

# Confluence of Cut-elimination and Rules commutations in Linear Logic

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# Introduction

Study the **identity / equality** of proofs:

*when are two proofs  $\pi$  and  $\rho$  equal?*

By the Curry-Howard isomorphism similar to:

*when are two  $\lambda$ -terms  $M$  and  $N$  equal?*

~ **syntactic equality** is generally not enough!

We want (at least) equality up to **cut-elimination /  $\beta$ -reduction**

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Considered framework: **Linear Logic**

→ fine-grained enough for this problem to be relevant  
*(Classical Logic equates all proofs!)*

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Motivations:

- instance of *when are two morphisms in a category equal?*
- relevant for **isomorphisms**: *when are two formulas  $A$  and  $B$  equal?*
- useful when looking for a **canonical** representative: proof-nets!

# Plan

- ▶ Equality of terms in  $\lambda$ -calculus
- ▶ Equality of proofs in Linear Logic
  - Quick sketch of Linear Logic
  - Why is equality more complicated than in  $\lambda$ -calculus?

# Simply typed $\lambda$ -calculus

Terms

$$M, N := x \mid \lambda x. M \mid M \ N$$

Types

$$A, B := O \mid A \rightarrow B$$

$\beta$ -reduction

$$(\lambda x. M) \ N \xrightarrow{\beta} M\{N/x\}$$

$\eta$ -expansion

$$M \xrightarrow{\eta} (\lambda x. M \ x)$$

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*Syntactic equality* is usually not enough:

- Church encoding:  $\underline{n} := \lambda f. \lambda x. \underbrace{f \ f \ \dots \ f}_{n \text{ times}} \ x$

$\underline{2} + \underline{2}$  should be *equivalent* to  $\underline{2} + (\underline{1} + \underline{1})$

- Quotient in category / denotational model:

$$M =_{\beta\eta} N \implies \llbracket M \rrbracket = \llbracket N \rrbracket$$

→ a useful notion of equality is up to **computations** =  $\beta\eta$  equivalence

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Here: **only on equality up to  $\beta$ -reduction** to simplify

# Checking equality of terms

Problem:

- $M =_{\beta} N$ ? Give a sequence of terms  $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} N$
- $M \neq_{\beta} N$ ? Prove such a sequence cannot exist!

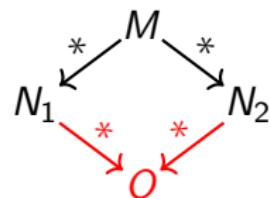
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Key results:

- $\beta$  is **strongly normalizing**  
(no infinite sequence of reductions)
- $\beta$  is **confluent**



**Corollary**

$$M =_{\beta} N \iff \beta(M) = \beta(N)$$

with  $\beta(\cdot)$  the unique normal form of the term

**Examples**

$$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1}) \quad \underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \neq \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$$

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# Linear Logic

## Formulas

$A, B :=   X   X^\perp$	<i>atom</i>
$  A \wp B   A \otimes B   \perp   1$	<i>multiplicative</i>
$  A \oplus B   A \& B   0   \top$	<i>additive</i>
$  ?A   !A$	<i>exponential</i>
$  \forall X A   \exists X A$	<i>quantifier</i>

## Involutive Negation / Orthogonality

$$(X^\perp)^\perp = X$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp \quad (A \otimes B)^\perp = A^\perp \wp B^\perp \quad \perp^\perp = 1 \quad 1^\perp = \perp$$

...

## Sub-systems

- MLL = atom + multiplicative
- MALL = atom + multiplicative + additive
- ...

# 16 Rules of Linear Logic

$$\begin{array}{c} \frac{}{\vdash A^\perp, A} (\text{ax}) \quad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta} (\text{cut}) \\[10pt] \frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} (?) \quad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} (\otimes) \quad \frac{\vdash \Gamma}{\vdash \perp, \Gamma} (\perp) \quad \frac{}{\vdash 1} (1) \\[10pt] \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma} (\&) \quad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} (\oplus_1) \quad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} (\oplus_2) \quad \frac{}{\vdash \top, \Gamma} (\top) \\[10pt] \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} (?d) \quad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} (?c) \quad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} (?w) \quad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} (!) \\[10pt] X \text{ not free in } \Gamma \quad \frac{\vdash A, \Gamma}{\vdash \forall X A, \Gamma} (\forall) \quad \frac{\vdash A\{B/X\}, \Gamma}{\vdash \exists X A, \Gamma} (\exists) \end{array}$$

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**Curry-Howard isomorphism:**  $\beta$ -reduction  $\approx$  cut-elimination

# Cut-elimination

Key steps (9)

*"true" computations*

$$\frac{\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \vdash A, \Gamma}{\vdash A, \Gamma} \text{ (cut)} \xrightarrow{\beta} \vdash A, \Gamma$$

$$\frac{\frac{\vdash B^\perp, A^\perp, \Gamma \quad \vdash A, \Delta \quad \vdash B, \Sigma}{\vdash B^\perp \wp A^\perp, \Gamma} \text{ (}\wp\text{)} \quad \frac{\rho \quad \tau}{\vdash A \otimes B, \Delta, \Sigma} \text{ (}\otimes\text{)}}{\vdash \Gamma, \Delta, \Sigma} \text{ (cut)} \xrightarrow{\beta} \frac{\frac{\vdash B^\perp, A^\perp, \Gamma \quad \vdash B, \Sigma}{\vdash A^\perp, \Gamma, \Sigma} \text{ (cut)} \quad \frac{\rho}{\vdash A, \Delta} \text{ (cut)}}{\vdash \Gamma, \Delta, \Sigma}$$

