ETH Zürich HS 2014
D-MATH
Prof. Martin Schweizer

Coordinator
Ariel Neufeld

Mathematical Finance
Solution 10

Solution 10-1

a) as $Z^1 = Z^2 = 1$ on $[\![0,t]\!]$ Z satisfies the desired properties on $[\![0,t]\!]$. Thus, we only have to check the desired properties on $(\!(t,\infty)\!)$. To see that Z is a martingale: $E[Z_u|\mathcal{F}_s] = E[Z_u^1 1_A + Z_u^2 1_{A^c}|\mathcal{F}_s] = E[Z_u^1|\mathcal{F}_s]1_A + E[Z_u^2|\mathcal{F}_s]1_{A^c} = Z_s$ for $t \leq s \leq u$; the case $s < t \leq u$ is similar. Indeed, we have with the same argument as above and using that Z satisfies the desired property on $[\![0,t]\!]$ that

$$E[Z_u|\mathcal{F}_s] = E[E[Z_u|\mathcal{F}_t]|\mathcal{F}_s] = E[Z_t|\mathcal{F}_s] = Z_s$$

(In fact $Z_t = Z_s = 1$). by first conditioning on $\mathcal{F}_t$. Clearly $Z > 0$, $E[Z_0] = 1$, and $Z|_{[0,t]} = 1$ so it remains to check that ZS is a sigma martingale: It is enough to show this on $(\!(t,\infty)\!)$. Due to exercice **8-3 a)**, this is equivalent to showing the existence of a sequence $(D_m) \uparrow \Omega \times [0,\infty)$ with $D_m \in \mathcal{P}$ such that $1_{D_m} \bullet (ZS) \in \mathcal{M}_{loc}$ for each m. Let $(D_m^i)_m$ be the corresponding sets such that $1_{D_m^i} \bullet (Z^i S) \in \mathcal{M}_{loc}$ for each m for $i = 1,2$. Define $D_m = D_m^1 \cap D_m^2$. By definition, $(D_m) \uparrow \Omega \times [0,\infty)$ with $D_m \in \mathcal{P}$. Then, we see that for each m

$$1_{D_m} \bullet (ZS) = 1_A\big(1_{D_m^2} \bullet (1_{D_m^1} \bullet Z^1 S)\big) + 1_{A^c}\big(1_{D_m^1} \bullet (1_{D_m^2} \bullet Z^2 S)\big).$$

By exercise **7-1 c)**, we obtain that $M^{1,m} := \big(1_{D_m^2} \bullet (1_{D_m^1} \bullet Z^1 S)\big)$ and $M^{2,m} := \big(1_{D_m^1} \bullet (1_{D_m^2} \bullet Z^2 S)\big)$ are both in $\mathcal{M}_{loc}$ for each m. But this implies that also

$$1_{D_m} \bullet (ZS) = 1_A M^{1,m} + 1_{A^c} M^{2,m} \in \mathcal{M}_{loc}$$

for each m, as we can argue, via localization, the same way we did for showing that Z is a martingale.

b) **i)** If $x \in \mathbb{R}$ and $\vartheta \in \Theta_{adm}$ with $x + \int_0^T \vartheta_t \, dS_t \geq H'$, then , as $H \leq H'$, $x + \int_0^T \vartheta_t \, dS_t \geq H$. Thus, by definition $\pi_s(H) \leq \pi_s(H')$.

ii) For $y := x - c$ we obtain that

$$\pi_s(H + c) = \inf\{x \,|\, x + \vartheta \bullet S_T \geq H + c\} = \inf\{y \,|\, y + \vartheta \bullet S_T \geq H\} + c = \pi_s(H) + c.$$

iii) If $x \geq \pi_s(H)$, then $\exists\, \vartheta \in \Theta_{adm}$ such that $x + \int_0^T \vartheta_t \, dS_t \geq H$. This implies that $\lambda x + \int_0^T \lambda \vartheta_t \, dS_t \geq \lambda H$, which gives us $\lambda \pi_s(H) \geq \pi_s(\lambda H)$. For the other inequality, observe that if $x < \pi_s(H)$ then we have $\forall\, \vartheta \in \Theta_{adm}$ that $x + \int_0^T \vartheta_t \, dS_t < H$. This implies that $\forall\, \vartheta \in \Theta_{adm}$ we have $\lambda x + \int_0^T \vartheta_t \, dS_t < \lambda H$. Thus, we obtain that $\lambda \pi_s(H) \leq \pi_s(\lambda H)$.

