

Cut-Expansion in Proof-Nets of Multiplicative Linear Logic

Rémi Di Guardia

IRIF (CNRS, Université Paris Cité)

Lumini, 9 October 2025



Introduction

Cut-reduction

$$\frac{\frac{\frac{\pi_1}{\vdash B^\perp, A^\perp, \Gamma} (\wp) \quad \frac{\pi_2 \quad \pi_3}{\vdash A, \Delta \quad \vdash B, \Sigma} (\otimes)}{\vdash A \otimes B, \Delta, \Sigma} (cut)}{\vdash \Gamma, \Delta, \Sigma}$$
 \longrightarrow $\frac{\frac{\pi_1 \quad \pi_3}{\vdash B^\perp, A^\perp, \Gamma \quad \vdash B, \Sigma} (cut) \quad \frac{\pi_2}{\vdash A^\perp, \Gamma, \Sigma} (cut)}{\vdash \Gamma, \Delta, \Sigma}$

Very useful

Example of result: can reduce to a cut-free proof (weak normalization)

Applications:

- No proof of falsity \perp or 0
- Proof search
- ...

Introduction

Cut-expansion

$$\frac{\frac{\pi_1}{\vdash B^\perp, A^\perp, \Gamma} (\wp) \quad \frac{\pi_2 \quad \pi_3}{\vdash A, \Delta \quad \vdash B, \Sigma} (\otimes)}{\vdash B^\perp \wp A^\perp, \Gamma} \text{ (cut)} \quad \leftarrow \quad \frac{\frac{\pi_1 \quad \pi_3}{\vdash B^\perp, A^\perp, \Gamma \quad \vdash B, \Sigma} (\text{cut}) \quad \frac{\pi_2}{\vdash A, \Delta} (\text{cut})}{\vdash A^\perp, \Gamma, \Sigma \quad \vdash \Gamma, \Delta, \Sigma}$$

Useful?

Example of result: can expand to a proof with only one cut-rule, at its root

Applications:

- Craig's interpolation – following [Sau25]¹ and [FOS25]²
- Denotational semantic – used in [EFP24]³

¹Interpolation as Cut-Introduction: On the Computational Content of Craig-Lyndon Interpolation

²On Correctness, Sequentialization and Interpolation

³Bayesian Networks and Proof-Nets: a proof-theoretical account of Bayesian Inference

Introduction

Cut-expansion

$$\frac{\frac{\frac{\pi_1}{\vdash B^\perp, A^\perp, \Gamma} (\wp) \quad \frac{\pi_2 \quad \pi_3}{\vdash A, \Delta \quad \vdash B, \Sigma} (\otimes)}{\vdash A \otimes B, \Delta, \Sigma} (cut)}{\vdash \Gamma, \Delta, \Sigma}$$
 ←
$$\frac{\frac{\pi_1 \quad \pi_3}{\vdash B^\perp, A^\perp, \Gamma \quad \vdash B, \Sigma} (cut) \quad \frac{\pi_2}{\vdash A, \Delta} (cut)}{\vdash A^\perp, \Gamma, \Sigma \quad \vdash \Gamma, \Delta, \Sigma}$$

Useful?

Example of result: can expand to a proof with only one cut-rule, at its root

Applications:

- Craig's interpolation – following [Sau25]¹ and [FOS25]²
- Denotational semantic – used in [EFP24]³

This talk: Simple proof of this result in *proof-nets*, adaptable enough to be transferred in most fragments of LL

¹Interpolation as Cut-Introduction: On the Computational Content of Craig-Lyndon Interpolation

²On Correctness, Sequentialization and Interpolation

³Bayesian Networks and Proof-Nets: a proof-theoretical account of Bayesian Inference

Plan

- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

Proof-nets for unit-free Multiplicative Linear Logic

- Formula

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- Orthogonal

$$(X^\perp)^\perp = X \quad (A \otimes B)^\perp = B^\perp \wp A^\perp \\ (A \wp B)^\perp = B^\perp \otimes A^\perp$$

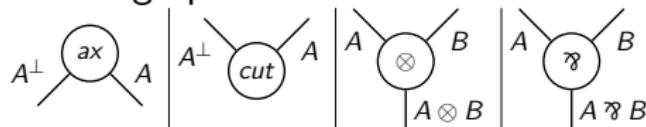
Proof-nets for unit-free Multiplicative Linear Logic

- Formula

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- Proof-Structure

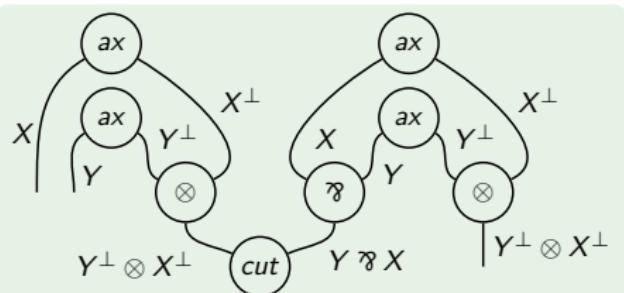
Partial graph built from:



Pending edges = sequent proved

- Orthogonal

$$(X^\perp)^\perp = X \quad (A \otimes B)^\perp = B^\perp \wp A^\perp \\ (A \wp B)^\perp = B^\perp \otimes A^\perp$$



Proof of $\vdash X, Y, Y^\perp \otimes X^\perp$

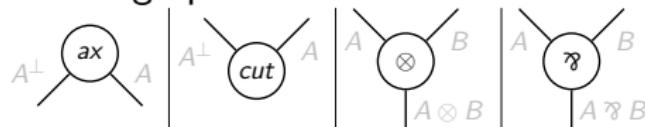
Proof-nets for unit-free Multiplicative Linear Logic

- Formula

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- Proof-Structure

Partial graph built from:

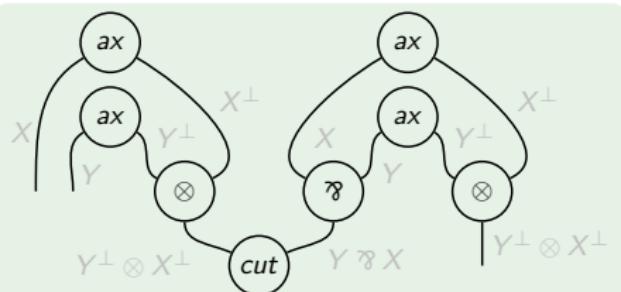


Pending edges = sequent proved

We do not care about edge labels.

- Orthogonal

$$(X^\perp)^\perp = X \quad (A \otimes B)^\perp = B^\perp \wp A^\perp \\ (A \wp B)^\perp = B^\perp \otimes A^\perp$$



Proof of $\vdash X, Y, Y^\perp \otimes X^\perp$

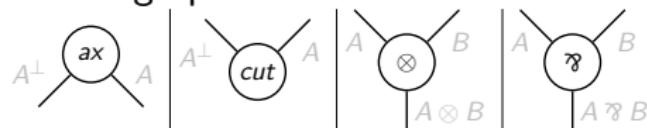
Proof-nets for unit-free Multiplicative Linear Logic

- Formula

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- Proof-Structure

Partial graph built from:



Pending edges = sequent proved

We do not care about edge labels.

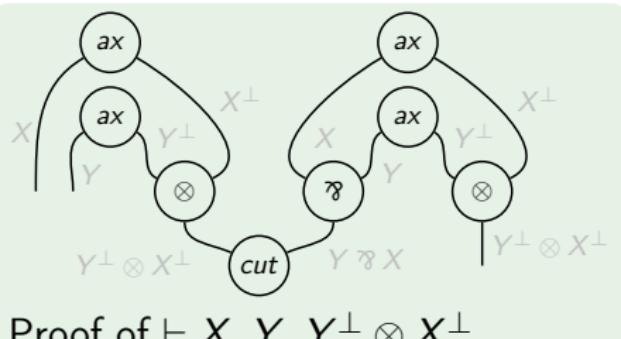
- Proof-Net

Danos-Regnier correctness criterion:
each cycle contains the two edges
above some \wp

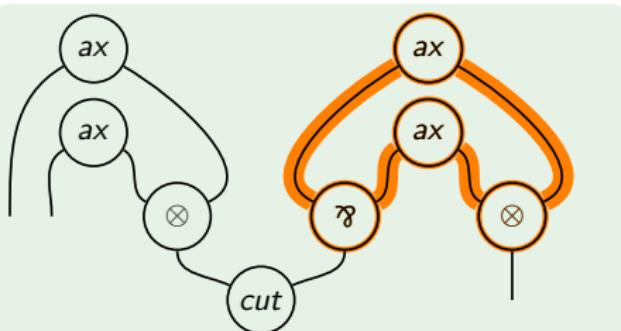
We work with *mix* to simplify, but
everything also holds without it.

- Orthogonal

$$(X^\perp)^\perp = X \quad (A \otimes B)^\perp = B^\perp \wp A^\perp \\ (A \wp B)^\perp = B^\perp \otimes A^\perp$$

