

Confluence of Cut-elimination and Rules commutations in Linear Logic

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Introduction

Study the **identity / equality** of proofs:

when are two proofs π and ρ equal?

By the Curry-Howard isomorphism similar to:

when are two λ -terms M and N equal?

\leadsto **syntactic equality** is generally not enough!

We want (at least) equality up to **cut-elimination / β -reduction**

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Considered framework: **Linear Logic**

\longrightarrow fine-grained enough for this problem to be relevant
(*Classical Logic equates all proofs!*)

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Motivations:

- instance of *when are two morphisms in a category equal?*
- relevant for **isomorphisms**: *when are two formulas A and B equal?*
- useful when looking for a **canonical** representative: proof-nets!

- ▶ Equality of terms in λ -calculus
- ▶ Equality of proofs in Linear Logic
 - Quick sketch of Linear Logic
 - Why is equality more complicated than in λ -calculus?

Simply typed λ -calculus

Terms

$$M, N := x \mid \lambda x.M \mid M N$$

Types

$$A, B := O \mid A \rightarrow B$$

β -reduction

$$(\lambda x.M) N \xrightarrow{\beta} M\{N/x\}$$

η -expansion

$$M \xrightarrow{\eta} (\lambda x.M x)$$

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Syntactic equality is usually not enough:

- **Church encoding:** $\underline{n} := \lambda f. \lambda x. \underbrace{f f \dots f}_n x$

$\underline{2} + \underline{2}$ should be *equivalent* to $\underline{2} + (\underline{1} + \underline{1})$

- **Quotient in category / denotational model:**

$$M \text{ $\beta\eta$ } N \implies \llbracket M \rrbracket = \llbracket N \rrbracket$$

\longrightarrow a useful notion of equality is up to **computations** = $\beta\eta$ equivalence

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- **Quotient in category / denotational model:**

$$M \stackrel{\beta\eta}{=} N \implies \llbracket M \rrbracket = \llbracket N \rrbracket$$

\longrightarrow a useful notion of equality is up to **computations** = $\beta\eta$ equivalence

Here: *only on equality up to β -reduction* to simplify

Checking equality of terms

Problem:

- $M =_{\beta} N$? Give a sequence of terms $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \dots \xleftarrow{\beta} N$
- $M \neq_{\beta} N$? Prove such a sequence cannot exist!

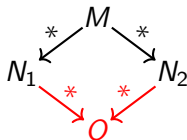
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Key results:

- β is **strongly normalizing**
(no infinite sequence of reductions)
- β is **confluent**



Corollary

$$M =_{\beta} N \iff \beta(M) = \beta(N)$$

with $\beta(\cdot)$ the unique normal form of the term

Examples

$$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1})$$

$$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \not= \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$$

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Linear Logic

Formulas

$A, B :=$	$ X X^\perp$	<i>atom</i>
	$ A \wp B A \otimes B \perp 1$	<i>multiplicative</i>
	$ A \oplus B A \& B 0 \top$	<i>additive</i>
	$?A !A$	<i>exponential</i>
	$ \forall X A \exists X A$	<i>quantifier</i>

Involutive Negation / Orthogonality

$$(X^\perp)^\perp = X$$
$$(A \wp B)^\perp = A^\perp \otimes B^\perp \quad (A \otimes B)^\perp = A^\perp \wp B^\perp \quad \perp^\perp = 1 \quad 1^\perp = \perp$$
$$\dots$$

Sub-systems

- MLL = atom + multiplicative
- MALL = atom + multiplicative + additive
- ...

16 Rules of Linear Logic

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)} \\
 \\
 \frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} \text{ (}\wp\text{)} \quad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \text{ (}\otimes\text{)} \quad \frac{\vdash \Gamma}{\vdash \perp, \Gamma} \text{ (}\perp\text{)} \quad \frac{}{\vdash 1} \text{ (1)} \\
 \\
 \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma} \text{ (}\&\text{)} \quad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \text{ (}\oplus_1\text{)} \quad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \text{ (}\oplus_2\text{)} \quad \frac{}{\vdash \top, \Gamma} \text{ (}\top\text{)} \\
 \\
 \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \text{ (?d)} \quad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \text{ (?c)} \quad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} \text{ (?w)} \quad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \text{ (!)} \\
 \\
 X \text{ not free in } \Gamma \quad \frac{\vdash A, \Gamma}{\vdash \forall X A, \Gamma} \text{ (}\forall\text{)} \quad \frac{\vdash A\{B/X\}, \Gamma}{\vdash \exists X A, \Gamma} \text{ (}\exists\text{)}
 \end{array}$$

