

# Bayesian Networks and Proof-Nets: the proof-theory of Bayesian Inference

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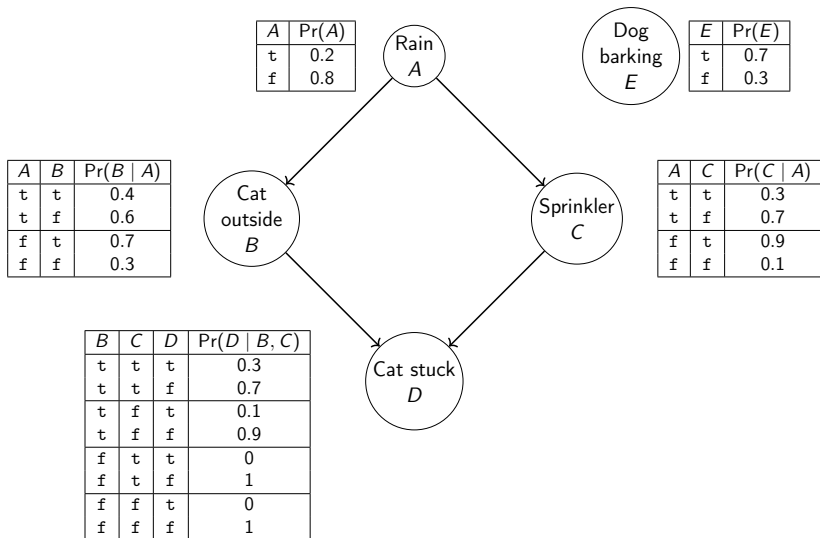


# Plan

- ▶ Bayesian Networks
- ▶ Bayesian Networks in Proof Nets of Linear Logic
- ▶ Example of Graphical Reasoning

