ETH Zürich HS 2014
D-MATH
Prof. Martin Schweizer

Coordinator
Ariel Neufeld

Mathematical Finance
Solution 4

Solution 4-1

a) Let $(\vartheta^n) \subseteq \Theta^a$ such that $G_T(\vartheta^n)$ converges to Y in L^0. By Proposition 3.4 of the lecture, we know that there exists a predictable process ϑ such that $Y = G_T(\vartheta)$. We need to check that $\vartheta \in \Theta^a$. $G_T(\vartheta^n)$ converges in L^0, there is a subsequence $G_T(\vartheta^{n_k})$ which converges P-a.s. to $G_T(\vartheta)$. Thus, as $G_T(\vartheta^n) \geq -a$ P-a.s. for all n, we obtain that $G_T(\vartheta) \geq -a$ P-a.s. Now, we claim that $G_.(\vartheta) \geq -a$ P-a.s. Assume by contradiction that this is not true. Then, define $k^* := \max\left\{k \in 1, ..., T-1 \,|\, G_k(\vartheta) + a \in L^0_- \setminus \{0\}\right\}$. We remind that k^* is deterministic. Now define

$$\hat{\vartheta} := \vartheta 1_{]\!]k^*, k^*+1]\!]}.$$

Then, we see that

$$G_T(\hat{\vartheta}) = G_{k^*+1}(\hat{\vartheta}) - G_{k^*}(\hat{\vartheta}) \in L^0_+ \setminus \{0\},$$

which implies that $\mathcal{C} \cap L^0_+ \neq \{0\}$. But by assumption, S satisfies (NA), which gives us a contradiction (see Theorem 3.1).

b) We construct an example to show that $\mathcal{G}_{adm}$ is not closed in L^0 even when (NA) is satisfied. Let $T = 2$ and let $\Omega = [-1,1] \times \{-1,0,1\}$ with its Borel σ-field $\mathcal{F}$; elements of Ω will be denoted by $\omega := (u,v)$ with $u \in [-1,1]$ and $v \in \{-1,0,1\}$, and we denote by $U(\omega) := u$ the first and by $V(\omega) := v$ the second coordinate. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \sigma(U), \mathcal{F}_2 = \mathcal{F}$, and let P be the measure on $(\Omega, \mathcal{F})$ such that U is uniformly distributed on $[-1,1]$ and the conditional distribution of V given U is $\frac{1}{2}\,\delta_{+1} + \frac{1}{2}|U|\,\delta_{-1} + (1 - \frac{1}{2} - \frac{1}{2}|U|)\,\delta_0$. Finally. set $S_0 = 0, \Delta S_1 = U$, and

$$\Delta S_2 = 1_{\{V=1\}} + 0 \cdot 1_{\{V=0\}} - \frac{1}{|U|} 1_{\{V=-1\}}.$$

By construction, we obtain that $E[\Delta S_1 \,|\, \mathcal{F}_0] = 0 = E[\Delta S_2 \,|\, \mathcal{F}_1]$. Thus the price process S is a martingale under P. In particular, (NA) holds true. Next, consider the sequence of investment strategy (ϑ^n) defined by

$$\vartheta^n_1 := 1,\ \vartheta^n_2 := \frac{1}{|U|}\, 1_{\{|U| \geq \frac{1}{n}\}}$$

By construction, each ϑ^n is predictable and satisfies P-a.s.

$$G_1(\vartheta^n) = \vartheta^n_1 \Delta S_1 = U \geq -1$$
$$G_2(\vartheta^n) = \vartheta^n_1 \Delta S_1 + \vartheta^n_2 \Delta S_2 = U + \frac{1}{|U|}\, 1_{\{|U| \geq \frac{1}{n}\}}\, 1_{\{V=1\}} - \frac{1}{|U|^2}\, 1_{\{|U| \geq \frac{1}{n}\}}\, 1_{\{V=-1\}} \geq -n^2 - 1,$$

which means that $(\vartheta^n) \subseteq \Theta_{adm}$. Moreover, we see that the sequence$(G_2(\vartheta^n))$ converges P-a.s., in particular also in L^0, to $Y := U + \frac{1}{|U|}\, 1_{\{V=1\}} - \frac{1}{|U|^2}\, 1_{\{V=-1\}}$. From Proposition 3.4, we know that $Y = G_2(\vartheta)$ for some predictable process ϑ. But since U is uniformly distributed on $[-1,1]$, we see that we cannot find a constant $a \in \mathbb{R}$ such that $G_2(\vartheta) \geq -a$ P-a.s., which means that $G_2(\vartheta) \notin \mathcal{G}_{adm}$. This shows that in this example, $\mathcal{G}_{adm}$ is not closed in L^0.

