Cut-Cut Commutations Are Not Superfluous

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Abstract

In the definition of a cut-elimination procedure, there often is a $cut - cut$ commutation allowing to swap two *cut*-rules. This allows for instance to fully eliminate a chosen *cut*-rule without reducing other such rules in a proof. One may wonder whether such $cut - cut$ steps are superfluous. We prove it is not the case in linear logic: some cut-free forms of a proof can only be reached using a $cut - cut$ commutation.

1 Introduction

We consider the sequent calculus of linear logic [Gir87]. As many logics, it has a *cut*-rule and a rewriting system called *cut-elimination*, which details how to reach a cut-free (or normal) proof starting from a proof with possibly many *cut*-rules. Cut-elimination has been studied expensively in this system and more generally in linear logic (but mainly for its proof-net syntax): mostly its normalization [Acc13; DG99; LM08; Tor03; PT10] but also its confluence [CP05; Di24]. In particular, it is well-known one can reach a normal form without using any $cut - cut$ commutations. One may wonder whether such $cut - cut$ steps are superfluous: can the same normal forms be reached with and without $cut - cut$ commutations? We answer this question negatively, exhibiting a simple counter-example. This counter-example is even one for the simpler multiplicative-exponential fragment of linear logic, that we will consider here for simplicity's sake.

2 Definitions

We define here the sequent calculus of multiplicative-exponential linear logic [Gir87], with its cut-elimination procedure. Formulas are given by the following grammar, where X is an atom in a given countable set:

$$A, B ::= X^+ \mid X^- \mid A \parr B \mid A \otimes B \mid \perp \mid 1 \mid ?A \mid !A$$

$$\frac{}{\vdash A^\perp, A}\ (ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

Figure 1: Rules of Linear Logic

We define on formulas a function $(\cdot)^\perp$ called **orthogonality**, also named negation or duality, through the following inductive definition:

$$\begin{aligned} (X^+)^\perp &= X^- & (X^-)^\perp &= X^+ \\ (A \parr B)^\perp &= B^\perp \otimes A^\perp & (A \otimes B)^\perp &= B^\perp \parr A^\perp \\ \perp^\perp &= 1 & 1^\perp &= \perp \\ (?A)^\perp &= !A^\perp & (!A)^\perp &= ?A^\perp \end{aligned}$$

Sequents are multisets of formulas written in the form $\vdash A_1, \ldots, A_n$. Rules of linear logic are given on Fig. 1, where A and B stand for arbitrary formulas, Γ and Δ for multisets of formulas. The notation $?\Gamma$ means that each formula of Γ is a ?-formula.

As in many systems with a *cut*-rule, the *cut*-rule is *admissible* in linear logic: the same sequents can be proved with and without the *cut*-rule. The procedure turning a proof into a *cut*-free one is called *cut-elimination*. As we will not use *ax*- $\otimes$- $\parr$- and $?_c$-formulas and rules, we give this rewriting system without them. For a full description, see [Di24, Chapter 1].

Definition 1. Cut-elimination is the rewriting system whose rules are described on Table 1, up to commuting the two branches of any *cut*-rule.

The up to commutation means one should also consider a version of each rule with the left and right premises of any *cut*-rule swapped. In particular, for the $cut - cut$ case, there are in fact 4 rewriting rules.

In the $?w - !$ case, the doubled $?_w$-rule means we apply a $?_w$-rule on each formula of $?\Delta$. The order in which these rules are applied has no importance; in other words, the $?w - !$ step is non-deterministic according to the order in which the $?_w$-rules are applied.

3 Cut-cut commutations allow to reach more normal forms

One may wonder whether the same normal forms can be reached with and without $cut - cut$ steps. This is not the case in multiplicative-exponential linear logic, and thus in linear logic in general.

$\bot - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot) \qquad \dfrac{}{\vdash 1}\,(1)}{\vdash \Gamma}\,(cut)$	$\xrightarrow{\beta}$	$\overset{\pi}{\vdash \Gamma}$
$?d - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash ?A^{\perp}, \Gamma}\,(?d) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \qquad \overset{\phi}{\vdash A, ?\Delta}}{\vdash \Gamma, ?\Delta}\,(cut)$
$?w - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A^{\perp}, \Gamma}\,(?w) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?\Delta}\,(?w)$
$\bot - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, \bot, \Gamma}\,(\bot) \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \bot, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \bot, \Gamma, \Delta}\,(\bot)$
$?d - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma}}{\vdash A^{\perp}, ?B, \Gamma}\,(?d) \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?d)$
$?w - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, ?B, \Gamma}\,(?w) \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?w)$
$! - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma}}{\vdash ?A^{\perp}, !B, ?\Gamma}\,(!) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash !B, ?\Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash B, ?\Gamma, ?\Delta}\,(cut)}{\vdash !B, ?\Gamma, ?\Delta}\,(!)$
$cut - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash B^{\perp}, \Gamma, \Delta}\,(cut) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, \Gamma, \Sigma}\,(cut) \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$

Table 1: Cut-elimination – $\bot$, 1, $?_d$, $?_w$ and ! cases

Lemma 2. *There exist a proof π and a cut-free proof ϕ such that π reduces by cut-elimination to ϕ using a $cut - cut$ elimination step, but π does* not *reduce to ϕ without using a $cut - cut$ elimination step.*

Proof. Set π the following proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash 1, \perp_c$}
\RightLabel{($?_d$)}
\UnaryInfC{$\vdash 1, ?\perp_c$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1_d$}
\RightLabel{(!)}
\UnaryInfC{$\vdash !1_d$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash !1_d, \perp_a$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1_c$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash ?\perp_d, 1_c$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash ?\perp_d, 1_c, ?\perp_b$}
\RightLabel{(!)}
\UnaryInfC{$\vdash ?\perp_d, !1_c, ?\perp_b$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash !1_c, \perp_a, ?\perp_b$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash 1, \perp_a, ?\perp_b$}
\end{prooftree}
$$

We add letters as indices on $\perp$- and 1-formulas to distinguish occurrences, and to make apparent that the upper *cut*-rule introduces formulas $!1_d$ and $?\perp_d$ while the bottom one introduces $?\perp_c$ and $!1_c$.