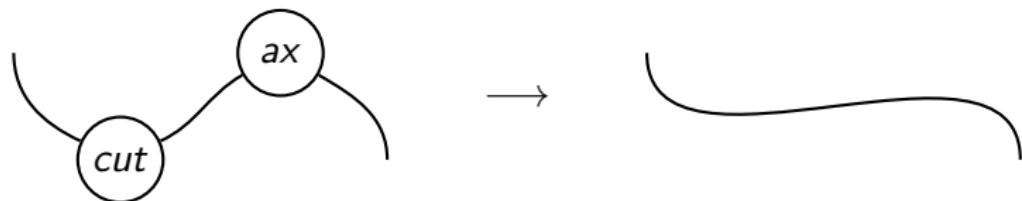


Proof of $\vdash X, Y, Y^\perp \otimes X^\perp$

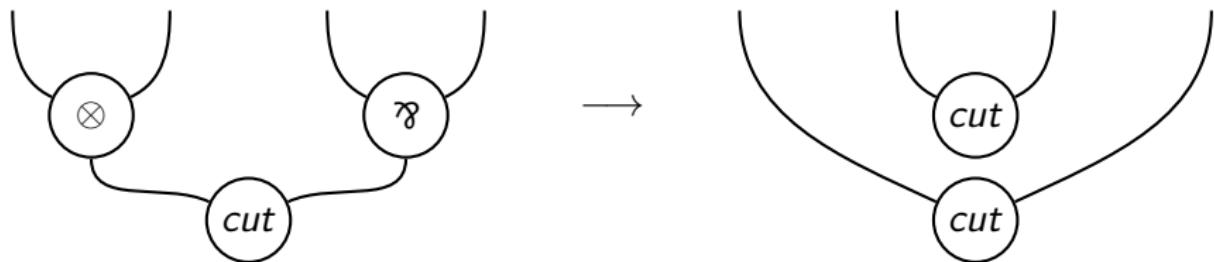


Cut-Reduction

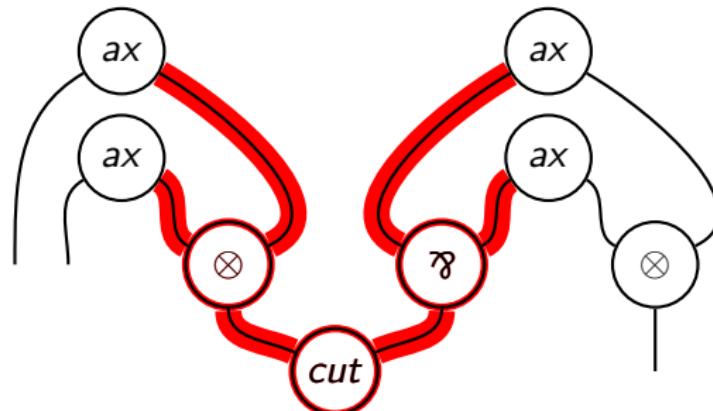
- **ax case**



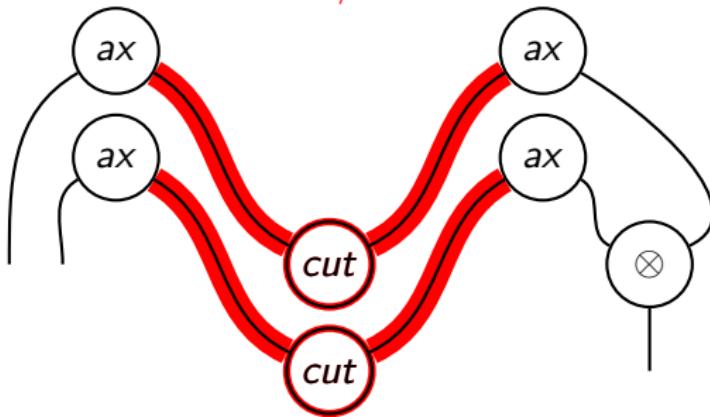
- $\wp - \otimes$ case



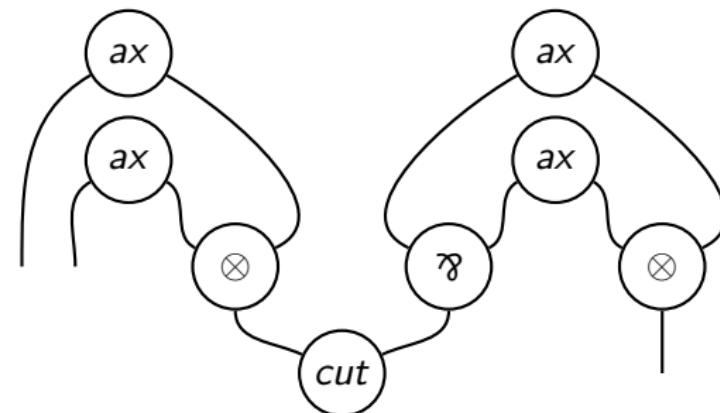
Cut-reduction on an example



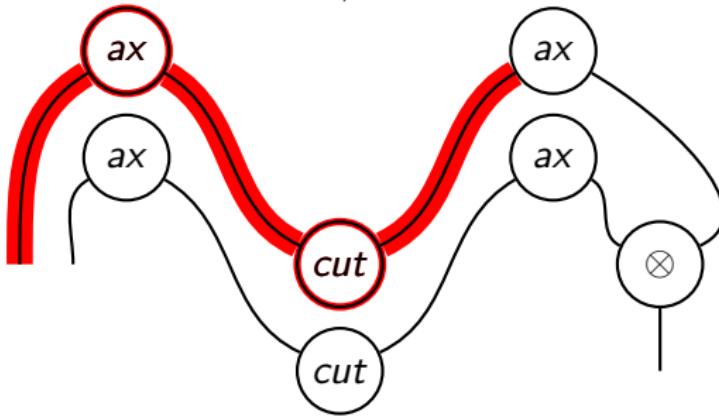
→



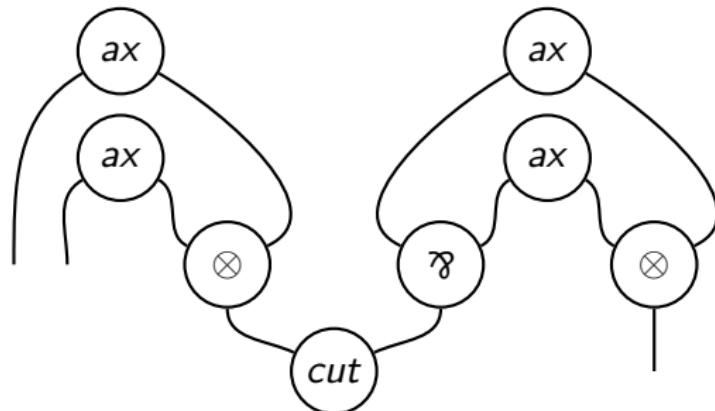
Cut-reduction on an example



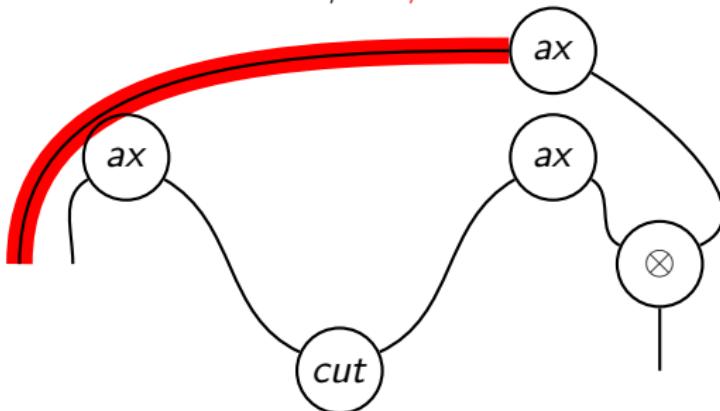
→



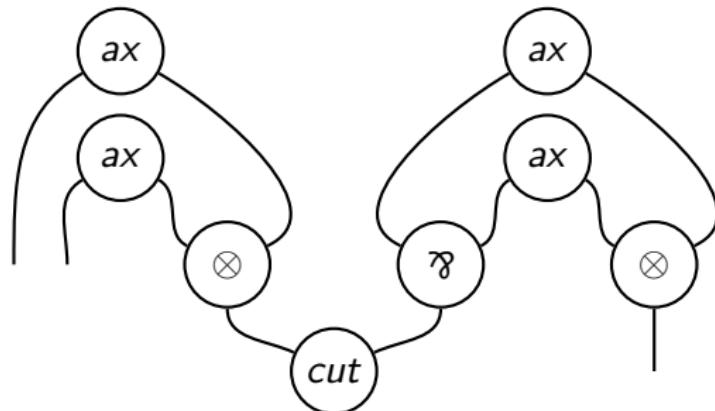
Cut-reduction on an example



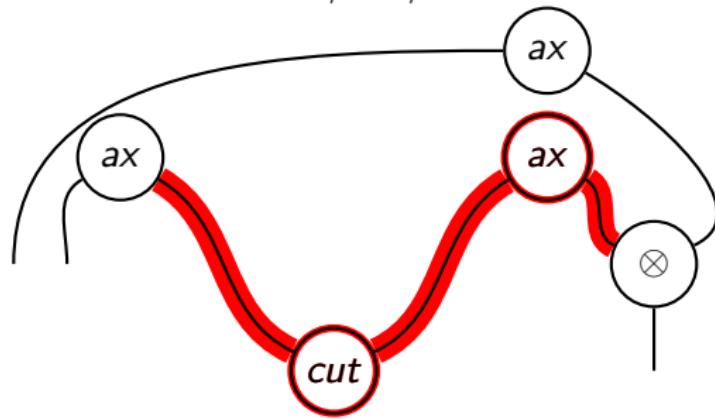
→ →



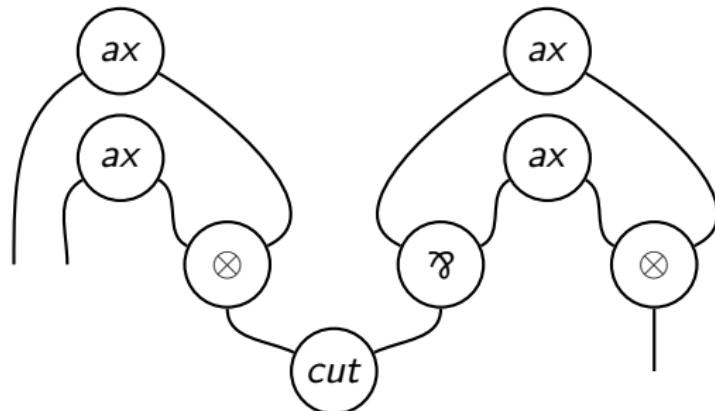
Cut-reduction on an example



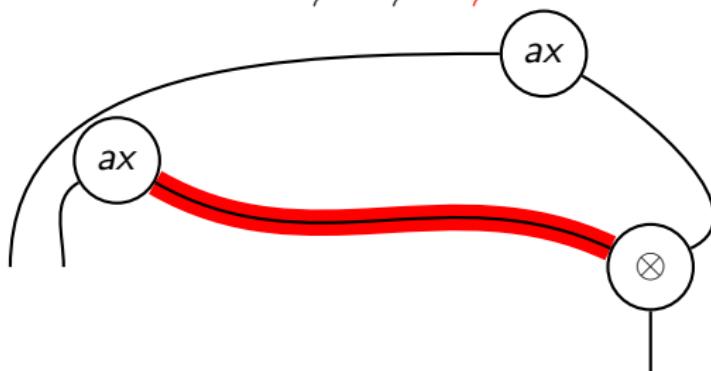
→ →



Cut-reduction on an example



→ → →



On Correctness and Cut-expansion

Has a cut? Can always reduce it.

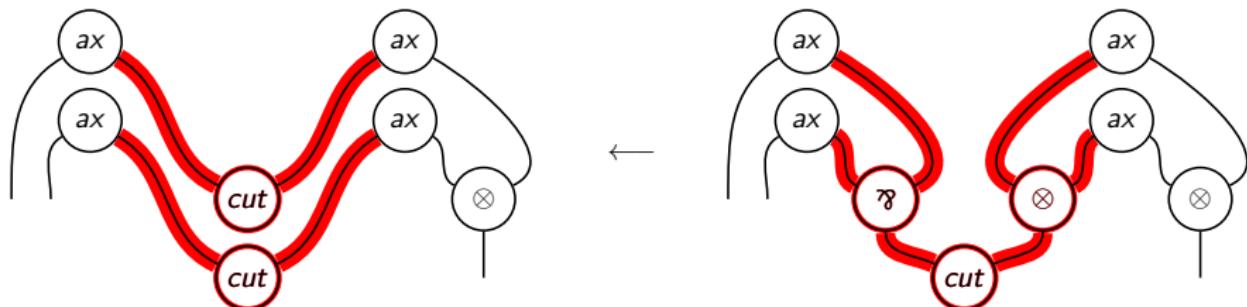
Lemma

If \mathcal{P} is a proof-net and $\mathcal{P} \rightarrow \mathcal{Q}$, then \mathcal{Q} is a proof-net.

Has 2 cuts? Can always expand them.

Fact

If \mathcal{P} is a proof-net and $\mathcal{P} \leftarrow \mathcal{Q}$, then \mathcal{Q} **may not be** a proof-net.



On Correctness and Cut-expansion

Has a cut? Can always reduce it.

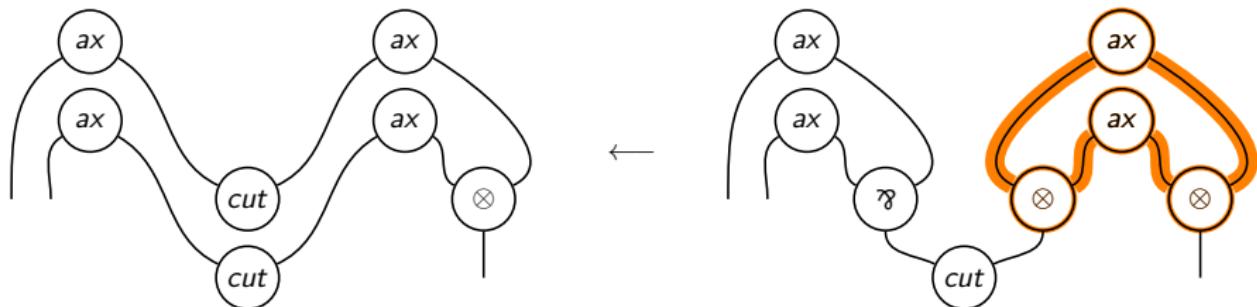
Lemma

If \mathcal{P} is a proof-net and $\mathcal{P} \rightarrow \mathcal{Q}$, then \mathcal{Q} is a proof-net.

Has 2 cuts? Can always expand them.

Fact

If \mathcal{P} is a proof-net and $\mathcal{P} \leftarrow \mathcal{Q}$, then \mathcal{Q} **may not be** a proof-net.



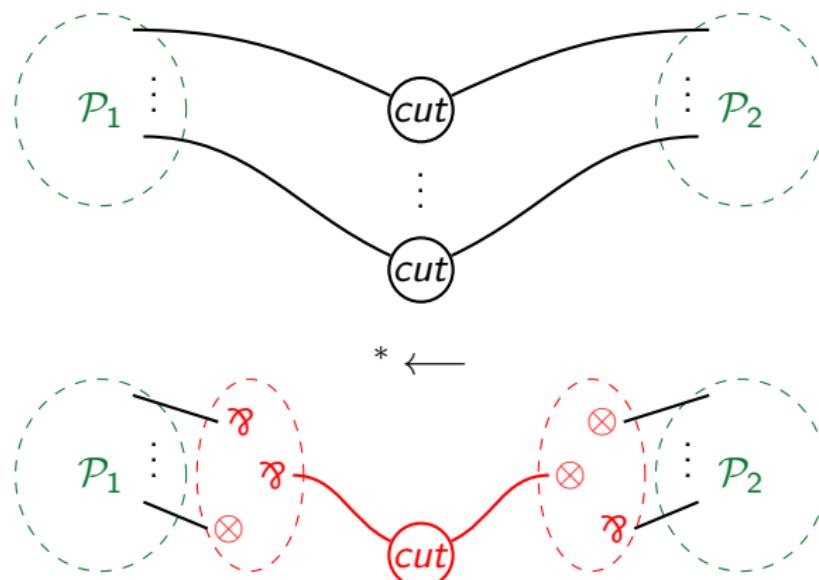
Plan

- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

Expanding to a unique cut

Proposition: Expanding to a unique cut

Take a proof-net \mathcal{P} made of two sub-graphs linked by $n \geq 1$ cut-vertices. There is a sequence of $n - 1$ $\wp - \otimes$ cut-expansion steps on these vertices yielding a **proof-net**.



