ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 11

Solution 11-1

a) Fix $Q \in \mathbb{P}_\sigma$ and introduce

$$\mathcal{Z}_t := \left\{ Z \,\middle|\, Z \text{ is density process with respect to } Q \text{ of some } R \in \mathbb{P}_\sigma \text{ with } R = Q \text{ on } \mathcal{F}_t \right\}$$

to obtain, as in the European case,

$$\overline{V}_t = \text{ess sup} \left\{ E_Q[U_\tau Z_\tau \,|\, \mathcal{F}_t] = \Gamma(Z,\tau) \,\middle|\, \tau \in \mathcal{T}_{t,T},\, Z \in \mathcal{Z}_t \right\}.$$

Moreover, the collection $\left\{ \Gamma(Z,\tau) \,\middle|\, \tau \in \mathcal{T}_{t,T},\, Z \in \mathcal{Z}_t \right\}$ is directed upward. Indeed, for $\Gamma(Z^i,\tau^i)$, $i = 1,2$, the set $A := \left\{ \Gamma_t(Z^1,\tau^1) \geq \Gamma_t(Z^2,\tau^2) \right\} \in \mathcal{F}_t$. We deduce from Exercise **10-1 a)** that $Z := Z^1 1_A + Z^2 1_{A^c}$ is in $\mathcal{Z}_t$, and that $\tau := \tau^1 1_A + \tau^2 1_{A^c} \in \mathcal{T}_{t,T}$. Moreover, by plugging in, we obtain that

$$\max\left(\Gamma_t(Z^1,\tau^1), \Gamma_t(Z^2,\tau^2) \right) = \Gamma_t(Z,\tau),$$

which proves the claim about directedness. As a consequence, we can write

$$\overline{V}_t = \operatorname*{ess\,sup}_{Z \in \mathcal{Z}_t, \tau \in \mathcal{T}_{t,T}} \Gamma(Z,\tau) = \uparrow \lim_{n\to\infty} E_Q\left[U_{\tau^n} Z^n_{\tau^n} \,\middle|\, \mathcal{F}_t \right]$$

(i.e. the limit is increasing) for some sequence $(Z^n, \tau^n) \in \mathcal{Z}_t \times \mathcal{T}_{t,T} \subseteq \mathcal{Z}_s \times \mathcal{T}_{s,T}$ for any $s \leq t$. Hence, by monotone convergence, we obtain

$$E_Q\left[\overline{V}_t \,\middle|\, \mathcal{F}_s \right] = \lim_{n\to\infty} E_Q\left[U_{\tau^n} Z^n_{\tau^n} \,\middle|\, \mathcal{F}_s \right] \leq \operatorname*{ess\,sup}_{Z \in \mathcal{Z}_s, \tau \in \mathcal{T}_{s,T}} E_Q\left[U_\tau Z_\tau \,\middle|\, \mathcal{F}_s \right] = \overline{V}_s.$$

Thus we have that $\overline{V}$ satisfies the Q-supermartingale property. To see that $\overline{V}$ is Q-integrable, use the supermartingale property for $s = 0$ and that $\mathcal{F}_0$ is trivial to obtain that

$$E_Q[\overline{V}_t] \leq E_Q[\overline{V}_0] = \overline{V}_0 = \sup_{Q \in \mathbb{P}_\sigma, \tau \in \mathcal{T}_{0,T}} E_Q[U_\tau] < \infty.$$

Adaptedness is clear. As $Q \in \mathbb{P}_\sigma$ was arbitrary, we obtain the desired result.

b) Let $V' \geq U$ be an RCLL adapted process which is a Q-supermartingale for each $Q \in \mathbb{P}_\sigma$. Then for any $\tau \in \mathcal{T}_{t,T}$ and any $Q \in \mathbb{P}_\sigma$, we have by the stopping theorem that

$$V'_t \geq E_Q[V'_\tau \,|\mathcal{F}_t] \geq E_Q[U_\tau \,|\mathcal{F}_t].$$

Taking $\text{ess sup}_{Z \in \mathcal{Z}_t, \tau \in \mathcal{T}_{t,T}}$ in the above inequality yields that $V'_t \geq \overline{V}_t$ P-a.s., for each $t \geq 0$, hence $V' \geq \overline{V}$ as both $V', \overline{V}$ have RCLL paths.

