ETH Zürich HS 2014
D-MATH
Prof. Martin Schweizer

Coordinator
Ariel Neufeld

Mathematical Finance
Solution 5

Solution 5-1

a) Assume first that $\mathcal{G}^1$ is bounded in L^0. That is,

$$\lim_{m\to\infty}\sup_{G\in\mathcal{G}^1} P[|G|\geq m]=0.$$

Observe now that for every sequence $\varepsilon_n \downarrow 0$ and every sequence of admissible strategies ϑ^n with $G(\vartheta^n)\geq -\varepsilon_n$, we have $\frac{\vartheta^n}{\varepsilon_n}\in\mathcal{G}^1$ for every n. Thus

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} P\left[\left|G_T\left(\frac{\vartheta^n}{\varepsilon_n}\right)\right|\geq m\right]=0.$$

This means that for every $\delta>0$ there exists some M independent of n such that

$$\delta\geq P\left[\left|G_T\left(\frac{\vartheta^n}{\varepsilon_n}\right)\right|\geq M\right]=P\big[|G_T(\vartheta^n)|\geq M\varepsilon_n\big].$$

Now, fix any $\eta>0$. Since $\varepsilon_n\downarrow 0$, we find some N such that for all $n\geq N$, we have $\varepsilon_n M\leq\eta$. Therefore, we obtain that for all $n\geq N$

$$P\big[|G_T(\vartheta^n)|\geq\eta\big]\leq P\big[|G_T(\vartheta^n)|\geq M\varepsilon_n\big]\leq\delta.$$

As $\delta>0$ was arbitrary, we obtain that for any $\eta>0$

$$\lim_{n\to\infty}P\big[|G_T(\vartheta^n)|\geq\eta\big]=0,$$

which proves the convergence of $(G_T(\vartheta^n))$ to 0 in L^0.

Conversely, assume by contradiction that $\mathcal{G}^1$ is not bounded in L^0. Then there exist $\alpha>0$ and a sequence $(\vartheta^k)\subseteq\mathcal{G}^1$ such that for all k,

$$P\big[|G_T(\vartheta^k)|\geq k\big]\geq\alpha>0.$$

Now, let $\delta>0$ and define a sequence $\varepsilon_k:=\frac{\delta}{k}$. Then $\varepsilon_k\downarrow 0$ and $G(\frac{\delta\vartheta^k}{k})\geq-\varepsilon_k$, but for any k

$$P\left[\left|G_T\Big(\frac{\delta\vartheta^k}{k}\Big)\right|\geq\delta\right]\geq\alpha>0$$

so that $(G_T(\frac{\delta\vartheta^k}{k}))$ does not converge to 0 in L^0.

b) (i) S_- is predictable being adapted and left-continuous. Consider the sequence of stopping times $T_n := \inf\{t \geq 0 \,|\, |S_t| \geq n + |s_0|\}$. $T_n \uparrow \infty$ P-a.s., as S has RCLL paths and therefore is bounded on compacts. Moreover, we see that for each n, the stopped process $S_-^{T_n} \leq n + |s_0|$ by definition of T_n.

(ii) As $S = S_- + \Delta S$, we get the result directly from (i).

(iii) For each n, consider the stopping time $T_n := \inf\{t \geq 0 \,|\, |S_t| \geq n + |s_0|\}$. We claim that each T_n is a predictable time. Indeed, as S is predictable, we obtain that

$$[\![T_n]\!] := \big\{(\omega, t) \,\big|\, T_n(\omega) = t\big\} = [\![0, T_n]\!] \cap S^{-1}\big([n + |s_0|, \infty)\big) \in \mathcal{P}.$$

For each n, take a foretelling sequence $(\tau_{k,n})_{k\in\mathbb{N}}$ for T_n and define the stopping time $\sigma_n := \max_{i,j=1,\cdots,n} \tau_{i,j}$. Then, $\sigma_n \uparrow \infty$ P-a.s. as $\tau_{k,n}$ converges to T_n and $T_n \uparrow \infty$ P-a.s.. Moreover, as $\tau_{k,n} < T_n$ we have $\sigma_n < T_n$ and hence, by definition of T_n, we have for each n that $|S_{\sigma_n}| \leq n + |s_0|$. This implies that S^{σ_n} is bounded for each n, hence S is locally bounded.