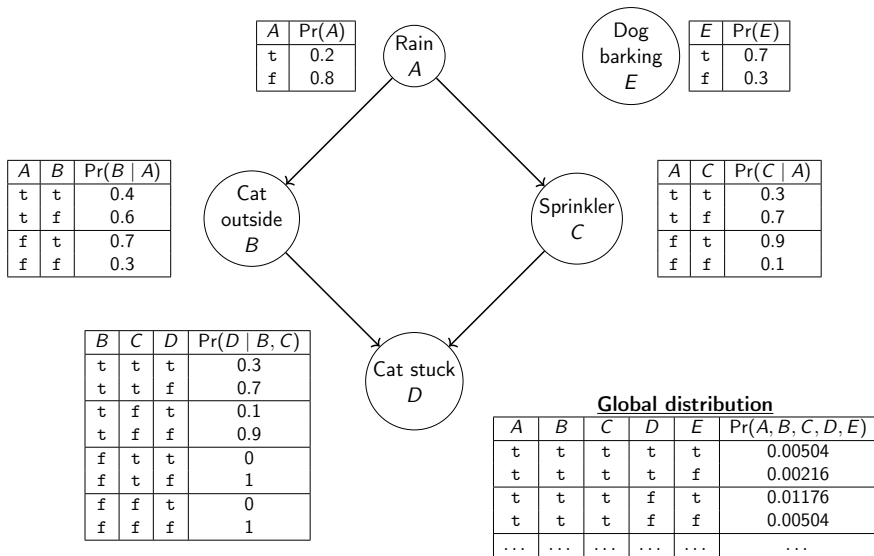
# Bayesian Networks

Bayesian Network = DAG + a (conditional) probability per vertex



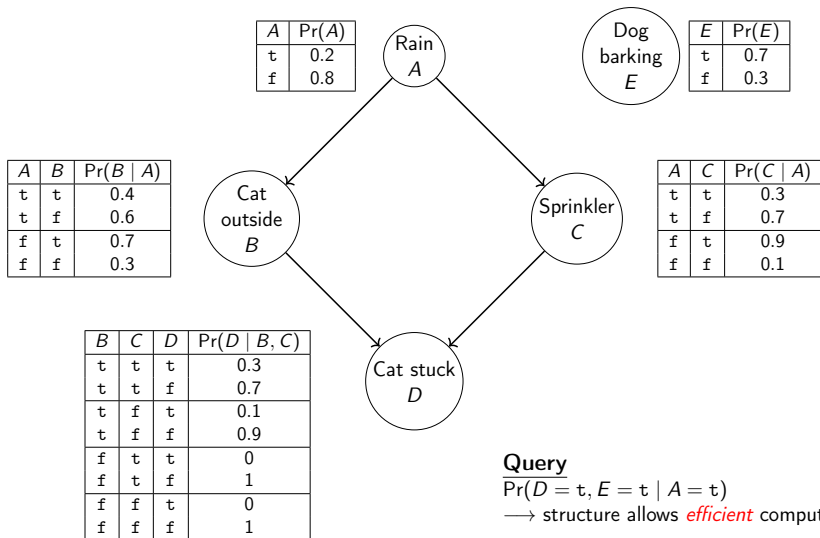
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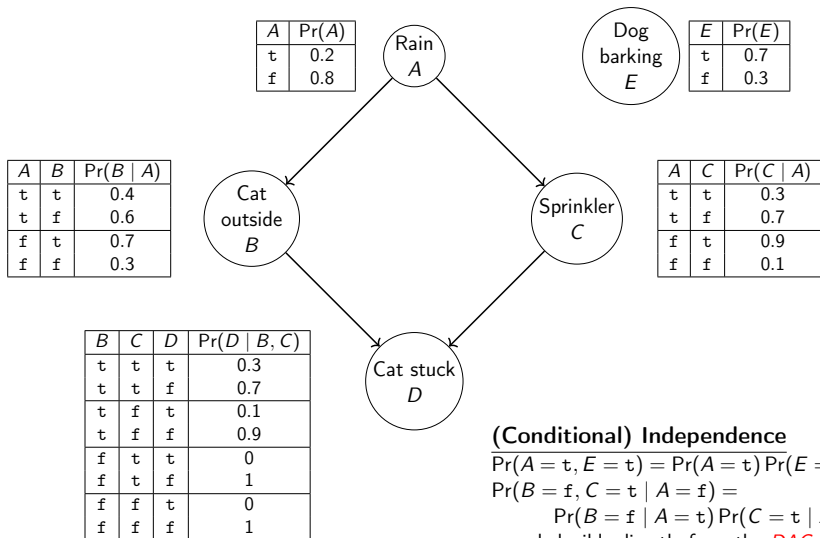
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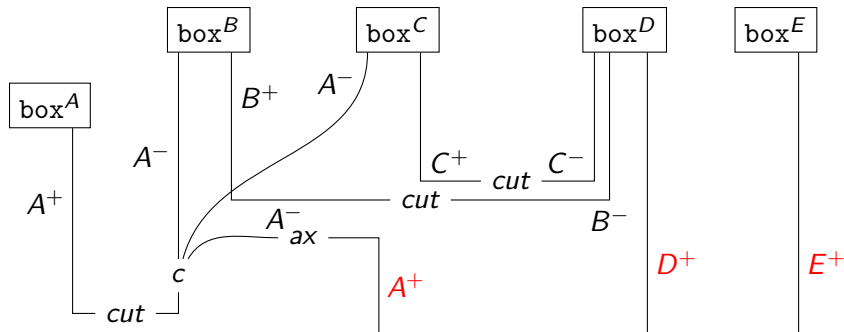
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# Bayesian Proof Nets

Bayesian Networks can be embedded in Proof Nets

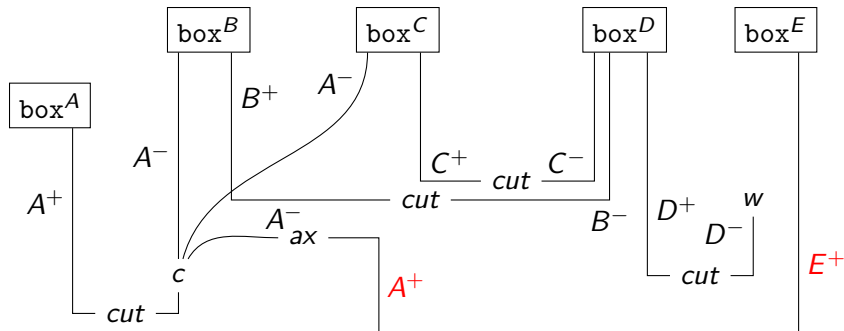


**Query** in the syntax = labels of the pending edges

→ here the semantics of the proof net  $P$  is  $\llbracket P \rrbracket = \Pr(A, D, E)$

