ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 9

Solution 9-1

Let Q be the martingale measure; we can write $\widetilde{S}_t = S_0 \exp(\sigma W_t^Q + (r - \frac{1}{2}\sigma^2)t)$ for a Q-Brownian motion W^Q. Using the risk neutral valuation formula, we have

$$\begin{aligned}
\widetilde{V}_t &= e^{-r(T-t)} E_Q[\widetilde{H}|\mathcal{F}_t] = e^{-r(T-t)} E_Q[1_{\{\widetilde{S}_T > \widetilde{K}\}}|\mathcal{F}_t] = e^{-r(T-t)} Q[\widetilde{S}_T > \widetilde{K}|\mathcal{F}_t] \\
&= e^{-r(T-t)} Q\left[\widetilde{S}_t \exp\left(\sigma(W_T^Q - W_t^Q) + (r - \frac{1}{2}\sigma^2)(T-t)\right) > \widetilde{K} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)} Q\left[-\sigma(W_T^Q - W_t^Q) < \ln\frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)\right]\Big|_{x=\widetilde{S}_t} \\
&= e^{-r(T-t)} Q\left[\xi < \frac{\ln\frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]\Big|_{x=\widetilde{S}_t} \\
&= e^{-r(T-t)} \Phi\left(\frac{\ln\frac{x}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\Big|_{x=\widetilde{S}_t} \\
&= \widetilde{v}(t, \widetilde{S}_t).
\end{aligned}$$

where $\xi = -(W_T^Q - W_t^Q)/\sqrt{T-t}$ has a standard Gaussian law and Φ is the standard normal c.d.f. As in the lecture, the strategy is given by the spatial derivative,

$$\widetilde{\vartheta}_t = \frac{\partial \widetilde{v}}{\partial x}(t, \widetilde{S}_t) = e^{-r(T-t)} \phi\left(\frac{\ln\frac{\widetilde{S}_t}{\widetilde{K}} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \frac{1}{\widetilde{S}_t \sigma\sqrt{T-t}}.$$

Here $\phi = \Phi'$ is the standard normal density.

Solution 9-2

Let Q be the martingale measure. Let q be the 'up'-probability under Q and Z_k the Q-(i.i.d.) returns of $\widetilde{S}$ with values in $\{1+d, 1+u\}$.

a) $\widetilde{V}_k(\widetilde{H}) = (1+r)^{k-T} E_Q[\widetilde{h}(\widetilde{S}_T)|\mathcal{F}_k] = (1+r)^{k-T} E_Q[\widetilde{h}(\widetilde{S}_k Z_{k+1} \cdots Z_T)|\mathcal{F}_k] =: \widetilde{v}(k,x)|_{x=\widetilde{S}_k}$ for $\widetilde{v}(k,x) = (1+r)^{k-T} E_Q[\widetilde{h}(x Z_{k+1} \cdots Z_T)]$ by the Q-i.i.d. property of the (Z_k).

b) Clearly $\widetilde{v}(T, \widetilde{S}_T) = \widetilde{h}(\widetilde{S}_T)$. Furthermore,

$$\begin{aligned}
\widetilde{v}(k,x) &= (1+r)^{k-T} E_Q[\widetilde{h}(x Z_{k+1} \cdots Z_T)] = (1+r)^{-1} E_Q[\widetilde{v}(k+1, x Z_{k+1})] \\
&= (1+r)^{-1} q \widetilde{v}(k+1, x(1+u)) + (1+r)^{-1}(1-q)\widetilde{v}(k+1, x(1+d)).
\end{aligned}$$

c) Expanding the recursion of **b)** we obtain

$$\widetilde{V}_0(H) = \widetilde{v}(0, \widetilde{S}_0) = (1+r)^{-T} \sum_{k=0}^{T} \binom{T}{k} q^k (1-q)^{T-k}\, \widetilde{h}\big(\widetilde{S}_0 (1+u)^k (1+d)^{T-k}\big).$$

Alternativ argument: $\prod_{j=1}^{T} Z_j$ has value $(1+u)^k(1+d)^{T-k}, k = 0, 1, \cdots, T$, if and only if there are k-times up-moves and $T-k$-times down-moves, and that has binomial distribution $\binom{T}{k} q^k(1-q)^k$ under Q.

d) For each t we should have, using **b)**,

$$\begin{aligned}\widetilde{\vartheta}_{k+1}(\widetilde{S}_{k+1} - \widetilde{S}_k) &= \widetilde{V}_{k+1}(\widetilde{H}) - \widetilde{V}_k(\widetilde{H}) - \widetilde{\eta}_{k+1}(\widetilde{B}_{k+1} - \widetilde{B}_k) \\ &= \widetilde{v}(k+1, \widetilde{S}_{k+1}) - \widetilde{v}(k, \widetilde{S}_k) - \widetilde{\eta}_{k+1}(\widetilde{B}_{k+1} - \widetilde{B}_k)\end{aligned}$$

