Simulation to investigate the self-organised criticality and scale invariance of the Drossel-Schwabl Forest Fire Model

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1 Abstract

A simulation was developed for the Drossel-Schwabl forest fire model to investigate its self-organised criticality. The model was found to partially show self-organised criticality. The cluster size of trees showed self-organised criticality with a power-law distribution of cluster size and frequency with $\alpha = 1.85 \pm 0.01$. The system was also found to reach a steady-state where the number of trees did not vary much with a relative standard deviation of 7.1% and a tree density of 0.42. Parameters such as the size of the forest, ratio of p to f and number of iterations of the simulation were varied. The tree-density was found to not vary much with these parameters supporting the self-organised criticality and scale invariance of the system. However the value of α varied quite significantly with these parameters relative to the tree-density. This indicates that the system does not have true self-organised criticality or scale invariance since it only fully displays these properties within a certain range of parameter values.

2 Introduction

The Drossel-Schwabl Forest Fire model is a grid of L^d cells where L is the side-length and d is the dimension. For this project, $d = 2$ and L=50.

In this grid each cell can be burning, empty or contain a tree. The model is defined by the following rules:

1. A burning cell will become an empty cell
2. A tree will burn if one or more of its neighbors are burning. A neighbor is defined as a cell directly above, to the right, to the left or below the cell.
3. The probability of a tree spontaneously igniting is f Once a tree starts burning all trees in its cluster will burn down.
4. The probability of an empty cell growing a tree is p

[2]

Another important parameter is, θ, given by the following equation.

$$\theta = \frac{p}{f} \tag{1}$$

3 Theory

A system is said to exhibit self-organized criticality when a measure of the system fluctuates about a state of marginal stability.

We can describe groups of trees connected by their top, right, bottom or left sides as clusters. As shown by other studies, the size and frequency of the clusters should obey the following power-law,

$$N = bs^{-\alpha} \tag{2}$$

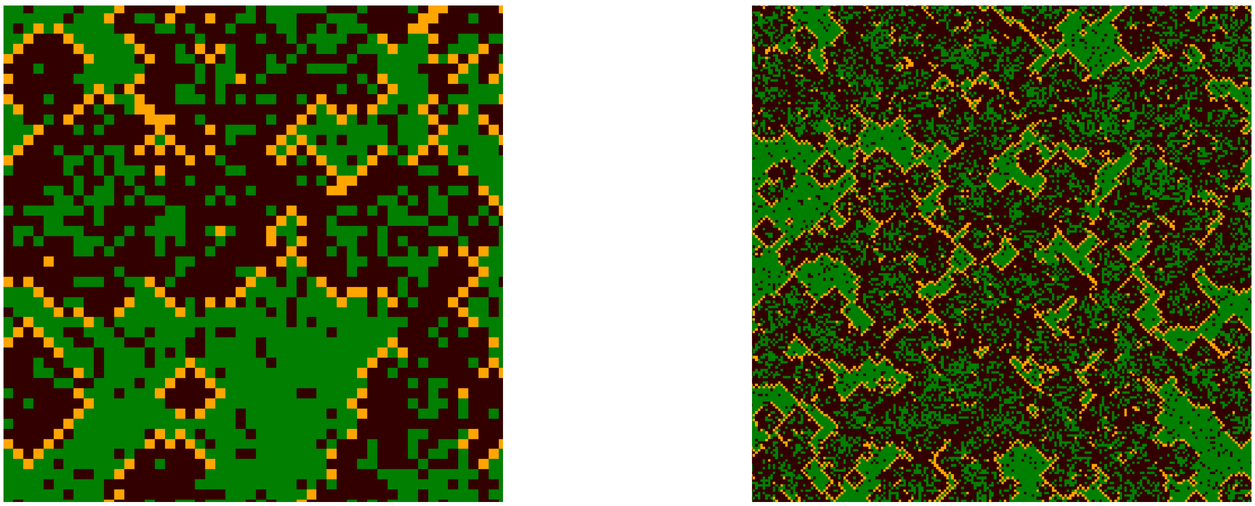
where N is the number of clusters, s is the size of the cluster (i.e. the number of trees) and α and b are constants that were determined. [2]

Equation 2 shows how the distribution of clusters does not have a characteristic length scale so is scale invariant. For larger forests we expect to have a similar distribution of cluster sizes though the cluster sizes themselves are all larger.

We can determine the density of trees in the steady state (where the number of burning sites, empty sites and tree sites remain fairly constant) and compare this with literature values.

4 Experimental Procedure

The simulation works by looping through the entire grid of cells at predetermined intervals and updating the states of each cell according to the rules given in section 2. This continues until a pre-determined number of iterations is reached. The figure below shows the forest fire simulation.



(a) 50x50 Forest

(b) 200x200 Forest

Figure 1: Simulation for a 50x50 and 200x200 forest. Orange sites are burning, green sites have a tree and black sites are empty

A link to an animation showing the live forest fire is also given in the appendix.

The clusters were found using an implementation of the Hoshen-Kopelman algorithm. The clustering was tested by running the simulation for a 10x10 forest and labelling the sites containing trees with a cluster label. Below is a picture from the test simulation.

