Confluence of Cut-elimination and Rules commutations in Linear Logic

Rémi Di Guardia

IRIF (CNRS, Université Paris Cité)

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Introduction

Study the **identity / equality** of proofs:

when are two proofs π and ρ equal?

By the Curry-Howard isomorphism similar to:

when are two λ-terms M and N equal?

$\rightsquigarrow$ **syntactic equality** is generally not enough!
We want (at least) equality up to cut-elimination / β-reduction

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Considered framework: **Linear Logic**
$\longrightarrow$ fine-grained enough for this problem to be relevant
(Classical Logic equates all proofs!)

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Motivations:

- instance of *when are two morphisms in a category equal?*
- relevant for **isomorphisms**: *when are two formulas A and B equal?*
- useful when looking for a **canonical** representative: proof-nets!

Plan

- **Equality of terms in λ-calculus**
- **Equality of proofs in Linear Logic**
  - Quick sketch of Linear Logic
  - Why is equality more complicated than in λ-calculus?

Simply typed λ-calculus

Terms

$$M, N \;:=\; x \mid \lambda x.M \mid M\ N$$

Types

$$A, B \;:=\; O \mid A \to B$$

β**-reduction**

$$(\lambda x.M)\ N \xrightarrow{\beta} M\{N/x\}$$

η**-expansion**

$$M \xrightarrow{\eta} (\lambda x.M\ x)$$

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Syntactic equality is usually not enough:

- **Church encoding:** $\underline{n} := \lambda f.\lambda x.\underbrace{f\ f\ \ldots\ f}_{n \text{ times}}\ x$

  $\underline{2} + \underline{2}$ should be *equivalent* to $\underline{2} + (\underline{1} + \underline{1})$
- **Quotient in category / denotational model:**
  $M =_{\beta\eta} N \implies [\![M]\!] = [\![N]\!]$

$\longrightarrow$ a useful notion of equality is up to computations $= \beta\eta$ equivalence

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Here: ***only on equality up to β-reduction*** to simplify

Checking equality of terms

Problem:

- $M =_\beta N$? Give a sequence of terms $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} N$
- $M \neq_\beta N$? Prove such a sequence cannot exist!

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Key results:

- β is **strongly normalizing**
  (no infinite sequence of reductions)
- β is **confluent**

$$\begin{tikzcd}
 & M \arrow[dl, "*"'] \arrow[dr, "*"] & \\
N_1 \arrow[dr, red, "*"'] & & N_2 \arrow[dl, red, "*"] \\
 & O &
\end{tikzcd}$$

Corollary

$$M =_\beta N \iff \beta(M) = \beta(N)$$

with $\beta(\cdot)$ *the unique normal form of the term*

Examples

$$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1}) \qquad \underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \neq \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$$

Plan

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  - Quick sketch of Linear Logic
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Linear Logic

Formulas

$$
\begin{array}{lll}
A, B \; := & \mid X \mid X^{\perp} & \textit{atom} \\
& \mid A \parr B \mid A \otimes B \mid \perp \mid 1 & \textit{multiplicative} \\
& \mid A \oplus B \mid A \mathbin{\&} B \mid 0 \mid \top & \textit{additive} \\
& \mid ?A \mid !A & \textit{exponential} \\
& \mid \forall X\ A \mid \exists X\ A & \textit{quantifier}
\end{array}
$$

Involutive Negation / Orthogonality

$$(X^{\perp})^{\perp} = X$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \quad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \quad \perp^{\perp} = 1 \quad 1^{\perp} = \perp$$

$$\ldots$$

Sub-systems

- MLL = atom + multiplicative
- MALL = atom + multiplicative + additive
- ...