Solution 4-2

a) We define the stopping time $\rho+ := \inf\{t \in \mathcal{D}_n \,|\, t \geq \rho\}$. First, we observe that for each $t_i \in \mathcal{D}_n$

$$E\big[X^{\rho+}_{t_{i+1}} - X^{\rho+}_{t_i} \,\big|\, \mathcal{F}_{t_i}\big] = E\big[(X_{t_{i+1}} - X_{t_i})\, 1_{\{t_i < \rho\}} \,\big|\, \mathcal{F}_{t_i}\big] = 1_{\{t_i<\rho\}}\, E\big[X_{t_{i+1}} - X_{t_i} \,\big|\, \mathcal{F}_{t_i}\big].$$

Thus, we obtain that

$$\mathrm{MV}(X^{\rho+}, \mathcal{D}_n) := E\Big[\sum_{t_i \in \mathcal{D}_n} \big|E[X^{\rho+}_{t_{i+1}} - X^{\rho+}_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big] = \sum_{t_i \in \mathcal{D}_n} E\Big[1_{\{t_i<\rho\}}\, \big|E[X_{t_{i+1}} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big].$$

By Jensen's inequality, we obtain for any two processes X' and X'' that

$$\big|\mathrm{MV}(X', \mathcal{D}_n) - \mathrm{MV}(X'', \mathcal{D}_n)\big| \leq E\Big[\sum_{t_i \in \mathcal{D}_n} \big|(X_{t_{i+1}} - X_{t_i}) - (X'_{t_{i+1}} - X'_{t_i})\big|\Big]$$

Take $X' := X^{\rho}$ and $X'' := X^{\rho+}$. Then, we see that the only (possibly) non-zero term above in the sum is the one for which $\rho \in [t_i, t_{i+1})$. Thus, we obtain that

$$\big|\mathrm{MV}(X^{\rho}, \mathcal{D}_n) - \mathrm{MV}(X^{\rho+}, \mathcal{D}_n)\big| \leq 2\|X\|_\infty.$$

Remark: In fact, this holds true for any partition π of $[0, T]$.

b) Let $n \in \mathbb{N}$ and let $0 = t_0 < \cdots < t_n = T$ be a finite partition of $[0, T]$. We have for all $i := 0, ..., n-1$ the existence of a sequence $(k_i^m)_m$ such that for each m, we have $k_i^m \in \mathcal{D}_m$, $k_i^m \leq k_{i+1}^m$, $k_i^m \geq t_i$ and $\lim_{m\to\infty} k_i^m = t_i$. Set $k_n^m := T$ for each m. Then we have for each m

$$\begin{aligned}
\mathrm{MV}(X, \pi) &= E\Big[\sum_{i=0}^{n-1} \big|E[X_{t_{i+1}} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big] \\
&\leq E\Big[\sum_{i=0}^{n-1} \big|E[X_{k_{i+1}^m} - X_{k_i^m} \,|\, \mathcal{F}_{k_i^m}]\big| + \big|E[X_{k_{i+1}^m} - X_{t_{i+1}} \,|\, \mathcal{F}_{t_{i+1}}]\big| + \big|E[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big] \\
&\leq E\Big[\sum_{j=0}^{2^m-1} \big|E[X_{jT/2^m} - X_{(j-1)T/2^m} \,|\mathcal{F}_{(j-1)T/2^m}]\big|\Big] \\
&\quad + E\Big[\sum_{i=0}^{n-1} \big|E[X_{k_{i+1}^m} - X_{t_{i+1}} \,|\, \mathcal{F}_{t_{i+1}}]\big| + \big|E[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big]. \\
&= \mathrm{MV}(X, \mathcal{D}_m) + E\Big[\sum_{i=0}^{n-1} \big|E[X_{k_{i+1}^m} - X_{t_{i+1}} \,|\, \mathcal{F}_{t_{i+1}}]\big| + \big|E[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big].
\end{aligned}$$

By dominated convergence, as X is bounded and right-continuous, we have

$$\lim_{m\to\infty} E\Big[\sum_{i=0}^{n-1} \big|E[X_{k_{i+1}^m} - X_{t_{i+1}} \,|\, \mathcal{F}_{t_{i+1}}]\big| + \big|E[X_{k_i^m} - X_{t_i} \,|\, \mathcal{F}_{t_i}]\big|\Big] = 0.$$

Thus, we obtain that

$$\mathrm{MV}(X, \pi) \leq \lim_{m\to\infty} \mathrm{MV}(X, \mathcal{D}_m).$$

As the partition was arbitrarily chosen, taking the sup over all the finite partitions in the above inequality yields that $\mathrm{MV}(X) \leq \lim_{m\to\infty} \mathrm{MV}(X, \mathcal{D}_m)$, and so $\mathrm{MV}(X) = \lim_{m\to\infty} \mathrm{MV}(X, \mathcal{D}_m)$.