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 \end{array}$$

Curry-Howard isomorphism: β -reduction \approx cut-elimination

Cut-elimination

Key steps (9)

"true" computations

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, A} (ax) \quad \frac{}{\vdash A, \Gamma} \pi \\
 \hline
 \vdash A, \Gamma \quad (cut) \quad \xrightarrow{\beta} \quad \vdash A, \Gamma
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\vdash B^\perp, A^\perp, \Gamma} \pi \quad \frac{}{\vdash A, \Delta} \rho \quad \frac{}{\vdash B, \Sigma} \tau \\
 \hline
 \vdash B^\perp \wp A^\perp, \Gamma \quad (\wp) \quad \vdash A \otimes B, \Delta, \Sigma \quad (\otimes) \quad \xrightarrow{\beta} \quad \frac{}{\vdash B^\perp, A^\perp, \Gamma} \pi \quad \frac{}{\vdash B, \Sigma} \tau \quad (cut) \\
 \hline
 \vdash \Gamma, \Delta, \Sigma \quad (cut) \quad \frac{}{\vdash A^\perp, \Gamma, \Sigma} \quad \frac{}{\vdash A, \Delta} \rho \quad (cut) \\
 \hline
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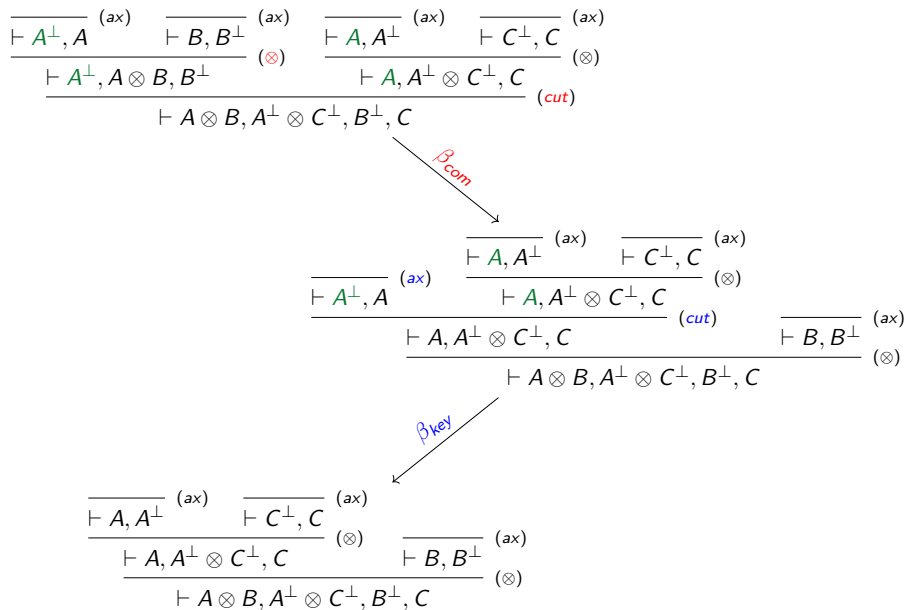
Commutative steps (15)

used to reach a key step

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, B, C, \Gamma} \pi \quad \frac{}{\vdash A, \Delta} \rho \\
 \hline
 \vdash A^\perp, B \wp C, \Gamma \quad (\wp) \quad \vdash A, \Delta \quad (cut) \quad \xrightarrow{\beta} \quad \frac{}{\vdash A^\perp, B, C, \Gamma} \pi \quad \frac{}{\vdash A, \Delta} \rho \quad (cut) \\
 \hline
 \vdash B \wp C, \Gamma, \Delta
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$$\begin{array}{c}
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 \hline
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Cut-elimination on an example



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- Cut-elimination is **strongly normalizing**?

- Cut-elimination is **confluent**?

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- Cut-elimination is **confluent**?

Not at all!