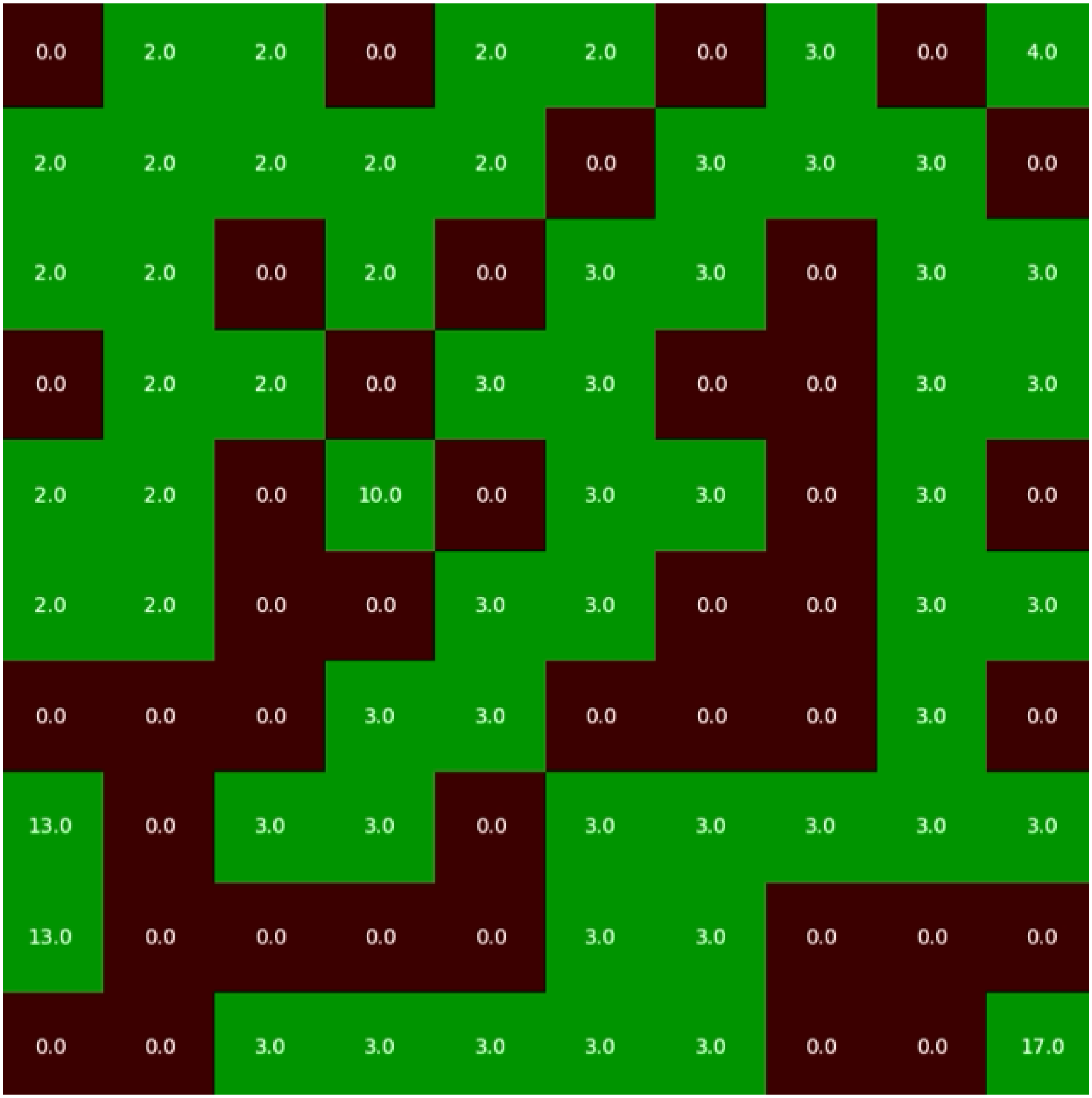


Figure 2: Cluster Labelling Test

As shown by figure 2, the clustering labelling passed the test simulation since trees connected to form a cluster are given the same label, a number. In this scenario we have 6 different clusters given by the labels 2, 3, 4, 10, 13, 17.

The simulation was repeated while changing the following parameters:

1. The value of f from 0.9 to 1×10^{-7}
2. The number of iterations from 50 to 1×10^{5}

3. The Forest Size from 1×10^2 to 4×10^4

A wide-range of parameter values were used as this allowed the conditions under which the Drossel-Schwabl exhibits self-organised criticality to be investigated and the behaviour of the model in different limits such as $f > p$ to be investigated.

5 Results and Interpretation

5.1 Main Results

The simulation was initially run for 3000 iterations of a 50x50 forest with $p = 0.1$, $f = 0.001$. The number of empty, burning, and tree sites was counted for each iteration, a plot of this is shown below.

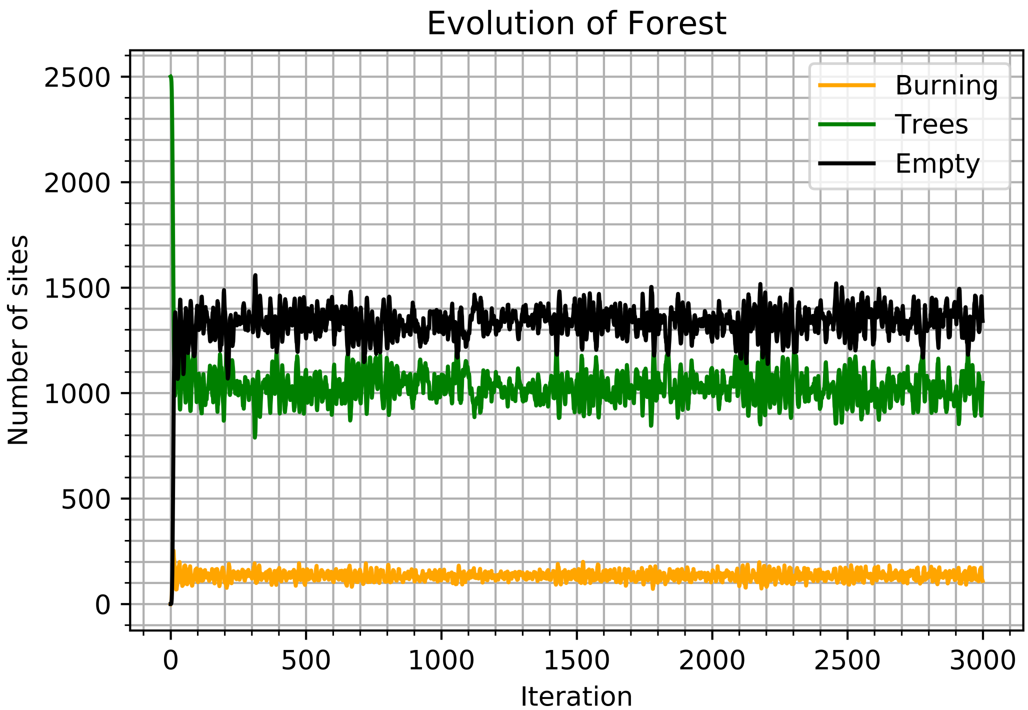


Figure 3: Number of burning, empty, tree sites in 50x50 grid

As seen by figure 3 there is an initial period of instability in the system. The number of trees initially increases from zero and reaches a point where the number of trees is greater than the number of empty sites. The number of trees continues to increase until the first fire starts that spreads through the forest, burning down the trees. The iteration when the number of trees again falls below the number of empty sites is recorded and used as a reference for the start of the steady-state of the forest. After this point the number of trees fluctuates about a position of marginal stability. The number of trees increases until fires spread and burn down parts of the forest, decreasing the number of trees again and repeating in a cycle.