c) <u>"$J \geq \overline{V}$"</u>: By definition, $J \geq U$, and for any $Q \in \mathbb{P}$, we have

$$J_k \geq \operatorname*{ess\,sup}_{R \in \mathbb{P}} E_R[J_{k+1} \,|\, \mathcal{F}_k] \geq E_Q[J_{k+1} \,|\, \mathcal{F}_k].$$

Hence J satisfies the Q-supermartingale property for all $Q \in \mathbb{P}$. Using the Q-supermartingale property, we obtain for any k that for any $\tau \in \mathcal{T}^d_{k,T}$ and any $Q \in \mathbb{P}$,

$$J_k \geq E_Q[J_\tau \,|\mathcal{F}_k] \geq E_Q[U_\tau \,|\mathcal{F}_k]$$

(remember, we are in a discrete-time setting). Thus, by taking $\operatorname{ess\,sup}_{Q\in\mathbb{P},\tau\in\mathcal{T}^d_{k,T}}$ we obtain that $J \geq \overline{V}$.

"$J \leq \overline{V}$": We use a backward induction argument. Observe that $J_T = U_T = \overline{V}_T$. Assume that $J_{k+1} \leq \overline{V}_{k+1}$. Then, we get for each $Q \in \mathbb{P}$ from the Q-supermartingale property of $\overline{V}$ that

$$E_Q[J_{k+1} \,|\, \mathcal{F}_k] \leq E_Q[\overline{V}_{k+1} \,|\, \mathcal{F}_k] \leq \overline{V}_k.$$

Since $Q \in \mathbb{P}$ was arbitrary, we obtain that

$$\operatorname*{ess\,sup}_{Q\in\mathbb{P}} E_Q[J_{k+1} \,|\, \mathcal{F}_k] \leq \overline{V}_k.$$

Therefore, as $U_k \leq \overline{V}_k$, we obtain that

$$J_k \leq \max\big(U_k, \overline{V}_k\big) = \overline{V}_k.$$

By induction, we obtain that $J \leq \overline{V}$.

Solution 11-2

a) First observe that since $t \in \mathcal{T}_{t,T}$, we always have $\widetilde{V}^{Eu} \leq \widetilde{V}^{Am}$. The key idea for the other inequality is the following: S is a Q^*-martingale and the call payoff function $x \mapsto (x-K)^+$ is convex, and a convex function applied to a martingale gives a submartingale, which grows in average and hence one does not want to exercise it earlier. Let us prove this rigorously. First, observe that as $\widetilde{B}$ is increasing, we have for any $s \leq t$ that

$$\frac{\widetilde{U}_t}{\widetilde{B}_t} = \Big(S_t - \frac{\widetilde{K}}{\widetilde{B}_t}\Big)^+ \geq \Big(S_t - \frac{\widetilde{K}}{\widetilde{B}_s}\Big)^+.$$

Therefore, as $x \mapsto (x-K)^+$ is convex, we have by Jensen's inequality and the fact that S is a Q^*-martingale that

$$E_{Q^*}\left[\frac{\widetilde{U}_t}{\widetilde{B}_t}\,\middle|\,\mathcal{F}_s\right] \geq E_{Q^*}\left[\Big(S_t - \frac{\widetilde{K}}{\widetilde{B}_s}\Big)^+\,\middle|\,\mathcal{F}_s\right] \geq \Big(S_s - \frac{\widetilde{K}}{\widetilde{B}_s}\Big)^+ = \frac{\widetilde{U}_s}{\widetilde{B}_s}.$$

So $\frac{\widetilde{U}}{\widetilde{B}}$ satisfies the submartingale property under Q^*. As it is clearly adapted and integrable under Q^*, we conclude that $\frac{\widetilde{U}}{\widetilde{B}}$ is a Q^*-submartingale. Now, fix any $t \in [0,T]$. Applying the stopping theorem, we obtain for any $\tau \in \mathcal{T}_{t,T}$ from the submartingale property that

$$E_{Q^*}\left[\frac{\widetilde{U}_T}{\widetilde{B}_T}\,\middle|\,\mathcal{F}_t\right] = E_{Q^*}\left[E_{Q^*}\left[\frac{\widetilde{U}_T}{\widetilde{B}_T}\,\middle|\,\mathcal{F}_\tau\right]\,\middle|\,\mathcal{F}_t\right] \geq E_{Q^*}\left[\frac{\widetilde{U}_\tau}{\widetilde{B}_\tau}\,\middle|\,\mathcal{F}_t\right].$$