Commutative steps (15)

*used to reach a key step*

$$\frac{\frac{\vdash A^\perp, B, C, \Gamma \quad \vdash A^\perp, B \wp C, \Gamma}{\vdash A^\perp, B \wp C, \Gamma} \text{ (}\wp\text{)} \quad \frac{\rho}{\vdash A, \Delta} \text{ (cut)}}{\vdash B \wp C, \Gamma, \Delta} \text{ (cut)} \xrightarrow{\beta} \frac{\frac{\vdash A^\perp, B, C, \Gamma \quad \vdash A, \Delta}{\vdash B, C, \Gamma, \Delta} \text{ (cut)} \quad \frac{\rho}{\vdash B \wp C, \Gamma, \Delta} \text{ (}\wp\text{)}}{\vdash B \wp C, \Gamma, \Delta} \text{ (}\wp\text{)}$$

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# Cut-elimination on an example

$$\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{}{\vdash B, B^\perp} \text{ (ax)} \quad \frac{}{\vdash A, A^\perp} \text{ (ax)} \quad \frac{}{\vdash C^\perp, C} \text{ (ax)}$$

$$\frac{}{\vdash A^\perp, A \otimes B, B^\perp} \text{ (⊗)} \quad \frac{}{\vdash A, A^\perp \otimes C^\perp, C} \text{ (⊗)}$$

$$\frac{}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} \text{ (cut)}$$

$\beta_{com}$

$$\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{}{\vdash A, A^\perp} \text{ (ax)} \quad \frac{}{\vdash C^\perp, C} \text{ (ax)}$$

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$$\frac{}{\vdash B, B^\perp} \text{ (ax)} \quad \frac{}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} \text{ (⊗)}$$

$\beta_{key}$

$$\frac{}{\vdash A, A^\perp} \text{ (ax)} \quad \frac{}{\vdash C^\perp, C} \text{ (ax)}$$

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- Cut-elimination is **confluent**?

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- Cut-elimination is **confluent**?

Not at all!

Cut-elimination is not confluent!

$$\frac{\frac{\frac{\vdash A^\perp, A \quad \vdash B^\perp, B}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{\vdash A^\perp, A \quad \vdash C^\perp, C}{\vdash A^\perp \otimes C^\perp, A, C} (\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} (cut)}$$

10. *Leucosia* *leucostoma* *leucostoma* *leucostoma*

3

B\*

$$\frac{\frac{\vdash A^\perp, A \quad \vdash C^\perp, C}{\vdash A, A^\perp \otimes C^\perp, C} (\otimes) \quad \vdash B^\perp, B}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} (\otimes)$$

$\neq$

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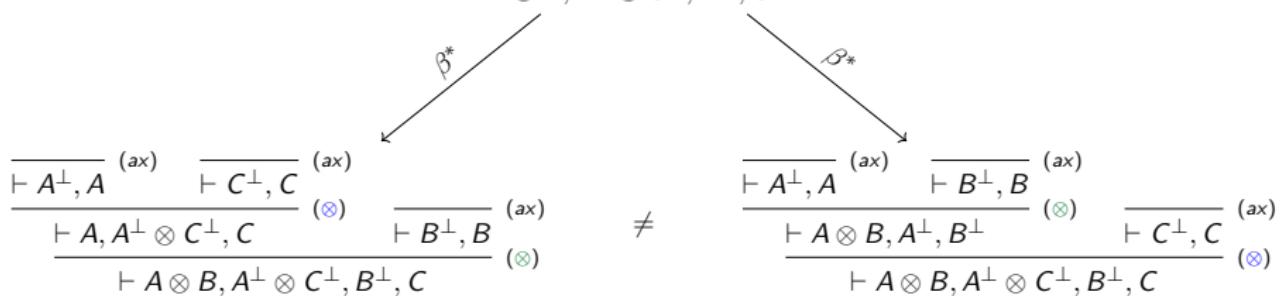
**Irreversible choice** at the beginning:

first commutative case with the left  $\otimes$ -rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

# Cut-elimination is not confluent!

$$\frac{\frac{\frac{\vdash A^\perp, A \quad \vdash B^\perp, B}{\vdash A \otimes B, A^\perp, B^\perp} (\otimes) \quad \frac{\vdash A^\perp, A \quad \vdash C^\perp, C}{\vdash A^\perp \otimes C^\perp, A, C} (\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C} (cut)}$$



### Irreversible choice at the beginning:

first commutative case with the **left**  $\otimes$ -rule or with the **right** one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

But confluence up to rule commutation!