16 Rules of Linear Logic

$$\frac{}{\vdash A^\perp, A}\ (ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

$$X \text{ not free in } \Gamma \quad \frac{\vdash A, \Gamma}{\vdash \forall X\ A, \Gamma}\ (\forall) \qquad \frac{\vdash AB/X, \Gamma}{\vdash \exists X\ A, \Gamma}\ (\exists)$$

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Curry-Howard isomorphism: β-reduction $\approx$ cut-elimination

Cut-elimination

Key steps (9) *"true" computations*

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \overset{\pi}{\vdash A,\Gamma}}{\vdash A,\Gamma}\,(cut) \quad\xrightarrow{\beta}\quad \overset{\pi}{\vdash A,\Gamma}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B^\perp, A^\perp,\Gamma}}{\vdash B^\perp \parr A^\perp,\Gamma}\,(\parr) \qquad \dfrac{\overset{\rho}{\vdash A,\Delta}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B,\Delta,\Sigma}\,(\otimes)}{\vdash \Gamma,\Delta,\Sigma}\,(cut) \quad\xrightarrow{\beta}\quad \dfrac{\dfrac{\overset{\pi}{\vdash B^\perp,A^\perp,\Gamma}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A^\perp,\Gamma,\Sigma}\,(cut)\qquad \overset{\rho}{\vdash A,\Delta}}{\vdash\Gamma,\Delta,\Sigma}\,(cut)$$

Commutative steps (15) *used to reach a key step*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, C,\Gamma}}{\vdash A^\perp, B \parr C,\Gamma}\,(\parr) \qquad \overset{\rho}{\vdash A,\Delta}}{\vdash B\parr C,\Gamma,\Delta}\,(cut) \quad\xrightarrow{\beta}\quad \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B,C,\Gamma}\quad \overset{\rho}{\vdash A,\Delta}}{\vdash B,C,\Gamma,\Delta}\,(cut)}{\vdash B\parr C,\Gamma,\Delta}\,(\parr)$$

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Cut-elimination on an example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash B, B^\perp$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash A^\perp, A \otimes B, B^\perp$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash A, A^\perp \otimes C^\perp, C$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C$}
\end{prooftree}
$$

$$\xrightarrow{\beta_{com}}$$

$$
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$$

$$\xrightarrow{\beta_{key}}$$

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- Cut-elimination is **strongly normalizing**?

- Cut-elimination is **confluent**?

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  Almost: an infinite reduction has an infinite suffix made only of

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- Cut-elimination is **confluent**?

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- Cut-elimination is **confluent**?

Not at all!

Cut-elimination is not confluent!

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A\otimes B, A^\perp, B^\perp}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A^\perp\otimes C^\perp, A, C}\,(\otimes)}{\vdash A\otimes B, A^\perp\otimes C^\perp, B^\perp, C}\,(cut)$$

$$\swarrow \beta^* \qquad\qquad \beta^* \searrow$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp\otimes C^\perp, C}\,(\otimes) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A\otimes B, A^\perp\otimes C^\perp, B^\perp, C}\,(\otimes) \quad \neq \quad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A\otimes B, A^\perp, B^\perp}\,(\otimes) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A\otimes B, A^\perp\otimes C^\perp, B^\perp, C}\,(\otimes)$$

Irreversible choice at the beginning:
first commutative case with the left $\otimes$-rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, . . .

Cut-elimination is not confluent!

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(\mathit{ax}) \qquad \dfrac{}{\vdash B^{\perp}, B}\,(\mathit{ax})}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(\mathit{ax}) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(\mathit{ax})}{\vdash A^{\perp} \otimes C^{\perp}, A, C}\,(\otimes)}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(\mathit{cut})$$

$$\swarrow_{\beta^*} \qquad\qquad \searrow_{\beta^*}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(\mathit{ax}) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(\mathit{ax})}{\vdash A, A^{\perp} \otimes C^{\perp}, C}\,(\otimes) \qquad \dfrac{}{\vdash B^{\perp}, B}\,(\mathit{ax})}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(\otimes) \quad \neq \quad \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(\mathit{ax}) \qquad \dfrac{}{\vdash B^{\perp}, B}\,(\mathit{ax})}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\,(\otimes) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(\mathit{ax})}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(\otimes)$$

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But confluence up to rule commutation!

