ETH Zürich HS 2014
D-MATH
Prof. Martin Schweizer

Coordinator
Ariel Neufeld

Mathematical Finance
Exercise Sheet 8

Please hand in until Wednesday, November 12th, 12:00 in your assistant's box in HG G 52.1

Exercise 8-1

Consider a stock price model based on a 2-dimensional Brownian motion (W, W') in the latter's own filtration, where the 1-dimensional (discounted) stock price follows

$$dS_t = \mu_t dt + \sigma_t dW_t$$

and $\mu_t = \mu(t, S_t, Y_t)$ and $\sigma_t = \sigma(t, S_t, Y_t) > 0$ are determined by continuous functions depending on the diffusion Y which follows

$$\begin{aligned} dY_t &= b(t, Y_t)dt + a(t, Y_t)dB_t, \\ Y_0 &= y_0. \end{aligned}$$

Here B is a Brownian motion correlated to W, defined by

$$B_t = \rho W_t + \sqrt{1-\rho^2}\, W'_t$$

for some constant $\rho \in (0, 1)$. We assume that $\sigma > 0$ and that the function $\frac{\mu}{\sigma}$ is uniformly bounded. The process Y is often called *stochastic factor* in this context.

a) Show that $d\langle B, W\rangle_t = \rho dt$ and $d\langle S, Y\rangle_t = a(t, Y_t)\sigma(t, S_t, Y_t)\rho dt$.

b) What is the general form of the density process of an ELMM Q for S?

c) Find the dynamics of S and Y under such a measure; that is, give stochastic differential equations for these processes involving only Brownian motions under Q, not under P.

Exercise 8-2

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Moreover, let $X = (X_t)_{t\in[0,T]}$ be a a compound Poisson process with jump intensity $\lambda > 0$ and jump distribution ν, i.e., $X_t := \sum_{k=1}^{N_t} Y_k$, where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with rate λ and $(Y_k)_{k\in\mathbb{N}}$ a sequence of random variables independent of N such that the Y_k are i.i.d. with distribution ν (with $\nu(\{0\}) = 0$). Suppose about the filtration $(\mathcal{F}_t)_{t\geq 0}$ that X is a Lévy process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Let $\widetilde{\lambda} > 0$ and let $\widetilde{\nu} \approx \nu$ be an equivalent probability measure on $\mathbb{R}$. Define the exponential Lévy process $Z = (Z_t)_{t\in[0,T]}$ by

$$Z_t := \exp\left(\sum_{k=1}^{N_t} \phi(Y_k) + (\lambda - \widetilde{\lambda})t\right),$$

where $\phi = \log\left(\frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{\nu}}{d\nu}\right)$.

a) Show that Z is a P-martingale.

b) Define the probability measure $Q \approx P$ on $\mathcal{F}_T$ by $dQ = Z_T\, dP$. Show that under Q, X is again a compound Poisson process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with rate $\widetilde{\lambda}$ and jump distribution $\widetilde{\nu}$.

Hint: Show that X is a Lévy process under Q for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and calculate the characteristic function of X_1 under Q to determine its law (assuming without loss of generality that $T \geq 1$).

Exercise 8-3

Let $Z > 0$ be a P-local martingale with $Z_0 = 1$ and let S be a semimartingale under P.

a) Show that $S \in \mathcal{M}_\sigma(P) \iff \exists\,(D_m) \uparrow \Omega \times [0, \infty)$, $D_m \in \mathcal{P}$ with $1_{D_m} \bullet S \in \mathcal{M}_{loc}(P)$ for each m.

b) Let ϑ be in $L(S)$. Show that if $ZS \in \mathcal{M}_\sigma(P)$ and ϑ is admissible for S, then $Z\,G(\vartheta)$ is a local P-martingale and P-supermartingale.