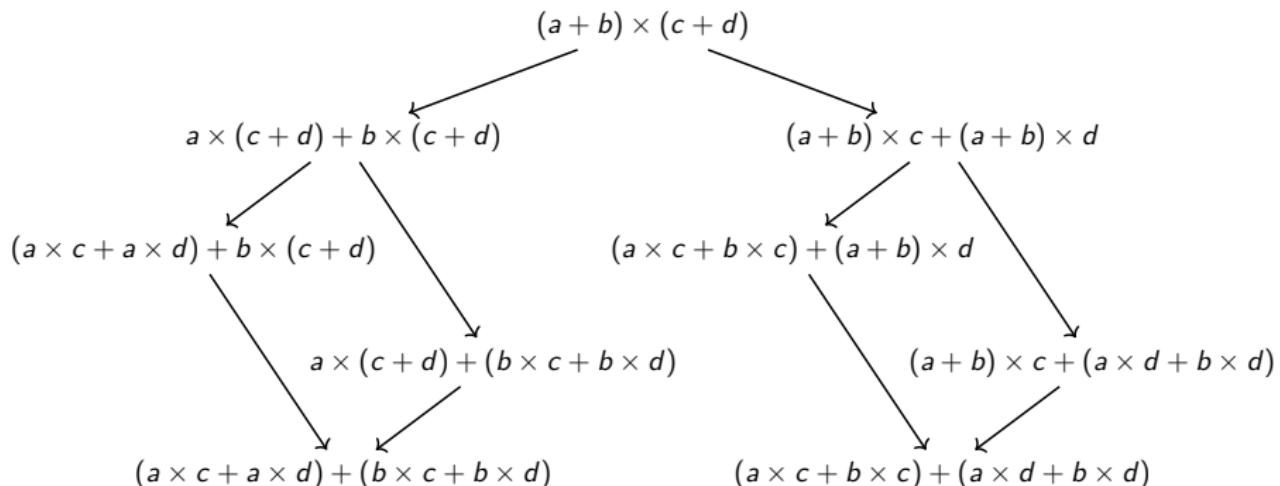
# Intuition: Confluence up to in distributivity

Exercices from junior high school: **distributivity** of  $\times$  over  $+$

$$a \times (b + c) \rightarrow (a \times b) + (a \times c)$$

$$(b + c) \times a \rightarrow (b \times a) + (c \times a)$$

Not confluent:



But confluent **up to** associativity and commutativity of  $+$

# Rule commutations (from a list of cases)

$$\frac{\frac{\pi}{\vdash C, \Gamma} \quad \frac{\rho}{\vdash A, D, \Delta} \quad \frac{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma} (\otimes) \quad \text{H} \quad \frac{\frac{\pi}{\vdash C, \Gamma} \quad \frac{\rho}{\vdash A, D, \Delta}}{\frac{\vdash A, C \otimes D, \Gamma, \Delta}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}} (\otimes) \quad \frac{\tau}{\vdash B, \Sigma}$$

$$\frac{\frac{\pi}{\vdash A, \Gamma} \quad \frac{\rho}{\vdash B, C, \Delta} \quad \frac{\tau}{\vdash B, D, \Delta}}{\vdash B, C \& D, \Delta} (\&) \quad \text{H} \quad \frac{\frac{\pi}{\vdash A, \Gamma} \quad \frac{\rho}{\vdash B, C, \Delta}}{\frac{\vdash A \otimes B, C, \Gamma, \Delta}{\vdash A \otimes B, C \& D, \Gamma, \Delta}} (\otimes) \quad \frac{\frac{\pi}{\vdash A, \Gamma} \quad \frac{\tau}{\vdash B, D, \Delta}}{\frac{\vdash A \otimes B, D, \Gamma, \Delta}{\vdash A \otimes B, C \& D, \Gamma, \Delta}} (\otimes)$$

$$\frac{}{\vdash \top, ?A, \Gamma} (\top) \quad \text{H} \quad \frac{\frac{\vdash \top, ?A, ?A, \Gamma}{\vdash \top, ?A, \Gamma} (\top)}{\vdash \top, ?A, \Gamma} (?_c)$$

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... and many many many more ...

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... and many many many more ...

! Non-trivial: **duplicates** / merges sub-proofs

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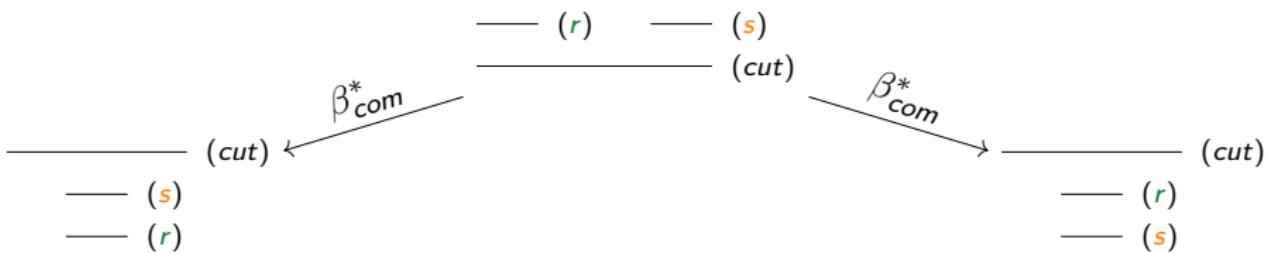
... and many many many more ...

