Cut-Cut Commutations Are Not Superfluous

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Abstract

In the definition of a cut-elimination procedure, there often is a $cut - cut$ commutation allowing to swap two cut-rules. This allows for instance to fully eliminate a chosen cut-rule without reducing other cut-rules in a proof. One may wonder whether such $cut - cut$ steps are superfluous. We prove it is not in linear logic: some cut-free forms of a proof can only be reached using a $cut - cut$ commutation.

1 Introduction

We consider the sequent calculus of linear logic [Gir87]. As many logics, it has a cut-rule and a rewriting system called *cut-elimination*, which details how to reach a cut-free (or normal) proof starting from a proof with possibly many cut-rules. Cut-elimination has been studied expensively in this system and more generally in linear logic (but mainly for its proof-net syntax): mostly its normalization [Acc13; DG99; LM08; Tor03; PT10] but also its confluence [CP05; Di24]. In particular, it is well-known one can reach a normal form without using any $cut - cut$ commutations. One may wonder whether such $cut - cut$ steps are completely superfluous: can the same normal forms be reached with and without $cut - cut$ commutations? We answer this question negatively, exhibiting a simple counter-example. This counter-example is even one for the simpler multiplicative-exponential fragment of linear logic, that we will consider here for simplicity's sake.

2 Definitions

We consider the sequent calculus of multiplicative-exponential linear logic [Gir87]. Formulas are given by the following grammar, with X an atom in a given countable set:

$$A, B ::= X^+ \mid X^- \mid A \parr B \mid A \otimes B \mid \bot \mid 1 \mid ?A \mid !A$$

We define on formulas a function $(\cdot)^\perp$ called **orthogonality**, also named negation or duality, through the following inductive definition:

$$\frac{}{\vdash A^\perp, A}\ (ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

Figure 1: Rules of Linear Logic

$$\begin{aligned} (X^+)^\perp &= X^- & (X^-)^\perp &= X^+ \\ (A \parr B)^\perp &= B^\perp \otimes A^\perp & (A \otimes B)^\perp &= B^\perp \parr A^\perp \\ \bot^\perp &= 1 & 1^\perp &= \bot \\ (?A)^\perp &= !A^\perp & (!A)^\perp &= ?A^\perp \end{aligned}$$

Sequents are sets of (occurrences of) formulas written as $\vdash A_1, \ldots, A_n$. Rules of linear logic are given on Fig. 1, where A and B stand for arbitrary formulas, Γ and Δ for sets of (occurrences of) formulas. By $?\Gamma$ we mean that each formula of Γ is a ?-formula.

As in many systems with a *cut*-rule, the *cut*-rule is *admissible* in this logic: the same sequents can be proved with and without the *cut*-rule. *Cut-elimination* is the procedure turning a proof into a *cut*-free one. As we will not use *ax*- $\otimes$- $\parr$- $?_c$- and $?_w$-rules, we give this rewriting system without those. For a full description, see [Di24, Chapter 1].

Definition 1. Cut-elimination is the rewriting system whose rules are described on Table 1, up to commuting the two branches of any *cut*-rule.

The "up to commutation" means we consider a version of each case with the left and right premises of any *cut*-rule swapped – *e.g.* the $cut - cut$ case gives 4 rewriting rules.

3 Cut-cut commutations allow to reach more normal forms

We show that removing the $cut - cut$ step in multiplicative-exponential linear logic (and thus in linear logic) narrows the set of normal forms.

Lemma 2. *There exist a proof π and a cut-free proof ϕ such that π reduces by cut-elimination to ϕ using a $cut - cut$ elimination step, but π does* not *reduce to ϕ without using a $cut - cut$ elimination step.*