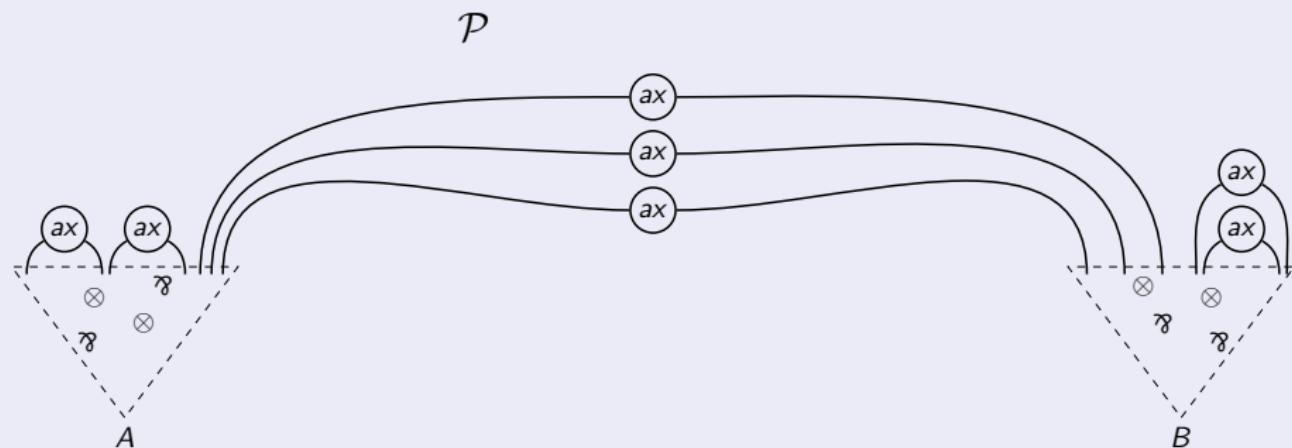
Application: Craig's Interpolation

Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P}^* \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.

New Proof.



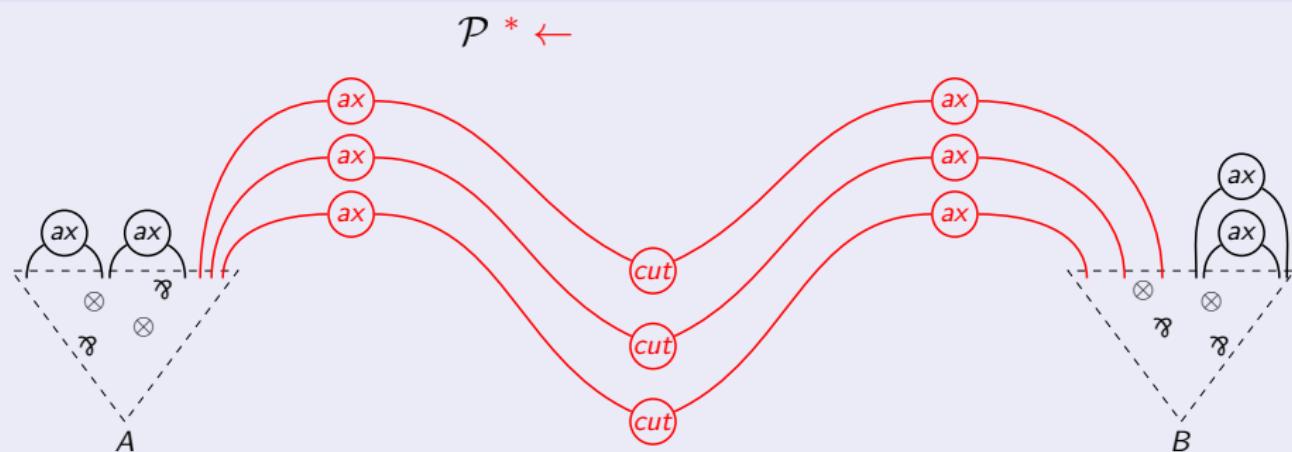
Application: Craig's Interpolation

Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P}^* \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.

New Proof.



Application: Craig's Interpolation

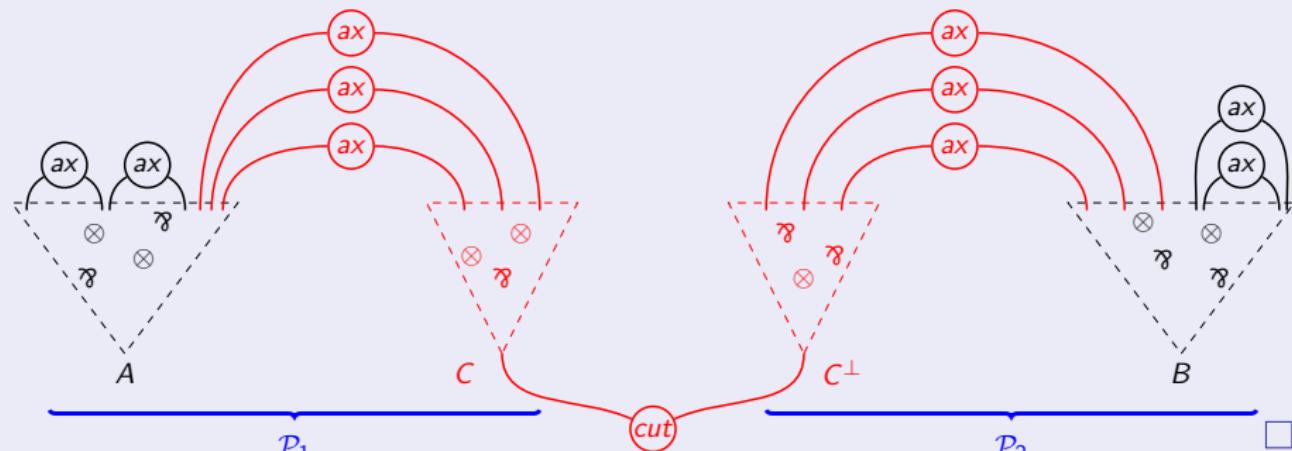
Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P}^* \leftarrow^* \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.

New Proof.

$$\mathcal{P}^* \leftarrow^* \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$$



Application: Craig's Interpolation

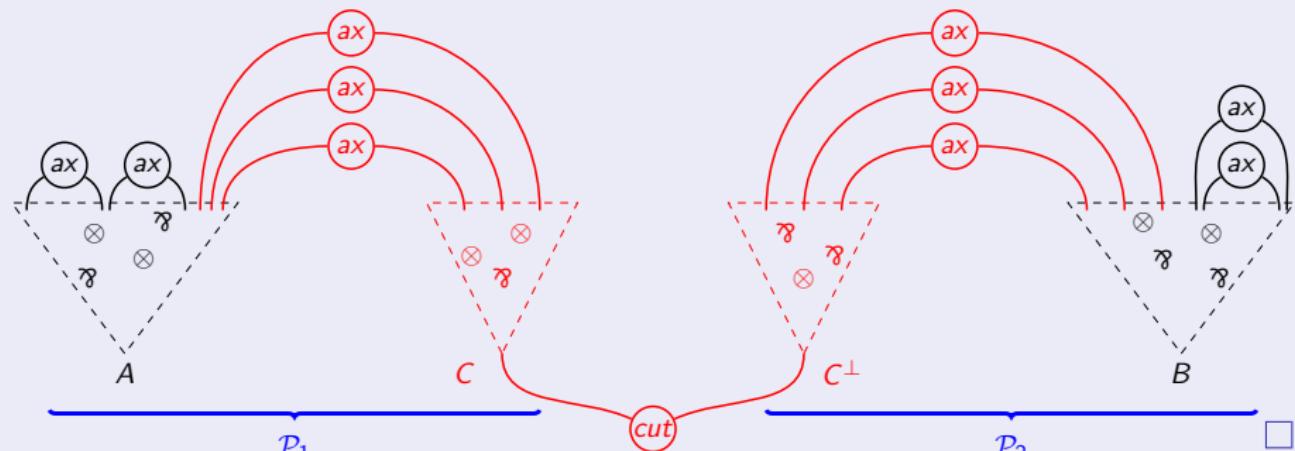
Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P} \xrightarrow{*} \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.
(Unless $\mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2$ with \mathcal{P}_1 of conclusion $\vdash A$ and \mathcal{P}_2 of conclusion $\vdash B$.)

New Proof.

$$\mathcal{P} \xrightarrow{*} \mathcal{P}_1 \text{ cut } \mathcal{P}_2$$



Differences with [FOS25]

Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a *connected* cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P}^* \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.

No mix-rule \implies connected proof-net.

[FOS25] also proves Craig's interpolation in proof-nets but:

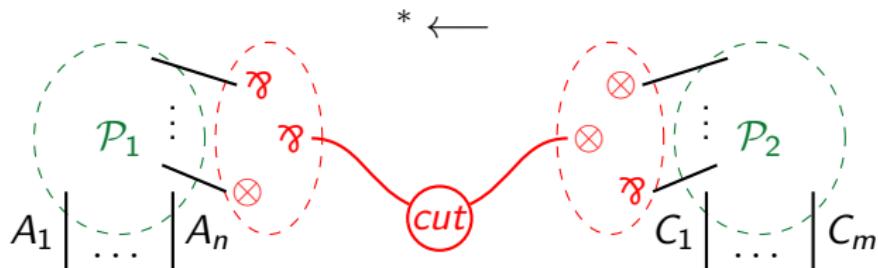
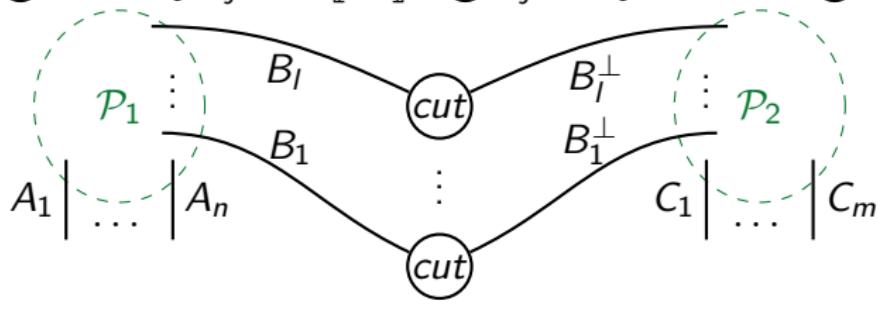
- it needs the **units** \perp and 1 , even if \mathcal{P} has none;
- it proceeds by **parsing** (*i.e.* goes top-down) so cannot have **mix**; meanwhile, we will proceed by **splitting vertices** (*i.e.* bottom-up).

Application: Denotational Semantic

Useful for **compositionality** (e.g. in [EFP24]): given *interpretations* $\llbracket \mathcal{P}_1 \rrbracket$ and $\llbracket \mathcal{P}_2 \rrbracket$, how to get the *interpretation* of the full net?

↪ Easy when **only one cut**, otherwise *typing* problem: given

$\llbracket \mathcal{P}_1 \rrbracket : \bigotimes A_i \rightarrow \wp B_j$ and $\llbracket \mathcal{P}_2 \rrbracket : \bigotimes B_j \rightarrow \wp C_k$, want $\bigotimes A_i \rightarrow \wp C_k$

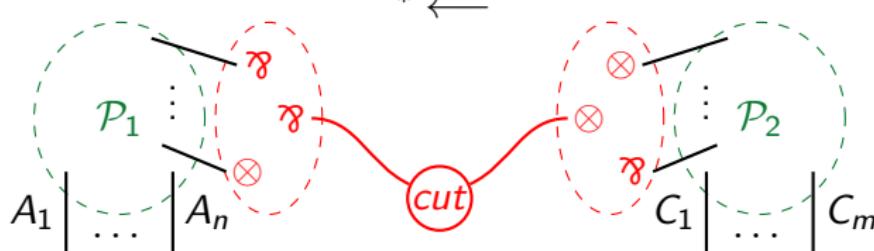
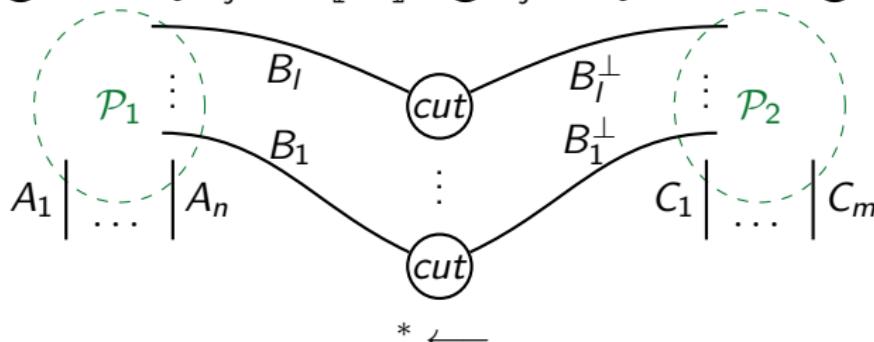


Application: Denotational Semantic

Useful for **compositionality** (e.g. in [EFP24]): given *interpretations* $\llbracket \mathcal{P}_1 \rrbracket$ and $\llbracket \mathcal{P}_2 \rrbracket$, how to get the *interpretation* of the full net?

