Mathematical Finance
Exercise Sheet 13

Please hand in until Wednesday, December 17th, 12:00 in your assistant's box in HG G 52.1

Exercise 13-1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with $\mathcal{F}_0$ P-trivial and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, and let $S = (S_t^1, \ldots, S_t^d)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued semimartingale satisfying NFLVR. Assume that there exists a unique equivalent σ-martingale measure $Q \approx P$ on $\mathcal{F}_T$. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function as in the lecture. We assume that $u(x) < \infty$ for some (and hence all) $x \in (0,\infty)$. We do **not** assume, however, that $AE_{+\infty}(U) < 1$. Define the functions $J, I, u, j : (0,\infty) \to \mathbb{R} \cup \{\infty\}$ and the sets $\mathcal{C}(x)$ and $\mathcal{D}(z)$, $x, z > 0$, as in the lecture.

a) Fix $z > 0$. Show that

$$h \le z\frac{dQ}{dP} \;\; P\text{-a.s. for all } h \in \mathcal{D}(z),$$

where $\frac{dQ}{dP}$ denotes the density of Q with respect to P on $\mathcal{F}_T$. Deduce that

$$j(z) = \inf_{h\in\mathcal{D}(z)} E[J(h)] = E\left[J\left(z\frac{dQ}{dP}\right)\right],$$

where $E[J(h)] := +\infty$ if $J^+(h) \notin L^1(P)$.

Hint: Suppose that there is $h \in \mathcal{D}(z)$ such that $A := \{h > z\frac{dQ}{dP}\}$ has $P[A] > 0$. Set $a := Q[A]$, and use that completeness of S is equivalent to the *predictable representation property* of S under Q, i.e. for every bounded Q-martingale M there exists $H \in L(S)$ such that $M = M_0 + H \bullet S$.

b) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j is in $C^1(z_0, \infty)$ and satisfies

$$j'(z) = E\left[\frac{dQ}{dP} J'\left(z\frac{dQ}{dP}\right)\right], \quad z \in (z_0, \infty).$$

Hint: Apply the fundamental theorem of calculus to the function $z \mapsto J\left(z\frac{dQ}{dP}\right)$ and take expectations.

c) Set $x_0 := \lim_{z\downarrow\downarrow z_0} -j'(z)$. Fix $x \in (0, x_0)$. Let $z_x \in (z_0, \infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I\left(z_x\frac{dQ}{dP}\right)$ is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} E[U(f)].$$

Hint: Show that $f^* \in \mathcal{C}(x)$ using Proposition 11.14 in the lecture notes and part **a)**. Then use a Taylor expansion and the strict concavity of U in $(0,\infty)$ to argue that $E[U(f) - U(f^*)] \le 0$ for all $f \in \mathcal{C}(x)$ and that the inequality is an equality if and only if $f = f^*$ P-a.s.

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Exercise 13-2

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,T]}$. Denote by $(\mathcal{F}_t^W)_{t\in[0,T]}$ the natural (completed) filtration of W. Let $\sigma > 0$ and $\mu, r \in \mathbb{R}$. Consider the *undiscounted* Black-Scholes market $(\widetilde{S}^0, \widetilde{S}^1) = (\widetilde{S}_t^0, \widetilde{S}_t^1)_{t\in[0,T]}$ given by the SDEs

$$d\widetilde{S}_t^0 = r\widetilde{S}_t^0\,dt, \quad \widetilde{S}_0^0 = 1, \quad \text{and} \quad d\widetilde{S}_t^1 = \widetilde{S}_t^1(\mu\,dt + \sigma\,dW_t), \quad \widetilde{S}_0^1 = s > 0.$$

Denote by $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$ the discounted stock price. Let $U : (0,\infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1)\setminus\{0\}$. We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$), where we use the notation from Chapter 11 in the lecture.

