Percolation Theory

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Motivation

Understand a simplistic (probabilistic) model for a porous stone. Develop a Toolbox for your daily percolation need.

Framework and Definitions

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- Edges are called **bonds**.
- We let $\mathbb{Z}^d$ refer to the lattice and its vertex set.

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- $\mathcal{F}$ is σ-algebra generated by events depending only on finitely many edges.
- **Percolation measure:** $\mathbb{P}_p(\text{Bernoulli}(p))^{\otimes E}$

Framework and Definitions

Definition

We declare each bond of the lattice $\mathbb{Z}^d$ to be ***open*** *with probability $p \in [0, 1]$ and* ***closed*** *otherwise (i.e. with probability $q = 1 - p$). Bonds are open or closed independently of all other bonds.*

Definition

A ***path*** *(of length n) in $\mathbb{Z}^d$ is a sequence of vertices $(x_1, \ldots, x_n)$ such that (x_i, x_{i+1}) is a bond of $\mathbb{Z}^d$. A path is called* ***open*** *if all its edges are open and in this case we say that the path connects x_0 with x_n. A path is called* ***closed*** *if all its edges are closed.*

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Remark

In percolation we often care about open paths, because such paths simulate where water can flow.

Definition

Consider the random subgraph of $\mathbb{Z}^d$ which contains only the open edges of $\mathbb{Z}^d$. The connected components of this graph are called ***open clusters****. We denote $C(x)$ to be the open cluster that contains the vertex x.*

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Remark

By translation invariance of the lattice and the probability measure, the distribution of $C(x)$ is independent of the choice of x. We let $C(0) = C$ be the open cluster that contains the origin.

Framework and Definitions

Definition

We define the ***percolation probability*** $\theta(p)$ *as the probability that the origin belongs to an infinite open cluster, i.e.*

$$\theta(p) := \mathbb{P}_p(|C| = \infty)$$

Broadbent-Hammersley Theorem

Broadbent-Hammersley Thm

Lemma

There exists a ***critical value*** $p_c = p_c(d)$ *such that* $\theta(p) = 0$ *for* $p < p_c$ *and* $\theta(p) > 0$ *for* $p > p_c$*. Moreover, the critical value is decreasing w.r.t. the dimension, i.e.* $p_c(d+1) \leq p_c(d)$*.*

Proof: Blackboard

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Remark

The one-dimensional case is not interesting because there we have $p_c = 1$.

Definition

*The parameter set $p < p_c$ is called the **sub-critical phase**, the set $p > p_c$ is called the **supercritical phase**.*

Definition

*A **self-avoiding path** of length n is a sequence of edges $e_1, \dots, e_n$ with $e_i \neq e_j$ for $i \neq j$ and such that e_i and e_{i+1} share an endpoint for every $1 \leq i < n$. Let $\sigma(n)$ denote the number of self-avoiding paths in $\mathbb{Z}^d$ of length n, we define the **connective constant** of $\mathbb{Z}^d$ as*

$$\lambda(d) = \lim_{n \to \infty} \sigma(n)^{1/n}.$$

Definition

Let G be a graph, we define its dual G^* as the graph which has as vertices the faces of G and as vertices pairs of faces which are adjacent.

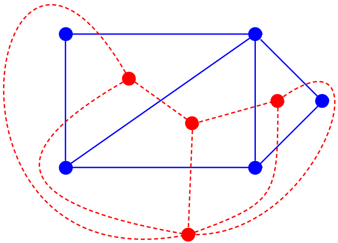

Figure: The red graph is the dual graph of the blue graph, and vice versa.

Broadbent-Hammersley Thm

Theorem (Broadbent-Hammersley)

For $d \geq 2$ we have

$$0 < \lambda(d)^{-1} \leq p_c(d) \leq p_c(2) < 1.$$

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Remark

Later: $p_c(2) \leq (1 - \lambda(2)^{-1}))$.