↪ Easy when **only one cut**, otherwise *typing* problem: given

$\llbracket \mathcal{P}_1 \rrbracket : \bigotimes A_i \rightarrow \wp B_j$ and $\llbracket \mathcal{P}_2 \rrbracket : \bigotimes B_j \rightarrow \wp C_k$, want $\bigotimes A_i \rightarrow \wp C_k$



[EFP24]'solution with the *same result* as here, but using **parsing** again!

Plan

- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

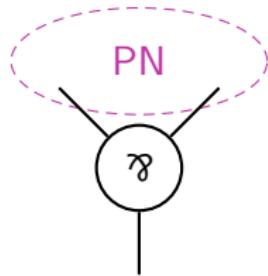
Reasonning

Proposition: Expanding to a unique *cut*

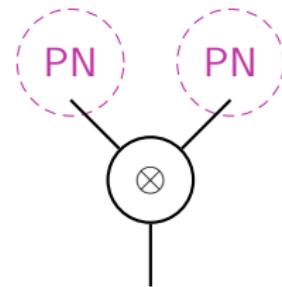
Take a proof-net \mathcal{P} made of two sub-graphs linked by $n \geq 1$ cut-vertices. There is a sequence of $n - 1$ \wp – \otimes cut-expansion steps on these vertices yielding a **proof-net**.

- Proof by induction on $\#\text{vertices}$
- Main consideration: when adding a \otimes , we ensure it creates no “bad cycle”
- Case study using a proof-net with a *cut* always has a **splitting vertex**:

