ETH Zürich HS 2014
D-MATH
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Mathematical Finance
Solution 12

Solution 12-1

Note that the argument only uses that U is increasing and concave.

a) Take $x \leq y$ and $\vartheta \in \mathcal{A}(x)$. Then $V_T^{y,\vartheta} \geq V_T^{x,\vartheta}$, thus $\vartheta \in \Theta^y_{adm}$ and since $U^-(V_T^{x,\vartheta}) \in L^1(P)$ also $U^-(V_T^{y,\vartheta}) \in L^1(P)$, and so $\vartheta \in \mathcal{A}(y)$. So, since $U(V_T^{x,\vartheta}) \leq U(V_T^{y,\vartheta})$,

$$E[U(V_T^{x,\vartheta})] \leq E[U(V_T^{y,\vartheta})] \leq u(y),$$

hence taking sup over $\vartheta \in \mathcal{A}(x)$ also yields $u(x) \leq u(y)$.

For concavity, take $x \leq y$ and $z = \lambda x + (1-\lambda)y$ with $\lambda \in [0,1]$. Then $x \leq z \leq y$, so $u(x) \leq u(z) \leq u(y)$, and w.l.o.g. $u(x) < \infty$. For a particular ϑ_1 and ϑ_2, we have

$$\lambda V_T^{x,\vartheta_1} + (1-\lambda)V_T^{y,\vartheta_2} = z + V^{z,\vartheta^*},$$

where z as above and

$$\vartheta^* := \lambda\vartheta_1 + (1-\lambda)\vartheta_2 \in \Theta^z_{adm}.$$

Since U is concave, we have that

$$U(V_T^{z,\vartheta^*}) \geq \lambda U(V_T^{x,\vartheta_1}) + (1-\lambda)U(V_T^{x,\vartheta_2}).$$

If $U^-(V_T^{x,\vartheta_1}) \in L^1(P)$ and $U^-(V_T^{y,\vartheta_2}) \in L^1(P)$ then also $U^-(V_T^{z,\vartheta^*}) \in L^1(P)$; therefore, $\vartheta^* \in \mathcal{A}(z)$ and

$$u(z) \geq E[U(V_T^{z,\vartheta^*})] \geq \lambda E[U(V_T^{x,\vartheta_1})] + (1-\lambda)E[U(V_T^{y,\vartheta_2})];$$

so taking sup over ϑ_1, ϑ_2 gives $u(z) \geq \lambda u(x) + (1-\lambda)u(y)$.

b) Assume by contradiction that there exists $x > 0$ such that $u(x) = \infty$. Then, as u is monotone by **a)**, we must have $x > x_0$. Thus, there exists $k > 1$ such that $kx_0 = x$. Choose $0 < \lambda < \frac{1}{k} < 1$ and then, take $c \in (0,1)$ such that $(1-\lambda)c + \lambda k = 1$. Due to the concavity uf u by **b)**, we then have

$$u(x_0) \geq (1-\lambda)\, u(cx_0) + \lambda\, u(x).$$

Moreover, due to monotonicity of u, we have $u(cx_0) < \infty$. Hence, we get that $u(x) < \infty$, which contradicts our assumption on x.

c) By **b)**, $u(x) < \infty$ for all $x > 0$. Suppose $U^+(V_T^{x,\vartheta}) \notin L^1(P)$ for some $x > 0$ and $\vartheta \in \mathcal{A}(x)$. Then, by definition of $\mathcal{A}(x)$, we must have that

$$E\big[U(V_T^{x,\vartheta})\big] = \infty.$$

But then, we have

$$u(x) \geq E\big[U(V_T^{x,\vartheta})\big] = \infty,$$

which gives us a contradiction.

Solution 12-2

a) "$\Rightarrow$": Seeking a contradiction, suppose that S fails NA. Then there exists $\vartheta \in \mathbb{R}^d \setminus \{0\}$ such that $\vartheta^{tr}\Delta S_1 \geq 0$ P-a.s. and $P[\vartheta^{tr}\Delta S_1 > 0] > 0$. In particular, $\vartheta \in \mathcal{A}(0)$. But then also for each $\lambda > 0$, $\lambda\vartheta \in \mathcal{A}(0)$, and so $\mathcal{A}(0)$ is not bounded and hence not compact. Since $\mathcal{A}(0) \subset \mathcal{A}(x)$, we arrive at a contradiction.

