Mathematical Finance
Exercise Sheet 6

Please hand in until Wednesday, October 29th, 12:00 in your assistant's box in HG G 52.1

Exercise 6-1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space, $S = (S^1, \dots, S_t^d)_{t\in[0,T]}$ a d-dimensional semimartingale and $Q \approx P$ on $\mathcal{F}_T$ an equivalent probability measure.

a) Assume that Q is a separating measure for S. Show that if S is (locally) bounded, then Q is an equivalent (local) martingale measure for S.

Hint: Consider suitable simple integrands.

b) Assume that Q is an equivalent σ-martingale measure for S. Show that it is also an equivalent separating measure.

Hint: Use the Ansel-Stricker theorem.

c) Now assume that $d = 1$, that $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural (completed) filtration of S and that $S = (S_t)_{t\in[0,T]}$ is of the form

$$S_t = \begin{cases} 0 & \text{for } 0 \le t < T, \\ X & \text{for } t = T, \end{cases}$$

where X is normally distributed with mean $\mu \neq 0$ and variance $\sigma^2 > 0$. Show that in this case the class $\mathcal{M}_{\text{sep}}$ of equivalent separating measures for S is strictly bigger than the class $\mathcal{M}_\sigma$ of equivalent σ-martingale measures for S.

Hint: Describe explicitly all predictable processes ϑ for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and compute then $G(\vartheta)$.

Exercise 6-2

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,T]}$. Denote by $(\mathcal{F}_t^W)_{t\in[0,T]}$ the natural (completed) filtration of W. Let $\mu = (\mu_t)_{t\in[0,T]}$ be a predictable process with $\int_0^T |\mu_s|\,ds < \infty$ P-a.s. and $\int_0^t \mu_s^2\,ds < \infty$ P-a.s. for all $t \in (0,T)$. Define the process $S = (S_t)_{t\in[0,T]}$ by the SDE

$$dS_t = S_t(\mu_t\,dt + dW_t), \quad S_0 = s_0 > 0.$$

Moreover, define the process $Z = (Z_t)_{t\in[0,T]}$ by the SDE

$$dZ_t = -Z_t\mu_t\,dW_t, \quad Z_0 = 1. \qquad (*)$$

a) Show that $(*)$ has a unique strong solution, which is nonnegative. (Note that Z_T can become 0.)

Hint: Use that $(*)$ has a unique strong solution on $[0,t]$ for fixed $t \in (0,T)$ and that this solution has an explicit formula. Paste these solutions together and use the supermartingale convergence theorem.

b) Show that Z is the unique nonnegative local P-martingale for the filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$ with $Z_0 = 1$ such that ZS is also a local P-martingale for the filtration $(\mathcal{F}_t^W)_{t\in[0,T]}$.

Hint: Use the product formula and Itô's representation theorem.

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c) Suppose that Z is a true martingale. Show that the process S satisfies NA.

Hint: Define $Q \ll P$ on $\mathcal{F}_T$ by $dQ := Z_T \, dP$. Then show that if $\vartheta \bullet S_T \geq 0$ P-a.s. for $\vartheta \in \Theta_{adm}$, then $\vartheta \bullet S \equiv 0$ under Q and use that $Q \approx P$ on $\mathcal{F}_t$ for all $t \in [0, T)$.

d) Show that S satisfies NFLVR if and only if Z is a true martingale with the additional property that $Z_T > 0$ P-a.s.

Hint: Use the fundamental theorem of asset pricing.

e) Assume that $(\mu_t)_{t\in[0,T]}$ is given by $\mu_t := \frac{1}{\sqrt{T-t}} 1_{]\!]0,\tau]\!]}$, where $\tau = \inf\{t \in (0, T) : \widetilde{Z}_t = 2\} \wedge T$ and where $\widetilde{Z} = (\widetilde{Z}_t)_{t\in[0,T]}$ is the unique strong solution of the SDE

$$d\widetilde{Z}_t = \frac{-\widetilde{Z}_t}{\sqrt{T-t}} \, dW_t, \quad \widetilde{Z}_0 = 1.$$

Using the above results, show that S satisfies NA but fails NFLVR.

Exercise 6-3

The aim of this exercise is to see that the fundamental theorem of asset pricing does not extend to market models with infinitely many assets in a straightforward manner.

Consider $\mathbb{T} = \{0, 1\}$, i.e. a one-period model, and $\Omega = \{1, 2, \ldots\}$ with filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^\Omega$, and any probability P such that $P[\{k\}] > 0$ for all $k \in \Omega$. We assume that there is a constant bank account $S^0 \equiv 1$ and that there are infinitely many assets i, $i = 1, 2, \ldots$, satisfying

$$S^i_0 = 1, \quad S^i_1(k) = 1 - \delta_{k,i} + \delta_{k,i+1}, \quad i = 1, 2, \ldots, \quad k \in \Omega,$$

where δ denotes Kronecker's delta. Since $\Delta S(k) \in \ell^\infty$ for all k, a strategy should take values in ℓ^1. As $T = 1$, a strategy is therefore simply a summable (deterministic) sequence $\vartheta = (\vartheta^i)_{i=0,1,\ldots}$, where ϑ^0 represents V_0.

a) Show that the no-arbitrage condition (NA) holds true, i.e. that

$$\vartheta \cdot \Delta S \geq 0 \quad \Longrightarrow \quad \vartheta \cdot \Delta S = 0.$$

b) Show that there exists no equivalent martingale measure for S.
