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D-MATH
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Mathematical Finance
Solution 3

Solution 3-1

a) First kind is “make something from nothing” and second kind is “repay debts from nothing”.

b) We argue by contradiction. The idea is: Start with nothing, deposit some money in your pocket, do the second kind arbitrage, clear the account, and stay with the money in the pocket. Formally: Let $\varphi = (\eta, \vartheta)$ be an arbitrage of the second kind and take $\varphi' = (\eta', \vartheta') := (\eta - V_0(\varphi), \vartheta)$, note $V_0(\varphi) \in L^0_- \setminus \{0\}$. Then for $\bar{S} := (B, S)$ we have $V_0(\varphi') = \varphi_0 \bar{S}_0 - V_0(\varphi) = 0$ and $V_T(\varphi') = \varphi_T \bar{S}_T - V_0(\varphi) \in L^+_0 \setminus \{0\}$.

c) We take a one period market with a bank account $B = 1$ and a stock S which can remain constant or go up strictly, e.g., $P[S_1 = S_0] = P[S_1 = S_0 + 1] = 1/2$. Let $V_0 = 0$, $\vartheta = 1$ and let (η, ϑ) be the self-financing strategy corresponding to (V_0, ϑ). Then we see that $V_0(\varphi) = 0$ and $V_1(\varphi) = S_1 - S_0 \in L^0_+ \setminus \{0\}$, which means that (η, ϑ) is an arbitrage of the first kind. Now, assume that there exists a self-financing strategy $\varphi = (\eta, \vartheta)$ which is an arbitrage of the second kind. In particular, we have $V_0(\varphi) \in L^0_- \setminus \{0\}$. By the self-financing property, we have $V_1(\varphi) = V_0(\varphi) + \vartheta_1(S_1 - S_0)$. But then, we see that

$$P\big[V_1(\varphi) \in L^0_- \setminus \{0\}\big] \geq P[S_1 = S_0] = \frac{1}{2},$$

which gives us a contradiction.

Solution 3-2

a) It clearly suffices to establish the more general hint. So let Y be a real-valued non-trivial random variable and $\mu \in (\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y)$.

First, define $\widetilde{P} \approx P$ on $\mathcal{F}$ by

$$\frac{d\widetilde{P}}{dP} = \frac{1}{1+|Y|} \Big/ E\left[\frac{1}{1+|Y|}\right] =: g_1(Y). \tag{1}$$

Then $Y \in L^1(\widetilde{P})$ as

$$E_{\widetilde{P}}[|Y|] = E\left[\frac{|Y|}{1+|Y|}\right] \Big/ E\left[\frac{1}{1+|Y|}\right] < \infty. \tag{2}$$

Next, set $A := \{Y \leq \mu\}$. Then by the fact that $\mu \in (\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y)$ (under P) and by the fact that $\operatorname{ess\,inf} Y$ and $\operatorname{ess\,sup} Y$ are invariant under an equivalent change of measure, $\widetilde{P}[A], \widetilde{P}[A^c] > 0$. So the numbers $\alpha := E_{\widetilde{P}}[Y \mid A]$ and $\beta := E_{\widetilde{P}}[Y \mid A^c]$ are well defined, finite and satisfy

$$\alpha = \frac{E_{\widetilde{P}}[Y 1_{\{Y \leq \mu\}}]}{\widetilde{P}[Y \leq \mu]} < \mu \quad \text{and} \quad \beta = \frac{E_{\widetilde{P}}[Y 1_{\{Y > \mu\}}]}{\widetilde{P}[Y > \mu]} > \mu. \tag{3}$$

Set $\lambda := \frac{\beta-\mu}{\beta-\alpha} \in (0,1)$ and define the probability measure Q on $\mathcal{F}$ by

$$Q = \lambda \widetilde{P}[\,\cdot \mid A] + (1-\lambda)\widetilde{P}[\,\cdot \mid A^c]. \tag{4}$$