"$\Leftarrow$": Seeking a contradiction, suppose that $\mathcal{A}(x)$ is not compact. Since $\mathcal{A}(x)$ is clearly closed, this means that $\mathcal{A}(x)$ is not bounded. Hence, there exists a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ in $\mathcal{A}(x)\setminus\{0\}$ such that $\lim_{n\to\infty}\|\vartheta_n\|_\infty = +\infty$. For $n \in \mathbb{N}$, define $\eta_n := \frac{\vartheta_n}{\|\vartheta_n\|_\infty}$. Then $\|\eta_n\|_\infty = 1$ by construction for each $n \in \mathbb{N}$. Since the unit ball (with respect to the maximum norm) in $\mathbb{R}^d$ is compact, there exists a subsequence, denoted also by $(\eta_n)_{n\in\mathbb{N}}$, converging to some $\eta \in \mathbb{R}^d$ with $\|\eta\|_\infty = 1$. Using that $\vartheta_n \in \mathcal{A}(x)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|\vartheta_n\|_\infty = +\infty$ gives

$$\eta^{tr}\Delta S_1 = \lim_{n\to\infty}\eta_n^{tr}\Delta S_1 = \lim_{n\to\infty}\frac{\vartheta_n^{tr}\Delta S_1}{\|\vartheta_n\|_\infty} \geq \liminf_{n\to\infty}\frac{-x}{\|\vartheta_n\|_\infty} \geq 0 \ \ P\text{-a.s.} \tag{1}$$

Since $\eta \neq 0$, it follows from the non-redundancy of S that $P[\eta^{tr}\Delta S_1 > 0] > 0$. Thus, η is an arbitrage opportunity, and we arrive at a contradiction.

b) "$\Rightarrow$": Seeking a contradiction, suppose that S fails NA. Then there exists $\vartheta \in \mathbb{R}^d \setminus \{0\}$ such that $\vartheta^{tr}\Delta S_1 \geq 0$ P-a.s. and $P[\vartheta^{tr}\Delta S_1 > 0] > 0$. Then by monotone convergence and by the fact that $U(\infty) = +\infty$,

$$\lim_{\lambda\to\infty} E[U(x+\lambda\vartheta^{tr}\Delta S_1)] = U(x)P[\vartheta^{tr}\Delta S_1 = 0] + U(\infty)P[\vartheta^{tr}\Delta S_1 > 0] = +\infty, \tag{2}$$

Since $\lambda\vartheta \in \mathcal{A}(x)$ for all $\lambda > 0$ as in part **a)**, this implies that $u(x) = +\infty$, and we arrive at a contradiction.

"$\Leftarrow$": Since $\mathcal{A}(x)$ is compact by part **a)**, there exists $c > 0$ such that $\|\vartheta\|_\infty \leq c$ for all $\vartheta \in \mathcal{A}(x)$. This together with concavity of U shows that for all $\vartheta \in \mathcal{A}(x)$,

$$U(x+\vartheta^{tr}\Delta S_1) \leq U(x) + U'(x)(\vartheta^{tr}\Delta S_1) \leq U(x) + cU'(x)\sum_{i=1}^d |\Delta S_1^i| =: Y. \tag{3}$$

Note that Y is integrable since $E[|\Delta S_1^i|] < \infty$ for $i \in \{1,\dots,d\}$ by hypothesis and by the fact that $\mathcal{F}_0$ is trivial. Thus

$$u(x) = \sup_{\vartheta\in\mathcal{A}(x)} E[U(x+\vartheta^{tr}\Delta S_1)] \leq E[Y] < \infty. \tag{4}$$

c) Note that $u(x) < \infty$ by part **b)**.

