ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 1

Solution 1-1

a) Fix $k \in \{0, \dots, T-1\}$. Then $(\vartheta_{k+1} - \vartheta_k)^{tr} S_k$ denotes the cost for changing the position in the risky assets in the interval $(k, k+1]$ (this cost is already determined at time k as we have to chose ϑ_{k+1} at time k as we cannot see into the future) and $\eta_{k+1} - \eta_k$ denotes the cost for changing the position in the riskless asset in the interval $(k, k+1]$ (this cost is not determined until time $k+1$ since we need not choose η_{k+1} until time $k+1$ as S^0 (i.e. the constant bank account equal to 1) is riskless). So φ being affordable means that the total cost for changing the position in all assets (the risky and the riskless) is not positive, i.e., one can afford this strategy without injecting additional wealth at any time.

b) First, assume that $\varphi = (\eta, \vartheta)$ is affordable. Define the process $K = (K_k)_{k=0,\dots,T}$ by

$$K_k := V_0(\varphi) - V_k(\varphi) + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j, \quad k = 0, \dots, T. \tag{1}$$

Then K is clearly adapted and null at 0. In order to show that it is increasing, it suffices to show that $\Delta K_{k+1} \geq 0$ for all $k = 0, \dots, T-1$. Fix $k \in \{0, \dots, T-1\}$. Then

$$\begin{aligned}\Delta K_{k+1} &= V_k(\varphi) - V_{k+1}(\varphi) + \vartheta_{k+1}^{tr} \Delta S_{k+1} \\ &= \eta_k + \vartheta_k^{tr} S_k - (\eta_{k+1} + \vartheta_{k+1}^{tr} S_{k+1}) + \vartheta_{k+1}^{tr}(S_{k+1} - S_k) \\ &= -\big((\vartheta_{k+1} - \vartheta_k)^{tr} S_k + (\eta_{k+1} - \eta_k)\big) \geq 0. \end{aligned} \tag{2}$$

For the converse, just work through the above equation backwards.

To establish the final claim, it suffices to note that K is predictable if and only if ΔK_{k+1} is $\mathcal{F}_k$-measurable for $k = 0, \dots, T-1$ and that the latter by (2) and predictability of ϑ is equivalent to η_{k+1} being $\mathcal{F}_k$-measurable for $k = 0, \dots, T-1$, i.e., η being predictable.

c) Let (V_0, ϑ, K) be as in the setup of the question. Define the process $\eta = (\eta_k)_{k=0,\dots,T}$ by

$$\eta_k := V_0 + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j - \vartheta_k^{tr} S_k - K_k. \tag{3}$$

Then η is clearly adapted and $\varphi := (\eta, \vartheta)$ satisfies

$$\begin{aligned} V_0(\varphi) &= \vartheta_0^{tr} S_0 + \eta_0 = \eta_0 = V_0, \\ V_k(\varphi) &= \eta_k + \vartheta_k^{tr} S_k = V_0 + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j - K_k \\ &= V_0(\varphi) + \sum_{j=1}^{k} \vartheta_j^{tr} \Delta S_j - K_k, \quad k = 1, \dots, T. \end{aligned} \tag{4}$$

If $\widetilde{\varphi} = (\widetilde{\eta}, \vartheta)$ is another affordable strategy satisfying $\widetilde{\eta}_0 = V_0$ and (4), then $\widetilde{\eta}_0 = \eta_0$, therefore $V(\widetilde{\varphi}) = V(\varphi)$ and hence $\widetilde{\eta} = \eta$.

Solution 1-2

We first observe that the portfolio value at time $t > 0$ is given by

$$V_t = \eta_t 1 + \vartheta_t S_t = -K 1_{\{S_t > K\}} + 1_{\{S_t > K\}} S_t = \max\big(0, S_t - K\big).$$

By definition, (η, ϑ) is self-financing if and only if for any $t > 0$,

$$V_t = \max\big(0, S_t - K\big) = \max\big(0, S_0 - K\big) + \int_0^t 1_{\{S_u > K\}}\, dS_u. \tag{5}$$