It is straightforward to check that $Q \approx P$. Moreover,

$$E_Q[Y] = \lambda E[Y \mid A] + (1-\lambda)E[Y \mid A^c] = \frac{\beta-\mu}{\beta-\alpha}\alpha + \frac{\beta-\mu}{\beta-\alpha}\beta = \mu. \tag{5}$$

Finally, $Q \approx P$ on $\mathcal{F}_T$ and

$$\frac{dQ}{dP} = \frac{dQ}{d\widetilde{P}}\frac{d\widetilde{P}}{dP} = g_2(Y)g_1(Y) =: g(Y). \tag{6}$$

b) "$\Leftarrow$". Assume that $\operatorname{ess\,inf} Y_k < 1 < \operatorname{ess\,sup} Y_k$ for all $k = 1, \dots, T$. Then by part (a), there exist measurable functions $g_1, \dots, g_T : (0,\infty) \to (0,\infty)$ such that

$$E[g_k(Y_k)] = 1, \tag{7}$$
$$E[g_k(Y_k)Y_k] = 1, \tag{8}$$
$$\tag{9}$$

$k = 1, \dots, T$. Define the process $Z = (Z_k)_{k=0,\dots,T}$ by

$$Z_k := \prod_{j=1}^{k} g_j(Y_j), \quad k = 0, \dots, T. \tag{10}$$

We claim that Z is a strictly nonnegative P-martingale with mean 1 for the filtration $(\mathcal{F}_k)_{k=0,\dots T}$. Indeed, strict nonnegativity is clear by definition of Z and adaptedness follows from the fact that $\mathcal{F}_k = \sigma(S_1, \dots, S_k) = \sigma(Y_1, \dots, Y_k)$, $k = 0, \dots, T$. To check the martingale property, we fix $k \in \{1, \dots, T\}$. Then by independence of Y_k and $\mathcal{F}_{k-1}$,

$$\begin{aligned} E[Z_k \mid \mathcal{F}_{k-1}] &= E[Z_{k-1}g_k(Y_k) \mid \mathcal{F}_{k-1}] = Z_{k-1}E[g_k(Y_k) \mid \mathcal{F}_{k-1}] \\ &= Z_{k-1}E[g_k(Y_k)] = Z_{k-1} \;\; P\text{-a.s.} \end{aligned} \tag{11}$$

Furthermore, as $Z_0 = 1$, we get the integrability condition as

$$E[Z_k] = E\big[E[Z_k \mid \mathcal{F}_{k-1}]\big] = E[Z_{k-1}] = \dots = E[Z_0] = 1.$$

Define $Q \approx P$ on $\mathcal{F}_T$ by $\frac{dQ}{dP} = Z_T$. We claim that S is a Q-martingale for the filtration $(\mathcal{F}_k)_{k=1,\dots,T}$. By Bayes' theorem, it is equivalent to check that ZS is a P-martingale for the filtration $(\mathcal{F}_k)_{k=0,\dots,T}$. Indeed, adaptedness of ZS is clear, and by nonnegativity of ZS, it suffices to check the P-martingale property. So fix $k \in \{1, \dots, T\}$. Then by independence of Y_k and $\mathcal{F}_{k-1}$ and (8),

$$\begin{aligned} E[Z_kS_k \mid \mathcal{F}_{k-1}] &= E[Z_{k-1}S_{k-1}g_k(Y_k)Y_k \mid \mathcal{F}_{k-1}] \\ &= Z_{k-1}S_{k-1}E[g_k(Y_k)Y_k \mid \mathcal{F}_{k-1}] \\ &= Z_{k-1}S_{k-1}E[g_k(Y_k)Y_k] = Z_{k-1}S_{k-1} \;\; P\text{-a.s.} \end{aligned} \tag{12}$$

Finally, since S is a Q-martingale it follows from the lecture that S satisfies NA.