a) Using Exercise **13-1 a)** show that

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad z \in (0,\infty).$$

b) Using Exercise **13-1 c)** show that $f_x^* := x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_T$, where $R = (R_t)_{t\in[0,T]}$ is given by $R_t = W_t + \frac{\mu-r}{\sigma}t$, is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} E[U(f)], \quad x \in (0,\infty).$$

c) Deduce that $f_x^* = V_T(x, \vartheta^x)$, where $\vartheta^x = (\vartheta_t^x)_{t\in[0,T]}$ is given by

$$\vartheta_t^x = \frac{x}{S_t^1}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t, \quad x \in (0,\infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad x \in (0,\infty).$$

d) For any x-admissible ϑ with $V(x,\vartheta) > 0$, denote by

$$\pi_t := \frac{\vartheta_t S_t^1}{V_t(x,\vartheta)}$$

the fraction of wealth that is invested in the stock. Show that the optimal strategy ϑ^x is given by the *Merton proportion*

$$\pi_t^* = \frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}.$$

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Exercise 13-3

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions and $S = (S_t)_{t\in[0,T]}$ an $\mathbb{R}^d$-valued continuous semimartingale. Consider the set

$$\mathcal{N} := \{N := 1 + \vartheta \bullet S \,|\, \vartheta \in L(S) \text{ and } N > 0 \ \ P\text{-a.s.}\}.$$

Call $N^* \in \mathcal{N}$ a *numéraire portfolio* if $\frac{N}{N^*}$ is a P-supermartingale for all $N \in \mathcal{N}$. Set

$$\ell := \sup_{N\in\mathcal{N}} E[\log N_T] \in [0, \infty], \tag{$*$}$$

where $E[\log N_T] := -\infty$ if $\log^- N_T \notin L^1(P)$. If $\ell < \infty$, then the unique optimiser of $(*)$, if it exists, is denoted by $N^{\log}$ and called the *growth-optimal portfolio.*

a) Show that if the numéraire portfolio N^* exists and $\ell < \infty$, then the growth optimal portfolio exists and $N^{\log} = N^*$.

Hint: Use that $\log x \geq \log y + 1 - y/x$ for all $x, y > 0$, because log is concave.

Suppose for the rest of the question that $\ell < \infty$ and that the growth optimal portfolio $N^{\log}$ exists.

b) Fix $N \in \mathcal{N}$. Show that

$$E\left[\frac{N_T}{N_T^{\log}}\right] \leq 1.$$

Hint: For $\epsilon \in (0,1)$, set $N^\epsilon := \epsilon N + (1-\epsilon)N^{\log} \in \mathcal{N}$ and show that $E\left[\frac{N_T^{\log} - N_T^\epsilon}{N_T^\epsilon}\right] \geq 0$. Then let $\epsilon \downarrow 0$.

c) Let $0 \leq s \leq T$, $A \in \mathcal{F}_s$ and $N^1, N^2 \in \mathcal{N}$ with associated ϑ^1, ϑ^2. Then the strategy of *switching from N^1 to N^2 at time s on A* is given by

$$\widetilde{\vartheta} := 1_{[\![0,s]\!]}\vartheta^1 + 1_{]\!]s,T]\!]}\left(1_A \frac{N_s^1}{N_s^2}\vartheta^2 + 1_{A^c}\vartheta^1\right).$$

Show that $\widetilde{\vartheta} \in L(S)$ and $\widetilde{N} := 1 + \widetilde{\vartheta} \bullet S \in \mathcal{N}$ with

$$\widetilde{N} = 1_{[\![0,s]\!]}N^1 + 1_{]\!]s,T]\!]}\left(1_A \frac{N_s^1}{N_s^2}N^2 + 1_{A^c}N^1\right).$$

d) Deduce that the numéraire portfolio exists and that $N^* = N^{\log}$.

Hint: Fix $N \in \mathcal{N}$ and $0 \leq s < t \leq T$. Set $A := \{E[N_t/N_t^{\log} \,|\, \mathcal{F}_s] > N_s/N_s^{\log}\}$ and consider $\widehat{N}$ corresponding to the strategy of switching first from $N^{\log}$ to N at time s on A and then back to $N^{\log}$ at time t on A. Then apply part **b)** to $\widehat{N}$.

Remark: The above result also holds true for a general (i.e. not necessarily continuous) semimartingale S (when adding the condition that $N_- > 0$ for each $N \in \mathcal{N}$.)