Cut-elimination is not confluent!

$$\begin{array}{c}
 \frac{\frac{\overline{\vdash A^\perp, A} \text{ (ax)}}{\vdash A \otimes B, A^\perp, B^\perp} \text{ (}\otimes\text{)}}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} \text{ (cut)} \quad \frac{\frac{\overline{\vdash B^\perp, B} \text{ (ax)}}{\vdash A^\perp \otimes C^\perp, A, C} \text{ (}\otimes\text{)}}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} \text{ (cut)} \\
 \swarrow \beta^* \qquad \searrow \beta^* \\
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 \end{array}$$

Irreversible choice at the beginning:

first commutative case with the **left** \otimes -rule or with the **right** one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

Cut-elimination is not confluent!

$$\begin{array}{c}
 \frac{\frac{}{\vdash A^\perp, A} (ax) \quad \frac{}{\vdash B^\perp, B} (ax)}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{\frac{}{\vdash A^\perp, A} (ax) \quad \frac{}{\vdash C^\perp, C} (ax)}{\vdash A^\perp \otimes C^\perp, A, C} (\otimes) \\
 \hline
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 \end{array}$$

$\swarrow \beta^* \qquad \searrow \beta^*$

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But confluence **up to rule commutation**!

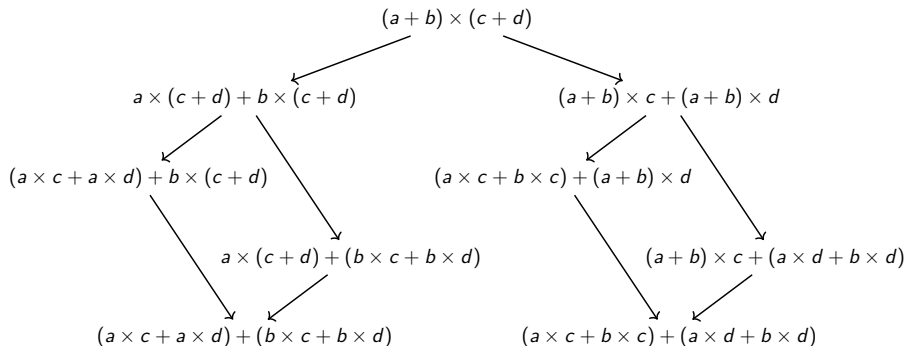
Intuition: Confluence up to in distributivity

Exercises from junior high school: **distributivity** of \times over $+$

$$a \times (b + c) \rightarrow (a \times b) + (a \times c)$$

$$(b + c) \times a \rightarrow (b \times a) + (c \times a)$$

Not confluent:



But confluent **up to** associativity and commutativity of $+$

Rule commutations (from a list of cases)

$$\frac{\frac{\frac{\pi}{\vdash C, \Gamma} \quad \frac{\frac{\rho}{\vdash A, D, \Delta} \quad \frac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma} (\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma} (\otimes) \quad \vdash \quad \frac{\frac{\frac{\pi}{\vdash C, \Gamma} \quad \frac{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta} (\otimes) \quad \frac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma} (\otimes)$$

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$$\frac{}{\vdash \top, ?A, \Gamma} (\top) \quad \vdash \quad \frac{\frac{}{\vdash \top, ?A, ?A, \Gamma} (\top)}{\vdash \top, ?A, \Gamma} (?_c)$$

$$\frac{}{\vdash \top, A \otimes B, \Gamma, \Delta} (\top) \quad \vdash \quad \frac{\frac{}{\vdash \top, A, \Gamma} (\top) \quad \frac{\pi}{\vdash B, \Delta}}{\vdash \top, A \otimes B, \Gamma, \Delta} (\otimes)$$

... and many many many more ...

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... and many many many more ...

! Non-trivial: **duplicates** / merges sub-proofs

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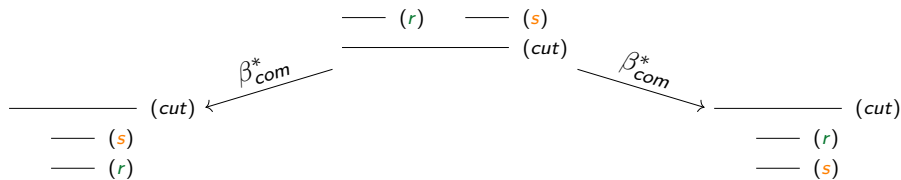
... and many many many more ...