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Broadbent-Hammersley Thm

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- Kesten: $p_c(2) = 1/2$ and $\theta(p_c) = 0$.
- Hara and Slade (1990) for $d \geq 19$ (lace expansion).
  - Lace expansion expected to only work for $d \geq 6$.
- State of the art: shown for $d \geq 11$.

Harris-FKG inequality

Definition

Let $\omega, \omega' \in \Omega$ be two configurations. We say $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all bonds $e \in \mathbb{Z}^d$. We say a RV $X : \Omega = \{0,1\}^E \to \mathbb{R}$ is increasing if for $\omega \leq \omega'$ we have $X(\omega) \leq X(\omega')$. An event $A \in \mathcal{A}$ is increasing if 1_A is increasing.

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Remark

Heuristically, increasing events are favoured by opening up edges.

Example

The following events are increasing:

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The following event is neither $\nearrow$ nor $\searrow$:

- $C = \{|C| = 5\}$.

Harris-FKG

Standard coupling of percolation:

- Let $(X_e)_{e\in E}$ be a seq of i.i.d. RV, $X_e \sim \text{unif}[0,1]$.
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Proposition (increasing coupling)

Fix $p \le p'$. There exists a probability measure $\mathbb{P}$ on $[0,1]^E$ which coincides with $\mathbb{P}_p$ on $\{0,1\}^E$ such that $\omega_p \le \omega_{p'}$ $\mathbb{P}$-a.s.

Harris-FGK

Lemma

If X is an increasing RV in $L^1(\mathbb{P}_p) \cap L^1(\mathbb{P}_q)$ then we have

$$\mathbb{E}_p(X) \leq \mathbb{E}_q(X), \text{ for } p \leq q.$$

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Remark

$A = \{|C| = \infty\}$ is increasing $\rightsquigarrow \theta(p) = \mathbb{P}_p(|C| = \infty)$ is increasing.

Proof of Lemma: Blackboard.

Theorem (FKG inequality)

For increasing random variables X, Y in $L^2(\Omega, \mathbb{P}_p)$, we have

$$\mathbb{E}_p(XY) \geq \mathbb{E}_p(X)\mathbb{E}_p(Y).$$

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Remark

We obtain for positive events $\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$ or equivalently $\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A)$.

Proof of Theorem: Blackboard.

Corollary

We can improve the bound of $p_c(2)$ as follows

$$p_c(2) \leq (1 - \lambda(2)^{-1}).$$

Proof: Blackboard.

Russo's formula

Definition

Let $A \in \mathcal{A}$ be an event and $\omega \in \Omega$ a configuration. We say that an edge $e \in \mathbb{Z}^d$ is ***pivotal*** *for the pair (A, ω) if $1_A(\omega) = 1_A(\widetilde{\omega_e})$ where $\widetilde{\omega_e}$ is the unique configuration which agrees with ω except at the edge e.*

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Remark

Pivotal edges are the edges that are essential for an event A to occur.

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e is pivotal for A iff, when e is removed from the lattice, one endvertex of e is in a finite open cluster containing the origin, the other endvertex is an infinite open cluster.

Question

Let $d = 2$, how would a pivotal edge for the event $x \longleftrightarrow y$ look like?

Theorem (Russo's formula)

Let A be an increasing event, depending only on finitely many edges of $\mathbb{Z}^d$. Then $p \mapsto \mathbb{P}_p(A)$ is differentiable, and

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{e \in F} \mathbb{P}_p(e \textit{ is pivotal for } A)$$

where $F \subset E$ is a finite subset.

Proof: Blackboard.

Exponential decay

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Theorem (Exponential decay)

For all $d \geq 2$, we have in the subcritical regime $p < p_c$, that there exists a constant $c = c(p) > 0$ such that for all $n \geq 1$

$$\mathbb{P}_p(0 \longleftrightarrow \partial B_n) \leq e^{-cn},$$

where $B_n := \{-n, \ldots, n\}^d$ denotes the box of size n around the origin.

Thank you