ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Exercise Sheet 7

Please hand in until Wednesday, October 29th, 12:00 in your assistant's box in HG G 52.1

Exercise 7-1

a) Let $\big(\mathcal{M}_{loc}\big)_{loc}$ be defined as

$$\big(\mathcal{M}_{loc}\big)_{loc} =: \Big\{ M = (M_t)_{t\geq 0} \,\Big|\, \exists \text{ localizing sequence } (\tau_n) \text{ such that } M^{\tau_n} \in \mathcal{M}_{loc} \text{ for each } n \Big\}.$$

Show that $\big(\mathcal{M}_{loc}\big)_{loc} = \mathcal{M}_{loc}$.

b) Let Z be a strictly positive local martingale with $Z_0 = 1$ and S be an adapted, continuous process such that ZS is a σ-martingale. Prove that ZS is a local martingale.

c) Let M be a local martingale and H predictable and locally bounded. Show that H is M-integrable (in the sense of semimartingales, i.e. $H \in L(M)$) and that $\int H\, dM$ is a local martingale.

Exercise 7-2

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions.

a) Let $X = (X_t)_{t\geq 0}$ be a nonnegative supermartingale, and define the stopping time τ_0 by

$$\tau_0 := \inf\{t > 0 : X_t \wedge X_{t-} = 0\}.$$

Show that $X = 0$ on $[\![\tau_0, \infty]\!]$ P-a.s.

Hint: For $n \in \mathbb{N}$, consider the stopping time $\tau_n := \inf\{t > 0 : X_t < 1/n\}$.

Remark: This result is known as the *minimum principle* for nonnegative supermartingales.

b) Let $X = (X_t)_{t\geq 0}$ be a strictly positive local martingale with $X_0 = 1$. Show that there exists a unique local martingale $M = (M_t)_{t\geq 0}$ null at 0 such that $X = \mathcal{E}(M)$. M is called the *stochastic logarithm* of X and denoted by $\mathcal{L}(X)$.

Hint: Use part **a)** and Exercise **7-1 c)**.

Please turn the page

Exercise 7-3

a) Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where $\mathbb{F}$ is the filtration generated by $S = (S_t)_{t\in[0,1]}$, which is given by

$$S_t = (1+2t)1_{\{t<\gamma\}} + 2\gamma 1_{\{t\geq\gamma\}},$$

where γ is uniformly distributed on $(0,1)$. Show that S satisfies NUPBR but fails NA.
Hint: Use the following special case of *Gronwall's inequality* for $h : [0,1] \to \mathbb{R}_+$:

$$\text{If } h(t) \leq 1 + \int_0^t 2h(u)\,du \ \ \forall t \in [0,1], \text{ then } \ h(t) \leq 1 + 2t\exp(2t) \ \ \forall t \in [0,1].$$

b) Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where $\mathbb{F}$ is the filtration generated by $S = (S_t)_{t\in[0,1]}$, which is given by

$$S_t = \left(1 + t - \frac{2}{3}t^2\right)1_{\{t<\gamma\}} + \left(\frac{1}{3} + 3\gamma - 2\gamma^2\right)1_{\{t\geq\gamma\}},$$

where γ is a $(0,1)$-valued random variable with distribution function $G(t) = t1_{\{t<\frac{1}{2}\}} + 1_{\{t\geq\frac{1}{2}\}}$, i.e., the law of γ is a mixture of a uniform distribution on $(0, \frac{1}{2})$ and a Dirac measure at $\frac{1}{2}$. Show that S satisfies NA but fails NUPBR.
Hint: Use the following special case of *Gronwall's inequality*:
Let $h, k, b : [0, \frac{1}{2}] \to \mathbb{R}_+$. Moreover, let k be nondecreasing and b continuous. Then:

$$\text{If } h(t) \leq k(t)\int_0^t g(u)\,b(u)\,du \ \ \forall t \in [0,1/2], \text{ then } \ h(t) = 0 \ \ \forall t \in [0,1/2].$$

Remark: This exercise was constructed in Herdegen/Herrmann.