Intuition: Confluence up to in distributivity

Exercices from junior high school: **distributivity** of $\times$ over $+$

$$a \times (b + c) \to (a \times b) + (a \times c)$$
$$(b + c) \times a \to (b \times a) + (c \times a)$$

Not confluent:

$$\begin{tikzcd}[column sep=tiny]
 & & & (a+b)\times(c+d) \arrow[dll] \arrow[drr] & & & \\
 & a\times(c+d)+b\times(c+d) \arrow[dl] \arrow[ddr] & & & & (a+b)\times c+(a+b)\times d \arrow[dl] \arrow[ddr] & \\
(a\times c+a\times d)+b\times(c+d) \arrow[ddr] & & & & (a\times c+b\times c)+(a+b)\times d \arrow[ddr] & & \\
 & & a\times(c+d)+(b\times c+b\times d) \arrow[dl] & & & & (a+b)\times c+(a\times d+b\times d) \arrow[dl] \\
 & (a\times c+a\times d)+(b\times c+b\times d) & & & & (a\times c+b\times c)+(a\times d+b\times d) &
\end{tikzcd}$$

But confluent up to associativity and commutativity of $+$

Rule commutations (from a list of cases)

$$\dfrac{\dfrac{\pi}{\vdash C, \Gamma} \quad \dfrac{\dfrac{\rho}{\vdash A, D, \Delta} \quad \dfrac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\pi}{\vdash C, \Gamma} \quad \dfrac{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \quad \dfrac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$$

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$$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top) \quad \mapsto \quad \dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_c)$$

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... and many many many more ...

Rule commutations (from a list of cases)

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. . . and many many many more . . .

Non-trivial: duplicates / merges sub-proofs

Rule commutations (from a list of cases)

$$\dfrac{\dfrac{}{\vdash C, \Gamma}\pi \qquad \dfrac{\dfrac{\rho}{\vdash A, D, \Delta} \quad \dfrac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes) \quad \longmapsto \quad \dfrac{\dfrac{\dfrac{\pi}{\vdash C, \Gamma} \quad \dfrac{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}(\otimes) \qquad \dfrac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes)$$

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... and many many many more ...

- Non-trivial: duplicates / merges sub-proofs
- Tricky: produces / deletes rules and sub-proofs

Rule commutations (from a general method)

Every pair $\begin{array}{l} \overline{\quad\quad}\ (s) \\ \overline{\quad\quad}\ (r) \end{array} \longmapsto \begin{array}{l} \overline{\quad\quad}\ (r) \\ \overline{\quad\quad}\ (s) \end{array}$ coming from:

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{l} \overline{\quad\quad}\ (r) \quad \overline{\quad\quad}\ (s) \\ \overline{\quad\quad\quad\quad\quad\quad\quad\quad}\ (cut) \end{array} \arrow[dl, "\beta^*_{com}"'] \arrow[dr, "\beta^*_{com}"] & \\
\begin{array}{c} \overline{\quad\quad\quad\quad\quad\quad}\ (cut) \\ \overline{\quad\quad}\ (s) \\ \overline{\quad\quad}\ (r) \end{array} & & \begin{array}{c} \overline{\quad\quad\quad\quad\quad\quad}\ (cut) \\ \overline{\quad\quad}\ (r) \\ \overline{\quad\quad}\ (s) \end{array}
\end{tikzcd}$$

$\approx \#|rules|^2$ commutations $\longrightarrow$ 93 equations in LL!

Rule commutations (from a general method)

Every pair $\dfrac{\dfrac{\qquad}{\qquad}\,(s)}{\qquad}\,(r) \quad \vdash\!\dashv \quad \dfrac{\dfrac{\qquad}{\qquad}\,(r)}{\qquad}\,(s)$ coming from:

$$\begin{tikzcd}[column sep=small]
 & \dfrac{\dfrac{\qquad}{}\,(r) \qquad \dfrac{\qquad}{}\,(s)}{\qquad\qquad}\,(cut) \arrow[dl, "\beta^*_{com}"'] \arrow[dr, "\beta^*_{com}"] & \\
\dfrac{\dfrac{\dfrac{}{\qquad}\,(s)}{\qquad}\,(r)}{\qquad\qquad}\,(cut) & & \dfrac{\dfrac{\dfrac{}{\qquad}\,(r)}{\qquad}\,(s)}{\qquad\qquad}\,(cut)
\end{tikzcd}$$

$\approx \#|rules|^2$ commutations $\longrightarrow$ 93 equations in LL!