In the steady-state the mean number of trees was found to be 1032 with a relative standard deviation, RSD, of 7.1%. The relatively low value of the RSD shows how the number of trees exhibits marginal stability as the amplitude of the fluctuations are low compared to the mean number of trees. The density of trees in the steady-state was found to be $\rho = 0.42$. This is comparable to the value found in previous studies of $\rho = 0.41$. [2]

The number of trees in each cluster was also calculated for each iteration and added to a running total of the number of times each cluster size occured, i.e. the frequency. A bar chart of this is shown below.

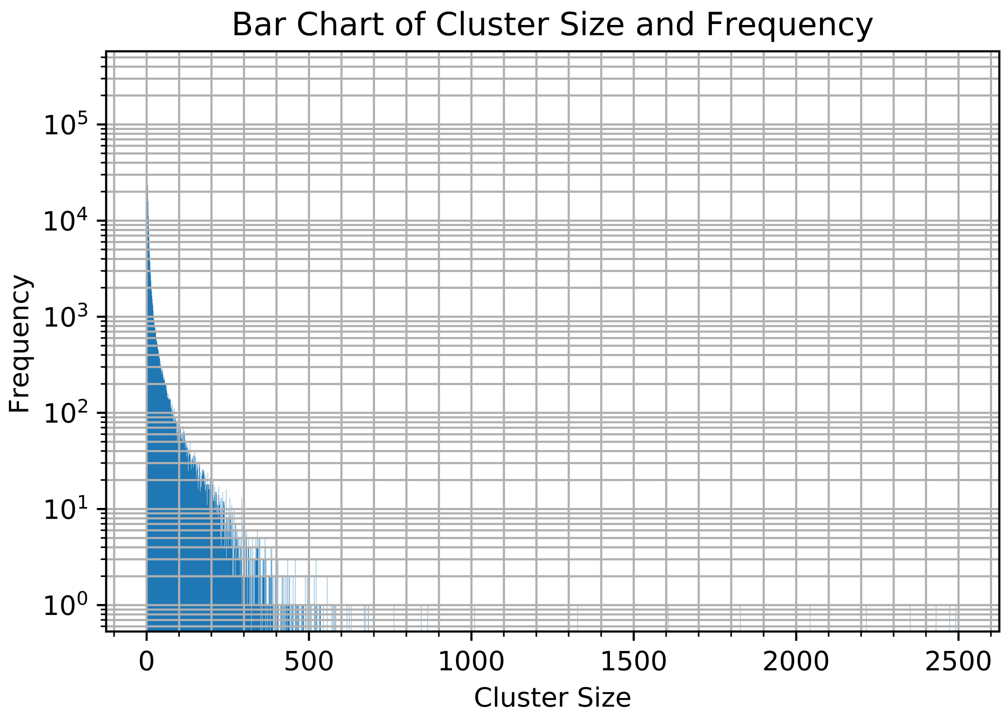


Figure 4: Cluster size and frequency in a 50x50 grid

As seen in figure 4, there is what appears to be a power-law decay in the frequency as cluster size increases following equation 2.

By taking the log of both sides of this equation, equation 2 becomes,

$$log(N) = log(bs^{-\alpha}) \tag{3}$$

$$log(N) = log(b) + log(s^{-\alpha}) \tag{4}$$

$$log(N) = -\alpha log(s) + log(b) \tag{5}$$

which is just a straight line of gradient $-\alpha$. α was then determined using the scipy module's leastsq function. Below is the log10-log10 plot of cluster size and frequency.

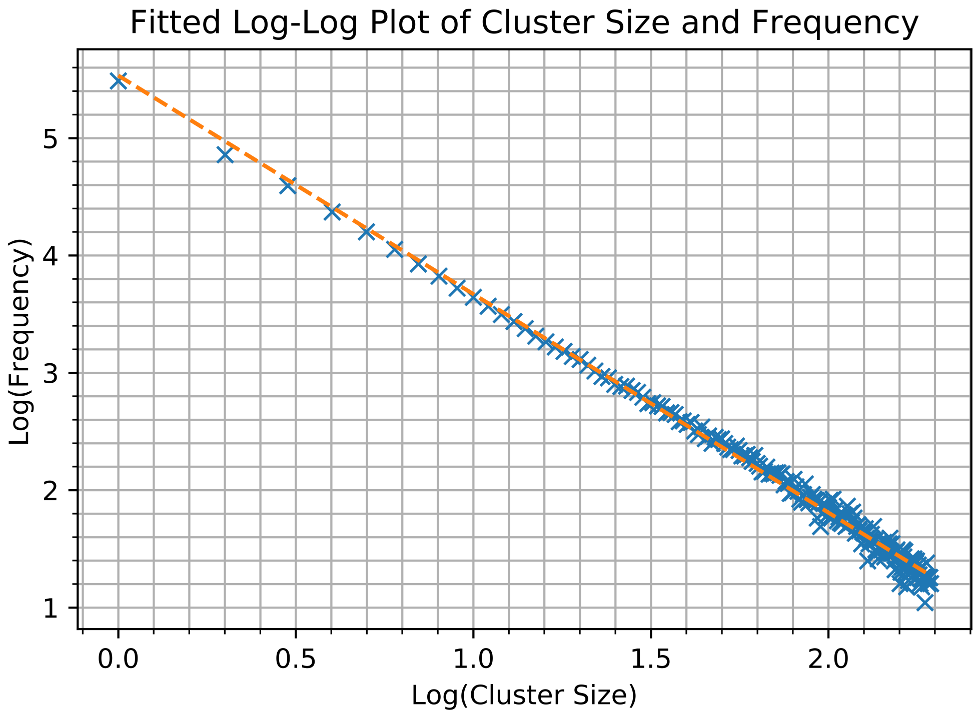


Figure 5: Log-Log plot of cluster size and frequency for a 50x50 grid

A value for α was obtained of $\alpha = 1.85 \pm 0.01$ and a reduced chi-squared value of $\chi_v^2 = 0.003$. Since the reduced chi-squared value is less than 1 this indicates a good fit. The low error in α of 0.54% shows how the cluster size distribution clearly follows the power-distribution in equation 2. This supports the self-organised criticality of the Drossel-Scwabl forest fire model as the cluster size exhibits critical behaviour.