# Bayesian Proof Nets

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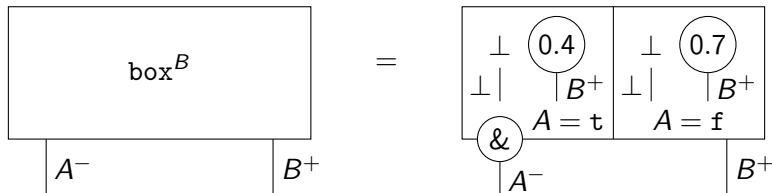
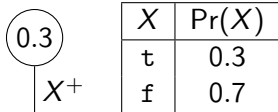
→ here the semantics of the proof net  $P$  is  $\llbracket P \rrbracket = \Pr(A, E)$



# What is inside a box?

Secretly, variables  $A^+, B^+, \dots$  are not atoms but **booleans** ( $= 1 \oplus 1$ ).

A box = **Bernoullis** + “if ... then ... else ...” ( $= \&$ ).



A	B	Pr(B   A)
t	t	0.4
t	f	0.6
f	t	0.7
f	f	0.3

# Conditional Independence Graphically

## Definition

Variables  $X$  and  $Y$  are **conditionally independent** given  $Z$  if

$$\Pr(X, Y \mid Z) = \Pr(X \mid Z) \Pr(Y \mid Z).$$

## Definition

In a Bayesian Proof Net,  $X$  and  $Y$  are **disconnected** by  $Z$  if there is no path between  $\text{box}^X$  and  $\text{box}^Y$  once removing all edges labeled by  $Z$ .

## Theorem

*If  $X$  and  $Y$  are disconnected by  $Z$  then  $X$  and  $Y$  are conditionally independent given  $Z$  in the associated distribution.*

## Graphical Proof

$$\Pr(X, Y, Z) = \langle \begin{array}{c} \text{---} R \text{---} \\ | \quad | \quad | \\ X \quad Z \quad Z \\ | \quad | \quad | \\ Z \quad \text{cut} \quad Z \\ | \quad | \quad | \\ Z \quad Y \end{array} \rangle = \langle \begin{array}{c} \text{---} R \text{---} \\ | \quad | \quad | \\ X^+ Z^+ Z^+ Z^+ \end{array} \rangle \langle \begin{array}{c} \text{---} S \text{---} \\ | \quad | \quad | \\ Z^- Z^- Z^- Y^+ \end{array} \rangle$$

$$\Pr(X,Z) = \left( \begin{array}{c} R \\ | - + | - + | \\ X^+ Z^+ Z^- Z^+ \end{array} \right) \left( \begin{array}{c} S \\ | - + | - + | \\ Z^- Z^- Z^+ \end{array} \right) \left[ Y_{cut}^+ Y_{cut}^- \right]$$

$$\Pr(Y,Z) = \left( \begin{array}{c} w \\ \boxed{\text{X}_{cut}^- \text{X}_+^+} \\ Z^+ Z^- Z^+ \end{array} \right) \left( \begin{array}{c} R \\ \text{---} \\ Z^- Z^- Z^+ Y^+ \end{array} \right) \left( \begin{array}{c} S \\ \text{---} \\ Z^- Z^- Z^+ Y^+ \end{array} \right)$$

$$\Pr(Z) = \left( \begin{array}{c} w \\ \boxed{\text{X}_{cut}}^+ \end{array} \begin{array}{c} R \\ Z^+Z^-Z^+ \end{array} \right) \left( \begin{array}{c} S \\ Z^-Z^-Z^+ \end{array} \begin{array}{c} Y^+ \\ \boxed{\text{Y}_{cut}}^- \end{array} \begin{array}{c} w \\ Y^- \end{array} \right)$$