Now we observe that the function $y \mapsto \max\big(0, y - K\big)$ is convex, but not in C^2. Thus we can apply the generalized Itô formula to obtain

$$g(S_t) = g(S_0) + \int_0^t D^- g(S_u)\, dS_u + \int_{-\infty}^{\infty} L_t(y)\, \mu(dy),$$

where

$$D^- g(y) = \lim_{\varepsilon \downarrow 0} \frac{g(y) - g(y - \varepsilon)}{\varepsilon} \quad \text{and} \quad \mu(a, b) = D^- g(b) - D^- g(a) \ \text{ is a signed measure}$$

and $L_t^S(y)$ is the so-called *local time* of S at level y up to time t. Since here $g(y) = \max\big(0, y - K\big)$, we get that

$$D^- g(y) = 1_{\{y > K\}}, \quad \mu(dy) = \delta_K(dy).$$

Thus we obtain from the generalized Itô formula that for any $t > 0$,

$$g(S_t) = \max\big(0, S_t - K\big) = \max\big(0, S_0 - K\big) + \int_0^t 1_{\{S_u > K\}}\, dS_u + L_t(K). \tag{6}$$

Thus, we see from the comparison of (5) with (6) that (η, ϑ) is self-financing if and ony if for any $t > 0$, $L_t(K)$ is equal to zero P-a.s. But we know (from the hints) that $L_t^S(K) \geq 0$ P-a.s. and $P\big[L_t^S(K) > 0\big] > 0$, and hence (η, ϑ) is not self-financing.

Solution 1-3

a) We can define a probability measure $\widetilde{P}$ equivalent to P on $\mathcal{F}_1$ by

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(2W)_t = \exp(2W_t - 2t), \quad 0 \leq t \leq 1,$$

where $(\mathcal{F}_t)_{0 \leq t \leq 1}$ denotes the P-augmentation of the natural filtration generated by W. According to Girsanov's theorem, $\widetilde{W}_t = W_t - 2t$ is then a $\widetilde{P}$-Brownian motion. Now let B be a 3-dimensional $\widetilde{P}$-Brownian motion with $B_0 = (1, 0, 0)$ and look at $\|B_t\|$. Since B_t never hits the origin $\widetilde{P}$-a.s., we can apply Itô's formula to obtain

$$d\,\|B_t\| = \sum_{i=1}^{3} \frac{B_t^i}{\|B_t\|} dB_t^i + \frac{1}{\|B_t\|} dt.$$

By Levy's characterization theorem,

$$\widehat{W}_t := \int_0^t \sum_{i=1}^{3} \frac{B_s^i}{\|B_s\|} dB_s^i$$

is a $\widetilde{P}$-Brownian motion. Thus we see that

$$d\|B_t\| = d\widehat{W}_t + \frac{1}{\|B_t\|}dt.$$

But by definition of $\widetilde{W}$, we know that

$$dX_t = d\widetilde{W}_t + \frac{1}{X_t}dt. \tag{7}$$

In other words, $(\Omega, \mathcal{F}, \mathbb{F}, \widetilde{P}, \widetilde{W}, X)$ and $(\Omega, \mathcal{F}, \mathbb{F}, \widetilde{P}, \widehat{W}, \|B\|)$ are both weak solutions of the SDE (7). Thus by uniqueness of the weak solution, we get that

$$\widetilde{P}\big[X_t > 0,\ 0 \le t \le 1\big] = \widetilde{P}\big[\,\|B_t\| > 0,\ 0 \le t \le 1\big] = 1,$$

as B never hits the origin $\widetilde{P}$-a.s. As P and $\widetilde{P}$ are equivalent on $\mathcal{F}_1$, we can deduce that $P[X_t > 0,\ 0 \le t \le 1] = 1$.

b) The cumulative gains process $G_t(\varphi) = \int_0^t \vartheta_t dS_t$ for $\vartheta_t = \frac{1}{S_t}$ satisfies

$$dG_t = \frac{1}{X_t}dt + dW_t = dX_t + 2\,dt, \quad G_0 = 0.$$

Thus, $G_t = X_t + 2t - 1$ which implies by **a)** that $P[G_t \ge -1,\ 0 \le t \le 1] = 1$.

c) We can choose η so that $\varphi = (\eta, \frac{1}{S})$ is self-financing. Then we have $V_t = 0 + \int_0^t \frac{1}{S_u}\,dS_u = G_t$. Thus $V_1 = G_1 = X_1 + 1 \ge 1$ a.s.