Without $cut - cut$ steps, cut-elimination on π leads to a unique normal form:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash 1, ?\perp_b$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash 1, \perp_a, ?\perp_b$}
\end{prooftree}
$$

For our purpose, it is enough to prove any normal form reached without a $cut - cut$ step has a $\perp$-rule at its root. This is the case: one first has to apply a $\perp - cut$ step on the upper *cut*-rule; then, whatever happens on the upper *cut*-rule, the first step involving the bottom *cut*-rule must be a $\perp - cut$ step, resulting in a $\perp$-rule at the root of the proof.

Meanwhile, consider the result of applying a $cut - cut$ step first:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1_d$}
\RightLabel{(!)}
\UnaryInfC{$\vdash !1_d$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash !1_d, \perp_a$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash 1, \perp_c$}
\RightLabel{($?_d$)}
\UnaryInfC{$\vdash 1, ?\perp_c$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1_c$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash ?\perp_d, 1_c$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash ?\perp_d, 1_c, ?\perp_b$}
\RightLabel{(!)}
\UnaryInfC{$\vdash ?\perp_d, !1_c, ?\perp_b$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash ?\perp_d, 1, ?\perp_b$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash 1, \perp_a, ?\perp_b$}
\end{prooftree}
$$

Here, one can apply a $?_d - !$ step on the upper *cut*-rule, then a $?_w - cut$ step still on this *cut*-rule, followed by a $?_w - cut$ step on the bottom *cut*-rule, leading to a proof with a $?_w$-rule at its root. In particular, one can reach the following normal form ϕ:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{($\perp$)}
\UnaryInfC{$\vdash 1, \perp_a$}
\RightLabel{($?_w$)}
\UnaryInfC{$\vdash 1, \perp_a, ?\perp_b$}
\end{prooftree}
$$

$\square$

4 Conclusion

Perhaps surprisingly, $cut - cut$ commutative steps of cut-elimination have a computational content: they allow to reach more normal forms in multiplicative-exponential linear logic, and so in linear logic. Nonetheless, our example seems hard to adapt without a contextual rule such as ! – which needs the sequent it is applied on to be of the shape $!A, ?\Gamma$. Hence, we conjecture that in the multiplicative-additive fragment of linear logic the same normal forms can be reached with and without $cut - cut$ commutative steps.

References

[Acc13] Beniamino Accattoli. "Linear Logic and Strong Normalization". In: *Rewriting Techniques and Applications*. Ed. by Femke van Raamsdonk. Vol. 21. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2013, pp. 39–54. DOI: `10.4230/LIPIcs.RTA.2013.39`.

[CP05] Robin Cockett and Craig Pastro. "A Language For Multiplicative-additive Linear Logic". In: *Electronic Notes in Theoretical Computer Science* 122 (2005). Proceedings of the 10th Conference on Category Theory in Computer Science (CTCS 2004), pp. 23–65. DOI: `/10.1016/j.entcs.2004.06.049`. URL: `https://www.sciencedirect.com/science/article/pii/S1571066105000320`.

[DG99] Roberto Di Cosmo and Stefano Guerrini. "Strong Normalization of Proof Nets Modulo Structural Congruences". In: *Rewriting Techniques and Applications*. Ed. by P. Narendran and M. Rusinowitch. Vol. 1631. Lecture Notes in Computer Science. Springer, 1999, pp. 75–89. DOI: `10.1007/3-540-48685-2_6`.

[Di24] Rémi Di Guardia. "Identity of Proofs and Formulas using Proof-Nets in Multiplicative-Additive Linear Logic". Thèse de Doctorat. Ecole normale supérieure de lyon - ENS LYON, Sept. 2024. URL: `https://theses.hal.science/tel-04830060`.

[Gir87] Jean-Yves Girard. "Linear logic". In: *Theoretical Computer Science* 50 (1987), pp. 1–102. DOI: `10.1016/0304-3975(87)90045-4`.

[LM08] Olivier Laurent and Roberto Maieli. "Cut Elimination for Monomial MALL Proof Nets". In: *Proceedings of the twenty-third annual symposium on Logic In Computer Science*. IEEE. IEEE Computer Society Press, June 2008, pp. 486–497.

[PT10] Michele Pagani and Lorenzo Tortora de Falco. "Strong normalization property for second order linear logic". In: *Theoretical Computer Science* 411.2 (2010), pp. 410–444.

[Tor03] Lorenzo Tortora de Falco. "Additives of linear logic and normalization- Part I: a (restricted) Church-Rosser property". In: *Theoretical Computer Science* 294.3 (Feb. 2003), pp. 489–524. DOI: `10.1016/S0304-3975(01)00176-1`.