Solution 5-2

a) ${}^{\mathcal{P}}({}^{\mathcal{O}}X)$ is predictable by definition. Let τ be any predictable time. As $\mathcal{F}_{\tau-} \subseteq \mathcal{F}_\tau$, we obtain by the tower property that

$$({}^{\mathcal{P}}({}^{\mathcal{O}}X))_\tau 1_{\{\tau<\infty\}} = E\big[({}^{\mathcal{O}}X)_\tau 1_{\{\tau<\infty\}} \,\big|\, \mathcal{F}_{\tau-}\big] = E\Big[E\big[X_\tau 1_{\{\tau<\infty\}} \,\big|\, \mathcal{F}_\tau\big] \,\Big|\, \mathcal{F}_{\tau-}\Big] = E\big[X_\tau 1_{\{\tau<\infty\}} \,\big|\, \mathcal{F}_{\tau-}\big].$$

Thus, by uniqueness of the predictable projection, we get that ${}^{\mathcal{P}}({}^{\mathcal{O}}X) = {}^{\mathcal{P}}X$.

b) By right-continuity of A, we have $A_{C_t} \geq t$. Thus if $C_t \leq s$ we get $A_s \geq A_{C_t} \geq t$ as A is increasing. On the other hand, if $A_s \geq t$ we get $C_t \leq s$ as C_t by definition is the inf over all times u with $A_u \geq t$. To see that C_t is a predictable stopping time, observe that

$$[\![C_t]\!] := \big\{(\omega, s) \,\big|\, C_t(\omega) = s\big\} = [\![0, C_t]\!] \cap A^{-1}\big([t, \infty)\big) \in \mathcal{P}.$$

Now let H be of the form $\xi\, 1_{[\![0,u)\!)}$ with $u \in [0, \infty]$ and ξ being a bounded $\mathcal{F}$-measuable random variable. From above, we see that for any $0 \leq t_1 < t_2 < \infty$, we have $A_{t_1} < s \leq A_{t_2}$ if and only if $t_1 < C_s \leq t_2$. Thus, we see that

$$\int_0^\infty H_s(\omega)\, dA_s(\omega) = \xi \int_0^\infty 1_{[0,u)}(s)\, dA_s = \xi \int_0^\infty 1_{[0,u)}(C_s)\, ds = \int_0^\infty H_{C_s}(\omega)\, 1_{\{C_s(\omega)<\infty\}}\, ds. \quad (1)$$

Now, we define

$$\mathcal{H} := \big\{\text{bounded processes } H \text{ for which (1) holds true}\big\}$$

as well as

$$\mathcal{M} := \big\{\xi\, 1_{[\![0,u)\!)} \text{ with } u \in [0, \infty] \text{ and } \xi \text{ a bounded } \mathcal{F}\text{-measurable random variable}\big\}.$$

From the above calculation, we see that $\mathcal{M} \subseteq \mathcal{H}$. Moreover, $\mathcal{M}$ is closed under multiplication. Furthermore, $\mathcal{H}$ is a vector space of bounded processes containing the constant process 1 and by monotnone convergence it is also closed under monotone bounded convergence. Thus, $(\mathcal{H}, \mathcal{M})$ satisfy the conditions needed for being able to apply the monotone class theorem. As every product measurable process is $\sigma(\mathcal{M})$-measurable, we are done.

c) Let τ be any stopping time. It is enough to show that

$$E\Big[\int_0^\infty M_\tau\, 1_{\{s\le\tau\}}\, dA_s\Big] = E\Big[\int_0^\infty M_{s-}\, 1_{\{s\le\tau\}}\, dA_s\Big].$$