! Non-trivial: **duplicates** / merges sub-proofs

! Tricky: **produces** / deletes rules and sub-proofs

# Rule commutations (from a general method)

Every pair  $\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} (\textcolor{brown}{s}) \\ (\textcolor{green}{r}) \end{array} \vdash \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} (\textcolor{green}{r}) \\ (\textcolor{brown}{s}) \end{array}$  coming from:



$\approx \#|rules|^2$  commutations  $\longrightarrow$  93 equations in LL!

# Rule commutations (from a general method)

Every pair  $\frac{\text{--- } (s)}{\text{--- } (r)}$   $\vdash$   $\frac{\text{--- } (r)}{\text{--- } (s)}$  coming from:

$$\frac{\text{--- } (r) \quad \text{--- } (s)}{\text{--- } (cut)} \quad \beta_{com}^* \quad \beta_{com}^*$$
$$\frac{\text{--- } (cut) \quad \text{--- } (cut)}{\text{--- } (cut) \quad \text{--- } (cut)}$$
$$\frac{\text{--- } (s) \quad \text{--- } (r)}{\text{--- } (cut)}$$

$\approx \#|rules|^2$  commutations  $\rightarrow$  93 equations in LL!

## Remarks

- $\vdash \subseteq =_\beta$
- $\vdash$  is the usual (cut-free) commutations **without**  $! - ?_c$  and  $! - ?_w$

$$\frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?B, ?\Gamma} (?) \quad \nvdash \quad \frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash A, ?B, ?\Gamma} (?)_c \quad \text{and} \quad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} (?) \quad \nvdash \quad \frac{\vdash A, ?\Gamma}{\vdash A, ?B, ?\Gamma} (?)_w$$
$$\frac{\vdash A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (?)$$
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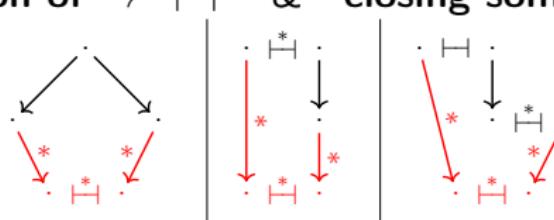
# Proving Confluence up to

## Definition: Church-Rosser modulo

→ is Church-Rosser modulo an equivalence relation  $\overset{*}{\vdash}$  when:



How to prove it? Several theorems in rewriting theory. Usual hypotheses:  
strong normalization of  $\rightarrow$  & closing some diagrams



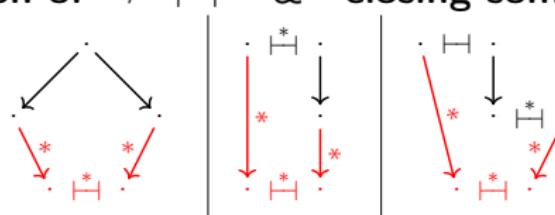
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How to prove it? Several theorems in rewriting theory. Usual hypotheses:  
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## Difficulties:

- $\overset{*}{\vdash}$  is too difficult to manipulate, we prefer  $\vdash$
- $\rightarrow \cdot \overset{*}{\vdash}$  is **not** strongly normalizing!

# $\rightarrow \cdot \vdash^*$ is not strongly normalizing!

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\frac{\vdash X^\perp, X}{\vdash X^\perp \& X} (\text{ax})}{\vdash X^\perp \& X} (?)}}{\vdash !(X^\perp \& X)} (!)}{\vdash ?(X^\perp \otimes X), \top} (\text{cut})} \vdash \top \\
 \uparrow \\
 \frac{\frac{\frac{\frac{\vdash X^\perp, X}{\vdash X^\perp \& X} (\text{ax})}{\vdash X^\perp \& X} (?)}}{\vdash !(X^\perp \& X)} (!)}{\vdash ?(X^\perp \otimes X), \top} (\text{cut})} \vdash \top
 \end{array} \quad \vdash \quad
 \begin{array}{c}
 \frac{\frac{\frac{\vdash X^\perp, X}{\vdash X^\perp \& X} (\text{ax})}{\vdash X^\perp \& X} (?)}}{\vdash !(X^\perp \& X)} (!) \quad \frac{\frac{\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top}{\vdash ?(X^\perp \otimes X), \top} (\text{?}_c)}{\vdash ?(X^\perp \otimes X), \top} (\text{cut})} \\
 \vdash \top
 \end{array}$$

$$\begin{array}{c}
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 \vdash \top
 \end{array}$$

We even have an infinity of key cut-elimination cases!

## Idea

The problem comes from the **production** of rules / sub-proofs.

# Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is *Church-Rosser modulo* rule commutation.

Theorem 2.2 from [AT12]

Let  $\vdash$ ,  $\rightarrow$  and  $\rightsquigarrow$  be relations such that  $\vdash$  is symmetric and  $\rightsquigarrow \subseteq \vdash$ .