5.2 Effect of Changing f

The simulation was run again for 3000 iterations of a 50x50 forest with $p = 0.1$ and f varying from 0.9 to 1×10^{-7}. The results are shown in the table and figures below.

Table 1: Effect of Changing f on α and ρ

theta	f	α	ρ	X_v^2	RSD
$1.1\text{x}10^{-1}$	$9\text{x}10^{-1}$	4.7± 0.5	0.09	0.05	6.6
1	$1\text{x}10^{-1}$	3.4± 0.2	0.23	0.04	5.1
$1\text{x}10^{2}$	$1\text{x}10^{-2}$	2.31± 0.04	0.35	0.01	6.3
$1\text{x}10^{3}$	$1\text{x}10^{-3}$	1.85± 0.01	0.42	0.003	6.8
$1\text{x}10^{4}$	$1\text{x}10^{-4}$	1.78±0.01	0.43	0.003	6.8
$1\text{x}10^{5}$	$1\text{x}10^{-5}$	1.79±0.01	0.44	0.003	7.0
$1\text{x}10^{6}$	$1\text{x}10^{-6}$	1.86±0.01	0.41	0.003	7.1
$1\text{x}10^{7}$	$1\text{x}10^{-7}$	2.00±0.07	0.99	0.006	3.1

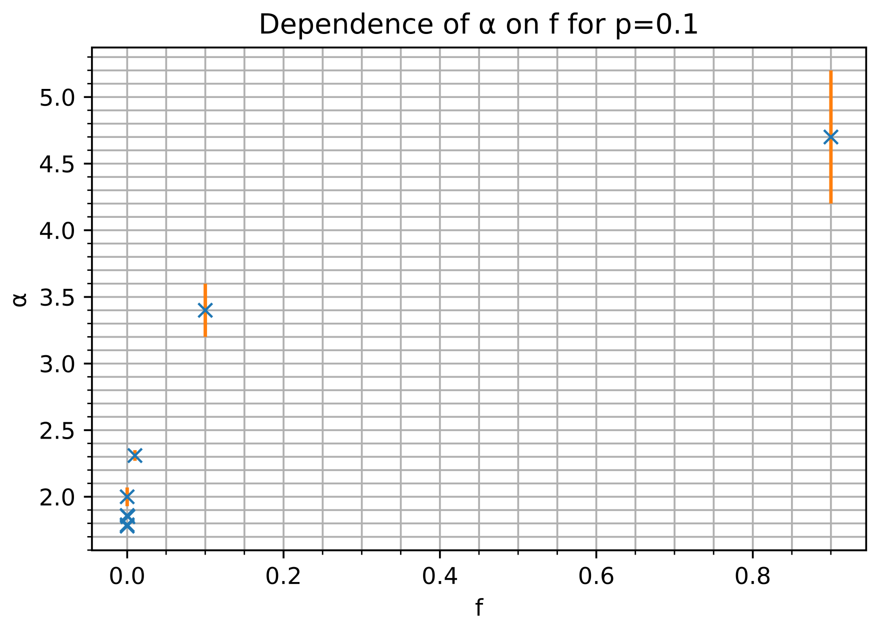


Figure 6: Plot of α against log(f)

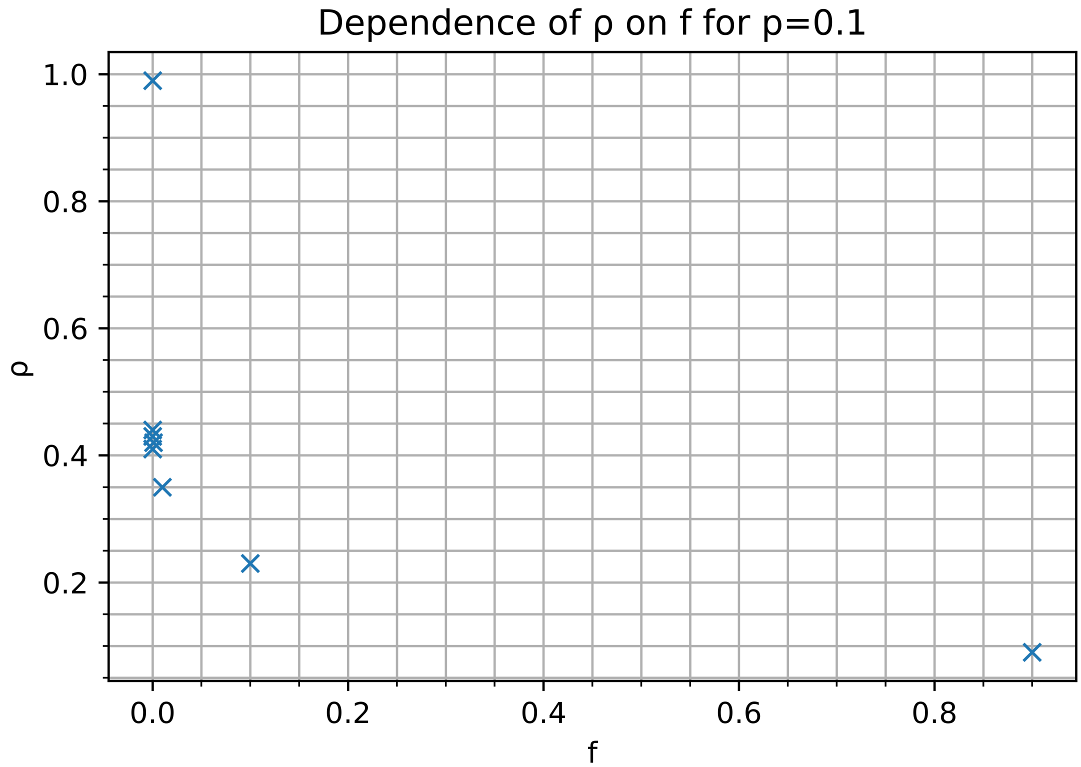


Figure 7: Plot of ρ against log(f)

As seen from figure 6, α follows a power law distribution. α increases as f approaches and becomes greater than p since fewer larger clusters are able to form due more frequent fires resulting in cluster frequency decreasing more rapidly with increasing cluster size. Stability is seen from f= 10^{-3} to 10^{-6} with α remaining within 2.2% of the mean. When $f << p$, at f=10^{-7}, α increases perhaps due to the grid being mostly full of trees so a smaller variety of cluster sizes can form.