!! Non-trivial: **duplicates** / merges sub-proofs

!! Tricky: **produces** / deletes rules and sub-proofs

Rule commutations (from a general method)

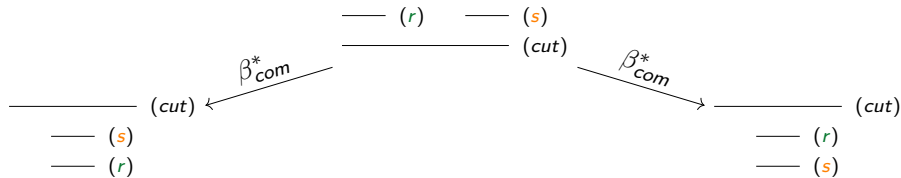
Every pair $\frac{\text{---} (s)}{\text{---} (r)} \vdash \frac{\text{---} (r)}{\text{---} (s)}$ coming from:



$\approx \#|rules|^2$ commutations \longrightarrow 93 equations in LL!

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Remarks

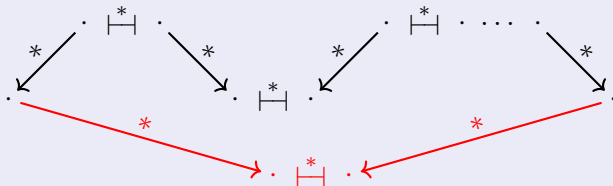
- $\vdash \subseteq =_{\beta}$
- \vdash is the usual (cut-free) commutations **without** $! - ?_c$ and $! - ?_w$

$$\frac{\frac{\pi}{\vdash A, ?B, ?B, ?\Gamma} \quad \frac{\vdash !A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (?!)}{\vdash !A, ?B, ?\Gamma} (?!)} \not\vdash \frac{\frac{\pi}{\vdash A, ?B, ?B, ?\Gamma} \quad \frac{\vdash !A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (?!)}{\vdash !A, ?B, ?\Gamma} (?!)} \quad \text{and} \quad \frac{\frac{\pi}{\vdash A, ?\Gamma} \quad \frac{\vdash !A, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (?!)}{\vdash !A, ?B, ?\Gamma} (?!)} \not\vdash \frac{\frac{\pi}{\vdash A, ?\Gamma} \quad \frac{\vdash !A, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (?!)}{\vdash !A, ?B, ?\Gamma} (?!)} (?!)$$

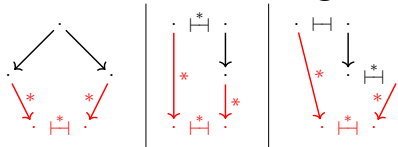
Proving Confluence up to

Definition: Church-Rosser modulo

\rightarrow is **Church-Rosser modulo** an equivalence relation \equiv^* when:



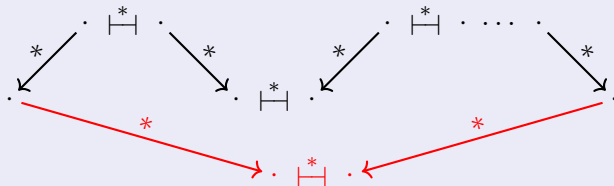
How to prove it? Several theorems in rewriting theory. Usual hypotheses:
strong normalization of \rightarrow & **closing some diagrams**



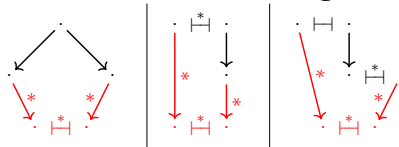
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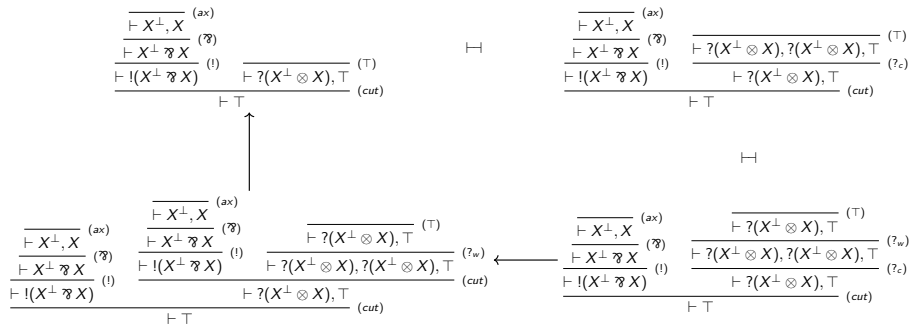
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Difficulties:

- \equiv^* is too difficult to manipulate, we prefer \equiv
- $\rightarrow \cdot \equiv^*$ is **not** strongly normalizing!