Suppose:

1  $\rightarrow \cdot \rightsquigarrow^*$  is strongly normalizing

2  $\leftarrow \cdot \rightarrow \subseteq (\rightarrow \cup \rightsquigarrow)^* \cdot \bar{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*$

3  $\vdash \cdot \rightarrow \subseteq (\bar{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*) \cup (\rightarrow \cdot (\rightarrow \cup \rightsquigarrow)^* \cdot \bar{\vdash} \cdot (\leftarrow \cup \rightsquigarrow)^*)$

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# Confluence up to rule commutation – SN

## Proposition

Set  $\rightsquigarrow$  the rule commutations without  $\top$ -commutations in the direction “creating rules”, plus the cut-cut step of cut-elimination.

Then  $\xrightarrow{\bar{\beta}} \cdot \rightsquigarrow^*$  is strongly normalizing, with  $\xrightarrow{\bar{\beta}} = (\xrightarrow{\beta} \text{ without cut-cut})$ .

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Strong Normalization Property  
for Second Order Linear Logic

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### Abstract

The paper contains the first complete proof of strong normalization (SN) for full second order linear logic (LL). Girard's original proof uses a standardization theorem which is not proven. We introduce sliced pure structures (sps), a very general version of Girard's proof-nets, and we apply to sps Gandy's method to infer SN from weak normalization (WN). We prove a standardization theorem for sps if WN without erasing steps holds for an sps, then it enjoys SN. A key step in our proof of standardization is a confluence theorem for sps obtained by using only a very weak form of correctness, namely acyclicity slice by slice. We conclude by showing how standardization for sps allows to prove SN of LL, using as weakly Girard's reducibility candidates.

**Key words:** (weak) strong normalization, confluence, standardization, linear logic, proof-nets, additive connectives, sliced pure structures

### 1. Introduction

In every abstract approach to computation, the distinction between terminating and non-terminating processes is crucial. A rewriting system enjoys *weak normalization* (WN) if every term in the system has a reduced form (number of reductions is finite).

In the  $\lambda$ -calculus, weak normalizing computations start from  $\lambda$ -terms that strongly exploit self-application: every  $\lambda$ -term can be applied to itself (see for example [13]). Termination fails for a  $\lambda$ -calculus (even in its weak form WN), but holds for some of its most remarkable subsystems: the simply typed  $\lambda$ -calculus and its extension Girard's system F ([6]). The proofs of WN for these calculi have a deep logical content: they correspond to proofs of consistency in the logical sense, as highlighted by the *proofs-as-programs* paradigm. This paradigm is also called *Curry-Howard isomorphism* and establishes a correspondence between a fragment of intuitionistic natural deduction

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This paper almost do it [PT10].  
“Just” check that some additions at the start go through the 61 pages of this technical proof using non-standard proof-nets!

# Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is *Church-Rosser modulo* rule commutation.

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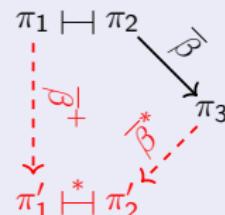
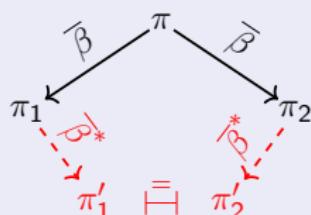
# Confluence up to rule commutation

Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is *Church-Rosser modulo rule commutation*.

Proof.

$$\begin{aligned} &\#(\text{cut steps})^2 \\ &\approx \#|\text{rules}|^2 \text{ cases} \end{aligned}$$



$$\begin{aligned} &\#(\text{cut steps}) \times \\ &\#(\text{commutations}) \\ &\approx \#|\text{rules}|^3 \text{ cases} \end{aligned}$$

□

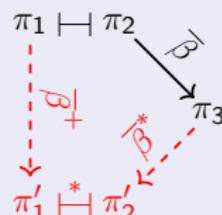
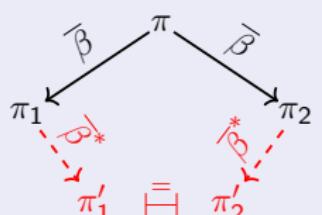
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□

Thousands of similar cases to check

→ horrible and tedious with pen and paper, better in a **proof assistant!**

But the **exchange** rule overcomplicates everything...

# Confluence up to rule commutation

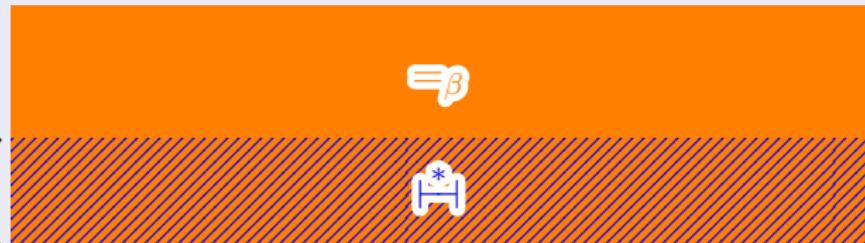
Theorem (Proved in MALL [CP05; DL23]; not yet written for LL)

Cut-elimination is *Church-Rosser modulo* rule commutation.