Remarks

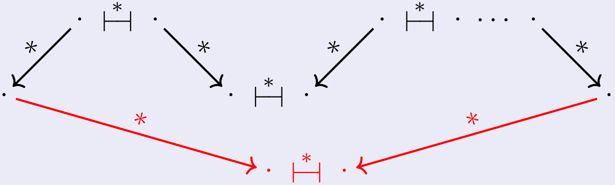
- $\vdash\!\dashv \; \subseteq \; =_\beta$
- $\vdash\!\dashv$ is the usual (cut-free) commutations without $! - ?_c$ and $! - ?_w$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad \not\vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \quad \text{and} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad \not\vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

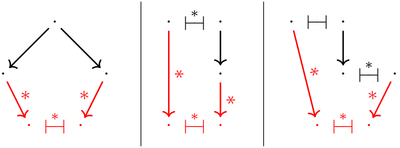
Proving Confluence up to

Definition: Church-Rosser modulo

$\rightarrow$ is **Church-Rosser modulo** an equivalence relation $\overset{*}{\vdash\!\dashv}$ when:



How to prove it? Several theorems in rewriting theory. Usual hypotheses:
strong normalization of $\rightarrow \cdot \overset{*}{\vdash\!\dashv}$ & **closing some diagrams**



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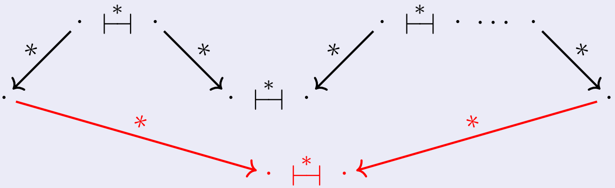

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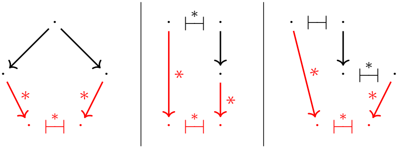

Difficulties:

- $\stackrel{*}{\vdash\!\dashv}$ is too difficult to manipulate, we prefer $\vdash\!\dashv$
- $\rightarrow \cdot \stackrel{*}{\vdash\!\dashv}$ is not strongly normalizing!

$\to \cdot \overset{*}{\vdash\!\dashv}$ is not strongly normalizing!

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (ax)}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{\scriptsize $(\parr)$}
\UnaryInfC{$\vdash X^\perp \parr X$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !(X^\perp \parr X)$}
\AxiomC{}
\RightLabel{\scriptsize $(\top)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash \top$}
\end{prooftree}
\quad \vdash\!\dashv \quad
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (ax)}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{\scriptsize $(\parr)$}
\UnaryInfC{$\vdash X^\perp \parr X$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !(X^\perp \parr X)$}
\AxiomC{}
\RightLabel{\scriptsize $(\top)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize $(?_c)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash \top$}
\end{prooftree}
$$

$$\vdash\!\dashv$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (ax)}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{\scriptsize $(\parr)$}
\UnaryInfC{$\vdash X^\perp \parr X$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !(X^\perp \parr X)$}
\AxiomC{}
\RightLabel{\scriptsize (ax)}
\UnaryInfC{$\vdash X^\perp, X$}
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\UnaryInfC{$\vdash X^\perp \parr X$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !(X^\perp \parr X)$}
\AxiomC{}
\RightLabel{\scriptsize $(\top)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize $(?_w)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash \top$}
\end{prooftree}
\quad \longleftarrow \quad
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (ax)}
\UnaryInfC{$\vdash X^\perp, X$}
\RightLabel{\scriptsize $(\parr)$}
\UnaryInfC{$\vdash X^\perp \parr X$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !(X^\perp \parr X)$}
\AxiomC{}
\RightLabel{\scriptsize $(\top)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize $(?_w)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize $(?_c)$}
\UnaryInfC{$\vdash ?(X^\perp \otimes X), \top$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash \top$}
\end{prooftree}
$$

We even have an infinity of *key* cut-elimination cases!

Idea

The problem comes from the production of rules / sub-proofs.

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*

Theorem 2.2 from [AT12]

Let $\vdash\!\dashv$, $\rightarrow$ *and* $\rightsquigarrow$ *be relations such that* $\vdash\!\dashv$ *is symmetric and* $\rightsquigarrow\ \subseteq\ \vdash\!\dashv$.
Suppose:

1. $\rightarrow \cdot \rightsquigarrow^{*}$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow\ \subseteq\ (\rightarrow \cup \rightsquigarrow)^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \leftsquigarrow)^{*}$
3. $\vdash\!\dashv \cdot \rightarrow\ \subseteq\ (\overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \leftsquigarrow)^{*}) \cup (\rightarrow \cdot (\rightarrow \cup \rightsquigarrow)^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \leftsquigarrow)^{*})$

Then $\rightarrow$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$.