$\rightarrow \cdot \vdash^*$ is not strongly normalizing!



We even have an infinity of *key* cut-elimination cases!

Idea

The problem comes from the **production** of rules / sub-proofs.

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is **Church-Rosser modulo** rule commutation.

Theorem 2.2 from [AT12]

Let \vdash , \rightarrow and \rightsquigarrow be relations such that \vdash is symmetric and $\rightsquigarrow \subseteq \vdash$.
Suppose:

1 $\rightarrow \cdot \rightsquigarrow^*$ is strongly normalizing

2 $\leftarrow \cdot \rightarrow \subseteq (\rightarrow \cup \rightsquigarrow)^* \cdot \overline{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*$

3 $\vdash \cdot \rightarrow \subseteq (\overline{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*) \cup (\rightarrow \cdot (\rightarrow \cup \rightsquigarrow)^* \cdot \overline{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*)$

Then \rightarrow is Church-Rosser modulo $\overline{\vdash}^*$.

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Confluence up to rule commutation – SN

Proposition

Set \rightsquigarrow the rule commutations *without \top -commutations in the direction “creating rules”*, plus the cut-cut step of cut-elimination.

Then $\xrightarrow{\bar{\beta}} \cdot \rightsquigarrow^*$ is strongly normalizing, with $\xrightarrow{\bar{\beta}} = (\xrightarrow{\beta} \text{ without cut-cut})$.

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Strong Normalization Property
for Second Order Linear Logic

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Abstract

The paper contains the first complete proof of strong normalization (SN) for full second order linear logic (LL); Girard's original proof uses a standardization theorem which is not proven. We introduce sliced pure structures (sps), a very general version of Girard's proof-nets, and we apply to sps Gandy's method to infer SN from weak normalization (WN). We prove a standardization theorem for sps: if WN without erasing steps holds for an sps, then it enjoys SN. A key step in our proof of standardization is a confluence theorem for sps obtained by using only a very weak form of correctness, namely acyclicity slice by slice. We conclude by showing how standardization for sps allows to prove SN of LL, using as usual Girard's reducibility candidates.

Key words: (weak strong) normalization, confluence, standardization, linear logic, proof-nets, additive connectives, sliced pure structures

1. Introduction

In every abstract approach to computation, the distinction between terminating and non-terminating processes is crucial. A rewriting system enjoys *weak normalization* (WN) if every term of the system can be executed in a finite number of steps.

In the λ -calculus, non terminating computations start from λ -terms that strongly exploit *self-application*: every λ -term can be applied to itself (see for example [13]). Termination fails for the λ -calculus (even in its weak form WN), but holds for some of its most remarkable subsystems: the simply typed λ -calculus and its extension Girard's system F ([6]). The proofs of WN for these calculi have a deep logical content: they correspond to proofs of consistency in the logical sense, as highlighted by the *proofs-as-programs* paradigm. This paradigm is also called *Curry-Howard isomorphism* and establishes a correspondence between a fragment of intuitionistic natural deduction

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This paper *almost* do it [PT10].
“Just” check that some additions at the start go through the 61 pages of this technical proof using non-standard proof-nets!

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is **Church-Rosser modulo** rule commutation.

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Then \rightarrow is Church-Rosser modulo \vdash^* .

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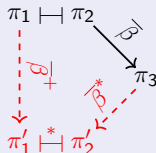
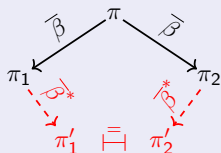
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Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

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Proof.

