ETH Zürich HS 2014
D-MATH
Prof. Martin Schweizer

Coordinator
Ariel Neufeld

Mathematical Finance
Exercise Sheet 5

Please hand in until Wednesday, October 22th, 12:00 in your assistant's box in HG G 52.1

Exercise 5-1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Moreover, let S be an $\mathbb{R}^d$-valued adapted process with RCLL paths.

a) Let S be a semimartingale and fix $T < \infty$. Set $\mathcal{G}^1 := \{G_T(\vartheta) \,|\, \vartheta \in \Theta_{adm} \text{ is 1-admissible}\}$. Prove that $\mathcal{G}^1$ is bounded in L^0, i.e.

$$\lim_{m\to\infty} \sup_{g\in\mathcal{G}^1} P[|g| \geq m] = 0,$$

if and only if for every sequence $\varepsilon_n \downarrow 0$ and every sequence of ε_n-admissible strategies (ϑ^n), we have $G_T(\vartheta^n) \to 0$ in L^0.

b) Assume that $S_0 = s_0$ is constant. Prove the following:

(i) S_- is locally bounded and predictable.

(ii) S is locally bounded if and only if ΔS is locally bounded.

(iii) If S is predictable, it is locally bounded.
Hint: Find a suitable sequence of predictable times (T_n) and use that every predictable time has a foretelling sequence (see Exercise **4-3**).

Exercise 5-2

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual conditions.

a) Let $(X_t)_{t\geq 0}$ be a product measurable process and such that ${}^{\mathcal{P}}X$ and ${}^{\mathcal{O}}X$ are well defined. Show that ${}^{\mathcal{P}}({}^{\mathcal{O}}X) = {}^{\mathcal{P}}X$.

b) Suppose that $(A_t)_{t\geq 0}$ is an RCLL, increasing predictable process with $A_0 = 0$ and define

$$C_t := \inf\big\{s \geq 0 \,\big|\, A_s \geq t\big\} \quad \text{for } t \geq 0.$$

Show that each C_t is a predictable stopping time, that $\{C_t \leq s\} = \{A_s \geq t\}$ and therefore

$$\int_0^\infty H_s(\omega)\, dA_s(\omega) = \int_0^\infty H_{C_s(\omega)}\, 1_{\{C_s(\omega)<\infty\}}\, ds$$

for all bounded measurable processes H.
Hint: Use the monotone class theorem.

c) Suppose that $(A_t)_{t\geq 0}$ satisfies the conditions in **b)** and is integrable, i.e. $E[A_\infty] < \infty$, and $(M_t)_{t\geq 0}$ is a bounded martingale. Show that for every stopping time τ,

$$E\Big[\int_0^\tau \big(M_\tau - M_{s-}\big)\, dA_s\Big] = 0.$$

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Exercise 5-3

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space. For an increasing, adapted, continuous process $A = (A_t)_{t\in[0,T]}$ null at 0, define the σ-finite measure $P\otimes A$ on $(\bar{\Omega}, \mathcal{P})$, where $\bar{\Omega} := \Omega\times[0,T]$ and $\mathcal{P}$ denotes the predictable σ-field, by

$$(P \otimes A)[\bar{N}] := E\left[\int_0^T 1_{\bar{N}}(\omega, s)\, dA_s\right], \quad \bar{N} \in \mathcal{P}.$$

a) Let A^1 and A^2 be increasing, adapted, continuous processes null at 0 with $P \otimes A^1 = P \otimes A^2$ on $(\bar{\Omega}, \mathcal{P})$. Show that $A^1 = A^2$.

Hint: Show that $A^1 - A^2$ is a local martingale.

b) Let B and C be increasing, adapted, continuous processes null at 0. Show that there exist a predictable process $H \in L(C)$ and $\bar{N} \in \mathcal{P}$ such that

$$B_t = \int_0^t H_s\, dC_s + \int_0^t 1_{\bar{N}}\, dB_s, \quad \text{and} \quad \int_0^t 1_{\bar{N}}\, dC_s = 0, \quad t \in [0,T].$$

Hint: Consider the Lebesgue decomposition of $P \otimes B$ with respect to $P \otimes C$.

c) Now let $S = (S_t)_{t\in[0,T]}$ be a (real-valued) continuous semimartingale with canonical decomposition $S = S_0 + M + A$, where $M \in \mathcal{M}^c_{0,loc}$ and A is adapted, continuous, of finite variation and null at 0. Show that if S satisfies NA for 0-admissible strategies, then there exists a predictable process $H \in L(\langle M\rangle)$ such that

$$A_t = \int_0^t H_s\, d\langle M\rangle_s, \quad t \in [0,T].$$

In other words, this means that S satisfies the so-called structure condition (SC').

Hint: Write $A = A^+ - A^-$, where A^+ and A^- are the positive and negative variation of A, respectively, use part **b)** and argue by contradiction.