iv) If $x^1, x^2 \in \mathbb{R}$ and $\vartheta^1, \vartheta^2 \in \Theta_{adm}$ satisfies both $x^1 + \int_0^T \vartheta_t^1 \, dS_t \geq H^1$ and $x^2 + \int_0^T \vartheta_t^2 \, dS_t \geq H^2$, then, with $\vartheta = \vartheta^1 + \vartheta^2 \in \Theta_{adm}$, we obtain that $x^1 + x^2 + \int_0^T \vartheta_t \, dS_t \geq H^1 + H^2$. This implies that $\pi_s(H^1) + \pi_s(H^2) \geq \pi_s(H^1 + H^2)$.

c) Assume **(i)**. Then $E_{Q^*}[H] = V_0(\vartheta)$ and $E_Q[H] \leq V_0(\vartheta)$ for all $Q \in \mathbb{P}_\sigma$, as admissible wealth processes are Q-supermartingales.

Assume **(ii)**. We know from the lecture that H can be superreplicated by some $\vartheta \in \Theta_{adm}$ from initial capital $V_0(\vartheta) = E_{Q^*}[H]$. As all admissible wealth processes are Q^*-supermartingales, $V_0(\vartheta) = E_{Q^*}[H] \leq E_{Q^*}[V_0(\vartheta) + (\vartheta \cdot S)_T] \leq V_0(\vartheta)$. Thus $E_{Q^*}[(\vartheta \cdot S)_T] = 0 = (\vartheta \cdot S)_0$ and $V(\vartheta)$ is a Q^*-supermartingale with constant Q^*-expectation, hence a Q^*-martingale. Moreover, $H = V_0(\vartheta) + (\vartheta \cdot S)_T$ Q^*-a.s. (and hence P-a.s., too).

Solution 10-2

a) For an equivalent martingale measure Q, q_u, q_m and q_d must satisfy $q_m, q_d, q_u > 0$ (so that Q is equivalent to P) as well as

$$q_u + q_m + q_d = 1 \tag{1}$$

as well as $E^Q[S_1|\mathcal{F}_0] = S_0$, which, as $\mathcal{F}_0$ is trivial and $S_1 = S_0 Z$, can written as the condition

$$q_d(1+d) + q_m(1+m) + q_u(1+u) = 1. \tag{2}$$

We see that for $d > 0$ and $u < 0$, the system of equations (1) and (2) does not have any solution. This means that in these cases, there is no equivalent martingale measure which is by the fundamental theorem of asset pricing (FTAP) equivalent to the existence of arbitrage. For $u > 0 > d$, the system of equations (1) and (2) has solutions of the form

$$q_d := \frac{-q_m(m-u) - u}{d-u} \in (0,1), \qquad q_u = 1 - q_m - \frac{-q_m(m-u) - u}{d-u} \in (0,1), \qquad q_m \in (0,1). \tag{3}$$

By the FTAP, this is equivalent to the absence of arbitrage.

b) Assume that H is attainable. This means, as $\mathcal{F}_0$ is trivial, that there exists constants $x, \vartheta \in \mathbb{R}$ such that

$$x + \vartheta(S_1 - S_0) = x + \vartheta S_0(Z-1) = (S_1 - K)^+ = (S_0\, Z - K)^+ \tag{4}$$

On $\{Z = 1+u\}$, equation (4) can be read as

$$x + \vartheta S_0 u = S_0(1+u) - K. \tag{5}$$

On $\{Z = 1+m\}$, equation (4) can be read as

$$x + \vartheta S_0 m = S_0(1+m) - K. \tag{6}$$

Thus, substracting (6) from (5) leads to $\vartheta = 1$ and $x = S_0 - K$. On the other hand, on $\{Z = 1+d\}$, equation (4) can be read as