First, we establish existence of ϑ^*. Let $(\vartheta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}(x)$ such that

$$\lim_{n\to\infty} E[U(x+\vartheta_n^{tr}\Delta S_1)] = u(x). \tag{5}$$

Since $\mathcal{A}(x)$ is compact by part **a)**, there exists a subsequence, denoted again by $(\vartheta_n)_{n\in\mathbb{N}}$, converging to some $\vartheta^* \in \mathcal{A}(x)$. Now by Fatou's lemma using (3), continuity of U in $[0,\infty)$ and the fact that $\vartheta^* \in \mathcal{A}(x)$,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} E[U(x+\vartheta_n^{tr}\Delta S_1)] \leq E\left[\limsup_{n\to\infty} U(x+\vartheta_n^{tr}\Delta S_1)\right] \\ &= E\left[U(x+(\vartheta^*)^{tr}\Delta S_1)\right] \leq u(x). \end{aligned} \tag{6}$$

Next, we establish uniqueness of ϑ^*. To this end, let $\widetilde{\vartheta}^* \in \mathcal{A}(x)$ be another maximiser of $\vartheta \mapsto E[U(x + \vartheta^{tr}\Delta S_1)]$. Set $\widehat{\vartheta}^* := \frac{1}{2}\vartheta^* + \frac{1}{2}\widetilde{\vartheta}^*$. Then $\widehat{\vartheta}^* \in \mathcal{A}(x)$ by convexity of $\mathcal{A}(x)$. By concavity of U on $[0,\infty)$,

$$U(x + (\widehat{\vartheta}^*)^{tr}\Delta S_1) \geq \frac{1}{2}U(x + (\vartheta^*)^{tr}\Delta S_1) + \frac{1}{2}U(x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1). \tag{7}$$

Moreover, by strict concavity of U on $(0,\infty)$, by strict concavity of U on $[0,\infty)$ in case that $U(0) > -\infty$ and by the fact that $x + (\vartheta^*)^{tr}\Delta S_1, x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1 > 0$ P-a.s. in case that $U(0) = -\infty$, the inequality in (7) is strict on $\{(\vartheta^*)^{tr}\Delta S_1 \neq (\widetilde{\vartheta}^*)^{tr}\Delta S_1\}$. On the other hand, by maximality of ϑ^* and $\widetilde{\vartheta}^*$, it follows that

$$E[U(x + (\widehat{\vartheta}^*)^{tr}\Delta S_1)] \leq \frac{1}{2}E[U(x + (\vartheta^*)^{tr}\Delta S_1)] + \frac{1}{2}E[U(x + (\widetilde{\vartheta}^*)^{tr}\Delta S_1)].$$

Thus, we may conclude that $(\vartheta^*)^{tr}\Delta S_1 = (\widetilde{\vartheta}^*)^{tr}\Delta S_1$ P-a.s. Now non-redundancy of S gives $\widetilde{\vartheta}^* = \vartheta^*$.

Solution 12-3

a) Fix $0 \leq a < b < c$. Then there exists $\lambda \in (0,1)$ such that $b = \lambda c + (1-\lambda)a$. By concavity of U,

$$\begin{aligned}\frac{U(b) - U(a)}{b-a} &= \frac{U(\lambda c + (1-\lambda)a) - U(a)}{\lambda(c-a)} \geq \frac{\lambda(U(c) - U(a))}{\lambda(c-a)} = \frac{U(c) - U(a)}{c-a} \\ &= \frac{(1-\lambda)(U(c) - U(a))}{(1-\lambda)(c-a)} \geq \frac{U(c) - U(\lambda c + (1-\lambda)a)}{(1-\lambda)(c-a)} = \frac{U(c) - U(b)}{c-b}.\end{aligned} \tag{8}$$

For $z < y' < y''$, setting $a := z$, $b := y'$ and $c := y''$ shows that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is decreasing on (z,∞), for $y' < y'' < z$, setting $a := y'$, $b := y''$ and $c := z$ shows that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is also decreasing on $(0,z)$, and for $y' < z < y''$, setting $a := y'$, $b := z$ and $c := y''$, establishes that $y \mapsto \frac{U(y)-U(z)}{y-z}$ is decreasing everywhere on $(0,\infty) \setminus \{z\}$.

b) Let $\eta \in \mathbb{R}^d \setminus \{0\}$ be arbitrary. Since ϑ^* is an interior point of $\mathcal{A}(x)$, $\vartheta^* + \epsilon\eta \in \mathcal{A}(x)$ for all $\epsilon > 0$ sufficiently small. For $\epsilon > 0$ sufficiently small, set

$$\Delta_\epsilon^\eta := \frac{U(x + (\vartheta^* + \epsilon\eta)^{tr}\Delta S_1) - U(x + (\vartheta^*)^{tr}\Delta S_1)}{\epsilon}. \tag{9}$$

