Mathematical Finance
Solution 6

Solution 6-1

a) First, assume that S is bounded. Note that then every simple strategy is admissible. Moreover, S is a uniformly integrable Q-martingale if and only if $E_Q[S_\tau - S_0] = 0$ for all stopping times (taking values in $[0,T]$). So let τ be an arbitrary stopping time, and consider the simple strategies $\vartheta^\pm := \pm 1_{]\!]0,\tau]\!]}$. Using that Q is an equivalent separating measure for S then gives

$$0 \geq E_Q[\vartheta^\pm \bullet S_T] = \pm E_Q[S_\tau - S_0]. \tag{1}$$

If S is locally bounded, then there exists an increasing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ taking values in $[0,T]$ with $\lim_{n\to\infty} P[\sigma_n = T] = 1$ such that S^{σ_n} is bounded for all $n \in \mathbb{N}$. It suffices to show that for each $n \in \mathbb{N}$, S^{σ_n} is a uniformly integrable Q-martingale. To this end, fix $n \in \mathbb{N}$. It suffices to show that for each stopping time τ with $\tau \leq \sigma_n$ P-a.s., $E_Q[S_\tau - S_0] = 0$. So let τ be such a stopping time, and consider as above the simple strategies $\vartheta^\pm := \pm 1_{]\!]0,\tau]\!]}$. Then both strategies are admissible since S is bounded on $[\![0,\sigma_n]\!]$ and $\tau \leq \sigma_n$ P-a.s., and the same argument as in the first step gives $E_Q[S_\tau - S_0] = 0$.

b) By assumption, there exist a strictly positive predictable process $\psi = (\psi_t)_{t\in[0,T]}$, an $\mathbb{R}^d$-valued (local) Q-martingale M, and an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random vector S_0 such that $S = S_0 + \psi \bullet M$. Let $\vartheta \in \Theta_{adm}$. Then by the associativity of the stochastic integral, $G(\vartheta) = \vartheta \bullet S = (\vartheta\psi) \bullet M$. Moreover, as $(\vartheta\psi) \bullet M$ is uniformly bounded from below by admissibility, it is a local Q-martingale by the Ansel-Stricker theorem. By Fatou's lemma, it is then also a Q-supermartingale, and hence

$$E_Q[G_T(\vartheta)] \leq E_Q[G_0(\vartheta)] = 0. \tag{2}$$

c) First, since $\mathcal{F}_t$ is P-trivial for all $t \in [0,T)$, a process $\xi = (\xi_t)_{t\in[0,T]}$ is adapted if and only if it is deterministic on $[0,T)$ and ξ_T is $\sigma(X)$-measurable. In particular, all left-continuous and adapted processes for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ are deterministic, and by a monotone class argument, the same is true for all predictable processes.

Next, if $\vartheta \in L(S)$ is arbitrary, then

$$G_T(\vartheta) = \vartheta \bullet S_T = \vartheta \bullet S_{T-} + \vartheta_T \Delta S_T = \lim_{t\uparrow T} \vartheta \bullet S_t + \vartheta_T X = 0 + \vartheta_T X = \vartheta_T X. \tag{3}$$

Since ϑ_T is deterministic and X is normally distributed, it follows that $\vartheta \in \Theta_{adm}$ if and only if $\vartheta_T = 0$. Thus, we may conclude that $G_T(\vartheta) = 0$ for all $\vartheta \in \Theta_{adm}$. Therefore the condition

$$E_Q[G_T(\vartheta)] \leq 0 \quad \text{for all } \vartheta \in \Theta_{adm}$$

is trivially satisfied for each probability measure $Q \approx P$ on $\mathcal{F}_T$. In particular, P itself is a separating measure.

Finally if $Q \approx P$ on $\mathcal{F}_T$ is an equivalent probability measure, by the first step (whose results remain unchanged by an equivalent change of measure), $M = (M_t)_{t\in[0,T]}$ is a Q-martingale null at 0 for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if and only if M_T is $\sigma(X)$-measurable, Q-integrable with mean 0 and $M_t = 0$ for all $t \in [0,T)$. Moreover, if $\psi \in L^Q(M)$, then as M is constant and equal to 0 on $[0,T)$,

$$\psi \bullet M_t = \begin{cases} 0 & \text{for } t < T, \\ \psi_T M_T & \text{for } t = T. \end{cases} \tag{4}$$

Note that as ψ_T is constant, $\psi \bullet M$ is a true Q-martingale, and thus Q is an equivalent σ-martingale measure for S if and only if it is an equivalent martingale measure. Since $E[S_T] = \mu \neq 0$, P is not a martingale measure and hence also not a σ-martingale measure.