$$x + \vartheta S_0 d = 0. \tag{7}$$

If $d = 0$, we obtain that $x = 0$, which implies that $0 = x = S_0 - K$. This contradicts the assumption that $S_0 \neq K$. If $d \neq 0$, we must have $\vartheta = \frac{-x}{S_0 d}$ for any initial value $x \in \mathbb{R}$. In particular, for $x = S_0 - K$, we obtain $\vartheta = \frac{-S_0+K}{S_0 d}$. Since we already have $\vartheta = 1$, we see that the equation

$$\frac{-S_0 + K}{S_0 d} = 1$$

must hold true. But this is only the case when $K = S_0(1+d)$, which was excluded in the assumption. Therefore $H = (S_1 - K)^+$ is not attainable.
Remark: We have not made any assumption about arbitrage in this part **b).**

c) *Case 1:* $\min_\Omega S_1 < K < \max_\Omega S_1$: Note that $H(\omega_d) = 0$. The formula is known from the lecture to be $\pi_s(H) = \sup_{Q\in\mathbb{P}_e} E_Q[H]$. Intuitively speaking we should put maximal weight on the extreme events and therefore $q_m = 0$ to attain the supremum at some $Q \in \mathbb{P}_a$. Such $Q \in \mathbb{P}_a$ will not be equivalent to P but clearly a limit of some $Q_n \in \mathbb{P}_e$.

The formal argument runs as follows: Denote $s_i^1 = S_1(\omega_i)$ for $i = d, m, u$. As $h(x) := (x-K)^+$ is convex, for any $Q' \in \mathbb{P}_e$, and for suitable $\lambda \in (0,1)$ such that $s_m^1 = \lambda s_d^1 + (1-\lambda)s_u^1$ (or equivalently, such that $1+m = \lambda(1+d) + (1-\lambda)(1+u)$, hence $\lambda = \frac{u-m}{u-d}$),

$$\begin{aligned}
E_{Q'}[h(S_1)] &= q'_d h(s_d^1) + q'_m h(\lambda s_d^1 + (1-\lambda)s_u^1) + q'_u h(s_u^1) \\
&\leq q'_d h(s_d^1) + q'_m \lambda h(s_d^1) + q'_m(1-\lambda)h(s_u^1) + q'_u h(s_u^1) \\
&= [q'_d + \lambda q'_m]h(s_d^1) + [q'_u + (1-\lambda)q'_m]h(s_u^1) =: q_d h(s_d^1) + q_u h(s_u^1) = E_Q[h(S_1)]
\end{aligned}$$

for $Q := (q_d, 0, q_u)$. We point out, using (3), that we have $q_d = q'_d + \lambda q'_m = \frac{u}{u-d}$ and that $q_u = q'_u + (1-\lambda)q'_m = \frac{-d}{u-d}$, which is independent of the choice of $Q' \in \mathbb{P}_e$, hence Q is well-defined. Clearly, Q is a probability measure as $q_d, q_u > 0$ and $q_d + q_u = 1$. Moreover, $Q \ll P$, but Q is not equivalent to P as Q does not have any mass in ω_m. To be a martingale measure, we need to have that $E_Q[Z] = 1$ or equivalently, that

$$q_d(1+d) + q_u(1+u) = 1,$$

which is satisfied. Now, we know from above that $\pi_s(H) := \sup_{Q' \in \mathbb{P}_e} E^{Q'}[H] \leq E_Q[H]$. We claim that $\sup_{Q' \in \mathbb{P}_e} E^{Q'}[H] = E_Q[H]$. For that purpose, consider for each n a measure Q^n with corresponding $q_d^n, q_m^n, q_u^n > 0$ satisfying (3) such that $(q_m^n)_{n \in \mathbb{N}}$ converges to 0. Clearly, from **a)**, we have $(Q^n) \subseteq \mathbb{P}_e$. Moreover, from the above calculation, we see that $\lim_{n\to\infty} E^{Q^n}[H] = E^Q[H]$. Thus, we see that

$$\sup_{Q' \in \mathbb{P}_e} E^{Q'}[H] \geq \lim_{n\to\infty} E^{Q^n}[H] = E^Q[H],$$

which implies that $\sup_{Q' \in \mathbb{P}_e} E^{Q'}[H] = E_Q[H]$. Thus, as $H(\omega_d) = 0$, we get that

$$\pi_s(H) = E^Q[H] = q_3(S_0(1+u) - K) + 0 = \frac{-d}{u-d}\big(S_0(1+u) - K\big).$$