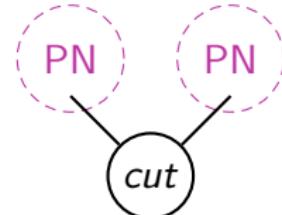
- \wp -vertex



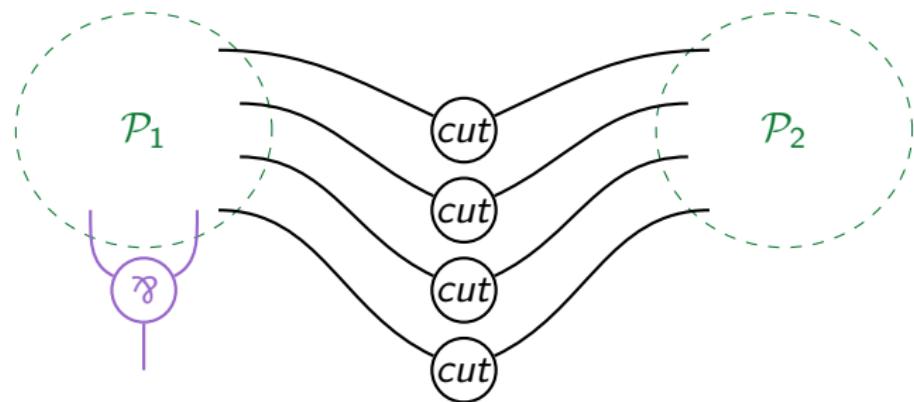
- \otimes -vertex



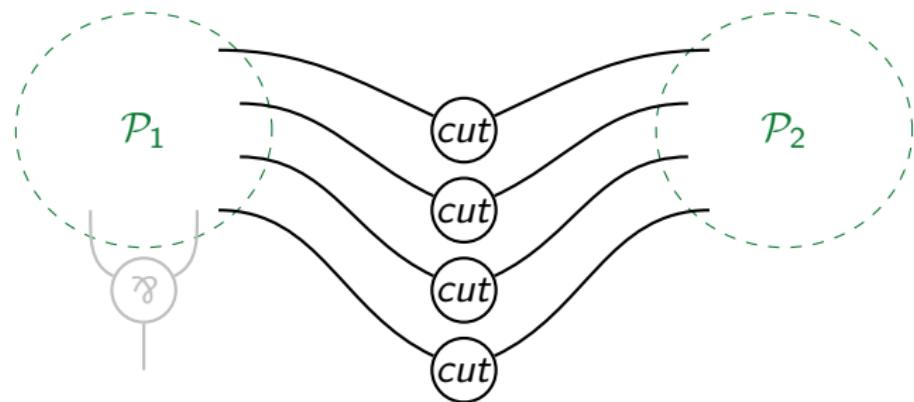
- *cut*-vertex



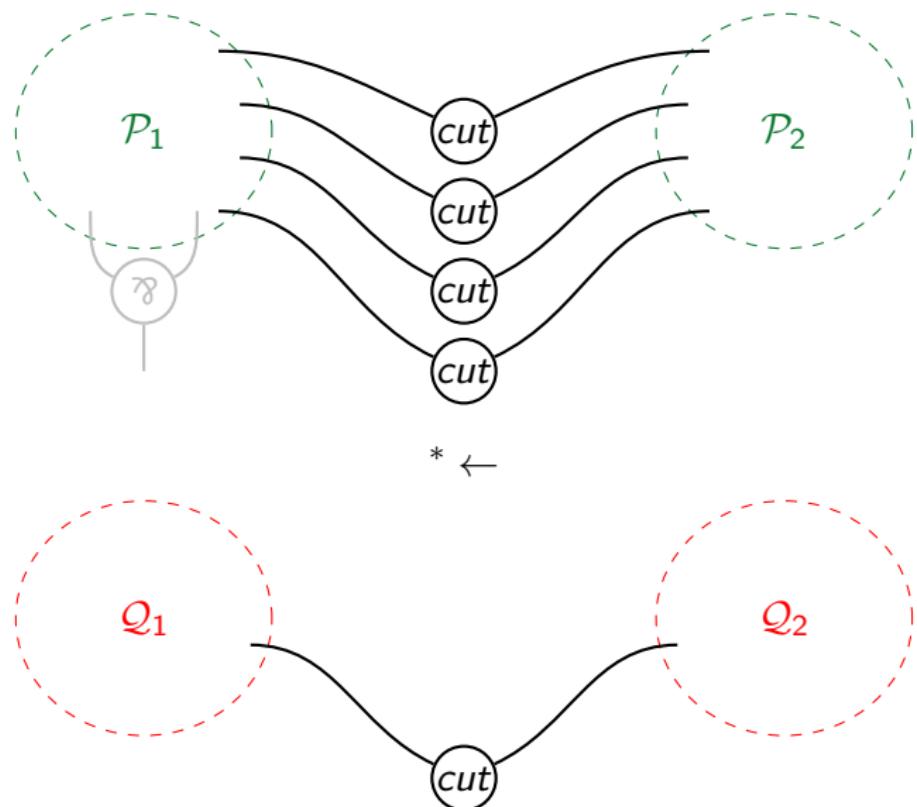
Splitting γ



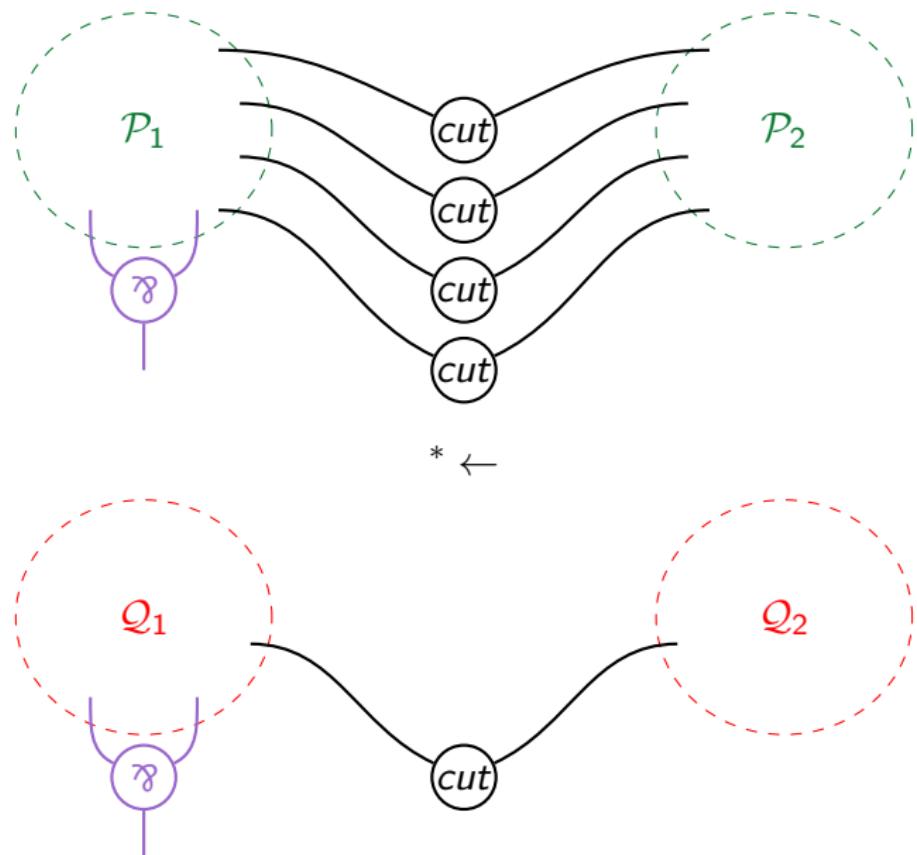
Splitting γ



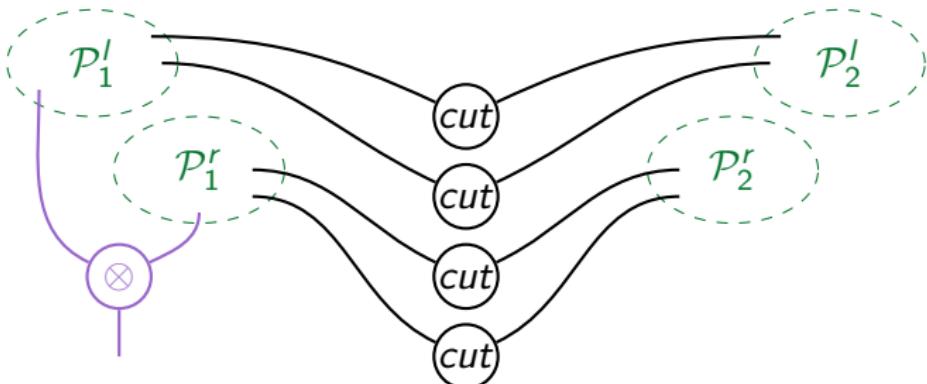
Splitting γ



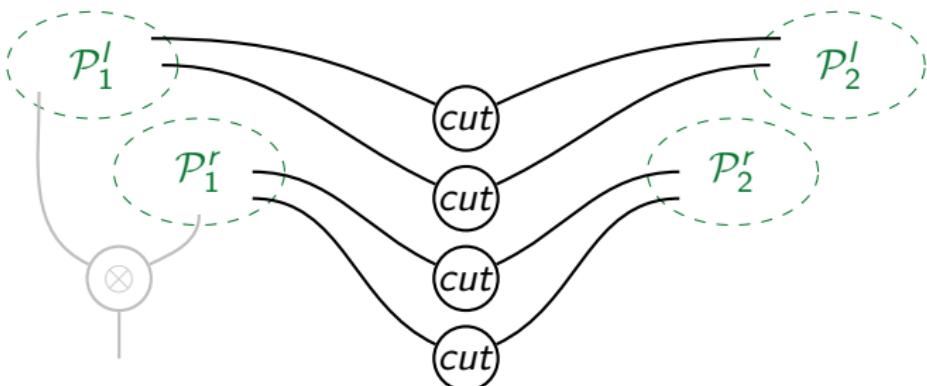
Splitting γ



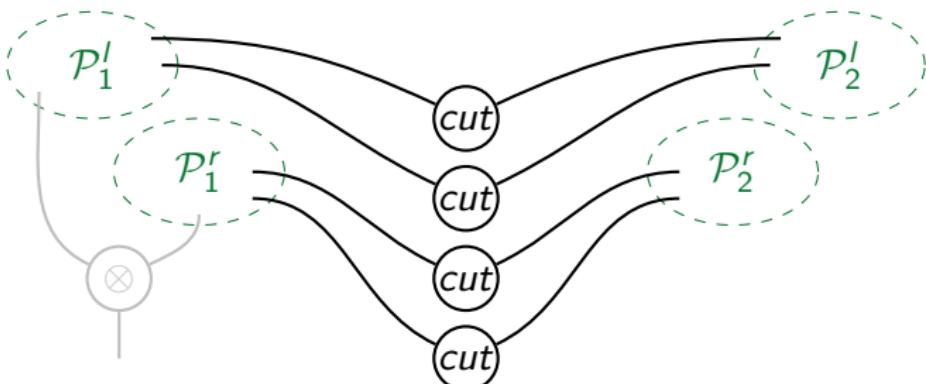
Splitting \otimes



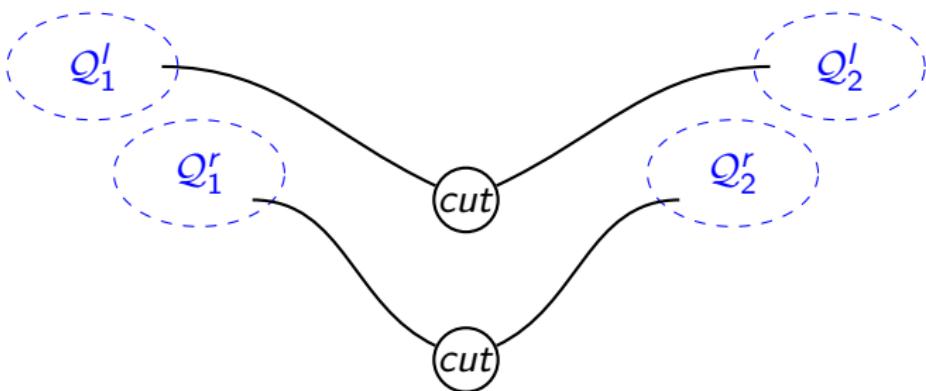
Splitting \otimes



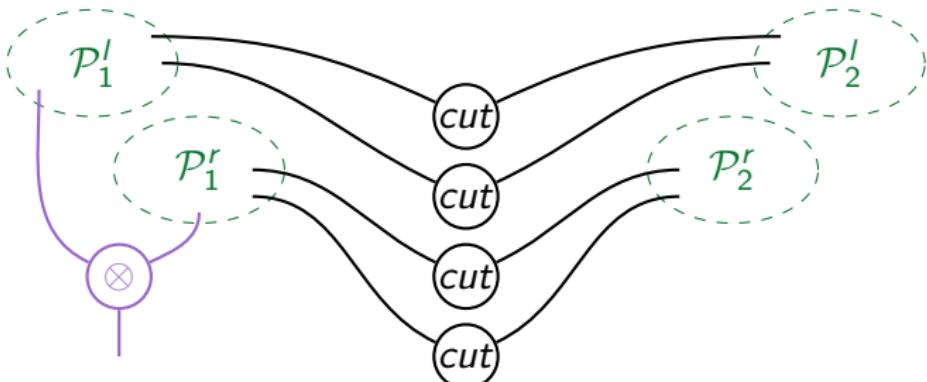
Splitting \otimes



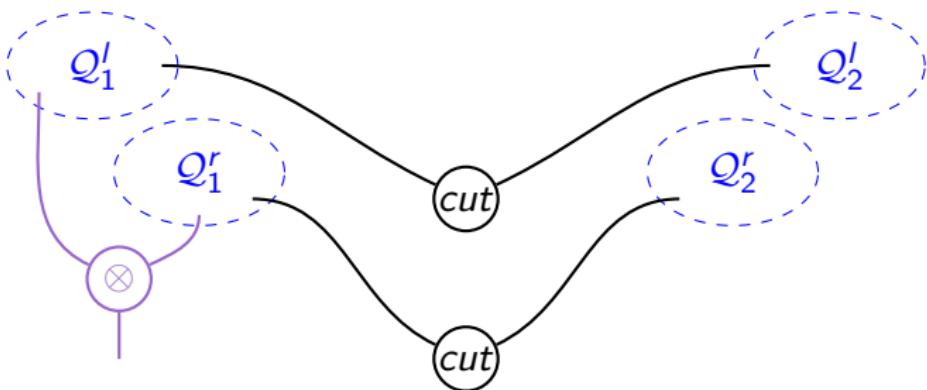
$* \leftarrow * \leftarrow$



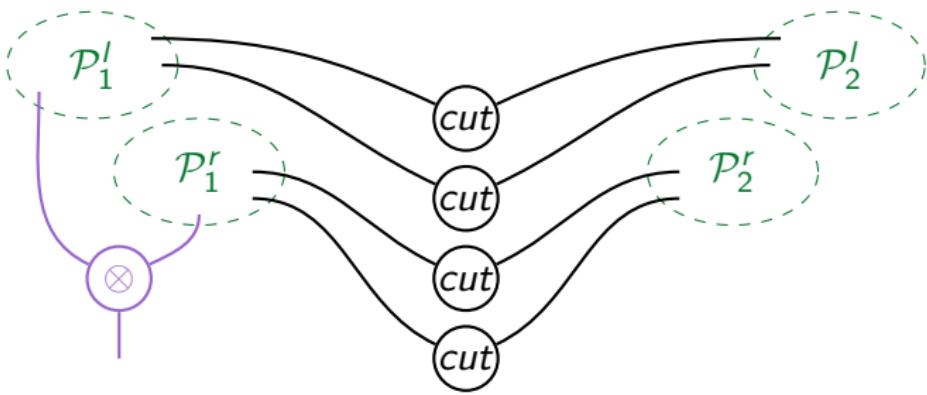
Splitting \otimes



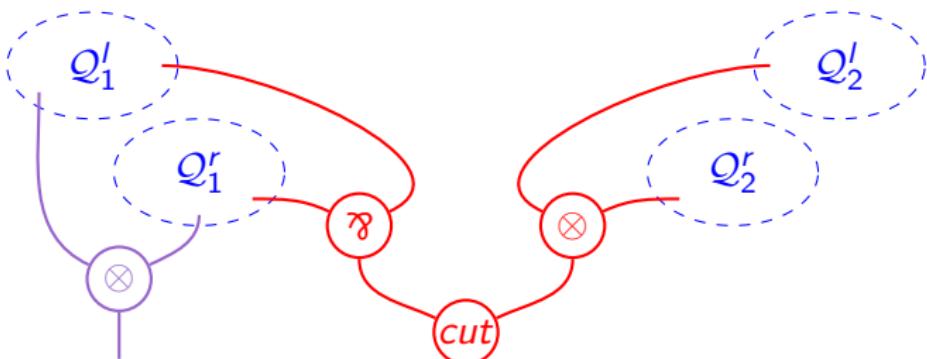
$* \leftarrow * \leftarrow$



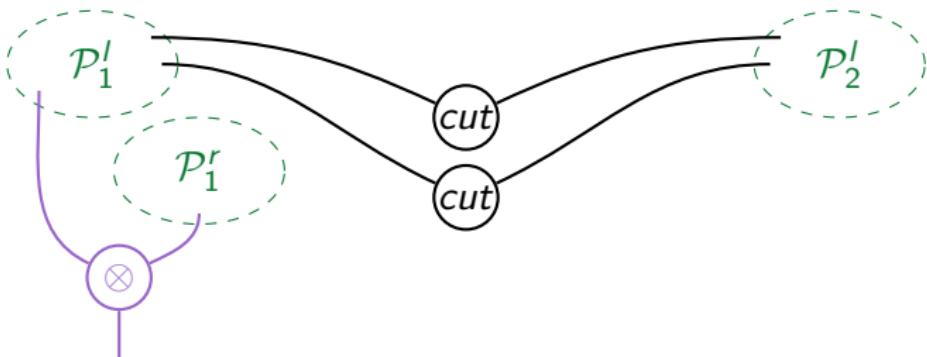
Splitting \otimes



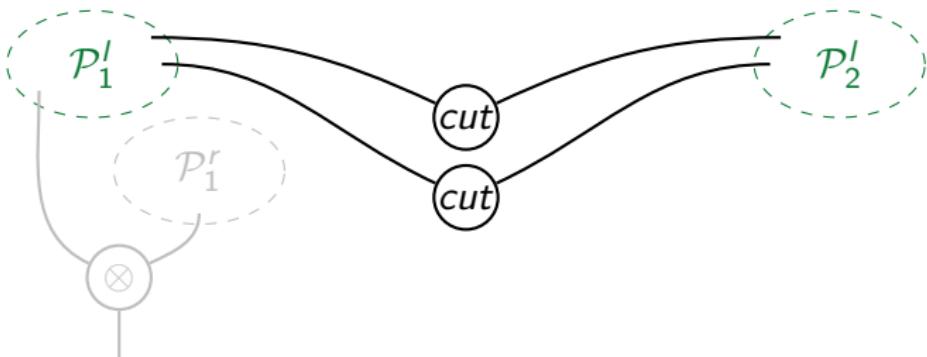
$* \leftarrow * \leftarrow \textcolor{red}{\leftarrow}$