Using **b)** and the predictable stopping theorem, which we can apply as each C_s is a predictable time, we obtain that

$$\begin{aligned}
E\Big[\int_0^\infty M_\tau\, 1_{\{s\le\tau\}}\, dA_s\Big] &= E\Big[\int_0^\infty M_\tau\, 1_{\{C_s\le\tau\}}\, 1_{\{C_s<\infty\}}\, ds\Big] \\
&= \int_0^\infty E\big[M_\tau\, 1_{\{C_s\le\tau\}}\, 1_{\{C_s<\infty\}}\big]\, ds \\
&= \int_0^\infty E\big[M_{C_s-}\, 1_{\{C_s\le\tau\}}\, 1_{\{C_s<\infty\}}\big]\, ds \\
&= E\Big[\int_0^\infty M_{C_s-}\, 1_{\{C_s\le\tau\}}\, 1_{\{C_s<\infty\}}\, ds\Big] \\
&= E\Big[\int_0^\infty M_{s-}\, 1_{\{s\le\tau\}}\, dA_s\Big].
\end{aligned}$$

Solution 5-3

For any semimartingale S and any S-integrable process H we write $H \bullet S$ for $\int H\, dS$ to keep the notation short.

a) Since $A^1 - A^2$ is adapted, continuous, of finite variation and null at 0, it suffices to show that $A^1 - A^2$ is a local martingale. To this end, define for each $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,T] : \max(A^1_t, A^2_t) \ge n\} \wedge T. \tag{2}$$

Then for each $n \in \mathbb{N}$, the stopped processes $(A^1)^{\tau_n}$ and $(A^2)^{\tau_n}$ are both uniformly bounded by n, and $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} P[\tau_n = T] = 1$. We proceed to show that for each $n \in \mathbb{N}$, the stopped process $(A^1 - A^2)^{\tau_n}$ is a uniformly integrable martingale. So fix $n \in \mathbb{N}$. It suffices to show that for each stopping time σ, $E[(A^1)^{\tau_n}_\sigma] = E[(A^2)^{\tau_n}_\sigma]$. So let σ be an arbitrary stopping time. Then

$$\begin{aligned}
E[(A^1)^{\tau_n}_\sigma] &= E\left[\int_0^T 1_{]\!]0,\sigma\wedge\tau_n]\!]}\, dA^1_s\right] = (P\otimes A^1)\Big[]\!]0,\sigma\wedge\tau_n]\!]\Big] \\
&= (P\otimes A^2)\Big[]\!]0,\sigma\wedge\tau_n]\!]\Big] = E\left[\int_0^t 1_{]\!]0,\sigma\wedge\tau_n]\!]}\, dA^2_s\right] \\
&= E[(A^2)^{\tau_n}_\sigma].
\end{aligned} \tag{3}$$

b) Consider the σ-finite measures $P\otimes B$ and $P\otimes C$ on $(\bar\Omega, \mathcal{P})$. Then by the Lebesgue decomposition theorem, there exist unique σ-finite measures $\nu_a \ll P\otimes C$ and $\nu_s \perp P\otimes C$ on $(\bar\Omega,\mathcal{P})$ such that $\nu_a + \nu_s = P\otimes B$. In particular, there exists a predictable set $\bar N \in \mathcal{P}$ such that $\nu_s = 1_{\bar N}(P\otimes B)$, $\nu_a = 1_{\bar\Omega\setminus\bar N}(P\otimes B)$ and $(P\otimes C)[\bar N] = 0$. Define the processes $B^1 = (B^1_t)_{t\in[0,T]}$, $B^2 = (B^2_t)_{t\in[0,T]}$ and $C^1 = (C^1_t)_{t\in[0,T]}$ by

$$B^1_t = \int_0^t 1_{\bar N}\, dB_s, \quad B^2_t = \int_0^t 1_{\bar\Omega\setminus\bar N}\, dB_s \quad \text{and} \quad C^1_t := \int_0^t 1_{\bar N}\, dC_s. \tag{4}$$