$\#(\text{cut steps})^2$
 $\approx \#|\text{rules}|^2$ cases



$\#(\text{cut steps}) \times$
 $\#(\text{commutations})$
 $\approx \#|\text{rules}|^3$ cases



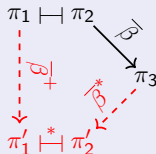
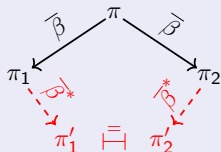
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□

Thousands of similar cases to check

→ horrible and tedious with pen and paper, better in a **proof assistant!**

But the **exchange** rule overcomplicates everything...

Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is *Church-Rosser modulo* rule commutation.

Corollary: Equality on cut-free proofs

Between cut-free proofs, $=_\beta$ is exactly \vdash^* .

cut-free proofs {



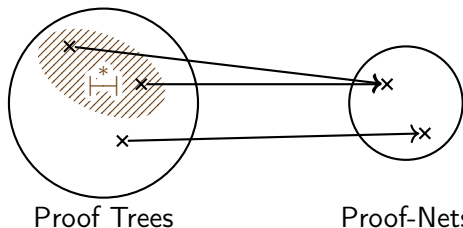
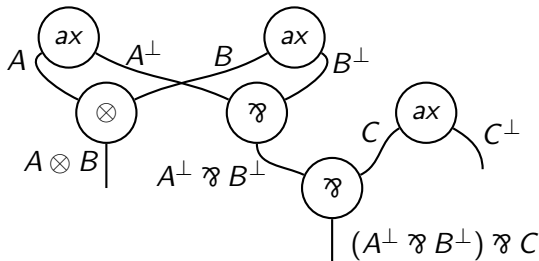
Consequences & Avail

- “Bureaucracy”: have to order all rules, but some order does not matter and **no canonical** choice

Consequences & Avail

- “Bureaucracy”: have to order all rules, but some order does not matter and **no canonical** choice
- **Proof-nets**: identify proofs exactly up to rule commutation \vdash^*

- ▶ \vdash is **equality of graphs**
- ▶ cut-elimination is **confluent** and has **only key steps**
- ▶ defined **only in some sub-systems** of LL



\vdash^* is better than $=_\beta$ but is not “nice”

Proof Equivalence problem: *given proofs π and ρ , does $\pi \vdash^* \rho$?*

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Sub-system	Complexity of Proof Equivalence	
ALL	in P [Hei11]	(using proof-nets)
unit-free MLL	in P	(using proof-nets)
unit-free MALL	in EXPTIME [HG05; HG16]	(using proof-nets)

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Lemma

$$\frac{\overline{\vdash !A \otimes T, T}^{(T)}}{\vdash !A \otimes T, T \oplus T}^{(\oplus_1)} \vdash^* \frac{\overline{\vdash !A \otimes T, T}^{(T)}}{\vdash !A \otimes T, T \oplus T}^{(\oplus_2)} \iff A \text{ is provable}$$

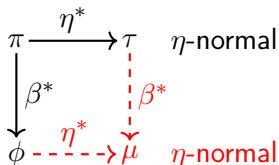
Proof.

A provable \implies use its proof to find a sequence of commutations

A not provable \implies can compute the full equivalence class (3 proofs) \square

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination



(holds without 2nd order quantifiers)

We may have more than cut-elimination ...

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!

$$\frac{\frac{\pi}{\vdash A, ?B, ?B, ?\Gamma} \quad (!)}{\vdash !A, ?B, ?B, ?\Gamma} \quad (!) \quad \equiv \quad \frac{\frac{\pi}{\vdash A, ?B, ?B, ?\Gamma} \quad (?_c)}{\vdash A, ?B, ?\Gamma} \quad (?_c) \quad \equiv \quad \frac{\vdash A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} \quad (!)$$

$$\frac{\frac{\pi}{\vdash A, ?\Gamma} \quad (!)}{\vdash !A, ?\Gamma} \quad (!) \quad \equiv \quad \frac{\frac{\pi}{\vdash A, ?\Gamma} \quad (?_w)}{\vdash A, ?B, ?\Gamma} \quad (?_w) \quad \equiv \quad \frac{\vdash A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} \quad (!)$$

$$\frac{\pi_B}{\vdash A[B/X], \Gamma} \quad (\exists) \quad \equiv \quad \frac{\pi_C}{\vdash A[C/X], \Gamma} \quad (\exists) \quad \text{when } \pi_B \text{ and } \pi_C \text{ are "witness irrelevant"}$$

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!
- One may want **other rewritings**, with interactions to check

$$\frac{\frac{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma} \quad (?_w) \quad \rightsquigarrow \quad \frac{\pi}{\vdash ?A, \Gamma}$$

$(?_c)$

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- **Axiom-expansion** and its interactions with cut-elimination
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Thank you!