Corollary: Equality on cut-free proofs

Between cut-free proofs,  $=_\beta$  is exactly  $\vdash^*$ .

cut-free proofs {



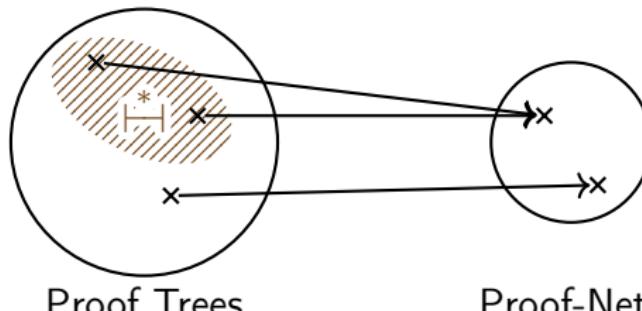
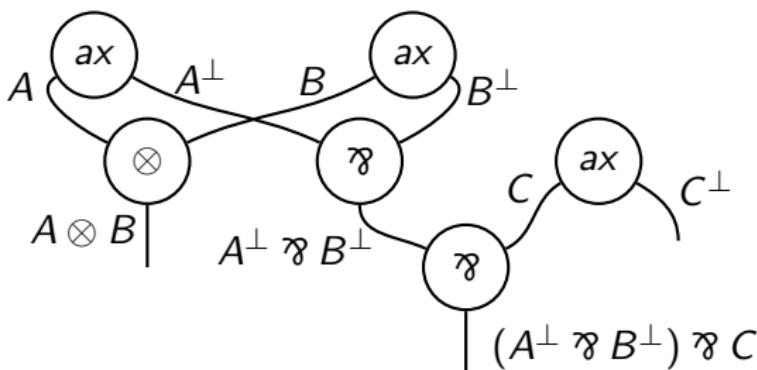
## Consequences & Avail

- “Bureaucracy”: have to order all rules, but some order does not matter and **no canonical** choice

# Consequences & Avail

- “Bureaucracy”: have to order all rules, but some order does not matter and **no canonical** choice
- **Proof-nets**: identify proofs exactly up to rule commutation  $\vdash^*$

- ▶  $\vdash$  is **equality of graphs**
- ▶ cut-elimination is **confluent** and has **only key steps**
- ▶ defined **only in some sub-systems** of LL



$\vdash^*$  is better than  $=_\beta$  but is not “nice”

Proof Equivalence problem: *given proofs  $\pi$  and  $\rho$ , does  $\pi \vdash^* \rho$ ?*

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Sub-system	Complexity of Proof Equivalence	
ALL unit-free MLL	in P [Hei11]	(using proof-nets)
	in P	(using proof-nets)
unit-free MALL	in EXPTIME [HG05; HG16]	(using proof-nets)

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unit-free MALL MALL	in EXPTIME [HG05; HG16] decidable	(using proof-nets) (finite number of cut-free proofs)

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MLL	PSPACE-complete [HH16]	(reduces to a graph rewriting pb)
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MALL	decidable	(finite number of cut-free proofs)

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MALL	decidable	(finite number of cut-free proofs)
LL	undecidable	(reduces to provability)

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### Lemma

$$\frac{\frac{\vdash !A \otimes T, T}{\vdash !A \otimes T, T \oplus T} (\top)}{\vdash !A \otimes T, T \oplus T} (\oplus_1) \quad \vdash^* \quad \frac{\frac{\vdash !A \otimes T, T}{\vdash !A \otimes T, T \oplus T} (\top)}{\vdash !A \otimes T, T \oplus T} (\oplus_2) \iff A \text{ is provable}$$

### Proof.

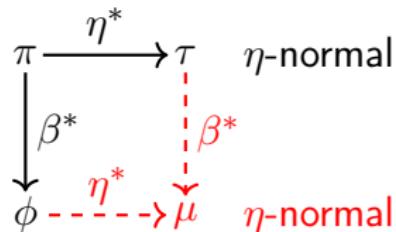
A provable  $\implies$  use its proof to find a sequence of commutations

A not provable  $\implies$  can compute the full equivalence class (3 proofs)



# We may have more than cut-elimination . . .

- Axiom-expansion and its interactions with cut-elimination



(holds without 2<sup>nd</sup> order quantifiers)

# We may have more than cut-elimination . . .

- Axiom-expansion and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!

$$\frac{\frac{\frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?B, ?\Gamma} (!)}{\vdash !A, ?B, ?\Gamma} (?)_c}}{\vdash !A, ?B, ?\Gamma} (?)_c \equiv \frac{\frac{\vdash A, ?B, ?B, ?\Gamma}{\vdash A, ?B, ?\Gamma} (?)_c}{\frac{\vdash A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (!)}$$

$$\frac{\frac{\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} (!)}{\vdash !A, ?B, ?\Gamma} (?)_w}}{\vdash !A, ?B, ?\Gamma} (?)_w \equiv \frac{\frac{\vdash A, ?\Gamma}{\vdash A, ?B, ?\Gamma} (?)_w}{\frac{\vdash A, ?B, ?\Gamma}{\vdash !A, ?B, ?\Gamma} (!)}$$

$$\frac{\frac{\pi_B}{\vdash A[B/X], \Gamma} (\exists)}{\vdash \exists X A, \Gamma} (\exists) \equiv \frac{\frac{\pi_C}{\vdash A[C/X], \Gamma} (\exists)}{\vdash \exists X A, \Gamma} (\exists) \text{ when } \pi_B \text{ and } \pi_C \text{ are "witness irrelevant"}$$

# We may have more than cut-elimination . . .

- Axiom-expansion and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!
- One may want **other rewritings**, with interactions to check

$$\frac{\vdash ?A, \Gamma}{\vdash ?A, ?A, \Gamma} \stackrel{(\text{?}_w)}{\rightsquigarrow} \frac{\vdash ?A, \Gamma}{\vdash ?A, \Gamma} \stackrel{(\text{?}_c)}{\rightsquigarrow} \vdash ?A, \Gamma$$

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Thank you!