Case 2: $K > \max_\Omega S_1$: Then $H = 0$ and thus $\pi_S(H) = 0$.
Case 3: $K < \min_\Omega S_1$:. Then $H = S_1 - K$ is attainable with admissible strategy $(V_0, \vartheta) = (S_0 - K, 1)$. In particular $\pi_S(H) = V_0 = S_0 - K$.

i) This is correct, the proof is part of the proof of Theorem 8.3 of the lecture (where one argues that there is at most one equivalent martingale measure.)

ii) This is wrong. Take the market as above with $d > 0$. Then, from a), we know that there is arbitrage, thus $\mathbb{P}_e = \emptyset$ by the fundamental theorem of asset pricing. But in b), we find an option H not being attainable.

Solution 10-3

a) Define the process $R = (R_t)_{t\in[0,T]}$ by

$$\begin{aligned} R_t &:= \mu t + \frac{\sigma}{\sqrt{\lambda}}\widetilde{N}_t = \mu t + \frac{\sigma}{\sqrt{\lambda}}(N_t - \lambda t) = (\mu - \sigma\sqrt{\lambda})t + \frac{\sigma}{\sqrt{\lambda}}N_t \\ &= \frac{\sigma}{\sqrt{\lambda}}(N_t - \ell t), \quad t \in [0,T], \end{aligned} \tag{8}$$

where $\ell := \lambda - \frac{\mu}{\sigma}\sqrt{\lambda}$. It follows from Exercise **9-3 b)** that S fails NA, and a fortiori NFLVR, if the paths of R are monotone, i.e., if $\ell \leq 0$. On the other hand, if $\ell > 0$, define the measure $Q^\lambda \approx P$ on $\mathcal{F}_T$ by

$$\frac{dQ^\lambda}{dP} = \exp\left(\sum_{k=1}^{N_T} \log\frac{\ell}{\lambda} + (\lambda - \ell)T\right). \tag{9}$$

Then it follows from Exercise **8-2** that under Q^λ, $R_t = \frac{\sigma}{\sqrt{\lambda}}(N^{Q^\lambda} - \ell t), t \geq 0$, where $N^{Q^\lambda} := N$ is a Poisson process with rate ℓ. Since R is a Q^λ-martingale, it follows from Exercise **9-3 a)** that S is so, too.

b) First observe that under Q^λ, $S = \mathcal{E}(R)$, where $R_t = \frac{\sigma}{\sqrt{\lambda}}(N^{Q^\lambda} - \ell t), t \geq 0$, with $N^{Q^\lambda} := N$ is a Poisson process with rate ℓ. Thus, using the given formula for $\mathcal{E}(R)$ and the facts that $\sum_{i=1}^{T} \Delta R_t = \sum_{i=1}^{T} \Delta N_t^{Q^\lambda} = N_T^{Q^\lambda}$, we obtain that

$$S_T = S_0 \exp\left(\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) N_T^{Q^\lambda} - \frac{\sigma\ell}{\sqrt{\lambda}} T\right).$$

Since S admits a unique equivalent martingale measure Q^λ (known from the hints), the risk-neutral price of $1_{\{S_T > K\}}$ is given by

$$\begin{aligned}
E_{Q^\lambda}[1_{\{S_T>K\}}] &= Q^\lambda[S_T > K] \\
&= Q^\lambda\left[S_0 \exp\left(\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right) N_T^{Q^\lambda} - \frac{\sigma\ell}{\sqrt{\lambda}} T\right) > K\right] \\
&= Q^\lambda\left[N_T^{Q^\lambda} > \frac{\log\frac{K}{S_0} + \frac{\sigma\ell}{\sqrt{\lambda}} T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right] \\
&= \overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned} \tag{10}$$

c) First, define $\widetilde{Q}^\lambda \approx Q^\lambda$ on $\mathcal{F}_T$ by $\frac{d\widetilde{Q}^\lambda}{dQ^\lambda} := S_T/S_0$. Note that