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Suppose:

1. $\to \cdot \leadsto^{*}$ *is strongly normalizing*
2. $\leftarrow \cdot \to\ \subseteq\ (\to \cup \leadsto)^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \reflectbox{$\leadsto$})^{*}$
3. $\vdash\!\dashv \cdot \to\ \subseteq\ (\overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \reflectbox{$\leadsto$})^{*}) \cup (\to \cdot (\to \cup \leadsto)^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \reflectbox{$\leadsto$})^{*})$

Then $\to$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$.

Confluence up to rule commutation – SN

Proposition

Set $\rightsquigarrow$ the rule commutations without $\top$-commutations in the direction "creating rules", plus the cut-cut step of cut-elimination.
Then $\xrightarrow{\overline{\beta}} \cdot \rightsquigarrow^*$ is strongly normalizing, with $\xrightarrow{\overline{\beta}} = \ (\xrightarrow{\beta}$ without cut-cut).

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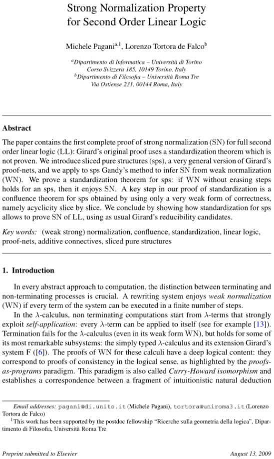

Strong Normalization Property
for Second Order Linear Logic

Michele Pagani[a,1], Lorenzo Tortora de Falco[b]

[a] *Dipartimento di Informatica – Università di Torino*
Corso Svizzera 185, 10149 Torino, Italy
[b] *Dipartimento di Filosofia – Università Roma Tre*
Via Ostiense 231, 00144 Roma, Italy

Abstract

The paper contains the first complete proof of strong normalization (SN) for full second order linear logic (LL): Girard's original proof uses a standardization theorem which is not proven. We introduce sliced pure structures (sps), a very general version of Girard's proof-nets, and we apply to sps Gandy's method to infer SN from weak normalization (WN). We prove a standardization theorem for sps: if WN without erasing steps holds for an sps, then it enjoys SN. A key step in our proof of standardization is a confluence theorem for sps obtained by using only a very weak form of correctness, namely acyclicity slice by slice. We conclude by showing how standardization for sps allows to prove SN of LL, using as usual Girard's reducibility candidates.

Key words: (weak strong) normalization, confluence, standardization, linear logic, proof-nets, additive connectives, sliced pure structures

1. Introduction

In every abstract approach to computation, the distinction between terminating and non-terminating processes is crucial. A rewriting system enjoys *weak normalization* (WN) if every term of the system can be executed in a finite number of steps.

In the λ-calculus, non terminating computations start from λ-terms that strongly exploit *self-application*: every λ-term can be applied to itself (see for example [13]). Termination fails for the λ-calculus (even in its weak form WN), but holds for some of its most remarkable subsystems: the simply typed λ-calculus and its extension Girard's system F ([6]). The proofs of WN for these calculi have a deep logical content: they correspond to proofs of consistency in the logical sense, as highlighted by the *proofs-as-programs* paradigm. This paradigm is also called *Curry-Howard isomorphism* and establishes a correspondence between a fragment of intuitionistic natural deduction

Email addresses: paganí@di.unito.it (Michele Pagani), tortora@uniroma3.it (Lorenzo Tortora de Falco)

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This paper *almost* do it [PT10].
"Just" check that **some additions at the start go through** the 61 pages of this technical proof using non-standard proof-nets!

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is ***Church-Rosser modulo*** *rule commutation.*

Theorem 2.2 from [AT12]

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3. $\vdash\!\dashv \cdot \rightarrow \ \subseteq \ (\overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \leftsquigarrow)^{*}) \cup (\rightarrow \cdot (\rightarrow \cup \rightsquigarrow)^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot (\leftarrow \cup \leftsquigarrow)^{*})$

Then $\rightarrow$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$.

