# PRML Note
# C08 Graphical Models

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- The properties of *probabilistic graphical models*

  1. Provide a simple way to visualize the structure of a probabilistic model and can be used to design and motivate new models.
  2. Insights into the properties of the model, including conditional independence properties, can be obtained by inspection of the graph.
  3. Complex computations, required to perform inference and learning in spphisticated models, can be expressed in terms of graphical manipulations, in which underlying mathematical expressions are carried along implicitly.

- The graph comprises vertices connected by edges, where the vertices represents a random variable and the edges express probabilistic relationships between the variables.

- The categories of the graphical models

  1. Bayesian networks: directed graphical models.
  2. Markov random fields: undirected graphical models.
  3. factor graph: be used to do the inference.

- Graph terminology: a graph $G = (V, E)$ can be represented by adjacency matrix, in which we write $\mathbf{L}(s,t) = 1$ if $(s,t) \in E$. We say the graph is undirected if $\mathbf{L} = \mathbf{L}^T$, otherwise it is directed. We usually assume $\mathbf{L}(s,s) = 0$ which means there are no self loops.

  1. *Parent*: For a directed graph, the parents of a node is the set of all nodes that feed into it: $Pa(s) \triangleq \{t : \mathbf{L}(t,s) = 1\}$.
  2. *Child*: For a directed graph, the children of a node is the set of all nodes that feed out of it: $Ch(s) \triangleq \{t : \mathbf{L}(s,t) = 1\}$.
  3. *Family*: For a directed graph, the family of a node is the node and its parents, $Fam(s) = \{s\} \cup Pas$.
  4. *Root*: For a directed graph, a root is a node with no parents.
  5. *Leaf*: For a directed graph, a leaf is a node with no children.

6. *Ancestors*: For a directed graph, the ancestors are the parents, grand- parents, etc of a node.
7. *Descendants*: For a directed graph, the descendants are the children, grand-children, etc of a node.
8. *Neighbors*: For any graph, we define the neighbors of a node as the set of all immediately connected nodes.
9. *Degree*: The degree of a node is the number of neighbors. For directed graphs, we speak of the in-degree and out-degree, which count the number of parents and children.
10. *Cycle or loop*: For any graph, we define a cycle or loop to be a series of nodes such that we can get back to where we started.
11. *DAG*: A directed acyclic graph is a directed graph with no directed cycles.
12. *Topological ordering*: For a DAG, a topological ordering or total ordering is a numbering of the noders such that parents have lower numbers than their children.
13. *Path or trail*: A path or trail $s \rightsquigarrow t$ is a series of directed edges leading from $s$ to $t$.
14. *Tree*: An undirected tree is an undirected graph with no cycles. A directed tree is a DAG in which there are no directed cycles. If we allow a node to have multiple parents, we call it a polytree.
15. *Forest*: A set of trees.
16. *Subgraph*: A subgraph $G_A$ is the graph created by using the nodes in A and their corresponding edges, $G_A = (V_A, E_A)$.
17. *Clique*: For an undirected graph, a clique is a set of nodes that are all neighbors of each other.

# 1 Bayesian Networks

- For a graph with $K$ nodes, the joint distribution is given by

  $$p(\boldsymbol{x}) = \prod_{k=1}^{K} p(x_k | Pa(x_k))$$

  This key equation expresses the *factorization* properties of the joint distribution for a directed graphical model.

- Three cases of 3-node graph, which is shown as the figure 1.
  - For figure 1a, if node $c$ is observed, node $a$ is independent with node $b$.

    $$p(a, b|c) = p(a|c)p(b|c)$$

  - For figure 1b, if node $c$ is observed, node $a$ is independent with node $b$.

    $$p(a, b|c) = p(a|c)p(b|c)$$

  - For figure 1c, if node $c$ isn't observed, node $a$ is independent with node $b$. This case is also called the *v-structure*.

    $$p(a, b|c) \neq p(a|c)p(b|c)$$

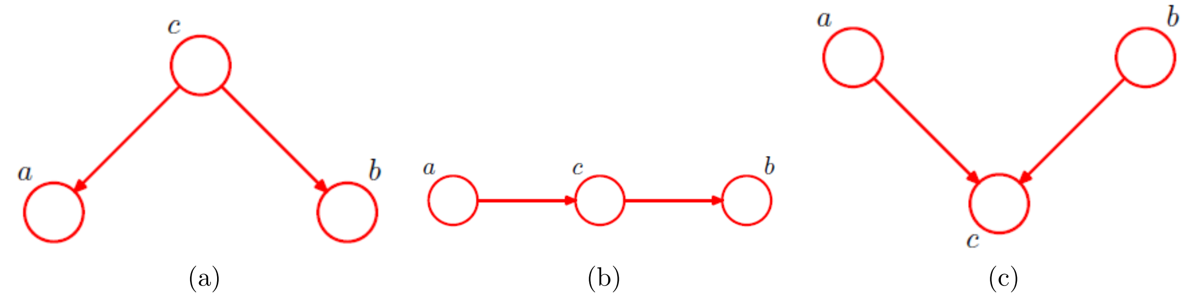


Figure 1: Three cases of 3-node graphs. For the figure 1a, the node $c$ is said to be tail-to-tail with respect to the path from node $a$ to $b$. For the figure 1b, the node $c$ is said to be head-to-tail with respect to the path from node $a$ to $b$. For the figure 1c, the node $c$ is said to be head-to-head with respect to the path from node $a$ to $b$.

## 2 Conditional Independence

## 3 Markov Random Fields

## 4 Inference in Graphical Models