There is a region of stability seen for α from f= 10^{-3} to 10^{-6} with α remaining within 2.2% of the mean value in this region despite f changing by many orders of magnitude. This supports the scale invariance of α and the system. As f increases from 10^{-4}, α increases since more frequent fires means smaller clusters form, so the frequency of clusters decreases more rapidly with cluster size.

As f decreases past 10^{-4}, α again increases since the forest is more full due to less fires so a smaller variety of cluster sizes form. These trends occur the fastest outside the region of stability.

As seen from figure 7, ρ follows a power law distribution with f, similar to that seen for the cluster size in equation 2. There is a region of stability from f= 10^{-3} to 10^{-6} with ρ remaining within 3.5 of the mean. This indicates scale invariance of ρ as ρ remains mostly constant despite the value of f changing by 3 orders of magnitude.

At values of f greater than 10^{-3}, the tree density rapidly decreases as the frequency of fires increases and limits the re-growth of the trees. For $f > p$, ρ approaches 0 as fires stop all but a few small clusters from forming.

At values of f less than 10^{-6}, the tree density rapidly increases since fires become very rare and the forest is able to completely fill with trees. Critical behaviour is lost here since the number of trees remains constant with small, rare dips due to fires that briefly lower the number of trees before the forest quickly becomes full again. This is shown in the figure below.

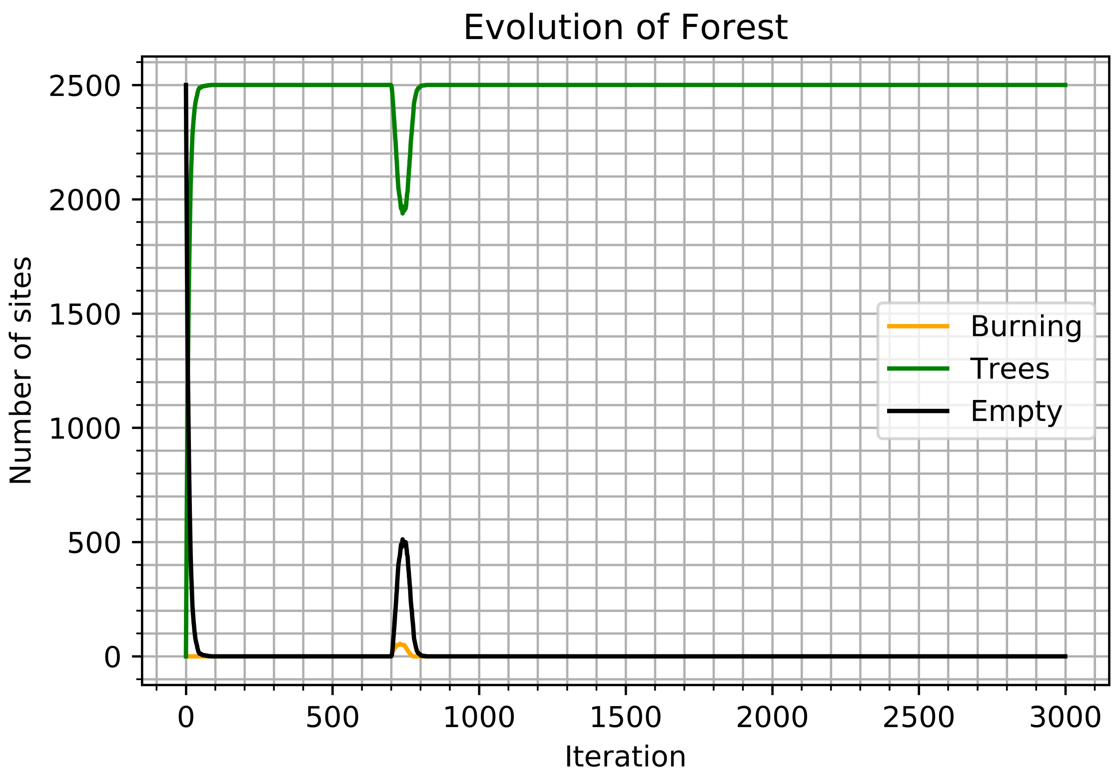


Figure 8: Evolution of the forest for f=1×10^{-7}

5.3 Effect of Changing number of iterations

The simulation was then run for a 50x50 forest with p = 0.1, f=0.001, and the number of iterations varying from 50 to 100 000. The results are shown in the table and figures below.

Table 2: Effect of Changing the number of iterations on α and ρ

Iterations	α	ρ	X_v^2	RSD
50	1.88±0.04	0.46	0.003	11.4
1.50×10^2	1.80± 0.03	0.43	0.005	9.9
5.00×10^2	1.77± 0.02	0.41	0.003	6.7
1.50×10^3	1.81 ± 0.01	0.42	0.002	7.2
3.00×10^3	1.86± 0.01	0.41	0.004	6.4
1.0000×10^4	1.86 ± 0.01	0.42	0.003	6.6
3.0000×10^5	2.16±0.02	0.41	0.01	6.7
1.00000×10^5	2.36±0.02	0.41	0.02	6.7