## Graphical Proof

$$\Pr(X, Y, Z) = \langle \begin{array}{c} \text{---} R \text{---} \\ | \quad | \quad | \\ X^+ \quad Z^- \quad Z^- \\ | \quad | \quad | \\ Z^- \quad \text{cut} \quad Z^- \\ | \quad | \quad | \\ Z^- \quad Z^- \quad Y^+ \end{array} \rangle = \langle \begin{array}{c} \text{---} R \text{---} \\ | \quad | \quad | \\ X^+ \quad Z^+ \quad Z^+ \end{array} \rangle \langle \begin{array}{c} \text{---} S \text{---} \\ | \quad | \quad | \\ Z^- \quad Z^- \quad Y^+ \end{array} \rangle$$

$$\Pr(X, Z) = \left( \begin{array}{c} \text{---} R \text{---} \\ | \quad | \quad | \quad | \\ X^+ Z^+ Z^+ Z^+ \end{array} \right) \left( \begin{array}{c} \text{---} S \text{---} \\ | \quad | \quad | \quad | \\ Z^- Z^- Z^+ \quad \boxed{Y^+ \quad Y^-}_{cut} \end{array} \right)$$

$$\Pr(Y, Z) = \left( \begin{array}{c} w \\ \boxed{X^- \quad X^+} \\ \text{cut} \end{array} \right) \begin{array}{c} \text{---} R \text{---} \\ Z^+ \quad Z^- \quad Z^+ \end{array} \left( \begin{array}{c} \text{---} S \text{---} \\ Z^- \quad Z^- \quad Z^+ \quad Y^+ \end{array} \right)$$

$$\Pr(Z) = \left( \begin{array}{c} w \\ \text{cut} \end{array} X^+ \right) \begin{array}{c} R \\ Z^+ Z^- Z^+ \end{array} \left( \begin{array}{c} S \\ Z^- Z^- Z^+ \end{array} Y^+ \text{cut} \right)^w$$

$$\Pr(X \mid Z) \Pr(Y \mid Z) = \frac{\Pr(X, Z)}{\Pr(Z)} \frac{\Pr(Y, Z)}{\Pr(Z)}$$

# Graphical Proof

$$\begin{aligned}
 \Pr(X, Y, Z) &= \langle \langle \text{Diagram 1} \rangle \rangle = \langle \langle \text{Diagram 2} \rangle \rangle \\
 \Pr(X, Z) &= \langle \langle \text{Diagram 3} \rangle \rangle \\
 \Pr(Y, Z) &= \langle \langle \text{Diagram 4} \rangle \rangle \\
 \Pr(Z) &= \langle \langle \text{Diagram 5} \rangle \rangle
 \end{aligned}$$

The diagrams are graphical models (Bayesian networks) with nodes  $X, Y, Z$  and plates  $R$  and  $S$ .  
 Diagram 1:  $X$  and  $Z$  are in plate  $R$ ;  $Z$  and  $Y$  are in plate  $S$ . Edges:  $X \rightarrow Z$ ,  $Z \rightarrow Y$ . A "cut" is indicated between  $Z$  in  $R$  and  $Z$  in  $S$ .  
 Diagram 2: Same as Diagram 1, but with different plate labels:  $X^+ Z^+ Z^+ Z^+$  for  $R$  and  $Z^- Z^- Z^+ Y^+$  for  $S$ .  
 Diagram 3: Same as Diagram 1, but with different plate labels:  $X^+ Z^+ Z^+ Z^+$  for  $R$  and  $Z^- Z^- Z^+ [Y^+ \text{ cut } Y^-^w]$  for  $S$ .  
 Diagram 4: Same as Diagram 1, but with different plate labels:  $[X^- \text{ cut } X^+]^w Z^+ Z^- Z^+$  for  $R$  and  $Z^- Z^- Z^+ Y^+$  for  $S$ .  
 Diagram 5: Same as Diagram 1, but with different plate labels:  $[X^- \text{ cut } X^+]^w Z^+ Z^- Z^+$  for  $R$  and  $Z^- Z^- Z^+ [Y^+ \text{ cut } Y^-^w]$  for  $S$ .

$$\begin{aligned}
 \Pr(X | Z) \Pr(Y | Z) &= \frac{\Pr(X, Z)}{\Pr(Z)} \frac{\Pr(Y, Z)}{\Pr(Z)} \\
 &= \frac{\langle \langle \text{Diagram 6} \rangle \rangle \langle \langle \text{Diagram 7} \rangle \rangle}{\langle \langle \text{Diagram 8} \rangle \rangle \langle \langle \text{Diagram 9} \rangle \rangle}
 \end{aligned}$$

