ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Exercise Sheet 9

Please hand in until Wednesday, November 19th, 12:00 in your assistant's box in HG G 52.1

Exercise 9-1

In the Black–Scholes model, prove that the so-called digital option with undiscounted payoff $\widetilde{H} = 1_{\{\widetilde{S}_T > \widetilde{K}\}}$, $\widetilde{K} > 0$, is attainable, and calculate its arbitrage-free price process and the replicating strategy.

Exercise 9-2

Consider the multiperiod Cox–Ross–Rubinstein binomial model for $\widetilde{B}_n = (1+r)^n$ and (undiscounted) stock price process $\widetilde{S}$, where $\widetilde{S}$ evolves between two consecutive periods as

$$\widetilde{S}_{n+1} = \widetilde{S}_n Z_{n+1}, \quad n = 0, \dots, T-1, \quad \widetilde{S}_0 > 0.$$

Here $(Z_n)_{n=1,\dots,T}$ are i.i.d. random variables, taking only the two values $1+u$ and $1+d$ for $u > d > -1$ with probabilities

$$P[Z_n = 1+u] = p \quad \text{and} \quad P[Z_n = 1+d] = 1-p, \quad p \in (0,1).$$

Consider the CRR model with parameters $u > r > d > -1$. Recall from Exercise **2-3** the unique martingale measure Q for the discounted stock price process S, and let $\widetilde{H}$ be a contingent claim of the form $\widetilde{H} = \widetilde{h}(\widetilde{S}_T)$, where $\widetilde{S}$ is the (undiscounted) stock price process.

a) Show that the arbitrage-free price of $\widetilde{H}$ at time k is of the form $\widetilde{V}_k^{\widetilde{H}} = \widetilde{v}(k, \widetilde{S}_k)$.

b) Find a backward recursion for the function $\widetilde{v}(k, x)$.

c) Deduce an explicit formula for the price of $\widetilde{H}$ at time $k = 0$.

d) Show that the replicating strategy for $\widetilde{H}$ has the form

$$\widetilde{\vartheta}_k = \frac{\widetilde{v}(k, (1+u)\widetilde{S}_{k-1}) - \widetilde{v}(k, (1+d)\widetilde{S}_{k-1})}{(u-d)\widetilde{S}_{k-1}}.$$

e) Show that the replicating strategy for a call option does not involve short positions, i.e., $\widetilde{\vartheta} \geq 0$. What is a general sufficient condition on h such that $\widetilde{\vartheta} \geq 0$?
Hint: Use **d)** and **b)**.

Please turn the page

Exercise 9-3

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Moreover, let $R = (R_t)_{t\in[0,T]}$ be a *simple jump-diffusion*, i.e., there exist a Brownian motion $W = (W_t)_{t\in[0,T]}$ and an independent compound Poisson process $X = (X_t)_{t\geq 0}$ with jump intensity $\lambda > 0$ and jump distribution ν (with $\nu(\{0\}) = 0$) such that $R_t = at + \sigma W_t + X_t$, $t \in [0,T]$, where $a \in \mathbb{R}$ and $\sigma \geq 0$. Suppose about the filtration that R is a Lévy process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, and suppose about ν that $\nu((-\infty,-1]) = 0$, i.e. the jumps of R are strictly greater than -1. Define the process $S = (S_t)_{t\in[0,T]}$ by $dS_t := S_{t-}\,dR_t$, $S_0 = s_0 > 0$, i.e., $S = s_0\mathcal{E}(R)$.

a) Suppose that R is a martingale for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Show that S is then also a martingale (and not only a local martingale) for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Hint: Use that for a Lévy process R with triplet (a, σ^2, ν), the stochastic exponential of R is given by

$$\mathcal{E}(R)_t := \exp\left(R_t - \frac{1}{2}\sigma^2 t\right) \prod_{0<s\leq t} (1 + \Delta R_s)\exp(-\Delta R_s).$$

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b) Show that in general S satisfies NA if and only if the paths of R are not monotone.

Hint: To establish "$\Leftarrow$", use Exercise **8-2 b)** to show first that there exists a measure $\widetilde{P} \approx P$ on $\mathcal{F}_T$ such that the process R is $\widetilde{P}$-integrable. Then distinguish the cases $\sigma > 0$ and $\sigma = 0$ and apply a second change of measure. Moreover, use that if A and B are independent stochastic processes under P and $Q \approx P$ with $\frac{dQ}{dP} = \Phi(B)$ for some measurable functional Φ, then A and B are still independent under Q and $\mathcal{L}(A \mid Q) = \mathcal{L}(A \mid P)$.