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# Back-Up: Rule Commutations & Provability

## Lemma

$$\frac{\frac{\overline{\vdash !A \otimes T, T} \quad (\top)}{\vdash !A \otimes T, T \oplus T} \quad (\oplus_1)}{\vdash !A \otimes T, T \oplus T} \quad \text{H}^* \quad \frac{\frac{\overline{\vdash !A \otimes T, T} \quad (\top)}{\vdash !A \otimes T, T \oplus T} \quad (\oplus_2)}{\vdash !A \otimes T, T \oplus T} \iff A \text{ is provable}$$

## Proof.

♦ If  $A$  is provable ( $\iff !A$  is provable)

$$\begin{array}{c} \frac{\frac{\overline{\vdash !A \otimes T_A, T} \quad (\top)}{\vdash !A \otimes T_A, T \oplus T} \quad (\oplus_i)}{\vdash !A \otimes T_A, T \oplus T} \quad \text{H} \quad \frac{\vdash !A \quad \overline{\vdash T_A, T} \quad (\top)}{\vdash !A \otimes T_A, T} \quad (\otimes) \quad \frac{\vdash !A \quad \overline{\vdash T_A, T} \quad (\top_A)}{\vdash !A \otimes T_A, T} \quad (\otimes) \\ \frac{\vdash !A \otimes T_A, T \oplus T \quad (\oplus_i)}{\vdash !A \otimes T_A, T \oplus T} \quad \frac{\vdash !A \otimes T_A, T \oplus T \quad (\otimes)}{\vdash !A \otimes T_A, T \oplus T} \\ \text{H} \quad \frac{\vdash !A \quad \overline{\vdash T_A, T \oplus T} \quad (\top_A)}{\vdash T_A, T \oplus T} \quad (\otimes) \quad \frac{\vdash !A \quad \overline{\vdash T_A, T \oplus T} \quad (\top_A)}{\vdash !A \otimes T_A, T \oplus T} \quad (\otimes) \\ \frac{\vdash !A \otimes T_A, T \oplus T \quad (\otimes)}{\vdash !A \otimes T_A, T \oplus T} \end{array}$$

# Back-Up: Rule Commutations & Provability

## Lemma

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## Proof.

♦ If  $A$  is not provable ( $\iff !A$  is not provable)

We can compute the full equivalence class in this case:

$$\frac{\frac{\overline{\vdash !A \otimes T_A, T} \text{ } (\top)}{\vdash !A \otimes T_A, T \oplus T} \text{ } (\oplus_i)}{\vdash !A \otimes T_A, T \oplus T} \text{ } (\oplus_i) \quad \vdash \quad \frac{\frac{\overline{\vdash !A, T} \text{ } (\top) \quad \overline{\vdash T_A} \text{ } (\top_A)}{\vdash !A \otimes T_A, T} \text{ } (\otimes)}{\vdash !A \otimes T_A, T \oplus T} \text{ } (\oplus_i) \quad \vdash \quad \frac{\frac{\overline{\vdash !A, T} \text{ } (\top)}{\vdash !A, T \oplus T} \text{ } (\oplus_i) \quad \overline{\vdash T_A} \text{ } (\top_A)}{\vdash !A \otimes T_A, T \oplus T} \text{ } (\otimes)}$$

Remark we use  $!A$  instead of  $A$  to prevent commutations in  $\overline{\vdash !A, T} \text{ } (\top)$ , as  $!$  is the sole rule not commuting with  $T$



# Back-up: Isomorphisms in Linear Logic

## Isomorphism $A \simeq B$

Proofs  $\pi$  of  $A \vdash B$  and  $\rho$  of  $B \vdash A$  such that

$$\frac{\pi \quad \rho}{A \vdash B \quad B \vdash A} \text{ (cut)} =_{\beta\eta o} \frac{}{A \vdash A} \text{ (ax)} \quad \text{and} \quad \frac{\rho \quad \pi}{B \vdash A \quad A \vdash B} \text{ (cut)} =_{\beta\eta o} \frac{}{B \vdash B} \text{ (ax)}$$

## Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
Commutativity	$A \otimes B \simeq B \otimes A$	$A \wp B \simeq B \wp A$
Neutrality	$A \otimes 1 \simeq A$	$A \wp \perp \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$	$A \wp (B \& C) \simeq (A \wp B) \& (A \wp C)$
Annihilation	$A \otimes 0 \simeq 0$	$A \wp \top \simeq \top$
Seely	$!(A \& B) \simeq !A \otimes !B$ $! \top \simeq 1$	$? (A \oplus B) \simeq ?A \wp ?B$ $?0 \simeq \perp$
Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$
		$\forall X \top \simeq \top$ $\forall X \forall Y A \simeq \forall Y \forall X A$
		$\exists X \exists Y A \simeq \exists Y \exists X A$