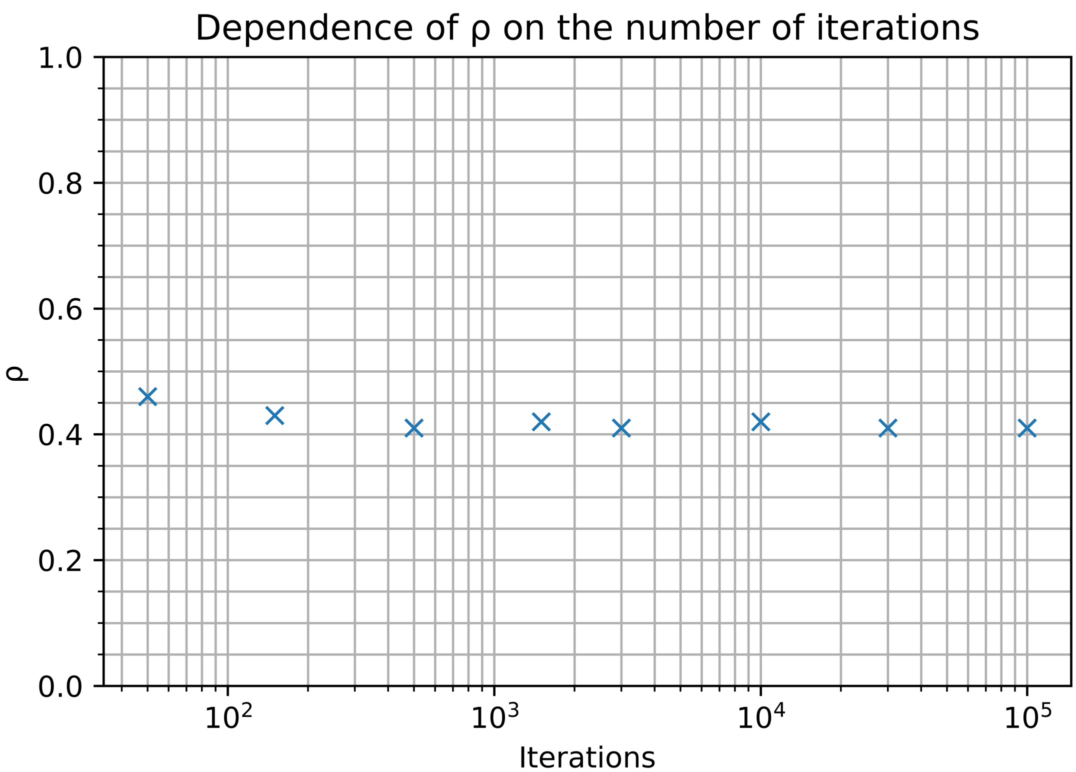


Figure 9: Plot of ρ against iterations

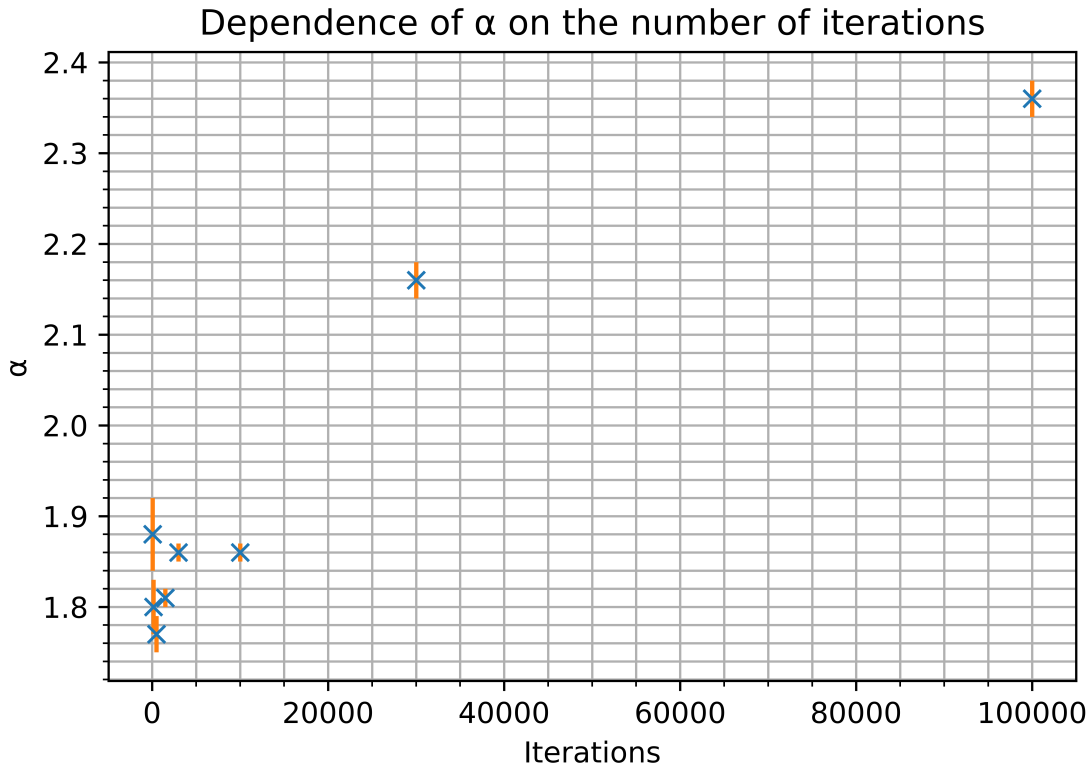


Figure 10: Plot of α against iterations

Figure 9 shows strong support for the scale invariance of ρ as ρ remained largely constant despite the significant range in the number of iterations. The mean value was found to be $\rho = 0.45$ with a very small relative standard deviation of 3.8% considering the number of iterations ranged by 5 orders of magnitude.

As seen from figure 10, α increases with the number of iterations perhaps due to the system being in the steady-state for a higher proportion of the time so there are more smaller size clusters than the larger clusters which are more common when the system is not in the steady-state.

5.4 Effect of Changing the Forest Size

The simulation was then repeated for different forest sizes with p=0.1, f=0.001, and the number of iterations fixed at 3000.