Then B^1, B^2 and C^1 are increasing, adapted, continuous and null at 0, $B^1 + B^2 = B$, $\nu_s = P\otimes B^1$ and $\nu_a = P\otimes B^2$, and for all $\bar A \in \mathcal{P}$,

$$(P\otimes C^1)[\bar A] = E\left[\int_{\bar A} 1_{\bar N}\, dC_s\right] = (P\otimes C)[\bar A\cap\bar N] = 0 = (P\otimes 0)[\bar A], \tag{5}$$

where 0 denotes the zero process. Thus by part **a)**, we may deduce that $C^1 \equiv 0$. Next, by the Radon–Nikodým theorem, there exists a predictable process $H \geq 0$ such that

$$\nu_a = P \otimes B^2 = H(P \otimes C). \tag{6}$$

We proceed to show that $H \in L(C)$ and hence $H(P \otimes C) = P \otimes (H \bullet C)$. To this end, define for each $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,T] : \max(B_t^2, C_t) \geq n\} \wedge T. \tag{7}$$

Then for each $n \in \mathbb{N}$, the stopped processes $(B^2)^{\tau_n}$ and C^{τ_n} are both uniformly bounded by n, and $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} P[\tau_n = T] = 1$. Hence, it suffices to show that $H \in L(C^{\tau_n})$ for all $n \in \mathbb{N}$. So fix $n \in \mathbb{N}$. Then by the definition of H,

$$\begin{aligned} E\left[\int_0^T H_s \, dC_s^{\tau_n}\right] &= E\left[\int_0^T H_s 1_{]\!]0,\tau_n]\!]} \, dC_s\right] = \int_{\bar{\Omega}} 1_{]\!]0,\tau_n]\!]} H \, d(P \otimes C) \\ &= \int_{\bar{\Omega}} 1_{]\!]0,\tau_n]\!]} \, d(P \otimes B^2) = E\left[\int_0^T 1_{]\!]0,\tau_n]\!]} \, dB_s^2\right] = E[B_{\tau_n}^2] \leq n. \end{aligned} \tag{8}$$

Finally, since $P \otimes (H \bullet C) = P \otimes B^2$, it follows from part **a)** that $B^2 = H \bullet C$, which together with the above establishes the claim.

c) Suppose that S satisfies NA.

First, write $\mathrm{Var}(A) = (\mathrm{Var}(A)_t)_{t\in[0,T]}$, $A^+ = (A_t^+)_{t\in[0,T]}$ and $A^- = (A_t^-)_{t\in[0,T]}$ for the total, the positive and the negative variation of A, respectively. Then $\mathrm{Var}(A)$, A^+ and A^- are all increasing, adapted, continuous and null at 0, and $A = A^+ - A^-$ and $\mathrm{Var}(A) = A^+ + A^-$. On the level of measures, this means that

$$P \otimes A^+, P \otimes A^- \ll P \otimes \mathrm{Var}(A) \quad \text{and} \quad (P \otimes A^+) + (P \otimes A^-) = P \otimes \mathrm{Var}(A). \tag{9}$$

Hence there exist $\bar{D}^+ \in \mathcal{P}$ and $\bar{D}^- = \bar{\Omega} \setminus \bar{D}^+$ such that

$$\begin{aligned} P \otimes A^+ &= 1_{\bar{D}^+}(P \otimes \mathrm{Var}(A)) = P \otimes (1_{\bar{D}^+} \bullet \mathrm{Var}(A)), \\ P \otimes A^- &= 1_{\bar{D}^-}(P \otimes \mathrm{Var}(A)) = P \otimes (1_{\bar{D}^-} \bullet \mathrm{Var}(A)). \end{aligned} \tag{10}$$

By part (a), it follows that $A^+ = 1_{\bar{D}^+} \bullet \mathrm{Var}(A)$ and $A^- = 1_{\bar{D}^-} \bullet \mathrm{Var}(A)$.