This equation on the event $\{Z_{k+1} = 1+u\}$ reads

$$\widetilde{\vartheta}_{k+1}(\widetilde{S}_k(1+u) - \widetilde{S}_k) = \widetilde{v}(k+1, (1+u)\widetilde{S}_k) - \widetilde{v}(k, \widetilde{S}_k) - \widetilde{\eta}_{k+1}(\widetilde{B}_{k+1} - \widetilde{B}_k),$$

and on $\{Z_{k+1} = 1+d\}$ it reads

$$\widetilde{\vartheta}_{k+1}(\widetilde{S}_k(1+d) - \widetilde{S}_k) = \widetilde{v}(k+1, (1+d)\widetilde{S}_k) - \widetilde{v}(k, \widetilde{S}_k) - \widetilde{\eta}_{k+1}(\widetilde{B}_{k+1} - \widetilde{B}_k).$$

Taking the difference we get $\widetilde{\vartheta}_{k+1}(u-d)\widetilde{S}_k = \widetilde{v}(k+1, (1+u)\widetilde{S}_k) - \widetilde{v}(k+1, (1+d)\widetilde{S}_k)$ and hence

$$\widetilde{\vartheta}_{k+1} = \frac{\widetilde{v}(k+1, (1+u)\widetilde{S}_k) - \widetilde{v}(k+1, (1+d)\widetilde{S}_k)}{(u-d)\widetilde{S}_k}, \tag{1}$$

which is indeed predictable. Going backwards the above steps we see that $\widetilde{\vartheta}$ indeed replicates $\widetilde{H}$. Equation (1) can be read as

$$\widetilde{\vartheta}_{k+1} = \frac{\widetilde{V}_{k+1}(\widetilde{H})|_{up} - \widetilde{V}_{k+1}(\widetilde{H})|_{down}}{\widetilde{S}_{k+1}|_{up} - \widetilde{S}_{k+1}|_{down}} = \frac{\Delta(\text{price of derivative } \widetilde{H})}{\Delta(\text{price of underlying})}$$

and is sometimes called discrete Delta hedging. In the Black–Scholes world, "Delta" is the derivative of the price of a financial product with respect to the price of the underlying.

e) From **b)** we see that if $\widetilde{h}$ is increasing, then $\widetilde{v}(k, \cdot)$ is increasing for each k. The formula from **d)** then shows that $\widetilde{\vartheta} \geq 0$.

Solution 9-3

a) Write $X_t = \sum_{k=1}^{N_t} Y_k$, where N is a Poisson process with rate λ and $(Y_k)_{k\in\mathbb{N}}$ a sequence of random variables independent of N such that the Y_i are i.i.d. with distribution ν. Note that W, N and $(Y_k)_{k\in\mathbb{N}}$ are independent. If R is a martingale, then in particular $E[R_T] = 0$, and hence $E[X_T] = -aT$. Since $E[X_T] = \lambda T E[Y_1]$, this gives $E[Y_1] = -\frac{a}{\lambda}$. Since $\mathcal{E}(R)$ is a nonnegative local martingale and hence a supermartingale, it suffices to show that $E[\mathcal{E}(R)_T] = 1$. Using the formula for $\mathcal{E}(R)$, the fact that X is a simple jump process, i.e. $X_t = \sum_{0<s\leq t} \Delta X_s$ and hence $\exp(X_t)\prod_{0<s\leq t}\exp(-\Delta X_s) = 1$, the fact that W, N and $(Y_k)_{k\in\mathbb{N}}$ are independent and that $\Delta Y_s = \Delta X_s$ gives

$$\begin{aligned}E[\mathcal{E}(R)_T] &= E\left[\exp(aT)\exp\left(\sigma W_T - \frac{1}{2}\sigma^2 T\right)\prod_{k=1}^{N_T}(1+Y_k)\right] \\ &= \exp(aT)E\left[\left(1 - \frac{a}{\lambda}\right)^{N_T}\right] = \exp(aT)\exp\left(\left(-\frac{a}{\lambda}\right)\lambda T\right) \\ &= 1.\end{aligned} \tag{2}$$

Remark: One can show in general that if R is a Lévy process and a local martingale, then $\mathcal{E}(R)$ is a martingale.

b) First, assume that the paths of R are not monotone. We try to find a measure under which Q is a Lévy process and a martingale.