As $\tau \in \mathcal{T}_{t,T}$ was arbitrary, we obtain that

$$E_{Q^*}\left[\frac{\widetilde{U}_T}{\widetilde{B}_T}\,\middle|\,\mathcal{F}_t\right] \geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_{Q^*}\left[\frac{\widetilde{U}_\tau}{\widetilde{B}_\tau}\,\middle|\,\mathcal{F}_t\right].$$

Therefore, we conclude that

$$\widetilde{V}^{Eu}_t = \widetilde{B}_t\, E_{Q^*}\left[\frac{\widetilde{U}_T}{\widetilde{B}_T}\,\middle|\,\mathcal{F}_t\right] \geq \widetilde{B}_t \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_{Q^*}\left[\frac{\widetilde{U}_\tau}{\widetilde{B}_\tau}\,\middle|\,\mathcal{F}_t\right] = \widetilde{V}^{Am}_t.$$

Thus, we have $\widetilde{V}^{Eu} = \widetilde{V}^{Am}$.

b) Denote by Q the unique ELMM (and by q its corresponding parameter). Using the recursion of exercise **9-2**, we have

$$\widetilde{V}^{Am}(k,\widetilde{S}_k) = \max\Big((\widetilde{K}-\widetilde{S}_k)^+, (1+r)^{-1}q\widetilde{V}^{Am}\big(k+1,\widetilde{S}_k(1+u)\big)+(1+r)^{-1}(1-q)\widetilde{V}^{Am}\big(k+1,\widetilde{S}_k(1+d)\big)\Big).$$

Let us denote $x = S_0$ and assume that $\widetilde{K} - x > 0$, otherwise the intrinsic value at time zero vanishes and so selling at time zero is never interesting. (Only if $d < 0$ the payoff can become positive, otherwise it remains always 0.) The investor keeps the put in the portfolio during the first time-step, i.e. he does not sell it at time 0, if

$$\widetilde{K} - x < (1+r)^{-1}q\widetilde{V}^{Am}(1, x(1+u)) + (1+r)^{-1}(1-q)\widetilde{V}^{Am}(1, x(1+d)).$$

c) This case arises, for example, when $\widetilde{K} \geq (1+u)^T S_0$ and $r > 0$. Then

$$\widetilde{V}_0^{Am} = \sup_\tau E_Q\Big[\frac{(\widetilde{K}-\widetilde{S}_\tau)^+}{\tilde{B}_\tau}\Big] = \sup_\tau E_Q\Big[\frac{(\widetilde{K}-\widetilde{S}_\tau)}{\tilde{B}_\tau}\Big] = \sup_\tau E_Q\Big[\frac{\widetilde{K}}{(1+r)^\tau}\Big] - E_Q[S_\tau] = \sup_\tau E_Q\Big[\frac{\widetilde{K}}{(1+r)^\tau}\Big] - S_0,$$

so $\tau^* \equiv 0$ is optimal and $\widetilde{V}_0^{Am} = \widetilde{K} - S_0$, whereas the European call is given by

$$\widetilde{V}_0^{Eu} = E_Q\Big[\frac{(\widetilde{K}-\widetilde{S}_T)^+}{\tilde{B}_T}\Big] = E_Q\Big[\frac{\widetilde{K}}{(1+r)^T}\Big] - S_0.$$

Thus $\widetilde{V}_0^{Am} > \widetilde{V}_0^{Eu}$.

Solution 11-3

In the following calculation, we use the fact that

i) $X \sim \mathcal{N}(0,\sigma^2)$ if and only if $M_X(t) = E[e^{tX}] = \exp\left(\frac{1}{2}\sigma^2 t^2\right)$.

ii) $X \sim \mathrm{Poi}(\lambda)$ if and only if $M_X(t) = E[e^{tX}] = \exp\left((e^t - 1)\lambda\right)$.