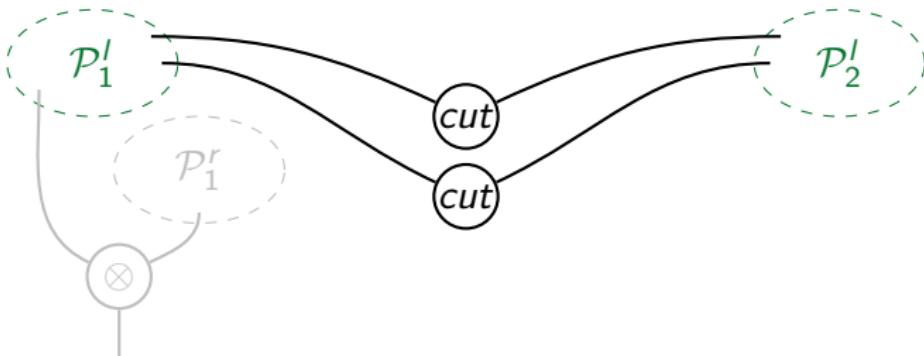
Splitting \otimes bis



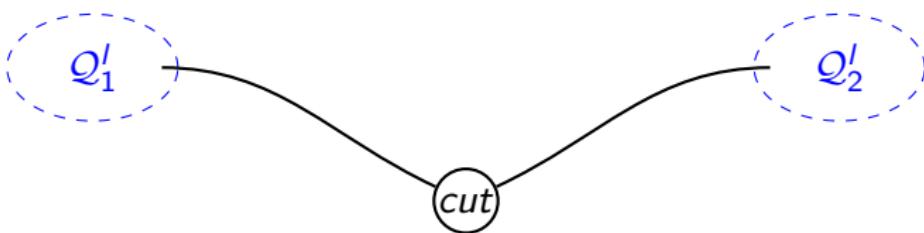
Splitting \otimes bis



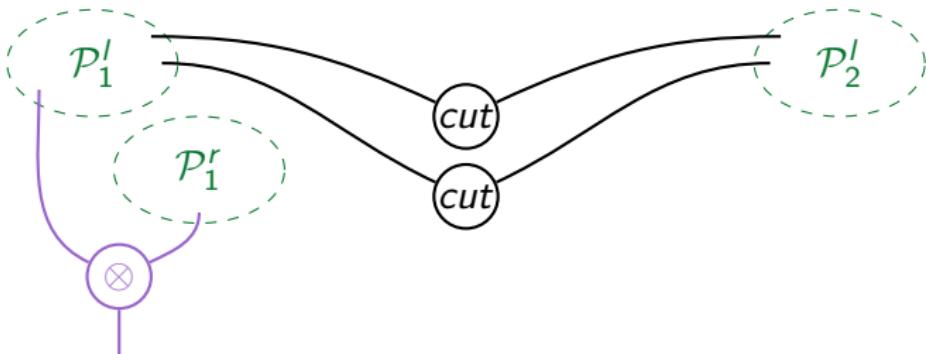
Splitting \otimes bis



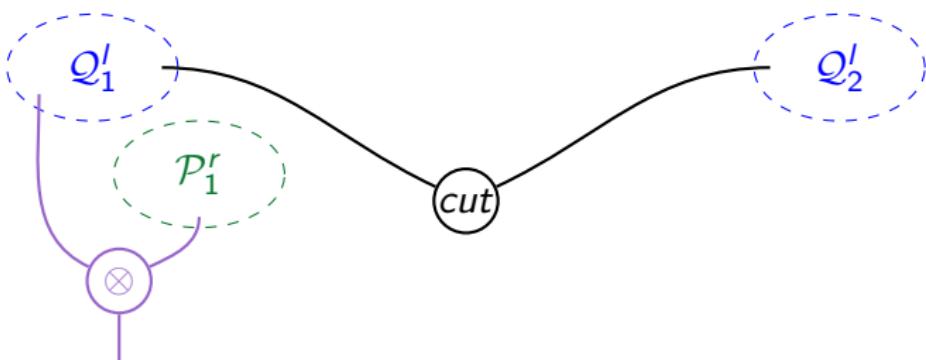
\leftarrow



Splitting \otimes bis

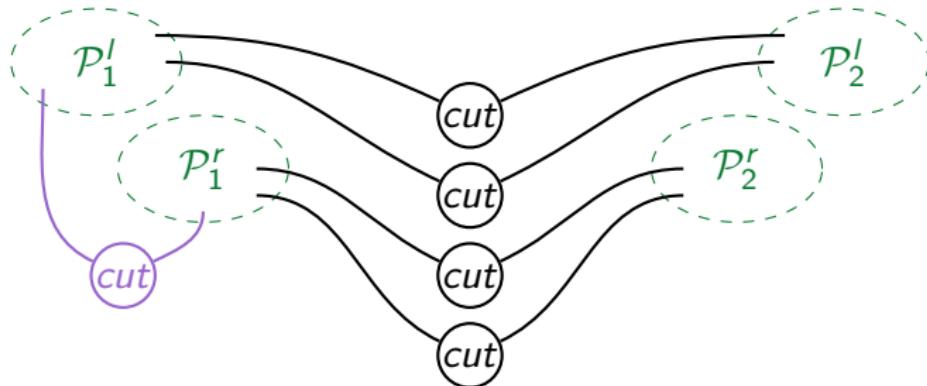


\leftarrow

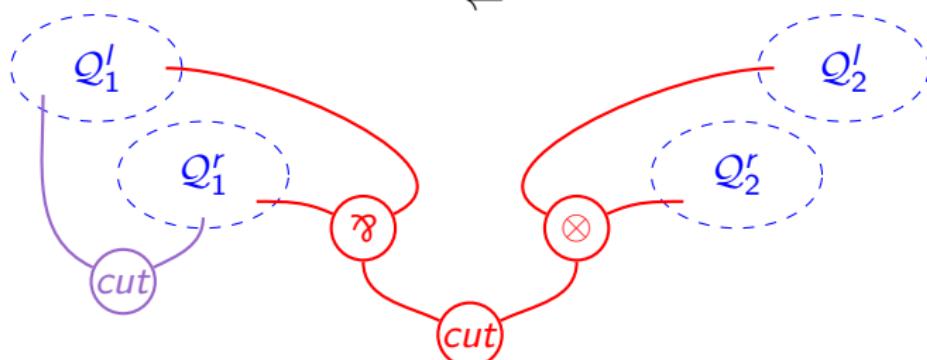


Splitting cut

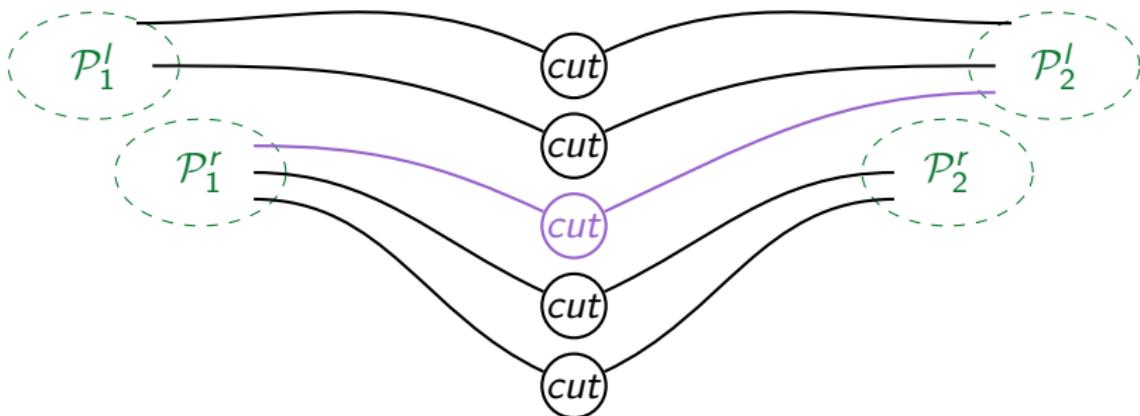
If this *cut* is not between the two sub-graphs: same as the \otimes case.



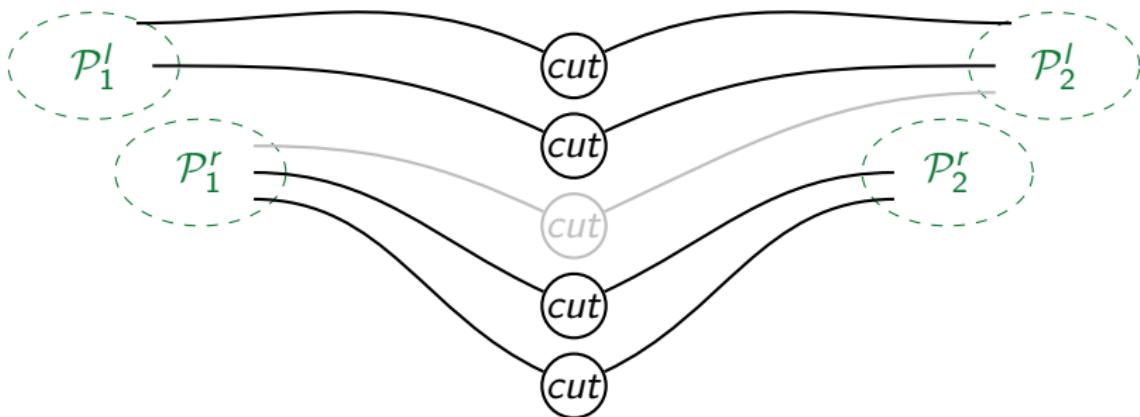
\leftarrow



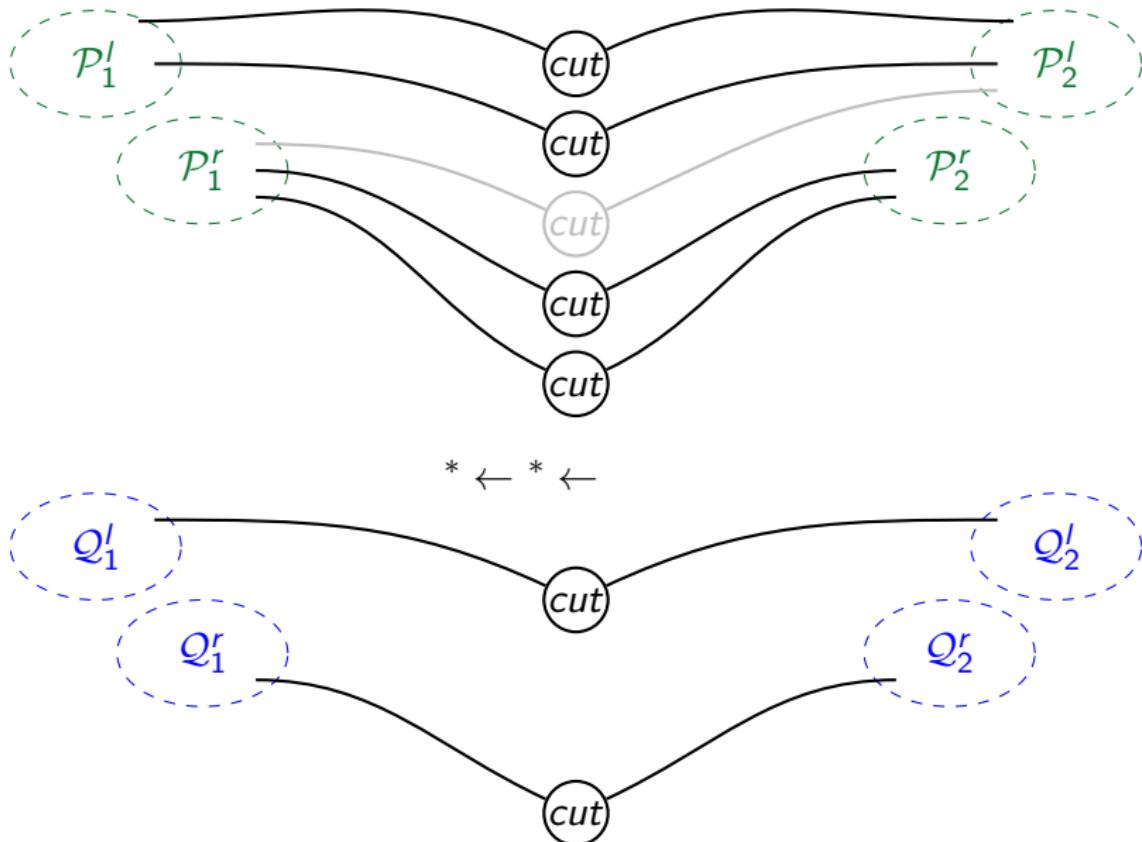
Splitting cut



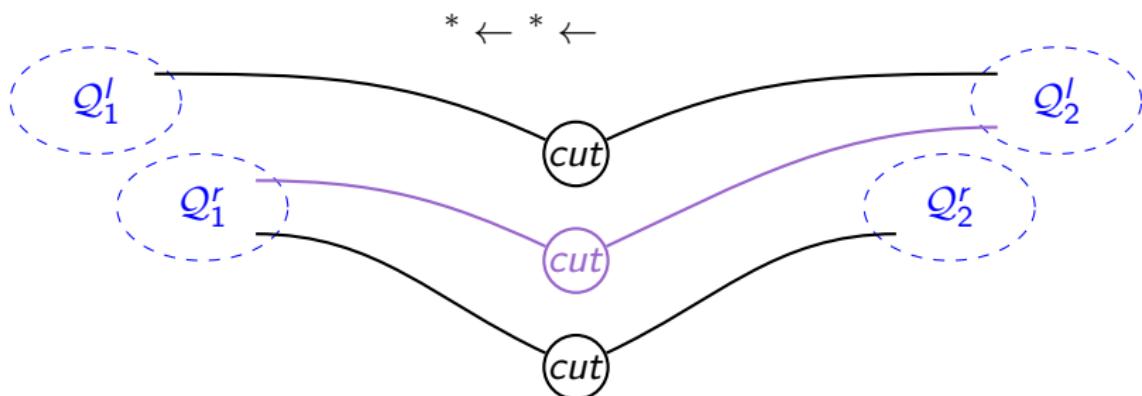
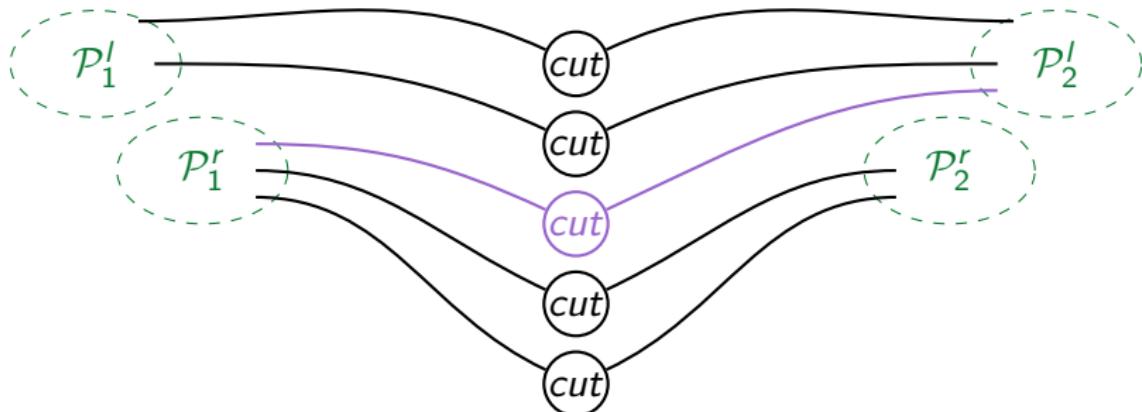
Splitting cut



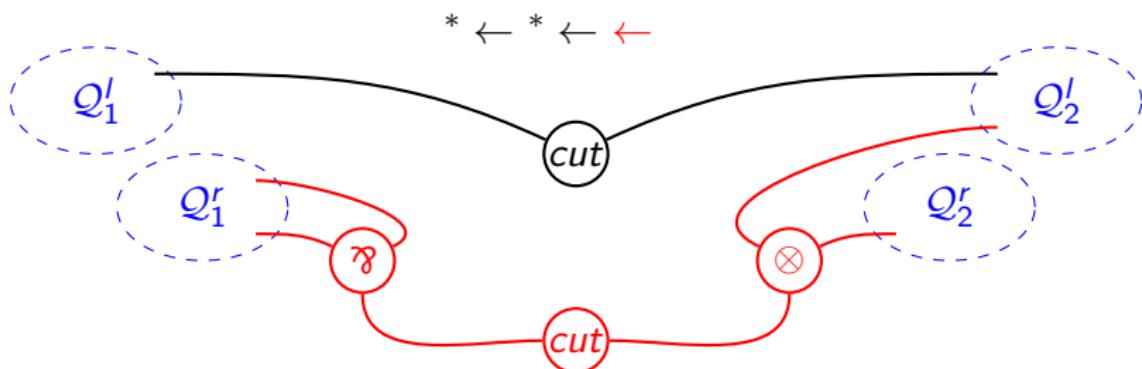
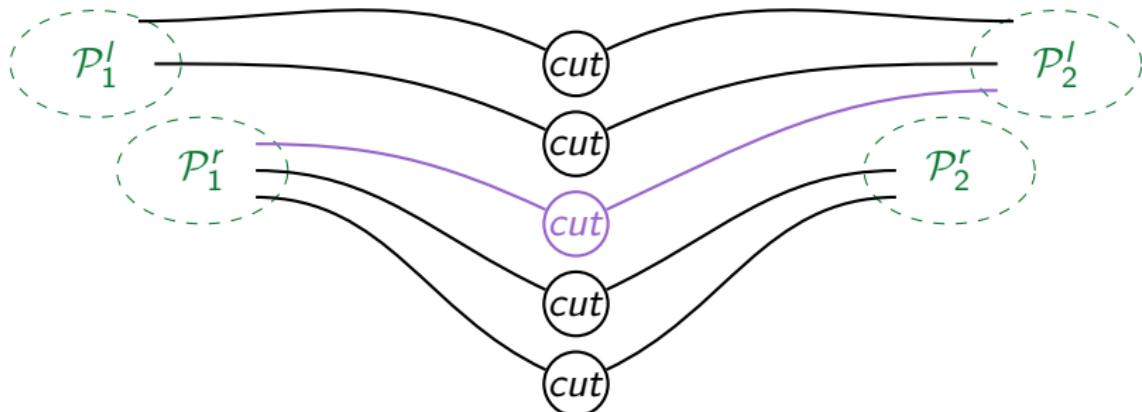
Splitting cut