Let $\widetilde{\nu} \approx \nu$ be an equivalent probability measure on $\mathbb{R}$ with finite mean $\widetilde{\mu}$ (if ν already has finite mean, we can take $\widetilde{\nu} = \nu$). Define as in Exercise **8-2** (with $\widetilde{\lambda} := \lambda$) the measure $\widetilde{P} \approx P$ on $\mathcal{F}_T$ by

$$\frac{d\widetilde{P}}{dP} := \exp\left(\sum_{k=1}^{N_T} \phi(Y_k)\right), \tag{3}$$

where $\phi = \log \frac{d\widetilde{\nu}}{d\nu}$. Then by the hint, W and X are still independent under $\widetilde{P}$ and W is a Brownian motion under $\widetilde{P}$. Hence, R is an integrable jump diffusion under $\widetilde{P}$ with mean function

$$E_{\widetilde{P}}[R_t] = at + 0 + \widetilde{\mu}\widetilde{\lambda}t = t(a + \widetilde{\mu}\widetilde{\lambda}), \quad t \in [0,T]. \tag{4}$$

Now we distinguish two cases. First, assume that $\sigma > 0$. Define the measure $Q \approx \widetilde{P}$ on $\mathcal{F}_T$ by

$$\frac{dQ}{d\widetilde{P}} := \mathcal{E}\left(-\frac{a + \widetilde{\mu}\widetilde{\lambda}}{\sigma} W\right)_T.$$

Then by Girsanov's theorem, it follows that $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,T]}$ defined by $\widetilde{W}_t = W_t + \frac{a+\widetilde{\mu}\widetilde{\lambda}}{\sigma}t$ is a Q-Brownian motion, and by the hint, $\widetilde{W}$ and X are still independent under Q and the distributions of X under $\widetilde{P}$ and Q coincide. Since $R_t = \sigma\widetilde{W}_t - \widetilde{\lambda}\widetilde{\mu}t + X_t$, this implies that R is under Q again an integrable jump-diffusion with mean function 0, and hence a Q-martingale.

Next, assume that $\sigma = 0$. Since R has paths which are not monotone, we are in one of the following three cases:

(a) $\widetilde{\nu}((-1,0)) > 0$ and $\widetilde{\nu}((0,\infty)) > 0$, i.e., R has positive and negative jumps.
(b) $\widetilde{\nu}((-1,0)) = 0$ and $a < 0$, i.e, R has only positive jumps but a strictly negative drift.
(c) $\widetilde{\nu}((0,\infty)) = 0$ and $a > 0$, i.e, R has only negative jumps but a strictly positive drift.

In each of the three cases, it suffices to find $\widehat{\lambda} > 0$ and an equivalent probability measure $\widehat{\nu} \approx \widetilde{\nu}$ on $\mathbb{R}$ with mean $\widehat{\mu}$ such that $\widehat{\lambda}\widehat{\mu} = -a$ and to define as in Exercise **8-2** the measure $Q \approx \widetilde{P}$ on $\mathcal{F}_T$ by

$$\frac{dQ}{d\widetilde{P}} := \exp\left(\sum_{k=1}^{N_T} \widehat{\phi}(Y_k) + (\widetilde{\lambda} - \widehat{\lambda})T\right), \tag{5}$$

where $\widehat{\phi} = \log\left(\frac{\widehat{\lambda}}{\widetilde{\lambda}}\frac{d\widehat{\nu}}{d\widetilde{\nu}}\right)$. Then it follows as in the first step that R is a jump-diffusion under Q with mean function

$$E_Q[R_t] = t(a + \widehat{\mu}\widehat{\lambda}) = 0, \quad t \in [0,T], \tag{6}$$

which implies that R is a Q-martingale.

To find $\widehat{\lambda}$ and $\widehat{\nu}$ as above, we proceed as follows: In the first case, choose $\widehat{\lambda} > 0$ such that $\frac{-a}{\widehat{\lambda}} \in (\inf \operatorname{supp} \nu, \sup \operatorname{supp} \nu)$, and let $\widehat{\nu} \approx \widetilde{\nu}$ be an equivalent probability measure on $\mathbb{R}$ with mean $\frac{-a}{\widehat{\lambda}}$. The existence of such a measure can be shown similarly as in Exercise **3-2 a)**. In the second and third case, set $\widehat{\lambda} := \frac{-a}{\widetilde{\mu}}$ and $\widehat{\nu} := \widetilde{\nu}$.

Conversely, assume that the paths of R are monotone. We only consider the case that they are nondecreasing, the argument for the nonincreasing case is similar. Then $\sigma = 0$, $a \geq 0$ and ν is concentrated on $(0,\infty)$. Hence by the formula in the hint, using the same arguments as in a), $S = s_0\mathcal{E}(R)$ satisfies

$$S_t = s_0 \exp(at) \prod_{k=1}^{N_t} (1 + Y_k), \quad t \in [0,T], \tag{7}$$

and so also S has nondecreasing paths. Consider the strategy $\theta = (\theta_t)_{t\in[0,T]}$ where $\theta \equiv 1$. Then its gains process $G(\theta)$ satisfies

$$G_t(\theta) = S_t - S_0, \quad t \in [0,T]. \tag{8}$$

Since S has nondecreasing paths and $P[S_T > S_0] \geq P[N_T \geq 1] > 0$, it follows that θ is 0-admissible and that S fails NA.