a) First, observe that $Z > 0$ and $Z_0 = 1$. In order to prove that Z defines an equivalent martingale measure, we have to verify that Z and ZS are martingales. For that purpose, fix any $0 \leq s \leq t < \infty$. Using that W is independent of N, that both $W_t - W_s$ and $N_t - N_s$ are independent of $\mathcal{F}_s$ with $W_t - W_s \overset{(d)}{=} W_{t-s}$ and $N_t - N_s \overset{(d)}{=} N_{t-s}$, we obtain that

$$E\Big[\frac{Z_t}{Z_s}\,\Big|\,\mathcal{F}_s\Big] = E\Big[\exp\Big(\beta\sigma W_{t-s} - \frac{1}{2}\beta^2\sigma^2(t-s)\Big)\Big]\,E\Big[\exp\Big(\ln\frac{\widetilde{\lambda}}{\lambda}\,N_{t-s}^\lambda + (\lambda-\widetilde{\lambda})(t-s)\Big)\Big]$$

Due to **i)** the first factor equals to 1, and due to **ii)**, as $N_{t-s} \sim \mathrm{Poi}(\lambda(t-s))$, the second factor equals 1, too. Thus

$$E\Big[\frac{Z_t}{Z_s}\,\Big|\,\mathcal{F}_s\Big] = 1,$$

which shows that Z is a martingale. Next, we show that ZS is a martingale. For that purpose, fix any $0 \leq s \leq t < \infty$. Using that W is independent of N, that both $W_t - W_s$ and $N_t - N_s$ are independent of $\mathcal{F}_s$ with $W_t - W_s \overset{(d)}{=} W_{t-s}$ and $N_t - N_s \overset{(d)}{=} N_{t-s}$, we obtain that

$$E\Big[\frac{Z_t S_t}{Z_s S_s}\,\Big|\,\mathcal{F}_s\Big] = E\Big[\exp\Big((\beta+1)\sigma W_{t-s} - \frac{1}{2}(\beta+1)^2\sigma^2(t-s)\Big)\Big]\,E\Big[\exp\Big((\ln\frac{\widetilde{\lambda}}{\lambda} + a)\,N_{t-s}^\lambda\Big)\Big]R(t,s),$$

where $R(t,s)$ is a deterministic function defined by

$$R(t,s) := \exp\Big(-\frac{1}{2}\beta^2\sigma^2(t-s) + \frac{1}{2}(\beta+1)^2\sigma^2(t-s) + (\lambda - \widetilde{\lambda})(t-s)\Big) = \exp\Big(\big(1 - e^a\frac{\widetilde{\lambda}}{\lambda}\big)\lambda(t-s)\Big),$$

where for the last equality we used the definition of β. The first factor equals to 1 due to **i)** and the second factor equals to $\frac{1}{R(t,s)}$ due to **ii)**, as as $N_{t-s} \sim \mathrm{Poi}(\lambda(t-s))$. Thus, we see that

$$E\Big[\frac{Z_t S_t}{Z_s S_s}\,\Big|\,\mathcal{F}_s\Big] = 1,$$

which proves that ZS is a martingale.

b) Clearly, $\pi_s(H) \le S_0$, as the constant strategy $\vartheta \equiv 1 \in \Theta_{adm}$ and

$$S_0 + \int_0^T 1\, dS_t = S_T \ge (S_T - K)^+.$$

According to Theorem 9.4 of the lecture, the superreplication price for any $H \in L^0_+(\mathcal{F}_T)$ is given by

$$\pi_s(H) = \sup_{Q\in\mathbb{P}_\sigma} E_Q[H].$$

We point out, that as $S \ge 0$, S is an local martingale under a measure Q if and only if S is a σ-martingale under Q (and hence we can focus on local martingale measures). Consider now the density process Z_t defining an equivalent martingale measures $Q^{\widetilde{\lambda}}$ for every $\widetilde{\lambda} > 0$

$$Z_t = \frac{dQ^{\widetilde{\lambda}}}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\beta\sigma W_t + \ln\left(\frac{\widetilde{\lambda}}{\lambda}\right) N_t^{\lambda} - \frac{1}{2}\beta^2\sigma^2 t + (\lambda - \widetilde{\lambda})t\right),$$

where β is given by

$$\beta = -\widetilde{\lambda}\frac{e^a - 1}{\sigma^2} - \frac{1}{2}.$$

Due to **a)**, we know that $Q^{\widetilde{\lambda}} \in \mathbb{P}$ for each $\widetilde{\lambda} > 0$. By Girsanov or by calculating the moment generating function $E_{Q^{\widetilde{\lambda}}}[e^{uX_t}] = E_P[e^{uX_t} Z_t]$ for $u \in \mathbb{R}$, where $X_t = \sigma W_t + aN_t^{\lambda}$, we find that S_t under $Q^{\widetilde{\lambda}}$ is given by

