ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Exercise Sheet 3

Please hand in until Wednesday, October 8th, 12:00 in your assistant's box in HG G 52.1

Exercise 3-1

Consider a financial market in finite discrete time and a self-financing strategy φ. We say that φ is an arbitrage opportunity of the first kind if

$$V_0(\varphi) \in L^0_- \quad \text{and} \quad V_T(\varphi) \in L^0_+ \setminus \{0\}.$$

The condition (NA$_1$) is the assumption that such φ do not exist. We say that φ is an arbitrage opportunity of the second kind if

$$V_0(\varphi) \in L^0_- \setminus \{0\} \quad \text{and} \quad V_T(\varphi) \in L^0_+.$$

a) Describe the two kinds of arbitrage in words.

b) Show that (NA$_1$) implies that there are no arbitrage opportunities of the second kind.

c) Prove by a counterexample that the converse to **b)** is wrong.

Exercise 3-2

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(Y_k)_{k=1,\dots,T}$ with $T \in \mathbb{N}$ a sequence of independent, non-trivial and strictly positive random variables. Define a discrete-time model $\bar{S} = (1, S_k)_{k=0,\dots,T}$ with time horizon T by $S_k = \prod_{j=1}^k Y_j$, $k = 0, \dots, T$, and let the filtration $(\mathcal{F}_k)_{k=0,\dots,T}$ be generated by S.

a) Fix $k \in \{1, \dots, T\}$, and assume that $\operatorname{ess\,inf} Y_k < 1 < \operatorname{ess\,sup} Y_k$. Show that there exists a probability measure $Q^k \approx P$ on $\mathcal{F}_T$ with $E_{Q^k}[Y_k] = 1$ and $\frac{dQ^k}{dP} = g_k(Y_k)$ for some measurable function $g_k : (0,\infty) \to (0,\infty)$.

Hint: Show more generally that if Y is a real-valued non-trivial random variable and $\mu \in (\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y)$, then there exists $Q \approx P$ on $\mathcal{F}$ such that $Y \in L^1(Q)$, $E_Q[Y] = \mu$ and $\frac{dQ}{dP} = g(Y)$ for a measurable function $g : \mathbb{R} \to (0,\infty)$.

Divide the proof into two steps: First, show that there exists $\widetilde{P} \approx P$ on $\mathcal{F}$ such that $Y \in L^1(\widetilde{P})$. Then, set $A := \{Y \le \mu\}$, use the two conditional probabilities $\widetilde{P}[\,\cdot\,|\,A]$ and $\widetilde{P}[\,\cdot\,|\,A^c]$ to construct $Q \approx \widetilde{P}$ with $Y \in L^1(Q)$ and $E_Q[Y] = \mu$, and put everything together.

b) Show that S satisfies NA if and only if $\operatorname{ess\,inf} Y_k < 1 < \operatorname{ess\,sup} Y_k$ for all $k = 1, \dots, T$.

Hint: For "$\Leftarrow$" construct an equivalent martingale measure using part (a). For "$\Rightarrow$", argue by contraposition and construct an explicit arbitrage strategy.

Please turn the page

Exercise 3-3

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a filtered probability space and $S = (S_t)_{0 \leq t \leq T}$ a $(P, \mathbb{F})$-semimartingale.

a) Let $\mathcal{C} = (G_T(\Theta_{adm}) - L^0_+) \cap L^\infty$. Show that if $Q \approx P$ with $E_Q[Y] \leq 0$ for all $Y \in \mathcal{C}$ then $E_Q[G_T(\vartheta)] \leq 0$ for all $\vartheta \in \Theta_{adm}$.

b) Show that the condition (NA), i.e. $G_T(\Theta_{adm}) \cap L^0_+ = \{0\}$, is equivalent to $\mathcal{C} \cap L^\infty_+ = \{0\}$.

c) Show that if $\mathcal{D} \subseteq L^0$, then D is bounded in L^0, i.e. $\lim\limits_{m \to \infty} \sup_{d \in \mathcal{D}} P[|d| \geq m] \to 0$, if and only if for every sequence $(\lambda_n) \subseteq (0, \infty)$ with $\lambda_n \downarrow 0$ we have $\lambda_n D^n \to 0$ in L^0 for every sequence $(D^n) \subseteq \mathcal{D}$.

Exercise sheets and further information are also available on:

http://www.math.ethz.ch/education/bachelor/lectures/hs2014/math/mf/uebungen