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Back-Up: Rule Commutations & Provability

Lemma

$$\frac{\overline{\vdash !A \otimes T, T}^{(T)}}{\vdash !A \otimes T, T \oplus T}^{(\oplus_1)} \vdash^* \frac{\overline{\vdash !A \otimes T, T}^{(T)}}{\vdash !A \otimes T, T \oplus T}^{(\oplus_2)} \iff A \text{ is provable}$$

Proof.

♦ If A is provable ($\iff !A$ is provable)

$$\begin{aligned} \frac{\overline{\vdash !A \otimes T_A, T}^{(T)}}{\vdash !A \otimes T_A, T \oplus T}^{(\oplus_i)} &\vdash \frac{\vdash !A \quad \overline{\vdash T_A, T}^{(T)}}{\vdash !A \otimes T_A, T}^{(\otimes)} \vdash \frac{\vdash !A \quad \overline{\vdash T_A, T}^{(T_A)}}{\vdash !A \otimes T_A, T}^{(\otimes)} \\ &\vdash \frac{\overline{\vdash T_A, T}^{(T_A)}}{\vdash !A \quad \overline{\vdash T_A, T \oplus T}^{(\oplus_i)}}^{(\oplus_i)} \vdash \frac{\vdash !A \quad \overline{\vdash T_A, T \oplus T}^{(T_A)}}{\vdash !A \otimes T_A, T \oplus T}^{(\otimes)} \end{aligned}$$

Back-Up: Rule Commutations & Provability

Lemma

$$\frac{\overline{\vdash !A \otimes \top, \top}^{(\top)}}{\vdash !A \otimes \top, \top \oplus \top}^{(\oplus_1)} \vdash^* \frac{\overline{\vdash !A \otimes \top, \top}^{(\top)}}{\vdash !A \otimes \top, \top \oplus \top}^{(\oplus_2)} \iff A \text{ is provable}$$

Proof.

◆ If A is not provable ($\iff !A$ is not provable)

We can compute the full equivalence class in this case:

$$\frac{\overline{\vdash !A \otimes \top_A, \top}^{(\top)}}{\vdash !A \otimes \top_A, \top \oplus \top}^{(\oplus_i)} \vdash \frac{\frac{\overline{\vdash !A, \top}^{(\top)}}{\vdash !A \otimes \top_A, \top}^{(\otimes)} \quad \overline{\vdash \top_A}^{(\top_A)}}{\vdash !A \otimes \top_A, \top \oplus \top}^{(\oplus_i)} \vdash \frac{\overline{\vdash !A, \top}^{(\top)}}{\vdash !A, \top \oplus \top}^{(\oplus_i)} \quad \overline{\vdash \top_A}^{(\top_A)} \quad (\otimes)$$

Remark we use $!A$ instead of A to prevent commutations in $\overline{\vdash !A, \top}^{(\top)}$, as $!$ is the sole rule not commuting with \top



Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\frac{A \vdash B \quad B \vdash A}{A \vdash A} \text{ (cut) } =_{\beta\eta o} \overline{A \vdash A} \text{ (ax)} \quad \text{and} \quad \frac{B \vdash A \quad A \vdash B}{B \vdash B} \text{ (cut) } =_{\beta\eta o} \overline{B \vdash B} \text{ (ax)}$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
Commutativity	$A \otimes B \simeq B \otimes A$ $A \wp B \simeq B \wp A$	$A \oplus B \simeq B \oplus A$ $A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$ $A \wp \perp \simeq A$	$A \oplus 0 \simeq A$ $A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$	$A \wp (B \& C) \simeq (A \wp B) \& (A \wp C)$
Annihilation	$A \otimes 0 \simeq 0$ $A \wp \top \simeq \top$	
Seely	$!(A \& B) \simeq !A \otimes !B$ $!\top \simeq 1$	$?(A \oplus B) \simeq ?A \wp ?B$ $?0 \simeq \perp$
Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$
	$\forall X \top \simeq \top$ $\forall X \forall Y A \simeq \forall Y \forall X A$	$\exists X 0 \simeq 0$ $\exists X \exists Y A \simeq \exists Y \exists X A$