$$S_t = S_0 \exp\left(\sigma W_t^{Q^{\widetilde{\lambda}}} + aN_t^{\widetilde{\lambda}} - \frac{\sigma^2 t}{2} - \widetilde{\lambda}(e^a - 1)t\right),$$

where $W_t^Q = W_t - \sigma\beta t$ is a $Q^{\widetilde{\lambda}}$-Brownian motion and $N_t^{\widetilde{\lambda}} := N_t^{\lambda}$ an independent $Q^{\widetilde{\lambda}}$-Poisson process with intensity $\widetilde{\lambda}$. Let us now calculate the call price under $Q^{\widetilde{\lambda}}$: it is

$$E_{Q^{\widetilde{\lambda}}}[(S_T - K)^+] = E_{Q^{\widetilde{\lambda}}}[S_T 1_{\{S_T > K\}}] - KE_{Q^{\widetilde{\lambda}}}[1_{\{S_T > K\}}] =: J_1^{\widetilde{\lambda}} - J_2^{\widetilde{\lambda}}.$$

For $J_2^{\widetilde{\lambda}}$, as W^Q is independent of $N_T^{\widetilde{\lambda}}$ under $Q^{\widetilde{\lambda}}$, we have

$$\begin{aligned}
J_2^{\widetilde{\lambda}} &= KE_{Q^{\widetilde{\lambda}}}\left[1_{\{S_0\exp(\sigma W_T^Q + aN_T^{\widetilde{\lambda}} - \frac{\sigma^2 T}{2} - \widetilde{\lambda}(e^a - 1)T) > K\}}\right] \\
&= KE_{Q^{\widetilde{\lambda}}}\left[E_{Q^{\widetilde{\lambda}}}\left[1_{\{S_0\exp(\sigma w + aN_T^{\widetilde{\lambda}} - \frac{\sigma^2 T}{2} - \widetilde{\lambda}(e^a - 1)T) > K\}}\right]\Big|_{W_T^Q = w}\right] \\
&= KE_{Q^{\widetilde{\lambda}}}\left[Q^{\widetilde{\lambda}}\left[aN_T^{\widetilde{\lambda}} > \ln\frac{K}{S_0} - \sigma w + \frac{\sigma^2 T}{2} + \widetilde{\lambda}(e^a - 1)T\right]\Big|_{W_T^Q = w}\right].
\end{aligned}$$

Denote by $D_{\widetilde{\lambda}T}(x)$ the $\mathrm{Poi}(\widetilde{\lambda}T)$-distribution function, i.e. $\nu[X \leq x]$ for $X \sim \mathrm{Poi}(\widetilde{\lambda}T)$ under a probability measure ν. Then, for $a > 0$, we see that

$$J_2^{\widetilde{\lambda}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T}} \, e^{-\frac{w^2}{2T}} \left(1 - D_{\widetilde{\lambda}T}\big(c(w) + \alpha\widetilde{\lambda}T\big)\right) dw,$$

with

$$c(w) := \frac{\ln \frac{K}{S_0} - \sigma w + \frac{\sigma^2 T}{2}}{a}, \qquad \alpha := \frac{e^a - 1}{a}. \tag{1}$$

Define by $\widehat{D}_{\widetilde{\lambda}T}(x) = \nu[X < x]$ for an $X \sim \mathrm{Poi}(\widetilde{\lambda}T)$ under a probability measure ν. For $a < 0$, we have

$$J_2^{\widetilde{\lambda}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T}} \, e^{-\frac{w^2}{2T}} \, \widehat{D}_{\widetilde{\lambda}T}\big(c(w) + \alpha\widetilde{\lambda}T\big) \, dw,$$

with $c(w), \alpha$ as in (1). Thus, using dominated convergence and the hint, we obtain that $J_2^{\widetilde{\lambda}}$ tends to 0 as $\widetilde{\lambda} \to \infty$. In order to calculate $J_1^{\widetilde{\lambda}}$, we introduce a probability measure $\widetilde{Q}^{\widetilde{\lambda}}$ defined by

$$\widetilde{Z}_t := \frac{d\widetilde{Q}^{\widetilde{\lambda}}}{dQ^{\widetilde{\lambda}}}\Big|_{\mathcal{F}_t} = \exp\left(\sigma W_t^Q + aN_t^{\widetilde{\lambda}} - \frac{\sigma^2 t}{2} - \widetilde{\lambda}(e^a - 1)t\right).$$