$\bot - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot) \quad \dfrac{}{\vdash 1}(1)}{\vdash \Gamma}(cut)$	$\xrightarrow{\beta}$	$\overset{\pi}{\vdash \Gamma}$
$?d - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash ?A^{\perp}, \Gamma}(?d) \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}(!)}{\vdash \Gamma, ?\Delta}(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, ?\Delta}}{\vdash \Gamma, ?\Delta}(cut)$
$\bot - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, \bot, \Gamma}(\bot) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \bot, \Gamma, \Delta}(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}(cut)}{\vdash \bot, \Gamma, \Delta}(\bot)$
$?d - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma}}{\vdash A^{\perp}, ?B, \Gamma}(?d) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}(cut)}{\vdash ?B, \Gamma, \Delta}(?d)$
$! - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma}}{\vdash ?A^{\perp}, !B, ?\Gamma}(!) \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}(!)}{\vdash !B, ?\Gamma, ?\Delta}(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma} \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}(!)}{\vdash B, ?\Gamma, ?\Delta}(cut)}{\vdash !B, ?\Gamma, ?\Delta}(!)$
$cut - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B^{\perp}, \Gamma, \Delta}(cut) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, \Gamma, \Sigma}(cut) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}(cut)$

Table 1: Cut-elimination – $\bot$, 1, $?_d$ and ! cases

Proof. Set π the following proof tree:

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash ?1}(?_d) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1, \bot}(\bot)}{\vdash 1, ?\bot}(?_d)}{\vdash ?1, ?\bot}(?_d)}{\vdash \bot, ?1, ?\bot}(\bot)}{\vdash !\bot, ?1, ?\bot}(!) \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash !1}(!)}{\vdash !1, \bot}(\bot)}{\vdash !\bot, ?1, \bot}(cut)}{\vdash ?1, \bot}(cut)$$

(This proof tree with an interactive cut-elimination procedure is available here using C1ick ⅋ c⊗LLec⊥.)

Without using $cut - cut$ steps, cut-elimination on π leads to a unique normal form:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{\scriptsize $(?_d)$}
\UnaryInfC{$\vdash ?1$}
\RightLabel{\scriptsize $(\bot)$}
\UnaryInfC{$\vdash ?1, \bot$}
\end{prooftree}
$$

For our purpose, it is enough to prove any normal form reached without a $cut - cut$ step has a $\bot$-rule at its root. This is the case: one first has to apply a $\bot - cut$ step on the upper cut-rule; then, whatever happens on the upper cut-rule, the first step involving the bottom cut-rule must be a $\bot - cut$ step, resulting in a $\bot$-rule at the root of the proof.

Meanwhile, consider the result of applying a $cut - cut$ step first:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{\scriptsize $(?_d)$}
\UnaryInfC{$\vdash ?1$}
\AxiomC{}
\RightLabel{\scriptsize (1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{\scriptsize $(\bot)$}
\UnaryInfC{$\vdash 1, \bot$}
\RightLabel{\scriptsize $(?_d)$}
\UnaryInfC{$\vdash 1, ?\bot$}
\RightLabel{\scriptsize $(?_d)$}
\UnaryInfC{$\vdash ?1, ?\bot$}
\RightLabel{\scriptsize $(\bot)$}
\UnaryInfC{$\vdash \bot, ?1, ?\bot$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !\bot, ?1, ?\bot$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash ?1, ?\bot$}
\AxiomC{}
\RightLabel{\scriptsize (1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{\scriptsize $(!)$}
\UnaryInfC{$\vdash !1$}
\RightLabel{\scriptsize $(\bot)$}
\UnaryInfC{$\vdash !1, \bot$}
\RightLabel{\scriptsize (cut)}
\BinaryInfC{$\vdash ?1, \bot$}
\end{prooftree}
$$

Here, one can apply a $?_d - !$ step on the upper cut-rule, then a $\bot - 1$ step still on this cut-rule, followed by a $?_d - cut$ step on the (bottom) cut-rule, leading to a proof with a $?_d$-rule at its root. In particular, one can reach the following normal form ϕ:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize (1)}
\UnaryInfC{$\vdash 1$}
\RightLabel{\scriptsize $(\bot)$}
\UnaryInfC{$\vdash 1, \bot$}
\RightLabel{\scriptsize $(?_d)$}
\UnaryInfC{$\vdash ?1, \bot$}
\end{prooftree}
$$

□

4 Conclusion

Perhaps surprisingly, $cut - cut$ commutative steps of cut-elimination are not totally useless: they allow to reach more normal forms in multiplicative-exponential linear logic, and so in linear logic. Nonetheless, our example seems hard to adapt without a contextual rule such as ! – which needs the sequent it is applied on to be of the shape $!A, ?\Gamma$. Hence, we conjecture that in the multiplicative-additive fragment of linear logic the same normal forms can be reached with and without $cut - cut$ commutative steps.

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