ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 2

Solution 2-1

a) If $u \le r$, then we also have $d < r$. Thus it is always more profitable to invest in the bank account than in the risky asset. Thus, take $V_0 = 0$, $\vartheta \equiv -1$ and let $\varphi := (\eta, \vartheta)$ be the corresponding self-financing strategy (see Lemma 1.1 from the lecture). Then, we see that for each $n \in \mathbb{N}$ we have

$$\eta_n = V_0 + \sum_{j=1}^{n} \vartheta_j \cdot \Delta S_j - \vartheta_n \cdot S_n \equiv s_0.$$

Moreover, as $u \le r$, we have that $0 \le R_j \le 1$ P-a.s. for each j. Thus, we obtain that

$$P[V_T(\varphi) \ge 0] = P[s_0 - s_0 \prod_{j=1}^{T} R_j \ge 0] = 1.$$

Moreover, as $r \ge u > d$, we see that

$$P[V_T(\varphi) > 0] \ge P\left[R_1 = \cdots = R_T = \frac{1+d}{1+r}\right] = (1-p)^T > 0.$$

If $d \ge r$, it is always more profitable to invest in the risky asset than in the bank account. Thus, take $V_0 = 0$, $\vartheta \equiv 1$ and let $\varphi := (\eta, \vartheta)$ be the corresponding self-financing strategy. Then, we see that $\eta \equiv -s_0$. Moreover, as $d \ge r$, we have that $R_j \ge 1$ P-a.s. for each j. Thus, we obtain that

$$P[V_T(\varphi) \ge 0] = P[-s_0 + s_0 \prod_{j=1}^{T} R_j \ge 0] = 1.$$

Moreover, as $u > d \ge r$, we see that

$$P[V_T(\varphi) > 0] \ge P\left[R_1 = \cdots = R_T = \frac{1+u}{1+r}\right] = p^T > 0.$$

b) The martingale condition is $E_Q[R_{j+1}|\mathcal{F}_j] = 1$ for all j, which implies $E_Q[R_j] = 1$. Thus, pick probability weights q_u, q_d such that

$$q_u \frac{1+u}{1+r} + q_d \frac{1+d}{1+r} = 1, \quad q_u + q_d = 1,$$

which is possible by the assumption that $u > r > d$. Let Q be the probability measure on Ω such that the R_j are independent and the up/down probabilities are given by q_u, q_d. Then S is a Q-martingale.

Solution 2-2

a) Define

$$k^* := \min\Big\{k \in \{1,\dots,N\} : G_{\tau_k}(\vartheta) \in L^0_+ \setminus \{0\}\Big\}, \tag{1}$$

and set $\sigma_0 := \tau_{k^*-1}$ and $\sigma_1 := \tau_{k^*}$. Observe that k^* is deterministic. Moreover, set

$$h := \begin{cases} h^{k^*} & \text{if } P[G_{\tau_{k^*-1}}(\vartheta) = 0] = 1, \\ h^{k^*} 1_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} & \text{if } P[G_{\tau_{k^*-1}}(\vartheta) = 0] < 1. \end{cases} \tag{2}$$

Note that $P[G_{\tau_{k^*-1}}(\vartheta) < 0] > 0$ in the second case by the definition of k^*. We claim that $\vartheta^* := h 1_{]\!]\sigma_0,\sigma_1]\!]} \in \mathbf{b}\mathcal{E}$ is an arbitrage opportunity. Indeed, in the first case,

$$G_T(\vartheta^*) = G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta) = G_{\tau_{k^*}}(\vartheta) \in L^0_+ \setminus \{0\}, \tag{3}$$

and in the second case,

$$\begin{aligned} G_T(\vartheta^*) &= \big(G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta)\big) 1_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} \\ &\geq -G_{\tau_{k^*-1}}(\vartheta) 1_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} \in L^0_+ \setminus \{0\}. \end{aligned} \tag{4}$$

