ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Exercise Sheet 1

Please hand in until Wednesday, September 24th, 12:00 in your assistant's box in HG G 52.1

Exercise 1-1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,\dots,T}, P)$ be a filtered probability space, and $\bar{S} = (1, S) = (1, S_k^1, \dots, S_k^d)_{k=0,\dots,T}$ a discrete-time model with time horizon $T \in \mathbb{N}$, i.e., S is adapted to $(\mathcal{F}_k)_{k=0,\dots,T}$. A stochastic process $\varphi := (\eta, \vartheta) = (\eta_k, \vartheta_k^1, \dots, \vartheta_k^d)_{k=0,\dots,T}$ is called a *strategy* if ϑ is predictable for $(\mathcal{F}_k)_{k=0,\dots,T}$ with $\vartheta_0 = 0$ and η is adapted to $(\mathcal{F}_k)_{k=0,\dots,T}$. It is called *affordable (for $\bar{S}$)* if

$$(\varphi_{k+1} - \varphi_k)^{tr} \bar{S}_k = (\vartheta_{k+1} - \vartheta_k)^{tr} S_k + (\eta_{k+1} - \eta_k) \leq 0, \quad k = 0, \dots, T-1.$$

a) Give an economic interpretation of an affordable strategy.

b) Show that a strategy $\varphi = (\vartheta, \eta)$ is affordable if and only if there exists an adapted, increasing process $K = (K_k)_{k=0,\dots,T}$ null at 0 such that

$$V_k(\varphi) := \varphi_k^{tr} \bar{S}_k = V_0(\varphi) + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j - K_k, \quad k = 0, \dots, T.$$

Moreover, show that K is predictable if and only if η is.

c) Show that for all triplets (V_0, ϑ, K), where V_0 is $\mathcal{F}_0$-measurable, $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=0,\dots,T}$ is predictable with $\vartheta_0 = 0$ and $K = (K_k)_{k=0,\dots,T}$ is an adapted, increasing process null at 0, there exists a unique affordable strategy $\varphi = (\eta, \vartheta)$ such that $\varphi_0^{tr} \bar{S}_0 = \eta_0 = V_0$ and $V_k(\varphi) = V_0(\varphi) + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j - K_k$ for $k = 1, \dots, T$.

Exercise 1-2

Let the financial market on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be described by a bank account $B = 1$ and one risky asset S being a geometric Brownian motion, i.e.

$$dS_t = S_t\big(\mu dt + \sigma dW_t\big), \quad S_0 = s > 0$$

for some given constants $\mu \in \mathbb{R}, \sigma > 0$. Fix $K \in \mathbb{R}$. We start with one share if $S_0 > K$ and with no share if $S_0 \leq K$. Whenever the stock price falls below K (or equals K), the share is sold, and whenever the price returns to a level strictly above K, one share is bought again. Thus, the amount held in the bank account is given by $\eta_t = -K 1_{\{S_t > K\}}$ and the amount held in the risky asset is given by $\vartheta_t = 1_{\{S_t > K\}}$. Show that this so-called *stop-loss start-gain strategy* (η, ϑ) is not a self-financing strategy.
Hint: Use the generalized Itô formula for convex functions $g : \mathbb{R} \to \mathbb{R}$ which are not necessarily in C^2, i.e., the formula

$$g(S_t) = g(S_0) + \int_0^t D^- g(S_u)\, dS_u + \int_{-\infty}^{\infty} L_t^S(y)\, \mu(dy),$$

where

$$D^- g(y) = \lim_{\varepsilon \downarrow 0} \frac{g(y) - g(y - \varepsilon)}{\varepsilon} \quad \text{and} \quad \mu[a, b) = D^- g(b) - D^- g(a) \quad \text{is a signed measure}$$

and $L_t^S(y)$ is the so-called *local time* of S at level y up to time t. Use that for any $t > 0$ and $y \in \mathbb{R}$, we have $L_t^S(y) \geq 0$ P-a.s. and $P\big[L_t^S(y) > 0\big] > 0$.

Exercise 1-3

Consider for the (discounted) stock price process S the continuous-time model under P with time horizon $T = 1$ given by

$$dS_t = S_t \left(\frac{1}{X_t} dt + dW_t \right), \quad S_0 > 0,$$

where $W = (W_t)_{0 \le t \le 1}$ is a standard P-Brownian motion and $X = (X_t)_{0 \le t \le 1}$ is given by

$$dX_t = \left(\frac{1}{X_t} - 2 \right) dt + dW_t, \quad X_0 = 1.$$

For the bank account, assume that $B_t = 1$ for all $t \in [0, 1]$.

a) Prove that $P\big[X_t > 0, 0 \le t \le 1\big] = 1$.
Hint: Use Girsanov's theorem to show that there exists a probability measure $\tilde{P}$ equivalent to P such that $\widetilde{W}_t := W_t - 2t$ is a $\tilde{P}$-Brownian motion. Show then that the norm of a 3-dimensional $\tilde{P}$-Brownian motion B with $B_0 = (1, 0, 0)$ coincides in law with

$$X_t = 1 + \int_0^t \frac{1}{X_s} ds + \widetilde{W}_t, \quad X_0 = 1.$$

You can further use that the solution of this SDE is unique in law and that a 3-dimensional Brownian motion never attains the origin.

b) Show that the cumulative gains process $G_t(\vartheta) = \int_0^t \vartheta_u dS_u$ for $\vartheta_t := \frac{1}{S_t}$ is a.s. bounded from below by -1.

c) Prove that $\vartheta = \frac{1}{S}$ induces an arbitrage opportunity, that is, a self-financing strategy $\varphi = \big(\eta, \frac{1}{S}\big)$ corresponding to $(V_0, \vartheta) = \big(0, \frac{1}{S}\big)$ with $V_0 = 0, V_1(\varphi) \ge 0$ P-a.s. and $P[V_1(\varphi) > 0] > 0$.

Exercise sheets and further information are also available on:

http://www.math.ethz.ch/education/bachelor/lectures/hs2014/math/mf/uebungen