* if X not free in A

Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\frac{\pi \quad \rho}{A \vdash B \quad B \vdash A} (cut) =_{\beta\eta\circ} \overline{A \vdash A}^{(ax)} \quad \text{and} \quad \frac{\rho \quad \pi}{B \vdash A \quad A \vdash B} (cut) =_{\beta\eta\circ} \overline{B \vdash B}^{(ax)}$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
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Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$ $\forall X \top \simeq \top$ $\exists X 0 \simeq 0$ $\forall X \forall Y A \simeq \forall Y \forall X A$ $\exists X \exists Y A \simeq \exists Y \exists X A$

* if X not free in A

With the $! - ?_c$, $! - ?_w$ and $?_c - ?_w$ commutations and reductions

$$\frac{\frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?B, ?\Gamma}^{(!)} \quad \frac{\vdash !A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma}^{(?_c)}}{\vdash !A, ?B, ?\Gamma}^{(?_c)} \quad \equiv \quad \frac{\frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma}^{(?_c)} \quad \frac{\frac{\vdash \bar{A}, ?\Gamma}{\vdash !A, ?\Gamma}^{(!)} \quad \frac{\vdash !A, ?\Gamma}{\vdash !A, ?B, ?\Gamma}^{(?_w)}}{\vdash !A, ?B, ?\Gamma}^{(!)} \quad \equiv \quad \frac{\frac{\vdash \bar{A}, ?\Gamma}{\vdash A, ?B, ?\Gamma}^{(?_w)} \quad \frac{\vdash \bar{A}, ?\Gamma}{\vdash ?A, ?A, \Gamma}^{(?_w)} \quad \frac{\vdash ?A, \Gamma}{\vdash ?A, \Gamma}^{(?_c)}}{\vdash ?A, \Gamma}^{(?_w)} \rightarrow \vdash ?A, \Gamma$$

Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\frac{\frac{\pi}{A \vdash B} \quad \frac{\rho}{B \vdash A}}{A \vdash A} (cut) =_{\beta\eta\circ} \overline{A \vdash A}^{(ax)} \quad \text{and} \quad \frac{\frac{\rho}{B \vdash A} \quad \frac{\pi}{A \vdash B}}{B \vdash B} (cut) =_{\beta\eta\circ} \overline{B \vdash B}^{(ax)}$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
Commutativity	$A \otimes B \simeq B \otimes A$ $A \wp B \simeq B \wp A$	$A \oplus B \simeq B \oplus A$ $A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$ $A \wp \perp \simeq A$	$A \oplus 0 \simeq A$ $A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$	$A \wp (B \& C) \simeq (A \wp B) \& (A \wp C)$
Annihilation	$A \otimes 0 \simeq 0$	$A \wp \top \simeq \top$
Seely	$!(A \& B) \simeq !A \otimes !B$ $! \top \simeq 1$	$?(A \oplus B) \simeq ?A \wp ?B$ $?0 \simeq \perp$
Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\forall X \top \simeq \top$ $\forall X \forall Y A \simeq \forall Y \forall X A$
Optional	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$	$\exists X 0 \simeq 0$ $\exists X \exists Y A \simeq \exists Y \exists X A$
	$\forall X A \simeq A^{\dagger}$ $\exists X A \simeq A^{\ddagger}$	$1 \simeq \perp^{\ddagger}$ $0 \simeq \top^{\clubsuit}$

* if X not free in A

$$\dagger \text{ if } \frac{\pi_B}{\vdash A[B/X], \Gamma} (\exists) \equiv \frac{\pi_C}{\vdash A[C/X], \Gamma} (\exists) \equiv \frac{\pi_C}{\vdash \exists X A, \Gamma} (\exists)$$

when π is "witness irrelevant"

$$\ddagger \text{ if } \frac{\pi}{\vdash \Gamma} \frac{-}{\vdash} (mix_0) \equiv \frac{\pi}{\vdash \Gamma} (mix_2) \equiv \frac{\pi}{\vdash \Gamma}$$

\clubsuit with $\vdash \Gamma$ (\emptyset)