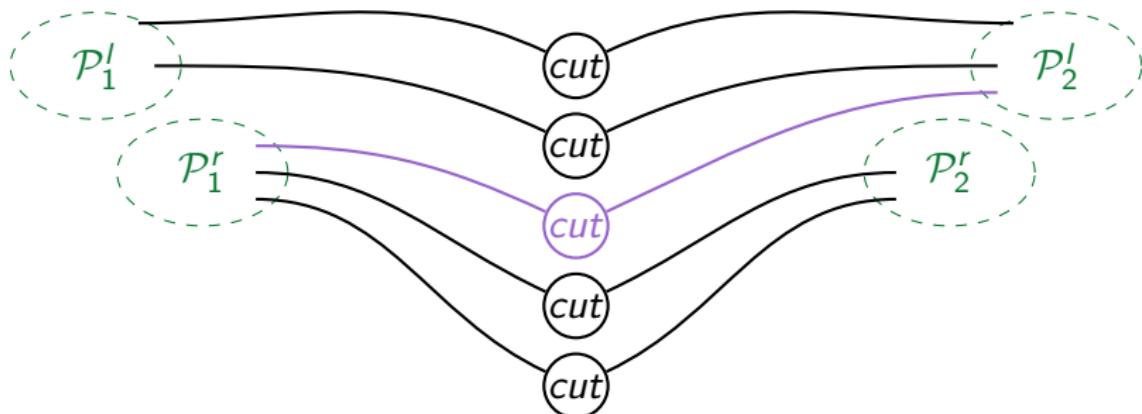
Splitting cut



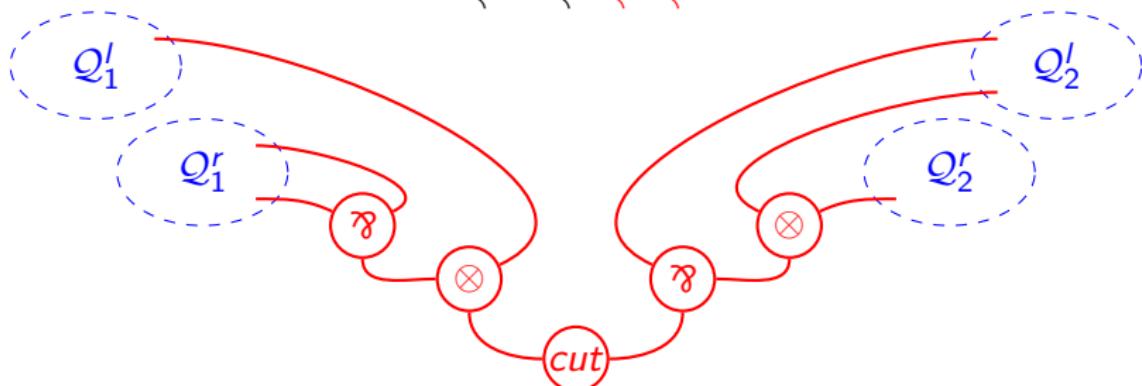
Splitting cut



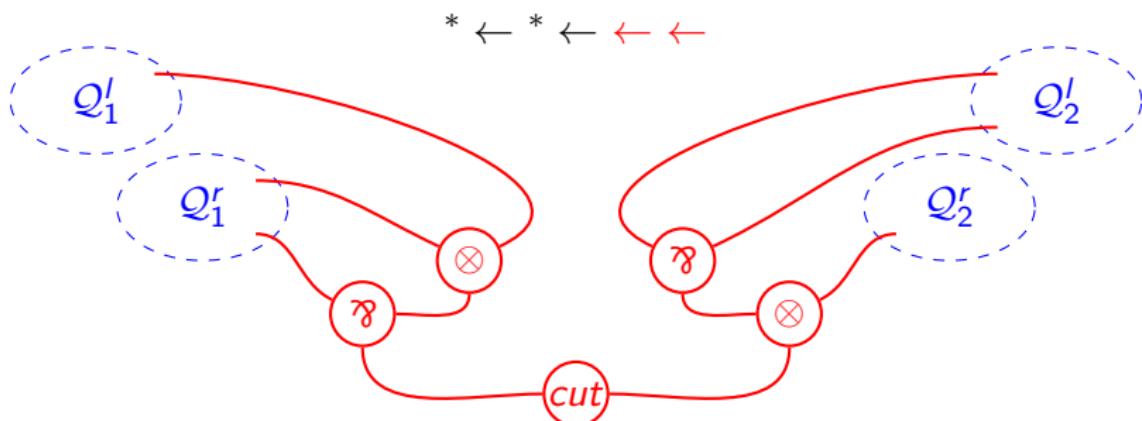
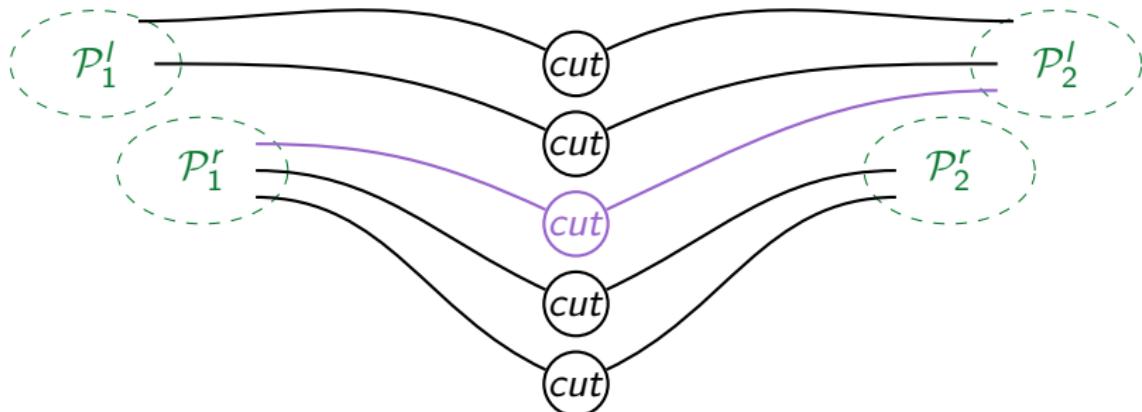
Splitting cut