Solution 6-2

a) First, we show existence of a strong solution. To this end, note that $\mu \in L^2_{loc}((W_s)_{s\in[0,t]})$ for all $t < T$ and define the process $\bar{Z} = (\bar{Z}_t)_{t\in[0,T)}$ by

$$\bar{Z}_t = \mathcal{E}\Big(-\int_0^\cdot \mu_s\,dW_s\Big)_t = \exp\Big(-\int_0^t \mu_s\,dW_s - \frac{1}{2}\int_0^t \mu_s^2\,ds\Big), \quad t < T. \tag{5}$$

Then for fixed $t \in (0,T)$ the process $\bar{Z}$ restricted to $[0,t]$ is a strictly positive continuous local martingale and the unique strong solution on $[0,t]$ to the SDE

$$dZ_t = -Z_t\mu_t\,dW_t, \quad Z_0 = 1. \tag{6}$$

By Fatou's lemma, it follows that $\bar{Z}$ is a strictly positive supermartingale on $[0,t]$ for each fixed $t < T$ and therefore also on $[0,T)$. By the supermartingale convergence theorem, $\lim_{t\uparrow T}\bar{Z}_t$ exists P-a.s. Define the process $Z = (Z_t)_{t\in[0,T]}$ by $Z_t := \bar{Z}_t$ for $t < T$ and $Z_T := \lim_{t\to T}\bar{Z}_t$. Clearly Z is continuous, adapted and nonnegative. To show that it is a local martingale, define for $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,T) : Z_t > n\} \wedge T. \tag{7}$$

Then $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} P[\tau_n = T] = 1$. We proceed to show that for each fixed $n \in \mathbb{N}$, Z^{τ_n} is a uniformly integrable martingale on $[0,T]$. Since Z is a supermartingale, this is equivalent to showing that $E[Z_{\tau_n}] = E[Z_0] = 1$. To this end, note that Z^{τ_n} is uniformly bounded by n. Moreover, for each fixed $m \in \mathbb{N}$, $Z^{\tau_n \wedge \frac{m-1}{m}}$ is a bounded local and hence uniformly integrable martingale on $[0, \frac{m-1}{m}]$. This implies in particular that $E[Z_{\tau_n \wedge \frac{m-1}{m}}] = 1$. Now dominated convergence shows that $E[Z_{\tau_n}] = 1$.

To show that Z satisfies (6) on $[0,T]$, let $M \in \mathcal{H}_0^{2,c}$ be arbitrary. Then the fact that Z satisfies (6) on $[0,t]$ for each fixed $t \in (0,T)$ gives

$$\langle Z, M\rangle_t = \int_0^t -Z_s\mu_s\,d\langle W, M\rangle_s, \quad t < T,$$
$$\langle Z, Z\rangle_t = \int_0^t Z_s^2\mu_s^2\,ds, \quad t < T,$$

Then monotone convergence gives

$$\int_0^T Z_s^2\mu_s^2\,ds = \langle Z, Z\rangle_T < \infty \ \ P\text{-a.s.}, \tag{8}$$

and this together with the Kunita-Watanabe inequality and dominated convergence gives

$$\langle Z, M\rangle_T = \int_0^T -Z_s\mu_s\,d\langle W, M\rangle_s \tag{9}$$

Since $M \in \mathcal{H}^{2,c}$ was arbitrary, Z solves (6) on $[0,T]$ by the definition of the stochastic integral.

Next, we show uniqueness. So suppose that Z^1 and Z^2 are solutions of (6) on $[0,T]$. Then they are a fortiori solutions of (6) on $[0,t]$ for each fixed $t \in (0,T)$. But for each fixed $t \in (0,T)$, the solution of (6) is unique, and so Z^1 and Z^2 coincide on $[0,t]$ for each fixed $t \in (0,T)$, and by continuity also on $[0,T]$.

b) Let $\widetilde{Z} = (\widetilde{Z}_t)_{t\in[0,T]}$ be a local P-martingale for the filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$ with $\widetilde{Z}_0 = 1$ such that $\widetilde{Z}S$ is also local P-martingale for the filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$. By Itô's representation theorem, we may assume that $\widetilde{Z}$ has continuous paths and that there exists a predictable process $\widetilde{H} \in L^2_{loc}(W)$ such that

$$\widetilde{Z}_s = 1 + \int_0^t \widetilde{H}_s \, dW_s, \quad t \in [0,T]. \tag{10}$$

Now the product rule gives

$$\widetilde{Z}_t S_t - \widetilde{Z}_0 S_0 = \int_0^t S_s \, d\widetilde{Z}_s + \int_0^t \widetilde{Z}_s S_s \, dW_s + \int_0^t \widetilde{Z}_s S_s \mu_s \, ds + \int_0^t \widetilde{H}_s S_s \, ds, \quad t \in [0,T]. \tag{11}$$