Solution 4-3

a) By monotone convergence, we can without loss of generality assume that X is bounded. Consider first $X = \xi \, 1_{[\![0,u))}$ with $u \in [0, \infty]$ and ξ a bounded $\mathcal{F}$-measurable random variable. Then, let M be an RCLL version of the martingale $E[\xi \,|\, \mathbb{F}]$. We define ${}^{\mathcal{O}}X := M \, 1_{[\![0,u))}$. Then ${}^{\mathcal{O}}X$ is adapted and RCLL, hence optional. Moreover, by the stopping theorem, we have for every stopping time τ that

$$E[X_\tau \, 1_{\{\tau<\infty\}} \,|\mathcal{F}_\tau] = 1_{\{\tau<\infty\}} \, 1_{\{0,\tau\wedge u\}} \, E[\xi \,|\, \mathcal{F}_\tau] = 1_{\{\tau<\infty\}} \, ({}^{\mathcal{O}}X)_\tau \quad P\text{-a.s.}$$

Now, we define

$$\mathcal{H} := \big\{\text{bounded processes } X \text{ for which } {}^{\mathcal{O}}X \text{ exists}\big\}$$

as well as

$$\mathcal{M} := \big\{\xi \, 1_{[\![0,u))} \text{ with } u \in [0, \infty] \text{ and } \xi \text{ a bounded } \mathcal{F}\text{-measurable random variable}\big\}.$$

From the above calculation, we see that $\mathcal{M} \subseteq \mathcal{H}$. Moreover, $\mathcal{M}$ is closed under multiplication. Furthermore, $\mathcal{H}$ is a vector space of bounded processes containing the constant process 1 and by monotnone convergence it is also closed under monotone bounded convergence. Thus, $(\mathcal{H}, \mathcal{M})$ satisfy the conditions needed for being able to apply the monotone class theorem. As every product measurable process is $\sigma(\mathcal{M})$-measurable, we are done.

b) Take (τ_n) a foretelling sequence for σ. Then, $\mathcal{F}_{\sigma-} = \bigvee_n \mathcal{F}_{\tau_n}$. Now, by the usual stopping theorem, we have $E[Y_\tau \,|\, \mathcal{F}_{\tau_n}] = Y_{\tau_n}$ for every n. For the left-hand side, use the martingale convergence theorem for $n \to \infty$ to get

$$E[Y_\tau \,|\mathcal{F}_{\sigma-}] = Y_{\sigma-} \quad P\text{-a.s.}$$

c) The argument is similar to **a)**. By monotone convergence, we can without loss of generality assume that X is bounded. Consider first $X = \xi \, 1_{[\![0,u))}$ with $u \in [0, \infty]$ and ξ a bounded $\mathcal{F}$-measurable random variable. Then, let M be an RCLL version of the martingale $E[\xi \,|\, \mathbb{F}]$. We define ${}^{\mathcal{P}}X := M_- \, 1_{[\![0,u))}$. Then ${}^{\mathcal{P}}X$ is predictable as u is a predictable time, therefore $[\![0, u)) \in \mathcal{P}$ and as M_- is adapted and left-continuous. Moreover, by the predictable stopping theorem, we have for every predictable stopping time σ that

$$E[X_\sigma \, 1_{\{\sigma<\infty\}} \,|\mathcal{F}_{\sigma-}] = 1_{\{\sigma<\infty\}} \, 1_{\{0,\sigma\wedge u\}} \, E[\xi \,|\, \mathcal{F}_{\sigma-}] = 1_{\{\sigma<\infty\}} \, ({}^{\mathcal{P}}X)_\sigma \quad P\text{-a.s.}$$

Now, we define

$$\mathcal{H} := \big\{\text{bounded processes } X \text{ for which } {}^{\mathcal{P}}X \text{ exists}\big\}$$

as well as

$$\mathcal{M} := \big\{\xi \, 1_{[\![0,u))} \text{ with } u \in [0, \infty] \text{ and } \xi \text{ a bounded } \mathcal{F}\text{-measurable random variable}\big\}.$$

As in **a)**, we see that $(\mathcal{H}, \mathcal{M})$ satisfy the conditions needed for being able to apply the monotone class theorem. As every product measurable process is $\sigma(\mathcal{M})$-measurable, we are done.