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Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*

Proof.

$\#(\text{cut steps})^2$
$\approx \#|\mathit{rules}|^2$ cases

$$
\begin{tikzcd}
& \pi \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\
\pi_1 \arrow[d, dashed, "\beta^*"'] & & \pi_2 \arrow[d, dashed, "\beta^*"] \\
\pi'_1 \arrow[rr, phantom, "\overset{=}{\vdash\!\dashv}"] & & \pi'_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1 \arrow[r, phantom, "\vdash\!\dashv"] \arrow[dd, dashed, "\beta^+"] & \pi_2 \arrow[dr, "\beta"] & \\
& & \pi_3 \arrow[dl, dashed, "\beta^*"] \\
\pi'_1 \arrow[r, phantom, "\overset{*}{\vdash\!\dashv}"] & \pi'_2 &
\end{tikzcd}
$$

$\#(\text{cut steps}) \times$
$\#(\text{communtations})$
$\approx \#|\mathit{rules}|^3$ cases

□

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$$\begin{tikzcd}
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\pi'_1 \arrow[rr, phantom, "\overset{=}{\longmapsto\!\!\dashv}"] & & \pi'_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1 \arrow[r, phantom, "\longmapsto\!\!\dashv"] \arrow[d, dashed, red, "\beta^+"'] & \pi_2 \arrow[dr, "\beta"] & \\
\pi'_1 \arrow[r, phantom, "\overset{*}{\longmapsto\!\!\dashv}"] & \pi'_2 & \pi_3 \arrow[l, dashed, red, "\beta^*"']
\end{tikzcd}$$

$\#(\text{cut steps})\times$
$\#(\text{communtations})$
$\approx \#|\mathit{rules}|^3$ cases

□

Thousands of similar cases to check
$\longrightarrow$ horrible and tedious with pen and paper, better in a **proof assistant**!

But the **exchange** rule overcomplicates everything. . .

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*

Corollary: Equality on cut-free proofs

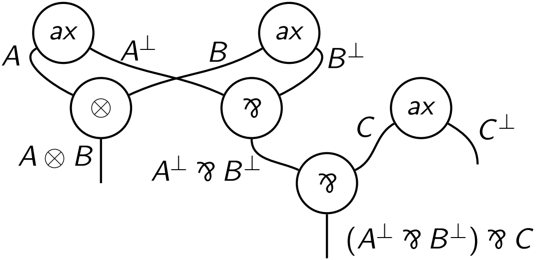
Between cut-free proofs, $=_\beta$ is exactly $\overset{*}{\vdash\!\!\!-\!\!\!\dashv}$.

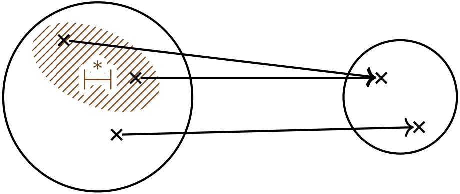
Consequences & Avail

- **"Bureaucracy":** have to order all rules, but some order does not matter and no canonical choice

Consequences & Avail

- **"Bureaucracy":** have to <u>order</u> all rules, but some order does not matter and no canonical choice
- **Proof-nets:** identify proofs <u>exactly</u> up to rule commutation $\overset{*}{\vdash\!\dashv}$
  - $\vdash\!\dashv$ is equality of graphs
  - cut-elimination is confluent and has only key steps
  - defined only in some sub-systems of LL





Proof Trees Proof-Nets

$\overset{*}{\longmapsto\!\dashv}$ is better than $=_\beta$ but is not "nice"

Proof Equivalence problem: *given proofs* π *and* ρ*, does* $\pi \overset{*}{\longmapsto\!\dashv} \rho$*?*

$\overset{*}{\vdash\!\dashv}$ is better than $=_\beta$ but is <u>not</u> "nice"

<u>Proof Equivalence problem:</u> *given proofs π and ρ, does $\pi \overset{*}{\vdash\!\dashv} \rho$?*

Sub-system	Complexity of Proof Equivalence	
ALL	in P [Hei11]	(using proof-nets)
unit-free MLL	in P	(using proof-nets)
unit-free MALL	in EXPTIME [HG05; HG16]	(using proof-nets)