\* if  $X$  not free in  $A$

# Back-up: Isomorphisms in Linear Logic

## Isomorphism $A \simeq B$

Proofs  $\pi$  of  $A \vdash B$  and  $\rho$  of  $B \vdash A$  such that

$$\frac{\pi \quad \rho}{A \vdash B \quad B \vdash A} \text{ (cut)} =_{\beta\eta o} \frac{}{A \vdash A} \text{ (ax)} \quad \text{and} \quad \frac{\rho \quad \pi}{B \vdash A \quad A \vdash B} \text{ (cut)} =_{\beta\eta o} \frac{}{B \vdash B} \text{ (ax)}$$

## Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
Commutativity	$A \otimes B \simeq B \otimes A$	$A \wp B \simeq B \wp A$
Neutrality	$A \otimes 1 \simeq A$	$A \wp \perp \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$	$A \wp (B \& C) \simeq (A \wp B) \& (A \wp C)$
Annihilation	$A \otimes 0 \simeq 0$	$A \wp \top \simeq \top$
Seely	$!(A \& B) \simeq !A \otimes !B$ $!T \simeq 1$	$?(A \oplus B) \simeq ?A \wp ?B$ $?0 \simeq \perp$
Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$
		$\forall X \top \simeq \top$ $\exists X 0 \simeq 0$
		$\forall X \forall Y A \simeq \forall Y \forall X A$ $\exists X \exists Y A \simeq \exists Y \exists X A$

\* if  $X$  not free in  $A$

With the  $! - ?_c$ ,  $! - ?_w$  and  $?_c - ?_w$  commutations and reductions

$$\frac{\vdash A, ?B, ?B, ?\Gamma \quad (!)}{\vdash !A, ?B, ?B, ?\Gamma \quad (?_c)} \equiv \frac{\vdash A, ?B, ?B, ?\Gamma \quad (\pi)}{\vdash !A, ?B, ?\Gamma \quad (!)} \quad \frac{\vdash A, ?\Gamma \quad (!)}{\vdash !A, ?\Gamma \quad (?_w)} \equiv \frac{\vdash A, ?\Gamma \quad (!)}{\vdash !A, ?B, ?\Gamma \quad (?_w)} \quad \frac{\vdash ?A, \Gamma \quad (?_w)}{\vdash ?A, ?A, \Gamma \quad (?_c)} \rightarrow \frac{\vdash ?A, \Gamma \quad (\pi)}{\vdash ?A, \Gamma \quad (?_c)}$$

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## Isomorphism $A \simeq B$

Proofs  $\pi$  of  $A \vdash B$  and  $\rho$  of  $B \vdash A$  such that

$$\frac{\pi \quad \rho}{A \vdash B \quad B \vdash A} \text{ (cut)} =_{\beta\eta\circ} \frac{}{A \vdash A} \text{ (ax)} \quad \text{and} \quad \frac{\rho \quad \pi}{B \vdash A \quad A \vdash B} \text{ (cut)} =_{\beta\eta\circ} \frac{}{B \vdash B} \text{ (ax)}$$

## Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$	$A \wp (B \wp C) \simeq (A \wp B) \wp C$ $A \& (B \& C) \simeq (A \& B) \& C$
Commutativity	$A \otimes B \simeq B \otimes A$	$A \wp B \simeq B \wp A$
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Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$	$A \wp (B \& C) \simeq (A \wp B) \& (A \wp C)$
Annihilation	$A \otimes 0 \simeq 0$	$A \wp \top \simeq \top$
Seely	$!(A \& B) \simeq !A \otimes !B$ $!1 \simeq 1$	$?!(A \oplus B) \simeq ?A \wp ?B$ $?0 \simeq \perp$
Quantifiers	$\forall X (A \& B) \simeq \forall X A \& \forall X B$ $\forall X B \wp A \simeq \forall X (B \wp A)^*$	$\exists X (A \oplus B) \simeq \exists X A \oplus \exists X B$ $\exists X B \otimes A \simeq \exists X (B \otimes A)^*$
Optional	$\forall X A \simeq A^{*\dagger}$	$\exists X A \simeq A^{*\dagger}$
		$1 \simeq \perp^\ddagger$
		$0 \simeq \top^\ddagger$

\* if  $X$  not free in  $A$

$$\textcolor{red}{†} \text{ if } \frac{\pi_B}{\vdash \exists X A, \Gamma} \text{ (}\exists\text{)} \equiv \frac{\vdash A[C/X], \Gamma}{\vdash \exists X A, \Gamma} \text{ (}\exists\text{)}$$

when  $\pi$  is “witness irrelevant”

$$\textcolor{red}{‡} \text{ if } \frac{\pi}{\vdash \Gamma \vdash \Gamma} \text{ (} mix_0 \text{)} \equiv \frac{\pi}{\vdash \Gamma} \text{ (} mix_2 \text{)}$$

♣ with  $\overline{\vdash \Gamma}$  (0)