
