b) Let $a \geq 0$ be such that $G(\vartheta) \geq -a$ P-a.s. By right-continuity of the paths of $G(\vartheta)$, it suffices to show $G_t(\vartheta) \geq -c$ P-a.s. for any $t \in [0,T)$. Seeking a contradiction, assume there is $t \in [0,T)$ such that $P[G_t(\vartheta) < -c] > 0$. But then $\vartheta^* := \vartheta 1_{\{G_t(\vartheta)<-c\}\times(t,T]}$ is predictable, is S-integrable (see hints) and satisfies

$$\begin{aligned} G(\vartheta^*) &= (G(\vartheta) - G_t(\vartheta)) 1_{\{G_t(\vartheta)<-c\}\times(t,T]} \geq -a + c, \\ G_T(\vartheta^*) &= (G_T(\vartheta) - G_t(\vartheta)) 1_{\{G_t(\vartheta)<-c\}} \geq (-c - G_t(\vartheta)) 1_{\{G_t(\vartheta)<-c\}}. \end{aligned} \tag{5}$$

But this shows both that ϑ^* is admissible and that S fails NA, in contradiction to the hypothesis.

Solution 2-3

a) Let τ_m be a localizing sequence. Note that $\sup_{m\in\mathbb{N},\, s\in\{0,\cdots t\}} |X^{\tau_m}_s| \leq \sum_{k=0}^t |X_k| \in L^1$, hence dominated convergence allows passage from $E[X^{\tau_m}_t | \mathcal{F}_{t-1}] = X^{\tau_m}_{t-1}$ to $E[X_t|\mathcal{F}_{t-1}] = X_{t-1}$.

b) Let (τ_m) be a localizing sequence and denote $X^{(m)}_t = X^{\tau_m}_t$. Proceeding inductively, assume that $E[X^-_t] < \infty$. As $X^{(m)-}_{\cdot}$ is a submartingale, $X^{(m)-}_{t-1} \leq E[X^{(m)-}_t | \mathcal{F}_{t-1}]$. Therefore

$$X^-_{t-1} 1_{\{\tau_m > t-1\}} = X^{(m)-}_{t-1} 1_{\{\tau_m>t-1\}} \leq E[X^{(m)-}_t 1_{\{\tau_m>t-1\}} \,|\, \mathcal{F}_{t-1}] \leq E[X^-_t | \mathcal{F}_{t-1}] 1_{\{\tau_m>t-1\}}.$$

Letting $m \to \infty$, we obtain $X^-_{t-1} \leq E[X^-_t|\mathcal{F}_{t-1}]$ and hence $E[X^-_{t-1}] < \infty$. As $E[X^-_T] < \infty$ by assumption, we have proved that X^- is integrable and a submartingale.

For the positive part, $X^+_t = X_t + X^-_t = \liminf_{m\to\infty} X^{(m)}_t + X^-_t$ combined with Fatou's lemma and the martingale/submartingale property yield

$$E[X^+_t] \leq \liminf_m E[X^{(m)}_t] + E[X^-_T] = E[X_0] + E[X^-_T].$$

Note that Fatou's lemma can indeed be used as the $X^{(m)}_t$ have a common lower bound $-\sum_{t=0}^T X^-_t \in L^1$. It remains to apply **a)**.

In particular, if X is bounded from below, the stopped process X^T is a martingale for any $T > 0$ and thus X is a martingale, too.

c) There is a localizing sequence (σ_n) such that X^{σ_n} is a martingale for all n. For every $n \in \mathbb{N}$, we define the stopping time $\tau_n = \inf\{k \geq 0 : |\vartheta_{k+1}| \geq n\}$ and then

$$E[\vartheta^{\tau_n}_{t+1}(X^{\sigma_n}_{t+1} - X^{\sigma_n}_t)|\mathcal{F}_t] = \vartheta^{\tau_n}_{t+1} E[(X^{\sigma_n}_{t+1} - X^{\sigma_n}_t)|\mathcal{F}_t] = 0,$$

so $(\int \vartheta \, dX^{\sigma_n})^{\tau_n} = (\int \vartheta \, dX)^{(\sigma_n \wedge \tau_n)}$ is a martingale.