ETH Zürich HS 2014
D-MATH
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# Mathematical Finance
## Exercise Sheet 10

*Please hand in until Wednesday, November 26th, 12:00 in your assistant's box in HG G 52.1*

**Exercise 10-1**

Consider a financial market model with $\mathbb{P}_\sigma \neq \emptyset$.

**a)** Fix $Q \in \mathbb{P}_\sigma$ and let

$$\mathcal{Z}_t := \left\{ \text{RCLL martingale } Z : \, Z = \frac{dR}{dQ}\Big|_{\mathbb{F}} \text{ for some } R \in \mathbb{P}_\sigma \text{ with } R = Q \text{ on } \mathcal{F}_t \right\}.$$

Prove that if $Z^1, Z^2 \in \mathcal{Z}_t$ and $A \in \mathcal{F}_t$, then $Z := Z^1 1_A + Z^2 1_{A^c} \in \mathcal{Z}_t$.

**b)** Prove, using only the definition of $\pi_s$, that $\rho := -\pi_s : L^\infty \to \mathbb{R}$ is a *coherent risk measure*. That is, for all $H, H' \in L^\infty$,

- **i)** $\pi_s(H) \leq \pi_s(H')$ if $H \leq H'$ $P$-a.s. (*monotonicity*),
- **ii)** $\pi_s(H + c) = \pi_s(H) + c$ for all $c \in \mathbb{R}$ (*cash invariance*),
- **iii)** $\pi_s(\lambda H) = \lambda \pi_s(H)$ for all $\lambda \geq 0$ (*positive homogeneity*),
- **iv)** $\pi_s(H + H') \leq \pi_s(H) + \pi_s(H')$ (*subadditivity*).

**c)** Let $H \in L^0_+(\mathcal{F}_T)$. Show that the following are equivalent:

- **(i)** $H = c + G_T(\vartheta)$ for some $c \in \mathbb{R}$ and $\vartheta \in \Theta_{adm}$ such that $G(\vartheta)$ is a martingale under some $Q^* \in \mathbb{P}_\sigma$, i.e., $H$ is *attainable*.
- **(ii)** $E_{Q^*}[H] = \sup_{Q \in \mathbb{P}_\sigma} E_Q[H] < \infty$ for some $Q^* \in \mathbb{P}_\sigma$.

**Exercise 10-2**

Consider the one-period trinomial model. This means $T = 1$, a constant bank account $B = 1$, and price process $(S_t)_{t \in \{0,1\}}$ has a constant $S_0 > 0$ and $S_1 = S_0 Z$ with $Z$ taking three possible values $1 + u > 1 + m > 1 + d$ under a given measure $P$. Moreover, assume $\Omega = \{\omega_u, \omega_m, \omega_d\}$, where $\omega_u = \{S_1 = S_0(1 + u)\}$, $\omega_m = \{S_1 = S_0(1 + m)\}$ and $\omega_d = \{S_1 = S_0(1 + d)\}$. Furthermore, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \sigma(S_1)$. For a probability measure $Q$, write $q_i = Q[\{\omega_i\}], i = u, m, d$. Assume that $P[\{\omega\}] > 0$ for all $\omega \in \Omega$.

**a)** For which $u > m > d$ is there arbitrage? In the case where (NA) holds true, characterize all equivalent local martingale measures in terms of $q_u$, $q_m$, $q_d$.

**b)** Let $H = (S_1 - K)^+$ be a call option with strike $K > 0$ satisfying $S_0(1+m) > K > S_0(1+d)$ and $S_0 \neq K$. Show that $H$ is not attainable.

**c)** From now on, assume that $u > 0 > d$. Let $H = (S_1 - K)^+$ be a call option with strike $K > 0$. Compute its superreplication price $\pi_s(H)$. Moreover, prove that there exists a martingale measure $Q_a$ absolutely continuous with respect to $P$ such that $\pi_s(H) = E^{Q_a}[H]$.

**d)** Now consider any finite discrete-time market with $\mathcal{F}_0$ being trivial. Prove or disprove:

- **i)** $(S, \mathbb{F})$ is complete implies that $\#(\mathbb{P}_e) \leq 1$.
- **ii)** $\#(\mathbb{P}_e) \leq 1$ implies that $(S, \mathbb{F})$ is complete.

***Please turn the page***

**Exercise 10-3**

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Poisson process $N = (N_t)_{t\in[0,T]}$ with rate $\lambda > 0$. Denote by $(\mathcal{F}_t^N)_{t\in[0,T]}$ the natural (completed) filtration of $N$. Define the process $S = (S_t)_{t\in[0,T]}$ by $dS_t = S_{t-}(\mu\, dt + \frac{\sigma}{\sqrt{\lambda}}\, d\widetilde{N}_t)$, $S_0 = s_0 > 0$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\widetilde{N}_t = N_t - \lambda t$, $t \geq 0$, denotes the compensated Poisson process.

**a)** Show that $S$ satisfies NFLVR if and only if $\mu < \sigma\sqrt{\lambda}$, and find an equivalent martingale measure $Q^\lambda$ for $S$ in that case.

**Hint:** Use Exercises **8-2** and **9-3**.

Assume for the rest of the question that $S$ satisfies NFLVR. Moreover, use without proof that the equivalent martingale measure $Q^\lambda$ from part (a) is unique. For $\rho > 0$ denote by $\overline{\Psi}_\rho$ the tail distribution function of a Poisson random variable with parameter $\rho$, i.e., $\overline{\Psi}_\rho(x) := P[X_\rho > x]$, where $X_\rho$ has a Poisson distribution with parameter $\rho$.

**b)** Show that the risk-neutral price of a *cash-or-nothing call option* with payoff $H = 1_{\{S_T > K\}}$ with maturity $T$ and strike $K > 0$ is given by

$$\overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

**Hint:** Use that for a Lévy process $R$ with triplet $(a, \sigma^2, \nu)$, the stochastic exponential of $R$ is given by

$$\mathcal{E}(R)_t := \exp\left(R_t - \frac{1}{2}\sigma^2 t\right)\prod_{0<s\leq t}(1 + \Delta R_s)\exp(-\Delta R_s).$$

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**c)** Show that the risk-neutral price of a *stock-or-nothing call option* with payoff $H = S_T 1_{\{S_T > K\}}$ with maturity $T$ and strike $K > 0$ is given by

$$S_0\overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

**Hint:** Define the measure $\widetilde{Q}^\lambda \approx Q^\lambda$ on $\mathcal{F}_T$ by $\frac{d\widetilde{Q}^\lambda}{dQ^\lambda} := S_T/S_0$, and work under this measure.

**d)** Derive the risk-neutral price $C_0^\lambda$ of a call option with payoff $H = (S_T - K)^+$ with maturity $T$ and strike $K$. Moreover, show that

$$\lim_{\lambda\to\infty} C_0^\lambda = S_0\Phi\left(\frac{\log\frac{S_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{S_0}{K} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right), \tag{1}$$

where $\Phi$ denotes the distribution function of a standard normal random variable. This means that for large $\lambda$, the arbitrage-free price in the Poisson model is very close to the Black–Scholes price with the same parameter $\sigma$.

**Hint:** Use that if $X_\rho$ has a Poisson distribution with parameter $\rho$, then $\frac{X_\rho - \rho}{\sqrt{\rho}}$ converges weakly to a standard normal random variable for $\rho \to \infty$. Moreover, use the fact that if $(F_n)_{n\in\mathbb{N}}$ is a sequence of distribution functions converging *pointwise* to a *continuous* distribution function $F$, then the convergence is also *uniform*.

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