Table 3: Effect of Changing the Forest Size

Forest Size	α	ρ	X_v^2	RSD
10x10	1.19 ± 0.04	0.55	0.019	36.2
25x25	1.75 ± 0.02	0.44	0.004	13.7
50x50	1.86 ± 0.01	0.42	0.003	6.6
200x200	2.19±0.02	0.40	0.01	3.5

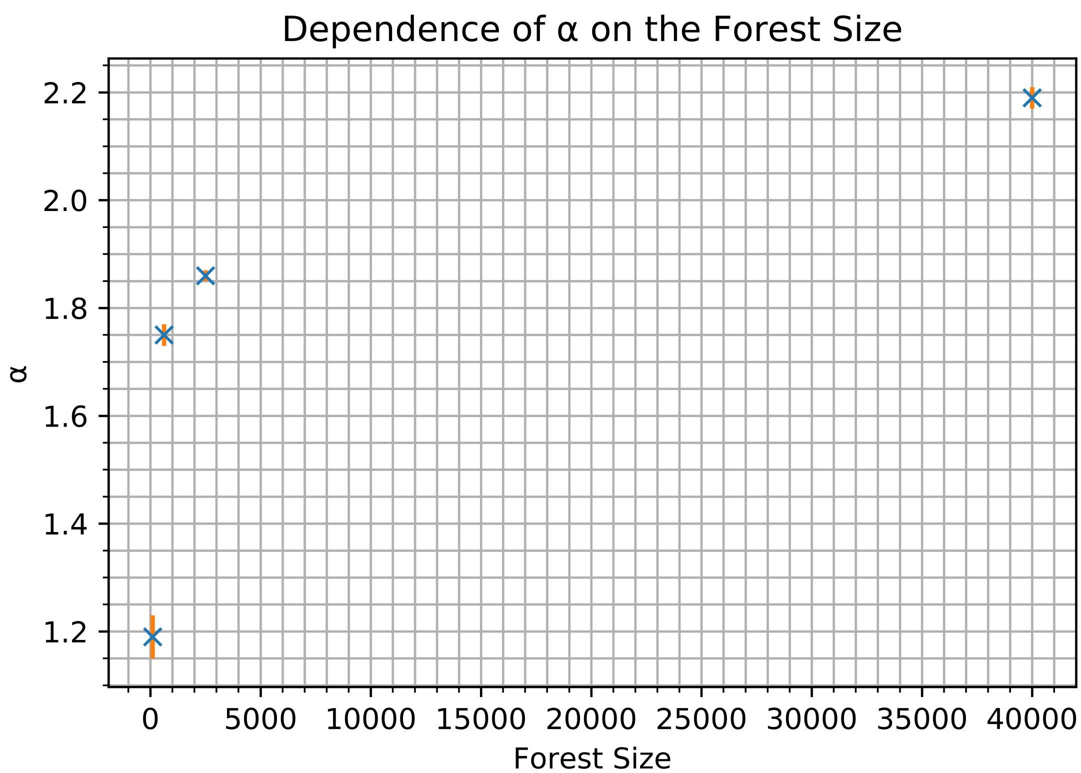


Figure 11: Plot of α against Forest Size

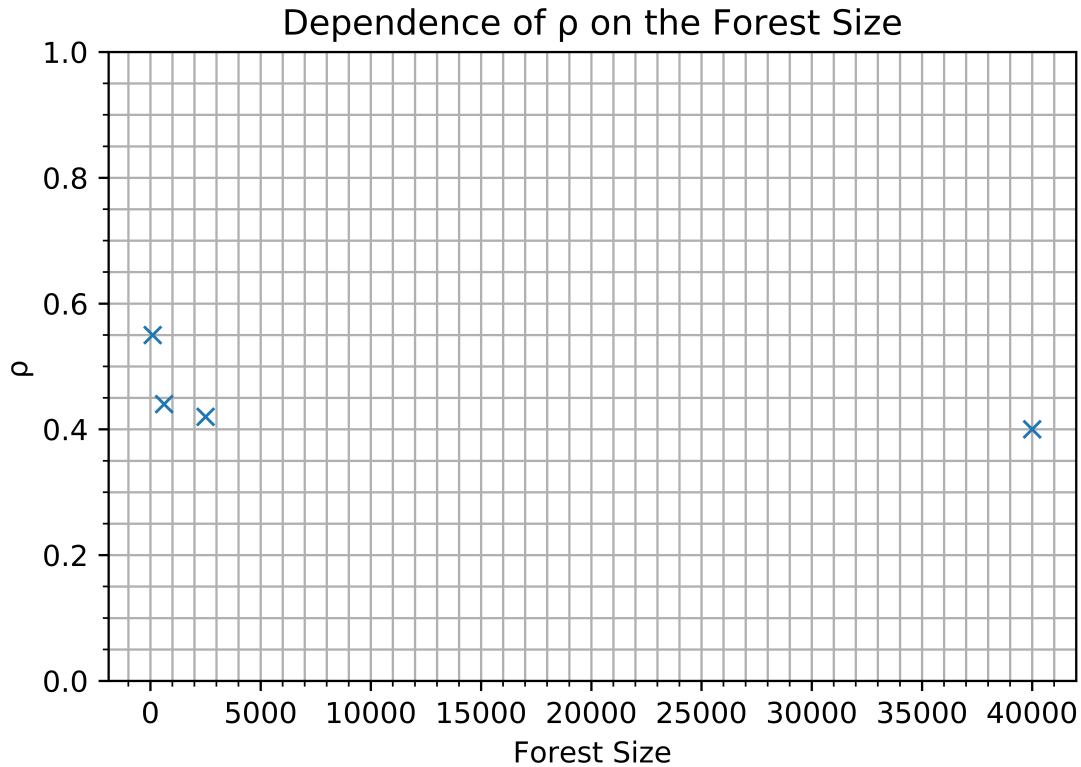


Figure 12: Plot of ρ against Forest Size

Figure 12 shows that ρ decreases slightly with the forest size. Perhaps due to the clusters being larger for larger forests meaning that fires reduce the number of trees more. This decrease is quite small for 25x25 to 200x200 forests with ρ remaining within 4.8% of the mean of 0.42. For a 10x10 forest ρ increases and the relative standard deviation increases significantly to 36.2%. This indicates large fluctuations in the number of trees which means that self-organised criticality is not seen, as shown by the figure below.

