Number of Subsets in a Set

Rémi Viné

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Statement In a probability space composed of Ω elements, there are 2^Ω subsets. Each subset is called an event and the probability mapping follows Kolmogorov axioms.

Proof At first, a basic intuition, provided that events can be deemed are independent from each other - which it should be in any well-defined sets of elements - is that there are Ω elements that can be in either case: (i) in the event or (ii) out of the event. Therefore, each element has two options. Independence of events implies that the number of events is the multiplication of each elements' number of possibilities: the number of subsets is then $2 \times 2 \times 2 \times \ ... \times 2 = 2^\Omega$.

More formally, the number of subsets in a set of Ω elements is: the empty set, plus all sets of 1 elements in Ω, plus all sets of 2 elements in Ω, plus all sets of 3 elements in Ω, ..., plus the universe set (composed of all Ω elements. Formally, this can be written as $\binom{\Omega}{0} + \binom{\Omega}{1} + \binom{\Omega}{2} + ... + \binom{\Omega}{\Omega} = \sum_{i=0}^{\Omega} \binom{\Omega}{i}$.[1]

Invoking the Binomial Theorem (see wikipedia page) leads to $(x + y)^\Omega = \sum_{i=0}^{\Omega} \binom{\Omega}{i} x^i y^{\Omega - i}$ and setting $x = 1$ and $y = 1$, the total number of subsets in a set composed of Ω elements is 2^Ω.[2] $\square$

[1] Recall that $\binom{\Omega}{i} = \frac{\Omega!}{i!(\Omega-i)!}$ is the binominal coefficient, or the combination of i element in a set composed of Ω, irrespective of the order within the subset of i elements.

[2] For another way to visualize the problem, one might rather look at Pascal's triangle, which capture the same information than the Binomial Theorem (consult this wikipedia page).

Examples To illustrate this result, take the simple case of a unique toss of a coin. Defining the probability space, the universe is: $\Omega = \{H, T\}$, where H stands for $Heads$, and T for $Tails$. Events are the following: $\{\varnothing\}, \{H\}, \{T\}, \{\Omega\}$. Hence the number of events for a probability space composed of two elements is $2^2 = 4$.

Similarly, for a Rock (R), Paper (P), Scissors (S) game, one can define the universe as $\Omega = \{R, P, S\}$. Events are: $\{\varnothing\}, \{R\}, \{P\}, \{S\}, \{R, P\}, \{R, S\}, \{P, S\}, \{\Omega\}$. Hence, this probability space is composed of $2^3 = 8$ elements. For these two examples, one can easily see how quickly this number inflates, typically: $2^{10} = 1024$ and $2^{20} = 1048576$. For a probability space with a very large number of elements, the number of elements becomes gigantic.