Since $\widetilde{Z}S - \widetilde{Z}_0 S_0$, $\int S \, d\widetilde{Z}$ and $\int \widetilde{Z}S \, dW$ are continuous local martingales null at 0, if follows that $\int(\widetilde{Z}S\mu + \widetilde{H}S) \, ds$ is a continuous local martingale null at 0. Since it is of finite variation, it must be constant 0. But this implies that for a.a. ω, $\widetilde{Z}S\mu + \widetilde{H}S$ is 0 a.e. on $[0,T]$. Since S is strictly positive the same is true for $\widetilde{Z}\mu + \widetilde{H}$. But this implies that $\int(\widetilde{Z}\mu + \widetilde{H})^2 \, ds$ is constant 0 and hence $\int \widetilde{H} \, dW = \int(-\widetilde{Z}\mu) \, dW$, which shows that $\widetilde{Z}$ solves (6) on $[0,T]$. By uniqueness of the solution, we may deduce that $\widetilde{Z} = Z$.

c) Define $Q \ll P$ on $\mathcal{F}_T$ by $dQ := Z_T \, dP$. Note that since Z is strictly positive on $[0,T)$, $Q \approx P$ on $\mathcal{F}_t$ for all $t \in (0,T)$. Moreover, S is a local Q-martingale by part (b). It suffices to show that all $\vartheta \in \Theta_{adm}$ with $\vartheta \bullet S_T \geq 0$ P-a.s. satisfy $\vartheta \bullet S_T = 0$ P-a.s. So let $\vartheta \in \Theta_{adm}$ with $\vartheta \bullet S_T \geq 0$ P-a.s. Then by absolute continuity, $\vartheta \bullet S_T \geq 0$ Q-a.s. and hence $\vartheta \bullet S \equiv 0$ by the fact that $\vartheta \bullet S$ is a Q-supermartingale (by Ansel-Stricker and Fatou) with $\vartheta \bullet S_0 = 0$. But since $Q \approx P$ on $\mathcal{F}_t$ for all $t \in [0,T)$, this implies that $\vartheta \bullet S_t = 0$ P-a.s. for all $t \in [0,T)$, and continuity of $\vartheta \bullet S$ gives $\vartheta \bullet S_T = 0$ P-a.s.

d) By the fundamental theorem of asset pricing, S satisfies NFLVR if and only if there exists a strictly positive P-martingale $\widetilde{Z} = (\widetilde{Z}_t)_{t\in[0,T]}$ with $\widetilde{Z}_0 = 1$ such that $\widetilde{Z}S$ is a local P-martingale. But if $\widetilde{Z}$ exists, then part b) shows that $\widetilde{Z} = Z$. This establishes the claim.

e) First, note that the process $\widetilde{Z}$ is well defined by part **a)** since

$$\int_0^T \frac{1}{\sqrt{T-s}} \, ds = 2 < \infty \quad \text{and} \quad \int_0^t \frac{1}{T-s} \, ds = \log\left(\frac{1}{T-t}\right) < \infty, \quad t \in (0,T). \tag{12}$$

Next, note that $Z = \widetilde{Z}^\tau$ and $\sup_{t\in[0,T]} Z_t \leq 2$ P-a.s., which shows that Z is a bounded local and hence true P-martingale. Moreover, $Z_\tau = 0$ on $\{\tau = T\}$ (since $\widetilde{Z}_T = 0$ P-a.s.) and $Z_\tau = 2$ on $\{\tau < T\}$, which implies that $P[Z_T = 0] = P[Z_T = 2] = 1/2$ by the fact that

$$1 = E[Z_T] = 0 \times P[Z_T = 0] + 2 \times P[Z_T = 2]. \tag{13}$$

Now the claim follows immediately from parts **c)** and **d)**.

Solution 6-3

a) Take $\vartheta \in \ell^1$ with $\vartheta \cdot \Delta S \geq 0$. Then we have

$$0 \leq \vartheta \cdot \Delta S(k) = \sum_{i \geq 1} \vartheta^i(-\delta_{k,i} + \delta_{k,i+1}) = \begin{cases} \vartheta^{k-1} - \vartheta^k, & k \geq 2; \\ -\vartheta^1, & k = 1. \end{cases}$$

This means that $0 \geq \vartheta^1 \geq \vartheta^2 \geq \cdots$, i.e. (ϑ^i) is a decreasing, nonpositive sequence.
Assume $\exists\, k$ s.t. $\vartheta \cdot \Delta S(k) > 0$, then $\exists\, k$ s.t. $\vartheta^k \neq 0$. But then ϑ cannot be a summable sequence.

b) Assume $Q \approx P$ is an EMM, i.e., $q_k := Q[\{k\}]$ satisfy $q_k > 0 \quad \forall k$, $\sum q_k = 1$, and

$$0 = E_Q[\Delta S^i] = \sum_{k \geq 1} q_k \Delta S^i(k) = -q_i + q_{i+1} \quad \forall i \geq 1$$

i.e. $(q_k)_{k \geq 1}$ is a constant sequence. This contradicts $q_k > 0 \quad \forall k$, $\sum q_k = 1$.