Next, if there exist predictable processes $H^+, H^- \in L(\langle M \rangle)$ such that

$$A_t^+ = \int_0^t H_s^+ \, d\langle M \rangle_s \quad \text{and} \quad A_t^- = \int_0^t H_s^- \, d\langle M \rangle_s, \quad t \in [0,T], \tag{11}$$

we are done by setting $H := H^+ - H^-$. So, seeking a contradiction, assume without loss of generality that there does not exist $H^+ \in L(\langle M \rangle)$ such that $A^+ = \int H^+ \, d\langle M \rangle$. Then by part **b)**, there exists $\widetilde{H}^+ \in L(\langle M \rangle)$ and $\bar{N}^+ \in \mathcal{P}$ such that

$$A_t^+ = \int_0^t \widetilde{H}_s^+ \, d\langle M \rangle_s + \int_0^t 1_{\bar{N}^+} \, dA_s^+ \quad \text{and} \quad \int_0^t 1_{\bar{N}^+} \, d\langle M \rangle_s = 0, \quad t \in [0,T], \tag{12}$$

with $P\left[\int_0^T 1_{\bar{N}^+} \, dA_s^+ > 0\right] > 0$. (Otherwise, we could set $H^+ := \widetilde{H}^+$.) We define the strategy $\theta = (\theta_t)_{t\in[0,T]}$ by $\theta := 1_{\bar{N}^+} 1_{\bar{D}^+}$. Then $\theta \in L(S)$ as it is predictable and bounded, and it satisfies $\theta \bullet M \equiv 0$ as

$$\theta \bullet \langle M \rangle = (1_{\bar{N}^+} 1_{\bar{D}^+}) \bullet \langle M \rangle = 1_{\bar{D}^+} \bullet (1_{\bar{N}^+} \bullet \langle M \rangle) = 1_{\bar{D}^+} \bullet 0 \equiv 0. \tag{13}$$

Moreover,

$$\begin{aligned}
\theta \bullet A &= \theta \bullet A^{+} - \theta \bullet A^{-} \\
&= (1_{\bar{N}^{+}} 1_{\bar{D}^{+}}) \bullet (1_{\bar{D}^{+}} \bullet \mathrm{Var}(A)) - (1_{\bar{N}^{+}} 1_{\bar{D}^{+}}) \bullet (1_{\bar{D}^{-}} \bullet \mathrm{Var}(A)) \\
&= 1_{\bar{N}^{+}} \bullet \Big((1_{\bar{D}^{+}} 1_{\bar{D}^{+}}) \bullet \mathrm{Var}(A) \Big) - 1_{\bar{N}^{+}} \bullet \Big((1_{\bar{D}^{+}} 1_{\bar{D}^{-}}) \bullet \mathrm{Var}(A) \Big) \\
&= 1_{\bar{N}^{+}} \bullet (1_{\bar{D}^{+}} \bullet \mathrm{Var}(A)) - 1_{\bar{N}^{+}} \bullet (0 \bullet \mathrm{Var}(A)) \\
&= 1_{\bar{N}^{+}} \bullet A^{+} - 1_{\bar{N}^{+}} \bullet 0 = 1_{\bar{N}^{+}} \bullet A^{+}.
\end{aligned} \tag{14}$$

Thus,

$$\theta \bullet S = 1_{\bar{N}^{+}} \bullet A^{+} \geq 0 \quad \text{and} \quad P[\theta \bullet S_T > 0] = P\left[\int_0^T 1_{\bar{N}^{+}} \, dA_s^{+} > 0 \right] > 0. \tag{15}$$

Thus θ is 0-admissible and S fails NA for 0-admissible strategies, in contradiction to the hypothesis.