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MLL	PSPACE-complete [HH16]	(reduces to a graph rewriting pb)
unit-free MALL	in EXPTIME [HG05; HG16]	(using proof-nets)
MALL	decidable	(finite number of cut-free proofs)

$\overset{*}{\longmapsto}$ is better than $=_\beta$ but is not "nice"

Proof Equivalence problem: *given proofs π and ρ, does $\pi \overset{*}{\longmapsto} \rho$?*

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Lemma

$$\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\ (\top)}{\vdash !A \otimes \top, \top \oplus \top}\ (\oplus_1) \quad \overset{*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\ (\top)}{\vdash !A \otimes \top, \top \oplus \top}\ (\oplus_2) \iff A \text{ is provable}$$

Proof.

A provable $\Longrightarrow$ use its proof to find a sequence of commutations

A not provable $\Longrightarrow$ can compute the full equivalence class (3 proofs) $\square$

We may have more than cut-elimination . . .

- **Axiom-expansion** and its <u>interactions</u> with cut-elimination

$$\begin{tikzcd} \pi \arrow[r, "\eta^*"] \arrow[d, "\beta^*"] & \tau \arrow[d, dashed, "\beta^*"] & \eta\text{-normal} \\ \phi \arrow[r, dashed, "\eta^*"] & \mu & \eta\text{-normal} \end{tikzcd}$$

(holds without 2$^{\text{nd}}$ order quantifiers)

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \quad\equiv\quad \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \text{ when } \pi_B \text{ and } \pi_C \text{ are “witness irrelevant”}$$

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!
- One may want **other rewritings**, with interactions to check

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\ (?_w)}{\vdash ?A, \Gamma}\ (?_c) \quad \rightsquigarrow \quad \overset{\pi}{\vdash ?A, \Gamma}$$

We may have more than cut-elimination ...

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!
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Thank you!

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Back-Up: Rule Commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_1) \;\;\overset{*}{\longmapsto\!\dashv}\;\; \dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \text{ is provable}$$

Proof.

♦ **If A is provable** ($\iff$ $!A$ is provable)

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \longmapsto\!\dashv \dfrac{\dfrac{\vdash\, !A \quad \dfrac{}{\vdash \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \longmapsto\!\dashv \dfrac{\dfrac{\vdash\, !A \quad \dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i)$$

$$\longmapsto\!\dashv \dfrac{\vdash\, !A \quad \dfrac{\dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash \top_A, \top \oplus \top}\,(\oplus_i)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes) \longmapsto\!\dashv \dfrac{\vdash\, !A \quad \dfrac{}{\vdash \top_A, \top \oplus \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Back-Up: Rule Commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{*}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \textit{ is provable}$$

Proof.

♦ **If A is not provable** ($\iff$ $!A$ is not provable)

We can compute the full equivalence class in this case:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top_A, \top}\,(\top)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \longmapsto\!\!\dashv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top) \qquad \dfrac{}{\vdash \top_A}\,(\top_A)}{\vdash\ !A \otimes \top_A, \top}\,(\otimes)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \longmapsto\!\!\dashv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top)}{\vdash\ !A, \top \oplus \top}\,(\oplus_i) \qquad \dfrac{}{\vdash \top_A}\,(\top_A)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Remark we use $!A$ instead of A to prevent commutations in $\dfrac{}{\vdash\ !A, \top}\,(\top)$, as $!$ is the sole rule not commuting with $\top$

□

Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

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Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

With the $! - ?_c$, $! - ?_w$ and $?_c - ?_w$ commutations and reductions

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \to \overset{\pi}{\vdash ?A, \Gamma}$$

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Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
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Quantifiers	$\forall X(A \& B) \simeq \forall X\ A \& \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$
Optional	$\forall X\ A \simeq A^{*\dagger}$	$\exists X\ A \simeq A^{*\dagger}$	$1 \simeq \bot^{\ddagger}$	$0 \simeq \top^{\clubsuit}$

$*$ if X not free in A

$\dagger$ if $\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \equiv \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists)$

when π is "witness irrelevant"

$\ddagger$ if $\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2) \equiv \overset{\pi}{\vdash \Gamma}$

$\clubsuit$ with $\dfrac{}{\vdash \Gamma}\,(\emptyset)$