ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Exercise Sheet 4

Please hand in until Wednesday, October 15th, 12:00 in your assistant's box in HG G 52.1

Exercise 4-1

For a financial market with discounted price process S, define as usual $G(\vartheta) = \int \vartheta\, dS$ when ϑ is predictable and the stochastic integral is well defined, i.e. $\vartheta \in L(S)$. For all $a \geq 0$, set

$$\Theta^a := \left\{\vartheta \in L(S) \,\middle|\, G.(\vartheta) \geq -a \;\; P\text{-a.s.}\right\},$$
$$\mathcal{G}^a := \left\{G_T(\vartheta) \,\middle|\, \vartheta \in \Theta^a\right\} = G_T(\Theta^a).$$

and

$$\Theta_{adm} := \bigcup_{a\geq 0} \Theta^a,$$
$$\mathcal{G}_{adm} := \bigcup_{a\geq 0} \mathcal{G}^a = G_T(\Theta_{adm}).$$

Suppose that S satisfies (NA). For the case of finite discrete time, show that

a) each $\mathcal{G}^a$ is closed in L^0 – but you are not allowed to use martingale measures.

b) $\mathcal{G}_{adm}$ is not closed in L^0.
Remark: This corrects an error in the course.

Exercise 4-2

Let $X = (X_t)_{0\leq t\leq T}$ be an adapted RCLL process and $\mathcal{D}_n := k2^{-n}T,\; k := 0, 1, ..., 2^n$, the n-th dyadic partition of $[0, T]$, for each $n \in \mathbb{N}$. Suppose that X is bounded.

a) Show that for each stopping time ρ with values in $[0, T]$ that

$$\mathrm{MV}(X^\rho, \mathcal{D}_n) \leq \sum_{t_i\in\mathcal{D}_n} E\Big[1_{\{t_i<\rho\}} \left|E\left[X_{t_{i+1}} - X_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right]\right|\Big] + 2\|X\|_\infty =: \mathrm{MV}(X^{\rho+}, \mathcal{D}_n) + 2\|X\|_\infty.$$

b) Show that $\mathrm{MV}(X) = \lim_{n\to\infty} \mathrm{MV}(X, \mathcal{D}_n)$.

Exercise 4-3

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual conditions and let $(X_t)_{t\geq 0}$ be a nonnegative product-measurable process.

a) Show that there is an optional process ${}^{\mathcal{O}}X$ such that for every stopping time τ

$$E[X_\tau\, 1_{\{\tau<\infty\}}\, |\mathcal{F}_\tau] = ({}^{\mathcal{O}}X)_\tau\, 1_{\{\tau<\infty\}} \quad P\text{-a.s.}$$

Remark: In fact, one can show that ${}^{\mathcal{O}}X$ is unique. It is called the *optional projection* of X.

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b) For every stopping time τ, we can define the σ-field

$$\mathcal{F}_{\tau-} := \mathcal{F}_0 \vee \sigma\big(A \cap \{t < \tau\};\, t \geq 0, A \in \mathcal{F}_t\big).$$

One can show that for any sequence of stopping times (τ_n) with $\tau = \sup_n \tau_n$ and $\tau_n < \tau$ on $\{0 < \tau_n < \infty\}$ for all n, we have $\mathcal{F}_{\tau-} = \bigvee_n \mathcal{F}_{\tau_n}$.

A stopping time σ is called *predictable* if its graph $[\![\sigma]\!] = \big\{(\omega, t) \in \Omega \times [0, \infty) \,\big|\, \sigma(\omega) = t\big\}$ is in $\mathcal{P}$. In particular, every deterministic time $t \in [0, \infty]$ is a predictable stopping time. One can show that a stopping time σ is predictable if and only if there exists a sequence of stopping time (τ_n) converging to σ with $\tau_n \leq \sigma$ and $\tau_n < \sigma$ on $\{\sigma > 0\}$. We call (τ_n) a *foretelling sequence* for σ.

Now let $(Y_t)_{t \in [0,\infty]}$ be a (right-closed) RCLL martingale. Show that for every predictable stopping time σ and any stopping time $\tau \geq \sigma$, we have

$$E[Y_\tau \,|\mathcal{F}_{\sigma-}] = Y_{\sigma-} \quad P\text{-a.s.} \tag{1}$$

Remark: The result in (1) is called the *predictable stopping theorem.*

c) Show that there is a predictable process ${}^{\mathcal{P}}X$, such that for every predictable stopping time σ

$$E[X_\sigma \, 1_{\{\sigma < \infty\}} \,|\mathcal{F}_{\sigma-}] = ({}^{\mathcal{P}}X)_\sigma \, 1_{\{\sigma < \infty\}} \quad P\text{-a.s.}$$

Remark: In fact, one can show that ${}^{\mathcal{P}}X$ is unique. It is called the *predictable projection* of X.

Remark: By decomposing $X = X^+ - X^-$ into its positive and negative parts, we see that we can generalize the above results as long as the (conditional) expectation of X is well defined.