Under $\widetilde{Q}^{\widetilde{\lambda}}$, S is given by

$$S_t = S_0 \exp\left(\sigma W_t^{\widetilde{Q}^{\widetilde{\lambda}}} + aN_t^{e^a\widetilde{\lambda}} + \frac{\sigma^2 t}{2} - \widetilde{\lambda}(e^a - 1)t\right),$$

where $W_t^{\widetilde{Q}^{\widetilde{\lambda}}} = W_t - \sigma t$ is a $\widetilde{Q}^{\widetilde{\lambda}}$-Brownian motion and $N^{e^a\widetilde{\lambda}} := N^{\lambda}$ is an independent $\widetilde{Q}^{\widetilde{\lambda}}$-Poisson process with intensity $e^a\widetilde{\lambda}$. Indeed, this can be seen by calculating the moment generating function $E_{\widetilde{Q}^{\widetilde{\lambda}}}[e^{u\widetilde{X}_t}] = E_{Q^{\widetilde{\lambda}}}[e^{u\widetilde{X}_t}\widetilde{Z}_t]$ for $u \in \mathbb{R}$, where $\widetilde{X}_t = \sigma W_t^{\widetilde{Q}^{\widetilde{\lambda}}} + aN_t^{e^a\widetilde{\lambda}} + \frac{\sigma^2 t}{2} - \widetilde{\lambda}(e^a - 1)t$, or by observing that $\widetilde{Z}$ has the same structure as Z (with $\widetilde{\widetilde{\lambda}} := \widetilde{\lambda}e^a$ and $\beta = 1$) and then use the above calculation to conclude.

Then we have for $J_1^{\widetilde{\lambda}}$

$$\begin{aligned}
J_1^{\widetilde{\lambda}} &= S_0 E_{\widetilde{Q}^{\widetilde{\lambda}}}[1_{\{S_T > K\}}] \\
&= S_0 E_{\widetilde{Q}^{\widetilde{\lambda}}}\left[E_{\widetilde{Q}^{\widetilde{\lambda}}}\left[1_{\{S_0 \exp(\sigma W_T^{\widetilde{Q}^{\widetilde{\lambda}}} + aN_T^{e^a\widetilde{\lambda}} + \frac{\sigma^2 T}{2} - \widetilde{\lambda}(e^a-1)T) > K\}}\right]\Big|_{W_T^{\widetilde{Q}} = w}\right] \\
&= S_0 E_{\widetilde{Q}^{\widetilde{\lambda}}}\left[\widetilde{Q}^{\widetilde{\lambda}}\left[aN_T^{e^a\widetilde{\lambda}} > \ln\frac{K}{S_0} - \sigma w - \frac{\sigma^2 T}{2} + e^a\widetilde{\lambda}\frac{e^a - 1}{e^a}T\right]\Big|_{W_T^{\widetilde{Q}} = w}\right].
\end{aligned}$$

Thus, for $a > 0$, we see that

$$J_1^{\widetilde{\lambda}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T}} \, e^{-\frac{w^2}{2T}} \left(1 - D_{e^a\widetilde{\lambda}T}\big(c(w) + \alpha\widetilde{\lambda}T\big)\right) dw,$$

with

$$c(w) := \frac{\ln \frac{K}{S_0} - \sigma w - \frac{\sigma^2 T}{2}}{a}, \qquad \alpha := \frac{e^a - 1}{ae^a}. \tag{2}$$

For $a < 0$, we have

$$J_1^{\widetilde{\lambda}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T}} \, e^{-\frac{w^2}{2T}} \, \widehat{D}_{e^a\widetilde{\lambda}T}\big(c(w) + \alpha\widetilde{\lambda}T\big) \, dw,$$

with $c(w), \alpha$ as in (2). Using again the hint and dominated convergence, $J_1^{\widetilde{\lambda}}$ tends to S_0 as $\widetilde{\lambda} \to \infty$. Thus

$$\pi_s((S_T - K)^+) \geq \lim_{\widetilde{\lambda} \to \infty} E_{Q^{\widetilde{\lambda}}}[(S_T - K)^+] = S_0,$$

which implies that $\pi_s((S_T - K)^+) = S_0$.