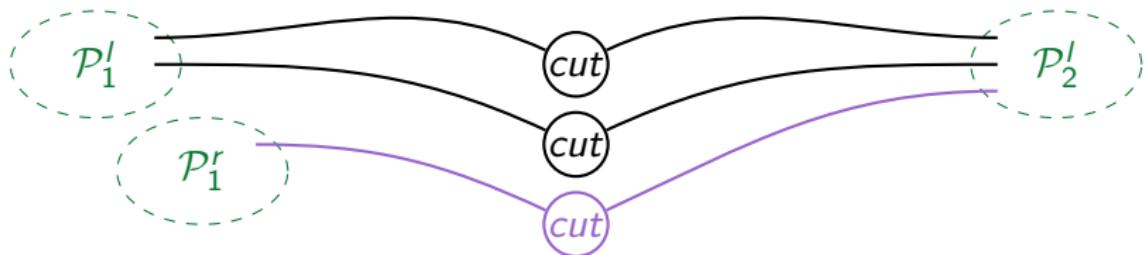
* ← * ← ← ←



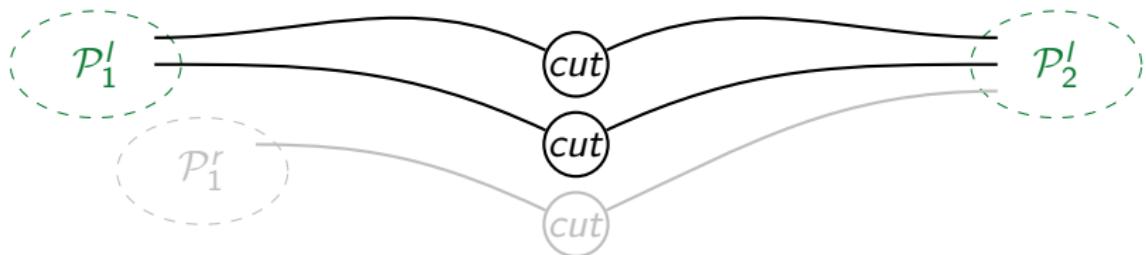
Splitting cut



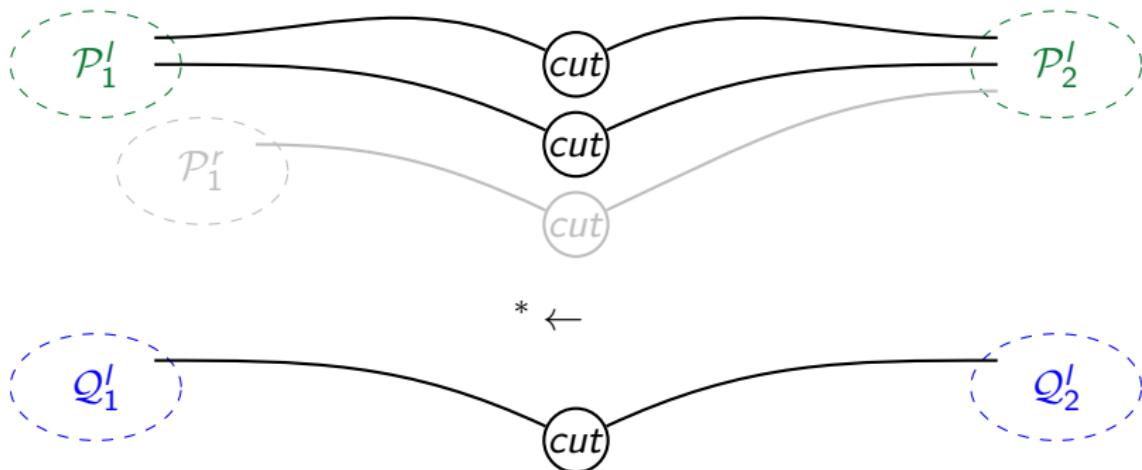
Splitting *cut bis*



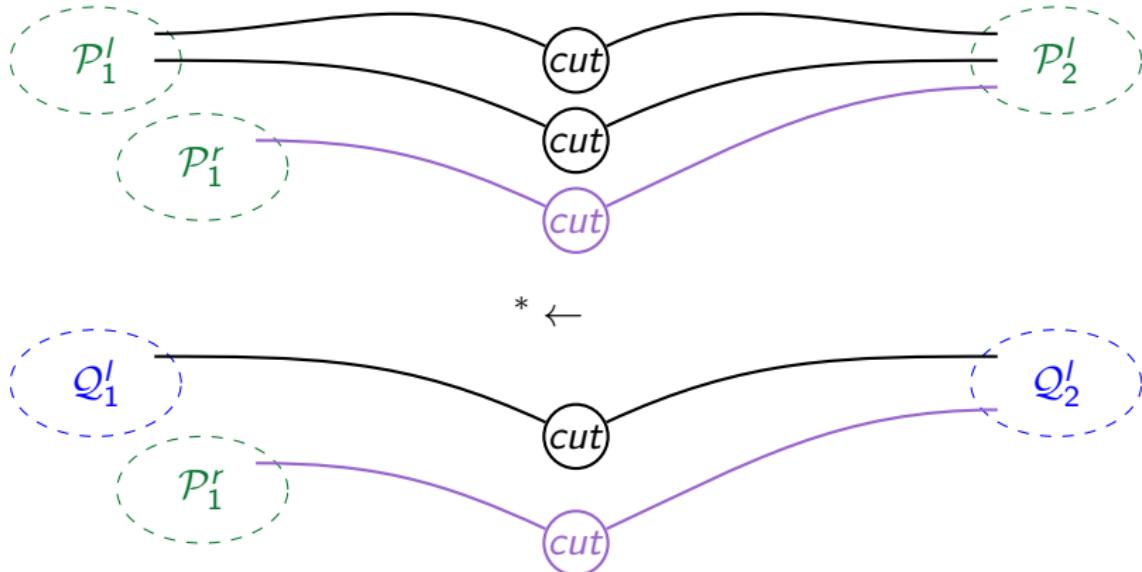
Splitting *cut bis*



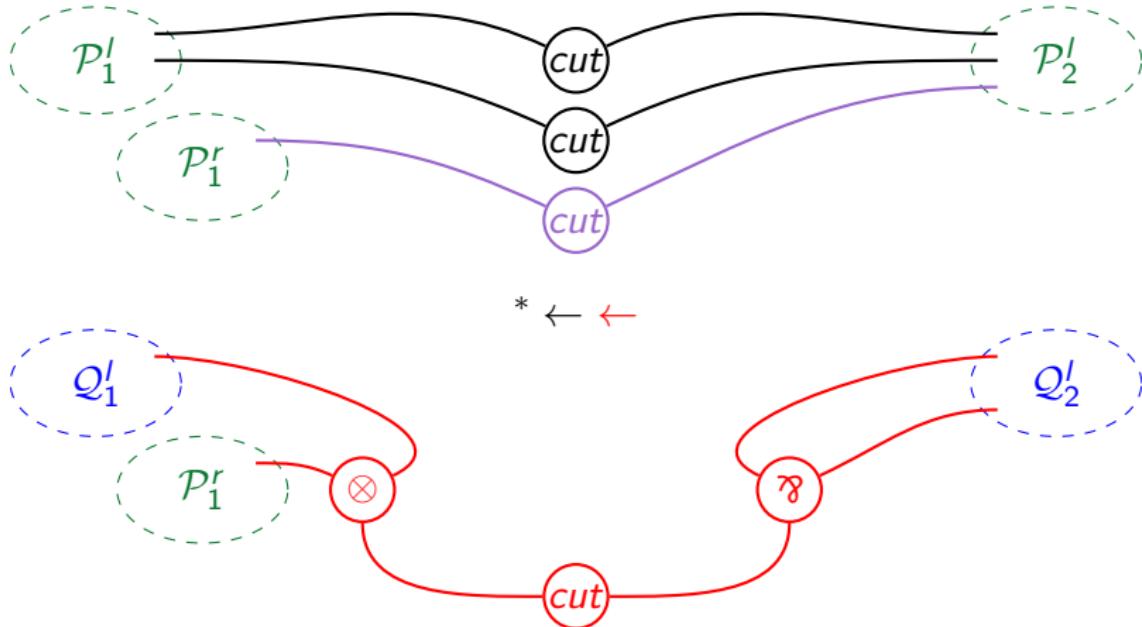
Splitting *cut bis*



Splitting *cut bis*



Splitting *cut bis*



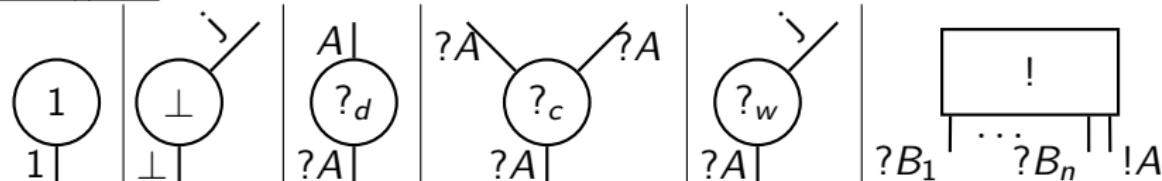
Plan

- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

In Proof-Nets for MLL with units & MELL

Same as for unit-free MLL!

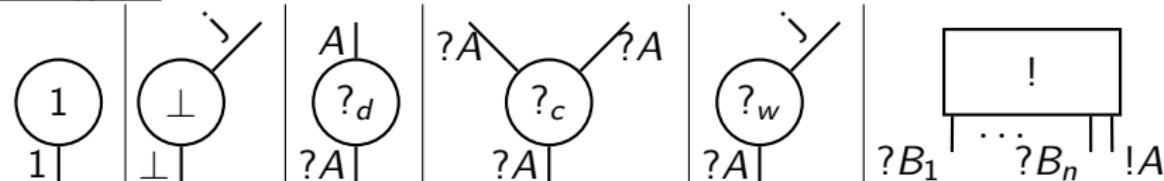
- Proof-nets with jumps for \perp and $?_w$ (or the mix-rules) and boxes for $!$:



In Proof-Nets for MLL with units & MELL

Same as for unit-free MLL!

- Proof-nets with jumps for \perp and $?_w$ (or the mix-rules) and boxes for $!$:



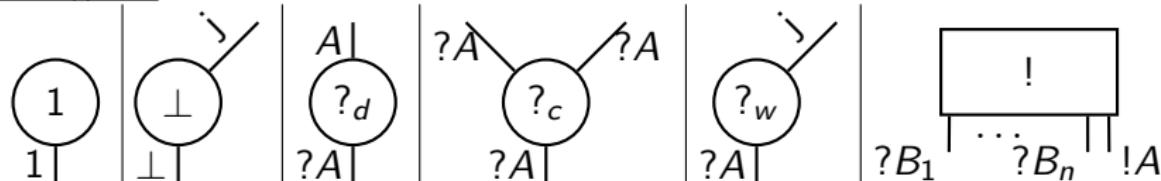
- Proof of the expansion to a unique cut

Same as before: \perp , $?_d$, $?_c$, $?_w$ treated as a \wp , 1 and $!$ irrelevant (like ax)

In Proof-Nets for MLL with units & MELL

Same as for unit-free MLL!

- Proof-nets with jumps for \perp and $?_w$ (or the mix-rules) and boxes for $!$:

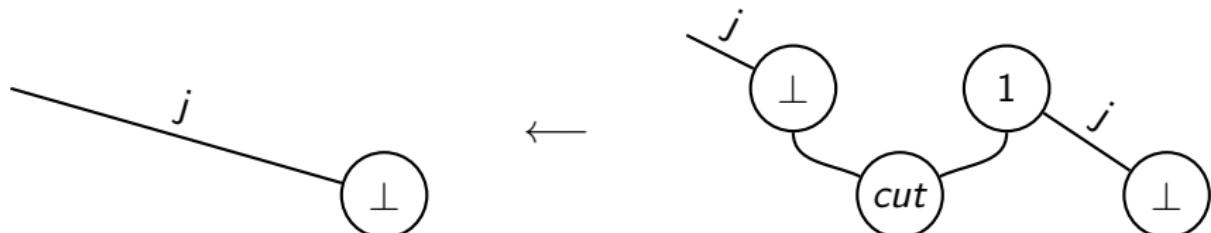


- Proof of the expansion to a unique cut

Same as before: \perp , $?_d$, $?_c$, $?_w$ treated as a \wp , 1 and $!$ irrelevant (like ax)

- Craig's interpolation just needs additional “pre-processing” steps:

- for a jump between A and B :



- for a $!$ -box using both A and B : a little bit more complex, uses an induction (on the net associated to the promotion), then move this cut out of the box (by a $?_d - !$ step followed by a $! - !$ step)

In Sequent Calculus for full LL: what are cut-nets?

No proof-net for full LL → go to sequent calculus

Definition: Derivation corresponding to a cut-net

A proof derivation π of $\vdash \Gamma$ is a **cut-derivation** if, denoting (A_i) the cut-formulas of π , there is a bipartition $\{B_1, B_2\}$ of $\Gamma \cup (\bigcup_i A_i)$ such that:

- all ax-rules are applied either only on sub-formulas of B_1 or only on sub-formulas of B_2 ; and similarly for !- and \top -rules;
 ↪ no ax, !-box or \top using both sides
- given a \perp -rule associated to B_i , the rule just above is not a *cut*-rule and has for principal formula a sub-formula of B_i or is an ax/!/ \top -rule between sub-formulas of B_i ; and similarly for a $?_w$ -rule;
 ↪ no \perp - or $?_w$ -jump going from one side to the other
- given an \exists -rule associated to B_i , its witness has no atoms introduced by a \forall -rule associated to B_{1-i} .
 ↪ no \forall -jump going from one side to the other

In Sequent Calculus for full LL: what are cut-nets?

No proof-net for full LL → go to sequent calculus

Definition: Derivation corresponding to a cut-net

A proof derivation π of $\vdash \Gamma$ is a **cut-derivation** if, denoting (A_i) the cut-formulas of π , there is a bipartition $\{B_1, B_2\}$ of $\Gamma \cup (\bigcup_i A_i)$ such that:

- all ax-rules are applied either only on sub-formulas of B_1 or only on sub-formulas of B_2 ; and similarly for !- and \top -rules;
 ↪ no ax, !-box or \top using both sides
- given a \perp -rule associated to B_i , the rule just above is not a *cut*-rule and has for principal formula a sub-formula of B_i or is an ax/!/ \top -rule between sub-formulas of B_i ; and similarly for a $?_w$ -rule;
 ↪ no \perp - or $?_w$ -jump going from one side to the other
- given an \exists -rule associated to B_i , its witness has no atoms introduced by a \forall -rule associated to B_{1-i} .
 ↪ no \forall -jump going from one side to the other

Deduce Kraig's interpolation as before: introduce a cut when one of the hypothesis of a *cut-derivation* is not respected.

In Sequent Calculus for full LL: proof

Proposition: Expanding to a unique cut

Take a cut-derivation π . There is a sequence of cut-expansion steps yielding a proof with only one cut-rule, at its root.

Similar proof as previously: case study on a last rule

!! & case: $\vdash A \& B, \Gamma_1, \Gamma_2$ where $\Gamma_2 = \emptyset$ and only one branch has a cut:

$$\frac{\frac{\vdash A, \Gamma_1, C_1 \quad \vdash C_1^\perp}{\vdash A, \Gamma_1} \text{ (cut)} \quad \vdash B, \Gamma_1}{\vdash A \& B, \Gamma_1} \text{ (&)}$$

How to move the *cut* at the root?

In Sequent Calculus for full LL: proof

Proposition: Expanding to a unique cut

Take a cut-derivation π . There is a sequence of cut-expansion steps yielding a proof with only one cut-rule, at its root.

Similar proof as previously: case study on a last rule

!! & case: $\vdash A \& B, \Gamma_1, \Gamma_2$ where $\Gamma_2 = \emptyset$ and only one branch has a cut:

$$\frac{\frac{\vdash A, \Gamma_1, C_1 \quad \vdash C_1^\perp}{\vdash A, \Gamma_1} \quad \frac{}{\vdash B, \Gamma_1} \quad (\text{cut})}{\vdash A \& B, \Gamma_1} \quad (\&)$$

How to move the cut at the root?

- Introduce a cut on $\perp - 1$ in the right branch: no more stable by fragment; or
- Eliminate the cut in the left branch: no more only cut-expansion.
~ The additive connectives need the multiplicative units to work well, but the multiplicative connectives do not! (Quantifiers also need them.)

Conclusion

- Simple visual proof of expansion to a unique *cut*, with and without mix
- Yields a proof of Craig's interpolation for unit-free MLL, MLL and MELL
- Useful for denotational semantic when typing an interface
- Adaptable in sequent calculus for full linear logic, but definitions less visual and some problems with & and quantifiers

Conclusion

- Simple visual proof of expansion to a unique *cut*, with and without mix
- Yields a proof of Craig's interpolation for unit-free MLL, MLL and MELL
- Useful for denotational semantic when typing an interface
- Adaptable in sequent calculus for full linear logic, but definitions less visual and some problems with & and quantifiers

Thank you!

References I

- [EFP24] Thomas Ehrhard, Claudia Faggian, and Michele Pagani. *Bayesian Networks and Proof-Nets: a proof-theoretical account of Bayesian Inference*. 2024. DOI: 10.48550/ARXIV.2412.20540. URL: <https://arxiv.org/abs/2305.18945>.
- [FOS25] Guido Fiorillo, Daniel Osorio Valencia, and Alexis Saurin. “On Correctness, Sequentialization and Interpolation”. In: *Ninth International Workshop on Trends in Linear Logic and Applications* 2025. Ed. by Lionel Vaux Auclair. 2025. URL: <https://lipn.univ-paris13.fr/TLLA/2025/abstracts/14-fiorillo-osoriovalencia-saurin.pdf>.

References II

- [Sau25] Alexis Saurin. “Interpolation as Cut-Introduction: On the Computational Content of Craig-Lyndon Interpolation”. In: *International Conference on Formal Structures for Computation and Deduction (FSCD)*. Ed. by Maribel Fernández. Vol. 337. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, July 2025, 32:1–32:21. DOI: 10.4230/LIPIcs.FSCD.2025.32. URL: <https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.FSCD.2025.32>.