$$S_T/S_0 = \mathcal{E}(R)_T = \exp\left(\sum_{k=1}^{N_T^{Q^\lambda}} \log\frac{\widetilde{\ell}}{\ell} + (\ell - \widetilde{\ell})T\right), \tag{11}$$

where $\widetilde{\ell} := \left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)\ell$. Now it follows from Exercise **8-2** that under $\widetilde{Q}^\lambda$,

$$R_t = \frac{\sigma}{\sqrt{\lambda}} N_t^{\widetilde{Q}^\lambda} - \frac{\sigma}{\sqrt{\lambda}} \ell t, \quad t \in [0, T], \tag{12}$$

where $N^{\widetilde{Q}^\lambda}$ is a Poisson process with rate $\widetilde{\ell}$.

Next, since S admits a unique equivalent martingale measure Q^λ (see hints), the arbitrage-free price of $S_T 1_{\{S_T>K\}}$ is given by $E_{Q^\lambda}[S_T 1_{\{S_T>K\}}]$. By Bayes' formula and the above and noting that under $\widetilde{Q}^\lambda$, the calculation is exactly the same as in part (b),

$$\begin{aligned}
E_{Q^\lambda}[S_T 1_{\{S_T>K\}}] &= E_{\widetilde{Q}^\lambda}[S_0 1_{\{S_T>K\}}] = S_0 \widetilde{Q}^\lambda[S_T > K] \\
&= S_0 \overline{\Psi}_{\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned} \tag{13}$$

d) First, it follows immediately from parts **b)** and **c)** that

$$\begin{aligned}
C_0^\lambda &= E_{Q^\lambda}[(S_T - K)^+] = E_{Q^\lambda}[S_T 1_{\{S_T>K\}}] - K E_{Q^\lambda}[1_{\{S_T>K\}}] \\
&= S_0 \overline{\Psi}_{\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right) \\
&\quad - K \overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right) T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).
\end{aligned} \tag{14}$$

Next, for $\rho > 0$, let F_ρ be the distribution function of $\frac{X_\rho - \rho}{\sqrt{\rho}}$, where X_ρ is Poisson distributed with parameter ρ. Moreover, set $\overline{F}_\rho := 1 - F_\rho$ and $\overline{\Phi} = 1 - \Phi$. Then by the hint, F_ρ converges pointwise to Φ as $\rho \to \infty$, and the convergence is even uniform as Φ is continuous. Thus $\overline{F}_\rho$ converges uniformly to $\overline{\Phi}$ as $\rho \to \infty$. Now the claim follows from the fact that $\overline{\Psi}_\rho(x) = \overline{F}_\rho\left(\frac{x-\rho}{\sqrt{\rho}}\right)$, the fact that $\overline{\Phi}(x) = \Phi(-x)$ and the limits

$$
\begin{aligned}
&\lim_{\lambda\to\infty} \log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\sqrt{\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T} = \sigma\sqrt{T},\\
&\lim_{\lambda\to\infty} \log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\sqrt{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T} = \sigma\sqrt{T},\\
&\lim_{\lambda\to\infty} \left(\left(\sigma\sqrt{\lambda}-\mu\right)T - \log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T\right) = \frac{\sigma^2}{2}T,\\
&\lim_{\lambda\to\infty} \left(\left(\sigma\sqrt{\lambda}-\mu\right)T - \log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T\right) = -\frac{\sigma^2}{2}T,
\end{aligned}
\tag{15}
$$

where we have used that

$$
\begin{aligned}
\log\left(1+\frac{\sigma}{\sqrt{\lambda}}\right) &= \frac{\sigma}{\sqrt{\lambda}} - \frac{\sigma^2}{2\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right),\\
\sqrt{\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}} &= \sqrt{\lambda}\sqrt{1+O\left(\frac{1}{\sqrt{\lambda}}\right)},\\
\sqrt{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)} &= \sqrt{\lambda}\sqrt{1+O\left(\frac{1}{\sqrt{\lambda}}\right)}.
\end{aligned}
\tag{16}
$$

Remark: More generally, one can show that if N^λ is a Poisson process with rate λ, then the normalised compensated Poisson process $\frac{\widetilde{N}^\lambda}{\sqrt{\lambda}}$ converges weakly to a standard Brownian motion. But this is of course much more difficult.