The diagrams in the fraction are graphical models with nodes  $X, Y, Z$  and plates  $R$  and  $S$ .  
 Diagram 6:  $X$  and  $Z$  are in plate  $R$ ;  $Z$  and  $Y$  are in plate  $S$ . Edges:  $X \rightarrow Z$ ,  $Z \rightarrow Y$ . A "cut" is indicated between  $Z$  in  $R$  and  $Z$  in  $S$ .  
 Diagram 7: Same as Diagram 6, but with different plate labels:  $X^+ Z^+ Z^+ Z^+$  for  $R$  and  $Z^- Z^- Z^+ [Y^+ \text{ cut } Y^-^w]$  for  $S$ .  
 Diagram 8: Same as Diagram 6, but with different plate labels:  $[X^- \text{ cut } X^+]^w Z^+ Z^- Z^+$  for  $R$  and  $Z^- Z^- Z^+ Y^+$  for  $S$ .  
 Diagram 9: Same as Diagram 6, but with different plate labels:  $[X^- \text{ cut } X^+]^w Z^+ Z^- Z^+$  for  $R$  and  $Z^- Z^- Z^+ [Y^+ \text{ cut } Y^-^w]$  for  $S$ .

# Graphical Proof

$$\begin{aligned}
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 \end{aligned}$$

Diagram 1: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 2: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 3: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

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Diagram 5: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

$$\begin{aligned}
 \Pr(X | Z) \Pr(Y | Z) &= \frac{\Pr(X, Z)}{\Pr(Z)} \frac{\Pr(Y, Z)}{\Pr(Z)} \\
 &= \frac{\langle \langle \text{Diagram 6} \rangle \rangle \langle \langle \text{Diagram 7} \rangle \rangle}{\langle \langle \text{Diagram 8} \rangle \rangle \langle \langle \text{Diagram 9} \rangle \rangle} \\
 &= \frac{\langle \langle \text{Diagram 10} \rangle \rangle \langle \langle \text{Diagram 11} \rangle \rangle}{\langle \langle \text{Diagram 12} \rangle \rangle \langle \langle \text{Diagram 13} \rangle \rangle} = \frac{\Pr(X, Y, Z)}{\Pr(Z)} = \Pr(X, Y | Z)
 \end{aligned}$$

Diagram 6: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

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Diagram 8: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 9: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 10: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

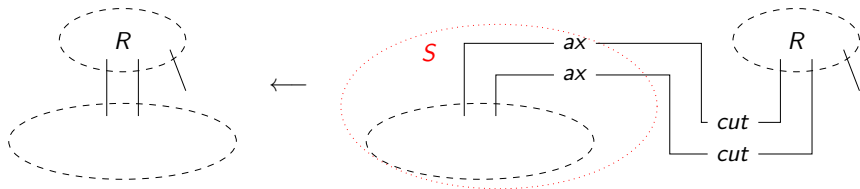
Diagram 11: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 12: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

Diagram 13: A causal graph with nodes X, Z, Y. X and Z are parents of Y. There is a directed edge from X to Z. A dashed oval labeled R encloses X and Z. A dashed oval labeled S encloses Z and Y. A 'cut' is indicated between Z and Y.

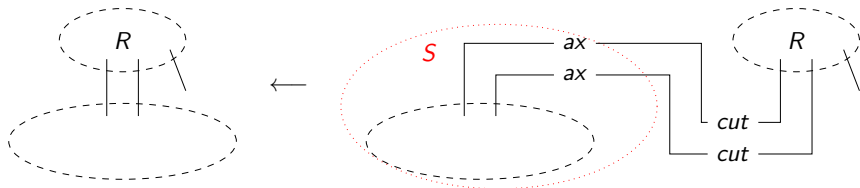
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- Can transfer **computations algorithms** (e.g. *Variable Elimination*) in Proof Nets, using the **usual rewriting rules** (cut-elimination).



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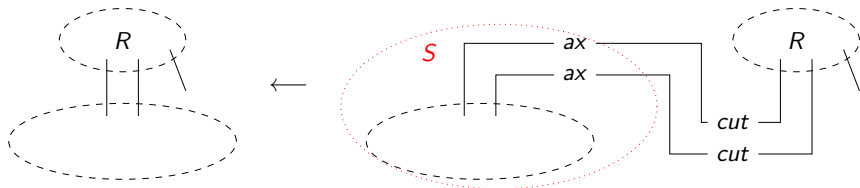


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# Thank you!