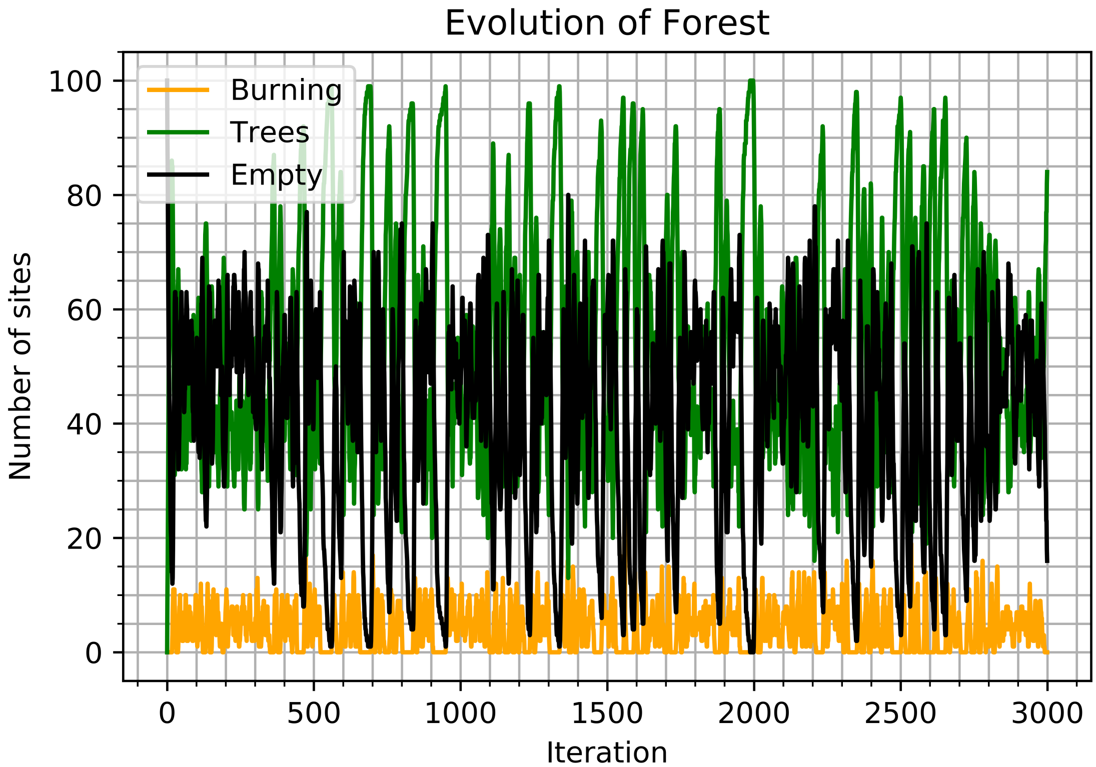


Figure 13: Evolution of a 10x10 Forest

This puts a rough minimum limit on the size of the forest needed for self-organised criticality of 25x25.

As seen in figure 11, α increases with forest size but at a decreasing rate. This is perhaps due to there being a lower proportion of sites on the edge. For a 10x10 forest there are 40 sites on the edge, corresponding to 40%. For a 200x200 forest there are 800 sites on the edge corresponding to 2%. It is easier for large clusters to form on the edges as they have fewer neighbours so are less likely to catch on fire from fire spreading than other clusters. This means there are relatively fewer larger clusters for a 200x200 forest and hence a larger value of α.

Investigations into the effect of changing the forest size were limited due to the efficiency of the simulation and processing power of the computer used. More efficient simulations on a more powerful computer would allow larger forest sizes to be used to see if α plateau's and investigate the behaviour of the forest in the high limit.

6 Conclusions

The Drossel-Schwabl Forest Fire model was shown by the simulation to partially exhibit self-organised criticality and scale invariance. The presence of a steady-state equilibrium where the number of trees did not vary much was seen in section 5.1, where the relative standard deviation was found to be 7.1%. The tree density in this state was found to be $\rho = 0.42$ which is comparable to the value found in previous studies of $\rho = 0.41$[2]. ρ showed scale invariance as it remained within 5% of the mean while the number of iterations was changed from 50 to 100 000, the value of f was changed from 10^{-3} to 10^{-6} and the forest size was changed from 25x25 to 200x200. The relative standard deviation of trees in the steady-state remained mostly below 10% indicating marginal stability and self-organised criticality in the number of trees.

Further evidence of self-organised criticality was seen by the distribution of frequency for different cluster sizes that followed the power-law distribution in equation 2. α was found in section 5.1 to be $\alpha = 1.85 \pm 0.01$.

The value of α was affected quite significantly by changing the parameter values. α increased as the number of iterations and forest size continued to increase towards infinity. α showed some stability for f ranging from 10^{-3} to 10^{-6} with a variation of only 2.2% from the mean.

By investigating the behaviour of the Drossel-Schwabl Forest Fire model further by varying other parameters it was seen that the model does exhibit true self-organised criticality or scale invariance as the parameter values have to be within certain ranges, summarised below.

1. f from 1×10^{-3} to 1×10^{-6}, corresponding to θ from 100 to 100 000
2. Number of iterations of at least 50
3. Forest Size of at least 25x25

The main aspect to be investigated further is the behaviour of α in the limit as the forest size approaches infinity and the number of iterations approaches infinity. Both of these were limited in this investigation due to constraints of the processing power of the computer available and the efficiency of the simulation. While the Hoshen-Kopelmann algorithim was improved to be more efficient, the labelling and counting of clusters remained the most time-consuming task. Investigating α in more detail would help determine if α reaches a limit or continues to increase with forest size and the number of iterations.

7 Acknowledgements

Thanks to my supervisors Alan Watson and Jurgen Thomas for overseeing the project and offering lots of helpful suggestions along the way

8 References

References

[1] H.J.Jensen, Self-organized criticality : emergent complex behaviour in physical and biological systems, CUP 1998

[2] Siegfried Clar et al 1996 J. Phys.: Condens. Matter 8 6803 Available from: https://iopscience.iop.org/article/10.1088/0953-8984/8/37/004/pdf/

[3] Siegfried Clar et al 1996 J. Phys.: Condens. Matter 8 6803 Available from: https://iopscience.iop.org/article/10.1088/0953-8984/8/37/004/pdf/

9 Appendices

A Clustering Test Video

https://1drv.ms/v/s!AvsCUj1LGvgN1menQCI9iE9l6CB4?e=ejaid9

B Main Simulation Code

https://1drv.ms/u/s!AvsCUj1LGvgN2DGzNWkVC647a1QN?e=No2THe

C Main Simulation Data Analysis Code

https://1drv.ms/u/s!AvsCUj1LGvgN2DL0YY5cfY9NC$_f$4?$e = gxSlG$6

D Other Code for Plots

https://1drv.ms/u/s!AvsCUj1LGvgN2hxnUNtrm4tMCUtg?e=EfIMql

E Data

https://1drv.ms/u/s!AvsCUj1LGvgN2DB6lAXJNshM7UoF?e=Vs3pop