"$\Rightarrow$": Assume there exists $k^* \in \{1, \dots T\}$ with $1 \le \operatorname{ess\,inf} Y_{k^*}$ or $1 \ge \operatorname{ess\,sup} Y_{k^*}$. We only consider the first case, the argument for the second one is analogous (a change of sign). Define the strategy $\theta = (\theta_k)_{k=0,\dots,T}$ by

$$\theta_k = \begin{cases} 0 & \text{if } k \neq k^*, \\ 1 & \text{if } k = k^*. \end{cases} \tag{13}$$

Then its gains process $G(\theta) = (G_k(\theta))_{k=0,\dots,T}$ satisfies

$$G_k(\theta) = \sum_{j=1}^{k} \theta_k \Delta S_k = \sum_{j=1}^{k} \theta_k (Y_k - 1) S_{k-1} = \begin{cases} 0 & \text{if } k < k^*. \\ (Y_{k^*} - 1) S_{k-1} & \text{if } k \geq k^*. \end{cases} \tag{14}$$

Since $S_{k-1} > 0$ P-a.s., $Y_{k^*} \geq 1$ P-a.s. and $P[Y_{k^*} > 1] > 0$ (as Y_{k^*} is nontrivial), it follows that θ is 0-admissible and that S fails NA.

Solution 3-3

a) We use $G_T(\vartheta) \wedge n$ to approximate $G_T(\vartheta)$. To this end we have to prove that $G_T(\vartheta) \wedge n \in \mathcal{C}$. Since $G_T(\vartheta) \wedge n$ is bounded from below (since ϑ is admissible) and bounded from above by n, it lies in L^∞. Furthermore, $G_T(\vartheta) \wedge n = G_T(\vartheta) - ((G_T(\vartheta) - n) \vee 0)$, where $(G_T(\vartheta) - n) \vee 0 \in L^0_+$. Thus $G_T(\vartheta) \wedge n \in \mathcal{C}$. By Fatou's lemma, since $G_T(\vartheta) \wedge n \geq -a$ for some $a \in \mathbb{R}_+$,

$$E_Q[G_T(\vartheta)] = E_Q[\lim_{n\to\infty} G_T(\vartheta) \wedge n] \leq \liminf_{n\to\infty} E_Q[G_T(\vartheta) \wedge n] \leq 0.$$

b) Denote by (NA') the condition $\mathcal{C} \cap L^\infty_+ = \{0\}$. First, assume that (NA') holds true. Let $G_T(\vartheta) := \int_0^T \vartheta_t \, dS_t \in (G_T(\Theta_{adm}) \cap L^0_+)$. We need to show that $G_T(\vartheta) = 0$. For every $n \in \mathbb{N}$ we know from the proof of **a)** that $G_T(\vartheta) \wedge n \in \mathcal{C}$, in particular in L^∞. Moreover, as $G_T(\Theta_{adm}) \geq 0$, we obtain that $G_T(\vartheta) \wedge n \in \mathcal{C} \cap L^\infty_+$. Thus, as (NA') holds true, we get that $G_T(\vartheta) \wedge n = 0$ for every n, in particular $G_T(\vartheta) = 0$.

Now, assume that (NA) holds true. Let $G_T(\vartheta) - Y \in \mathcal{C} \cap L^\infty_+$, where $Y \in L^0_+$ and ϑ is admissible. We need to show that $G_T(\vartheta) - Y = 0$. As $Y \in L^0_+$ we obtain that $G_T(\vartheta) \in (G_T(\Theta_{adm}) \cap L^0_+)$. Thus, by the (NA) condition, we obtain that $G_T(\vartheta) = 0$. As a consequence, we have that $-Y \in L^0_+$, which implies that $Y = 0$.

c) First, let $\mathcal{D}$ be bounded in L^0. Then, for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P\big[|\lambda_n D^n| \geq \varepsilon\big] \leq \lim_{n\to\infty} \sup_{D\in\mathcal{D}} P\Big[|D| \geq \frac{\varepsilon}{\lambda_n}\Big] = 0.$$

For the other direction, assume that $\mathcal{D}$ is not bounded in L^0. This means that there exists $\delta > 0$ such that for all $\forall n \, \exists D^n \in \mathcal{D}$ with $P\big[|D^n| \geq n\big] \geq \delta$. As a consequence, the sequence $(\frac{1}{n} D^n)$ don't converge to 0 in L^0.