Then on $\{\eta^{tr}\Delta S_1 \neq 0\}$,

$$\Delta_\epsilon^\eta = (\eta^{tr}\Delta S_1)\frac{U(x + (\vartheta^* + \epsilon\eta)^{tr}\Delta S_1) - U(x + (\vartheta^*)^{tr}\Delta S_1)}{\epsilon\eta^{tr}\Delta S_1}, \tag{10}$$

and by part **a)**, this increases monotonically to $(\eta^{tr}\Delta S_1)U'(x + (\vartheta^*)^{tr}\Delta S_1) > -\infty$ as $\epsilon \downarrow 0$. In particular, for $\eta := \vartheta^*$, using that $U' < +\infty$ on $(0,\infty)$ and $(\vartheta^*)^{tr}\Delta S_1 = -x < 0$ on $\{x + (\vartheta^*)^{tr}\Delta S_1 = 0\}$, this gives $U'(x + (\vartheta^*)^{tr}\Delta S_1) < \infty$ P-a.s.

On the other hand, on $\{\eta^{tr}\Delta S_1 = 0\}$, $\Delta_\epsilon^\eta \equiv 0$, and this trivially increases monotonically to $(\eta^{tr}\Delta S_1)U'(x + (\vartheta^*)^{tr}\Delta S_1)$ as $\epsilon \downarrow 0$.

Now by the fact that U is increasing, by the fact that $U(0) > -\infty$ and by optimality of ϑ^*, for $\epsilon > 0$ sufficiently small,

$$\frac{U(0) - U(x + (\vartheta^*)^{tr}\Delta S_1)}{\epsilon} \leq \Delta_\epsilon^\eta. \tag{11}$$

Thus, $\Delta_\epsilon^\eta \in L^1(P)$ for ϵ sufficiently small, and so by the above and monotone convergence,

$$(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(P)$$

and

$$E[(\eta^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1)] \leq 0. \tag{12}$$

The final claim follows by setting $\eta := (1,0,\dots,0)$, $\eta = (-1,0,\dots,0)$, $\eta := (0,1,0,\dots,0)$, $\eta := (0,-1,0,\dots,0)$, …, $\eta := (0,\dots,0,1)$ and $\eta := (0,\dots,0,-1)$.

c) Using that $U'(x+(\vartheta^*)^{tr}\Delta S_1) \in (0,\infty)$ P-a.s. by the strict concavity of U on $(0,\infty)$ and part **b)** and that $E[U'(x+(\vartheta^*)^{tr}\Delta S_1)\Delta S_1^i] = 0$ for all $i \in \{1,\dots,d\}$, it suffices to show that $U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(P)$. Since U' is decreasing on $(0,\infty)$, it even suffices to show that

$$U'(x+(\vartheta^*)^{tr}\Delta S_1)1_{\{x+(\vartheta^*)^{tr}\Delta S_1 \leq x/2\}} \in L^1(P). \tag{13}$$

Since $((\vartheta^*)^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1) \in L^1(P)$ by part **b)**,

$$\begin{aligned}
&E[U'(x+(\vartheta^*)^{tr}\Delta S_1)1_{\{x+(\vartheta^*)^{tr}\Delta S_1 \leq x/2\}}] \\
&\qquad = E[U'(x+(\vartheta^*)^{tr}\Delta S_1)1_{\{(\vartheta^*)^{tr}\Delta S_1 \leq -x/2\}}] \\
&\qquad \leq \frac{E[-((\vartheta^*)^{tr}\Delta S_1)U'(x+(\vartheta^*)^{tr}\Delta S_1)1_{\{(\vartheta^*)^{tr}\Delta S_1 \leq -x/2\}}]}{x/2} \\
&\qquad \leq \frac{2}{x}E[|(\vartheta^*)^{tr}\Delta S_1|U'(x+(\vartheta^*)^{tr}\Delta S_1)] < \infty.
\end{aligned} \tag{14}$$